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## GAPS IN INTERVALS OF $N$ -EXPANSIONS

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### Abstract

For  $N \in \mathbb{N}_{\geq 2}$  and  $\alpha \in \mathbb{R}$  such that  $0 < \alpha \leq \sqrt{N} - 1$ , the continued fraction map  $T_\alpha : [\alpha, \alpha + 1] \rightarrow [\alpha, \alpha + 1]$  is defined as  $T_\alpha(x) := N/x - d(x)$ , where  $d : [\alpha, \alpha + 1] \rightarrow \mathbb{N}$  is defined by  $d(x) := \lfloor N/x - \alpha \rfloor$ . A maximal open interval  $(a, b) \subset I_\alpha$  is called a *gap* of  $I_\alpha$  if for almost every  $x \in I_\alpha$  there is an  $n_0(x) \in \mathbb{N}$  such that  $x_n \notin (a, b)$  for all  $n \geq n_0$ . In this paper, all conditions are given in which  $I_\alpha$  is gapless. For  $\alpha = \sqrt{N} - 1$  it is shown that the number of gaps is a finite, monotonically non-decreasing and unbounded function of  $N$ .

### 1. Introduction

Recently, H. Nakada and the authors of the present paper studied  $N$ -expansions with finitely many digits in [1]. These  $N$ -expansions are continued fraction expansions, introduced in [2] (see [3]), and defined as follows.

**Definition 1.** For  $N \in \mathbb{N}_{\geq 2}$  and  $\alpha \in \mathbb{R}$  such that  $0 < \alpha \leq \sqrt{N} - 1$ , let  $I_\alpha := [\alpha, \alpha + 1]$  and  $I_\alpha^- := [\alpha, \alpha + 1)$ . We define the  $N$ -expansion map  $T_\alpha : I_\alpha \rightarrow I_\alpha^-$  as

$$T_\alpha(x) := \frac{N}{x} - d(x), \tag{1}$$

where  $d : I_\alpha \rightarrow \mathbb{N}$  is defined by

$$d(x) := \left\lfloor \frac{N}{x} - \alpha \right\rfloor, \quad \text{if either } x \in (\alpha, \alpha + 1] \text{ or both } x = \alpha \text{ and } \frac{N}{\alpha} - \alpha \notin \mathbb{Z}.$$

So as to avoid having cylinders consisting of one point, we also have

$$d(\alpha) = \left\lfloor \frac{N}{\alpha} - \alpha \right\rfloor - 1, \quad \text{if } \frac{N}{\alpha} - \alpha \in \mathbb{Z}.$$

For a fixed  $\alpha \in (0, \sqrt{N} - 1]$  and  $x \in I_\alpha$ , we define

$$d_n = d_n(x) := d(T_\alpha^{n-1}(x)) \text{ for } n \in \mathbb{N}.$$

Throughout this paper, we assume that  $N \in \mathbb{N}_{\geq 2}$ . We will write  $\alpha_i$  for  $T_\alpha^i(\alpha)$  and  $\beta_i$  for  $T_\alpha^i(\alpha + 1)$ , with  $i \in \mathbb{N}$ ; we also often write  $\beta$  for  $\alpha + 1$ . Applying (1), for every  $x \in I_\alpha$  we obtain a continued fraction expansion of the form

$$x = T_\alpha^0(x) = \frac{N}{d_1 + T_\alpha(x)} = \frac{N}{d_1 + \frac{N}{d_2 + \frac{N}{d_3 + \ddots}}}.$$

Throughout this paper, we will write  $x = [d_1, d_2, d_3, \dots]_{N, \alpha}$  to represent the above equation. For reasons of legibility, we will usually omit suffixes such as ‘(N)’, ‘(N, α)’ or ‘(N, d)’. We will call the numbers  $d_i$ , with  $i \in \mathbb{N}$ , the *partial quotients* or *digits* of this *N-continued fraction expansion* of  $x$ . Since  $0 \notin I_\alpha$ , this expansion is infinite for every  $x \in I_\alpha$ . We write  $x_n := T_\alpha^n(x)$ , with  $n \in \mathbb{N} \cup \{0\}$ . The sequence  $x_n$ ,  $n = 0, 1, 2, \dots$ , is called the *orbit of x under  $T_\alpha$* . When  $x = [d_1, d_2, d_3, \dots]$  and there are smallest  $h, k \in \mathbb{N}$  such that  $d_{h+i} = d_{h+nk+i}$  for all  $n \in \mathbb{N}$  and  $i \in \{0, \dots, k - 1\}$ , we call the expansion *eventually periodic* with *period length  $|x| = k$*  and denote the periodic part as  $\overline{d_h, \dots, d_{h+k-1}}$ . If  $h = 1$ , we write  $x = [d_1, \dots, d_k]$ . In this latter case,  $x$  is called a *periodic point* with a *purely periodic expansion* and *periodic orbit*.

Let  $N \in \mathbb{N}_{\geq 2}$  and  $\alpha \in (0, \sqrt{N} - 1]$ . Then in each *cylinder set*  $\Delta_i := \{x \in I_\alpha; d(x) = i\}$  of rank 1, with  $d_{\min} \leq i \leq d_{\max}$ , where  $d_{\max} := d(\alpha)$  is the largest partial quotient<sup>1</sup> and  $d_{\min} := d(\alpha + 1)$  the smallest one, the map  $T_\alpha$  obviously has one *fixed point*  $f_i$ . From now on, we will write simply ‘cylinder set’ for ‘cylinder set of rank 1’. It is easy to see that

$$f_i = f_i(N) := \frac{\sqrt{4N + i^2} - i}{2}, \text{ for } d_{\min} \leq i \leq d_{\max}.$$

Note that  $N/\alpha - \alpha \in \mathbb{Z}$  if and only if for some  $d \geq 2$  we have that  $d + 1 = \max\{d_i\}$  for any  $\alpha_0 < \alpha$ , i.e.,  $\Delta_{d+1} \neq \emptyset$  for  $\alpha_0 < \alpha$ , and  $\alpha = f_{d+1}$ . Moreover, we can write  $f_i = [\tilde{i}]$ . Note that fixed points are periodic points with period length 1.

Given  $N$ , we let  $\alpha_{\max} = \sqrt{N} - 1$  be the largest value of  $\alpha$  we consider, so as to avoid 0 being a partial quotient as well. Since  $T'_\alpha(x) = -N/x^2$  and because

<sup>1</sup>Note that the number of occurring digits is finite for  $\alpha > 0$ .

$0 < \alpha \leq \sqrt{N} - 1$ , we have  $|T'_\alpha(x)| > 1$  on  $I_\alpha^-$ . From this, it follows that the fixed points act as *repellers* and that the maps  $T_\alpha$  are *expanding* when  $0 < \alpha \leq \sqrt{N} - 1$ .

As in the previous paper (see [1]), we are interested in subsets of  $I_\alpha$  that we will call *gaps*, defined as follows.

**Definition 2.** A maximal open interval  $(a, b) \subset I_\alpha$  is called a *gap* of<sup>2</sup>  $I_\alpha$  if for almost every<sup>3</sup>  $x \in I_\alpha$  there is an  $n_0(x) \in \mathbb{N}$  such that  $x_n \notin (a, b)$  for all  $n \geq n_0$ .

Figure 1 is an example of an interval  $I_\alpha$  having two gaps. In this case, we have  $N = 100$  and  $\alpha \approx 8.983$ . It appears that in this example the boundaries of the gaps are the first two images of  $\alpha$  and  $\beta$ , respectively, under  $T_\alpha$ . In Theorem 5, we show that the boundaries between gaps and non-gaps are *always* images of  $\alpha$  and  $\beta$ .

