On Estimating the RMS Delay Spread from the Frequency-Domain Level Crossing Rate

Klaus Witrisal, *Student Member, IEEE*

***Abstract*—It is shown that the level crossing rate (LCR) of a Rayleigh distributed stochastic process is proportional to the second centralized moment of the normalized power spectrum of the underlying complex Gaussian process. This proportionality factor is independent of the power spectrum. The relation can be applied for estimating the rms delay spread of time-dispersive (= frequency-selective) radio channels from swept-frequency power measurements, where the rms delay spread is proportional to the LCR in the frequency-domain, independent of the channel impulse response.**

***Index Terms*—Channel measurement, level crossing rate, Rayleigh channels, rms delay spread.**

I. Introduction

In previous works by the author [1], [2], it has been shown that a strict proportional relationship exists between the level crossing rate of a frequency-selective radio channel in the frequency domain $(\mathrm{LCR_f},\, N_R(r'))$, and the channel's root mean square (rms) delay spread τ_{rms}. This relation is written as

$$N_R(r') = \tau_{\mathrm{rms}} f(r',\, K,\, u). \tag{1}$$

The proportionality factor $f(r', K, u)$ is a function of the threshold level at which the $\mathrm{LCR_f}$ is observed (r'), the Ricean K-factor of the channel (K), and channel parameters (expressed by u). It was suggested to use this relation for estimating the channel's τ_{rms} using swept-frequency power measurements, from which the $\mathrm{LCR_f}$ can be determined [1]–[3].

The above relationship was proven upon a second order, wide sense stationary (WSS) stochastic model of the frequency selective radio channel [2]. A non-zero-mean complex Gaussian process is underlying this model; therefore, the formula can be applied to channels having a Ricean fading distribution. Rayleigh channels are a special case among these, having $K = 0$.

The Ricean K-factor can be determined prior to applying (1). However, the impact of the current channel on $f(r', K, u)$ (expressed by u) remains an uncertainty, and might thus be a source of systematic estimation errors.

In this paper it is shown for Rayleigh fading channels that *the proportionality factor* between the $\mathrm{LCR_f}$ and τ_{rms} *is independent of the channel impulse response* (IR). This new finding strongly supports the claim that the $\mathrm{LCR_f}$ is a valuable means for estimating τ_{rms} in a simple way.

Analytical results from previous studies have shown that such an independent factor does not exist for the more general Ricean case. However, the factor has been compared for widely varying channel models, suggesting that the model's impact is very small and can thus be neglected [2]. For instance, the difference between $f(r', K, u)$ for a rectangular delay power profile and an exponentially decaying one is less than 4% at any given K-factor.

The result of the present paper can be generalized to any band-limited, Rayleigh distributed, WSS stochastic process. It is proven that there exists a fixed function of r', relating the level crossing rate to the second centralized moment of the normalized power spectrum (or periodogram) of the (zero-mean) complex Gaussian process underlying the Rayleigh process. The paper focuses on the application of the relation to channel measurements, without loss of generality.

The paper is organized as follows. The mathematical model of the radio channel is introduced in Section II, followed by the outline of the proof in SectionIII. Details are found in the two Appendixes. The paper is concluded by a discussion of the approximations introduced in the proof (Section IV).

II. Definitions and Mathematical Modeling

The time-dispersive (= frequency-selective) radio channel is described by its complex, lowpass equivalent IR

$$h(\tau) = \sum_{i=0}^{n-1} \beta_i \delta(\tau - \tau_i) \tag{2}$$

where $\{\beta_i\}$ are the complex-valued ray amplitudes and $\{\tau_i\}$ are the rays' respective delay times. The time-variability of a physical radio channel is neglected in this equation. The frequency-selectivity of the above-described channel is seen from the frequency dependency of its transfer function (TF)

$$H(f) = \sum_{i=0}^{n-1} \beta_i e^{-j2\pi f \tau_i}. \tag{3}$$

The similarity of this equation to Rice's sum of sinusoids suggests that $H(f)$ could be the underlying complex Gaussian process of any Rayleigh process $R(f) = |H(f)|$ (compare [4, (3.7-2)–(3.7-4)], [5]).

The rms delay spread is considered as the most important single parameter for defining the time-extent of a time-dispersive radio channel. It is defined from the IR as

$$\tau_{\mathrm{rms}} = \sqrt{\overline{\tau^2} - \overline{\tau}^2} \tag{4}$$

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The author is with the Department of Electrical Engineering, Delft University of Technology, NL-2628 CD Delft, The Netherlands (e-mail: K.Witrisal@ITS.TUDelft.NL).

