SOME CONTRIBUTIONS TO PERCOLATION THEORY

and related fields

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL DELFT, OP GEZAG VAN DE RECTOR MAGNIFICUS, PROF.DR. J.M.DIRKEN, IN HET OPENBAAR TE VERDEDIGEN TEN OVERSTAAN VAN HET COLLEGE VAN DEKANEN OP 8 JANUARI 1985 TE 16.00 UUR

DOOR

JACOB VAN DEN BERG,

GEBOREN TE VIANEN,

DOCTORANDUS IN DE WISKUNDE

Dit proefschrift is goedgekeurd door de promotor Prof.dr. M.S. Keane.

Copyrights

CONTENTS

Summary

Percolation theory studies the stochastics of a network whose nodes and/or connections randomly belong to one of two states. It was introduced by Broadbent and Hammersley (1957) to model the spread of a gas or fluid through a porous medium (the medium consists of a network of channels which are randomly passable or blocked). The subject soon appeared to be useful in the description of many cooperative phenomena, such as semi-conduction, reliability of large communication networks and the formation of polymers, and it has many relations with the Ising model of ferromagnets.

This thesis consists of seven articles and short notes on percolation theory and related subjects , preceded by an introduction and a short comment on each of the articles. The aim of the introduction is two-fold. In the first place, by presenting examples of concrete phenomena, it gives the non-specialist and even the non-mathematician an idea of the nature of *percolation theory. In the second place we sketch the history of percolation* theory and present rough outlines of the proofs of some of the main results in Bernoulli percolation. This gives the interested mathematician an idea of the methods used in percolation theory without having to go through all the details, and may help in the study of the literature. No attention is paid to renormalisation methods, although these methods are very interesting, since the accent of this work is on rigorous results

At the end of the introduction we try to indicate what seems to be the emphasis of actual research and we give a short introduction to first-passage percolation.

1. Introduction

1.1. *Examples*

Example 1. Reliability of large communication networks.

Consider a large telephone network. Suppose that some of the connections are broken. If the fraction of broken connections is small, there is a reasonable probability, that from a given location communication is still possible with locations arbitrarily far away. However, if the fraction is above a certain critical value, the network breaks into many "islands", on each of which internal communication is still possible, but which are isolated from the others. Apparently, by varying the fraction of broken connections a so-called percolation transition occurs. The term "percolation" refers to the next example, which was, in fact, the motivation for Broadbent and Hammersley to study these phenomena and introduce the subject into the mathematical literature.

Example 2. Absorption of fluid by a porous stone.

Suppose a large porous stone is surrounded by a fluid (or gas). The empty space inside the stone is considered as a collection of relatively large holes interconnected by narrow channels. Suppose the fluid can only flow through channels of a minimum width. If the fraction of sufficiently wide channels is very small, only the part of the stone very close to the surface will be wetted, but if it is above a critical value the fluid will percolate, i.e. there will be a non-zero overall density of wet volume.

Reports of Schlumberger-Doll show that these kinds of models are actually interesting for practical research concerning oil-reservoirs, see, e.g., Koplik, Wilkinson and Willemsen (1983).

Example 3. Critical phenomena in a dilute ferromagnet.

A dilute ferromagnet may be modelled as follows: a mixture of magnetic and non-magnetic atoms is randomly distributed among the nodes of a cubic lattice. Let p be the concentration of magnetic atoms. If the temperature is very low, neighbouring magnetic atoms will have parallel magnetic spins. If we neglect interactions between atoms at larger distance the following will happen: for low values of p there will be small clusters of magnetic atoms. Inside a cluster the atoms all have the same spin, but different clusters may have spins in opposite directions and the overall magnetic spin will be zero. However, if the concentration p increases, the above clusters grow and above a certain concentration an infinite cluster occurs causing a non-zero overall magnetic spin.

The relationship between percolation theory and the theory of ferromagnets is much more subtle than appears from this example. Kasteleyn and Fortuin (1969) have proved exact relations between percolation and the Ising model of ferromagnets.

There are many other phenomena where percolation plays a role, such as spread of disease in an orchard, propagation of fire in a forest, the formation of polymers, and semi-conduction. There are several publications where these and other examples are discussed, e.g. Frisch and Hammersley

(1963), de Gennes (1976), Stauffer (1979) and Essam (1980). The last also contains many results from simulation, renormalisation group techniques and numerical methods.

1.2. The mathematical model.

Roughly speaking, percolation theory studies the connectivity properties of random networks, i.e. networks from which a certain fraction of the connections and/or the nodes is randomly taken away. In particular it is interesting to know whether infinite connected subgraphs can occur in the remaining network; this phenomenon corresponds with long-range communication (example 1 in § 1.1), porosity (example 2), and macroscopic magnetic effects (example 3).

So consider a finite or countably infinite connected graph G, which consists of sites (nodes) interconnected by bonds. (In most cases G is a regular lattice imbedded in Z^d, e.g. the 3-dimensional cubic lattice). According to some random mechanism a certain fraction of the sites and/or the bonds is open while the others are closed (if only the sites (bonds) are concerned we speak of site (bond)-percolation; if both are concerned we speak of mixed percolation). The terms "open" and "closed" refer to example 2 in § 1.1, where the sites are the holes and the bonds are the channels, which, if they are too narrow, are closed to the fluid. In the example of the dilute ferromagnet the random mechanism concerns the sites (atoms) and "open" means magnetic, "closed" non-magnetic. The bonds are abstract connections between neighbouring atoms.

Most results in the literature are concerned with models in which different sites (bonds) behave independently. Moreover we will assume that each bond is passable in two directions (although many results have been found for so-called oriented percolation, see e.g. Durrett (1984)), and that G is locally finite, i.e. that each site has finitely many bonds. (As to long-range percolation, see, e.g., Grimmett, Keane and Marstrand (1984), and Newman (1984)).

Before we go further we need some definitions:

A path from site s to site s' is a finite sequence $s=s_1,b_1,s_2,b_2 \ldots,b_n,s_{n+1}=s'$, where each s_i is a site and b_i is a bond between s_i and s_{i+1}. The number n is the length of the path.

A path is self-avoiding (s.a.) if all s_i' s are different.

A path, or more generally a subgraph, is said to be open if all its bonds (or, in the case of site-percolation, all its sites) are open.

An open cluster is a maximal connected open subgraph (so two sites s and s' belong to the same open cluster iff there is an open path from s to s').

Analogously, a closed path, subgraph and cluster are defined.

The distance between two sites s and s' is the length of the shortest path from s to s'.

Now let all bonds (or sites) be indepedently open with probability p and closed with probability 1-p. For each site s define:

(1.1) $P_n(p,s)$ = the probability that there exists a site at distance $\geq$ n of s, which belongs to the same open cluster as s.

Further, the percolation probability function is defined as:

(1.2) $\theta(p,s) = \lim_{n\to\infty} P_n(p,s)$

(clearly, this is the probability that s belongs to an infinite open cluster).

Finally, define

(1.3) $P_H = \inf\{p: \theta(p,s) > 0\}$,

which is called the critical percolation probability.

Remarks:

i) As observed by Broadbent and Hammersley (1957), P_H does not depend on s because G is connected.

ii) The critical probability for bond-percolation is, in general, different from that for site-percolation (see Hammersley (1961) and Kesten (1982, ch. 10)).

1.3. Short history of percolation theory and sketches of proofs of some important results.

As remarked in § 1.1. Broadbent and Hammersley introduced percolation theory in 1957 as a model of the spread of a fluid or gas in a random medium, e.g.

a porous stone. Their first important observation was that, for many graphs, P_H is non- trivial, i.e. not equal to 0 or 1. Hammersley (1957, 1959) gave upper and lower bounds for critical probabilities.

Soon, attention was paid particularly to the regular two-dimensional lattices (the square lattice, denoted by S, the triangular lattice T and the hexogonal lattice H). A concept which soon appeared to be very useful was duality. The dual L^d of a planar lattice L is obtained by putting one site in each face of L and connecting sites which lie in adjacent faces (see fig. 1). The bonds of L are then in 1-1 correspondence with those of L^d, and each configuration on L (i.e. specification of open and closed bonds) induces a configuration on L^d by calling a bond of L^d open if and only if the corresponding bond of L is open.

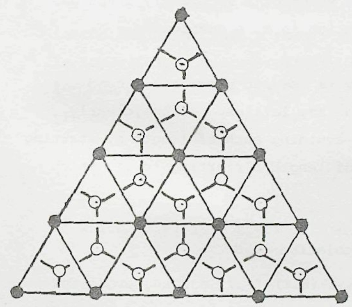

fig.1.

Example of a pair of dual lattices: the triangular (——). and the hexagonal lattice(----).

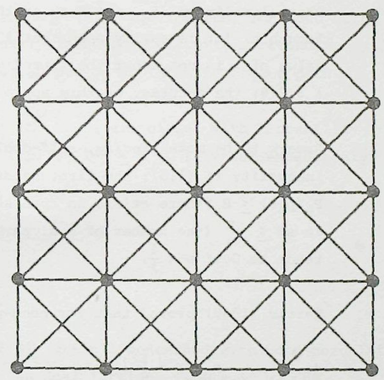

fig.2.

The matching of the square lattice.

It is easily seen that the triangular and hexagonal lattices are dual to each other and that the square lattice is self-dual.

The following fact, which is intuitively obvious follows from results of Whitney (1933).

(1.4) <u>Lemma</u>.

Each finite open cluster in L is surrounded by a closed circuit (by this we mean a circuit of which all bonds are closed) in L^d. (Of course, the same holds with L and L^d exchanged.)

Hammersley used self-duality of the square lattice S to show, for bond percolation on this lattice,

$$(1.5) \qquad \frac{1}{\lambda} \leq P_H(S) \leq 1 - \frac{1}{\lambda} \quad ,$$

where $\lambda = \limsup \sqrt[n]{f_n}$, where f_n is the number of self-avoiding paths of length n. (It is easily seen that λ is between 2 and 3; although the exact value of λ is not known the approximate value of λ is usually given as $\lambda \approx 2.6$; the accuracy of this approximation is also unknown.)

<u>Remark</u> to be more precise, self-duality is used to prove the second inequality in (1.5); the first holds for any lattice, because, clearly, $P_n(p,s) \leq P$ (there exists an open self-avoiding path of length n, starting in s) $\leq p^n$ (the number of s.a. paths of length n starting in s), which tends to 0 if $p < \frac{1}{\lambda}$.

Harris (1960) proved that for bond-percolation on S

$$(1.6) \qquad P_H(S) \geq \frac{1}{2} \; .$$

His proof makes extensive use of duality arguments and also concerns a correlation inequality which has become one of the basic tools in percolation. To state the inequality we need some definitions: An event A is called <u>increasing</u> (or positive) if, whenever a configuration belongs to A, each configuration which can be obtained from the first by changing one or more closed bonds (or, in the case of site percolation, sites) into open bonds (sites), also belongs to A. If we exchange, in the above definition, "open" and "closed", we get the definition of a decreasing (or negative) event. It is clear that if A is increasing its complement is decreasing and vice versa.

Harris' correlation inequality says:

(1.7) Lemma.

If A and B are both increasing (or both decreasing) then $P(A \cap B) \geq P(A)\,P(B)$.

<u>Remarks</u>

(i) Fortuin, Kasteleyn and Ginibre (1971) have extended this result to a class of models (including the Ising-model) where the sites (or bonds) do not necessarily behave independently, and therefore (1.7) is usually called the FKG-inequality.

(ii) Van den Berg and Kesten (1984) have obtained an inequality which says that the probability that two increasing events occur "disjointly" is <u>smaller</u> than the product of the individual probabilities.

Fisher (1961) applied Harris' method of proving (1.6) to other sufficiently regular planar lattices. The idea is that $\theta(p,L) > 0$ implies, for each site, the (a.s.) existence in L of arbitrarily large open circuits surrounding that site. But then, by (1.4), there can not be an infinite closed path in the dual (for such a path would intersect one of the above circuits which is impossible). Hence, noticing that the bonds are closed with probability 1-p, we may conclude that $p > p_H(L)$ implies $1-p \leq p_H(L^d)$, which immediately yields:

$$(1.8) \qquad p_H(L) + p_H(L^d) \geq 1 \; ,$$

of which (1.6) is a special case.

The above results ((1.4)-(1.6) and (1.8)) are all concerned with bond-percolation As to site-percolation, the role of the dual lattice is played by the so-called matching lattice, introduced by Sykes and Essam (1964): Consider a mosaic (which, roughly speaking, is a planar lattice built up of non-overlapping polygons which together cover the whole plane; (so S,T and H are examples of mosaics). Choose a (possibly empty) subset of polygons and draw all diagonals in these polygons. Call the lattice thus obtained L and the lattice obtained by drawing all diagonals in the complementary subset of polygons L*. L and L* form a so-called matching pair.

Remarks:

(i) If L is the original lattice (which happens by taking the first set of polygons empty) then L* is the lattice obtained by drawing all diagonals in all polygons. So the matching of the square lattice is the lattice in fig. 2.

(ii) Triangles have no diagonals, so each triangular lattice is self-matching.

Apparently, L and L* have the same sites, and a configuration on one of the lattices induces a configuration on the other. Sykes and Essam showed that for site-percolation on a matching pair L, L* the following (again intuitively obvious) analog of (1.4) holds.

(1.9) Each open cluster in L is surrounded by a closed circuit in L*.

(And, of course, the same holds with L and L* exchanged).

Further, for site-percolation the following analog of (1.8) holds:

(1.10) $p_H(L) + p_H(L^*) \geq 1$.

In particular, for the triangular lattice T:

(1.11) $p_H(T) \geq \frac{1}{2}$.

Remark

It can be shown (Fisher (1961)) that each bond-percolation problem on a certain graph is equivalent to a site-percolation problem on the so-called covering graph, and that the covering graph of a dual pair of planar lattices form a matching pair. Given this fact, (1.8) is contained in (1.10).

Sykes and Essan (1964) made plausible (by analogy with statistical mechanics), that in (1.8) and (1.10) equality holds, which yields immediately $p_H = \frac{1}{2}$ for bond-percolation on S and site-percolation on T. By using an additional relation between bond-percolation on T and on H (so-called star-triangle overlapping) they show that for a certain value of p (namely the root of the cubic equation $1-3p + p^3 = 0$) $\theta(p,T) > 0$ is equivalent to $\theta(1-p,H) > 0$. But if $p_H(T) + p_H(H) = 1$, this can only happen if p is exactly $p_H(T)$. Hence $p_H(T)$ must be the root of the above equation, which appears to be $2\sin(\frac{\pi}{18})$, and $p_H(H) = 1 - 2\sin(\frac{\pi}{18})$.

However, their proof of equality in (1.8) and (1.10) was based on an assumption (namely that the mean number of clusters per site is, as a function of p, always singular at p_H but nowhere else), which they were not able to prove and which is still open (see Grimmett (1981) and Kesten (1982, ch. 9)).

As to the conjectured equality in (1.6), (1.8), (1.10) and (1.11), mathematically speaking, not much progress was made between 1964 (the year of publication of Sykes' and Essam's work) and the late seventies. Independently, Russo (1978) and Seymour and Welsh (1978) put new life into the subject. They considered so-called sponge-crossing probabilities: define, for a 2-dimensional lattice of which the sites are in Z^2, the "sponge" T(m,n) as the subgraph of which all sites and bonds are in the rectangular region $0 \leq x \leq n$, $0 \leq y \leq m$. By a left-right crossing of T(m,n) we mean a path inside T(m,n) from its lift to its right edge.

Define:

(1.12) $P(m,n,p) = P$ [there exists an open left-right crossing of T(m,n)].

In addition to p_H define two other critical probabilities:

(1.13) $p_S = \inf\{p: \limsup_{n\to\infty} P(n,n,p) > 0\}$,

and

(1.14) $p_T = \inf\{p: E(|W|) = \infty\}$,

where $E(|W|)$ is the expected size of the open cluster W belonging to a specified site, say 0. It follows immediately that $p_T \leq p_H$.

The Russo-Seymour-Welsh (RSW) theorem states the following relations between the three critical probabilities: For site percolation on a matching pair of sufficiently "nice" lattices L and L*:

(1.15) $p_T(L) = p_S(L)$,

$p_H(L) + p_T(L^*) = 1$,

and the same with L and L* exchanged.

The analog for bond-percolation on a sufficiently nice pair of dual planar lattices is:

(1.16) $p_T(L) = p_S(L)$,

$p_H(L) + p_T(L^d) = 1$,

and, again, the same with L and L^d exchanged.

We shall give a rough outline of Russo's proof of (1.15) with L being the square lattice. The more general result can be proved analogously.

First Russo proves that, if $p > p_H(S)$, $\lim_{n\to\infty} P(n,n,p) = 1$ which, by using (1.10), is not very difficult. The most technical part is to show that the latter also implies $\lim_{n\to\infty} P(n,3n,p) = 1$. Once we have this result it is fairly easy that for each $a \in \mathbb{N}\setminus\{0\}$ the probability of an open circuit in the annulus $A(3^n a, 3^{n+1} a)$ tends to 1 if $n\to\infty$. ($A(k,l)$ is the part of the plane where both coordinates have absolute value between k and l). This is illustrated by *figure 3* and the following observations:

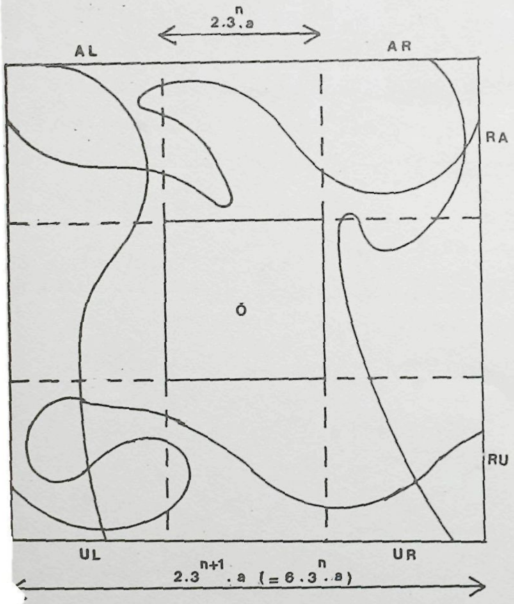

fig. 3.

The annulus $A(3^n a, 3^{n+1} a)$ *containing a circuit which consists of parts of:*

a path from LU to RU
,, ,, ,, UR ,, AR
,, ,, ,, RA ,, LA
,, ,, ,, AL ,, UL.

To have an open circuit in above-mentioned annulus it is sufficient to have, inside the annulus, an open path from LU to RU below the inner square, from UR to AR right from the inner square etc. By the FKG-inequality (1.7) the probability of this is larger than the product of the individual probabilities which is $P^4(2.3^n a, 6.3^n a, p)$ which tends to 1 if $n\to\infty$ (by the previous step in Russo's proof). From this it can be proved that the expected size of a closed cluster in S^* is finite, as follows: Define
$\lambda(a) = \sup\{P$ [there is no open L-circuit in the annulus $A(3^n a, 3^{n+1} a)]: n \in \mathbb{N}\}$.
It is clear, by (1.9), that if an open L-circuit exists in $A(3^i a, 3^{i+1} a)$ for some $i \le n$, then there cannot be a closed path in L^* from $0 = (0,0)$ to any site outside the outer edge of the last annulus, $A(3^n a, 3^{n+1} a)$. It is also clear that the expected number of sites in $A(3^{n+1} a, 3^{n+2} a)$ which are in the closed L^*-cluster belonging to 0 is at most P [$\exists$ closed L^*-path from 0 to the outer edge of $A(3^n a, 3^{n+1} a)$] multiplied by the number of sites in $A(3^{n+1} a, 3^{n+2} a)$, which is at most $\lambda(a)^{n+1} . 4 . 9^{n+2} a^2$, $n \ge 0$. So the expected size of the closed L^*-cluster belonging to 0 is at most: (the number of sites in the region $|x| \le 3a$, $|y| \le 3a$) + $36\, a^2 \sum_{n=0}^{\infty} (9.\lambda(a))^{n+1}$, which holds for each $a \in \mathbb{N}\setminus\{0\}$. By taking a such that $\lambda(a) < \frac{1}{9}$ (which is possible, because $\lambda(a) \to 0$ if $a \to \infty$), the above series converges, so the expected L^*-cluster belonging to 0 is finite and, consequently, $1-p \le p_T(L^*)$. Apparently, $p > p_H(L)$ implies $1-p \le p_T(L^*)$. Hence $p_H(L) + p_T(L^*) \ge 1$. The other part, i.e. $p_H(L) + p_T(L^*) \le 1$ is easier. One way is to use a theorem of Hammersley (1958) which yields (as remarked in Van den Berg (1981)) that finite expected cluster size implies that the $P_n(p)$, $n \in \mathbb{N}$ (see (1.1)) decrease exponentially so that the relevant probabilities can be sufficiently estimated. With a small amount of effort the above proofs also yield the result $p_S = p_T$.

By the RSW theorem ((1.15),(1.16)), Sykes' and Essam's conjecture ($p_H(L) + p_H(L^*) = 1$ for site percolation, and $p_H(L) + p_H(L^d) = 1$ for bond percolation) would be true if and only if:

(1.17) $p_H = p_T = p_S$.

That this is indeed the case has been proved by Kesten (1980) for bond percolation on the square lattice. In his article Kesten shows that, for $p < \frac{1}{2}$, $\lim_{n\to\infty} P(n,n,p) = 0$ which yields, by definition of P_s (and reminding $p_S = p_T \le p_H$), by (1.16) and by selfduality of S the desired result $p_H = p_T = p_S = \frac{1}{2}$. His proof is roughly as follows.

Suppose $p \leq \frac{1}{2}$. Then, given the event that there exists an open left-right crossing of $T(n,n)$, it is, if n is large, rather likely that there are many so-called pivotal bonds. (In this context a bond is called pivotal (or critical) if there is an open left-right crossing which contains the bond, but no open left-right crossing which does not contain this bond). More precisely this means that:

(1.18) there exists an $\alpha > 0$ such that for each k the (conditional) probability that there are at least k pivotal bonds (with respect to the event that there exists an open left-right crossing of $T(n,n)$) is larger than α if n is sufficiently large.

The proof of this is rather technical.

Further, given the events that there exists an open left-right crossing and that there are at least k pivotal bonds, the conditional probability that there is still an open left-right crossing if the open bonds independently remain open with probability p_2 and are closed with probability $1-p_2$ is, clearly, at most p_2^k. So the conditional probability that, after the second stage (keeping open bonds open with probability p_2) there exists an open left-right crossing of $T(n,n)$ given such a crossing exists after the first stage (making bonds open with probability p, closed with probability 1-p) is, for each k, at most $1 - P$ [there are at least k pivotal bonds] + P [there are at least k pivotal bonds]. p_2^k, which, by virtue of (1.18), can be put smaller than $1-\alpha$ by taking n sufficiently large and fitting k. Summarizing we get (by choosing β between $1-\alpha$ and 1, and remarking that first making bonds open with probability p_1 and subsequently keeping them open with probability p_2, gives, statistically the same result as making them open with probability $p_1 p_2$):

(1.19): $\exists \beta < 1 \ \forall p_1 \leq \frac{1}{2} \ \forall p_2 < 1 \ \exists N \ \forall n > N$

$$P(n,n,p_1 p_2) < \beta P(n,n,p_1) .$$

Now suppose $p < \frac{1}{2}$. Of course, for each $m \in \mathbb{N}$ there exists a $p_2 < 1$ such that $p < \frac{1}{2} p_2^m$. Repeated application of (1.18) then gives that $P(n,n,p) < \beta^m$ for n sufficiently large. This holds for each m, so $\lim_{n\to\infty} P(n,n,p) = 0$.

