CONTINUOUS LOCAL MARTINGALES AND STOCHASTIC INTEGRATION IN UMD BANACH SPACES

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Abstract. Recently, van Neerven, Weis and the author, constructed a theory for stochastic integration of UMD Banach space valued processes. Here the authors use a (cylindrical) Brownian motion as an integrator. In this note we show how one can extend these results to the case where the integrator is an arbitrary real-valued continuous local martingale. We give several characterizations of integrability and prove a version of the Itô isometry, the Burkholder-Davis-Gundy inequality, the Itô formula and the martingale representation theorem.

1. Introduction

There are various approaches to stochastic integration of Banach space valued processes. We are in particularly interested in the stochastic integral $\int \phi(t)\, dM(t)$ for an E-valued strongly progressively measurable process ϕ, where E is a Banach space and M is a local martingale. Already in the case where ϕ is an E-valued function and M is a standard Brownian motion some serious problems occur. In [26], Yor has constructed a bounded measurable function $\phi : [0,1] \to l^p$, where $1 \leq p < 2$, which cannot be stochastically integrated with respect to a standard real-valued Brownian motion W. On the other hand, for Banach spaces with type 2 (for example L^p with $p \in [2,\infty)$), Hoffmann-Jørgensen and Pisier [10] have observed that every function $\phi \in L^2(0,T;E)$ is stochastically integrable with respect to W and there is a constant C independent of ϕ such that

$$\mathbb{E}\Big\| \int_0^T \phi(t)\, dW(t)\Big\|^2 \leq C^2 \|\phi\|^2_{L^2(0,T;E)}.$$

A few years later, Rosiński and Suchanecki [23] have characterized the stochastic integrability for functions $\phi : [0,T] \to E$ and $M = W$. McConnell [16] has used decoupling methods to give sufficient conditions for stochastic integrability of strongly progressively measurable processes $\phi : [0,T] \times \Omega \to E$ and $M = W$ in the case where E is a UMD space (for example L^p with $p \in (1,\infty)$).

In the case that E has martingale type 2 there is a different approach to study stochastic integrability of strongly progressively measurable processes (cf. [1, 2,

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5, 21]). In this approach a sufficient condition for stochastic integrability is $\phi \in L^2(0,T;E)$ almost surely, and in this case one has the one-sided estimate

$$\mathbb{E}\Big\| \int_0^T \phi(t)\, dW(t)\Big\|^2 \leq C^2 \mathbb{E}\|\phi\|^2_{L^2(0,T;E)}, \tag{1.1}$$

where C is independent of ϕ. Although this gives a wide class of integrable processes, it is not the right class in the following sense. The L^2-condition is sufficient, but not necessary (except if E is isomorphic to a Hilbert space (see [23])), and the estimate (1.1) is only one-sided. We will now explain conditions for stochastic integrability which are necessary and sufficient, and such that a two-sided estimate holds.

In [20], van Neerven and Weis have obtained necessary and sufficient conditions for stochastic integrability in terms of γ-radonifying operators in the case that $M = W$. The space of stochastically integrable functions turns out to be $\gamma(0,T;E)$ and it is shown that the following version of the Itô isometry holds:

$$\mathbb{E}\Big\| \int_0^T \phi(t)\, dW(t)\Big\|^2 = \|\phi\|^2_{\gamma(0,T;E)}.$$

The γ-spaces are useful tools in proving properties of stochastic integrals and in studying existence and uniqueness of stochastic differential equations in Banach spaces (cf. [2, 6, 20, 25]). The spaces turned out to be useful in various other areas of mathematics as H^∞-calculus [7, 14], control theory [9] and wavelet decomposition [11]. Recently, in [13] some embedding results for γ-spaces have been obtained in the case the space has type p or cotype q for some $p \in [1,2]$ and $q \in [2,\infty]$.

In [19], using decoupling techniques related to [8, 16], these characterizations have been extended to the case of processes $\phi : [0,T]\times\Omega \to E$ and E is a UMD Banach space. It has been shown that the space of stochastically integrable processes is given by the adapted and strongly measurable processes that satisfy $\phi \in \gamma(0,T;E)$ a.s., and for all $p \in (1,\infty)$ the following Itô isomorphism has been proved

$$c_p^p \|\phi\|^p_{L^p(\Omega;\gamma(0,T;E))} \leq \mathbb{E}\Big\| \int_0^T \phi(t)\, dW(t)\Big\|^p \leq C_p^p \|\phi\|^p_{L^p(\Omega;\gamma(0,T;E))}, \tag{1.2}$$

where $c_p, C_p \geq 0$ are constants independent of ϕ.

The processes which one can integrate with the martingale type 2 theory are the processes with paths in $L^2(0,T;E)$ a.s. This class of processes is smaller than the class of processes with paths in $\gamma(0,T;E)$ a.s. (except if E is isomorphic to a Hilbert space). Secondly, among the spaces $E = L^p$, our theory is applicable for all $p \in (1,\infty)$. The theory for martingale type 2 spaces only applies to $E = L^p$ with $p \in [2,\infty)$. The L^p-spaces are important in the study of stochastic evolution equation and the applications to such equations are work in progress (cf. [18]).

A natural question is whether the theory from [19], in particular (1.2), can be extended to the case where the Brownian motion is replaced by an arbitrary continuous local martingale M. The goal of this paper is to construct such a stochastic integration theory and to find a precise description of the integrable processes. We show that the space of stochastically integrable processes is the set of all strongly progressively measurable processes that satisfy $\phi \in \gamma(\mathbb{R}_+, [M]; E))$ a.s. Moreover, for all $p \in (1,\infty)$ the following Itô isomorphism is obtained:

$$c_p^p \|\phi\|^p_{L^p(\Omega;\gamma([0,T],[M];E))} \leq \mathbb{E}\Big\| \int_0^T \phi(t)\, dM(t)\Big\|^p \leq C_p^p \|\phi\|^p_{L^p(\Omega;\gamma([0,T],[M];E))}. \tag{1.3}$$

Here $[M]$ is the quadratic variation of M. In the special case that E is a Hilbert space, $\gamma([0,T],[M];E) = L^2(0,T;E)$ isometrically, and the estimate (1.3) reduces to a well-known inequality. The proofs below are based on time change arguments as occur in some known proofs in the real-valued case. The procedure in the Banach space setting is however more difficult.

As an application of the main result Theorem 3.3, an Itô formula and a martingale representation theorem are proved in Section 4.

2. Preliminaries

A Banach space E is a *UMD space* if for some (equivalently, for all) $p \in (1,\infty)$ there exists a constant $\beta_{p,E} \geq 1$ such that for every $n \geq 1$, every martingale difference sequence $(d_j)_{j=1}^n$ in $L^p(\Omega;E)$, and every $\{-1,1\}$-valued sequence $(\varepsilon_j)_{j=1}^n$ we have

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n \varepsilon_j d_j\Big\|^p\Big)^{\frac{1}{p}} \leq \beta_{p,E}\,\Big(\mathbb{E}\Big\|\sum_{j=1}^n d_j\Big\|^p\Big)^{\frac{1}{p}}.$$

UMD spaces are reflexive. Examples of UMD spaces are all Hilbert spaces and the spaces $L^p(S)$ for $1<p<\infty$ and σ-finite measure spaces (S,Σ,μ). If E is a UMD space, then $L^p(S;E)$ is a UMD space for $1<p<\infty$. For an overview of the theory of UMD spaces we refer the reader to [4, 24] and references given therein.

