The numerical calculation of shear properties of members

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1 Introduction

The calculation of cross-section A and moments of inertia I_y and I_z of a prismatic member is usually quite simple. The calculation of the shear properties such as the torsional rigidity GI_x, the shear force areas k_yA and k_zA and possible eccentricities e_y and e_z of shear centre C may be much more difficult. A calculation using the finite element method can solve these problems.

Based upon the assumption of undisturbed warping, a potential equation for the axial displacement u_x can be formulated [4]. Typical of the torsion problem is the boundary condition which is dependent on the shape of the cross-section.

In addition to the assumption of undisturbed warping we assume a linear distribution of strain ε_{xx} over the cross-section. Considering the shear forces another potential equation in u_x can be formulated. From the solution we can calculate the shear force areas k_yA and k_zA relating the shear forces Q_y and Q_z and the averaged shear deformations $\bar{\Psi}_y$ and $\bar{\Psi}_z$ as follows:

$$Q_y = k_y GA\bar{\Psi}_y$$

$$Q_z = k_z GA\bar{\Psi}_z$$

Possible eccentricities e_y and e_z of the shear force centre C can be obtained from the same calculation.

The numerical elaboration of the differential equation shows a straight-forward method for the calculation of the shear properties of prismatic members without limitations as to irregular shapes or holes, symmetry conditions or inhomogenuities of the cross-section.

2 The torsion problem

Assume the principal axes of inertia y and z and the eccentricities e_y and e_z of the shear centre C. The torsion causes an axial displacement u_x in the cross-section and a rigid body rotation ϕ_x about the shear centre C. (Fig. 1).

Assuming an undisturbed warping we obtain the following deformations

$$\begin{aligned} u_x &= u_x(y, z) \\ u_y &= -\theta_0 x(z - e_z) \\ u_z &= \theta_0 x(y - e_y) \end{aligned} \tag{1}$$

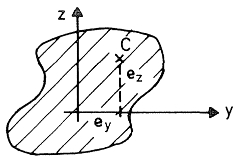

Fig. 1.

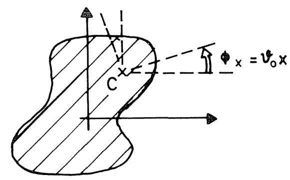

Fig. 2. Rotation about C.

where $\theta_0 x$ is the angle of rotation of the cross-section at a distance x from the origin (Fig. 2).

The corresponding shear deformations are

$$\gamma_{xy} = u_{x,y} - \theta_0 (z - e_z) \tag{2}$$

$$\gamma_{xz} = u_{x,z} + \theta_0 (y - e_y)$$

Substitution of this into the equilibrium condition for stresses in the X-direction

$$\sigma_{yx,y} + \sigma_{zx,z} = 0$$

and using the shear modulus G in Hookes law yields the potential equation

$$G u_{x,yy} + G u_{x,zz} = 0 \tag{3}$$

The boundary conditions require zero shear stresses, or

$$p_x - \sigma_{xn} = 0 \tag{4}$$

where surface load p_x equals zero. Substitution of the constitutive equations gives for (4)

$$-G(u_{x,n} + u_{n,x}) = 0$$

Following the shape of the boundary we can write for u_n

$$u_n = u_y \cos\alpha + u_z \sin\alpha$$

Substitution of (1) for the displacements u_y and u_z boundary condition (4) yields

$$-G u_{x,n} + G\theta_0 \{(z - e_z)\cos\alpha - (y - e_y)\sin\alpha\} = 0 \tag{5}$$

Differential equation (3) with boundary condition (5) is a potential equation of the Neumann type. The numerical solution procedure will be outlined in sections 4 and 5.

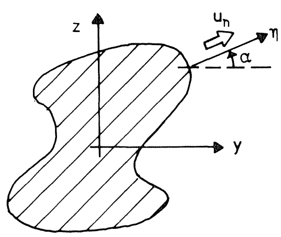

Fig. 3.

3 The shear force problem

The shear forces Q_y and Q_z and torsional moment M_x act at the shear centre C of the cross-section. The shear force C does not necessarily coincide with the member axis where the bending moments M_y and M_z and normal force N act at the cross-section. Eccentricities e_y and e_z, defining the distance from the shear centre C to the member axis, may exist.

To elaborate the shear force deformation we consider the deformations caused by bending about the principal axes of inertia with constant shear forces Q_y and Q_z and bending moments M_y and M_z. Since we have no rotation about the shear centre we may assume

$$
\begin{aligned}
u_y(y, z) &= u_y^m \\
u_z(y, z) &= u_z^m
\end{aligned}
\tag{6}
$$

where u_y^m and u_z^m are the displacements of the member axis.