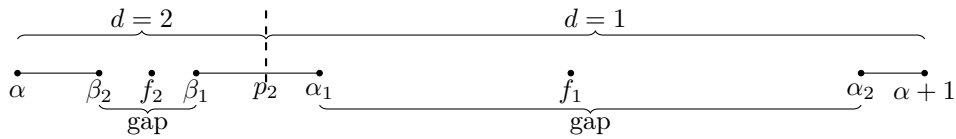


Figure 1: An interval  $I_\alpha$  with two gaps. Here  $N = 100$  and  $\alpha \approx 8.983$

Each pair of consecutive cylinder sets  $(\Delta_i, \Delta_{i-1})$  is separated by a *discontinuity point*  $p_i(N, \alpha)$  of  $T_\alpha$ , satisfying  $N/p_i - i = \alpha$ , so  $p_i = N/(\alpha + i)$ . In Figure 1,  $\alpha$  is such that  $\alpha_3 = p_2$ , implying  $\alpha \approx 8.983$ . A cylinder set  $\Delta_i$  is called *full* if  $T_\alpha(\Delta_i) = I_\alpha^-$  (or  $T_\alpha(\Delta_d) = I_\alpha$  in case  $d = d(\alpha)$  and  $T_\alpha(\alpha) = \beta$ ). When a cylinder set is not full, it contains either  $\alpha$  (in which case  $T_\alpha(\alpha) < \alpha + 1$ ) or  $\alpha + 1$  (in which case  $T_\alpha(\alpha + 1) > \alpha$ ), and is called *incomplete*. On account of our definition of  $T_\alpha$ , cylinder sets will always be an interval, and *never* consist of one single point. An interval  $I_\alpha$  together with its cylinder sets, associated fixed points and discontinuity points, is called an *arrangement* of  $I_\alpha$ , depending on  $N$ . An arrangement is called *full* when its cylinder sets are all full. Finally, a cylinder set is called *gapless* if it has empty intersection with any gap.

In [1], some results are obtained that can quite easily be extended to a complete overview of gaplessness; we write  $d := d(\alpha)$ .

**Theorem 1.** *Let  $I_\alpha = \Delta_d \cup \Delta_{d-1}$ . Then  $I_\alpha$  is gapless if and only if  $\alpha_1 \geq f_{d-1}$  and  $\beta_1 \leq f_d$ .*

*Proof.* The condition for gaplessness is proved in [1, Theorem 5]. So let  $I_\alpha = \Delta_d \cup \Delta_{d-1}$  and suppose  $\alpha_1 < f_{d-1}$ . Since  $T_\alpha$  is expanding and  $\alpha_1 < f_{d-1}$ , we find

<sup>2</sup>We will usually omit the addition ‘of  $I_\alpha$ ’.

<sup>3</sup>Here we use ‘for almost all  $x$ ’ (and not ‘for all  $x$ ’) because we want to exclude fixed points and pre-images of fixed points, i.e., points that  $T_\alpha$  maps to a fixed point, which may never leave an interval  $(a, b)$ . All ‘for all’ statements in this paper are with respect to Lebesgue measure.

that  $f_{d-1} < \alpha_2$ . For the same reason, we have that  $(\beta_1, \alpha_3) \subset [\alpha, \alpha_1]$ . Consequently,  $T_\alpha^n(\Delta_d) \subset I_\alpha \setminus (\alpha_1, \alpha_2)$  for  $n \geq 1$ , with  $f_{d-1} \in (\alpha_1, \alpha_2)$ . Now let  $x \in I_\alpha \setminus \{f_{d-1}\}$ . Then there is a smallest non-negative integer  $n_0$  (possibly 0) such that  $T_\alpha^{n_0}(x) \in \Delta_d$ . Since  $T_\alpha^n(\Delta_d) \subset I_\alpha \setminus (\alpha_1, \alpha_2)$  for  $n \geq 1$ , it follows that  $T_\alpha^n(x) \notin (\alpha_1, \alpha_2)$  for  $n \geq n_0$ . We conclude that  $(\alpha_1, \alpha_2)$  is a gap containing  $f_{d-1}$ . The reasoning is completely similar in case  $\beta_1 > f_d$ .  $\square$

**Theorem 2.** *Let  $I_\alpha = \Delta_d \cup \dots \cup \Delta_{d-m}$ , with  $m \in \{2, 3\}$ . Then  $I_\alpha$  is gapless if and only if  $\alpha_1 \geq f_{d-1}$  or  $\beta_1 \leq f_{d-m+1}$ .*

*Proof.* The condition for gaplessness is proved in [1, Theorem 6]. We consider here the case  $m = 2$ ; the case  $m = 3$  is proved in a similar way. So let  $I_\alpha = \Delta_d \cup \Delta_{d-1} \cup \Delta_{d-2}$ , and suppose that  $\alpha_1 < f_{d-1} < \beta_1$ . Then  $T_\alpha((p_d, \alpha_1)) = [\alpha_2, \alpha + 1)$  and  $T_\alpha([\beta_1, p_{d-1})) = (\alpha, \beta_2]$  (\*). Let  $a = \max\{\alpha_1, \beta_2\}$  and  $b = \min\{\alpha_2, \beta_1\}$  and let  $x \in I_\alpha \setminus \{f_{d-1}\}$ . Then there exists a smallest  $n_0$  (possibly 0) depending on  $x$ , for which  $T_\alpha^{n_0}(x) \in I_\alpha \setminus (a, b)$ . But then it follows from (\*) that we have  $T_\alpha^n(x) \in I_\alpha \setminus (a, b)$  for all  $n \geq n_0$ , i.e.,  $T_\alpha^n(x) \notin (a, b)$ .  $\square$

In [1], the following two results were obtained (see [1, Theorems 7 and 4]).

**Theorem 3.** *Let  $N \in \mathbb{N}_{\geq 2}$  and  $I_\alpha = \Delta_d \cup \Delta_{d-1} \cup \Delta_{d-2} \cup \Delta_{d-3}$ . Then there is a gap in  $I_\alpha$  if and only if  $N = 2k^2 + 2k - i$ , with  $k > 1$  and  $i \in \{1, 2, 3\}$ . Moreover, if there is a gap in  $I_\alpha$ , the gap contains  $f_{d-1}$  and  $f_{d-2}$ , while  $\Delta_d$  and  $\Delta_{d-3}$  are gapless.*

**Theorem 4.** *Let  $I_\alpha$  consist of five cylinders or more. Then  $I_\alpha$  has no gaps.*

Knowing all about *gaplessness*, we want to start a thorough investigation of the occurrence of gaps. We will make a lot of use of the graphs of  $T_\alpha$ , which are drawn in the square<sup>4</sup>  $\Upsilon_{N,\alpha} := I_\alpha \times I_\alpha^-$ . This square is divided into rectangular sets of points  $\square_i := \{(x, y) \in \Upsilon_\alpha : d(x) = i\}$ . We identify these two-dimensional *fundamental regions*  $\square_i$  with the one-dimensional cylinder sets  $\Delta_i$  we already use. It is obvious that the graph of  $T_\alpha$  has one fixed point  $F_i := (f_i, f_i)$  in each  $\square_i$ . We will denote the dividing line between  $\square_i$  and  $\square_{i-1}$  by  $\ell_i$ , which is the line segment  $\{p_i\} \times [\alpha, \alpha + 1)$ , where  $p_i$  is the discontinuity point between  $\Delta_i$  and  $\Delta_{i-1}$ . Depending on the fullness of a cylinder set  $\Delta_i$ , we will call the concerning branch of the graph of  $T_\alpha$  in  $\square_i$  *complete* or *incomplete*. Finally, we will use the word ‘arrangement’ in a similar way for  $\Upsilon_\alpha$  together with its cylinder sets, fixed points and dividing lines as for  $I_\alpha$ . Figure 2 shows the two-dimensional version of Figure 1. In grey, we have drawn the domain for *cobweb plots* of points outside the gaps of  $I_\alpha$ , which we will call the *roadmap* for  $I_\alpha$  or  $\Upsilon_{N,\alpha}$ .