Let $(\gamma_n)_{n\geq 1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega',\mathscr{F}',\mathbb{P}')$ (we reserve the notation $(\Omega,\mathscr{F},\mathbb{P})$ for the probability space on which our processes live) and let H be a separable real Hilbert space. A bounded operator $R \in \mathscr{L}(H,E)$ is said to be *γ-radonifying* if there exists an orthonormal basis $(h_n)_{n\geq 1}$ of H such that the Gaussian series $\sum_{n\geq 1}\gamma_n\, Rh_n$ converges in $L^2(\Omega';E)$. We then define

$$\|R\|_{\gamma(H,E)} := \Big(\mathbb{E}'\Big\|\sum_{n\geq 1}\gamma_n\, Rh_n\Big\|^2\Big)^{\frac{1}{2}}.$$

This number does not depend on the sequence $(\gamma_n)_{n\geq 1}$ and the basis $(h_n)_{n\geq 1}$, and defines a norm on the space $\gamma(H,E)$ of all γ-radonifying operators from H into E. Endowed with this norm, $\gamma(H,E)$ is a Banach space, which is separable if E is separable. If $R \in \gamma(H,E)$, then $\|R\| \leq \|R\|_{\gamma(H,E)}$. If E is a Hilbert space, then $\gamma(H,E) = \mathscr{L}_2(H,E)$ isometrically.

Let (S,Σ,μ) be a separable measure space. We say that a function $\phi : S \to E$ belongs to $L^2(S,\mu)$ *scalarly* if for all $x^* \in E^*$, $\langle \phi, x^*\rangle \in L^2(S,\mu)$. A function $\phi : S \to E$ is said to *represent* an operator $R \in \gamma(L^2(S,\mu),E)$ if ϕ belongs to $L^2(S,\mu)$ scalarly and for all $x^* \in E^*$ and $f \in L^2(S,\mu)$ we have

$$\langle Rf, x^*\rangle = \int_S f(s)\langle \phi(s), x^*\rangle\, d\mu(s).$$

The above notion will be abbreviated by $\phi \in \gamma(S,\mu;E)$. If μ is the Lebesgue measure we will also write $\gamma(L^2(\mathbb{R}_+),E)$ and $\gamma(\mathbb{R}_+;E)$ for $\gamma(L^2(\mathbb{R}_+,\mu),E)$ and $\gamma(\mathbb{R}_+,\mu;E)$ respectively. Here $\mathbb{R}_+ = [0,\infty)$.

Let M be a real valued continuous local martingale. For almost all $\omega\in\Omega$, the quadratic variation process $[M](\cdot,\omega)$ is continuous and increasing, so we can associate a Lebesgue-Stieltjes measure with it, which we will also denote by $[M](\cdot,\omega)$. We say that $\phi : \mathbb{R}_+\times\Omega \to E$ is *scalarly in* $L^2(\mathbb{R}_+,[M])$ *a.s.* if for all $x^* \in E^*$, for almost all $\omega \in \Omega$, $\langle \phi(\cdot,\omega), x^*\rangle \in L^2(\mathbb{R}_+, [M](\cdot,\omega))$. For such a process ϕ and a family $X = (X(\omega) : \omega \in \Omega)$ with $X(\omega) \in \gamma(L^2(\mathbb{R}_+, [M](\cdot,\omega); E)$ for almost all $\omega \in \Omega$, we say that ϕ *represents* X if for all $x^* \in E^*$, for almost all $\omega \in \Omega$,

$$\langle \phi(\cdot,\omega), x^*\rangle = X^*(\omega)x^* \text{ in } L^2(\mathbb{R}_+, [M](\cdot,\omega)).$$

In the case that M is a Brownian motion the above notion of representability reduces to the one in [19], since for almost all $\omega \in \Omega$, $[M](t,\omega) = t$.

The following relation between the above two representability concepts can be proved as [19, Lemma 2.7].

Lemma 2.1. *Let E be a separable real Banach space. Let $\phi : \mathbb{R}_+ \times \Omega \to E$ be strongly measurable. For each $\omega \in \Omega$, let $X(\omega) \in \gamma(L^2(\mathbb{R}_+, [M](\cdot,\omega)), E)$. If X is represented by ϕ, then for almost all $\omega \in \Omega$, $X(\omega)$ is represented by $\phi(\cdot,\omega)$. In particular, $\phi \in \gamma(\mathbb{R}_+, [M]; E)$ almost surely.*

3. Definitions and characterizations of the stochastic integral

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a complete probability space with a filtration $\mathscr{F} := (\mathscr{F}_t)_{t\in\mathbb{R}_+}$ that satisfies the usual conditions. Let M be a real-valued continuous local martingale with $M(0) = 0$.

We say that $\phi : \mathbb{R}_+ \times \Omega \to E$ is an *elementary progressive process* if it is of the form

$$\phi = \sum_{n=1}^N \mathbf{1}_{(t_{n-1},t_n]}\xi_n,$$

where $0 \le t_0 \le t_1, \dots, \le t_N < \infty$ and for each $n = 1, \dots, N$, ξ_n are $\mathscr{F}_{t_{n-1}}$-measurable E-valued random variables. For such ϕ we define the stochastic integral as an element of $L^0(\Omega; C_b(\mathbb{R}_+; E))$ as

$$\int_0^t \phi(s)\, dM(s) = \sum_{n=1}^N (M(t_n \wedge t) - M(t_{n-1} \wedge t))\xi_n.$$

We will extend this definition of the stochastic integral below.

It is immediate that the stochastic integral definition can be extended to all strongly progressively measurable processes $\phi : \mathbb{R}_+ \times \Omega \to E$ that take values in a finite dimensional subspace of E, and satisfy ϕ is in $L^2(\mathbb{R}_+, [M]; E)$ a.s. (or ϕ is scalarly in $L^2(\mathbb{R}_+, [M])$ a.s.). For more general ϕ the stochastic integral is constructed in Theorem 3.3.

Before characterizing the stochastic integral in general, we consider the case of Brownian motions. The next result is an infinite time interval version of a special case of the results in [19].

Proposition 3.1. *Let E be a UMD space. For a strongly measurable and adapted process $\phi : \mathbb{R}_+ \times \Omega \to E$ which is scalarly in $L^2(\mathbb{R}_+)$ a.s. the following assertions are equivalent:*

(1) *there exists a sequence $(\phi_n)_{n\ge1}$ of elementary progressive processes such that:*
   (i) *for all $x^* \in E^*$ we have $\lim_{n\to\infty} \langle \phi_n, x^*\rangle = \langle \phi, x^*\rangle$ in $L^0(\Omega; L^2(\mathbb{R}_+))$,*
   (ii) *there exists a process $\zeta \in L^0(\Omega; C_b(\mathbb{R}_+; E))$ such that*

$$\zeta = \lim_{n\to\infty} \int_0^{\cdot} \phi_n(t)\, dW(t) \quad \text{in } L^0(\Omega; C_b(\mathbb{R}_+; E)).$$

(2) *There exists a process* $\zeta \in L^0(\Omega; C_b(\mathbb{R}_+; E))$ *such that for all* $x^* \in E^*$ *we have*

$$\langle \zeta, x^* \rangle = \int_0^{\cdot} \langle \phi(t), x^* \rangle \, dW(t) \quad \textit{in } L^0(\Omega; C_b(\mathbb{R}_+)).$$

(3) $\phi \in \gamma(\mathbb{R}_+; E)$ *almost surely;*

Furthermore, for all $p \in (1, \infty)$ *we have*

$$\mathbb{E} \sup_{t \in \mathbb{R}_+} \|\zeta(t)\|^p \eqsim_{p,E} \mathbb{E}\|\phi\|^p_{\gamma(\mathbb{R}_+;E)}. \tag{3.1}$$

A process $\phi : \mathbb{R}_+ \times \Omega \to E$ satisfying the equivalent conditions of the proposition will be called *stochastically integrable* with respect to W. The process ζ is called the *stochastic integral process* of ϕ with respect to W, notation

$$\zeta = \int_0^{\cdot} \phi(t) \, dW(t).$$

The process ζ is a continuous local martingale that starts at 0.

To prove Proposition 3.1, some results from [19] have to be extended to the infinite time setting. This can be done without major difficulties and we leave this to the reader. For instance the following extension of [19, Proposition 2.12] is needed, which we also use in the proof of our main result below.