Russo (1981) applied Kesten's arguments to prove $p_H(L) + p_H(L^*) = 1$ (for site percolation) and $p_H(L) + p_H(L^d) = 1$ (for bond percolation) more generally. Wierman completed Sykes' and Essam's "proof" that the

critical probabilities for bond percolation on the triangular and hexagonal lattices are $2 \sin(\frac{\pi}{18})$ and $1-2\sin(\frac{\pi}{18})$ respectively.

Summarizing we have, for site percolation on a matching pair of "sufficiently nice" lattices L and L*:

(1.20) $p_H(L) + p_H(L^*) = 1,$

and for a dual pair of sufficiently nice planar lattices:

(1.21) $p_H(L) + p_H(L^d) = 1.$

In particular, for bond percolation on S and site percolation on T.

(1.22) $p_H = \frac{1}{2}$.

Further, for bond percolation on T and H:

(1.23) $p_H(T) = 2\sin(\frac{\pi}{18}) = 1-p_H(H)$.

And for all these lattices

(1.24) $p_H = p_S = p_T$.

The key in the proof of (1.20) and (1.21) is that it is impossible, except for one value of p, that the sequence $P(n,n,p)$, $n = 1,2, \ldots,$ is bounded away from 0 and 1. Russo (1982) discovered a more general phenomenon which he called "approximate zero-one law" and which gives an alternative proof of (1.20) and (1.21).

Remarks

Sykes and Essam (1964) also considered some 2- and 3-parameter problems. For bond-percolation on S, with horizontal bonds open with probability p_1 and vertical bonds open with probability p_2 they obtained (making the same kind of assumption as in the 1-parameter case) that the critical region is given by the equation

(1.25) $p_1 + p_2 = 1,$

and for the 3-parameter bond problem on T:

(1.26) $p_1 + p_2 + p_3 - p_1 p_2 p_3 = 1.$

An exact proof of (1.25) has been given by Kesten (1982). (See also Tibi (1984) who gives an alternative proof based on a multi-parameter version of Russo's approximate 0-1 law). Kesten also proved (1.26) under the assumption $p_1 = p_2$.

1.4. Some of the main problems in actual research

a) Estimation of critical probabilities of other lattices

There seems not be to much hope that other interesting critical probabilities than those mentioned in § 1.3 can be exactly calculated, and it even appears to be difficult to find reasonably sharp rigorous estimates. A problem which often occurs in this respect is to show that the critical probability of a certain subgraph is strictly larger than that of the total graph. For instance, consider site percolation on the square lattice S. By (1.21) $p_H(S) + p_H(S^*) = 1$. But S is a subgraph of S* so we would expect $p_H(S) > p_H(S^*)$ and hence $p_H(S) > \frac{1}{2}$. This special case has been proved by Higuchi (1982). Kesten (1982, ch. 10) proves a more general result but many cases are still open. As to Higuchi's result, this has recently been improved by Tóth (1984) who showed $p_H(S) > 0.502$ (by elegant combinatorial arguments), but this is still far from the value 0.59 ... expected by numerical extrapolation methods and simulation results. For site percolation on the cubic lattice Campanino and Russo (1984) have recently proved $p_H < \frac{1}{2}$.

b. Smoothness properties and power estimates

For many functions occuring in 2-dimensional percolation smooth behaviour outside p_H has been proved (see, e.g. Russo (1978), Grimmett (1981), Kesten (1982, ch. 9)). However, for lattices of higher dimension almost nothing has been proved rigorously and for 2-dimensional lattices the behaviour at and near p_H is one of the most interesting problems at the moment. For many functions, for instance $\theta(p)$, it is believed that they behave near p_H as a power of $p-p_H$ and that the exponents depend on the dimension but not on the details of the lattice. Kesten (1981, 1982, ch. 8) has shown that several functions are indeed bounded between two powers, but in all the cases there is a large difference between the exponent in the upper and the one in the lower bound . Some improvements have been made by Aizenman and Newman (1984) and Van den Berg and Kesten (1984).

c) The number of infinite open clusters

For many 2-dimensional lattices it was already observed by Harris (1960) and Fisher (1961), that above p_H there exists exactly one infinite infinite open cluster. For higher dimensions this is an open problem, though it is, e.g. true for bond percolation on Z^d with $p > \frac{1}{2}$ (Kesten). Newman and Schulman (1981) show for a class of periodic lattices imbedded in Z^d that, for each p, the number of infinite clusters is, with probability one, 0, 1 or ∞. Van den Berg and Keane (1984) show a relationship between the number of infinite clusters and the continuity of the percolation probability function.

d) Percolation in three and higher dimensions

In fact this has already been mentioned among the problems in (a)-(c), but we prefer to mention it separately to accentuate that complete new tools are required. Many proofs of 2-dimensional percolation are based on the fact that certain paths necessarily intersect, which is no longer the case in higher dimensions. Related to this is the concept of duality which is not clear in higher dimensions. Aizenman, Chayes, Chayes, Chayes, Fröhlich, have studied random surfaces which may lead to more insight.

1.5. First-passage percolation

We will only make a few remarks on first-passage percolation, because only one article in this thesis is concerned with first-passage percolation and only deals with a very special problem which can be explained without further knowledge of the subject.
In example 2 of § 1.1 it is shown that percolation can be used to model the spread of a fluid through a porous medium. However, the model describes where the fluid can flow but not how much time it takes. In 1965 Hammersley and Welsh introduced the following model which does involve time:

Consider a graph G. To each bond b of G a non-negative random variable e_b (called the time-coordinate of b) is associated which represents the time needed for a particle to travel along b from on of its endpoints to the other. Generally it is assumed that the e_b's are independent and have the same distribution. The travel time of a path is the sum of the time coordinates of the individual bonds in the path. For each pair of sites s_1, s_2, the shortest travel time from s_1 to s_2, denoted by $t(s_1,s_2)$ is defined as the infinum of the travel times of all paths from s_1 to s_2. Most of the results have been stated for the square lattice but also hold for many other

2-dimensional lattices.

The main result of Hammersley and Welsh (1965) is that, if the time coordinates have finite mean, $\frac{t((0,0),(n,0))}{n} \to \gamma$ in probability, where

$$\gamma = \inf \frac{E[t((0,0),(n,0))]}{n}$$

The result (the proof of which uses the observation that the passage time process is subadditive) was strengthened by Kingman (1968) who proved by his subadditive ergodic theorem that a.s. convergence and convergence in L_1 hold, and refined by several other people (see e.g. Cox and Durrett (1981) and Smythe and Wierman (1978)). As to recent results and main problems see e.g. Grimmett and Kesten (1982) and Kesten (1984).

2. Short comment on the articles

A. Percolation theory on pairs of matching lattices (1981)

The intention of "Percolation theory on pairs of matching lattices" (1981) was to show that Sykes' and Essam's assumption about the singularity of k could be replaced by the assumption that the expected cluster size is finite for $p < P_H$. This implies the required result without using the matching relation (2). However, when I wrote this paper I was not aware of the results of Russo, Seymour and Welsh (see pp. 9-11 in this thesis). Because of this, and Kesten's result (see pp. 11-12), the paper missed its main purpose. Yet, some parts of it are of interest. In the first place it clearly shows the strength of Hammersley's (1957) theorem $P_{nm} < F_n^m$, which has received less attention in the literature than it deserves. Further, the example of a fully triangulated planar graph of which the critical probability is 1 , shows the importance of periodicity.

The estimate $P_A \approx 0.5925$ for site percolation on the square lattice coincides with most of the approximations which have appeared later in the literature.

B. A note on percolation theory (1982)

In this article we pose the question whether for each graph G and each $p \geq P_H(G)$ there exists a subgraph of G of which the critical probability equals p . It is shown that this is true if $P_p\{$ there exists exactly one infinite open cluster $\} = 1$. This follows from the observation that if there is (a.s) exactly are infinite open cluster, the P_H of this cluster is (a.s.) equal to $P_H(G)/p$. This observation is also used in paper D in this thesis, and recently Georgii (1984) used it to describe the characteristics of infinite open clusters near the percolation threshold.

Grimmett (1983) gives, for bond percolation on the square lattice, a more constructive answer to the question raised above.

C. A counterexample to a conjecture of J.M. Hammersley and D.J.A. Welsh concerning first-passage percolation (1983)

In one of their introductory articles on first-passage percolation on the square lattice Hammersley and Welsh conjectured that the expected cylinder time from (o,o) to (n,o) is increasing in n . Our paper shows a counterexample. The cylinder condition plays a crucial role in this counterexample and we think the conjecture is true if we drop this condition. M. Keane and the author have solved several analogous problems for simpler graphs (including a problem of Joshi (1978)).

D. (With M. Keane) On the continuity of the percolation probability function (1984)

Consider percolation on a graph G and let, for a certain site s, $\theta(P)$ denote the probability that s belongs to an infinite open cluster. It is not difficult to show that always $\theta(P^+) = \theta(P)$. An important problem is under which conditions also $\theta(P^-) = \theta(P)$. Our paper makes a connection with another problem, namely the problem of how many and what kind of infinite open clusters occur. We prove that $\theta(P) - \theta(P^-) = P_p\{$ s belongs to an infinite open cluster of which the critical probability equals 1 $\}$. Further, as observed in article B, if $p > P_H$ and $P_p\{$ there is exactly one infinite open cluster $\} = 1$, then the critical probability of this infinite cluster is (a.s) equal to P_H/p which is strictly smaller than 1, and hence $\theta(P^-) = \theta(P)$.

For nearest neighbour bond percolation on Z^d, $d \geqslant 3$, with $p \geqslant \frac{1}{2}$ there is indeed a unique infinite open cluster (as has been proved by Kesten), so that in these cases θ is continuous in the interval $[\frac{1}{2},1]$. (For 2-dimensional percolation much more is known (see Russo (1978)). An interesting problem is whether our result can be extended to multi-parameter percolation. For instance, if a part of the bonds (or sites) is open with probability P_1, and the others are open with probability P_2, is it always true that θ is continuous at (P_1,P_2) whenever $\theta > 0$ in an environment of (P_1,P_2) and $P_{P_1,P_2}\{$ there is an unique infinite open cluster $\} = 1$?

E. Disproof of the conjectured subexponentiality of certain functions in percolation theory (1984)

As observed in article A, Hammersley's result $P_{nm} \leqslant F_n^m$ is important because it implies (for a large class of graphs) that if the expected size of the open cluster containing a specified site is finite, P_n is exponentially bounded from above. (See also Aizenman and Newman (1983) and corollary (3.18) in article F).

Hammersley conjectured that a stronger result would hold, namely $F_{n+m} \leqslant F_n F_m$. Our paper shows a counterexample. We do not know whether the conjecture is true for homogenous percolation models.

F. (With H. Kesten) Inequalities with applications to percolation and reliability

An important "notion" of this paper is SNBU (Strongly New Better than Used). In reliability theory this refers mostly to random life lengths. We show that it is also interesting to interpret the SNBU property in terms of random outputs of certain products. Roughly speaking, consider a certain device producing n items, the outputs of which are represented by the random variables $X_1,\ldots,X_n$.

Moreover, each individual in a group of persons has a list of wishes (each wish being of the form "I want at least a quantity W_1 of the first item, W_2 of the second item,.... W_n of the n^{th} item") and is satisfied if at least one of his wishes is fulfilled.

Definition (1.1) is equivalent to saying that the probability distribution of the output vector $(X_1,\ldots,X_n)$ is SNBU if and only if the group as a whole is always (i.e. for arbitrary numbers of individuals and for all lists of wishes of the form mentioned above) better off (i.e. has a larger probability that each individual can be satisfied) if all individuals receive an independent copy of the device of their own but are not allowed to exchange items with each other, rather than if there is only one device, the output of which is distributed among the individuals in such a way that as many as possible are satisfied.

In these terms problem 1.11 in our paper is equivalent to the following question: is the composition of independent SNBU devices again SNBU? This problem is, in fact, more general than conjecture 3.9 (which, however, is interesting in itself; see paper G). Theorem 1.6 (iii) states that the answer to this question is affirmative if each of the devices produces only one type of item. We also have an (unpublished) proof for the case that at most one device produces more types of item. The binary case of theorem 1.6 (iii) (i.e. the case that each device produces only one type of item and its output is 0 or 1) is treated in section 3 and interesting applications to percolation theory are shown. Remark 3.5 (b) shows the connection with theorem 1.6 (i) of Campanino and Russo (1984). It appears that this theorem has been obtained earlier, see e.g. McDiarmid (1980), who speaks of the "clutter theorem", and Hammersley (1961).

G. (With U. Fiebig) <u>On a combinatorial conjecture concerning disjoint occurrence of events</u>

We would like to make some additional remarks concerning the $\square$-operation and conjecture (2.6).

a) Consider conjecture (2.6). The condition that μ is a product measure is crucial, i.e., if this condition does not hold, then (2.7) is false (as is easy to show). This is even so if we restrict ourselves in (2.7) to monotone events.

b) The following very simple case shows in several respects how misleading intuition can be.

Let

(1) $\Omega = \{0,1\}^2$,

(2) $A = [0 \ast] \cup [\ast 0]$,

(3) $B = [0 \ast] \cup [\ast 1]$,

and

(4) $\mu = \mu_1 \times \mu_2$,

where

μ_1 and μ_2 are probability measures on Ω.

i) As remarked by Ahlswede (see p. G19), the probability that two given events occur disjointly does not always increase by splitting a coordinate (by "increase" we mean "strictly increase or remain unchanged"). The easiest counterexample follows from (1)-(4) above, where we have

(5) $A^{\ast} = [0 \ast \ast] \cup [\ast 0 \ast]$,

(6) ${}^{\ast}B = [0 \ast \ast] \cup [\ast \ast 1]$.

Note that the cylinders in the r.h.s. of (2), (3), (5) and (6) are exactly the maximal cylinders of $A, B, A^{\ast}$ and ${}^{\ast}B$ respectively. Hence (by lemma 3.2.ii)

(7) $A \square B = ([0 \ast] \cap [\ast 1]) \cup ([\ast 0] \cap [0 \ast]) = [0 1] \cup [0 0] = [0 \ast]$.

(8) $A^{\ast} \square {}^{\ast}B = ([0 \ast \ast] \cap [\ast \ast 1]) \cup ([\ast 0 \ast] \cap [0 \ast \ast]) \cup ([\ast 0 \ast] \cap [\ast \ast 1]) =$

$= [0 \ast 1] \cup [0 0 \ast] \cup [\ast 0 1]$,

which does not contain the element (0,1,0).

Hence, if $\mu_1(0) = 1$, $\mu_2(0) > 0$ and $\mu_2(1) > 0$, then

(9) $\mu(A \square B) = 1$,

while

(10) $\overset{\ast}{\mu}(A^{\ast} \square {}^{\ast}B) = \mu([0 \ast \ast] \setminus [0 1 0]) < 1$.

ii) One may think (because of lemma 3.2,i) that, for all events D,E,F,

(11) $D \square (E \square F) = \{ C \cap C' \cap C'' \mid C, C'$ and C'' are mutually perpendicular cylinders of D,E and F respectively$\}$.

However, this would imply that the $\square$-operation is associative which is *not true as we have the following* counterexample.

iii) Take A,B and Ω as in (1) - (3).

Clearly,

(12) $A \square (B \square B) = A \square [0 1] = \emptyset$,

while

(13) $(A \square B) \square B = [0 \ast] \square ([0 \ast] \cup [\ast 1]) = [0 1]$.

Hence the $\square$-operation is not associative.

iv) For $\Omega = \{0,1\}^n$ and $A_1, B_1, \ldots, A_k, B_k$ increasing subsets of Ω the following holds (*see* (3.6) *in article F*):

(14) $\mu(A_1 \square B_1 \cup A_2 \square B_2 \cup \ldots \cup A_k \square B_k) \leq$
$\leq (\mu \times \mu)(A_1 \times B_1 \cup A_2 \times B_2 \cup \ldots \cup A_k \times B_k)$,

where μ is a product probability measure on Ω.

We shall now show that (14) is not true for arbitrary events. Consider (1) - (4). We have

(15) $A \square \Omega \cup \Omega \square B = A \cup B = \Omega$,

while

(16) $A \times \Omega \cup \Omega \times B = (\Omega \times \Omega) \setminus [1 1 1 0]$.

Hence, the probability of the l.h.s. of (15) equals 1 and (if μ_1 and μ_2 are non-trivial) the probability of the l.h.s. of (16) is strictly smaller than 1.

References

Aizenman, M., Chayes, J.T. and L., Fröhlich, J. and Russo, L. (1983) On a sharp transition from area law to perimeter law in a system of random surfaces, Comm. Math. Phys. 92, 19 - 69.

Aizenman, M. and Newman, C.M., Tree graph inequalities and critical behavior in percolation models, to appear in J. Stat. Phys.

van den Berg, J. (1981) Percolation theory on pairs of matching lattices, J. Math. Phys. 22, 152 - 157.

van den Berg, J. and Keane M. (1984) On the continuity of the percolation probability function, Contemp. Math. 26, 61 - 65.

van den Berg, J. and Kesten, H., Inequalities with applications to percolation and reliability, to appear in J. Appl. Prob.

Broadbent, S.R. and Hammersley, J.M. (1957) Percolation processes, Proc. Cambr. Phil. Soc. 53, 629 - 641 and 642 - 645.

Campanino, M. and Russo, L., An upper bound on the critical probability for the three dimensional cubic lattice, to appear in Ann. Probab.

Cox, J.T. and Durrett, R. (1981) Some limit theorems for percolation processes with necessary and sufficiënt conditions,Ann.Prob.9,583-603.

Durrett, R. (1984) Oriented percolation in two dimensions, to appear in Ann. Prob.

Durrett, R. and Nguyen, B. (1984) Thermodynamic inequalities for percolation, preprint.

Essam, J.W. (1980) Percolation Theory, Rep. Prog. Phys. 43, 833 - 912.

Fisher, M.E. (1961) Critical probabilities for cluster size and percolation problems, J. Math. Phys. 2, 620 - 627.

Fortuin, C.M., Kasteleyn, P.W. and Ginibre, J. (1971) Correlation inequalities on some partially orderd sets, Comm. Math. Phys. 22, 89 - 103.

Frisch, H.L. and Hammersley, J.M. (1963) Percolation processes and related topics, J. Soc. Ind. Appl. Math. 11, 894 - 918.

de Gennes, P.G. (1976) La percolation, un concept unificateur, La recherche 72.

Georgii, H.O. (1984) Presentation at the Workshop on Stochastic Spatial Processes, Heidelberg, Sept. 1984.

Grimmett, G.R. (1981) On the differentiability of the number of clusters per vertex in the percolation model, J. London Math. Soc.(2) 23, 372 - 384.

Grimmett, G.R. (1983) Bond percolation on the square lattice, and the treshold between one-dimensional and two-dimensional behaviour, J. Phys. A: Math. Gen. 16, 599 - 604.

Grimmett, G.R., Keane M.S. and Marstrand, J.M. (1983) On the connectedness of a random graph, Math. Proc. Camb. Phil. Soc. 96, 151 - 165.

Grimmett, G.R. and Kesten, H. (1982) First-passage percolation, network flows and electrical resistances, to appear.

Hammersley, J.M. (1957) Percolation processes. Lower bounds for the critical probability, Ann. Math. Stat. 28, 790 - 795.

Hammersley, J.M. (1959) Bornes supérieures de la probabilité critique dans un processus de filtration, pp. 17 - 37 in Le calcul des probabilités et ses applications, CNRS, Paris.

Hammersley, J.M. (1961) Comparison of atom and bond percolation, J. Math. Phys. 2, 728 - 733.

Hammersley, J.M. and Welsh, D.J.A. (1965), in: Bernoulli-Bayes-Laplace Anniversary Volume, J. Neyman and L.M. Le Cam, editors, Springer-Verlag, Berlin.

Harris, T.E. (1960) A lower bound for the critical probability in a certain percolation process,Proc. Camb. Phil. Soc. 56, 13 - 20.

Higuchi, Y. (1982) Coëxistence of the infinite (*)clusters; a remark on the square lattice site percolation, Z. Wahrsch. verw. Geb.

Kasteleyn, P.W. and Fortuin, C.M. (1969) Phase transition in lattice systems with random local properties, Proc. Intern. Conf. Stat. Mech. Kyoto, 1968 or: J. Phys. Soc. Japan 26, Supplement, 11 - 14.

Kesten, H. (1980)The critical probability of bond percolation on the square lattice equals ½, Comm. Math. Phys. 74, 41 - 59.

Kesten, H. (1981) Analyticity properties and power law estimates of functions in percolation theory, J. Stat. Phys. 25, 717 - 756.

Kesten,H. (1982) Percolation theory for mathematicians, Birkhäuser, Boston.

Kesten,H. (1984) Aspects of first-passage percolation, Lecture Notes at Saint-Flour Summer School on Probability.

Koplik, J. Wilkinson, D. and Willemsen, J.F. (1983) Percolation and capillary fluid displacement. In: The Mathematics and Physics of Disordered Media, ed. B.D. Hughes and B.W. Ninham, pp. 169 - 183. *Springer LNM* 1035.

Newman, C. M., (1984) to appear.

Newman, C.M. and Schulman, L.S. (1981) Infinite clusters in percolation models, J. Stat. Phys. 26, 613 - 628.

Russo, L. (1978) A note on percolation, Z. Wahrsch.verw. Geb. 43, 39 - 48.

Russo, L. (1981) On the critical percolation probabilities, Z. Wahrsch. verw. Geb. 56, 229-237.

Seymour, P.D. and Welsh, D.J.A. (1978) Percolation probabilities on the square lattice, Ann. Discrete Math. 3, 227 - 245.

Smythe, R.T. and Wierman, J.C. (1978) First-passage percolation on the square lattice, LNM 671, Springer-Verlag.

Stauffer, D. (1979) Scaling theory of percolation clusters, Physics Reports 54, No. 1, 1 - 74.

Sykes, M.F. and Essam, J.W. (1964), Exact critical percolation probabilities for site and bond problems in two dimensions, J. of Math. Phys. 5, 1117 - 1127.

Tibi, D. (1984) Thèse Troisième Cycle , Université Pierre et Marie Curie, Paris.

Whitney, H. (1933) Planar graphs, Fund. Math. 21, 73 - 84.

Wierman, J.C. (1981) Bond percolation on honeycomb and triangular lattices, Adv. Appl. Prob. 13, 293 - 313.

Wierman, J.C. (1982) Percolation theory, Ann. Prob. 10, 509 - 524.

JMAPA,800400800444

Percolation theory on pairs of matching lattices

J. van den Berg

Mathematical Institute, University of Utrecht, Utrecht, The Netherlands

(Received 2 October 1979; accepted for publication 4 January 1980)

An important magnitude in percolation theory is the critical probability, which is defined as the supremum of those values of the occupation-probability p, for which only finite clusters occur. In 1964 Sykes and Essam obtained the relation $P_c^{(s)}(L) + P_c^{(s)}(L^*) = 1$, where L and L^* are a pair of matching lattices and $P_c^{(s)}$ denotes the critical probability (site-case). The proof was not complete, but based on certain assumptions about the mean number of clusters. Though Sykes and Essam suggested that the above relation holds for all mosaics (i.e., multiply-connected planar graphs) and decorated mosaics, we have constructed a counterexample. Subsequently, for a more restricted class of graphs, an alternative derivation of the Sykes–Essam relation is given, this time based on the usual assumption that below the critical probability the mean cluster size is finite. The latter assumption is also used to prove for some nontrivial subgraphs of the simple quadratic lattice S, that their critical probability is equal to $P_c^{(s)}(S)$. Finally, for a certain class of lattices, sequences of numbers are constructed, which converge to the critical probability. In the case of the site process on S, the number with highest index we found, is 0.5925 ± 0.0002, which seems to be a reasonable estimate of $P_c^{(s)}(S)$.