Our approach sometimes requires a shift of focus from gaps to gapless intervals of  $I_\alpha$ , the union of which is called the *attractor* of  $I_\alpha$ , denoted by  $A_\alpha$ . The maximal

<sup>4</sup>We have  $\Upsilon_{N,\alpha} := I_\alpha \times I_\alpha$  in case  $N/\alpha - \alpha \in \mathbb{Z}$ .



and  $T_\alpha^2(\Delta_{d-2}) \setminus (\Delta_{d-2} \cup [\beta_1, p_{d-1}] \cup \Delta_d) = (p_d, \beta_2]$ , the image of which is  $[\beta_3, \beta)$ . Second,  $f_d \leq \beta_2 \leq p_d$ , in which case  $T_\alpha^2(\Delta_{d-2}) \setminus (\Delta_{d-2} \cup [\beta_1, p_{d-1}] \cup \Delta_d) = \emptyset$ . If this is the case, all orbits except the fixed points pass through  $[\beta_1, p_{d-1}]$  and  $\Delta_d$ . Third,  $\beta_2 < f_d$ . Then  $T_\alpha^2(\Delta_{d-2}) \setminus (\Delta_{d-2} \cup [\beta_1, p_{d-1}]) = [\alpha, \beta_2]$ .

We conclude that all orbits except the fixed points will eventually pass through  $\Delta_d, \Delta_{d-2}, [\alpha, \alpha_3], [\alpha_2, \beta], [\beta_3, \beta)$  and  $[\alpha, \beta_2]$ . The establishment of the actual attractor  $A_\alpha$  will then depend on the position of the images of these last four intervals, all having points in the orbits of images of  $\alpha$  and  $\beta$  as endpoints. These intervals will expand under  $T_\alpha$  up to the point that an image contains a discontinuity point  $p \in \{p_d, p_{d-1}\}$ . Since we just saw that the image of an interval with  $p$  as left endpoint has  $\beta$  as right endpoint and an interval with  $p$  as right endpoint has  $\alpha$  as left endpoint, all orbits except the fixed points will eventually only be in intervals with images of  $\alpha$  and  $\beta$  as endpoints. This finishes the proof of Theorem 5.  $\square$

When studying gaps and attractors, we are mainly interested in the *characteristic part* of continued fraction expansions and their orbits, defined as follows.

**Definition 3.** Let  $x = [d_1, d_2, d_3, \dots]_{N,\alpha}$  be the  $N$ -expansion of  $x$ . The *characteristic part* of this expansion, CPE in short, is  $x_n = T_\alpha^n(x) = [d_{n+1}, d_{n+2}, \dots]_{N,\alpha}$ , where  $n \in \mathbb{N} \cup \{0\}$  is the smallest number such that  $x_n \in A_\alpha$ .

Investigating gaps and attractors involves determining not only the orbits of  $\alpha$  and  $\beta$ , but also comparing them with certain purely periodic orbits. The reason for this is that if one of the points of a periodic orbit is in a gap, all of its points are. In view of the previous theorems on gaplessness, we can confine our investigation of gaps to arrangements with two and three cylinders. For these arrangements we will find a formula for the number of gaps  $G(N, \alpha)$ . In this paper we will confine ourselves to the cases where  $\alpha = \alpha_{\max}$ . In a forthcoming paper, we will consider arrangements with two cylinders where  $\alpha < \alpha_{\max}$ .

## 2. Gaps in $I_\alpha$ when $\alpha = \alpha_{\max}$

Throughout the rest of this paper, we take  $\alpha = \sqrt{N} - 1$ . Applying Theorems 1 and 2, it is easily found that all arrangements for  $N \in \{2, 3, 4, 5, 6, 7, 8\}$  are gapless. Nonetheless, our approach involves that we consider the cases  $N \in \{4, 5, 6, 7, 8\}$  as well. Note that in case  $N = 4$ , we have a full arrangement. For  $N \geq 5$ , we have

$$d = d_{\max} = \left\lfloor \frac{N}{\sqrt{N} - 1} - (\sqrt{N} - 1) \right\rfloor = \left\lfloor \frac{2\sqrt{N} - 1}{\sqrt{N} - 1} \right\rfloor = 2$$

and

$$d_{\min} = \left\lfloor \frac{N}{\sqrt{N}} - (\sqrt{N} - 1) \right\rfloor = 1.$$



This means that we are solely dealing with arrangements  $I_\alpha = \Delta_2 \cup \Delta_1$ , the right one of which is full, yielding  $\beta_1 = \alpha$ . Therefore, referring to the proof of Theorem 5, we know that,  $\beta$  excluded, the endpoints of all intervals contained in  $A_\alpha$  are images of  $\alpha$  only.

The main goal of this paper is proving the following theorem, stating that in arrangements where  $\alpha = \alpha_{\max}$ , the number of gaps is an unbounded, stepwise, monotonically non-decreasing function of  $N$ .

**Theorem 6.** *Let  $N \in \mathbb{N}_{\geq 4}$ . Set  $\alpha := \sqrt{N} - 1$  and  $\alpha_i := T_\alpha^i(\alpha)$ , for  $i \geq 1$ . Define  $a := 2, 1, 1$  and  $b := 2, 2$  as strings of partial quotients so as to recursively define the  $N$ -expansions  $c_n$  as follows. First,  $c_0 := [\overline{1, 2}]$ ,  $c_1 := [\overline{1}]$ ,  $c_2 := [\overline{2}]$ ,  $c_3 := [\overline{a}]$  and  $c_4 := [\overline{a, b}]$ . For  $n \geq 5$ , if  $c_{n-1} = [\overline{j, k}]$ , then  $c_n = [\overline{j, k, j, \ell}]$ , with  $j$  a string of partial quotients and  $\{k, \ell\} = \{a, b\}$ . Let  $m \in \mathbb{N}$  be such that*

$$\alpha_{|c_i|} < \alpha_{2^{i-1}} \text{ for } i \in \{2, \dots, m\} \text{ and } \alpha_{|c_{m+1}|} > \alpha_{2^m}.$$

*Let  $\mathcal{G}_N$  be the collection of gaps of  $I_\alpha$ . Then every gap  $G \in \mathcal{G}_N$  contains exactly one point from one of the orbits in  $\bigcup_{i=1}^m \{c_i\}$ . Conversely, each of the points of the orbits<sup>5</sup> in  $\bigcup_{i=1}^m \{c_i\}$  is contained in some (unique)  $G \in \mathcal{G}_N$ . The number of gaps  $G(N) = |\mathcal{G}_N|$  is therefore  $\sum_{i=1}^m |c_i|$ , which can be expressed more explicitly as*

$$G(N) = \begin{cases} \frac{2^{m+1} - 1}{3}, & \text{when } m \text{ is odd;} \\ \frac{2^{m+1} - 2}{3}, & \text{when } m \text{ is even.} \end{cases}$$

*In particular,  $G$  is a finite, monotonically non-decreasing and unbounded function of  $N$ .*

Our approach of the proof of Theorem 6 is based on extending a property of fixed points to periodic points, or rather considering the fixed points as special cases of periodic points. This property is that if an interval  $J \subset I_\alpha$  contains a periodic point, the interval expands along its orbit. We will use this for a classification of attractors according to the number of disjoint intervals they consist of. We define

$$C_0 := \{4\} \text{ and } C_n := \{N \in \mathbb{N}_{\geq 5} \mid c_i \notin A_\alpha \text{ for } 0 \leq i < n, \text{ and } c_n \in A_\alpha\}. \quad (2)$$

We claim that for each class  $C_n$  there is a unique periodic point  $c_n$  such that all gaps of  $I_\alpha$  contain exactly one orbit point of  $\bigcup_{i=1}^{n-1} c_i$ . We call this point  $c_n$  the *characteristic point* of  $C_n$ . For the orbit points of  $c_n$ , we will use the notation  $c'_n = T_\alpha(c_n), \dots, c'''_n = T_\alpha^3(c_n)$  and  $c_n^{(i)} = T_\alpha^{(i)}(c_n)$  for  $i \geq 4$ . We will shortly introduce some equations and inequalities as tools for our proof. Then we will

<sup>5</sup>Note that for  $N = 2, \dots, 8$  the sets  $\mathcal{G}_N$  and  $\bigcup_{i=1}^m \{c_i\}$  are empty.

explain our approach for the determination of the attractors for  $4 \leq N \leq 532$ , divided into five classes, corresponding with the following values:  $N = 4$ ;  $5 \leq N \leq 8$ ;  $9 \leq N \leq 32$ ;  $33 \leq N \leq 132$  and  $133 \leq N \leq 532$ , with characteristic points  $c_0$  through  $c_4$ , respectively. Finally, we will generalize our approach and prove Theorem 6 by induction.