For a strongly measurable map $X : \Omega \to \gamma(L^2(\mathbb{R}_+), E)$ we define $X^*x^* \in L^0(\Omega; L^2(\mathbb{R}_+))$ as $(X^*x^*)(\cdot, \omega) = (X(\omega))^*x^*$, where we identified $L^2(\mathbb{R}_+)$ with its dual. We say that $X : \Omega \to \gamma(L^2(\mathbb{R}_+), E)$ is *elementary progressive* if it is represented by an elementary progressive process $\phi : \mathbb{R}_+ \times \Omega \to E$.

Lemma 3.2. *For a strongly measurable map* $X : \Omega \to \gamma(L^2(\mathbb{R}_+), E)$ *the following assertions are equivalent:*

(1) *There exist elementary progressive elements* $X_1, X_2, \ldots,$ *such that* $X = \lim_{n\to\infty} X_n$ *in* $L^0(\Omega; \gamma(L^2(\mathbb{R}_+), E))$.

(2) *For all* $x^* \in E^*$, *we have* X^*x^* *is adapted.*

Proof. (1) ⇒ (2) is obvious.

(2) ⇒ (1): The only thing we have to prove is that [19, Propositions 2.12] may be extended to an infinite time horizon. For each $T > 0$ let $P^T : L^2(\mathbb{R}_+) \to L^2(0, T)$ be defined as $P^T f = f|_{(0,T)}$. Let $Y^T = X \circ P^T$. By the right ideal property, we have $Y^T \in L^0(\Omega; \gamma(L^2(0, T), E))$. It follows from [19, Proposition 2.12] that we can find elementary progressive $X_n \in L^0(\Omega; \gamma(L^2(0, n), E))$ such that

$$\|X_n - Y^n\|_{L^0(\Omega;\gamma(L^2(0,n),E))} < \frac{1}{n}.$$

It follows from [19, Proposition 2.4] that $X = \lim_{n\to\infty} Y^n$ in $\gamma(L^2(\mathbb{R}_+), E))$ almost surely and we may conclude the result. □

We can now formulate the main result of this paper.

Theorem 3.3. *Let* E *be a UMD space. For a strongly progressively measurable process* $\phi : \mathbb{R}_+ \times \Omega \to E$ *which is scalarly in* $L^2(\mathbb{R}_+, [M])$ *a.s. the following assertions are equivalent:*

(1) *there exists a sequence* $(\phi_n)_{n\geq 1}$ *of elementary progressive processes such that:*

(i) *for all* $x^* \in E^*$ *we have* $\lim_{n\to\infty} \langle \phi_n, x^* \rangle = \langle \phi, x^* \rangle$ *in* $L^0(\Omega; L^2([M]))$,

(ii) *there exists a process* $\zeta \in L^0(\Omega; C_b(\mathbb{R}_+; E))$ *such that*

$$\zeta = \lim_{n\to\infty} \int_0^{\cdot} \phi_n(t)\, dM(t) \quad \textit{in } L^0(\Omega; C_b(\mathbb{R}_+; E)).$$

(2) *There exists a process* $\zeta \in L^0(\Omega; C_b(\mathbb{R}_+; E))$ *such that for all* $x^* \in E^*$ *we have*

$$\langle \zeta, x^* \rangle = \int_0^{\cdot} \langle \phi(t), x^* \rangle\, dM(t) \quad \textit{in } L^0(\Omega; C_b(\mathbb{R}_+)).$$

(3) $\phi \in \gamma(\mathbb{R}_+, [M]; E)$ *almost surely;*

Furthermore, for all $p \in (1, \infty)$ *we have*

$$\mathbb{E} \sup_{t\in\mathbb{R}_+} \|\zeta(t)\|^p \eqsim_{p,E} \mathbb{E}\|\phi\|^p_{\gamma(\mathbb{R}_+,[M];E)}. \tag{3.2}$$

A process $\phi : \mathbb{R}_+ \times \Omega \to E$ satisfying the equivalent conditions of the theorem will be called *stochastically integrable* with respect to M. The process ζ is called the *stochastic integral process* of ϕ with respect to M, notation

$$\zeta = \int_0^{\cdot} \phi(t)\, dM(t).$$

In the same way as in [19, Proposition 5.8] one can show that ζ is a continuous local martingale that starts at 0.

Remark 3.4. The norm $\gamma(\mathbb{R}_+, [M]; E)$ plays the rôle of the classical quadratic variation and (3.2) may be seen as a version of the Itô isometry or as a stochastic integral version of the Burkholder-Davis-Gundy inequalities. We do not know whether (3.2) holds for $p \in (0, 1]$.

For the proof of Theorem 3.3 we need some additional results. We start with a lemma on time changes in spaces of γ-radonifying operators.

Lemma 3.5. *Let* E *be a real Banach space. Let* $A : \mathbb{R}_+ \to \mathbb{R}$ *with* $A(0) = 0$ *be increasing and continuous and let* μ *be the Lebesgue-Stieltjes measure corresponding to* A*. Let* $S := \lim_{t\to\infty} A(t) \le \infty$ *and define* $\tau : \mathbb{R}_+ \to \overline{\mathbb{R}}$ *as*

$$\tau(s) = \begin{cases} \inf\{t \ge 0 : A(t) > s\} & \textit{for } 0 \le s < S, \\ \infty & \textit{for } s \ge S. \end{cases}$$

Let $\phi : \mathbb{R}_+ \to E$ *be strongly measurable and let* $\psi : \mathbb{R}_+ \to E$ *be defined as*

$$\psi(s) = \begin{cases} \phi(\tau(s)) & \textit{for } 0 \le s < S, \\ 0 & \textit{for } s \ge S. \end{cases}$$

Then we have $\phi \in \gamma(\mathbb{R}_+, \mu; E)$ *if and only if* $\psi \in \gamma(\mathbb{R}_+; E)$*. In that case,*

$$\|\phi\|_{\gamma(\mathbb{R}_+,\mu;E)} = \|\psi\|_{\gamma(\mathbb{R}_+;E)}. \tag{3.3}$$

Recall the substitution rule: for a strongly measurable $f : \mathbb{R}_+ \to E$, we have $f \in L^1(\mathbb{R}_+, \mu; E)$ if and only if $f \circ \tau \in L^1(0, S; E)$, and in that case

$$\int_{\mathbb{R}_+} f(t)\, d\mu(t) = \int_{[0,S)} f(\tau(s))\, ds. \tag{3.4}$$

Proof. First notice that for all $s \in \mathbb{R}_+$, $A(\tau(s)) = s \wedge S$ and for all $t \in \mathbb{R}_+$,

$$\tau(A(t)) = \sup\{r \geq t : A(r) = A(t)\}.$$

Let $(f_n)_{n\geq 1}$ be an orthonormal basis for $L^2(0,S)$. For each $n \geq 1$, let $f_n^A : \mathbb{R}_+ \to \mathbb{R}$ be defined as $f_n^A(t) = f_n(A(t))$. We claim that $(f_n^A)_{n\geq 1}$ is an orthonormal basis for $L^2(\mathbb{R}_+, \mu)$. First of all it follows from (3.4) that for all $m, n \geq 1$

$$\int_{\mathbb{R}_+} f_m^A(t) f_n^A(t)\, d\mu(t) = \int_{[0,S)} f_m(s) f_n(s)\, ds = \delta_{mn}.$$

Hence $(f_n^A)_{n\geq 1}$ is an orthonormal system. Let $f \in L^2(\mathbb{R}_+, \mu)$ be such that for all $n \geq 1$,

$$\int_{\mathbb{R}_+} f(t) f_n^A(t)\, d\mu(t) = 0.$$

We have to show that $f = 0$, μ-almost everywhere. Take any representative of f and denote it again by f. Define $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}$ as $\tilde{f}(t) = f(\tau(A(t)))$. Since $\tau(A(t)) \neq t$ is possible only if A is constant near t, we have $\tilde{f} = f$, μ-almost everywhere. It follows from (3.4) that for all $n \geq 1$,

$$\int_{[0,S)} \tilde{f}(\tau(s)) f_n(s)\, ds = \int_{\mathbb{R}_+} \tilde{f}(s) f_n^A(s)\, d\mu(s) = \int_{\mathbb{R}_+} f(s) f_n^A(s)\, d\mu(s) = 0.$$

Since $(f_n)_{n\geq 1}$ is an orthonormal basis for $L^2(0,S)$, we obtain that $\tilde{f} \circ \tau = 0$, λ-almost everywhere. From (3.4), it follows that

$$\int_{\mathbb{R}_+} \mathbf{1}_{\{f(t)\neq 0\}}\, d\mu(t) = \int_{\mathbb{R}_+} \mathbf{1}_{\{\tilde{f}(t)\neq 0\}}\, d\mu(t) = \int_{[0,S)} \mathbf{1}_{\{\tilde{f}(\tau(s))\neq 0\}}\, ds = 0,$$

and hence $f(t) = 0$, μ-almost everywhere. We may conclude that the claim is true.