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where $\overline{\tau^m} = (\sum_{i=0}^{n-1} \tau^m |\beta_i|^2)/(\sum_{i=0}^{n-1} |\beta_i|^2)$, $m = \{1, 2\}$ and $\tau_{\rm rms}$ is seen to be the second centralized moment of the normalized power delay profile. For simplified notation we introduce $p_i = |\beta_i|^2/(\sum_{i=0}^{n-1} |\beta_i|^2)$, yielding $\overline{\tau^m} = \sum_{i=0}^{n-1} \tau^m p_i$.

III. Outline of the Proof

The proof is based on the calculation of the level crossing rate for a discrete-time (or discrete-frequency—as required in our specific case) stochastic process. The probability of a level crossing between adjacent samples is the probability that the current sample's magnitude r_n is larger than a specified threshold value, $r_n \geq r$, while the preceding sample r_{n-1} was smaller, $r_{n-1} < r$. The $\mathrm{LCR_f}$ is thus written as

$$N_R(r) = P\,(R_n \geq r,\, R_{n-1} < r)\,/f_s \tag{5}$$

where f_s [Hz] is the sampling *interval* in the frequency-domain, and R_n and R_{n-1} denote correlated random variables. Knowledge of the bivariate cumulative distribution function (CDF) of $\{R_n,\, R_{n-1}\}$, denoted $F_{R_n,\, R_{n-1}}(r_1,\, r_2)$, is required to obtain the $\mathrm{LCR_f}$.

$$\begin{aligned} P\,(R_n \geq r, R_{n-1} < r) &= P\,(R_{n-1} < r) - P\,(R_n < r, R_{n-1} < r) \\ &= F_{R_{n-1}}(r) - F_{R_n, R_{n-1}}(r, r) \end{aligned} \tag{6}$$

where $F_{R_{n-1}}(r) = F_{R_n,\, R_{n-1}}(\infty,\, r)$ is the CDF of (any) sample r_{n-1}. Using an expression of the bivariate Rayleigh CDF given in [6, eq. (10-10-3)], the probability (6) becomes

$$\begin{aligned} &P\,(R_n \geq r,\, R_{n-1} < r) \\ &= e^{-r'^2} \times \left[Q\left(\sqrt{\frac{2}{1-\rho_c}}r',\, \sqrt{\frac{2\rho_c}{1-\rho_c}}r'\right) \right. \\ &\qquad \left. - Q\left(\sqrt{\frac{2\rho_c}{1-\rho_c}}r',\, \sqrt{\frac{2}{1-\rho_c}}r'\right)\right] \end{aligned} \tag{7}$$

where r' is the normalized threshold level $r' = r/\sqrt{2\psi_0}$; ρ_c is the correlation coefficient of the squared magnitudes $0 \leq \rho_c < 1$ defined as $\rho_c = \mathrm{cov}(R_n^2,\, R_{n-1}^2)/\sqrt{\mathrm{var}(R_n^2)\mathrm{var}(R_{n-1}^2)}$; and $Q(a,\, b)$ is the Marcum's Q-function (see [, (2-1-123)]). ρ_c is related to the auto-covariance coefficients of the underlying complex Gaussian process Z_n (where $|Z_n| = R_n$) as $\rho_c = |\psi_1|^2/\psi_0^2$, where $\psi_m = (1/2)E\{Z_n Z_{n+m}^*\}$.

It is seen that the crossing probability (7) is solely determined by the correlation coefficient ρ_c and by r'. Calculating ρ_c based on the stochastic or deterministic model of a Rayleigh distributed process [e.g., for the channel description (2)] thus leads to the level crossing rate.

It is shown in Appendix A that using (2) and (4), the coefficient ρ_c can be approximated by (Approx. A)

$$\rho_c \cong 1 - 4\pi^2 f_s^2 \tau_{\rm rms}^2 \tag{8}$$

if the product $f_s\tau_{\rm rms} \ll 1/2\pi$, which essentially means that the sampling theorem must be honored. This result is valid regardless of the current channel model, i.e., regardless of the delay-magnitude structure of the channel IR. Therefore, the $\mathrm{LCR_f}$ is independent of the channel model.

Equation (8) allows an interesting observation, suggesting that the $\mathrm{LCR_f}$ is proportional to $\tau_{\rm rms}$. Provided that the sampling theorem is not violated (i.e., strictly speaking, for $f_s \to 0$), the $\mathrm{LCR_f}$ must be independent of the sampling interval f_s. Therefore, the probability equation (7) must be proportional to f_s, to yield a constant $\mathrm{LCR_f}$ with (5). This implies that (7) is also proportional to $\tau_{\rm rms}$, because it is seen from (8) that f_s and $\tau_{\rm rms}$ have the same influence on ρ_c. Thus the $\mathrm{LCR_f}$ is proportional to $\tau_{\rm rms}$.