Since the composition of the attractors depends completely on the expansiveness of  $T_\alpha$ , and all expansions take place in  $\Delta_1$  or  $\Delta_2$ , we provide some useful tools for estimating expansion factors of intervals. As a start, let interval  $M \subset I_\alpha$  be any interval not containing  $p_2$ , with endpoints  $j$  and  $k$ , where  $j < k$ , or having  $p_2$  as an endpoint. Then the expansion factor  $g(M)$  under one iteration of  $T_\alpha$  is the geometric mean of  $|T'_\alpha(j)|$  and  $|T'_\alpha(k)|$ , which we can write as follows<sup>6</sup>:

$$|T_\alpha(M)| = \sqrt{|T'_\alpha(j)| \cdot |T'_\alpha(k)|} \cdot |M| = \frac{k-j}{kj} N = \frac{N}{jk} \cdot |M|,$$

so that

$$g(j, k) = \frac{N}{jk}. \tag{3}$$

For  $M \subset [f_2, p_2]$ , we define  $g_\ell(M)$  as the multiplication factor under *two* iterations of  $T_\alpha$ . We find:

$$g_\ell(M) = \frac{N^2}{(N-2j)(N-2k)}. \tag{4}$$

Taking  $j$  and  $k$  as small as possible and as large as possible, respectively, i.e., taking limits in Equation (4) of both  $j$  and  $k$  to  $f_2$  and  $p_2$ , respectively, we find a lower and an upper bound for  $g_\ell$ .

$$\frac{(\sqrt{N+1}+1)^4}{N^2} < g_\ell(M) < \frac{(\sqrt{N}+1)^4}{(N-1)^2}. \tag{5}$$

For  $M \subset (p_2, f_1]$ , we define  $g_2(M)$  as the multiplication factor under *two* iterations of  $T_\alpha$ . We find:

$$g_2(M) = \frac{N^2}{(N-j)(N-k)}. \tag{6}$$

Taking  $j$  and  $k$  as small as possible and as large as possible, respectively, in Equation (6), i.e., taking limits to  $p_2$  and  $f_1$ , respectively, we find a lower and an upper bound for  $g_2$ .

$$\left(1 + \frac{1}{\sqrt{N}}\right)^2 < g_2(M) < \left(1 + \frac{1}{\sqrt{N}} + \frac{1}{N}\right)^2. \tag{7}$$

For  $M \subset (p_2, f_1]$ , we define  $g_r(M)$  as the multiplication factor under *three* iterations of  $T_\alpha$ , provided that  $T_\alpha^2(M) \subset \Delta_2$ , i.e.,  $k \leq N/(N/(p_2+1)+1) = (N^2 + N\sqrt{N} +$

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<sup>6</sup>This formula is not limited to  $\alpha = \alpha_{\max}$ .

$N)/(N\sqrt{N} + 2N + \sqrt{N} + 1)$ . We find:

$$g_r(M) = \frac{N^3}{((N + 1)j - N)((N + 1)k - N)}. \tag{8}$$

Taking  $j$  and  $k$  as large as possible and as small as possible, respectively, in Equation (8), i.e., taking limits to  $N/(N/(p_2 + 1) + 1)$  and  $p_2$ , respectively, we find a lower and an upper bound for  $g_r$ .

$$\frac{(\sqrt{N} + 1)^4}{N^2} < g_r(M) < \frac{(\sqrt{N} + 1)^4}{(N - 1)^2}. \tag{9}$$

**Remark 1.** Note that all bounds are decreasing functions of  $N$ , reflecting the decrease of expansiveness of  $T_\alpha$  on  $I_\alpha$  as  $N$  increases.

**Remark 2.** For  $N$  increasing, with  $N \geq 33$ , the gaps containing the fixed points increase and all disjoint maximal subsets of  $A_\alpha$  are images of the maximal subset of  $A_\alpha$  containing  $p_2$ . As a consequence, the upper bounds approximate the actual factors generally (much) better than the lower bounds do.

**Remark 3.** Note that in the double inequalities (5), (7) and (9) only one side is optimal/sharp. The other bounds are chosen so as to ease computations, and are sufficient for this purpose; see also Remark 4. Moreover, the upper bound we use for  $g_\ell$  is equal to the upper bound we use for  $g_r$ . Finally, the lower bound we use for  $g_r$  is actually  $(T'_\alpha(p_2))^2$ . The lower bound for  $g_\ell$  is larger than that, but for our purposes it is useful to always use the lower bound of  $g_r$  only.

**Remark 4.** If for  $x \in (p_2, f_1)$  such that  $x_2 < p_2$  (implying  $p_2 < x < N/(N/(p_2 + 1) + 1)$ ) we take  $J_r = (p_2, x]$  and  $J_\ell = [2p_2 - x, p_2]$  (implying  $|J_\ell| = |J_r|$ ), we have

$$g_2(J_r) = \frac{N + \sqrt{N}}{N - x},$$

and since  $g((j, k)) = N/(jk)$  (see Equation (3)), we find

$$g(J_\ell) = \frac{(\sqrt{N} + 1)^2}{2N - x(\sqrt{N} + 1)}. \tag{10}$$

It is not hard to show that  $g(J_\ell) > g_2(J_r)$ , where the difference between the two factors is comparatively small. With some intricate but otherwise basic computations<sup>7</sup> it can be found that for  $x = \alpha_1$  the factor  $g(J_\ell)$  is then only

$$\frac{2(\sqrt{N} - 1)}{N^2 - N\sqrt{N} - 2\sqrt{N}} \cdot 100$$

percent larger than  $g_2(J_r)$ ; for  $N \geq 35$  this percentage is even smaller than 1.

<sup>7</sup>For almost all computations in this paper we made use of Wolfram's *Mathematica*.

We will now continue with the preparation of the proof of Theorem 6. First, we note that by definition of  $A_\alpha$ , for  $x \in I_\alpha$  and interval  $J \subset A_\alpha$ , we have  $x \in A_\alpha$  if and only if there is an integer  $n$  – equal to 0 if  $x \in J$  – such that  $x \in T_\alpha^n(J)$ . If  $J$  contains  $p_2$ , say  $J = J_\ell \cup J_r$ , where  $p_2$  is the right endpoint of  $J_\ell$  and the left endpoint of  $J_r$ , the two subsets expand with different expansion factors. Note that both  $T_\alpha(J_\ell)$  and  $T_\alpha^2(J_r)$  have  $\alpha$  as left endpoint, so expansion of  $J$  after multiple iterations of  $T_\alpha$  depends on the expansion of the larger one of  $T_\alpha(J_\ell)$  and  $T_\alpha^2(J_r)$  or, equivalently, on the expansion of  $J' = T_\alpha(J_\ell) \cup T_\alpha^2(J_r)$ . Now let  $m$  be an integer with either  $m = 0$  or  $m > 0$  such that  $p_2 \in T_\alpha^m(J')$  and  $p_2 \notin T_\alpha^i(J')$  for  $0 \leq i < m$ . We define

$$E(J) = \frac{\max\{|T_\alpha^{m+1}(J_\ell)|, |T_\alpha^{m+2}(J_r)|\}}{\max\{|J_\ell|, |J_r|\}}.$$

Then  $E(J) > 2$  is clearly a sufficient condition for  $J$  to expand after  $m+1$  or  $m+2$  iterations of  $T_\alpha$ . In the rest of this paper we will always use  $J, J_\ell, J_r, J'$  and  $m$  in the previous sense.

**Remark 5.** Although  $E(J) > 2$  is not a very sharp condition for the growth of  $J$  under  $T_\alpha$ , it is useful for the determination of the classes  $C_2, C_3$  and  $C_4$  (see (2)). Once we have done that, we are able to focus on the induction step of the proof of Theorem 6.