"$\Rightarrow$" Let $I_\phi^A \in \gamma(L^2(\mathbb{R}_+, \mu), E)$ be the operator that ϕ represents. It follows from (3.4) that for all $x^* \in E^*$ we have

$$\|\langle \psi, x^*\rangle\|_{L^2(0,S)} = \|\langle \phi, x^*\rangle\|_{L^2(\mathbb{R}_+,\mu)}, \tag{3.5}$$

so ψ is scalarly in $L^2(0,S)$. Hence, by the closed graph theorem we may define $I_\psi \in \mathscr{L}(L^2(0,S), E^{**})$ as

$$\langle x^*, I_\psi f\rangle = \int_{[0,S)} f(s)\langle \psi(s), x^*\rangle\, ds, \quad x^* \in E^*, f \in L^2(0,S).$$

From (3.4) we deduce that for all $n \geq 1$ and $x^* \in E^*$,

$$\langle x^*, I_\psi f_n\rangle = \int_{[0,S)} f_n(s)\langle \psi(s), x^*\rangle\, ds = \int_{\mathbb{R}_+} f_n^A(t)\langle \phi(t), x^*\rangle\, dt = \langle I_\phi^A f_n^A, x^*\rangle. \tag{3.6}$$

Since I_ϕ^A takes values in E we conclude that $I_\psi f_n \in E$ for all $n \geq 1$. Since E can be seen as a closed subspace of E^{**} it follows that for $f \in L^2(0,S)$, $I_\psi f = \sum_{n\geq 1} [f, f_n]_{L^2(0,S)} I_\psi f_n$ converges in E. We obtain that $I_\psi \in \mathscr{L}(L^2(0,S), E)$. Moreover by (3.6),

$$\sum_{n\geq 1} \gamma_n I_\psi f_n = \sum_{n\geq 1} \gamma_n I_\phi^A f_n^A, \tag{3.7}$$

and hence the result and (3.3) follow.

"$\Leftarrow$" As before (3.5) holds and we may define $I_\phi^A \in \mathscr{L}(L^2(\mathbb{R}_+,\mu), E^{**})$ as

$$\langle x^*, I_\phi^A f\rangle = \int_{\mathbb{R}_+} f(t)\langle \phi(t), x^*\rangle\, d\mu(t), \quad x^*\in E^*, f\in L^2(\mathbb{R}_+,\mu).$$

It follows from (3.4) that for all $x^*\in E^*$ and $f\in L^2(\mathbb{R}_+,\mu)$ we have

$$\langle x^*, I_\phi^A f\rangle = \int_{\mathbb{R}_+} f(t)\langle \phi(t), x^*\rangle\, d\mu(t) = \int_{[0,S)} f(\tau(s))\langle \psi(s), x^*\rangle)\, ds = \langle I_\psi(f\circ\tau), x^*\rangle.$$

Since I_ψ takes values in E we obtain that I_ϕ^A takes values in E. Moreover we may conclude that (3.7) holds. This proves $\phi\in\gamma(\mathbb{R}_+,\mu;E)$. □

Recall the following results (cf. [12] or [15]).

Theorem 3.6 (Dambis, Dubins and Schwartz). *Define*

$$\tau_s = \inf\{t\geq 0 : [M]_t > s\}, \quad \mathscr{G}_s := \mathscr{F}_{\tau_s}, \ s\in[0,\infty).$$

Then there exist a probability space $(\overline{\Omega}, \overline{\mathscr{A}}, \overline{\mathbb{P}})$ and a Brownian motion W with respect to an extension of $\mathscr{G} := (\mathscr{G}_s)_{s\in\mathbb{R}_+}$ such that almost surely

$$W = M\circ\tau \text{ on } [0,[M]_\infty) \text{ and } M = W\circ[M].$$

Moreover, $(\overline{\Omega}, \overline{\mathscr{A}}, \overline{\mathbb{P}})$ may be taken of the form $(\Omega\times[0,1], \mathscr{A}\otimes\mathscr{B}_{[0,1]}, \mathbb{P}\otimes\lambda$, where λ is the Lebesgue measure on $[0,1]$. The extension of $\mathscr{G}$ can be chosen as $\overline{\mathscr{G}} = \mathscr{G}\otimes\mathscr{H}$ for a certain filtration $\mathscr{H}$ on $([0,1],\mathscr{B}_{[0,1]})$.

Proposition 3.7 (Kazamaki)**.** *With the notations of Theorem 3.6, we have the following time-change formula for stochastic integrals. If $\phi:\mathbb{R}_+\times\Omega\to\mathbb{R}$ is $\mathscr{F}$-progressively measurable and satisfies*

$$\int_0^\infty |\phi(s)|^2\, d[M]_s < \infty, \text{ almost surely,}$$

then the process

$$\psi(s) = \begin{cases} \phi(\tau_s) & \text{if } 0\leq s<[M]_\infty, \\ 0 & \text{if } [M]_\infty \leq s<\infty. \end{cases} \tag{3.8}$$

is $\mathscr{G}$-adapted and satisfies almost surely $\int_0^\infty |\psi(r)|^2\, dr<\infty$ and

$$\int_0^t \phi(r)\, dM(r) = \int_0^{[M]_t} \psi(r)\, dW(r), \quad t\in\mathbb{R}_+, \tag{3.9}$$

$$\int_0^{\tau_s} \phi(r)\, dM(r) = \int_0^{s} \psi(r)\, dW(r), \quad s\in\mathbb{R}_+. \tag{3.10}$$

Finally, we need the next lemma for weak limits of processes.

Lemma 3.8. *Let E be a reflexive Banach space. Let $\zeta:\mathbb{R}_+\times\Omega\to E$ be a strongly measurable process such that almost surely $\{\zeta(t) : t\in\mathbb{R}_+\}$ is bounded. If for all $x^*\in E^*$, $\lim_{t\to\infty}\langle\zeta(t),x^*\rangle$ exists almost surely, then $\zeta_\infty :=$ weak-$\lim_{t\to\infty}\zeta(t)$ exists almost surely and is strongly measurable.*

Proof. Since ζ is strongly measurable, we may assume that E is separable. By the reflexivity of E, we can find a dense sequence $(x_n^*)_{n\geq 1}$ in E^*. For each $n \geq 1$, let Ω_n be such that $P(\Omega_n) = 1$ and for all $\omega \in \Omega_n$, $\lim_{t\to\infty}\langle \zeta(t,\omega), x_n^*\rangle$ exists. Let

$$\Omega_0 = \bigcap_{n\geq 1} \Omega_n \cap \{\omega \in \Omega : \zeta(\cdot,\omega) \text{ is bounded}\}.$$

Then it follows from an easy three-ε-argument that $\lim_{t\to\infty}\langle \zeta(t,\omega), x^*\rangle$ exists for all $x^* \in E^*$ and all $\omega \in \Omega_0$. For each $\omega \in \Omega_0$, define $x_\omega^{**} \in E^{**} = E$ as

$$\langle x^*, x_\omega^{**}\rangle = \lim_{t\to\infty} \langle \zeta(t,\omega), x^*\rangle$$

and $x_\omega^{**} = 0$ for $\omega \in \complement\Omega_0$. We may define $\zeta_\infty : \Omega \to E$ as $\zeta_\infty(\omega) = x_\omega^{**}$. The Pettis measurability theorem ensures that ζ_∞ is strongly measurable. □

We can now prove our main result.