PACS numbers: 05.50. + q, 02.50.Cw

1. INTRODUCTION

Percolation problems arise in many branches of science and engineering. Concerning physics, the most interesting example is the dilute ferromagnet, where the concentration of magnetic particles is p and the concentration of nonmagnetic impurities is $1 - p$. Below a certain value of p, the so-called critical concentration P_c, there are only finite clusters of magnetic particles and therefore no spontaneous magnetism occurs at any temperature. On the other hand, if $p > P_c$, spontaneous magnetism will occur below a certain temperature.

Generally, percolation can be described mathematically as follows. A graph G consists of abstract points, called vertices (or sites or atoms) and connections between some of these points, called bonds. These bonds may be oriented, in which case they connect in only one direction, or nonoriented. In this paper we only deal with nonoriented graphs, i.e., graphs of which all bonds are nonoriented.

With the graph G we now relate a so-called random coloring as follows: Each vertex of G has, independently of all other vertices, a fixed probability p of being colored black, and $q = 1 - p$ of being colored white. For such a realization of this random coloring we distinguish two section-graphs of G, one, called G_b, containing all black, and the other, G_w, containing all white vertices of G.

Percolation theory studies the properties of G_b and G_w. Especially, in the case that G is infinite, we are interested in the critical value P_c of p, above which infinite black clusters appear.

A related model is that in which the bonds of G, instead of the vertices, are randomly colored. This model and the model above are known as the bond- and the site-percolation process respectively. It appears that the site process is the more general one, because the bond process on a graph G is, in a certain sense, equivalent with the site process on the covering graph G^c of G. Therefore, quite often certain results are proved for the site- and then translated to the bond-case.

In 1964 Sykes and Essam[1] published some interesting results for two-dimensional percolation processes. We shall use much of their terminology. A more general introduction to the subject is to be found in, e.g., Refs. 2 and 3.

Remark: In this article we shall only deal with lattices which are mosaics or decorated mosaics.

One of the main results of Sykes and Essam is the relation

$$P_c^{(s)}(L) + P_c^{(s)}(L^*) = 1, \tag{1}$$

where L and L^* are a pair of matching lattices and $P_c^{(s)}$ denotes the critical probability for the site-percolation process. This relation follows from the fact that the mean number of black L clusters per vertex differs from the mean number of white L^* clusters per vertex by a finite polynomial $\phi(p)$ (where p is, as it will be throughout this article, the probability of a given vertex being black), in formula:

$$k(p;L) = k(1 - p;L^*) + \phi(p). \tag{2}$$

Now Sykes and Essam derive (1) immediately from (2) by the assumption (which has not been proved) that in the domain $0<p<1$ the function k is singular at $P_c^{(s)}$ and nowhere else.

Next they remark that the triangular lattice T is self-matching, which implies, by (1), that:

$$P_c^{(s)}(T) = \tfrac{1}{2}, \tag{3}$$

and that (3) more generally holds for any lattice of which all faces are triangular. However, it is easy to construct such a lattice for which (3) is not true, as follows.

Figure 1(a) shows a sequence of triangles $A_0, A_1, A_2, \cdots$, each of which (except A_0) has six vertices on its perimeter,

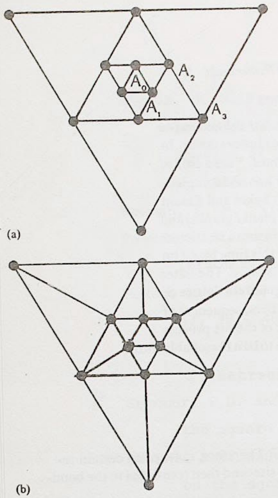

FIG. 1(a) Lattice, consisting of a sequence of nested triangles $A_0, A_1, A_2,\cdots$. (b) The lattice obtained by triangulation of the lattice in (a). It appears that the critical probability (site-case) of this lattice is, in contrast with the Sykes–Essam relation, not $\frac{1}{2}$ but 1.

one at each corner and one at the center of each of its edges.

The fully triangulated graph G in Fig. 1(b) is obtained by drawing a bond in every nontriangular face of Fig. 1(a).

Considering the site-percolation process on G, we note that, if $p<1$; for each $i>0$ the probability of the event that all six vertices of A_i are white is $q^6>0$. Further, we observe that any pair of the triangles with odd indices $A_1, A_3, A_5,\cdots$, has no common vertex, hence the number of white vertices on the perimeter of one of these triangles is independent of that on the others. But then it follows from a well-known law of probability theory that there is with probability 1 at least one such A_i of which all six vertices are white. It is obvious that such a triangle blocks all possible black walks starting in one of the corners of A_0. This is the case for every $p<1$, so we may conclude that for this graph, which obviously is a mosaic, $P_c^{(s)}=1$, so that (3) and therefore (1) does not hold.

In Sec. 2 relation (1) will be derived for a restricted class of lattices in a way that is totally different from that of Sykes and Essam. The proof is based on the following assumption:

Assumption 1: If $p<P_c^{(s)}$ then the mean number of vertices that can be reached from a given vertex via black walks (i.e., the mean size of black clusters) is finite.

Though not proved, this assumption is not unusual. It is even the main idea behind the method of estimating the critical probability by means of cluster-size expansion (see Domb, Sykes[4]).

It will appear that, besides (1), the assumption has other interesting consequences. In Sec. 3, e.g., we shall use it to prove for a certain class of subgraphs of the simple quadratic lattice S, that their critical probability is the same as for S itself. In Sec. 4 assumption 1, combined with a theorem of Hammersley, leads to another mathematical approach of a method to estimate the critical probability for certain lattices. This method is rather similar to the renormalization group method used by Reynolds *et al.*[5,6]

2. AN ALTERNATIVE DERIVATION OF: $P_c^{(s)}(L)+P_c^{(s)}(L^*)=1$.

We shall first discuss some definitions and arguments which lead to Lemma 1. Then we are ready to prove (1) for certain lattices.

Let v be a vertex of some graph G.

$N^n(v)$ denotes the set of all vertices of G that can be reached from v in n or fewer steps.

Further we define:

$$B^0(v)=N^0(v)=\{v\},$$

$$B^n(v)=N^n(v)\backslash N^{n-1}(v).$$

We shall call $B^n(v)$ the sphere with center v and radius n. Now consider the site-percolation process on G of which every vertex is colored black with probability p and white with probability $1-p$. Let $S(p;v)$ be the mean number of vertices that can be reached from v by black walks, and denote by $S_n(p;v)$ the mean number of such vertices which lie in $B^n(v)$, $n=0,1,2,\cdots$. It is clear that

$$S(p;v)=\sum_{n=0}^{\infty}S_n(p;v). \tag{4}$$

Further let $P_n(p;v)$ be the probability of the event that at least one vertex outside $N^n(v)$ can be reached from v by a black walk.

Every walk from v to a vertex outside $N^n(v)$ obviously visits some vertex of $B^n(v)$ and the probability that at least one vertex of the latter kind can be reached from v by a black walk is not larger than $S_n(p;v)$, so that

$$P_n(p;v)<S_n(p;v); \tag{5}$$

this, combined with (4) and assumption 1 gives:

Lemma 1: If $p<P_c^{(s)}$, then $\Sigma_{n=0}^{\infty}P_n(p;v)<\infty$.

For reasons of simplicity we shall first study as an example the site-process on the simple quadratic lattice S, for which with the help of Lemma 1 we shall prove (1). Afterwards the results will be generalized. Because for this lattice the functions S_n, S, and P_n do not depend on v, we shall omit this parameter.

For $p<P_c^{(s)}(S)$, it follows, by Lemma 1, that the series $\Sigma P_n(p)$ converges and so, for some $M\in N$ and positive real number r:

$$\sum_{n=M+1}^{\infty}P_n(p)=r<1. \tag{6}$$

Denote by W_∞, W, and C the events that the vertex $0=(0,0)$ belongs to an infinite white S^* cluster, that all vertices $(0,0),(0,-1),\ldots,(0,-M)$ are white. and that the vertex

0 is black or surrounded by a black S circuit, respectively.

From the matching–property (see Appendix 1 of Ref. 1 for a proof) it follows that either W_∞ or C occurs. We also note the following: If all vertices $(0,0),(0,-1),\ldots,(0,-M)$ are white, then the event C can only occur if there is a black S walk from a vertex on the Y axis below $(0,-M)$ to a vertex on the Y axis above 0. Further, for each positive n, all vertices on the positive Y axis lie outside $N^n((0,-n))$, so that the probability of the event that at least one of these vertices can be reached from $(0,-n)$ by a black walk is smaller than $P_n(p)$. Therefore, if for events E_1 and E_2 $\Pr\{E_1|E_2\}$ denotes the conditional probability of E_1, given E_2, it follows for $p<P_c^{(s)}(S)$:

$$\Pr\{C|W\}<\sum_{n=M+1}^{\infty}P_n(p)=r<1, \tag{7}$$

and hence

$$\Pr\{W_\infty\}\geqslant\Pr\{W\}\Pr\{W_\infty|W\}$$
$$=q^{M+1}(1-\Pr\{C|W\})>q^{M+1}(1-r)>0. \tag{8}$$

So we have proved that, for $p<P_c^{(s)}(S)$, there is a positive probability that a given vertex belongs to an infinite white S^* cluster. In other words, if $p<P_c^{(s)}(S)$, then $1-p>P_c^{(s)}(S^*)$. This immediately yields, by taking $p=P_c^{(s)}(S)-\epsilon$, with ϵ positive and arbitrarily small:

$$P_c^{(s)}(S)+P_c^{(s)}(S^*)\leqslant1. \tag{9}$$

Fisher,[7] generalizing Harris' method,[8] proved that for a certain class of lattices, to which S belongs, $P_c^{(b)}(L)+P_c^{(b)}(L^D)\geqslant1$, where L^D is the dual lattice of L and $P_c^{(b)}$ denotes the critical probability for the bond-percolation process. This result can be extended to the site-case, so that we have, for S,

$$P_c^{(s)}(S)+P_c^{(s)}(S^*)\geqslant1, \tag{10}$$

which, combined with (9), yields the wanted relation:

$$P_c^{(s)}(S)+P_c^{(s)}(S^*)=1. \tag{11}$$

When we call two vertices v_1 and v_2 equivalent if, for all n and p, $S_n(p;v_1)=S_n(p;v_2)$, then we can generalize the above result as follows:

Theorem 1: Let L be a lattice wh h has only a finite number of classes of equivalent vertic and which possesses a pair of orthogonal symmetry-axes. nen

$$P_c^{(s)}(L)+P_c^{(s)}(L^*)=1.$$

The proof of Theorem 1 is simila o that of the special case of the simple quadratic lattice (s also Fisher[7]).

3. SOME NONTRIVIAL SECTION-G PHS OF S WITH CRITICAL PROBABILITY $P_c^{(s)}(S)$.

In this section it will first be sho that $P_c^{(s)}(S(\frac{1}{2}\pi))=P_c^{(s)}(S)$, where $S(\frac{1}{2}\pi)$ denotes the drant of S with vertex-set $\{(n,m)|n,m\geqslant0\}$. Analogously $(\frac{1}{2}\pi)$ will denote the quadrant of S^* with the same vertex as $S(\frac{1}{2}\pi)$.

From the matching-property (se f. 1) it follows that the vertex $0=(0,0)$ belongs to an inf white cluster of $S^*(\frac{1}{2}\pi)$ if and only if there is no black n $S(\frac{1}{2}\pi)$ from some vertex $(n,0)$ to some vertex $(0,m),n,m$ t is trivial that the probability of the latter event is smaller than the probability of the corresponding event for S, which, in the case that $p<P_c^{(s)}(S)$, can be proved (in a similar way as in Sec. 2) to be smaller than 1. So we have that, for $p<P_c^{(s)}(S)$ (which, by (11), is equivalent with $1-p>P_c^{(s)}(S^*)$, $1-p\geqslant P_c^{(s)}(S^*(\frac{1}{2}\pi))$. Hence it follows that

$$P_c^{(s)}(S^*(\tfrac{1}{2}\pi))\leqslant P_c^{(s)}(S^*). \tag{12}$$

On the other hand, because $S^*(\frac{1}{2}\pi)$ is a subgraph of S^*, it is clear that the critical probability of the first cannot be smaller than that of the second; hence

$$P_c^{(s)}(S^*(\tfrac{1}{2}\pi))=P_c^{(s)}(S^*). \tag{13}$$

The analog of (13) for S is obtained by changing the roles of S and S^*.

In the same way we can prove the following theorem:

Theorem 2: Let u be a positive real number and let S' be a connected subgraph of S containing the section-graph of S with vertex-set

$$\{(n,m)|0\leqslant n;0\leqslant m\leqslant un\},$$

then

$$P_c^{(s)}(S')=P_c^{(s)}(S).$$

Remark: It is noted that similar results hold for many other lattices, particularly for the triangular and the honeycomb lattice.

4. ESTIMATES OF THE CRITICAL PROBABILITY

In this section for a certain class of lattices we shall construct sequences of numbers which converge to the critical probability. As in the last two sections, we shall first take as an example the simple quadratic lattice S.

Let $K(n)$ be the so-called "box" with $(n+1)\times(n+1)$ vertices (see Fig. 2).

By the upper, the lower, the left, and the right side of $K(n)$ we mean the sets $\{(0,n),(1,n),\ldots,(n,n)\}$, $\{(0,0),(1,0),\ldots,(n,0)\}$, $\{(0,0),(0,1),\ldots,(0,n)\}$, and $\{(n,0),(n,1),\ldots,(n,n)\}$, respectively.

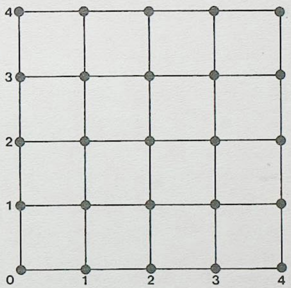

FIG. 2. The box $K(4)$ of the simple quadratic lattice.

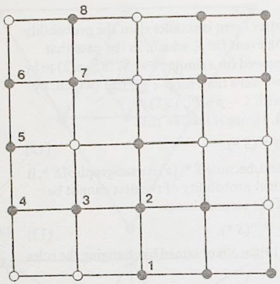

FIG. 3. Example of a coloring of the box in Fig. 2. In this example the event B_4 and not W_4^* occurs. The numbered vertices mark a black S walk from the lower to the upper side.

Let $B_n (W_n)$ be the event that there is a black (white) S walk, entirely lying in $K(n)$, from the lower to the upper side of the box.

Analogously $B_n^* (W_n^*)$ denotes the event that there is a black (white) S^* walk, entirely lying in $K(n)$, from the left to the right side.

Further we define:

$$f_n(p) = \Pr\{B_n\}; f_n^*(p) = \Pr\{B_n^*\}. \tag{14}$$

From this definition and the fact that the probability that a vertex is white is $1-p$, it follows that

$$\Pr\{W_n\} = f_n(1-p); \Pr\{W_n^*\} = f_n^*(1-p). \tag{15}$$

Because of the matching-property either B_n or W_n^* takes place (see e.g., Fig. 3). Hence, by (14) and (15):

$$f_n(p) = 1 - f_n^*(1-p). \tag{16}$$

It will be shown that, for $p < P_c^{(s)}(S)$, the sequence $f_n(p)$ tends to zero. Analogously, if $p < P_c^{(s)}(S^*)$, then $f_n^*(p)$ tends to zero. For this we shall first state a stronger version of Lemma 1.

Consider the site-percolation process on a graph G. Let v be a vertex of G. Define the following functions [with $B^n(v)$ as defined in Sec. 2]: $E_n(p;v) \equiv$ the mean number of vertices in $B^n(v)$ that can be reached from v by at least one black walk of which all vertices, except the last one [which, of course, is in $B^n(v)$], are in $N^{n-1}(v)$. Further,

$$F_n(p) \equiv \sup_v E_n(p;v).$$

In the case that G is a so-called medium[9] the following holds: If, for certain n and p, $F_n(p) = \lambda < 1$, then, for each nonnegative integer m and each vertex v:

$$F_m(p;v) < \lambda^{[m/n]}, \tag{17}$$

where $[m/n]$ denotes the integer part of (m/n) and with $P_m(p;v)$ as defined in Sec. 2. This theorem, which is due to Hammersley,[10] was formulated and proved by him for the bond-case,[11] but it is easily seen that also the above site-version holds.

Now if the medium G contains only a finite number of classes of equivalent vertices (equivalent used in the same sense as in Sec. 2), then it follows from assumption 1 that in the case that p is smaller than the critical probability, for each vertex v, $\Sigma_{m=0}^{\infty} E_m(p;v) < \infty$. Hence, (because of the finite number of equivalence classes) in that case there will be some n such that $F_n(p) < 1$. Next, application of Hammersley's theorem leads to the following lemma:

Lemma 2: If $p < P_c^{(s)}(G)$, then there exists a $\lambda(p) < 1$, such that for all m and all vertices v:

$$P_m(p;v) < \lambda^m(p).$$

Remark: If, in Hammersley's theorem, $[m/n]$ would be replaced by (m/n), then Lemma 2 follows trivially from the above reasonings (take $\lambda(p) = \lambda^{1/n}$), with λ as in Hammersley's theorem). The presence of the [ ]-function makes only a slight change of the proof necessary.

We are now ready to prove the statement about the limiting behavior of $f_n(p)$: From the definition it is clear that $f_n(p)$ is smaller than the probability of the event that there is a black S walk, not necessarily lying entirely in the box $K(n)$, from some vertex $(i,0)$ to some vertex (j,n) $(0 \leqslant i, j \leqslant n)$. Further, for each i and j, the vertex (j,n) lies outside $N^{n-1}((i,0))$ so that, for $p < P_c^{(s)}(S)$, it follows from Lemma 2 that

$$f_n(p) < \sum_{i=0}^{n} P_{n-1}(p;(i,0)) < (n+1)\lambda^{n-1}(p), \tag{18}$$

so that $f_n(p) \to 0$ for $n \to \infty$.

Of course the same arguments hold for S^*, i.e., if $p < P_c^{(s)}(S^*)$ then, for $n \to \infty$,

$$f_n^*(p) \to 0. \tag{19}$$

But, from (11), $p < P_c^{(s)}(S^*)$ is equivalent with

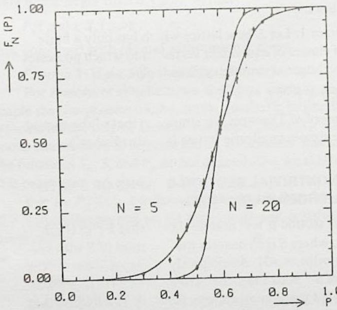

FIG. 4. The function $f_n(p)$ for $n = 5$ and $n = 20$.

$1 - p > P_c^{(s)}(S)$. Hence, by combining (16), (18), and (19), we have the following theorem.

Theorem 3: Denote by $f_n(p)$ the probability of the event that there is a black S walk, which connects the lower and the upper side of the box $K(n)$ and which does not leave this box. Then, for $n \to \infty$

$$f_n(p) \to 0, \text{ for } p < P_c^{(s)}(S),$$

$$f_n(p) \to 1, \text{ for } p > P_c^{(s)}(S).$$

Of course, by symmetry, an analogous theorem holds for S^*.

Remark: The substance of this theorem is already mentioned in earlier papers, e.g., by Reynolds *et al.*[5,6] (who show even more, namely that the "unstable" fixed points of the f_n's converge to the critical probability), but our proof is new. Their theory is based on scaling-arguments, which are very interesting but rather heuristic. On the other hand, our approach does not give insight in the theory of critical exponents. The interested reader is also referred to work by Kirkpatrick.[12]

Though Theorem 3 says nothing about the limiting-behavior of $f_n(p)$ in the case that $p = P_c^{(s)}(S)$, we do have the following theorem:

Theorem 4: Let r be any real number in the open interval (0,1) and let g_n:[0,1]→[0,1] be the inverse function of f_n, then:

$$\lim_{n \to \infty} g_n(r) = P_c^{(s)}(S).$$

This theorem follows from Theorem 3 and the fact that every $f_n(p)$ is continuous (it is a polynomial) and increasing in p, while, for each n, $f_n(0) = 0$ and $f_n(1) = 1$.

Every polynomial f_n is computable (because for every n there is only a finite number of ways in which the vertices of $B(n)$ can be colored black and white); hence Theorem 4 indeed provides sequences of numbers which converge to the critical probability. Unfortunately, even for rather small n, it takes very much time to calculate f_n. For various values of n and p, estimates of $f_n(p)$ are made by Monte Carlo simulations (see e.g., Fig. 4). These values lead to estimates of $g_n(r)$. Though every number between 0 and 1 is allowed, we made the most natural choice and took $r = \frac{1}{2}$.

TABLE I.

n ↕	$f_n(p)$ (p ⟷)	0.590	0.591	0.592	0.593	0.594	0.595
	80	0.454 ± 0.009					0.560 ± 0.009
	120	0.433 ± 0.012	0.462 ± 0.012	0.495 ± 0.012	0.527 ± 0.012	0.533 ± 0.012	0.567 ± 0.012
	160	0.405 ± 0.012	0.450 ± 0.012	0.477 ± 0.012	0.526 ± 0.012	0.539 ± 0.012	0.584 ± 0.012

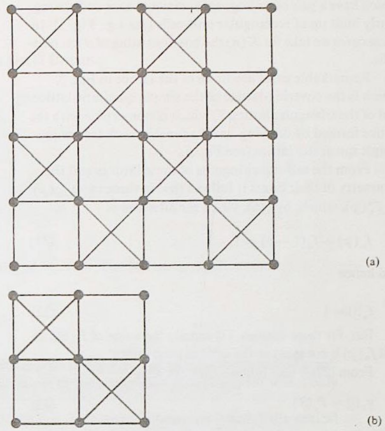

FIG. 5(a) The box $K(2)$ of the covering-lattice of S. (b) Unit-cell of the lattice in (a).

Linear interpolation in the intervals $[p_{n,1}, p_{n,2}]$, where $p_{n,1}$ $(p_{n,2})$ is the largest (smallest) p in Table 1 such that the uncertainty region of $f_n(p)$ lies entirely below (above) $\frac{1}{2}$ (that is: $p_{120,1} = 0.591$, $p_{120,2} = 0.593$; $p_{160,1} = 0.592$, $p_{160,2} = 0.593$), yields:

$$\begin{aligned} g_{80}(\tfrac{1}{2}) &= 0.5922 \pm 0.0003, \\ g_{120}(\tfrac{1}{2}) &= 0.5922 \pm 0.0003, \\ g_{160}(\tfrac{1}{2}) &= 0.5925 \pm 0.0002. \end{aligned} \tag{20}$$

The results (20) give the impression that the last value, 0.5925 ± 0.0002, is a reasonable estimate for the critical probability. This estimate is within the uncertainty region of the less precise result of Sykes *et al.*,[13] who obtained $P_c^{(s)}(S) = 0.593 \pm 0.002$, and a little smaller than the estimate of Reynolds *et al.*,[6] who found $0.5935^{+0.0005}_{-0.0010}$.

Finally it should be remarked that analogs of Theorems 3 and 4 hold for many other lattices, specifically for those

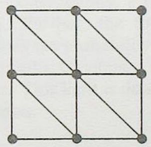

FIG. 6. By drawing one diagonal in each face of the simple quadratic lattice, we obtain this lattice, which is isomorphic with the regular triangular lattice.

which have a pair of orthogonal symmetry-axes and are regularly built up of rectangular unit-cells (see e.g., Fig. 5). In these cases we take for $K(n)$ the box consisting of $n \times n$ unit-cells.

Remarkable cases are those of the lattice in Fig. 5, which is the covering-lattice of the simple quadratic lattice, and of the triangular lattice T, which is isomorphic with the lattice formed by drawing one diagonal in each face of the simple quadratic lattice (see Fig. 6).

From the self-matchingness of these lattices and the symmetry of their boxes it follows that in these cases $f_n(p) = f_n^*(p)$, which, by (16), yields for all n and p:

$$f_n(p) + f_n(1-p) = 1, \tag{21}$$

and hence

$$f_n(\tfrac{1}{2}) = \tfrac{1}{2}. \tag{22}$$

But, for these lattices, $\frac{1}{2}$ is exactly the value of $P_c(S)$, so that $f_n(p)$ is constant at the critical probability.