We will show that for all  $N$  in  $C_2, C_3$  and  $C_4$ , the characteristic points of the related arrangements are  $c_2, c_3$  and  $c_4$ , respectively. Although this approach (with minor adaptations) is also valid for  $C_0$  and  $C_1$ , we grant these classes only two observations. First, although all arrangements for  $4 \leq N \leq 8$  are gapless, the fullness of the arrangement for  $N = 4$  makes an important difference: the periodic points  $c_0 = [1, 2]$  and  $c'_0 = [2, 1]$  exist in  $I_\alpha$  only if  $N = 4$ . Secondly, only for  $4 \leq N \leq 8$  the periodic point  $c_1$  is not contained in a gap. This is mainly why  $C_0$  consists of  $N = 4$  only and  $C_1$  consists of  $5 \leq N \leq 8$ .

For all  $N \geq 9$  the interval  $(\alpha_1, \alpha_2)$  is a gap; see Figure 3. As a consequence, the attractor  $A_\alpha$  consists of two components. Moreover, in CPEs the sequence 2, 1 is always followed by the digit 1, implying that, apart from  $c_2 = f_2$  (with period length 1), periodic point  $c_3 = [2, 1, 1]$  has the smallest period length.

Next assuming that  $N \geq 9$ , we investigate the expansion of open intervals  $J \subset (I_\alpha \setminus (\alpha_1, \alpha_2))$ . If  $c_2 = [2] \in J_\ell$ , there is an  $n$  such that  $\Delta_2 \subset T_\alpha^n(J)$ . Since  $T_\alpha(\Delta_2) = \Delta_2 \cup (p_2, \alpha_1]$ , we find that  $J$  expands to  $I_\alpha \setminus (\alpha_1, \alpha_2)$ . If  $c_2 \notin J'$  and  $p_2 \in T_\alpha(J')$ , then  $(p_2, \alpha_1] \subset T_\alpha(J')$ . Since  $T_\alpha((p_2, \alpha_1]) = [\alpha_2, \beta)$  and  $T_\alpha([\alpha_2, \beta)) = (\alpha, \alpha_3]$ , we find that  $c_2$  is contained in an image of  $J$  only if  $\alpha_3 \geq c_2$ , which is equivalent to  $\alpha_3 \geq \alpha_4$ . Disregarding that  $N \in \mathbb{N}$ , we solve  $\alpha_3 = \alpha_4$  and find  $N = 32.9777\dots$ . Note that in this case, we would have  $\alpha_3 = \alpha_4 = c_2 = f_2$ , so we could have also solved  $\alpha_3 = f_2$ . If  $p_2 \notin T_\alpha(J')$ , then  $m \geq 3$ , yielding  $E(J) > (1 + 1/\sqrt{N})^6$ , which is larger than 2 for  $N \leq 66$  and amply larger than 2 for  $N = 32$ . We conclude that

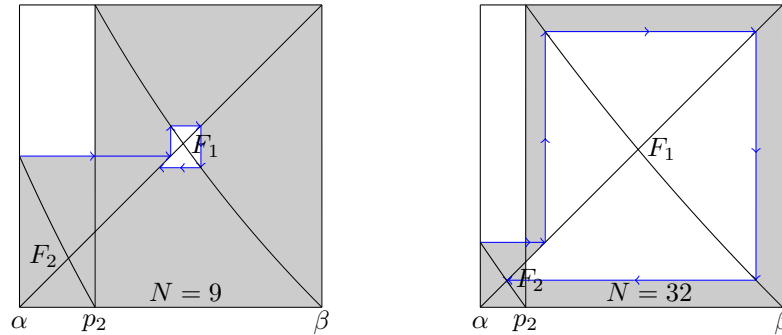


Figure 3: Arrangements for  $N = 9$  and  $N = 32$

any interval  $J \subset (I_\alpha \setminus (\alpha_1, \alpha_2))$  expands to containing  $c_2$  for  $9 \leq N \leq 32$  and that for  $N \geq 33$  there is a gap  $(\alpha_3, \alpha_4)$ , so  $C_2 = \{9, \dots, 32\}$ .

**Remark 6.** It is not hard to find that the length of the gap  $(\alpha_1, \alpha_2)$ , existing for all  $N \geq 9$ , is

$$1 - \frac{N}{N\sqrt{N} - 3N + 4\sqrt{N} - 2}.$$

This is an increasing function of  $N$ , with limit 1 if  $N \rightarrow \infty$ , implying  $|\Delta_2| \rightarrow 0$  if  $N \rightarrow \infty$ . As a consequence, all other gaps will eventually become infinitely small. In particular, the length of the gap  $(\alpha_3, \alpha_4)$ , existing for all  $N \geq 33$ , is given by

$$\frac{N^3 - 8N^2\sqrt{N} + 16N^2 - 20N\sqrt{N} + 16N - 8\sqrt{N} + 4}{N^3\sqrt{N} - 5N^3 + 12N^2\sqrt{N} - 18N^2 + 20N\sqrt{N} - 16N + 8\sqrt{N} - 4},$$

which has a maximum of approximately 0.0566 for  $N = 104$ , after which it decreases to 0 for  $N \rightarrow \infty$ .

In the following, the next straightforward results will become increasingly useful.

**Lemma 1.** Let  $x = [d_1, \dots, d_k, 1, \dots]$  and  $x' = [d_1, \dots, d_k, 2, \dots]$ . Then  $x < x'$  if and only if  $k$  is odd.

The proof of Lemma 1 follows immediately from the inequality  $|x - x'| < 1$  and the equivalence of  $a < b$  and  $c/a > c/b$  for positive numbers  $a, b$  and  $c$ . From this lemma the next two corollaries follow almost immediately.

**Corollary 1.** Let  $c_0, c_1, c_2 \dots$  be the sequence of periodic points as defined in Theorem 6. Then  $k > \ell \geq 1$  if and only if  $c_k < c_\ell$ .

**Corollary 2.** Let  $c_n$  be a periodic point as defined in Theorem 6. Then for  $n \geq 1$  the number  $c_n$  is the smallest of all points in the orbit of  $c_n$ .

As of  $N = 33$ , we have  $(\alpha_1, \alpha_2)$  and  $(\alpha_3, \alpha_4)$  as gaps. As a consequence, for  $N \geq 33$  each attractor has a component containing  $p_2$  that is bounded on both sides by a gap; we will denote this interval by  $P_2$ . For  $C_3$  – yet to be determined – the attractor  $A_\alpha$  consists of three components, each one containing one of the points in the orbit of the characteristic point  $c_3 = [2, 1, 1]$ . We will see that this is typical for odd  $n$ , whereas for even  $n$  the rightmost component does not contain an orbit point of  $c_n$ . The class  $C_2 = \{9, \dots, 32\}$ , for example, has two components, but the single orbit point  $c_2 = f_2$  is in the left-hand component only.

As  $N$  increases, the maximum number of consecutive digits 2 in CPEs becomes more limited. It decreases from 21, for  $N = 33$ , to 3 for  $N \geq 69$ . We note that for  $N \geq 69$ , we have  $\alpha_6 > p_2$ , implying that all CPEs consist of sequences 1, 1, 2 and 2, 2, while every sequence 1, 2, 2 is followed by 2, 1, 1. Some more observations are useful. In order to find the largest  $N$  in  $C_2$ , we could have confined ourselves to checking for which  $N$  we have  $f_2 = c_2 \in T_\alpha^2((p_2, \alpha_1]) = (\alpha_0, \alpha_3]$  and finding that this is the case for  $N \leq 32$ . Where we assumed  $J'$  (see page 9) to be such that  $p_2 \notin T_\alpha(J')$ , we found expansion factors larger than 2 involved for  $N \leq 32$ , excluding any gaps but the one containing  $c_2$ . We will apply these considerations for  $N \geq 33$  as follows: we take  $J = [\alpha_4, c'_3]$  (similar to taking  $J = (c_2, \alpha_1]$  in  $C_2$ ) and find that  $c_3 \in T_\alpha(J_\ell)$  if and only if  $\alpha_5 \geq c_3$ , which is equivalent to  $\alpha_5 \geq \alpha_8$ . Once more disregarding that  $N$ -expansions involve positive integers for  $N$ , we solve  $\alpha_5 = c_3 = \alpha_8$  (with ‘solution’  $N = 132.876\dots$ ) and find that for  $33 \leq N \leq 132$  we have  $\alpha_5 > \alpha_8$ , while for  $N \geq 133$  we have  $\alpha_5 < \alpha_8$ ; see Figure 4, where we have only drawn the bottom left parts of the arrangements for visual reasons.