Proof of Theorem 3.3. Since ϕ is strongly measurable, we may assume that E is separable. Define $\psi : \mathbb{R}_+ \times \Omega \to E$ as

$$\psi(s) = \begin{cases} \phi(\tau_s) & \text{if } 0 \leq s < [M]_\infty, \\ 0 & \text{if } [M]_\infty \leq s < \infty. \end{cases} \tag{3.11}$$

Notice that against functionals from E^*, (3.11) coincides with (3.8). By Proposition 3.7 and the Pettis measurability theorem, ψ is strongly measurable and $\mathscr{G}$-adapted. Moreover, from the substitution rule (3.4) it follows that pointwise in Ω for all $x^* \in E^*$,

$$\|\langle \psi, x^*\rangle\|_{L^2(\mathbb{R}_+)} = \|\langle \phi, x^*\rangle\|_{L^2(\mathbb{R}_+,[M])} \tag{3.12}$$

if one of the expressions is finite. In particular ψ is scalarly in $L^2(\mathbb{R}_+)$ a.s.

Let W, $(\overline{\Omega}, \overline{\mathscr{A}}, \overline{\mathbb{P}})$ and $\overline{\mathscr{G}}$ be as in Theorem 3.6. We will prove the result by showing that (1), (2) and (3) for ϕ are equivalent with (1), (2) and (3) in Proposition 3.1 for ψ. (Notation $(k,\,\phi) \Leftrightarrow (k,\,\psi)$ for $k = 1,2,3$).

$(1,\,\phi) \Rightarrow (1,\,\psi)$: Assume (1) holds for a sequence of elementary progressive processes $(\phi_n)_{n\geq 1}$. For all $n \geq 1$, define $\psi_n : [0,\infty] \times \Omega \to E$ as

$$\psi_n(s) := \begin{cases} \phi_n(\tau_s) & \text{if } 0 \leq s < [M]_\infty, \\ 0 & \text{if } [M]_\infty \leq s < \infty. \end{cases}$$

Then it follows from the Pettis measurability theorem and Proposition 3.7 that each ψ_n is strongly measurable and strongly adapted and since ϕ_n is elementary progressive it follows from (3.10) that for all $n \geq 1$ for all $s \in \mathbb{R}_+$, almost surely we have

$$\zeta_{\psi_n}(s) := \int_0^s \psi_n(r)\, dW(r) = \int_0^{\tau_s} \phi_n(r)\, dM(r).$$

By the assumption, $(\zeta_{\psi_n})_{n\geq 1}$ is a Cauchy sequence in $L^0(\Omega; C_b(\mathbb{R}_+;E))$, hence it is convergent to some $\zeta_\psi \in L^0(\Omega; C_b(\mathbb{R}_+;E))$. By (3.4) and Theorem 3.3 (1) (i) it follows that for all $x^* \in E^*$ we have $\lim\limits_{n\to\infty} \langle \psi_n, x^*\rangle = \langle \psi, x^*\rangle$ in $L^0(\Omega; L^2(\mathbb{R}_+))$. Since each ψ_n takes values in a finite dimensional subspace of E, we may approximate it to obtain a sequence of elementary progressive processes $(\hat{\psi}_n)_{n\geq 1}$ that satisfies Proposition 3.1 (1) (i) and (ii).

$(1,\,\psi) \Rightarrow (1,\,\phi)$: Let Proposition 3.1 (1) be satisfied for ψ on the extended probability space $\overline{\Omega}$. Then it follows from Proposition 3.1 that $\psi \in \gamma(\mathbb{R}_+;E)$,

$\overline{\mathbb{P}}$-almost surely. By the special choice of $\overline{\Omega}$ and by Fubini's theorem we may conclude that $\psi \in \gamma(\mathbb{R}_+; E)$, $\mathbb{P}$-almost surely. Now [19, Remark 2.8] assures that $\psi \in L^0(\Omega; \gamma(\mathbb{R}_+; E))$. By Lemma 3.2 we can find elementary progressive processes $(\psi_n)_{n\geq 1}$ in $L^0(\Omega; \gamma(\mathbb{R}_+; E))$ such that $\psi = \lim_{n\to\infty} \psi_n$ in $L^0(\Omega; \gamma(\mathbb{R}_+; E))$. In view of the right ideal property, $\psi = \lim_{n\to\infty} \mathbf{1}_{[0,[M]_\infty)}\psi_n$ in $L^0(\Omega; \gamma(\mathbb{R}_+; E))$ and it follows that we may assume that $\psi_n(s) = 0$ for $s \geq [M]_\infty$. For each $n \geq 1$, define $\phi_n : \mathbb{R}_+ \times \Omega \to E$ as $\phi_n = \psi_n \circ [M]$. Then $(\phi_n)_{n\geq 1}$ is a sequence of strongly progressively measurable processes. Moreover, $\phi_n \circ \tau = \psi_n$. By substitution it follows that for all $x^* \in E^*$,

$$\|\langle \phi, x^*\rangle - \langle \phi_n, x^*\rangle\|_{L^0(\Omega;L^2([M]))} = \|\langle \psi, x^*\rangle - \langle \psi_n, x^*\rangle\|_{L^0(\Omega;L^2(\mathbb{R}_+))}.$$

Since the latter converges to 0 we obtain (1) (i). By the Itô homeomorphism from [19, Theorem 5.5] we have,

$$\int_0^\cdot \psi(t)\, dW(t) = \lim_{n\to\infty} \int_0^\cdot \psi_n(t)\, dW(t) \ \text{ in } L^0(\Omega; C_b(\mathbb{R}_+; E)).$$

Since the ψ_n are elementary progressive processes one easily checks that, almost surely for all $t \in [0, T]$,

$$\int_0^{[M]_t} \psi_n(t)\, dW(t) = \int_0^t \phi_n(t)\, dM(t).$$

It follows that $(\int_0^t \phi_n(t)\, dM(t))_{n\geq 1}$ is a Cauchy sequence in $L^0(\Omega; C_b(\mathbb{R}_+; E))$. Now as in the proof of (1, ϕ) $\Rightarrow$ (1, ψ), we may conclude (1) (ii) via an approximation argument.

(2, ϕ) $\Rightarrow$ (2, ψ): Let $\zeta : [0, \infty) \times \Omega \to E$ be the given integral process. Let $\zeta_\psi : [0, \infty) \times \Omega \to E$ be defined as

$$\zeta_\psi(s) = \begin{cases} \zeta(\tau_s). & \text{if } 0 \leq s < [M]_\infty, \\ \text{weak}-\lim_{t\to\infty} \zeta(t) & \text{if } [M]_\infty \leq s < \infty. \end{cases}$$

The weak limit exists almost surely and is strongly measurable by Lemma 3.8. The result would follow immediately if $\zeta_\psi \in L^0(\Omega; C_b(\mathbb{R}_+; E))$. This is not clear, since the trajectories of $s \mapsto \tau_s$ are not necessarily continuous. Instead, we do the following argument to show that Proposition 3.1 (2) holds for ψ. Afterwards, in Corollary 3.9 we will show that almost surely ζ_ψ has continuous trajectories.

It follows from Proposition 3.7 that ζ_ψ is weakly continuous almost surely. Choose $(x_n^*)_{n\geq 1}$ dense in the closed unit ball B_{E^*}. Let Ω_0 with $P(\Omega_0) = 1$ be such that for all $\omega \in \Omega_0$, $\langle \zeta_\psi(\cdot, \omega), x_n^*\rangle$ is continuous. For each $k \geq 1$ define

$$T_k = \inf\{t > 0 : \|\zeta_\psi(t)\| \geq k\}.$$

Since $\|\zeta_\psi\| = \sup_{n\geq 1} |\langle \zeta_\psi, x_n^*\rangle|$ is progressively measurable, each T_k is a stopping time. We claim that $\|\zeta_\psi^{T_k}\| \leq k$ on Ω_0. Indeed, for all $n \geq 1$ we have $|\langle \zeta_\psi^{T_k}, x_n^*\rangle| \leq k$ on Ω_0, and taking the supremum over all n gives the claim. In particular, we have $\zeta_\psi(T_k) \in L^2(\Omega; E)$.