From (22) it also follows that, for all n

$$g_n(\tfrac{1}{2}) = P_c(S). \tag{23}$$

So if we take $r = \frac{1}{2}$ then, for these lattices, Theorem 4 yields an exact result.

Added in proof: In Sec. 4 a theorem of Hammersley is used to prove that, for $p < P_c^{(s)}(S)$, $\lim_{n\to\infty} (n+1)P_{n-1}(p;v) = 0$ and hence $\lim_{n\to\infty} f_n(p) = 0$ [see Lemma 2 and (18)]. It is possible to derive this result directly, i.e., without using Hammersley's theorem, namely as follows: $P_n(p;v)$ (see definition in Sec. 2) is obviously decreasing in n. Further, if $p < P_c^{(s)}(S)$, then, by Lemma 1 (Sec. 2), $\Sigma P_n(p;v) < \infty$. Hence, if $p < P_c^{(s)}(S)$, then, with $[n/2]$ denoting the integer part of $n/2$:

$$0 \leqslant nP_n(p;v) < 2 \sum_{n/2}^{n} P_m(p;v) \to 0, \text{ for } n \to \infty.$$

ACKNOWLEDGMENTS

The author is indebted to F. Zijderhand, undergraduate student of physics, for programming and operating the computer, to Professor J.W. Cohen and Dr. O.J. Boxma for making valuable comments on the text, and to the referee for drawing his attention to literature about renormalization group methods in percolation theory.

[1] M.F. Sykes and J.W. Essam, J. Math. Phys. 5, 1117 (1964).
[2] J.W. Essam, *Phase Transitions and Critical Phenomena*, *Vol.* 2, edited by C. Domb and M.S. Green (Academic, New York, 1974), pp. 197–270.
[3] H.L. Frisch and J.M. Hammersley, J. S.I.A.M. 11, 894 (1963).
[4] C. Domb and M.F. Sykes, Phys. Rev. 122, 77 (1961).
[5] P.J. Reynolds, W. Klein and H.E. Stanley, J. Phys. C 10, L. 167 (1977).
[6] P.J. Reynolds, H.E. Stanley, and W. Klein, J. Phys. A 11, L. 199 (1978).
[7] M.E. Fisher, J. Math. Phys. 2, 620 (1961).
[8] T.E. Harris, Proc. Cambridge Philos. Soc. 56, 13 (1960).
[9] A medium is a graph satisfying the following conditions: (i) The number of bonds leading from any vertex is finite. (ii) Each finite set of vertices contains a vertex from which a bond leads to a vertex not in that set.
[10] J.M. Hammersley, Ann. Math. Stat. 28, 790 (1957).
[11] In 1957 Hammersley's study of percolation processes was resctricted to the bond-case.
[12] S. Kirkpatrick, *Ill-Condensed Matter*, Les Houches. 1978 lectures, edited by Balian *et al.*
[13] M.F. Sykes, D.S. Gaunt, and M. Glen, J. Phys. A 9, 97 (1976).

J. Phys. A: Math. Gen. 15 (1982) 605–610. Printed in Great Britain

A note on percolation theory

J van den Berg†
Department of Mathematics, University of Utrecht, The Netherlands

Received 6 May 1981

Abstract. In percolation theory the critical probability $P_c(G)$ of an infinite connected graph G is defined as the supremum of those values of the occupation probability for which only finite clusters occur.

An interesting question is the following: is each number between 0 and 1 the critical probability of some graph? It will be shown that the answer is positive.

A remarkable intermediate result is that for an important class of graphs the following holds: for each $p \geqslant P_c(G)$ there exists a subgraph of G with critical probability equal to p.

1. Introduction

Percolation theory, introduced by Broadbent and Hammersley in 1957, has become a fascinating field. It has many applications, especially in physics, where it gives insight in cooperative phenomena (e.g. spontaneous magnetism in a dilute ferromagnet) but also in biology (epidemics in a large orchard), geology and chemistry. Many such examples are described in Frisch and Hammersley (1963) and Shante and Kirkpatrick (1971).

Let G be an infinite non-oriented connected graph of which each vertex is the starting point of only a finite number of bonds. To this graph the following random mechanism is attached. Each bond is, independently of all other bonds, undammed with a fixed probability p and dammed with probability $1-p$. The terms dammed and undammed have been introduced by Broadbent and Hammersley for reasons of clearness (they describe the process as water, which is supplied to a given vertex and spreads from there through the undammed bonds). However, we prefer to use the terminology of Sykes and Essam (1964), and replace the words undammed and dammed by black and white respectively. Consequently, a walk is said to be black (white) if all its bonds are black (white). Further, the following definitions are important. For each vertex v, $P_n(p;v)$ denotes the probability that there are at least n vertices that can be reached from v via black walks. Obviously, $P_n(p;v)$ is decreasing in n and hence the limit $\lim_{n\to\infty} P_n(p;v)$ exists. This limit is denoted by $P_\infty(p;v)$.

The critical probability is defined as follows:

$$P_c(v) = \sup\{p \mid P_\infty(p;v) = 0\}. \tag{1}$$

Broadbent and Hammersley, who dealt with the more general case of partially oriented and not necessarily connected graphs, proved that if v_1 and v_2 are two vertices such that there exists a walk from v_1 to v_2 and also a walk *vice versa*, then $P_c(v_1) = P_c(v_2)$.

† Present address: Physics Laboratory TNO, Room 3506, Oude Waalsdorperweg 63, 2597 AK The Hague, The Netherlands.

0305-4470/82/020605+06$02.00

Hence, since we consider only connected graphs, in our cases all vertices of a graph G have the same critical probability, which we denote by $P_c(G)$.

In another percolation model not the bonds but the vertices are randomly coloured. In this so-called site model we can give analogous definitions as for the bond model.

In general the critical probability for the bond process is not equal to that of the site process. Therefore, to make a distinction, we denote them by $P_c^{(b)}(G)$ and $P_c^{(s)}(G)$ respectively.

It can be shown (Fisher 1961) that the bond-percolation process on a graph G is equivalent with the site process on the so-called covering graph G^c of G, i.e.

$$P_c^{(b)}(G) = P_c^{(s)}(G^c). \tag{2}$$

We now turn to the central question of this paper: is any number p $(0 \leq p \leq 1)$ the critical probability of some graph G? It will be shown that this is indeed the case. From (2) it follows that it is sufficient to give a proof for the bond model. This proof is based on some well known results concerning the bond-percolation process on the square lattice, which we shall discuss in § 2.

2. The bond percolation process on the square lattice

The square lattice, denoted by S, consists of vertices $\{(n, m)|n, m \in \mathbb{Z}\}$, which all have one bond with each of their four neighbours.

The so-called dual lattice S^d of S is constructed as follows (see figure 1). Put one point in the centre of each face of S. These points $\{(n+\frac{1}{2}, m+\frac{1}{2})|n, m \in \mathbb{Z}\}$ form the vertex set of S^d. As we see, this graph S^d is again a square lattice, so that S and its dual are isomorphic. (This is generally not the case, e.g. the dual of the triangular lattice is the honeycomb lattice.) Therefore S is said to be self-dual.

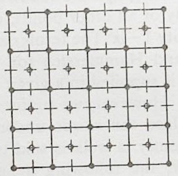

Figure 1. S and its dual S^d.

Each bond of S crosses exactly one bond of S^d so that the bond set of S is in 1–1 correspondence with that of S^d. So each colouring of the bonds of S induces a colouring of the bonds of S^d.

The following lemma is intuitively obvious. A proof is to be found in Whitney (1933).

Lemma 1. Each finite black cluster of S is surrounded by a white circuit of S^d. (This remains true after changing the terms black–white and/or the terms S–S^d.)

We shall now pay attention to the value of the critical probability $P_c^{(b)}(S)$ of S. Hammersley (1957), using the self-duality of S, proved that $e^{-v} \leq P_c^{(b)}(S) \leq 1-e^{-v}$, where v is the so-called connective constant of S. The lower bound has been improved by Harris (1960), who showed that $P_c^{(b)}(S) \geq \frac{1}{2}$. Although for a long time there had been many indications that in the last expression even equality holds (see e.g. Sykes and Essam, 1964), only recently a correct mathematical proof has been given, namely by Kesten (1980).

So we have

$$P_c^{(b)}(S) = \tfrac{1}{2}. \tag{3}$$

It has been proved by Harris (1960) that, for $p > P_c^{(b)}(S)$, almost surely (AS) there exists exactly one infinite black cluster. Hence, by (3) we have

Lemma 2. If $p > \frac{1}{2}$, then (AS) there is exactly one infinite black cluster in S.

Because this lemma plays an important role in the rest of the paper we let the proof (in a slightly different form) follow here. First note that the set of bonds of S is countable. When we denote the colour black by the number 1 and white by 0, then we can associate each bond b_i with a random variable x_i, which has the value 1 with probability p and the value 0 with probability $1-p$, and such that $\{x_i|i \in \mathbb{N}\}$ is a set of independent random variables. In these terms the event that there exists at least one infinite black cluster in S is a tail event of the sequence (x_i), $i \in \mathbb{N}$ (because, for each n, the existence of such a cluster does not depend on the colours of the bonds $b_0, b_1, \ldots, b_n$). Hence, by Kolmogorov's 0–1 law, the probability of this event is either 0 or 1. Now for $p > \frac{1}{2}$ this probability is, by (3), larger than 0 and therefore equal to 1.

The fact that, for $p > \frac{1}{2}$ (AS) not more than one infinite cluster exists can be seen as follows. Let v_1 and v_2 belong to the infinite black clusters C_1 and C_2 respectively. The probability of a bond to be white is $1-p$, which is smaller than $\frac{1}{2}$, so that (AS) all white clusters in S^d are finite.

But then it can be derived from lemma 1 that (AS) each finite set of vertices of S^d is surrounded by a black circuit in S, so (AS) there exists a black circuit in S which has both vertices v_1 and v_2 in its interior. It is obvious that this circuit connects C_1 and C_2, hence these clusters are one and the same.

3. A proof for the interval $[\frac{1}{2}, 1]$

In § 2 it has been stated that, for p larger than $\frac{1}{2}$, there exists (AS) exactly one infinite black cluster in S. It will appear that (AS) the critical probability $P_c^{(b)}$ of this cluster is equal to $\frac{1}{2}/p$. Then, by varying p in the interval $\langle\frac{1}{2}, 1]$, we can, for any value in $[\frac{1}{2}, 1\rangle$, 'create' a subgraph of S of which the critical probability is equal to that value. Subsequently, by a kind of trick, namely multiplication of the bonds of S, this result can be extended to the region $\langle 0, 1\rangle$. Next, only the trivial numbers 0 and 1 rest. As to the value 1, the easiest example of a graph with this critical probability is the linear chain consisting of vertices $v_1, v_2, v_3, \ldots$ and one bond between any pair (v_n, v_{n+1}). (In fact this graph can be considered as the section graph of S with vertex set $\{(x, 0)|x \in \mathbb{N}\}$.) Finally, the tree-like medium in figure 2 with vertex set $\{v_{n,m}n \geq 1, m \leq n!\}$, contains, for each k, the Bethe lattice of order k, so that its critical probability is, for each k, not larger than $1/k$ and hence equal to 0.

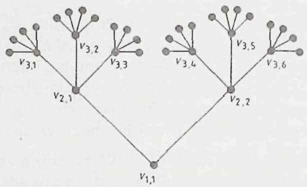

Figure 2. Example of a graph with critical probability 0.

We shall now prove the statement at the beginning of this section that, for $p > \frac{1}{2}$, the critical probability of the infinite black cluster is (AS) equal to $\frac{1}{2}/p$.

Let p_1 be a number in the interval $\langle \frac{1}{2}, 1]$ and let $\{b_i | i \in \mathbb{N}\}$ be the set of bonds of S. To this set corresponds a set $X = \{x_i | i \in \mathbb{N}\}$ of independent random variables, such that $\Pr\{x_i = 1\} = 1 - \Pr\{x_i = 0\} = p_1$.

The value 1 (0) of each random variable x_i corresponds with the state black (white) of its corresponding bond b_i. Further, let p_2 be any number in $[0, 1]$ and let $Y = \{y_i | i \in \mathbb{N}\}$ be a set of independent random variables such that $\Pr\{y_i = 1\} = 1 - \Pr\{y_i = 0\} = p_2$ and X and Y are independent sets of random variables. Finally, define $Z = \{z_i | z_i = x_i y_i;\ i \in \mathbb{N}\}$.

The black subgraph corresponding to the x_i is called B', and the one corresponding to the z_i is called B''.

By the results in § 2 the following statements hold.

(i) Because $p_1 > \frac{1}{2}$, B' contains (AS) exactly one infinite cluster (see lemma 2), which we call C.

(ii) B'' is a subgraph of B' and contains (AS) no or exactly one infinite cluster. In the last case that cluster is a subgraph of C.

(iii) If $p_2 < \frac{1}{2}/p_1$, then, for all i, $\Pr\{z_i = 1\} = p_1 p_2 < \frac{1}{2}$ and hence (AS) B'' consists only of finite clusters.

(iv) On the other hand, if $p_2 > \frac{1}{2}/p_1$, then, for all i, $\Pr\{z_i = 1\} > \frac{1}{2}$ and hence (AS) B'' contains an infinite cluster, which, as stated in (ii), is a subgraph of C.

Now from the above it follows by definition that, (AS) the critical probability of C is indeed equal to $\frac{1}{2}/p_1$. Hence the class of those subgraphs of S which have critical probability $\frac{1}{2}/p_1$ is not empty. Next, by varying p_1 in the interval $\langle \frac{1}{2}, 1]$, and noting the example of a graph with critical probability 1 at the beginning of this section, we obtain the following theorem.

Theorem 1. Let p be a number in the interval $[\frac{1}{2}, 1]$. Then there exists a connected subgraph L of the square lattice with critical probability $P_c^{(b)}(L) = p$.

Remark. If G is a planar lattice, regularly built up of unit cells and possessing a pair of orthogonal symmetry axes, then it can be shown (see Fisher 1961), that $P_c^{(b)}(G) + P_c^{(b)}(G^d) \geq 1$, where G^d denotes the dual lattice of G. From this, by using the arguments in the proof of lemma 2, it can be proved that the following generalisation of that lemma holds: if $p > P_c^{(b)}(G)$ then there exists exactly one infinite black cluster in G. This, in its turn leads to a generalisation of theorem 1.

Each $p \geq P_c^{(b)}(G)$ is the critical probability of some subgraph of G.
An interesting question is whether this holds for all lattices.

4. Extension of the result in § 3 to the interval [0, 1]

Let S^n be the graph obtained by replacing each bond of S by n parallel bonds, $n \geq 1$ (see figure 3). For each colouring of the bonds of S^n a colouring of the bonds of S can be defined as follows: each bond of S is coloured black if at least one of the bonds of the corresponding n-tuple in S^n is black, otherwise it is coloured white. Hence, if p is the probability that a bond of S^n is coloured black, then the probability of a bond of S to be black is $1 - (1-p)^n$. Further, note that there is an infinite black cluster in S if and only if there is one in S^n. From these reasonings it follows that $p \geq P_c^{(b)}(S^n)$ if and only if $1 - (1-p)^n \geq P_c^{(b)}(S)$, which equals $\frac{1}{2}$, so that

$$P_c^{(b)}(S^n) = 1 - [1 - P_c^{(b)}(S)]^{1/n} = 1 - (\tfrac{1}{2})^{1/n}. \qquad (4)$$

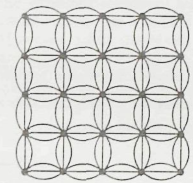

Figure 3. The lattice S^3.

Now we can apply the ideas of § 3 to S^n, which leads to the following theorem.

Theorem 2.

$$P_c^{(b)}(S^n) = 1 - (\tfrac{1}{2})^{1/n}.$$

Further, if the probability p that a bond of S^n is black, is larger than $P_c^{(b)}(S^n)$, then (AS) there exists exactly one infinite black cluster in S^n and the critical probability of that cluster is equal to $P_c^{(b)}(S^n)/p$.

Now because $\lim_{n\to\infty} P_c^{(b)}(S^n) = 0$, the following theorem follows by varying n and p in theorem 2 (and again noting the example of the graph with critical probability 1 in § 3).

Theorem 3. For each p in the interval $\langle 0, 1]$ there exists, for a certain n, an infinite connected subgraph of S^n, of which the critical probability (bond case) is equal to p.

Theorem 3, together with the example of a graph with critical probability 0 (figure 2), completes the work.

Remark. If we do not want to deal with graphs with multiple bonds, like the S^n, we can handle them as follows. Define (instead of S^n) S^{n^*} as the graph obtained by replacing each bond of S by an n-tuple of series of two bonds (figure 4). It is easily seen that the critical probability of S^{n^*} is equal to $(P_c^{(b)}(S^n))^{1/2}$ and a straightforward repeat of the arguments, earlier applied to S^n, leads to an analogue of theorem 3.

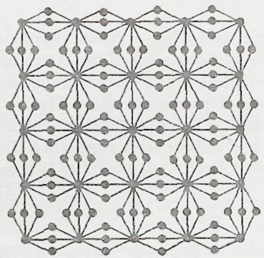

Figure 4. The lattice S^{3^*}.

Acknowledgment

The author is indebted to Professor P W Kasteleyn for helpful comments.

References

Broadbent S R and Hammersley J M 1957 *Proc. Camb. Phil. Soc.* **53** 629–41
Fisher M E 1961 *J. Math. Phys.* **2** 620–7
Frisch H L and Hammersley J M 1963 *J. SIAM* **11** 894–913
Hammersley J M 1957 *Le calcul des Probabilités et ses Applications* (Paris: Colloques Internationaux du CNRS)
Harris T E 1960 *Proc. Camb. Phil. Soc.* **56** 13–20
Kesten H 1980 *Commun. Math. Phys.* **74** 41–59
Shante V K S and Kirkpatrick S 1971 *Adv. Phys.* **20** 325–57
Sykes M F and Essam J W 1964 *J. Math. Phys.* **5** 1117–27
Whitney H 1933 *Fund. Math.* **21** 73–84

Adv. Appl. Prob. **15**, 465–467 (1983)
Printed in N. Ireland

A COUNTEREXAMPLE TO A CONJECTURE OF J. M. HAMMERSLEY AND D. J. A. WELSH CONCERNING FIRST-PASSAGE PERCOLATION

J. VAN DEN BERG,* *Delft University of Technology*

Abstract

Consider first-passage percolation on the square lattice. Hammersley and Welsh, who introduced the subject in 1965, conjectured that the expected minimum travel time from $(0,0)$ to $(n,0)$ along paths contained in the cylinder $0 \leqq x \leqq n$ is always non-decreasing in n. However, when the bonds have time-coordinate 1 with probability p and 0 with probability $1-p$ $(0<p<1)$, then, for p sufficiently small, we get a counterexample.

EXPECTED CYLINDER TIMES

1. Introduction

First-passage percolation was first introduced by Hammersley and Welsh in 1965. For more recent literature see e.g. Smythe and Wierman (1977).

Consider the square lattice S. To each bond b of S, independent of all other bonds, a random non-negative number is assigned, called the time coordinate, which can be considered as the time needed to travel along b from one of its endpoints to the other. The travel time of a path is defined as the sum of the time coordinates of its bonds. Hammersley and Welsh defined the cylinder time t_n as the infimum of the travel times of all cylinder paths from $(0,0)$ to $(n,0)$ (i.e., paths which, except for the starting point, are contained in the cylinder $0<x\leqq n$). They conjectured that the expectation τ_n of t_n is always non-decreasing in n. Before we go further, we slightly modify the definitions by replacing the $<$ sign in the above cylinder condition by $\leqq$, and we denote the analogs of t_n and τ_n by t_n' and τ_n' respectively. It is clear that, for each n, $\tau_{n+1}=\tau_1+\tau_n'$. In the next section we give an example where $\tau_1'>\tau_2'$, so that $\tau_2>\tau_3$, thus showing that the conjecture is false.

2. The counterexample

Consider the case where the time-coordinates are 1 with probability p and 0 with probability $1-p$, $0<p<1$.

Received 21 March 1983.
* Postal address: Department of Mathematics, Delft University of Technology, Julianalaan 132, 2628 BL Delft, The Netherlands.
Investigations supported by the Netherlands Foundation for Mathematics SMC with financial aid from the Netherlands Organization for the Advancement of Pure Research (ZWO).

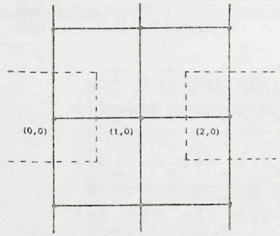

Figure 1. The two cut sets of size 3, which block the cylinder paths from (0, 0) to (2, 0)

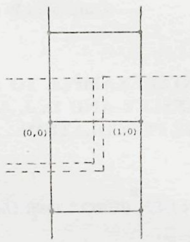

Figure 2. Two cut sets of size 3 which block cylinder paths from (0, 0) to (1, 0). The other two are obtained by reflection in the line $x = \frac{1}{2}$

Lemma. For $p \downarrow 0$,

$$P[t'_1 = 1] = P[t'_1 \neq 0] = 4p^3 + o(p^3). \tag{2.1}$$

$$P[t'_2 \geqq 1] = P[t'_2 \neq 0] = 2p^3 + o(p^3). \tag{2.2}$$

$$P[t'_2 = 2] = o(p^3). \tag{2.3}$$

Before we prove the lemma, we show its consequences. It is clear that t'_1 is either 0 or 1 and that t'_2 can only have the values 0, 1 and 2. Hence, by the lemma

$$\tau'_1 = E[t'_1] = P[t'_1 = 1] = 4p^3 + o(p^3),$$

and

$$\tau'_2 = E[t'_2] = P[t'_2 \geqq 1] + P[t'_2 \geqq 2] = 2p^3 + o(p^3),$$

and clearly, for p sufficiently small, τ'_1 is larger than τ'_2, as we stated in the introduction.

The lemma can be shown as follows. As to (2.2), $t'_2 \neq 0$ if and only if there is a so-called cut set (a set of bonds which blocks all cylinder paths from (0, 0) to (2, 0)), with all bonds having time-coordinate 1. The smallest of these cut sets consist of three bonds, and there are exactly two of them (see Figure 1). It is easily seen that the probability that for at least one of them all bonds have time-coordinate 1, is $2p^3 + o(p^3)$. All other cut sets are larger and the probability that for at least one of them all bonds have time-coordinate 1 is $o(p^3)$. (This follows from the facts that the cut sets correspond to certain paths in the dual lattice (see Whitney (1933)) and the number of paths of length n, starting at a given site, is exponentially bounded in n).

An analogous argument holds with respect to the event $[t'_1 = 1]$. Again the smallest cut sets have size 3, but now there are four smallest cut sets (see Figure 2) which is responsible for the factor 4 in (2.1).

Finally, (2.3) holds because, clearly, there are three disjoint cylinder paths from (0, 0) to (2, 0), and the probability that each of them contains at least two bonds with time coordinate 1 is the product of the individual probabilities which obviously are $o(p)$.

References

Hammersley, J. M. and Welsh, D. J. A. (1965) in *Bernoulli Bayes Laplace Anniversary Volume*, ed. J. Neyman and L. M. Le Cam, Springer-Verlag, Berlin, 61–110.

Smythe, R. T. and Wierman, J. C. (1978) *First-Passage Percolation on the Square Lattice*. Lecture Notes in Mathematics **671,** Springer-Verlag, Berlin.

Whitney, H. (1933) Planar graphs. *Fundamenta Math.* **21,** 73–84.