We will assume, following the bending theory, that the strain ε_{xx} is distributed linearly over the cross-section. With bending moment M_y and curvature $\varkappa_y$ we assume

$$\varepsilon_{xx} = z\varkappa_y$$

Substitution of the moment curvature relation $M_y = EI_y\varkappa_y$ gives

$$\varepsilon_{xx} = \frac{M_y}{EI_y} z \tag{7a}$$

The shear strains γ_{xy} and γ_{xz} are now

$$
\begin{aligned}
\gamma_{xy} &= u_{x,y} + u_{y,x}^m \\
\gamma_{xz} &= u_{x,z} + u_{z,x}^m
\end{aligned}
\tag{7b}
$$

These relations (7a) and (7b) are elaborated in the axial equilibrium condition. Assuming an uniaxial stress strain relation for σ_{xx} we obtain for $\sigma_{xx,x}$

$$\sigma_{xx,x} = E\varepsilon_{xx,x} = \frac{M_{y,x}}{I_y} z = \frac{Q_z}{I_y} z \tag{8}$$

Substitution of (8) together with (7a) and (7b) in the axial equilibrium equation gives the potential equation

$$Gu_{x,yy} + Gu_{x,zz} + \frac{Q_z}{I_y} z = 0 \tag{9}$$

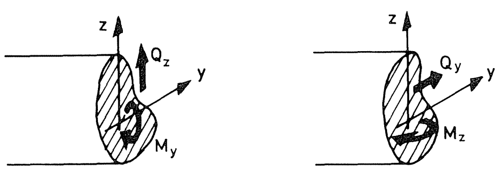

Fig. 4. Shear forces and bending moments.

Boundary condition (4) is valid also for this problem. After substitution of (6) we have the boundary condition

$$-Gu_{x,n} - Gu^{m}_{n,x} = 0 \tag{10}$$

To reduce the problem we introduce displacement u^*_x as follows

$$u^*_x = u_x - u^m_x + y\phi^m_z - z\phi^m_y \tag{11}$$

where ϕ^m_y and ϕ^m_z are the rotations of the cross-section due to bending.

Hence we obtain the potential equation

$$Gu^*_{x,yy} + Gu^*_{x,zz} + \frac{Q_z}{I_y} z = 0 \tag{12a}$$

and boundary condition

$$Gu^*_{x,n} = 0 \tag{13}$$

Similarly we obtain with shear force Q_y and bending moment M_z the potential equation

$$Gu^*_{x,yy} + Gu^*_{x,zz} + \frac{Q_y}{I_z} y = 0 \tag{12b}$$

and, of course, the same boundary condition (13).

4 Galerkin's residual method

The fundamental degree of freedom of the torsional problem is, according to (3) and (4), the displacement $u_x(y, z)$. Application of Galerkin's residual method requires for an approximation $\tilde{u}_x$ that

$$G = \iint \delta\tilde{u}_x G(\tilde{u}_{x,yy} + \tilde{u}_{x,zz})\, dA +$$
$$+ \oint \delta\tilde{u}_x \left[G\theta_0\{(z-e_z)\cos\alpha - (y-e_y)\sin\alpha\} - G\tilde{u}_{x,n}\right] dS = 0$$

for every kinematically admissible variation $\delta\tilde{u}_x$.

Application of Green's theorem gives the condition that

$$\iint \{\delta\tilde{\Psi}\}^T[G]\{\tilde{\Psi}\}\, dA = G\theta_0 \oint \delta\tilde{u}_x\{(z-e_z)\cos\alpha - (y-e_y\sin\alpha)\}\, dS \tag{15}$$

where

$$\{\tilde{\Psi}\} = \begin{bmatrix} \tilde{u}_{x,y} \\ \tilde{u}_{x,z} \end{bmatrix} \quad [G] = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}$$

Similarly we will require for the shear force problem

$$\iint \{\delta\tilde{\Psi}\}^T[G]\{\tilde{\Psi}\}\, dA = \frac{Q_z}{I_y} \iint \delta\tilde{u}_x z\, dA \tag{16}$$

for every kinematically admissible variation $\delta\tilde{u}_x$ of $\tilde{u}^*_x$

For shear force Q_y we require

$$\iint \{\delta\tilde{\Psi}\}^{\mathrm{T}}[G]\{\tilde{\Psi}\}\,\mathrm{d}A = \frac{Q_y}{I_z}\iint \delta\tilde{u}_x y\,\mathrm{d}A \tag{17}$$

for every kinematically admissible variation $\delta\tilde{u}$.

The finite element method gives us the tools to solve the conditions (15), (16) and (17).

5 The finite element method

Using the finite element method, we can transform Galerkin's variational conditions into a system of algebraic equations. To perform this step we use discrete displacements $\{u\}$ as degrees of freedom. Per element we chose an interpolation of $\tilde{u}_x(y, z)$ as follows

$$\tilde{u}_x(y, z) = [N^e(y, z)]\{\tilde{u}^e\} \tag{18}$$

with $\{\tilde{u}^e\}$ a set of discrete displacements of element e.