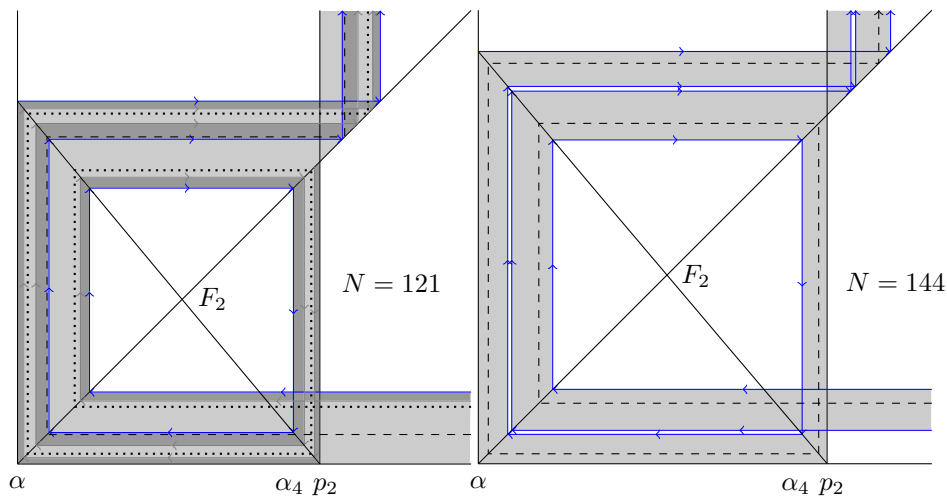


Figure 4: Two arrangements, one with  $c_3 \in A_\alpha$ , one with  $c_3 \notin A_\alpha$

In the left arrangement of Figure 4 we have drawn the orbit of  $c_3$  with dashed line segments and the orbit of  $c_4$  with dotted line segments. The dark grey stripes illustrate the expansion of an interval  $[x, p_2]$ , with  $c_4^{(4)} < x < p_2$ . This is to visualize our approach as follows: if the expansion under  $T_\alpha$  of an interval  $K = [x, p_2] \subset J_\ell$  that contains  $c_4^{(4)}$  is such that  $J_\ell \subset T_\alpha^n(K)$  for some positive integer  $n$ , only for  $33 \leq N \leq 132$  we have  $c_3 \in T_\alpha(J_\ell)$ . If  $c_4 \notin K$ , such as illustrated with the small dark grey stroke left of  $p_2$  in the left arrangement of Figure 4, we rule out gaps other than  $[\alpha_3, \alpha_4]$  and  $[\alpha_1, \alpha_2]$  by computing  $n \in \mathbb{N}$  such that  $p_2 \in T_\alpha^n(K)$ . We find  $n \geq 2|c_4| = 10$ , involving an expansion factor larger than  $(1 + 1/\sqrt{N})^{16}$ , which is amply larger than 2 for  $33 \leq 132$ . We conclude that  $C_3 = \{33, \dots, 132\}$ . Note that, for visual reasons, in Figure 4 only the images of  $K$  up to its seventh expansion are drawn.

Before we investigate the cases  $N \geq 133$ , we make some generalizing remarks. Only if  $N = 4$ , we have  $\{c_0 = [2, 1], c'_0 = [1, 2]\} \cap I_\alpha \neq \emptyset$ . This means that for  $N \geq 5$  all CPEs have a limited number of consecutive sequences 1, 2. Then, as of  $N = 9$ , all CPEs also have a limited number of consecutive digits 1 and, as of  $N = 33$ , also a limited number of consecutive digits 2. It follows immediately that for  $N \geq 33$  the periodic point in  $A_\alpha$  with shortest period length is  $c_3 = [2, 1, 1]$ . We already noted that in CPEs with  $N \geq 69$  digit 1 only occurs in pairs and each pair is followed by either one or three digits 2. If now, for  $N \geq 133$ , the number of consecutive sequences 2, 1, 1 is also limited, it follows that every CPE has a sequence 2, 1, 1, 2, 2. This implies that there is a set of  $N \geq 133$  for which the periodic point with shortest period length is  $c_4 = [2, 1, 1, 2, 2]$ ; in the right arrangement of Figure 4, where  $N = 144$ , we have added the orbit of  $c_4$ .

Assuming that  $N \geq 133$ , we know that  $[\alpha_5, \alpha_8]$ ,  $[\alpha_3, \alpha_4]$ ,  $[\alpha_9, \alpha_6]$ ,  $[\alpha_1, \alpha_2]$  and  $[\alpha_7, \alpha_{10}]$  are gaps, which implies that all attractors have at least six components. To determine  $C_4$ , we take  $J = (c_4^{(4)}, \alpha_9]$  (similar to taking  $J = (c_2, \alpha_1]$  in  $C_2$  and  $J = [\alpha_4, c'_3)$  in  $C_3$ ) and find that  $c_4 \in T_\alpha^2(J_r)$  if and only if  $\alpha_{11} \geq c_4$ , which is equivalent to  $\alpha_{11} \geq \alpha_{16}$ . Once more disregarding that  $N$ -expansions involve positive integers for  $N$ , we solve  $\alpha_{11} = c_4 = \alpha_{16}$  (with ‘solution’  $N = 532.497\dots$ ) and find that for  $133 \leq N \leq 532$  we have  $\alpha_{11} > \alpha_{16}$ , while for  $N \geq 533$  we have  $\alpha_{11} < \alpha_{16}$ ; see Figure 5. The existence of more than five gaps for  $133 \leq N \leq 532$  is ruled out as follows: if an interval  $K = (p_2, x] \subset J_r$  containing  $c_5^{(9)}$  expands such that  $J_r \subset T_\alpha^n(K)$  for some positive integer  $n$ , then  $c_4 \in T_\alpha^2(J_r)$  only for  $133 \leq N \leq 532$ . If  $c_5 \notin K$ , then  $p_2 \in T_\alpha^n(K)$  only if  $n \geq 2|c_5| = 22$ , involving an expansion factor larger than  $(1 + 1/\sqrt{N})^{32}$ , which is amply larger than 2 for  $133 \leq 532$ . We conclude that  $C_4 = \{133, \dots, 532\}$ . Note that in  $C_4$ , the orbit of  $c_4$  has one point in all components except the rightmost one, which is because  $c_4^{(4)}$  is the pre-image of  $c_4$ .