Contemporary Mathematics
Volume 26, 1984

ON THE CONTINUITY OF THE PERCOLATION PROBABILITY FUNCTION

by

J. van den Berg[+] and M. Keane

ABSTRACT. Let G be a countably infinite, connected, locally finite graph, and let s_0 be a designated vertex of G. Denote by $\theta(p)$ the percolation probability function for bond percolation on the pointed graph (G,s_0). We show that

$$\theta(p-) = \theta(p) - \Pr\left\{\begin{array}{l} s_0 \text{ belongs to an infinite p-open} \\ \text{connected component of } G\text{, which itself} \\ \text{has critical probability one.}\end{array}\right\}$$

As a corollary, we deduce that if p is strictly larger than the critical probability for bond percolation on G, and if for p there is a unique infinite p-open cluster (e.g. if G is $\mathbb{Z}^2$ with nearest neighbor bonds), then θ is continuous at p.

§1. DEFINITIONS AND NOTATIONS

Let S be a finite or a countably infinite set, and let B be a collection of two-element subsets of S. Thus $G = (S,B)$ is a (finite or infinite) undirected graph. The elements of S are called <u>sites</u> (= vertices) of G and those of B <u>bonds</u> (= edges) of G. If the bond b contains the sites s and t, then we say that b <u>links</u> s and t (or t and s).

A path π in G is a (finite or infinite) sequence $\pi = (b_1,b_2,\ldots)$ of elements of B such that there exists a sequence $\pi^s = (s_0,s_1,\ldots)$ of elements of S such that for each $i \geq 1$, b_i links s_{i-1} and s_i. Clearly, π^s is determined uniquely by π. If a path $\pi = (b_1,b_2,\ldots,b_n)$ is finite, with $\pi^s = (s_0,s_1,\ldots, s_n)$, then s_0 is called the <u>initial site</u> of π and s_n the terminal site of π. If π is infinite, then s_0 is the initial site, and it is convenient to call ∞ the terminal site (although it is not a site). If π is a path with initial site s and terminal site t, then we say that π joins s with t, and write $s \overset{\pi}{\to} t$.

Let $s_0 \in S$. The connected component of G containing s_0 is the subgraph of G whose sites are s_0 together with all $t \in S$ for which there

[+]Financially supported by the Netherlands Organization for the Advancement of Pure Research (ZWO).

exists a path π which joins s_0 and t, and whose bonds are all bonds in B which link any of these sites with another of these sites. G is connected if for some s_0 ($\equiv$ for each s_0), the connected component of G containing s_0 is G.

§2. PERCOLATION PROBABILITY AND CRITICAL PROBABILITY.

In this paragraph, we suppose that $G = (S,B)$ is a given infinite connected graph. Let $(X_b)_{b \in B}$ be a collection of independent identically distributed random variables indexed by the bonds of G, with common distribution given by

$$\Pr\{X_b \le x\} = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x \le 1. \end{cases}$$

Let $0 \le p \le 1$. For a given realization of the process $(X_b)_{b \in B}$, we say that the bond $b \in B$ is p-open if $X_b < p$, and p-closed if $X_b \ge p$. A path $\pi = (b_1, b_2, \ldots)$ is p-open if for each b_i in π, b_i is p-open. We denote by $G^p = (S^p, B^p)$ the (random) subgraph of G given by $S^p = S$ and

$$B^p = \{b \in B: b \text{ is p-open}\} .$$

Now choose and fix a site $s_0 \in S$, so that we are dealing with a pointed graph $\dot{G} = (S,B,s_0)$ and random pointed subgraphs $\dot{G}^p = (S^p, B^p, s_0)$. Let $\dot{G}^p_0$ be the (pointed) connected component of $\dot{G}^p$ containing s_0. We call $\dot{G}^p_0$ the p-open cluster (graph) containing s_0.

DEFINITION

1) The <u>percolation probability</u> function of the pointed graph $\dot{G} = (S,B,s_0)$ is given by

$$\theta(\dot{G},p) = \theta(p) = \Pr\{\dot{G}^p_0 \text{ is infinite}\} .$$

2) The <u>critical probability</u> of the graph $\dot{G}$ (see remark 4) is given by

$$p_H(\dot{G}) = p_H = \sup\{p: \theta(p) = 0\} = \inf\{p: \theta(p) > 0\} .$$

REMARKS

1. These definitions are due to Broadbent and Hammersley [2] where we have modified the notation to suit our purposes.
2. It is clear that $\theta(p)$ is a non-decreasing function of p.

3. It is not hard to deduce that if the graph G is locally finite ($\equiv$ each $s \in S$ is contained in only a finite number of bonds), then $\theta(p)$ is continuous from the right (see [7]).
4. In general, the percolation probability function depends on the choice of s_0. However (see e.g. [2]), it is quickly shown that the critical probability p_H is the same for each choice of s_0 as G is supposed connected.
5. Note that if G is locally finite,

$$\theta(p) = \Pr\{\text{there exists a p-open path } \pi \text{ with } s_0 \overset{\pi}{\to} \infty\} .$$

6. Though we restrict to bond percolation on undirected graphs, it is easily seen that analogs of the results in section 3 hold for site percolation, and for percolation on directed graphs.

§3. STATEMENT OF THE RESULTS.

Recently, some interest has been shown in determining the value of p_H ([5], [8], [9]) and the behavior of the function $\theta(p)$ and related functions for a variety of graphs. Except for regular two-dimensional graphs (see [7] and [8]) and examples where percolation is identical with "infinite life" in birth-death processes, it does not seem to be known whether the function $\theta(p)$ is continuous, although this is expected to be true for a wide class of graphs (e.g. Z^d with nearest neighbor bonds, $d \ge 3$). On the other hand, Harris [4] has shown for Z^2 that for p above the critical probability p_H ($= \frac{1}{2}$, Kesten [5]), the random graph G^p possesses exactly one infinite connected component with probability one, (for a generalization to regular two-dimensional graphs see Fisher [3]), and Newman-Schulman ([6]), have investigated the possibility of existence of more than one infinite connected component of G^p in a general setting

We hope that the following result, which links the continuity of θ^p with the number and types of infinite connected components, will help to clarify the situation.

THEOREM. <u>Let</u> $\dot{G} = (S,B,s_0)$ <u>be a countably infinite, connected, locally finite graph. Then for any</u> $0 \le p \le 1$,

$$\theta(p-) = \theta(p) - \Pr\{\dot{G}^p_0 \text{ is infinite and its critical probability is one}\} .$$

COROLLARY. <u>Let</u> $\dot{G}$ <u>be as in the theorem. Furthermore, suppose that</u> $p > p_H(\dot{G})$ <u>is such that with probability one,</u> G^p <u>contains exactly one infinite connected component. Then</u> $\theta(p)$ <u>is continuous at</u> p.

Note that regular two-dimensional graphs([3] and [4]) satisfy the conditions of the corollary (see also Russo ([7] and [8]), who proves in addition continuity at $p = p_H(\dot{G})$ and differentiability at $p \neq p_H(\dot{G})$). Kesten (private communication), has shown that for $p > \frac{1}{2}$, $\mathbb{Z}^3$ with nearest neighbor bonds satisfies the conditions of the corollary.

§4. PROOFS

To prove the theorem, we must show that

$$(*) \qquad \lim_{\substack{p'\uparrow p \\ p' \neq p}} \theta(p') = \Pr\{\dot{G}_0^p \text{ is infinite and } p_H(\dot{G}_0^p) < 1\}$$

Let $\dot{H} = (\tilde{S},\tilde{B},s_0)$ be a pointed infinite connected subgraph of G (with point s_0), and note that under the condition $\dot{G}_0^p = \dot{H}$, the joint (conditional) distribution of the process $(X_b)_{b \in \tilde{B}}$ is i.i.d. with common distribution given by

$$\Pr\{X_b \le x \mid \dot{G}_0^p = \dot{H}\} = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{x}{p} & \text{if } 0 \le x \le p \\ 1 & \text{if } x \ge p \end{cases}$$

Noting that

$$\lim_{\substack{p'\uparrow p \\ p' \neq p}} \theta(p') = \Pr\{\exists\, p' < p \text{ with } \dot{G}_0^{p'} \text{ infinite}\},$$

and conditioning both sides of (*) by $\dot{G}_0^p = \dot{H}$, we see that it is sufficient to show that

$$\Pr\{\exists\, p' < p \text{ with } \dot{G}_0^{p'} \text{ infinite} \mid \dot{G}_0^p = \dot{H}\}$$

$$= \Pr\{p_H(\dot{G}_0^p) < 1 \mid \dot{G}_0^p = \dot{H}\}$$

$$= \begin{cases} 0 & \text{if } p_H(\dot{H}) = 1 \\ 1 & \text{if } p_H(\dot{H}) < 1 \end{cases}$$

Using now the information on the joint distribution of the process $(X_b)_{b \in \tilde{B}}$ under the condition $\dot{G}_0^p = \dot{H}$, this translates to the requirement that

$$\Pr\{\exists\, p'' < 1 \text{ with } \dot{H}_0^{p''} \text{ infinite}\} = \begin{cases} 0 & \text{if } p_H(\dot{H}) = 1 \\ 1 & \text{if } p_H(\dot{H}) < 1 \end{cases}.$$

The case $p_H(\dot{H}) = 1$ is obvious, and if $p_H(\dot{H}) < 1$, then choose $\tilde{p}$ with

$p_H(\dot{H}) < \tilde{p} < 1$. By definition of $p_H(\dot{H})$, for almost every realization the random subgraph $H^{\tilde{p}}$ of H contains an infinite connected component, since this event is a tail event with positive probability. This component may not contain s_0, but since H is connected, there is a finite path π joining s_0 with an infinite component, and since we may assume with no loss of generality that $X_b < 1$ for all b, it follows that $\dot{H}_0^{p''}$ is infinite for some $p'' < 1$ ($p'' = \max(\tilde{p}, \max_{b \in \pi} X_b)$) almost surely. This finishes the proof of the theorem.

To prove the corollary, we show that under the given hypotheses,

$$\Pr\{\dot{G}_0^p \text{ is infinite and } p_H(\dot{G}_0^p) = 1\} = 0.$$

Choose $\tilde{p}$ with $p_H(\dot{G}) < \tilde{p} < p$. Then for almost every realization, $G^{\tilde{p}}$ contains an infinite connected component (by the 0-1 law for tail events) which is contained in the (unique) infinite connected component of G^p. Thus, choosing $\dot{H}$ and conditioning on $\dot{G}_0^p = \dot{H}$ as in the proof of the theorem, we see that

$$p_H(\dot{G}_0^p) = \frac{p_H(\dot{G})}{p}$$

almost surely (see also [1]), and this (together with remark 3) proves the corollary.

REFERENCES

[1] van den Berg, J., (1982). J. Phys. A: Math. Gen. 15, 605-610.

[2] Broadbent, S.R. and Hammersley, J.M., (1957), Proc. Camb. Phil. Soc. 53, 629-641.

[3] Fisher, M.E., (1961), J. Math. Phys. 2, 620-627.

[4] Harris, T.E., (1960). Proc. Camb. Phil. Soc. 56, 13-20.

[5] Kesten, H., (1980), Commun. Math. Phys. 74, 41-59.

[6] Newman, C.M. and Schulman, L.S., (1981). J. Stat. Phys. 26, 613-628.

[7] Russo, L., (1978). Z. Wahrscheinlichkeitstheorie verw. Gebiete 43, 39-48.

[8] Russo, L., (1981). Z Wahrscheinlichkeitstheorie verw. Gebiete 56, 229-237.

[9] Wierman, J. C., (1981). Adv. Appl. Prob. 13, 298-313.

DEPARTMENT OF MATHEMATICS
DELFT UNIVERSITY OF TECHNOLOGY
JULIANALAAN 132, 2628 BL DELFT,
THE NETHERLANDS

Adv. Appl. Prob. **16**, 690–691 (1984)
Printed in N. Ireland

DISPROOF OF THE CONJECTURED SUBEXPONENTIALITY OF CERTAIN FUNCTIONS IN PERCOLATION THEORY

J. VAN DEN BERG,* *Delft University of Technology*

Abstract

Consider bond-percolation on a graph G with sites $S(G)$. We disprove the conjecture of Hammersley (1957) that the function $n \to \sup_{s \in S(G)} E$ [the number of sites s' at distance n from s which can be reached from s by an open path which, except for s', only passes through sites at distance smaller than n from s] is always subexponential.

1. Introduction

Percolation theory has been introduced by Broadbent and Hammersley (1957). For a recent introduction to the subject see Kesten (1982), Chapter 1.

Let G be a locally finite graph (i.e. the number of bonds incident to any site is finite) and denote the set of sites of G by $S(G)$. Let the bonds of G, independent of each other, be open with probability p and closed with probability $1-p$. The length of a path is the number of bonds it contains. The distance between two sites is the length of the shortest path which connects them. Define, for $s \in S(G)$:

$N^n(s)$ is the set of sites at distance $\leqq n$ from s.
$B^n(s)$ is the set of sites at distance n from s.
$E_n(s)$ is the expected number of sites $s' \in B^n(s)$ for which there exists an open path from s to s' which, except for s', only passes through sites in $N^{n-1}(s)$.
Finally, define $F_n = \sup_{s \in S(G)} E_n(s)$.
Though E_n and F_n also depend on p, we omit this parameter.

Hammersley (1957) conjectured that $F_{n+m} \leqq F_n F_m$ always. In the next section we show that there exists a case for which $F_2 > F_1^2$ so that the conjecture is false.

2. The counterexample

Consider, for a positive integer r, the graph with $1+r+r^2$ sites denoted by c; s_i, $1 \leqq i \leqq r$, and $s_{i,j}$, $1 \leqq i, j \leqq r$; and with bonds (c, s_i), $1 \leqq i \leqq r$; (s_i, s_j), $1 \leqq i, j \leqq r$, $i \neq j$; and $(s_i, s_{i,j})$, $1 \leqq i, j \leqq r$. This graph can be imagined as a central site c, surrounded by and

Received 18 April 1984.

* Postal address: Department of Mathematics, Delft University of Technology, Julianalaan 132, 2628 BL Delft, The Netherlands.

Research supported by the Netherlands Foundation for Mathematics SMC with financial aid from the Netherlands Organization for the Advancement of Pure Research (ZWO).

connected with a complete graph on r sites, each of which having a bond to r other sites which have no further connections.

Now consider bond-percolation on this graph with p the probability of a bond to be open. It is clear that, for each site s, $E_1(s)$ equals p times the number of bonds incident to s and this is maximal if s is one of the s_i's, in which case it equals $2rp$. So

(2.1) $$F_1 = 2rp.$$

Further, F_2 is at least $E_2(c)$ which, by symmetry, equals the number of sites at distance 2 from c multiplied by the probability of the event that at least one of them, say s_{11}, can be reached from c by an open path. (By the structure of the graph the condition of containing no sites, except s_{11}, outside $N^1(c)$ is automatically fulfilled.) Note that this event occurs if and only if the bond (s_1, s_{11}) is open (which happens with probability p) and there exists, inside the complete graph on the set $\{c, s_1, s_2, \cdots, s_r\}$ an open path from c to s_1. Denote the probability of the latter event by $P(p, r)$. Using independence we get

(2.2) $$F_2 \geqq r^2 p P(p, r).$$

Hence, by (2.1) and (2.2)

(2.3) $$\frac{F_2}{F_1^2} \geqq \frac{P(p, r)}{4p}.$$

It is easily seen that for fixed p

(2.4) $$\lim_{r\to\infty} P(p, r) = 1, \qquad 0 < p \leqq 1.$$

Now fix p between 0 and $\frac{1}{4}$. Then, for r sufficiently large, the right-hand side of (2.3) is larger than 1, in contradiction to the conjecture.

Remarks.

(i) With the help of the finite graphs above it is easy to obtain a counterexample concerning an infinite connected graph. For example, connecting the site c with an infinite chain does not increase the value F_1.

(ii) One might think that the conjecture is true if, in the definition of $E^n(s)$, all open paths of which all sites are in $N^n(s)$ are allowed. However, consider the tree consisting of a site c which is connected with six sites $s_1, s_2, \cdots, s_6$, each s_i in its turn being connected with six sites $s_{i,1}, s_{i,2}, \cdots, s_{i,6}$. Add to this tree, for each $j \leqq 6$, the bonds $(s_{1,j}, s_{2,j})$, $(s_{3,j}, s_{4,j})$ and $(s_{5,j}, s_{6,j})$. For the graph thus obtained it is easily verified that $F_1 = 7p$ and according to the new definition of E_n, $F_2 \geqq E_2(c) = 36P$ [there exists an open path from c to $s_{1,1}$] $\geqq 36(p^2 + p^3 - p^5)$ which, if $p = \frac{1}{2}$, appears to be larger than $49p^2$.

References

Broadbent, S. R. and Hammersley, J. M. (1957) Percolation processes. *Proc. Camb. Phil. Soc.* **53**, 629–641; 642–645.

Hammersley, J. M. (1957) Percolation processes: lower bounds for the critical probability. *Ann. Math. Statist.* **28**, 790–795.

Kesten, H. (1982) *Percolation Theory for Mathematicians*. Birkhäuser, Boston.

(To appear in Journal of Applied Probability)

Inequalities with applications to percolation and reliability

J. van den Berg and H. Kesten

Abstract

A probability measure μ on $\mathbb{R}_+^n$ is defined to be Strongly New Better than Used (SNBU) if $\mu(A+B) \leq \mu(A)\,\mu(B)$ for all increasing subsets $A, B \subset \mathbb{R}_+^n$. For $n = 1$ this is equivalent to being New Better than Used (NBU distributions play an important role in reliability theory). We derive an inequality concerning products of NBU probability measures, which has as a consequence that if $\mu_1, \mu_2, \ldots, \mu_n$ are NBU probability measures on $\mathbb{R}_+$, then the product-measure $\mu = \mu_1 \times \mu_2 \times \ldots \times \mu_n$ on $\mathbb{R}_+^n$ is SNBU. A discrete analog (i.e., with $\mathbb{N}$ instead of $\mathbb{R}_+$) also holds.

Applications are given to reliability and percolation. The latter are based on a new inequality for Bernoulli sequences, going in the opposite direction of the FKG-Harris inequality. The main

application (3.15) gives a lower bound for the tail of the cluster size distribution for bond-percolation at the critical probability. Further applications are simplified proofs of some known results in peroclation. A more general inequality (which contains the above as well as the FKG-Harris inequality) is conjectured and connections with a recent inequality of Campanino and Russo are indicated.(+)

(+) It appears that Campanino's and Russo's inequality has been used before by Hammersley and McDiarmid. See also discussion in Ch. 2 of this thesis.

1. Definitions and main results

Because our main theorem holds for $\mathbb{R}_+ = [0,\infty)$ as well as $\mathbb{N} = \{0,1,2,\dots\}$ we shall use the same symbol R to denote either one of these sets.

If $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$, then $x \geq y$ means $x_i \geq y_i$, $i=1,\dots,n$. A function f on R^n is called increasing if $x \geq y$ implies $f(x) \geq f(y)$. A subset A of R^n is called increasing if its indicator function (denoted by I_A) is increasing. If A and B are two subsets of R^n, then $A + B \equiv \{a + b | a \in A, b \in B\}$. It follows from Dellacherie and Meyer [7], Theorem III.18 and Sect. III. 33a that $A + B$ is universally measurable when A, B are Borel sets of $\mathbb{R}^n_+$. In particular $A + B$ belongs to the completion of the Borel σ-field of $\mathbb{R}^n_+$ with respect to each probability measure.

A probability measure μ on R^n is Strongly New Better than Used (SNBU) if

(1.1) $$\mu(A + B) \leq \mu(A)\, \mu(B),$$

for all increasing Borel sets $A, B \subset R^n$.

For $n = 1$ and $R = \mathbb{R}_+$ this is equivalent to the usual definition of a New Better than Used (NBU) distribution. Therefore, in the one-dimensional case, we will say NBU instead of SNBU (see also section 2).

Let $n \geq 2$. For an increasing set $A \subset R^n$ and $i,j \leq n$, $i \neq j$, we define the image of A under (i,j)-identification as the set of all $x \in R^n$ for which there exists an $a \in A$ such that $x_i \geq a_i + a_j$ and $x_k \geq a_k$, $k \neq i,j$.

This definition is illustrated by the following example: Suppose someone receives a certain amount n_a of apples, n_p of pears and n_c of citrons. He is satisfied if, for a certain increasing set $A \subset \mathbb{N}^3$, $(n_a,n_p,n_c) \in A$. However, if he changes his mind, and wants each pear to be replaced by an apple, then he is satisfied if $(n_a,n_p,n_c) \in A^*$ where A^* is the above defined image of A under (1,2)-identification.

The above definition has the following natural extension. Let A be an increasing subset of R^n and let $\mathfrak{F}$ be a partition of the set $\{1,2,\ldots,n\}$. Choose for each class $F \in \mathfrak{F}$ a representative $i_F \in F$. Now <u>the image of A under identification according to the pair</u> $(\mathfrak{F}, \{i_F : F \in \mathfrak{F}\})$ is defined as the set of all $x \in R^n$ for which there exists an $a \in A$ such that for each class F: $x_{i_F} \geq \sum_{j \in F} a_j$. Again Theorem III.18 and Sect. III.33a of Dellacherie and Meyer [7] show that for a Borel set A of $\mathbb{R}_+^n$ its image under identification belongs to the completion of the Borel sets with respect to any probability measure.

(1.3) <u>Lemma</u>. Let $\mu_1,\ldots,\mu_n$ <u>be NBU probability measures on</u> R <u>and let</u> $i, j \leq n$, $i \neq j$ <u>be such that</u> $\mu_i = \mu_j$. <u>Then for all increasing Borel sets</u> $A \subset R^n$

(1.4) $$\mu(A) \geq \mu(A^*),$$

<u>where</u> A^* <u>denotes the image of</u> A <u>under</u> (i,j)-<u>identification</u>, <u>and</u> μ <u>is the product-measure</u> $\mu_1 \times \mu_2 \times \ldots \times \mu_n$ on R^n.

<u>Proof:</u> Without loss of generality we may assume $i = 1$, $j = 2$. In terms of random variables (1.4) is equivalent to saying that if $X_1, X_2, \ldots, X_n$ are independent random variables whose distribution on R is NBU, and X_1 and X_2 are identically distributed, then

(1.5) $$P[(X_1,X_2,\ldots,X_n) \in A] \geq P[(X_1,X_3,X_4,\ldots,X_n) \in A'],$$

where $A' = \{(x_1+x_2,x_3,x_4,\ldots,x_n) : (x_1,x_2,\ldots,x_n) \in A\} \subset R^{n-1}$. This inequality can now be proved as follows: Given $X_3 = x_3$, $X_4 = x_4,\ldots,X_n = x_n$, the conditional probability of the event in the left hand side of (1.5) is, for each x_1,x_2 with $(x_1,x_2,x_3,\ldots,x_n) \in A$, larger than or equal to $P[X_1 \geq x_1, X_2 \geq x_2]$. Since X_1 and X_2 are i.i.d. this probability equals $P[X_1 \geq x_1]\,P[X_1 \geq x_2]$. Hence the above mentioned conditional probability is at least

$$\sup\,\{P[X_1 \geq x_1]\,P[X_1 \geq x_2] : (x_1,x_2,x_3,\ldots,x_n) \in A\}$$

On the other hand, the conditional probability of the event in the right hand side of (1.5) is exactly $P[X_1 \in \{x_1+x_2 : (x_1,x_2,x_3,\ldots,x_n) \in A\}]$ which, because X_1 is a one-dimensional random variable, equals $\sup\,\{P[X_1 \geq x_1+x_2] : (x_1,x_2,x_3,\ldots,x_n) \in A]\}$ and this is, by the NBU property, at most $\sup\,\{P[X_1 \geq x_1]\,P[X_1 \geq x_2] : (x_1,x_2,x_3,\ldots,x_n) \in A\}$. □

(1.6) <u>Theorem</u>

(i) <u>Let</u> $\mu_1,\mu_2,\ldots,\mu_n$ <u>be NBU probability measures on</u> R <u>and let</u> $\mathfrak{F}$ <u>be a partition of the index set</u> $\{1,\ldots,n\}$, <u>with the</u>

property that μ_i's with indices in the same class are identical. Further, choose for each class $F \in \mathfrak{F}$ a representative $i_F \in F$ and let, for an increasing Borel set $A \subset R^n$, A^* denote the image of A under identification according to $(\mathfrak{F}, \{i_F : F \in \mathfrak{F}\})$. Then:

(1.7) $$\mu(A^*) \leq \mu(A),$$

where μ is the product-measure $\mu_1 \times \mu_2 \times \ldots \times \mu_n$ on R^n.