From (18) we obtain $\{\tilde{\Psi}^e\}$ by differentiation with respect to y and z.

This yields

$$\{\tilde{\Psi}^e(y, z)\} = [B^e(y, z)]\{u^e\} \tag{19}$$

Substitution of (18) and (19) into the contributions to Galerkin's variational conditions yields a "stiffness" matrix $[K^e]$

$$\iint \{\delta\tilde{\Psi}^e\}^{\mathrm{T}}[G]\{\tilde{\Psi}^e\}\,\mathrm{d}A = \{\delta u^e\}^{\mathrm{T}}[K^e]\{u^e\} \tag{20}$$

$$\text{with } [K^e] = \iint [B^e]^{\mathrm{T}}[G][B^e]\,\mathrm{d}A$$

and "loading" conditions

$$\oint G\delta\tilde{u}_x^e(z\cos\alpha - y\sin\alpha)\,\mathrm{d}S = \{\delta u^e\}^{\mathrm{T}}\{f_1^e\}$$

$$\text{with } \{f_1^e\} = \oint G[N^e]^{\mathrm{T}}(z\cos\alpha - y\sin\alpha)\,\mathrm{d}S \tag{21}$$

$$\oint G\delta\tilde{u}_x^e \sin\alpha\,\mathrm{d}S = \{\delta u^e\}^{\mathrm{T}}\{f_2^e\} \text{ with } \{f_2^e\} = \oint G[N^e]^{\mathrm{T}}\sin\alpha\,\mathrm{d}S$$

$$\oint G\delta\tilde{u}_x^e \cos\alpha\,\mathrm{d}S = \{\delta u^e\}^{\mathrm{T}}\{f_3^e\} \text{ with } \{f_3^e\} = \oint G[N^e]^{\mathrm{T}}\cos\alpha\,\mathrm{d}S$$

$$\iint \delta\tilde{u}_x^e z\,\mathrm{d}A = \{\delta u^e\}^{\mathrm{T}}\{f_4^e\} \text{ with } \{f_4^e\} = \iint [N^e]^{\mathrm{T}}z\,\mathrm{d}A$$

$$\iint \delta\tilde{u}_x^e y\,\mathrm{d}A = \{\delta u^e\}^{\mathrm{T}}\{f_5^e\} \text{ with } \{f_5^e\} = \iint [N^e]^{\mathrm{T}}y\,\mathrm{d}A$$

Application of the variational condition for the torsional problem results in the algebraic equations

$$[K]\{u\} = \theta_0\{f_1\} + e_y\theta_0\{f_2\} - e_z\theta_0\{f_3\} \tag{22}$$

For the shear force problems we obtain the equations

$$[K]\{u\} = \frac{Q_z}{I_y}\{f_4\} \tag{23a}$$

and

$$[K]\{u\} = \frac{Q_y}{I_z}\{f_5\} \tag{23b}$$

For further elaborations we may avail ourselves of the solutions $\{u_i\}$ of the systems of equations

$$[K]\{u_i\} = \{f_i\} \qquad i = 1, 2, 3, 4, 5 \tag{24}$$

6 Elaboration to shear properties

Shear force areas

The shear force areas k_yA and k_zA determine the relations between the shear forces Q_y and Q_z and the averaged shear deformations $\bar{\Psi}_y$ and $\bar{\Psi}_z$ as follows

$$Q_y = k_y G A \bar{\Psi}_y \tag{25}$$

$$Q_z = k_z G A \bar{\Psi}_z$$

With respect to the shear deformation $\bar{\Psi}_y$ and $\bar{\Psi}_z$ we require that the work done by the shear forces is the same as the work done by the shear stresses, thus

$$\tfrac{1}{2}Q_z\bar{\Psi}_z = \tfrac{1}{2}\iint \{\tilde{\Psi}\}^T[G]\{\tilde{\Psi}\}\,\mathrm{d}A = \tfrac{1}{2}\left(\frac{Q_z}{I_y}\right)^2 \{u_4\}^T\{f_4\} \tag{26}$$

From this it follows that

$$\bar{\Psi}_z = \frac{Q_z}{I_y^2}\{u_4\}^T\{f_4\}$$

and

$$k_z G A = \frac{I_y^2}{\{u_4\}^T\{f_4\}} \tag{27a}$$

In the same way we obtain

$$k_y G A = \frac{I_z^2}{\{u_5\}^T\{f_5\}} \tag{27b}$$

Eccentricities shear centre

Assuming that shear force Q_z acts at the shear centre, we obtain a torsional moment M_x with respect to the member axis:

$$M_x = Q_z e_y = \iint (\tilde{\sigma}_{xz}y - \tilde{\sigma}_{xy}z)\,\mathrm{d}A \tag{28}$$