Now fix $k \geq 1$ and $x^* \in E^*$. By (3.10) we have almost surely for all $t \in \mathbb{R}_+$,

$$\langle \zeta_\psi^{T_k}(t), x^*\rangle = \int_0^t \mathbf{1}_{[0,T_k]} \langle \psi(s), x^*\rangle\, dW(s).$$

Since $\langle \zeta_\psi^{T_k}, x^*\rangle$ is bounded it follows from the Burkholder-Davis-Gundy inequalities that $\mathbf{1}_{[0,T_k]}\langle \psi, x^*\rangle \in L^2(\Omega; L^2(\mathbb{R}_+))$. Hence $\mathbf{1}_{[0,T_k]}\psi$ is scalarly in $L^2(\Omega; L^2(\mathbb{R}_+))$ and for all $x^* \in E^*$, almost surely,

$$\langle \zeta_\psi(T_k), x^*\rangle = \int_0^\infty \mathbf{1}_{[0,T_k]}\langle \psi(s), x^*\rangle \, dW(s).$$

Therefore, we may apply the infinite time horizon case of [19, Theorem 3.6] to conclude that for all $k \geq 1$, $\mathbf{1}_{[0,T_k]}\psi \in L^2(\Omega; \gamma(\mathbb{R}_+; E))$. Since for all $\omega \in \Omega$, $T_k(\omega) = \infty$ for all k large enough, we deduce that $\psi = \lim_{k\to\infty} \mathbf{1}_{[0,T_k]}\psi \in L^0(\Omega; \gamma(\mathbb{R}_+; E))$. It follows from Proposition 3.1 that ψ is stochastically integrable and we may define $\tilde{\zeta}_\psi \in L^0(\Omega; C_b(\mathbb{R}_+; E))$ as

$$\tilde{\zeta}_\psi(t) = \int_0^t \psi(s) \, dW(s).$$

We conclude that Proposition 3.1 (2) holds for ψ and $\tilde{\zeta}_\psi$.

(2, ψ) $\Rightarrow$ (2, ϕ): Let ζ_ψ be the stochastic integral process of ψ with respect to W. Let $\zeta : [0, \infty) \times \Omega \to E$ be defined as $\zeta = \zeta_\psi \circ [M]$. Then $\zeta \in L^0(\Omega; C_b(\mathbb{R}_+; E))$ and it follows from (3.9) that for all $x^* \in E^*$, for all $t \in \mathbb{R}_+$, almost surely we have

$$\begin{aligned}\langle \zeta(t), x^*\rangle &= \langle \zeta_\psi([M]_t), x^*\rangle = (\langle \zeta_\psi, x^*\rangle)([M]_t)\\ &= \int_0^{[M]_t} \langle \psi(r), x^*\rangle \, dW(r) = \int_0^t \langle \phi(r), x^*\rangle \, dM(r).\end{aligned}$$

This proves (2).

(3, ϕ) $\Leftrightarrow$ (3, ψ): This follows from Lemma 3.5. Moreover, it follows from (3.3) that for almost all $\omega \in \Omega$, we have

$$\|\phi(\cdot, \omega)\|_{\gamma(\mathbb{R}_+, [M](\cdot,\omega); E)} = \|\psi(\cdot, \omega)\|_{\gamma(\mathbb{R}_+; E)}. \tag{3.13}$$

This shows that $\omega \mapsto \|\phi(\cdot, \omega)\|_{\gamma(\mathbb{R}_+, [M](\cdot,\omega); E)}$ is measurable and (3.2) follows from $\zeta(t) = \zeta_\psi([M]_t)$, (3.1) and (3.13). □

Proposition 3.7 has the following extension to E-valued processes.

Corollary 3.9. *Let E be a UMD space. Under the assumptions and the notation of Theorem 3.6, we have the following time-change formula for stochastic integrals. If $\phi : \mathbb{R}_+ \times \Omega \to E$ is strongly $\mathscr{F}$-progressively measurable and satisfies, $\phi \in \gamma(\mathbb{R}_+, [M]; E)$ almost surely, then the process $\psi : \mathbb{R}_+ \times \Omega \to E$ defined as* (3.11) *is $\mathscr{G}$-adapted and satisfies, $\psi \in \gamma(\mathbb{R}_+; E)$ almost surely, and the E-valued versions of* (3.9) *and* (3.10) *hold.*

Proof. In Theorem 3.3 we have already showed that ψ is $\mathscr{G}$-adapted and almost surely, $\psi \in \gamma(\mathbb{R}_+; E)$. Also the E-valued version of (3.9) has been obtained there. From (3.9) we deduce that

$$\int_0^{\tau_s} \phi(r) \, dM(r) = \int_0^{[M]_{\tau_s}} \psi(r) \, dW(r) = \int_0^{s\wedge[M]_\infty} \psi(r) \, dW(r) = \int_0^s \psi(r) \, dW(r)$$

and the E-valued version of (3.10) follows. □

To end this section we give the following useful convergence result for the stochastic integral.

Corollary 3.10. *Let E be a UMD space. For each $n \geq 1$ let $\phi_n : \mathbb{R}_+ \times \Omega \to E$ be stochastically integrable and let $\zeta_n \in L^0(\Omega; C_b(\mathbb{R}_+; E))$ denote its stochastic integral. Then we have $\phi_n \to 0$ in $L^0(\Omega; \gamma(\mathbb{R}_+, [M]; E))$ if and only if $\zeta_n \to 0$ in $L^0(\Omega; C_b(\mathbb{R}_+; E))$.*

Proof. This follows from [19, Theorem 5.5], Lemma 3.5 and Corollary 3.9. □

4. Applications

Several results in [19] can be extended to the case of general continuous local martingales. Below we state and prove some of the fundamental results.

The following characterization for stochastic integrability follows for instance from [19, Corollary 3.11], Lemma (3.5), Theorem 3.3 and the substitution rule (3.4).

Corollary 4.1. *Let E be UMD Banach function space over a σ-finite measure space (S, Σ, μ) and let $p \in (1, \infty)$. Let $\phi : \mathbb{R}_+ \times \Omega \to E$ be a strongly progressively measurable process. Then ϕ is stochastically integrable with respect to M if and only if almost surely*

$$\Big\| \Big(\int_{\mathbb{R}_+} |\phi(t, \cdot)|^2 \, d[M]_t \Big)^{\frac{1}{2}} \Big\|_E < \infty.$$

In this case, for all $p \in (1, \infty)$ we have

$$\mathbb{E} \sup_{t \in \mathbb{R}_+} \Big\| \int_0^t \phi(t) \, dM(t) \Big\|^p \eqsim_{p,E} \mathbb{E} \Big\| \Big(\int_{\mathbb{R}_+} |\phi(t, \cdot)|^2 \, d[M]_t \Big)^{\frac{1}{2}} \Big\|_E^p.$$

Via the canonical embedding $L^2(\mathbb{R}_+, \mu; E) \hookrightarrow \gamma(L^2(\mathbb{R}_+, \mu); E)$ for a measure μ for type 2 spaces, and the reversed embedding for cotype 2 spaces we obtain

Corollary 4.2. *Let E be a UMD space and let $p \in (1, \infty)$.*

1. *If E has type 2, then every strongly progressively measurable process ϕ such that $\phi \in L^2(\mathbb{R}_+, [M]; E)$ almost surely is stochastically integrable with respect to M and we have*

$$\mathbb{E} \sup_{t \in \mathbb{R}_+} \Big\| \int_0^t \phi(t) \, dM(t) \Big\|^p \lesssim_{p,E} \mathbb{E} \|\phi\|^p_{L^2(\mathbb{R}_+, [M]; E)}.$$

2. *If E has cotype 2, then every H-strongly measurable stochastically integrable process ϕ belongs to $L^2(\mathbb{R}_+, [M]; E)$ almost surely and we have*

$$\mathbb{E} \|\phi\|^p_{L^2(\mathbb{R}_+, [M]; E)} \lesssim_{p,E} \mathbb{E} \sup_{t \in \mathbb{R}_+} \Big\| \int_0^t \phi(t) \, dM(t) \Big\|^p.$$

Of course, (1) (and (2)) can also be obtained from the fact that a UMD space with (co)type 2 is a martingale (co)type 2 space.