We will now commence the induction part of our proof, in which we will also show that for  $n \in \mathbb{N}$  the orbit of  $c_n$  has one point in each component of  $C_n$  in case  $n$  is odd, and one point in each component except the rightmost one in case

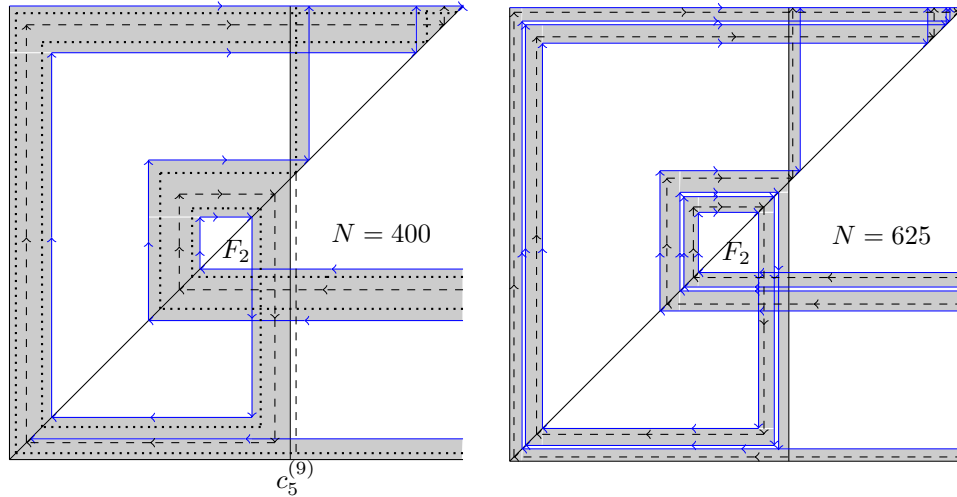


Figure 5: The arrangements, not to scale, for  $N = 400$ , where  $\alpha_{11} > \alpha_{16}$ , and  $N = 625$ , where  $\alpha_{11} < \alpha_{16}$

$n$  is even. First we observe that from Theorem 5 it follows that for  $N \geq 33$  we have  $P_2 = [\alpha_i, \alpha_j]$ , for certain  $i, j \in \mathbb{N}$ . Moreover,  $P_2$  is an image of the leftmost component  $K_\ell$  of  $A_\alpha$ , i.e., the component with  $\alpha$  as left endpoint. Note that in terms of  $J, J_\ell, J_r, J'$  (see page 9) we have  $J = P_2, J_\ell = [\alpha_i, p_2], J_r = (p_2, \alpha_j]$  and  $J' = K_\ell$ . Let  $x_0 \in K_\ell$  be in the orbit of a periodic point with the *smallest length possible*. Then there is an  $n \in \mathbb{N}$  such that  $x_0^* := T_\alpha^n(x_0) \in P_2$  and  $T_\alpha^i(x_0) \notin P_2$  for  $0 \leq i < n$ . Then either  $x_0^* \in [\alpha_i, p_2]$  or  $x_0^* \in (p_2, \alpha_j]$ . In the first case, we have  $T_\alpha(x_0^*) \in K_\ell$  and  $T_\alpha(x_0^*) = x_0$ , since the orbit is the shortest possible; in the second case, we have  $T_\alpha^2(x_0^*) = x_0$ . Note that  $x_0$  is the root of the equation  $T_\alpha^{n+1}(x) = x$  or  $T_\alpha^{n+2}(x) = x$ , with  $x \in K_\ell$ . Solving these is complicated<sup>8</sup>, and we will avoid this by solving equations of the form  $\alpha_n = \alpha_m$  instead, as we did for  $C_3$  and  $C_4$ . For  $33 \leq N \leq 132$  we have  $x_0 = c_3$  and  $x_0^* = c_3'$ , while for  $133 \leq N \leq 532$  we have  $x_0 = c_4$  and  $x_0^* = c_4^{(4)}$ . We will show that  $x_0$ , similarly defined for larger  $N$ , is a characteristic point for  $N \geq 533$  as well.

So, let  $N \geq 533$  and  $C = \{N_1, \dots, N_k\}$  be the class of all  $N$  such that  $P_2 =$

<sup>8</sup>Heuristically, the existence of these shortest periodic orbits for large  $N$  can be understood as follows: we could start with any  $x$  somewhere in the middle of the leftmost component  $K_\ell$ . After  $n$  iterations of  $T_\alpha$  the orbit of  $x$  has moved along with the images of  $K_\ell$ . Depending on whether  $T_\alpha^n(x) \in [\alpha_i, p_2]$  or  $T_\alpha^n(x) \in (p_2, \alpha_j]$ , we find that either  $T_\alpha^{n+1}(x) \in K_\ell$  or  $T_\alpha^{n+2}(x) \in K_\ell$ . If neither is equal to  $x$ , we choose some  $x'$  slightly left or right from  $x$  to perform the same procedure so as to find that the image after  $n + 1$  or  $n + 2$  iterations of  $T_\alpha$  on  $x'$  yields an image in  $K_\ell$  closer to  $x'$  than the image of  $x$  in  $K_\ell$  is to  $x$ . It will then be clear what digits should be used to draw up the equation  $T_\alpha^{n+1}(x) = x$  or  $T_\alpha^{n+2}(x) = x$ , with  $x \in K_\ell$ .



$[\alpha_i, \alpha_j]$ ; referring to Remark 1, we can take  $N_{i+1} = N_i + 1$  for  $i \in \{1, \dots, k - 1\}$ . Let  $a := 2, 1, 1$  and  $b := 2, 2$ . Let  $s$  be a string of partial quotients beginning with  $a$ . First, suppose that  $x_0 = \overline{[s, b]}$  is the periodic point with smallest possible period length in  $A_\alpha$ . Then  $x_0^* = x_0^{(|x_0|-1)} \in (\alpha_i, p_2)$ , and  $x_0$  is in the component of  $A_\alpha$  with  $\alpha$  as left endpoint. Let  $C' = \{N_{k+1}, \dots, N_t\}$  be the class of all  $N$  such that each orbit point of  $x_0$  is contained in a gap. Let  $M = (x_0^*, p_2)$ . Then

$$T_\alpha(M) = (\alpha, x_0), \dots, T_\alpha^{|x_0|}(M) = (\alpha_{|x_0|-1}, x_0^*), T_\alpha^{|x_0|+1}(M) = (x_0, \alpha_{|x_0|}) \dots,$$

where  $M$  expands along the orbit of  $x_0$ . Note that  $p_2 \in T_\alpha^{2|x_0|}(M)$  and  $p_2 \notin T_\alpha^i(M)$  for  $0 \leq i < 2|x_0|$ . This means that two possible expansions exist for points in  $M$ :  $[s, b, s, b, \dots]$  and  $[s, b, s, a, \dots]$ . For the periodic point with smallest possible period length in  $A'_\alpha$  (the attractor in  $C'$ ) only  $\overline{[s, b, s, a]}$  qualifies, since  $\overline{[s, b, s, b]} = \overline{[s, b]} = x_0$  does not.

Next, suppose that  $x_0 = \overline{[s, a]}$  is the periodic point with smallest possible period length in  $A_\alpha$ . Then  $x_0^* = x_0^{(|x_0|-2)} \in (p_2, \alpha_j)$ , and  $x_0$  is in the component of  $A_\alpha$  with  $\alpha$  as left endpoint. Let  $C' = \{N_{k+1}, \dots, N_p\}$  be the class of all  $N$  such that each orbit point of  $x_0$  is contained in a gap. Let  $M = (p_2, x_0^*)$ . Then

$$T_\alpha^2(M) = (\alpha, x_0), \dots, T_\alpha^{|x_0|}(M) = (x_0^*, \alpha_{|x_0|-2}), T_\alpha^{|x_0|+2}(M) = (\alpha_{|x_0|}, x_0) \dots,$$

where  $M$  expands along the orbit of  $x_0$ . Note that  $p_2 \in T_\alpha^{2|x_0|}(M)$  and  $p_2 \notin T_\alpha^i(M)$  for  $0 \leq i < 2|x_0|$ . This means that two possible expansions exist for points in  $M$ :  $[s, a, s, a, \dots]$  and  $[s, a, s, b, \dots]$ . For the periodic point with smallest possible period length in  $A'_\alpha$  (the attractor in  $C'$ ) only  $\overline{[s, a, s, b]}$  qualifies, since  $\overline{[s, a, s, a]} = \overline{[j, a]} = x_0$  does not.

We have thus obtained the following lemma.

**Lemma 2.** *Let  $N \geq 33$  and let  $C_n$  be the class with  $P_2 = [\alpha_i, \alpha_j]$ , where  $P_2$  is the component containing  $p_2$ . Let  $c_n^*$  be the orbit point of  $c_n$  in  $P_2$ .*

*If  $n$  is odd, then  $c_n^* \in (p_2, \alpha_j]$ ,  $c_{n+1}^* \in [\alpha_i, p_2)$  and  $P_2 = [\alpha_i, \alpha_{i+|c_{n+1}|}]$  for  $N \in C_{n+1}$ .*

*If  $n$  is even, then  $c_n^* \in [\alpha_i, p_2]$ ,  $c_{n+1}^* \in [p_2, \alpha_j)$  and  $P_2 = [\alpha_{j+|c_{n+1}|}, \alpha_j]$  for  $N \in C_{n+1}$ .*

The only thing left to prove Theorem 6 is finding an explicit formula for  $|c_n|$ . Note that by definition, we have  $|c_{n+1}| = 2|c_n| + (-1)^n$  for  $n \in \mathbb{N}$ , so

$$|c_n| = 2(2|c_{n-2}| + (-1)^{n-2}) + (-1)^{n-1} = 4|c_{n-2}| + (-1)^{n-2}.$$

Applying induction, we obtain the following.