(ii) Let $\nu_1, \nu_2, \ldots, \nu_n$ be NBU probability measures on R. Denote by ν the product-measure $\nu_1 \times \nu_2 \times \ldots \times \nu_n$ on R^n, and let $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_k$ be increasing Borel sets of R^m. (Hence, $\bigcup_{1 \leq i \leq k} (A_i \times B_i)$ is a subset of R^{2m} and $\nu \times \nu$ is a probability measure on R^{2m}).

Then

(1.8) $$\nu\Big(\bigcup_{1 \leq i \leq k} (A_i + B_i)\Big) \leq (\nu \times \nu)\Big(\bigcup_{1 \leq i \leq k} (A_i \times B_i)\Big).$$

(iii) Let $\mu_1, \mu_2, \ldots, \mu_n$ be NBU probability measures on R and $\mu = \mu_1 \times \mu_2 \times \ldots \times \mu_n$ the product measure. Then, for all increasing Borel sets $A, B \subset R^n$,

(1.9) $$\mu(A + B) \leq \mu(A)\ \mu(B),$$

i.e., μ is SNBU.

Proof. (i) follows by applying lemma 1.3 successively to all pairs (i,j) with, for some class $F \in \mathfrak{F}$, $i = i_F$ and $j \in F$, $j \neq i$.

(ii) If we take, in (i), $n = 2m$, $\mu_1 = \nu_1, \mu_2 = \nu_2, \ldots, \mu_m = \nu_m, \mu_{m+1} = \nu_1, \mu_{m+2} = \nu_2, \ldots, \mu_{2m} = \nu_m$ (hence $\mu = \nu \times \nu$). $\mathfrak{F}$ the partition with classes $\{1, m+1\}$, $\{2, m+2\}, \ldots, \{m, 2m\}$, and set of representatives $\{1, 2, \ldots, m\}$ and $A = \bigcup_{1 \leq i \leq k} (A_i \times B_i)$, then according to (1.7) we get

$$(\nu \times \nu)(A^*) \leq (\nu \times \nu)(A).$$

This reduces to (1.8) because, as is easily seen,

$A^* = \Big(\bigcup_{1 \leq i \leq k} (A_i + B_i)\Big) \times R^m$, so that $(\nu \times \nu)(A^*) = \nu\Big(\bigcup_{1 \leq i \leq k} (A_i + B_i$

(iii) follows immediately from (ii) by taking $k = 1$. □

(1.10) Remarks.

(a) Originally we had a different proof, of part (iii) of the above theorem only. However, we noticed that the special case of (iii) with all μ_i concentrated on $\{0,1\}$ can also be derived from Theorem 2.1 of Campanino and Russo [6], (which is more general than that special case of (iii)). Campanino and Russo's formulation led us to the more general result (i) (from which Theorem 2.1 of Campanino and Russo can again be retrieved; see also Remark 3.5(b)).

(b) We have also proved that if μ is an SNBU probability measure on R^n and ν is an NBU probability measure on R, then the product measure $\mu \times \nu$ on R^{n+1} is SNBU (the proof of this involves some more technicalities than that of (iii)).

However, the following problem, which arises naturally in the context of the above results is still unsolved:

(1.11) Problem. Let μ and ν be SNBU probability measures on R^n and R^m respectively. Is the product measure $\mu \times \nu$ on R^{n+m} always SNBU?

We note that the following variant is not hard to prove. Call a measure μ on R^n ENBU (extended NBU) if

$$(1.12) \qquad \mu\left(\bigcup_i (A_i+B_i)\right) \le (\mu\times\mu)\left(\bigcup_i (A_i \times B_i)\right)$$

for any family of increasing sets A_i and B_i. It holds in general that if μ and ν are ENBU on R^n and R^m, respectively, then $\mu \times \nu$ is ENBU on R^{n+m}.

One final comment. If there is only one pair A_1, B_1 in (1.12), then (1.12) reduces to (1.1). Thus ENBU is stronger than SNBU. On the other hand (1.8) shows that any product of one-dimensional NBU measures is actually ENBU.(+)

(+) Just before this thesis was finished, we saw that the reverse also holds: each ENBU measure is a product of one - dimensional NBU measures.

2. Applications to reliability.

In reliability theory (for a description of the subject see e.g., Barlow and Proschan [3]) a non-negative random variable T is called NBU if its corresponding probability measure on $\mathbb{R}_+$ is NBU which means that (see section 1), for all $t_1, t_2 \ge 0$,

$$(2.1) \qquad P[T > t_1 + t_2 | T > t_1] \le P[T > t_2],$$

or equivalently,

$$(2.2) \qquad P[T > t_1 + t_2] \le P[T > t_1]\, P[T > t_2] .$$

Marshall and Shaked [14] introduced a multivariate extension of (2.2) by defining a random vector $T = (T_1,\ldots,T_n)$ to be Multivariate New Better than Used (MNBU) if, for all increasing Borel sets $A \subset \mathbb{R}_+^n$ and all $\lambda,\mu \ge 0$,

$$(2.3) \qquad P[T \in (\lambda+\mu)A] \le P[T \in \lambda A]\, P[T \in \mu A],$$

where $\lambda A \equiv \{\lambda a : a \in A\}$. The main result in their paper was that if S and T are MNBU and if T and S are independent, then (S,T) is also MNBU (compare with problem 1.11). This yielded the following

Corollary: If $T_1,\ldots,T_n$ are independent NBU random variables, then

(i) $T = (T_1,\ldots,T_n)$ is MNBU,

(ii) $g(T_1,\ldots,T_n)$ is NBU, whenever g is a non-negative measurable subhomogeneous increasing function.

(a function g on $\mathbb{R}^n_+$ is called <u>subhomogeneous</u> if $g(\alpha x) \leq \alpha g(x)$ *for all* $x \in \mathbb{R}^n_+$ *and all* $\alpha \geq 1$). This corollary is improved by the following corollary of theorem 1.6(iii).

(2.4) <u>Corollary</u>. If $T_1,\dots,T_n$ are independent one-dimensional NBU random variables, then

(a) *For all increasing* Borel sets $A, B \subset \mathbb{R}^n_+$

$$P[(T_1,\dots,T_n) \in A+B] \leq P[(T_1,\dots,T_n) \in A]\ P[(T_1,\dots,T_n) \in B]$$

(b) $g(T_1,\dots,T_n)$ <u>is NBU whenever</u> $g:\mathbb{R}^n_+ \to \mathbb{R}_+$ <u>is a measurable increasing function with the property</u>

(2.5) $$g^{-1}(a+b,\infty) \subset g^{-1}(a,\infty) + g^{-1}(b,\infty),\ \forall a,\ b > 0,$$

<u>where</u> $g^{-1} A \equiv \{x | g(x) \in A\}$.

<u>Proof</u>: (a) follows immediately from theorem 1.6(iii).

(b) *Suppose* $T_1,\dots,T_n$ and g fulfill the conditions. Then:

$$P[g(T_1,\dots,T_n) > s+t] = P[(T_1,\dots,T_n) \in g^{-1}(s+t,\infty)]$$
$$\leq P[(T_1,\dots,T_n) \in g^{-1}(s,\infty) + g^{-1}(t,\infty)]$$
$$\leq P[(T_1,\dots,T_n) \in g^{-1}(s,\infty)]\ P[(T_1,\dots,T_n) \in g^{-1}(t,\infty)]$$
$$= P[g(T_1,\dots,T_n) > s]\ P[g(T_1,\dots,T_n) > t].$$ □

(2.6) <u>Remarks</u>. (a) *implies (i) because* $(\lambda+\mu)A \subset \lambda A + \mu A$.
(b) implies (ii) because each increasing non-negative subhomogeneous function has the property (2.5), which can be seen as follows:

Let $g:\mathbb{R}^n_+ \to \mathbb{R}_+$ be increasing and subhomogeneous and let, *for certain* $a, b > 0$, $x \in g^{-1}(a+b,\infty)$, i.e., $g(x) > a+b$. *Then* $g((a+b)^{-1} ax) \geq (a+b)^{-1} ag(x) > a$ and, analogously, $g((a+b)^{-1} bx) > b$. Hence $x = (a+b)^{-1} ax + (a+b)^{-1} bx$ is the sum of an element of $g^{-1}(a,\infty)$ and an element of $g^{-1}(b,\infty)$.

(b) In studies of NBU random variables these variables usually represent life lengths. However, the following interpretation of corollary 3.1(a), in which the variable represent amounts of certain products, might also be interesting: Suppose two people, say A and B, have to share the random output of a certain producer. *A wants at least an amount* a, *B at least* an amount b. If the output has an NBU distribution, then, by the definition of NBU, the following statement holds: the probability that the output can be shared such that A and B are both satisfied is not larger than the product of the probability that A would be satisfied if he had the total output for himself and the analogous probability for B. Now consider the case of n producers with independent random outputs, each having an NBU distribution. If A(B) wants at least an amount $a_1(b_1)$ of the first product, $a_2(b_2)$ of the second product etc. then by the independence of the variables, it is obvious that the probability that A and B are both satisfied is still no larger than the product of the probability that A respectively B, are satisfied. However, corollary 2.4(a) says that the statement still holds in the case that A and B are, within certain limits, willing to obtain somewhat less of one product in exchange for somewhat more of some of the other products.

3. Applications to Bernoulli sequences and percolation.

Let $\Omega = \{0,1\}^n$. An event in Ω is called increasing or positive if its indicator function is an increasing function on Ω (i.e., increasing in each coordinate separately). An event is called decreasing or negative if its complement is increasing.

If A and B are positive events we denote by $A \circ B$ the event that A and B "occur disjointly". More precisely, $A \circ B$ is defined as follows: Each $\omega = (\omega_1, \ldots, \omega_n) \in \Omega$ is uniquely determined by the set $K(\omega) \subset \{1, \ldots, n\}$ of all indices i for which $\omega_i = 1$. Now $\omega \in A \circ B$ if and only if there exists a $K' \subset K(\omega)$ such that ω', determined by $K(\omega') = K'$, belongs to A, and ω'', determined by $K(\omega'') = K(\omega) \setminus K'$ belongs to B.

Example. If A is the event {at least k_1 of the ω_i's are equal to 1} and B the event {at least k_2 of the ω_i's are equal to 1}, then $A \circ B$ is the event {at least $k_1 + k_2$ of the ω_i's are equal to 1}. (see below for further examples).

It is clear that $A \circ B$ is contained in $A \cap B$. Further, notice that $A \circ B = B \circ A$ and $A \circ (B \circ C) = (A \circ B) \circ C$.

Now let P be the probability measure on Ω under which $\omega_1, \ldots, \omega_n$ are independent and $P[\omega_i = 1] = 1 - P[\omega_i = 0]$. Harris [11] proved that

$$(3.1) \qquad P[A \cap B] \geq P[A]\, P[B],$$

if A and B are both positive events,

or, equivalently,

$$(3.2) \qquad P[A \cap B] \leq P[A]\, P[B],$$

if A is positive and B is negative.

This inequality, which is one of the basic tools in percolation theory, is now usually considered as a special case of the FKG inequality first proven in [9]. We now show that the inequality (3.1) is reversed if $A \cap B$ is replaced by $A \circ B$. This new inequality turns out to be a special case of Theorem 1.6 (iii).

(3.3) Theorem. If A and B are positive events, then

$$(3.4) \qquad P[A \circ B] \leq P[A]\, P[B].$$

Proof: In order to use Theorem 1.6 (iii) we imbed the state space Ω in $\mathbb{N}^n = \{0, 1, \ldots\}^n$. We still use P to denote the image measure under this imbedding. Thus $P[\mathbb{N}^n \setminus \Omega] = 0$ and $P[\{x\}]$ is unchanged if $x \in \Omega$. Further, we replace each positive event $A \subset \Omega$ by the smallest increasing subset $\tilde{A}$ of $\mathbb{N}^n$ containing A. Thus A is replaced by

$$\tilde{A} = \{y \in \mathbb{N}^n : \exists\, x \in A \text{ such that } x \leq y\}.$$

This operation does not change the probability of A because only a set of probability zero is added. One now easily sees that $A \circ B$ differs from $\tilde{A} + \tilde{B}$ by a set of probability zero. In fact $z = (z_1, \ldots, z_n) \in \tilde{A} + \tilde{B}$ can have positive mass only if each z_i equals 0 or 1. Thus, if $z = x + y$,

$x \in \tilde{A}$, $y \in \tilde{B}$, then one must actually have $x \in A$, $y \in B$ and the ones among the coordinates of x and y cannot occur at the same place (since $x_i = y_i = 1$ implies $z_i = 2$). Finally, noting that a probability measure on $\mathbb{N}$ with all mass concentrated on $\{0,1\}$ is always NBU, the theorem follows directly from Theorem 1.6 (iii).

(3.5) Remarks.

(a) Analagously, a special case of theorem 1.6 (ii) is that for positive events $A_1, B_1, A_2, B_2, \ldots, A_k, B_k \subset \Omega$,

$$(3.6) \quad P[A_1 \circ B_1 \cup A_2 \circ B_2 \cup \ldots \cup A_k \circ B_k] \leq$$
$$\leq (P \times P)\,[A_1 \times B_1 \cup A_2 \times B_2 \cup \ldots \cup A_k \times B_k].$$

Roughly speaking, this means that the probability that, for at least one i, A_i and B_i occur disjointly, is smaller than the probability that, for at least one i, A_i and B_i occur on independent copies of the probability space.

(b) In the same way the following result of Campanino and Russo [6] can be derived as a special case of theorem 1.6(i): Let $\mathfrak{F}$ be a partition of $\{1, \ldots, n\}$ (Campanino and Russo state the result also for the countable case, i.e., with $\Omega = \{0,1\}^{\mathbb{N}}$, but that extension is rather straightforward) and let $\mathcal{C}$ be a family of subsets of $\{1, \ldots, n\}$ such that for each $C \in \mathcal{C}$ and $F \in \mathfrak{F}$ $C \cap F$ contains at most one element. Consider, for a given $p \in [0,1]$, two probability measures P_p and $P_{p,\mathfrak{F}}$ on Ω under both of which each ω_i is equal to 1 with probability p and equal to 0 with probability 1-p $(i=1, \ldots, n)$. Under P_p the

ω_i, $i=1, \ldots, n$ are independent. Under $P_{p,\mathfrak{F}}$, all ω_i's with indices in the same class are equal with probability 1, while the families $V_F := \{\omega_i : i \in F\}$, $F \in \mathfrak{F}$, are independent. Now let A be the event that, for at least one $C \in \mathcal{C}$, $\omega_i = 1$ for all $i \in C$. Then

$$(3.7) \qquad P_{p,\mathfrak{F}}[A] \leq P_p[A].$$

In order to show that this follows from Theorem 1.6(i) imbed Ω again in $\mathbb{N}^n$, and replace A by $\tilde{A}$, exactly as in the proof of Theorem 3.3. Denote the image of P_p under the imbedding of Ω in $\mathbb{N}^n$ by $\tilde{P}_p$. Choose a representative i_F for each class $F \in \mathfrak{F}$, and form $(\tilde{A})^*$ from $\tilde{A}$ by identification according to $(\mathfrak{F}, \{i_F\})$. One can verify that

$$\tilde{P}_p\,[(\tilde{A})^*] = P_p\,[(\bigcup_{C \in \mathcal{C}} \{x : x_i = 1 \text{ for each } i \in C\})^*]$$

$$= \tilde{P}_p\,[\bigcup_{C \in \mathcal{C}} \{x : x_{i_F} = 1 \text{ for each } F \text{ with } F \cap C \neq \emptyset\}]$$

$$= P_{p,\mathfrak{F}}[\bigcup_{C \in \mathcal{C}} \{x : x_i = 1 \text{ for each } i \in C\}] = P_{p,\mathfrak{F}}[A]$$

(In the second equality we use the fact that $C \cap F$ is either empty or consists of a single element only.) Thus by (1.7)

$$P_{p,\mathfrak{F}}\,[A] = \tilde{P}_p[(\tilde{A})^*] \leq \tilde{P}_p[\tilde{A}] = P_p[A],$$

which is just (3.7).

Conversely, it is possible to derive (3.4) and (3.6) from (3.7) by applying (3.7) in the space Ω^2 with suitable choices of $\mathfrak{J}$ and $\mathfrak{C}$.

(c) Ahlswede and Daykin [1] have presented a rather general theory of correlation inequalities including the FKG -inequality. However, it seems that (3.4) does not fit in this framework and it might be the first step in a new direction (see also (d)).

(d) The operation "o" has been defined for positive events only. However, define for arbitrary events A and $B \subset \Omega$ the event $A \square B$ as follows: First, for $\omega = (\omega_1, \ldots, \omega_n) \in \Omega$ and $K \subset \{1, \ldots, n\}$, let $C(K,\omega)$ denote the cyliner event $\{\omega' : \omega' \in \Omega$ and $\omega_i' = \omega_i$ for all $i \in K\}$. Let $\overline{K}$ denote $\{1, \ldots, n\} \setminus K$. Now define

(3.8) $$A \square B = \{\omega : \exists\, K \subset \{1, \ldots, n\} \text{ such that } C(K,\omega) \subset A \text{ and } C(\overline{K},\omega) \subset B\}.$$

Clearly $A \square B \subset A \cap B$. We have the following conjecture

(3.9) $$P[A \square B] \leq P[A]\, P[B] \text{ for all events } A \text{ and } B.$$

It is easily seen that if A is positive and B negative, $A \square B$ is exactly $A \cap B$, and if A and B are both positive it equals $A \circ B$, so that (3.9) includes the FKG-Harris inequality as well as our inequality (3.9). Moreover, if the answer to problem (1.11) is affirmative for the case that μ or ν is a probability measure on $\mathbb{N}^2$, concentrated on

the elements (1,0) and (0,1), then (3.9) follows in a way comparable with the derivation of Theorem 3.3 from Theorem 1.6(iii).

Examples and applications in percolation theory.

Let $\mathcal{G}$ be a finite or countably infinite graph. A path from site s to site s' is a finite sequence of the form $(s_1 = s, e_1, s_2, e_2, \ldots, s_{n-1}, s_n = s')$, where each e_i is an edge connecting the sites s_i and s_{i+1}. There is no loss of generality for our purposes if we restrict outselves to paths which are self-avoiding (which means that all s_i's in the above sequence are different). The length of a path is the number of edges it contains. Now suppose that the edge is open (or passable) with probability p_e and closed with probability $1 - p_e$, and that all these events for different edges are independent. A path or, more generally, a subgraph, is said to be open if all its edges are open. An open cluster is a maximal connected open subgraph of $\mathcal{G}$. Percolation theory (introduced by Broadbent and Hammersley [5]) studies questions like: what is the probability of the existence of an open path between two specified sites, and (in the case where $\mathcal{G}$ is infinite) do there exist, with positive probability, infinite open clusters? The above case is called bond-percolation. If, instead of the edges, the sites of $\mathcal{G}$ are randomly open or closed, one speaks of site-percolation. For a recent introduction to these problems see, e.g., [12], Ch. 1 . Also models have been studied in which the edges are only passable in one direction (see e.g. [8]).

The following special case of theorem 3.3 is useful in percolation theory (see also (3.12) below).

(3.10) Corollary. Let, for some $k \geq 2$, $V_1, V_2, \ldots, V_k$ be sets of paths of a graph $\mathcal{G}$. Assume that all the edges (sites) of $\mathcal{G}$ are independently open or closed. Call two paths disjoint if they have no edge (site) in common. Let E_i, $i = 1, \ldots, k$, be the event that at least one of the paths in V_i is open. Then:

(3.11) P [There exist pair wise disjoint open paths $\pi_1 \varepsilon V_1, \pi_2 \varepsilon V_2, \ldots, \pi_k \varepsilon V_k] \leq P[E_1]\, P[E_2] \ldots P[E_k]$.

Proof: We may restrict ourselves to the case where $\mathcal{G}$ is finite (by obvious limit arguments). Now if we take $\Omega = \{0,1\}^E$, where E is the set of edges of $\mathcal{G}$ ($\Omega = \{0,1\}^S$, where S is the set of sites of $\mathcal{G}$) and take $\omega_e = 1$ or 0 ($\omega_s = 1$ or 0) according as the edge e (site s) is open or closed, then it is not difficult to see that the event in the left-hand side of (3.11) corresponds with $E_1 \circ E_2 \circ \ldots \circ E_k$ and the result follows by repeated application of theorem 3.3. □

(3.12) Remark. By using (3.6) or (3.7) one can also derive a similar result in first-passage percolation (see [13], Sect. 4).

The following result is a simple proof of the first "tree graph bound" of Aizenman and Newman ([2], Prop. 4.1). Their bounds for higher connectivity functions can be derived in the same way. Let $t(v,w) = P$ [v is connected to w by an open path].

(3.13) Corollary. Consider bond-percolation on a graph $\mathcal{G}$. Let s_1, s_2 and s_3 be sites of $\mathcal{G}$. Then

(3.14) $P[s_1, s_2$ and s_3 belong to the same open cluster$]$

$$\leq \sum_{s \text{ a site of } \mathcal{G}} t(s_1,s)\, t(s_2,s)\, t(s_3,s)$$

Proof. The result follows by using Corollary 3.10 and the observation that s_1, s_2 and s_3 belong to the same open cluster if and only if there exists a site s (which may be equal to one of the s_i's) such that there are disjoint open paths from s_1 to s, from s_2 to s and from s_3 to s, respectively. □

The nicest application is an improvement of a result for critical percolation in two dimensions. As an example we consider bond percolation on the square lattice, which is the graph with sites $\{(n,m) \mid n,m \in \mathbb{Z}\}$. (It is easy to derive analogous results for other two-dimensional lattices). On this graph each site (n,m) has exactly four edges incident to it, namely those between (n,m) and the sites $(n \pm 1, m \pm 1)$. *Suppose all edges are independently open with probability* p *and denote the corresponding probability measure by* P_p. Let B_n be the event that there exists an open path from the origin to some site at distance $\geq n$ from the origin. (The distance from (n_1, n_2) to (m_1, m_2) is defined as $|n_1 - m_1| + |n_2 - m_2|$). Clearly $P_p[B_n]$ is decreasing in n. It is known

([12], p. 54 and Theorem 5.1) that for $p < \frac{1}{2}$ there exists a $\lambda(P) < 1$ such that $P_p[B_n] < \lambda^n(p)$, while for $p > \frac{1}{2}$ $\lim_{n\to\infty} P_p[B_n] > 0$. When p is equal to the critical probability $\frac{1}{2}$ then $P_p[B_n]$ tends to 0, but not exponentially. Smythe and Wierman ([16] p. 61) gave an easy proof of $P_{\frac{1}{2}}[B_n] \geq \frac{1}{2n}$. Later Kesten ([12] Theorem 8.2) showed that there exist $C, \gamma > 0$ such that $P_{\frac{1}{2}}[B_n] > C\, n^{-1+\gamma}$. However, the value of γ which follows from his calculations appears to be very small. It is believed that $P_{\frac{1}{2}}[B_n] \sim Cn^{-\delta}$ for some $C > 0$, $0 < \delta < 1$ (see [17]). Even though we cannot prove such a power law, the following result greatly improves the estimates for γ obtainable from [12]. The proof uses a refinement of Smythe and Wierman's idea and Cor. 3.10.