We say that a strongly progressive measurable process $\phi : \mathbb{R}_+ \times \Omega \to E$ is *locally stochastically integrable* with respect to M if for all $T \in \mathbb{R}_+$, we have that $\phi \mathbf{1}_{[0,T]}$ is scalarly in $L^2(\mathbb{R}_+, [M])$ a.s. and is stochastically integrable with respect to M. In that case we may define the stochastic integral as

$$\int_0^t \phi(s) \, dM(s) = \int_0^t \phi(s) \mathbf{1}_{[0,T]}(s) \, dM(s),$$

if $t \in [0,T]$ and $T \in \mathbb{R}_+$. This is well-defined and $\int_0^\cdot \phi(s)\,dM(s)$ has an a.s. pathwise continuous version.

We will say that $\phi \in \gamma_{\mathrm{loc}}(\mathbb{R}_+,[M];E)$ a.s. if for every $T \in \mathbb{R}_+$, $\phi\mathbf{1}_{[0,T]}$ is in $\gamma_{\mathrm{loc}}(\mathbb{R}_+,[M];E)$ a.s. It is a direct consequence of Theorem 3.3 that ϕ is locally stochastically integrable with respect to M if and only if $\phi \in \gamma_{\mathrm{loc}}(\mathbb{R}_+;E)$ a.s.

Theorem 4.3 (Itô formula)**.** *Let E and F be UMD spaces. Let M be a continuous local martingale and let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be adapted, almost surely continuous and locally of finite variation. Assume that $f : \mathbb{R}_+ \times E \to F$ is of class $C^{1,2}$. Let $\phi : \mathbb{R}_+ \times \Omega \to E$ be a strongly progressively measurable process which is locally stochastically integrable with respect to M and assume that the paths of ϕ belong to $L^2_{loc}(\mathbb{R}_+,[M];E)$ almost surely. Let $\psi : \mathbb{R}_+ \times \Omega \to E$ be strongly progressively measurable with paths in $L^1_{loc}(\mathbb{R}_+,A;E)$ almost surely. Let $\xi : \Omega \to E$ be strongly $\mathscr{F}_0$-measurable. Define $\zeta : \mathbb{R}_+ \times \Omega \to E$ as*

$$\zeta = \xi + \int_0^\cdot \psi(s)\,dA(s) + \int_0^\cdot \phi(s)\,dM(s)$$

Then $s \mapsto D_2 f(s,\zeta(s))\phi(s)$ is locally stochastically integrable with respect to M and almost surely we have, for all $t \in [0,T]$,

$$\begin{aligned} f(t,\zeta(t)) - f(0,\zeta(0)) &= \int_0^t D_1 f(s,\zeta(s))\,ds + \int_0^t D_2 f(s,\zeta(s))\psi(s)\,dA(s) \\ &\quad + \int_0^t D_2 f(s,\zeta(s))\phi(s)\,dM(s) \\ &\quad + \tfrac{1}{2}\int_0^t \big(D_2^2 f(s,\zeta(s))\big)(\phi(s),\phi(s))\,d[M](s). \end{aligned}$$

In the case that E and F have type 2, $M = W$ and $A(s) = s$, the above Itô formula is a special case of [21, Theorem 74] (also see [3]).

In particular, Theorem 4.3 can be applied to the case that $E = X \times X^*$, where X is a UMD space, $F = \mathbb{R}$, and $f : X \times X^* \to \mathbb{R}$ is given by $f(x,x^*) = \langle x, x^*\rangle$. We do not know how such a result could be obtained with the Itô formula from [3, 21] unless X is isomorphic to a Hilbert space.

Proof. It suffices to prove the result on an arbitrary bounded interval $[0,T]$, so take ψ and ϕ to be 0 on (T,∞). To proof the result it suffices to reduce to the case where ξ,ψ and ϕ take values in a finite dimensional subspace of E. The only non-trivial extension of the arguments in [17] is to construct stochastically integrable processes $(\phi_n)_{n\geq 1}$ taking values in a finite dimensional subspace of E such that $\phi = \lim_{n\to\infty} \phi_n$ in $\gamma(\mathbb{R}_+,[M];E) \cap L^2(\mathbb{R}_+,[M];E)$ almost surely.

Let $\tilde{\phi}$ be the process ψ from Corollary 3.9. Then it follows from a substitution that $\tilde{\phi} \in \gamma(\mathbb{R}_+;E) \cap L^2(\mathbb{R}_+;E)$. It follows from [17] that $\tilde{\phi}$ may be approximated almost surely in $\gamma(\mathbb{R}_+;E) \cap L^2(\mathbb{R}_+;E)$ by a sequence of elementary processes $\tilde{\phi}_n$. As in the proof of Theorem 3.3 we may assume that $\tilde{\phi}_n(s) = 0$ for $s \geq [M]_\infty$. Let $\phi_n(t) = \tilde{\phi}_n([M]_t)$. Then $\phi_n \circ \tau = \tilde{\phi}_n$. It follows from Lemma 3.5 that $\phi = \lim_{n\to\infty} \phi_n$ in $\gamma(\mathbb{R}_+,[M];E)$ almost surely. It follows from (3.4) that $\phi = \lim_{n\to\infty} \phi$ in $L^2(\mathbb{R}_+,[M];E)$ almost surely. The rest of the arguments are similar as in [17]. $\square$

Let M be a real-valued continuous local martingale and let $\mathscr{F}$ be a filtration that satisfies the usual conditions and to which M is adapted. We say that the pair $(M, \mathscr{F})$ satisfies *the martingale representation property* if for any real-valued $\mathscr{F}$-local martingale ζ, there is a progressively measurable process ϕ with almost all paths in $L^2_{\rm loc}(\mathbb{R}_+, [M])$ and such that for all $t \in \mathbb{R}_+$, we have

$$\zeta(t) = \zeta(0) + \int_0^t \phi(s)\, dM(s) \quad a.s.$$

For a detailed study on martingale representation properties, we refer to [22].

As an application of the results of Section 3, we extend the representation property to E-valued local martingales. It should be observed that the proof critically depends on the fact that we are able to give two-sided estimates in Theorem 3.3 and necessary and sufficient convergence results in Corollary 3.10.

Theorem 4.4 (Martingale representation theorem)**.** *Let E be a UMD space with cotype 2 and let $p \in (1, \infty)$. Assume that the pair $(M, \mathscr{F})$ satisfies the martingale representation theorem. The following assertions hold:*

(1) *If $\xi \in L^p(\Omega, \mathscr{F}_\infty; E)$ has mean zero, then there exists a strongly progressively measurable process ϕ which is in $L^p(\Omega; \gamma(\mathbb{R}_+, [M]; E))$ such that*

$$\int_0^\infty \phi(s)\, dM(s) = \xi \quad a.s.$$

(2) *If ζ is an E-valued $\mathscr{F}$-local martingale, then it has a version with continuous paths and there is an a.s. unique strongly progressively measurable process ϕ which is in $\gamma_{loc}(\mathbb{R}_+, [M]; E)$ a.s. such that for all $t \in \mathbb{R}_+$, we have*

$$\zeta(t) = \zeta(0) + \int_0^t \phi(s)\, dM(s) \quad a.s.$$

Proof. (1): Let $L^p_{\gamma,\rm progr}(E)$ denote the subspace of progressively measurable processes in $L^p(\Omega; \gamma(\mathbb{R}_+, [M]; E))$. Let $L^p_0(\Omega, \mathscr{F}_\infty; E)$ denote the closed subspace of random variables in $L^p(\Omega, \mathscr{F}_\infty; E)$ with mean zero. Define $I_p : L^p_{\gamma,\rm progr}(E) \to L^p_0(\Omega; E)$ as

$$I_p \phi = \int_0^\infty \phi(t)\, dM(t).$$

This is well-defined by Theorem 3.3 and it follows from (3.2) and Doob's maximal inequality that it is an isomorphism onto its range in $L^p_0(\Omega, \mathscr{F}_\infty; E)$.