Substitution of (11) into the shear deformation yields for M_x

$$M_x = \iint (G\tilde{u}^*_{x,z}y - G\tilde{u}^*_{x,y}z)\,\mathrm{d}A$$

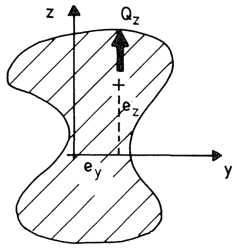

Fig. 5

Application of Green's theorem gives

$$M_x = Q_z e_y = \oint G\tilde{u}_x^* \,(y \sin\alpha - z\cos\alpha)\, dS \tag{29}$$

With reference to "loading" case $\{f_1\}$ we find numerically (21)

$$Q_z e_y = -\frac{Q_z}{I_y}\{u_4\}^T\{f_1\}$$

from which it follows that

$$e_y = -\frac{1}{I_y}\{u_4\}^T\{f_1\} \tag{30a}$$

and in the same way

$$e_z = \frac{1}{I_z}\{u_5\}^T\{f_1\} \tag{30b}$$

Torsional rigidity

For the torsion problem we use the "loading" combination $\theta_0\{f_6\}$

$$\theta_0(\{f_1\} + e_y\{f_2\} - e_z\{f_3\}) = \theta_0\{f_6\}$$

The torsional moment M_x is again

$$M_x = \iint (\tilde{\sigma}_{xz} y - \tilde{\sigma}_{xy} z)\, dA = GI_x\theta_0$$

Substitution of (2) into the shear strains results in

$$M_x = \iint (G\tilde{u}_{x,z} y - G\tilde{u}_{x,y} z)\, dA + G\theta_0(I_y + I_z) \tag{31}$$

Application of Green's theorem results in

$$M_x = \oint G\tilde{u}_x (y\sin\alpha - z\cos\alpha)\, dS + G\theta_0(I_y + I_z) \tag{32}$$

Where $\tilde{u}_x$ is solved with "loading" combination $\theta_0\{f_6\}$.

Assuming $\{u_6\}$ to be the solution with "loading" $\{f_6\}$, we obtain for M_x

$$M_x = -\theta_0\{u_6\}^T\{f_1\} + G\theta_0(I_y + I_z) = GI_x\theta_0 \tag{33}$$

From (33) it follows that

$$GI_x = GI_y + GI_z - \{u_6\}^T\{f_1\} \tag{34}$$

Summarizing

$$k_y GA = \frac{I_z^2}{\{u_5\}^T\{f_5\}}$$

$$k_z GA = \frac{I_y^2}{\{u_4\}^T\{f_4\}}$$

$$e_y = -\frac{\{u_4\}^T\{f_1\}}{I_y}$$

$$e_z = \frac{\{u_5\}^T\{f_1\}}{I_z}$$

$$GI_x = GI_y + GI_z - \{u_1\}^T\{f_1\} - e_y\{u_2\}^T\{f_1\} + e_z\{u_3\}^T\{f_1\}$$

7 Examples

A square cross-section is subdivided into four 8-node elements. The finite element method (using reduced integration rules) gives the following results:

$$I_x = 0.1417$$
$$k_y A = k_z A = 0.842$$

Exact values [1] are

$$I_x = 0.1406$$
$$k_y A = k_z A = 0.833$$

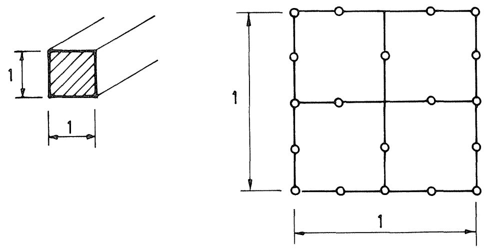

Fig. 6. Finite element mesh for the cross-section.

An L-shaped cross-section is subdivided into three 8-node elements. The finite element mesh gives the following results

$$I_x = 0.1146$$
$$e_y = -0.1158$$
$$e_z = -0.1158$$

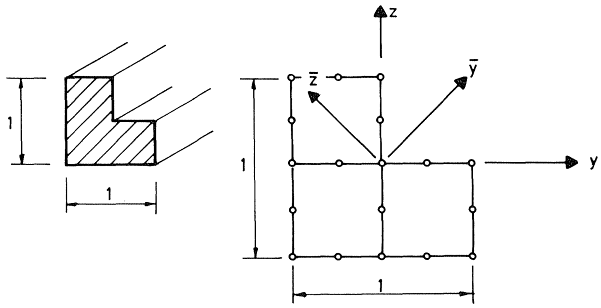

Fig. 7.

7 References

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A graduation thesis on this subject, containing many more details, will be published shortly.