(Another proof can be based on the (known) inequality (3.17).)

(3.15) Corollary.

$$P_{\frac{1}{2}}[B_n] \geq \frac{1}{2\sqrt{n}} .$$

Proof: *Consider the subgraph* $S(n)$ *of* S *which consists of* the part of S situated in the rectangle $0 \leq x \leq 2n$, $0 \leq y \leq 2n-1$. It is well-known from duality arguments (see [15], or [16], p. 31) that the $P_{\frac{1}{2}}$- probability that there exists an open path which lies in $S(n)$ and which connects the left-hand edge of $S(n)$ with its right-hand edge equals $\frac{1}{2}$. Further it is clear that such a path passes through at least one of the sites $\{n\} \times [0, 2n-1]$. Hence at least one of the $2n$ sites in the above set has two disjoint open connections

with the left-and right-hand edge of $S(n)$, respectively. Also, the distance between a site in $\{n\} \times [0, 2n-1]$ and a site in the left- or right edge of $S(n)$ is always $\geq n$. Consequently, by Cor. 3.10

$$\frac{1}{2} \leq \sum_{i=0}^{2n-1} P_{\frac{1}{2}}\, [(n,i) \text{ is connected by two disjoint open paths to the left and right edge of } S(n)] \leq 2n\{P_{\frac{1}{2}}[B_n]\}^2. \qquad \square$$

Lastly we give a new and simplified proof of a result of Hammersley [10]. *First consider* bond-percolation on a graph $\mathcal{G}$. By the distance between two sites of $\mathcal{G}$ we mean the minimal number of edges in any path which connects these sites. For any site s of $\mathcal{G}$ define

$N_n(s)$ = collection of sites at distance $\leq n$ from s,

$B_n(s)$ = collection of sites at distance exactly n from s

$P_n(s) = P$ [$\exists$ open path from s to a site in $B_n(s)$] if $n \geq 1$, and $P_0(s) = 1$.

We say that a path belongs to $N_n(s)$ if all sites of the path, *except for its endpoint*, lie in $N_n(s)$, and we define, for $n \geq 1$,

$E_n(s)$ = expected number of sites $s' \in B_n(s)$ for which there exists an open path from s to s' belonging to $N_{n-1}(s)$.

We take $E_o(s) = 1$. Finally, for $n \geq 0$ we set

(3.16) $$P_n = \sup_s P_n(s), \quad E_n = \sup_s E_n(s).$$

Hammersley [10] has proved that

(3.17) $$P_n \leq (E_m)^{\lfloor n/m \rfloor}$$

Where $\lfloor n/m \rfloor$ is the integer part of n/m. A direct consequence of this result is that if the expected size of the open cluster is finite, then the radius of the open cluster has a distribution with an exponentially bounded tail (see also [12], Sect. 5.1 and [2], Sect. 5 for a stronger result). Here we give an easy proof of the following inequality which is somewhat stronger than (3.17) (since by induction (3.18) will imply $P_{nm} \leq (E_m)^n$).

(3.18) Corollary.

$$P_{n+m} \leq E_m P_n \quad , n, m \geq 0.$$

Proof: If n or m equals zero the result is trivial. Assume $n,m > 0$ and fix s. Suppose there exists an open path from s to $B^{n+m}(s)$. Denote by s' the first site on the path (starting from s) which lies in $B_m(s)$. Then, clearly, there exist two disjoint open paths, the first from s to s' and belonging to $N_{m-1}(s)$, and the second from s' to $B_{n+m}(s)$.

Furthermore it is clear that $B_{n+m}(s)$ has distance at least n from s', so that the second path passes through $B_n(s')$. Thus

$$P_{n+m}(s) \leq \sum_{s' \varepsilon B_m(s)} P\,[\exists \text{ two disjoint open paths, one from } s \text{ to } s' \text{ and belonging to } N_{m-1}(s)\text{, and the other from } s' \text{ to some site in } B_n(s')]$$

By Cor. 3.10 this expression is at most

$$\sum_{s' \varepsilon B_m(s)} P\,[\exists \text{ open path from } s \text{ to } s' \text{ which belongs to } N_{m-1}(s)]\, P_n(s') \leq E_m(s)\, P_n.$$

This holds for all s, so that (3.18) follows. □

If one considers site percolation then (3.18) remains valid (and the proof goes through practically unchanged) provided one redefines P_n and E_n as follows: $N_n(s)$, $B_n(s)$ and (3.16) remain as before, but

$P_n(s) = P\,[\exists$ open path from a neighbor of s to a site of $B_n(s)]$, $n \geq 1$, $P_0(s) = 1$,

$E_n(s)$ = expected number of sites $s' \varepsilon B_n(s)$ for which there exists an open path from a neighbor of s to s' and belonging to $N_{n-1}(s)$, $n \geq 1$, $E_0(s) = 1$.

Another application of Cor. 3.10 is to be found in van den Berg [4], where it is used to prove that for one-parameter bond-percolation on $\mathbb{Z}^2$ the site (0,0) always has at least as high a probability to be connected by an open path to (1,0) as to (2,0).

Acknowledgements. We wish to thank R. Ahlswede for many helpful discussions concerning problem 1.11 and conjecture 3.9. The first author's research was supported by the Netherlands Foundation for Mathematics SMC with financial aid from the Netherlands Organization for the Advancement of Pure Research (ZWO). The second author is grateful to the Technical University of Delft for its support and hospitality during his sabbatical leave in 1983, when this research was begun.

References

[1] R. Ahlswede and D.E. Daykin, Inequalities for a pair of maps $S \times S \to S$ with S a finite set, Math. Z. 165 (1979) 269-289.

[2] M. Aizenman and C.M. Newman, Tree graph inequalities and critical behavior in percolation models, to appear in J. Stat. Phys.

[3] R.E. Barlow and F. Proschan, Statistical theory of reliability and life testing, probability models. Holt, Rinehart and Winston, 1975.

[4] J. van den Berg (1984) to appear

[5] S.R. Broadbent and J.M. Hammersley, Percolation processes, Proc. Cambr. Phil. Soc. 53 (1957)) 629-641 and 642-645.

[6] M. Campanino and L. Russo, An upper bound on the critical probability for the three dimensional cubic lattice, to appear in Ann. Probab.

[7] C. Dellacherie and P.A. Meyer, Probabilities and potential (english translation), North-Holland Publ. Co., 1978.

[8] R. Durrett, Oriented percolation in two dimensions, Ann. Probab. 12 (1984).

[9] C.M. Fortuin, P.W. Kasteleyn and J. Ginibre, Correlation inequalities on some partially ordered sets, Comm. Math. Phys. 22 (1971) 89-103.

[10] J.M. Hammersley, Percolation processes. Lower bounds for the critical probability, Ann. Math. Statist. 28 (1957) 790-795.

[11] T.E. Harris, A lower bound for the critical probability in a certain percolation process, Proc. Cambr. Phil. Soc. 56 (1960) 13-20.

[12] H. Kesten, Percolation theory for mathematicians, Birkhäuser-Boston, 1982.

[13] H. Kesten, Aspects of first-passage percolation. Lectures at St. Flour summer school in probability, 1984.

[14] A.W. Marshall and M. Shaked, A class of multivariate new better than used distributions, Ann. Probab. 10 (1982) 259-264.

[15] P.W. Seymour and D.J.A. Welsh, Percolation probabilities on the square lattice, Ann Discrete Math. 3 (1978) 227-245.

[16] R.T. Smythe and J.C. Wierman, First-passage percolation on the square lattice, Lecture notes in mathematics, vol. 671, Springer-Verlag, 1978.

[17] D. Stauffer, Scaling theory for percolation clusters, Phys. Reports 54, no. 1 (1979) 1-74.

Dept. of Mathematics
Delft University of Technology
2628 B L Delft
The Netherlands

Dept. of Mathematics
Cornell University
Ithaca, NY 14853
U.S.A.

On a Combinatorial Conjecture Concerning Disjoint Occurrence of Events.

J. van den Berg and U. Fiebig

Abstract

Recently Van den Berg and Kesten have obtained a correlation-like inequality for Bernoulli sequenes. This inequality, which goes in the opposite direction of the FKG inequality, states that the probability that two monotone (i.e. increasing or decreasing) events "occur disjointly" is smaller than the product of the individual probabilities. They conjecture that the monotonicity condition is immaterial, i.e. that the inequality holds for all events.

In the present paper we try to make clear the intuitive meaning of the conjecture and prove some non-trivial special cases, one of which, a pure correlation inequality, is an extension of the FKG-Harris inequality.

1. Introduction.

In [2] a conjecture is stated which has the intuitive interpretation given by the following example.

(1.1) Example

Suppose two children make a list of their wishes for Christmas. The first child is satisfied if he gets at least one of the combinations in the following list:

1) a green teddy-bear and a blue car.
2) a red teddy-bear.
3) a blue car and a blue football.

The second child has the following list:

1') a blue teddy-bear and a blue car.
2') a red teddy-bear.
3') a red football.
4') a blue football.

Now suppose Santa Claus takes two boxes and puts in each of them a teddy-bear, a football and a car. However, he doesn't consider the colours and chooses the toys randomly from large sacks, each sack containing one type of toy in several colours. We assume that this happens in such a way that the six colours in the two boxes may be considered as independent random variables, and that the contents of the two boxes are stochastically identical (i.e. the colour of the football in the first box has the same distribution as that of the football in the second box etc.)
Consider the following two options:

a) Santa Claus gives only one box to the two children and they must try to share the contents of this box in such a way that both are satisfied, i.e. get at least one of the combinations on their respective lists. It is easy to check that this is only possible if the box contains at least one of the following compositions of combinations of the first and the second list:

1 x 3' : a green teddy-bear, a blue car and a red football.
1 x 4' : a green teddy-bear, a blue car and a blue football.
2 x 3' : a red teddy-bear and a red football.
2 x 4' : a red teddy-bear and a blue football.
3 x 2' : a blue car, a blue football and a red teddy-bear.

b) This option is as follows: Both children receive a box but they are not

allowed to exchange toys. In this case the box given to the first child must contain at least one of the combinations 1,2,3, and the box given to the second child must contain at least one of the combinations 1',2',3',4'. (Note that these events are independent).

When for each type of toy the probability distribution of its colours is known, one can calculate, for both options, the probability that both children are satisfied. The conjecture in [2] is equivalent to saying that this probability for the second option is larger than for the first option and that this holds for arbitrary numbers of different toys and possible colours, for all probability distributions of the colours and for any pair of lists of wishes.

The investigations which led to the conjecture in [2] were motivated by the following percolation problem:

(1.2) Example

Let each bond b of a locally finite graph G, independent of the other bonds be open with probability p_b and closed with probability $1-p_b$. A path from s to s' is a sequence $s=s_1,b_1,s_2,b_2,\dots , b_{n-1}, s_n=s'$, where $s_1,s_2\dots, s_n$ are sites of G and each b_i is a bond of G connecting s_i and s_{i+1}, $i=1,\dots .n-1$. An open path is a path of which all bonds are open. Two paths are disjoint if they have no bonds in common.

Let V_1,V_2,W_1 and W_2 be sets of sites of G. Further, let A be the event that there exists an open path from a site of V_1 to a site of V_2, and B the corresponding event with respect to W_1 and W_2. It follows from a result of Harris(1960) that A and B are positively correlated, i.e. $P(A\cap B)\geq P(A)P(B)$. (We come back to Harris' result in section 4.). Now the problem is whether, on the other hand, the probability that there exist two disjoint open paths of which one goes from a site of V_1 to a site of V_2 and the other from a site of W_1 to a site of W_2, is smaller than $P(A)P(B)$. In [2] it is shown that this is indeed the case. However, the following related problem is unsolved: Consider again the above graph G. This time the bonds are not randomly open or closed, but they have a random direction. More precisely, if b is a bond with endpoints s_1,s_2 then it has, independent of the other bonds, probability $p_b(s_1,s_2)$ to be directed from s_1 to s_2 and probability $p_b(s_2,s_1)=1-p_b(s_1,s_2)$ to be directed from s_2 to s_1. A directed path from s to s' is a sequence as shown above, with the additional property that each b_i is directed from s_i to s_{i+1}, $i=1,\ldots, n-1$.

The problem, analogous to the one for the open-closed case, is now whether the probability that there exist two disjoint directed paths of which one goes from a side of V_1 to a site of V_2 and the other from a site of W_1 to a site of W_2 is, again, smaller than the product of the individual probabilities.

These two problems (the solved open-closed problem and the unsolved random-direction problem) represent special cases of the conjecture.

In section 2 we give a formal description of the conjecture after introducing the necessary definitions and notation. We also present an attractive special case which, as shown in section 3, turns out to be equivalent to the full conjecture. In section 3 we also show some other equivalent forms of the conjecture, try to make clear the relation to the examples in section 1, introduce additional definitions and notation, and give some general results concerning the conjecture.

In section 4 we state our main result, theorem 4.2, which consists of four non-trivial, proved, special cases of the conjecture. The first is an extension of the special case proved in [2], and also contains Harris' inequality mentioned in example 2. In section 4 we further give some corollaries and examples.

The proofs of the four cases of theorem 4.2 are rather long and, except for the first two cases whose proofs are related, completely different. Therefore they are given in three different sections, section 5,6 and 7.

2. Formal statement of the conjecture.

Let $\Omega = S_1 \times S_2 \times \ldots \times S_n$ with $S_1, S_2, \ldots, S_n$ finite subsets of $\mathbb{N}$.

Realisations (i.e. elements of Ω) are denoted by $\omega = (\omega_1, \ldots, \omega_n)$. The support of an event (a subset of Ω) is defined as the set of all indices on which it depends. More precisely, if $A \subset \Omega$ then

(2.1) $\operatorname{supp}(A) := \{i \mid 1 \leq i \leq n, \exists \omega, \omega' \in \Omega \,\forall j \neq i\; \omega_j = \omega'_j;\; \omega \in A,\; \omega' \notin A\}$.

Two events A and B are said to be perpendicular to each other, denoted by $A \perp B$, if $\operatorname{supp}(A) \cap \operatorname{supp}(B) = \emptyset$. For $\omega \in \Omega$ and $K \subset \{1,\ldots,n\}$ we define the cylinder

(2.2) $[\omega]_K := \{\omega' \mid \omega' \in \Omega, \omega'_i = \omega_i \text{ for all } i \in K\}$.

(2.3) Remarks i) Though $[\omega]_K$ depends on Ω we omit this parameter.

ii) Note that this notation is not unique; ω may be replaced by each $\omega' \in [\omega]_K$.

If $A, B \subset \Omega$ we say that ω is a disjoint realisation of A and B if ω is an element of both A and B but "for disjoint reasons". Formally the set $A \,\square\, B$ of disjoint realisations of A and B is defined as:

(2.4)

$$A \,\square\, B := \{\omega \in \Omega \mid \exists K, L \subset \{1,\ldots,n\}\; K \cap L = \emptyset,\; [\omega]_K \subset A \text{ and } [\omega]_L \subset B\}.$$

(2.5) Remark. Note that we again omit the parameter Ω.

Our subject, the conjecture stated in [2], is the following:

(2.6) Conjecture

Let $n \in \mathbb{N}\setminus\{0\}$. Let S_i be a finite subset of $\mathbb{N}$ and μ_i a probability measure on S_i, $i=1,\ldots,n$. Further, define $\Omega = S_1 \times S_2 \times \ldots \times S_n$ and $\mu = \mu_1 \times \mu_2 \times \ldots \times \mu_n$. Then

(2.7) $\forall A, B \subset \Omega \;\; \mu(A \square B) \leq \mu(A)\mu(B)$.

The special case that, for each i, $S_i = \{0,1\}$ and $\mu_i(0) = \mu_i(1) = \frac{1}{2}$ gives:

(2.8) Conjecture

If $\Omega = \{0,1\}^n$ $(n \in \mathbb{N}\setminus\{0\})$, then

(2.9) $\forall A, B \subset \Omega, \;\; |A \square B|\; 2^n \leq |A|\,|B|$,

where $|\cdot|$ denotes cardinality.

It will be shown in section 3 that the above special case is equivalent to the full conjecture 2.6.

We finish this section with the following example:

(2.10) Example

Let $\Omega=\{0,1\}^4$, $A=\{(\omega_1,\omega_2,\omega_3,\omega_4)\in\Omega|$ at least two ω_i's are equal to $0\}$ and $B=\{(\omega_1,\omega_2,\omega_3,\omega_4)\in\Omega|$ at least one ω_i equals 0 and at least one ω_i equals $1\}$.

Then $A\Box B=\{(\omega_1,\omega_2,\omega_3,\omega_4)\in\Omega|$ exactly three of the ω_i's are equal to $0\}$.

Further $|A| = 11$, $|B| = 14$ and $|A\Box B| = 4$ which, multiplied by 2^4 is indeed smaller than $|A||B|$. (This example falls under case b of theorem 4.2.)

3. General results concerning the -operation.

Several results in this section, especially in the beginning, are almost trivial. However, they may help to get familar with the $\Box$-operation and make it possible to shorten the proofs of the more interesting results.

We start by stating some properties of the $\Box$-operation leading to equivalent definitions of $A\Box B$ and, subsequently, to equivalents of (2.7).

Next we show the connection with the examples in section 1. We also prove, as announced in section 2, that the special case (2.8) implies the full (2.6). We do this with the help of a more general principle which will be used throughout in the sections 5-7 and is therefore presented as a separate lemma (lemma 3.4). Finally, we prove another useful result (lemma 3.9) and introduce some additional notation and definitions. It is easily seen that the $\Box$-operation has the following properties and we omit the proof:

(3.1) Lemma

(i) $A\Box B \subset A \cap B$,

(ii) If $A\perp B$ then $A\Box B = A\cap B$.

(iii) $A\Box B = B\Box A$.

(iv) $(A_1 \cup A_2)\Box B \supset ((A_1\Box B) \cup (A_2\Box B))$.

Using these properties, several definitions of $A\Box B$, equivalent to the one in section 2, can be given. First we define the following: A set C is called a maximal cylinder of A if: $C\subset A$, C is a cylinder and there is no cylinder $C'\subset A$ with $C \subsetneq C'$.

(3.2) Lemma

(i) $A\Box B = \cup\{C\cap C' | C$ is a cylinder of A, C' is a cylinder of B and $C\perp C'\}$.

(ii) $A\Box B = \cup\{C\cap C' | C$ is a maximal cylinder of A, C' is a maximal cylinder of B, $C\perp C'\}$.

(iii) $A\Box B = \cup\{A'\cap B' | A'\subset A, B'\subset B$ and $A'\perp B'\}$.

Proof(i) Follows immediately from definition (2.4) and the definition of "$\perp$".

(ii) It is clear that the r.h.s. of (i) does not change if we restrict ourselves to maximal cylinders.

(iii) By (i) it is obvious that the l.h.s of (iii) is contained in the r.h.s. We prove the other direction as follows: By lemma 3.1(iii,iv), $A\Box B \supset \cup\{A'\Box B' | A'\subset A, B'\subset B\}$ which contains, of course, $\cup\{A'\Box B' | A'\subset A, B'\subset B, A'\perp B'\}$ which, by lemma 3.1(ii) is equal to the r.h.s. of 3.2(iii).

Using the above lemma we get several equivalents of conjecture 2.6:

(3.3) Lemma

The following statements (i,ii,iii,iv) are equivalent to (2.7):

(i) $\mu(\bigcup_{1\leq i\leq m} (C_i\cap C_i')) \leq \mu(\bigcup_{1\leq i\leq m} C_i)\, \mu(\bigcup_{1\leq i\leq m} C_i')$, where

$m\in\mathbb{N}\setminus\{0\}$, $C_i, C_i'\subset\Omega$ cylinders, $C_i\perp C_i'$, $i=1,\dots,m$.

(ii) $\mu(\bigcup_{1\leq i\leq m} (A_i\cap B_i)) \leq \mu(\bigcup_{1\leq i\leq m} A_i)\, \mu(\bigcup_{1\leq i\leq m} B_i)$, where:

$m\in\mathbb{N}\setminus\{0\}$; $A_i,B_i\subset\Omega$, $A_i\perp B_i$, $i=1,\dots,m$.

(iii) $\mu(\cup\{C_i\cap C_j' | i\in I,\ j\in I',\ C_i\perp C_j'\}) \leq \mu(\bigcup_{i\in I} C_i)\, \mu(\bigcup_{i\in I'} (C_i'))$, where

I, I' are finite index sets, and $C_i, i\in I$ and $C_j, j\in I'$ are cylinders in Ω.

(iv) $\mu(\cup\{A_i\cap B_j | i\in I,\ j\in I',\ A_i\perp B_j\}) \leq \mu(\bigcup_{i\in I} A_i)\, \mu(\bigcup_{i\in I'} B_i)$, where

I, I' are finite index sets, and $A_i, i\in I$ and $B_j, j\in I'$ are subsets of Ω.

Proof (2.7) <==>(i): (2.7.) implies (i) by taking $A = \bigcup_{1\leq i\leq m} C_i$,

$B = \bigcup_{1\leq i\leq m} C_i'$ and noting that, by 3.2.(i), the l.h.s. of (i) is contained in A□B. Conversely, (i) implies (2.7) by taking, for $(C_i,C_i')_{1\leq i\leq m}$ all possible pairs (C,C') with C a cylinder of A,C' a cylinder of B and C⊥C', and again using 3.2(i).

(2.7) <==> ii: As the above proof; this time use 3.2(iii) instead of 3.2.(i).

(2.7) <==> (iii): Analogous to the first case.

(2.7) <==> (iv) : Analogous to the second case.

We shall now briefly discuss the examples 1.1 *and* 1.2 *in the light of the* above definitions and results. As to example 1.1, let, if there are n different types of toys, $S_1,S_2,\ldots, S_n$ represent the sets of the possible colours, and take $\Omega = S_1 x S_2 x \ldots x S_n$.

The combinations 1,2,3,...,k on the first list and 1',2',3',..., ℓ' on the second list correspond with cylinders $C_1,C_2,C_3\ldots,C_k$ and $C_1',C_2',C_3'\ldots, C_\ell'$ respectively. Further, the set of compositions 1",2",3",... corresponds exactly with $\{C_i \cap C_j' \mid 1\leq i\leq k,\ 1\leq j\leq \ell,\ C_i \perp C_j'\}$ and now, noting lemma 3.3(iii), *it is clear that the example, in its general setting* (i.e. arbitrary number of different toys etc.) is indeed an interpretation of the conjecture.

As to example 1.2, assume that G is a finite graph. (Otherwise we can use obvious limit arguments.) Now let $\Omega=\{0,1\}^{|E|}$, where E is the set of bonds of G and take, for $\omega\in\Omega$, ω_i=1 or 0 according as the bond b_i is open or closed (or, in the random-direction case, according as the direction of b_i. Then *the events* {*there exists an open (directed) path from a site of* V_1 to a site of V_2} and {there exists an open (directed) path from a site of W_1 to a site of W_2} can be considered as sets A,B⊂Ω. It is easy to check that the event {there exist two disjoint open (directed) paths of which one goes from a site of V1 to a site of V2 and the other from a site of W1 to a site of W_2} corresponds with A□B, which clarifies the connection with the conjecture.

Lemma 3.3 yields rather trivial equivalents of conjecture 2.6. More interesting is the equivalence of this conjecture to conjecture (2.8). This equivalence will be proved by using the following lemma which is also useful in many other applications.