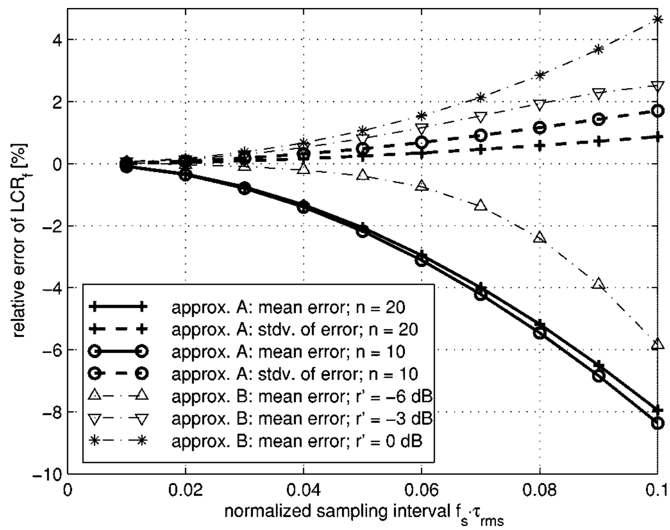

Fig. 1. Computational and simulation results depicting the errors introduced by the approximations.

In Appendix B, it is shown mathematically that (7) can be approximated as (Approx. B)

$$P(R_n \geq r,\, R_{n-1} < r) \cong \frac{r' e^{-r'^2}}{\sqrt{\pi}} \sqrt{1-\rho_c} \tag{9}$$

where the approximation is exact in the limit $\rho_c \to 1$. This condition is fulfilled strictly for $f_s \to 0$, i.e., for an infinitely small sampling interval, and approximately, if the sampling theorem holds. With (8) and (5), the $\mathrm{LCR_f}$ for Rayleigh fading channels becomes

$$N_R(r') \cong 2\sqrt{\pi} r' e^{-r'^2} \tau_{\rm rms}. \tag{10}$$

It should be noted that (10) is identical to the result of the continuous-frequency analysis presented in [1], [2], for $K = 0$.

IV. Discussion and Computational Results

Fig. 1 depicts computational and simulation results supporting the discussion of the approximations introduced.

Approx. A, leading to ρ_c [see. (8)], is evaluated by simulating sets of impulse responses[1] and comparing their exact ρ_c-values as obtained from (11) to the approximation. Using (9) and (5), these values were transformed to $\mathrm{LCR_f}$-values to allow for a more practical comparison. Relative errors are shown. The errors' small standard deviations indicate that ρ_c is largely independent of the structure of the IR. All errors increase as the sampling interval is increased. Note that to honor the sampling theorem, $f_s < (1/(2\tau_{\max})) \cong (1/(20\tau_{\rm rms}))$ should be given. In

[1]Each IR consists of n rays with unit-variance, Rayleigh distributed magnitudes, and arrival times being uniformly distributed within a unit time interval. The IR's were then normalized with respect to power and $\tau_{\rm rms}$.

this range of f_s, the systematic error due to approx. A is less than $\sim$2%.

The systematic error of approx. B [eq. (9)] compared to (7) is less than $\sim$1%, for the same f_s and for thresholds $r' = \{-6, -3, 0\}$ dB. Larger negative errors are evident for smaller r', since the sampling interval gets more impact as the fades get deeper and narrower.

It is concluded that the final expression (10) is a practical equation for deriving the LCR of Rayleigh processes.

APPENDIX A

The correlation coefficient $\rho_c = |\psi_1|^2/\psi_0^2$ is derived from the autocorrelation function of the underlying complex Gaussian process, $\psi_m = (1/2)E\{Z_n Z_{n+m}^*\}$. In the case of the frequency selective radio channel, this function is called the spaced-frequency correlation function, being

$$\psi_m = \frac{1}{2}E\{H(f)H^*(f+mf_s)\} = \frac{1}{2}\sum_{i=0}^{n-1} p_i e^{j2\pi\tau_i m f_s}. \quad (11)$$

To calculate ψ_m for one particular channel realization, the expectation operator should be considered as the average over the frequency f. But this equation also holds for a set of channels with common stochastic properties, or for any Rayleigh process, where the right-hand side of the equation is the Fourier transform of its (normalized) power spectrum (cf., [4, eq. (3.7-11)], [5]). The squared magnitude of ψ_m is

$$\begin{aligned} |\psi_m|^2 = \psi_m\psi_m^* &= \frac{1}{4}\left(\sum_{i=0}^{n-1} p_i e^{j2\pi\tau_i m f_s}\right)\left(\sum_{k=0}^{n-1} p_k e^{-j2\pi\tau_k m f_s}\right) \\ &= \frac{1}{4}\left[\sum_{i=0}^{n-1} p_i^2 + 2\sum_{i=0}^{n-1}\sum_{k=i+1}^{n-1} p_i p_k \cos 2\pi(\tau_i - \tau_k) m f_s\right]. \end{aligned} \quad (12)$$