Fix $\omega \in \Omega$ and write $A = [M](\cdot, \omega)$. Since E has cotype 2, $\gamma(L^2(\mathbb{R}_+, A); E)$ embeds continuously into $L^2(\mathbb{R}_+, A; E)$. Therefore, each element in $\gamma(L^2(\mathbb{R}_+, A); E)$ is represented by a function, i.e. $\gamma(L^2(\mathbb{R}_+, A); E) = \gamma(\mathbb{R}_+, A; E)$. It follows that $\gamma(\mathbb{R}_+, A; E)$ is a Banach space and therefore $L^p(\Omega; \gamma(\mathbb{R}_+, [M]; E))$ is a Banach space. It is straightforward to check that $L^p_{\gamma,\rm progr}(E)$ is a closed subspace of $L^p(\Omega; \gamma(\mathbb{R}_+, [M]; E))$.

By the above properties I_p has a closed range in $L^p_0(\Omega, \mathscr{F}_\infty; E)$. We claim that I_p is surjective. To prove this, it suffices to show that I_p has dense range. One easily checks that the random variables of the form $\sum_{n=1}^N (\mathbf{1}_{A_n} - \mathbb{P}(A_n)) \otimes x_n$ with each $A_n \in \mathscr{F}_\infty$ and $x_n \in E$, form a dense subspace of $L^p_0(\Omega, \mathscr{F}_\infty; E)$. By linearity, for each $A \subset \mathscr{F}_\infty$, it suffices to find a progressively measurable process $\phi \in L^p(\Omega; L^2(\mathbb{R}_+, [M]))$ such that

$$\int_0^\infty \phi(t)\, dM(t) = \mathbf{1}_A - \mathbb{P}(A) \quad a.s.$$

Define $\zeta(t) = \mathbb{E}(\mathbf{1}_A - \mathbb{P}(A)|\mathscr{F}_t)$. Then ζ is a real-valued martingale, and by the assumptions on $(M, \mathscr{F}_\infty)$ there exists a progressively measurable ϕ with paths in $L^2(\mathbb{R}_+, [M])$ a.s. such that for all $t \in \mathbb{R}_+$, we have

$$\int_0^t \phi(s)\, dM(s) = \zeta(t) \quad a.s.$$

This shows that ζ has a pathwise continuous version, say $\tilde{\zeta}$. By Doob's maximal inequality and the contractiveness of the conditional expectation one has

$$\mathbb{E} \sup_{t\in\mathbb{R}_+} |\tilde{\zeta}(t)|^p \leq C_p \sup_{t\in\mathbb{R}_+} \mathbb{E}|\tilde{\zeta}(t)|^p = C_p \sup_{t\in\mathbb{R}_+} \mathbb{E}|\zeta(t)|^p \leq C_p \mathbb{E}|\mathbf{1}_A - P(A)|^p < \infty.$$

By Burkholder-Davis-Gundy inequality we obtain that

$$\|\phi\|_{L^p(\Omega;L^2(\mathbb{R}_+,[M]))} \eqsim_p \mathbb{E} \sup_{t\in\mathbb{R}_+} |\tilde{\zeta}(t)|^p < \infty.$$

This proves the result.

(2): We may assume $\zeta(0) = 0$.

Step 1: First assume that there is a $T > 0$ such that $\zeta_T = \zeta_t \in L^1(\Omega; E)$ for all $t > T$. Then clearly, $\zeta_\infty = \zeta_T$ exists in $L^1(\Omega, \mathscr{F}_\infty; E)$. We show that ζ has a continuous version.

Choose $(\xi_n)_{n\geq1}$ in $L^2(\Omega, \mathscr{F}_\infty; E)$ such that $\zeta_\infty = \lim_{n\to\infty} \xi_n$. By (1), the martingales $(\zeta^n)_{n\geq1}$ defined by $\zeta^n_t = \mathbb{E}(\xi_n|\mathscr{F}_t)$ have a version with bounded and continuous paths, say $\tilde{\zeta}^n$. It follows from Doob's maximal inequality that for all $\varepsilon > 0$ and $n, m \geq 1$,

$$\mathbb{P}(\sup_{t\in\mathbb{R}_+} \|\tilde{\zeta}^n(t) - \tilde{\zeta}^m(t)\| > \varepsilon) \leq \varepsilon^{-1}\mathbb{E}\|\xi_n - \xi_m\|.$$

This shows that $(\tilde{\zeta}^n)_{n\geq1}$ is a Cauchy sequence in $L^0(\Omega; C_b(\mathbb{R}_+; E))$. Its limit is the required version of ζ.

Step 2: Under the assumption of Step 1, we show that there is a strongly progressively measurable ϕ with paths in $\gamma(\mathbb{R}_+, [M]; E)$ a.s. such that for all $t \in \mathbb{R}_+$, we have

$$\int_0^t \phi(s)\, dM(s) = \zeta(t) \quad a.s. \tag{4.1}$$

Let ζ denote the version constructed in Step 1. For each $n \geq 1$ define a stopping time τ_n as

$$\tau_n = \inf\{t \geq 0 : \|\zeta_t\| \geq n\} \wedge n.$$

It follows from (1) that there is a sequence $(\phi_n)_{n\geq1}$ of strongly progressively measurable processes with paths in $\gamma(\mathbb{R}_+[M]; E)$ a.s. such that

$$\int_0^\cdot \phi_n(t)\, dM(t) = \zeta^{\tau_n}.$$

Clearly, $(\zeta^{\tau_n})_{n\geq1}$ converges to ζ in $L^0(\Omega; C_b(\mathbb{R}_+; E))$. It follows from Corollary 3.10 that $(\phi_n)_{n\geq1}$ is a Cauchy sequence in $L^0(\Omega; \gamma(\mathbb{R}_+, [M]; E))$ and therefore it converges to some $\phi \in L^0(\Omega; \gamma(\mathbb{R}_+, [M]; E))$ (here we use the completeness of $\gamma(\mathbb{R}_+, [M]; E)$ again). Now (4.1) follows from Corollary 3.10.

Step 3: We prove the general case. The uniqueness follows from Corollary 3.10. Let $(\tau_n)_{n\geq1}$ be a localizing sequence for ζ. For each $n \geq 1$, the martingale $\eta^n = \zeta^{n\wedge\tau_n}$ satisfies the above properties with $T = n$. Since the result holds for each η^n, we obtain a sequence $(\phi_n)_{n\geq 1}$ in $L^0(\Omega;\gamma(\mathbb{R}_+,[M];E))$ such that for all $t\in\mathbb{R}_+$, we have

$$\eta^n(t) = \int_0^t \phi_n(s)\,dM(s) \quad a.s.$$

By the uniqueness it follows that for all $1\leq m\leq n$, a.s., for a.a. $t<\tau_m$, $\phi_n(t) = \phi_m(t)$. Therefore, we may take $\phi(t)=\phi_n(t)$ for $t<\tau_n$. □

Remark 4.5.
The assumption that E has cotype 2 in Theorem 4.4 is only needed for technical reasons. Namely under this assumption every operator in $\gamma(L^2(\mathbb{R}_+,[M](\cdot,\omega)),E)$ can be represented by a function. In general one obtains in (1) that there exists an operator X in the completion of $L^p_{\gamma,\mathrm{progr}}(E)$ in $L^p(\Omega;\gamma(L^2(\mathbb{R}_+,[M]);E))$ such that $\widehat{I_p}(X)=\xi$ a.s. Here $\widehat{I_p}$ denotes the unique continuous extension of I_p to the completion of $L^p_{\gamma,\mathrm{progr}}(E)$. A similar result holds in (2).

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