**Lemma 3.** *Let  $c_n$  be a periodic point as defined in Theorem 6, with  $n \geq 1$ . If  $n$  is even, then  $|c_n| = \frac{2^n - 1}{3}$ ; if  $n$  is odd, then  $|c_n| = \frac{2^n + 1}{3}$ .*

*Proof.* What follows is the final step of the proof of Theorem 6. First, there are no gaps in  $I_\alpha$  for  $4 \leq N \leq 8$ . For  $9 \leq N \leq 32$ , there is only one gap in  $I_\alpha$ , containing  $c_1$ . This satisfies the formula for the number of gaps for odd  $m$ ,  $G(N) = \frac{2^{m+1}-1}{3}$ , according to which  $G(N) = \frac{2^2-1}{3} = 1$  if  $9 \leq N \leq 32$ . For  $33 \leq N \leq 132$ , there are two gaps in  $I_\alpha$ , one containing  $c_1$ , the other  $c_2$ . This satisfies the formula for the number of gaps in  $I_\alpha$  for even  $m$ ,  $G(N) = \frac{2^{m+1}-2}{3}$ , according to which  $G(N) = \frac{2^3-2}{3} = 2$  if  $33 \leq N \leq 132$ . Finally, for  $133 \leq N \leq 532$ , there are five gaps in  $I_\alpha$ , one containing  $c_1$ , one containing  $c_2$ , and the other three gaps each containing one point of the orbit of  $c_3 = [2, 1, 1]$ . This satisfies the formula for the number of gaps in  $I_\alpha$  for odd  $m$ , according to which  $G(N) = \frac{2^4-1}{3} = 5$  for  $133 \leq N \leq 532$ .

As for the induction step for  $N \geq 533$ , we note that the specific composition of CPEs for  $N \geq 69$  with nothing but blocks of sequences 2, 2 and 2, 1, 1 allows us to apply the inequalities (5) and (9). Regarding Remark 1, we know that for  $N \in C_{m+1}$  the expansiveness of  $T_\alpha$  is such that  $P_2$  has no images under  $T_\alpha$  containing  $c_i^*$  for  $3 \leq i \leq m$ . Since the difference between the number of components and the number of gaps equals 1, we see that for the number of gaps  $G$  we have  $G(N) = |c_{m+1}|$  if  $m$  is odd, since in that case every component of  $A_\alpha$  except the rightmost one contains a point in the orbit of  $c_{m+1}$ ; if  $m$  is even, we have  $G(N) = |c_{m+1}| - 1$ , since in that case every component of  $A_\alpha$  contains a point in the orbit of  $c_{m+1}$ . Applying Lemma 3, we indeed find the formula for  $G(N)$ :

$$G(N) = \begin{cases} \frac{2^{m+1} - 1}{3}, & \text{when } m \text{ is odd;} \\ \frac{2^{m+1} - 2}{3}, & \text{when } m \text{ is even.} \end{cases}$$

□

As we noted before (see Remark 2), the function  $(\sqrt{N} + 1)^4 / (N - 1)^2$  provides a good indication of the expansion factors of iterations of  $T_\alpha$  related to the sequences 2, 2 and 2, 1, 1. As to this approximating factor we note that each class  $C_{n+1}$  involves a factor twice as large as  $C_n$ . While for all  $N$  in  $C_2 = \{9, \dots, 32\}$  and for  $N = 33$  as well we have  $(\sqrt{N} + 1)^4 / (N - 1)^2 > 2$ , for all  $N$  in  $C_3 \setminus \{33\} = \{34, \dots, 132\}$  and for  $N = 133$  as well we have  $((\sqrt{N} + 1)^4 / (N - 1)^2)^2 > 2$ , and for all  $N$  in  $C_4 \setminus \{133\} = \{134, \dots, 532\}$  and for  $N = 533$  as well we have  $((\sqrt{N} + 1)^4 / (N - 1)^2)^4 > 2$ .

It seems that the equation

$$\left( \frac{(\sqrt{N} + 1)^4}{(N - 1)^2} \right)^{2^{n-2}} = 2, \text{ implying } N = N(n) := \left( \frac{2^{\frac{1}{2^{2^n-2+1}} + 1}}{2^{\frac{1}{2^{2^n-2+1}} - 1}} \right)^2, \text{ with } n \in \mathbb{N}_{\geq 2},$$

provides very good indications for the boundary values of  $N$  between two classes  $C_n$  and  $C_{n+1}$  - once more disregarding that  $N \in \mathbb{N}$ . In Table 1 the number of gaps in

$I_\alpha$  for  $4 \leq N \leq 34\,919\,576$  is shown. Indeed, it is striking how well  $N(n)$  approaches the maximum value for each class. We also note that the ratio  $|C_{n+1}|/|C_n|$  very rapidly approaches 4 as  $n$  increases. It is  $4.0012515\dots$  for  $n = 5$ , for example, already 4 for  $n \in \{3, 6, 7, 10\}$ ,  $3.9999804$  for  $n = 8$  and  $4.0000004\dots$  for  $n = 11$ .

$n$	$ C_n $	up to $N = \dots$	$\lfloor N(n) \rfloor$	# gaps	condition
1	4	8	not defined	0	$\alpha_1 > \alpha_2$
2	24	32	33	1	$\alpha_3 > \alpha_4$
3	100	132	133	2	$\alpha_5 > \alpha_8$
4	400	532	533	5	$\alpha_{11} > \alpha_{16}$
5	1598	2 130	2 131	10	$\alpha_{21} > \alpha_{32}$
6	6394	8 524	8 525	21	$\alpha_{43} > \alpha_{64}$
7	25 576	34 100	34 101	42	$\alpha_{85} > \alpha_{128}$
8	102 304	136 404	136 405	85	$\alpha_{171} > \alpha_{256}$
9	409 214	545 618	545 619	170	$\alpha_{341} > \alpha_{512}$
10	1 636 855	2 182 473	2 182 474	341	$\alpha_{683} > \alpha_{1\,024}$
11	6 547 420	8 729 893	8.729894	682	$\alpha_{1\,365} > \alpha_{2\,048}$
12	26 189 683	34 919 576	34 919 577	1 365	$\alpha_{2\,731} > \alpha_{4\,096}$
$\geq 13$	not computed	not computed	139 678 308	$\geq 2\,730$	$\alpha_{2\,731} < \alpha_{4\,096}$

Table 1: The relation between the number of gaps and  $N$

Above, in Remark 6, we observed that as  $N$  tends to infinity, the ‘first’ gap (i.e.,  $(\alpha_1, \alpha_2)$ ) tends to ‘swallow up’  $I_\alpha$ . At the same time Theorem 6 states that the number of gaps grows also to infinity as  $N$  tends to infinity. So we have the remarkable fact that although the length of  $I_\alpha \setminus (\alpha_1, \alpha_2)$  tends to zero as  $N \rightarrow \infty$ , it will be punctuated by an increasing number of ever smaller gaps.

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**References**

[1] J. de Jonge, C. Kraaikamp and H. Nakada, Orbits of N-expansions with a finite set of digits, *Monatsh. Math.* **198** (1) (2022), 79-119.  
 [2] C. Kraaikamp and N. Langeveld, Invariant measures for continued fraction algorithms with finitely many digits, *J. Math. Anal. Appl.* **454** (1) (2017), 106-126.  
 [3] F. Schweiger, Remarks on continued fractions with finitely many digits, *Acta Arith.* **203** (3) (2022), 239-250.