For $f_s \to 0$ (for $|\psi_1|^2$) and $f_s = 0$ (for ψ_0^2), the cos-term can be replaced by $\cos x = 1 - x^2/2$ and $\cos 0 = 1$ yielding

$$\begin{aligned} |\psi_1|^2 &\cong \frac{1}{4}\left[\sum_{i=0}^{n-1} p_i^2 + 2\sum_{i=0}^{n-1}\sum_{k=i+1}^{n-1} p_i p_k \left(1 - 2\pi^2(\tau_i - \tau_k)^2 f_s^2\right)\right] \\ &= \frac{1}{4}\left[4\psi_0^2 - 4\pi^2 f_s^2 \sum_{i=0}^{n-1}\sum_{k=i+1}^{n-1} p_i p_k (\tau_i - \tau_k)^2\right]. \end{aligned} \quad (13)$$

Noting that, due to the normalization of the $\{p_i\}$, $\psi_0^2 = 1/4$, we obtain $\rho_c \cong 1 - 4\pi^2 f_s^2 \sum_{i=0}^{n-1}\sum_{k=i+1}^{n-1} p_i p_k (\tau_i - \tau_k)^2$. It remains to be shown that

$$\tau_{\text{rms}}^2 = \sum_{i=0}^{n-1}\sum_{k=i+1}^{n-1} p_i p_k (\tau_i - \tau_k)^2 \quad (14)$$

in order to get (8). The proof of this equation is a bit cumbersome, but trivial.

APPENDIX B

In order to find an asymptotic expression for (7) in the limit $\rho_c \to 1$, we use the relation of the Marcum's Q-function to the CDF of a Ricean random variable (see [7, (2-1-142)])

$$1 - Q(a, b) = \int_0^b u e^{-(u^2+a^2)/2} I_0(au)\, du. \quad (15)$$

When au becomes large, $I_0(au)$ may be replaced by its asymptotic expression, as suggested in [4, eq. (3.10–19)], [5]. This yields the following approximation for the Ricean CDF, being valid for $ab \gg 1$ and $a \gg |b - a|$ (see [4], [5]), which is fulfilled for $\rho_c \to 1$.

$$\begin{aligned} 1 - Q(a, b) &\cong \frac{1}{2} + \frac{1}{2}\,\text{erf}\frac{b-a}{\sqrt{2}} \\ &\quad - \frac{1}{\sqrt{8\pi}a} e^{-(b-a)^2/2}\left[1 - \frac{b-a}{4a} + \frac{1+(b-a)^2}{8a^2}\right]. \end{aligned} \quad (16)$$

Replacing the error function by the first terms of its power series expansion, the most important terms of (16) can be identified,

$$\text{erf}\frac{b-a}{\sqrt{2}} = \frac{2}{\sqrt{\pi}}\left[\frac{b-a}{\sqrt{2}} - \frac{(b-a)^3}{6\sqrt{2}} + \cdots\right]. \quad (17)$$

For the two Q-functions in (7), $b - a$ and a are

$$\begin{aligned} (b-a)_{I,II} &= \mp(1 - \sqrt{\rho_c})\sqrt{\frac{2\rho_c}{1-\rho_c}}r' \cong \mp\sqrt{\frac{1-\rho_c}{2}}r' \ll 1 \\ a_{I,II} &= \begin{Bmatrix} 1 \\ \sqrt{\rho_c} \end{Bmatrix}\sqrt{\frac{2}{1-\rho_c}}r' \cong \sqrt{\frac{2}{1-\rho_c}}r' \gg 1 \end{aligned} \quad (18)$$

respectively, where the approximations and the inequities hold for $\rho_c \to 1$. Keeping the most significant terms yields

$$\begin{aligned} Q(a, b)_{I,II} &\cong \frac{1}{2} - \frac{1}{\sqrt{2\pi}}(b-a)_{I,II} + \frac{1}{\sqrt{8\pi}a_{I,II}} \\ &\cong \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1-\rho_c}{\pi}}r' + \sqrt{\frac{1-\rho_c}{\pi}}\frac{1}{4r'}. \end{aligned} \quad (19)$$

Plugging this result into (7), the approximation (9) is obtained.

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