(3.4)Lemma Let, for $1\leq i\leq n$, $1\leq j\leq m$, S_i and T_j be finite subsets of ℕ, and μ_i and ν_j probability measures on S_i and T_j respectively. Further, let $\mu=\mu_1 x \mu_2 x \ldots x \mu_n$ and $\nu= \nu_1 \times \nu_2 \times \ldots \times \nu_n$ be the respective product measures on $\Omega(=S_1 x S_2 x \ldots x S_n)$ and $\Omega'(=T_1 x T_2 x \ldots x T_n)$. Finally, let A and B be subsets of Ω, and f: Ω'→Ω a map with the following properties: (i and(ii or ii')).

i) $\mu(\omega) = \nu(f^{-1}(\omega))$ for all $\omega\in\Omega$.

ii) If C_1 and C_2 are cylinders contained in Ω, and $C_1 \perp C_2$, then $f^{-1}(C_1) \perp f^{-1}(C_2)$.

ii') If C_1 and C_2 are maximal cylinders of A and B respectively and $C_1 \perp C_2$ then $f^{-1}(C_1) \perp f^{-1}(C_2)$.

Then (denoting $f^{-1}(A)$ by A' and $f^{-1}(B)$ by B'): $\mu(A\Box B)\leq\mu(A)\mu(B)$ if $\nu(A'\Box B') \leq \nu(A')\nu(B')$.

Remarks a) Note that we do not in(ii) and (ii') require that $f^{-1}(C_1)$ and $f^{-1}(C_2)$ are cylinders.

b) Note that (ii') is weaker than (ii) so that the latter is superfluous, since we require(i) and ((ii) or (ii')). However we also state (ii) because in many cases treated in this article, this stronger condition does hold.

Proof By lemma 3.2(ii) we have $f^{-1}(A\Box B) = f^{-1}(\cup\{C_1\cap C_2 \mid C_1$ is a maximal cylinder of A, C_2 is a maximal cylinder of B, $C_1\perp C_2\})$ which, of course $=\cup\{f^{-1}(C_1)\cap f^{-1}(C_2) \mid C_1$ is a maximal cylinder of A, C_2 is a maximal cylinder of B, $C_1\perp C_2\}$ which, by property ii' of f, is contained in $\cup\{A_1' \cap B_1' \mid A_1'\subset A', B_1'\subset B', A_1'\perp B_1'\}$ which, by lemma 3.2 (iii), is equal to A'□B'.

Hence $\mu(A\Box B) = \nu(f^{-1}(A\Box B)) \leq \nu(A'\Box B') \leq \nu(A')\nu(B') = \mu(A)\mu(B)$.

Lemma 3.5

The Conjectures 2.6 and 2.8 are equivalent.

Proof We only have to prove that if conjecture 2.8 is true then conjecture 2.6 is also true, since the other direction is trivial. So suppose conjecture 2.8 is true. Let $\Omega=S_1 \times S_2 \times \ldots \times S_n$ with $S_i=\{s_{i1}, s_{i2}, \ldots, s_{i,k_i}\}$, $i=1,\ldots,n$.

Further let, for $1 \leq i \leq n$, μ_i be a probability measure on S_i, and $\mu=\mu_1 \times \mu_2 \times \ldots \times \mu_n$. Define $\rho_{i,j}=\mu_i(s_{i,j})=\mu\{\omega_i=s_{i,j}\}$. Since we have a finite system, it is clear that, for each $A\subset\Omega$, $\mu(A)$ is a continuous function of $(\rho_{i,j})$ $1\leq i\leq n$, $1\leq j\leq k_i$. Using this and the fact that every $\rho_{i,j}$ can be approximated to arbitrary precision by numbers of the form $\ell.2^{-M}$, $\ell, M\in \mathbb{N}$, it is clearly sufficient to consider the case that there exist integers M and $C_{i,j}$, $1\leq i\leq n$, $1\leq j\leq k_i$, such that $\rho_{i,j}=C_{i,j}\,2^{-M}$. So assume that the $\rho_{i,j}$'s are indeed of this form.
Now consider, for each $i=1,2,\ldots,n$, the set $\{0,1\}^M$, and order the elements of this set, e.g. lexicographically: $(0,0,\ldots,0)$, $(1,0,\ldots,0)$, $(0,1,\ldots,0)$ $(1,1,0,\ldots,0)$ etc. (We refer to this ordering in section 5).
Define the map $f_i:\{0,1\}^M \rightarrow S_i$ as follows: The first $C_{i,1}$ elements (with respect to the above ordering) are all mapped to s_{i1}, the next $C_{i,2}$ elements to $s_{i,2}$ etc. Now apply lemma (3.4) with Ω and μ as above and $\Omega'=\{0,1\}^{nM}$ (i.e. $T_i=\{0,1\}$, $1\leq i\leq nM$), ν the uniform distribution on Ω' and $f:\Omega'\rightarrow\Omega$ as defined by :

$$f(\omega'_1,\ldots,\omega'_M,\ \omega'_{M+1},\ldots,\ \omega'_{2M},\ldots,\ldots,\omega'_{(n-1)M+1},\ldots,\ \omega'_{nM})$$
$$=(f_1(\omega'_1,\ldots\ \omega'_M),\ \ f_2(\omega'_{M+1},\ldots,\ \omega'_{2M}),\ldots,\ldots,\ f_n(\omega'_{(n-1)M+1},\ldots,\omega'_{nM})).$$

We finish this section with some additional notation and with two lemmas which are useful in the proofs of the results in section 4.

(3.6) Notation Let n,m be positive integers.
Denote, for $k_i\in\mathbb{N}$, $\omega^i\in\mathbb{N}^{k_i}$, $A_i\subset\mathbb{N}^{k_i}$, $(i=1,\ldots,m)$:

(3.7) $(\omega^1,\ldots,\omega^m) := (\omega^1_1,\ldots,\omega^1_{k_1},\ldots,\omega^m_1,\ldots,\omega^m_{k_m})$.

(3.8) $[A_1,\ldots,A_m] := A_1 \times A_2 \times \ldots \times A_m$.

Further, for $S_1,\ldots,S_n \subset \mathbb{N}$ finite, $\Omega=\prod_{i=1}^{n} S_i$, ℓ, $r\geq 0$, $\ell+r\leq n$, $A\subset \prod_{i=1}^{\ell} S_i$, $B\subset \prod_{i=\ell+r+1}^{n} S_i$:

(3.9) $[A, *^r, B] := [A, \prod_{i=\ell+1}^{\ell+r} S_i, B]$.

Remarks
a) Of course, when we use the notation of (3.9) the S_i, $\ell+1\leq i\leq \ell+r$, are assumed to be known.
b) If no confusion is possible we omit the commas in (3.8) and (3.9), and the "r" in (3.9).
c) If an A_i in (3.8) consists of one element ω, we write "ω" instead of "$\{\omega\}$".
d) The notation (3.9) can be extended in an obvious way to more "*"'s.

(3.10) Lemma

Let, for $i=1,\ldots,n$, S_i be a finite subset of $\mathbb{N}$ and μ_i a probability measure on S_i. Let $\Omega = \prod_{i=1}^{n} S_i$ and $\mu=\prod_{i=1}^{n}\mu_i$. Define, for π a permutation of $\{1,\ldots,n\}$, $\omega=(\omega_1,\ldots,\omega_n)\in\Omega$, and $D\subset\Omega$:

$\pi(\omega)=(\omega_{\pi(1)},\ldots,\omega_{\pi(n)})$ and $\pi(D)=\{\pi(\omega)\,|\,\omega\in D\}$.

Further, let $S'_i=S_{\pi(i)}$ and $\mu'_i=\mu_{\pi(i)}$, $i=1,\ldots n$, $\Omega'=\prod_{i=1}^{n} S'_i$, and $\mu'=\prod_{i=1}^{n}\mu'_i$.

Then, for all $A,B\subset\Omega$:

(3.11) $\mu'(\pi(A))=\mu(A)$, $\mu'(\pi(B))=\mu(B)$, $\mu'(\pi(A)\Box\pi(B))=\mu(A\Box B)$.

Proof The proof is straightforward.

(3.12) Lemma

Let $A,B\subset\prod_{i=1}^{n} S_i$, $K_A\subset \mathrm{supp}(A)\setminus \mathrm{supp}(B)$, $K_B\subset \mathrm{supp}(B)\setminus \mathrm{supp}(A)$, $K_{AB}=K_A\cup K_B$, $K=\{1,\ldots,n\}\setminus K_{AB}$. By the preceding lemma we may assume for our purpose that, for certain $r,s,t\geq 0$: $K=\{1,\ldots,r\}$, $K_A=\{r+1,\ldots,r+s+1\}$, $K_B=\{r+s+1,\ldots,r+s+t=n\}$, and $K_{AB}=\{r,\ldots,n\}$.

Define, for $\omega' \in \prod_{i\in K_{AB}} S_i$, and $D \subset \Omega$,

$$D(\omega') := \{\omega \in \prod_{i \in K} S_i \mid (\omega,\omega') \in D\}.$$

Let μ_i be a probability measure on S_i $(i=1,2,\dots,n)$,

$\mu = \prod_{i=1}^{n} \mu_i$, and $\tilde\mu = \prod_{i\in K}\mu_i$.

If:

(3.13) $\forall \omega' \in \prod_{i\in K_{AB}} S_i \quad \tilde\mu(A(\omega')\square B(\omega')) \leq \tilde\mu(A(\omega'))\,\tilde\mu(B(\omega'))$,

then:

(3.14) $\mu(A\square B) \leq \mu(A)\,\mu(B)$.

Proof Define, in addition to the above, for $\omega_A \in \prod_{i\in K_A} S_i$ and $\omega_B\in\prod_{i\in K_B} S_i$:

$A(\omega_A) := \{\tilde\omega \in \prod_{i\in K} S_i \mid [\tilde\omega\ \omega_A\ *] \subset A\}$ and $B(\omega_B) := \{\tilde\omega \in \prod_{i\in K} S_i \mid [\tilde\omega * \omega_B] \subset B\}$.

It is easily seen that if $\omega' = (\omega_A,\omega_B) \in \prod_{i\in K_{AB}} S_i$ then $A(\omega_A) = A(\omega')$,

$B(\omega_B) = B(\omega')$, and $(A\square B)(\omega') = A(\omega')\square B(\omega') = A(\omega_A)\square B(\omega_B)$.

Let $\mu_A = \prod_{i\in K_A}\mu_i$, $\mu_B = \prod_{i\in K_B}\mu_i$, and $\mu_{AB} = \mu_A \times \mu_B$.

If the condition in (3.13) holds, then:

$$\mu(A\square B) = \sum_{\omega'} \mu([A\square B(\omega')\ \omega']) =$$
$$= \sum_{\omega'} \mu_{AB}(\omega')\,\tilde\mu(A(\omega')\square B(\omega')) \leq \sum_{\omega'}\mu_{AB}(\omega')\,\tilde\mu(A(\omega'))\,\tilde\mu(B(\omega')) =$$
$$= \sum_{\omega_A}\sum_{\omega_B} \mu_A(\omega_A)\,\mu_B(\omega_B)\,\tilde\mu(A(\omega_A))\,\tilde\mu(B(\omega_B)) =$$
$$= \sum_{\omega_A}\mu_A(\omega_A)\,\tilde\mu(A(\omega_A)) \sum_{\omega_B}\mu_B(\omega_B)\,\tilde\mu(B(\omega_B)) =$$

$$= \sum_{\omega_A}\mu([A(\omega_A)\ \omega_A\ *]) \sum_{\omega_B}\mu([B(\omega_B) * \omega_B]) =$$
$$= \mu(A)\mu(B),$$

where ω' is summed over $\prod_{i\in K_{AB}} S_i$, ω_A over $\prod_{i\in K_A} S_i$ and ω_B over $\prod_{i\in K_B} S_i$.

4. Statement of main results

We state in theorem 4.2 four special cases of conjecture 2.6 which are proved in the sections 5-7. The theorem is followed by a short discussion of each of the cases.

(4.1) Remark. We can also prove the special case that the maximal cylinders of A or B are mutually disjoint. The proof is straighforward.(Use lemma 3.2.(ii) and the fact that each set is the union of its maximal cylinders.) Further, we have a (rather complicated)proof for the case that $\Omega=\{0,1\}^n$, μ is the uniform distribution an Ω, and A or B has at most 3 maximal cylinders.

For the first case of theorem 4.2 we need two definitions.

Let, as usual, $S_1,\dots,S_n$ be finite subsets of $\mathbb{N}$ and

$\Omega = \prod_{i=1}^{n} S_i$. If $\omega,\omega' \in \Omega$ then $\omega \geq \omega'$ means $\omega_i \geq \omega'_i$, $i=1,\dots,n$. A set $A \subset \Omega$ is called increasing or positive if $\omega' \in A$ whenever $\omega' \in \Omega$, $\omega' \geq \omega$ and $\omega \in A$.

Analogously,A is decreasing or negative if $\omega' \in A$ whenever $\omega' \in \Omega$, $\omega' \leq \omega$ and $\omega \in A$. The events A and B in the open-closed case of 2.1 are examples of increasing events. However the corresponding events in the random-direction case can be represented neither as increasing nor as decreasing events.

(4.2) Theorem

Let, for $1\leq i\leq n$, S_i be a finite subset and μ_i a probability measure on S_i.

Let $\Omega = S_1\times\dots\times S_n$, $\mu=\mu_1\times\dots\times\mu_n$,

and $A,B\subset\Omega$. In each of the following cases we have:

(4.3) $\mu(A\Box B) \leq \mu(A)\mu(B)$.

(a) There exist increasing $D,F\subset\Omega$ and decreasing $E,G \subset \Omega$ such that $A=D\cap E$ and $B=F\cap G$.

(b) $\Omega = \{0,1\}^n$ and A,B are both permutation invariant (i.e if the coordinates of an element of A(B) are permuted, the result is again an element of A(B).).

(c) There are cylinders C_i, $i\in I$ and C_i', $i\in J$ such that $A = \bigcup_{i\in I} C_i$, $B = \bigcup_{i\in I'} C_i'$ and for all $i\in I$, $j\in I'$: $C_i \perp C_j'$ or $C_i \cap C_j' = \emptyset$.

(d) There are cylinders C_i, $i\in I$ such that $A = \bigcup_{i\in I} C_i$, C_i is a maximal cylinder of A, $i\in I$, and for all $i,j\in I$: $C_i \perp C_j$ or $\mathrm{supp}(C_i) = \mathrm{supp}(C_j)$.

Proof: The cases (a) and (b) are proved in section 5, the case (c) in section 6, and the case (d) in section 7.

(4.3) Discussion of Theorem 4.2.

(a) Note that this result includes the case that A and B are both increasing and the case that A is increasing and B decreasing. The first has been proved, for $\Omega=\{0,1\}^n$ by Van den Berg and Kesten (1984), who obtained it as a special case of a result concerning so-called NBU measures. They had several other (unpublished) more direct proofs. One of these, which we call the splitting method, is closely related to the proof of the clutter theorem (see [4] and [6]) and can be refined to prove a and b.

If A is increasing and B decreasing and $\Omega=\{0,1\}^n$ then $A\Box B = A\cap B$ and we get, by (a), $\mu(A\cap B) \leq \mu(A)\mu(B)$. This is equivalent to Harris' inequality. *In fact Harris' inequality says that increasing events are positively* correlated to each other but, since the complement of an increasing event is decreasing, this is the same as saying that an increasing and a decreasing *event are always negatively correlated to each other. See also the discussion* of (c) which also contains Harris' inequality as a special case.

(b) In spite of serious attempts we have not been able to prove the permutation-invariant case more generally, i.e. for $\Omega=\{0,1,\ldots,k\}^n$, $k\geq 2$. That result would have the interesting consequence that the multinomial distribution is SNBU (see [2] for a definition of SNBU).

(c) Note that, in this case, $A\Box B = A\cap B$ so that we have a correlation inequality. As a special case we have $\Omega = \{0,1\}^n$, A increasing and B decreasing (because the maximal cylinders of an increasing event are always of the form $[w^1]_K$ and those of a decreasing event of the form $[\omega^0]_K$ where ω^1 is the element $(1,1,\ldots,1)$ and ω^0 the element $(0,0,\ldots,0)$); this reduces again to Harris' inequality which was also obtained as a special case *of (a). Harris' inequality has been extended by* Fortuin, Kasteleyn and Ginibre [3] to a larger class of probability measures on $\{0,1\}^n$. The FKG inequality in turn is contained in a rather general theory developed by *Ahlswede and Daykin* [1]. *However, apart from some common special cases,* like Harris' inequality, there does not seem to be a relationship between conjecture 2.6 and [1].

Another example of (c) is the following:

(4.5) Example

Define, for positive integers l,m,

$$B_{\ell,m} := \{(x,y)\in\mathbb{N}^2 \mid 0\leq x\leq \ell,\ 0\leq y\leq m\}.$$

The boundary of $B_{\ell,m}$ is defined as

$$\delta(B_{\ell,m}) := \{(x,y)\in B_{\ell,m} \mid x=0 \text{ or } x=\ell \text{ or } y=0 \text{ or } y=m\},$$

and the interior of $B_{\ell,m}$ *as*

$$\mathrm{int}(B_{\ell,m}) := B_{\ell,m} \setminus \delta(B_{\ell,m}).$$

The sets $B_{\ell,m}, \ell,m \in \mathbb{N}\setminus\{0\}$, and their images under translations $\mathbb{Z}^2 \to \mathbb{Z}^2$ are called boxes. Now suppose that each site $s\in\mathbb{Z}^2$ *is, independent of the other sites, black with probability* p_s and white with probability $1-p_s$. A box is called black (white) if its boundary is black and its interior is white (black). Let V be a finite region in

$\mathbb{Z}^2$, e,g, *for certain positive integer r*,
$V = \{(x,y)\in\mathbb{Z}^2 \mid |x|,|y|<r\}$.
Further let A be the event {there exists a black box in V} and B the event {there exists a white box in V}. It is not difficult to check that this falls under case (c) in our theorem, so we get $P(A \cap B) \leq P(A)\,P(B)$. Note that Harris' inequality cannot be applied here because neither A, nor B, is increasing or decreasing.

(d) This case has the following interesting consequence:
(4.6) Corollary
Let $x_1, x_2, \dots, x_n$ be independent random variables with values in $\mathbb{R}$ (or another set, it turns out that the set is immaterial).
Let for $1 \leq i \leq n$, A_i, B_i and C_i be subsets of $\mathbb{R}$.
Then:
(4.7) $P\{\text{there are three different } i,j,k \leq n \text{ such that } x_i \in A_i,\ x_j \in B_j \text{ and } x_k \in C_k\} \leq$
$\leq P\{\exists\, i\ x_i \in A_i\}\ P\{\exists\, i\ x_i \in B_i\}\ P\{\exists\, i\ x_i \in C_i\}$.

Remarks (i) the result can be extended to four types of sets or more (i.e. A_i's, B_i's, C_i's , D_i's etc.) We show how (4.7) follows from our theorem; the above mentioned extension can be proved by induction on the number of *different types of sets*.
(ii) For the case with two types of sets, and for the case that, for each i, the sets with index i are mutually disjoint, there is a more direct proof. However, *if there are no additional conditions, we don't know a proof of* (4.7) which is more direct than that of the full theorem 2.1(d) of which it is a corollary.
Proof of the corollary. First we remark that the l.h.s. of (4.7) is completely determined by the following probabilities:
(4.8) $P_i(q,r,s) := P\{x_i \in A(q) \cap B(r) \cap C(s)\}$, $q,r,s \in \{0,1\}, 1 \leq i \leq n$, where, for a set V, V(0) denotes V and V(1) denotes V^c. Therefore it is sufficient to prove the corollary for the case that each x_i can only have a finite number of values. Hence it is equivalent to the following:
(4.9) Let, for $i=1,2,\dots,n$, S_i be a finite subset of $\mathbb{N}$, μ_i a probability measure on S_i and A_i, B_i, C_i subsets of S_i. Further, let $\Omega = S_1 \times \dots \times S_n$ and $\mu = \mu_1 \times \mu_2 \times \dots \times \mu_n$.

Then:
(4.10) $\mu\{(\omega_1,\dots,\omega_n) \mid \text{there exist different } i,j,k \text{ such that } \omega_i \in A_i, \omega_j \in B_j \text{ and } \omega_k \in C_k\} \leq \mu\{(\omega_1,\dots,\omega_n) \mid \exists i\ \omega_i \in A_i\} \times$
$\times\ \mu\{(\omega_1 \dots \omega_n) \mid \exists j\ \omega_j \in B_j\}\ \mu\{(\omega_1,\dots,\omega_n) \mid \exists k\ \omega_k \in C_k\}$.

Proof of (4.9) Define $A := \{(\omega_1 \dots, \omega_n) \mid \exists i\ \omega_i \in A_i\}$, and define B and C analogously. It is easy to see that the event of the l.h.s. of (4.10) is a subset of $A\square(B\square C)$, and that A,B and C have the property mentioned in case d of theorem 4.6. Now the result follows by applying this theorem twice.

5. Proof of Theorem 4.2, case a and b.

In case b we have $\Omega = \{0,1\}^n$, and for a it is sufficient, by virtue of lemme 3.4 (take f as in the proof of lemma 3.5, noting that $f^{-1}(A)$ is increasing (decreasing) if A is increasing (decrasing)) to restrict ourselves to the binary case. The proofs are based on the splitting method mentioned in *section 4, which we shall explain here. First some definitions.* (Mind the notation 3.6 - 3.9 which will be used frequently). In the following we always assume n>1.

(5.1) Definition
If $A \subset \{0,1\}^n$, then

$$A^1 = \{\omega \mid \omega \in \{0,1\}^{n-1},\ (\omega,1) \in A\}$$
$$A^0 = \{\omega \mid \omega \in \{0,1\}^{n-1},\ (\omega,0) \in A\},$$

$$A^{01} = A^0 \cap A^1 = \{\omega \mid \omega \in \{0,1\}^{n-1},\ [\omega\ *] \subset A\}.$$

Apparently, for $V \subset \{0,1\}^{n-1}$

$V \subset A^1 \iff [V\ 1] \subset A.$
$V \subset A^0 \iff [V\ 0] \subset A.$
$V \subset A^{01} \iff [V\ *] \subset A.$

The following observations are frequently used:

(5.2) Observations.

i) If $\tilde{\omega} \in \{0,1\}^{n-1}$, $A \subset \{0,1\}^n$, $\omega_n \in \{0,1\}$, $\omega = (\tilde{\omega},\omega_n)$

and $K \subset \{1,\dots, n-1\}$, then $[\omega]_K \subset A \iff [\tilde{\omega}]_K \subset A^{01}$.

ii) $A^i \cap B^i = (A \cap B)^i$, where i denotes 0,1 or 01.

iii) $A = [A^1\ 1] \cup [A^0\ 0]$.

(5.3) Lemma For $A,B \subset \{0,1\}^n$:

i) $(A\square B)^1 = (A^1\square B^{01}) \cup (B^1\square A^{01})$.

ii) $(A\square B)^0 = (A^0\square B^{01}) \cup (B^0\square A^{01})$.

iii) $(A\square B)^{01} = (A^1\square B^{01} \cap B^0\square A^{01}) \cup (A^1\square B^{01} \cap A^0\square B^{01})$

$\cup (B^1\square A^{01} \cap A^0\square B^{01}) \cup (B^1 \square A^{01} \cap B^0\square A^{01})$.

Proof i) We prove that the l.h.s. is contained in the r.h.s. The reverse can be proved analogously.

$\omega = (\omega_1,\dots,\omega_{n-1}) \in (A\square B)^1 \Rightarrow (\omega,1) \in A\square B \Rightarrow$

$\exists K,L \subset \{1,\dots,n\}\ \ K\cap L = \emptyset, [\omega\ 1]_K \subset A,\ [\omega\ 0]_L \subset B$.

For K,L as above, al least one of these sets does not contain n. Suppose $n \notin L$. We show that this implies $\omega \in A^1\square B^{01}$ (Analogously (note the symmetry) $n \notin K$ implies $\omega \in B^1\square A^{01}$).

$n \notin L \Rightarrow L \subset \{1,\dots,n-1\} \Rightarrow$ (see 5.2(i)) $\Rightarrow$

$\Rightarrow [\omega]_L \subset B^{01}$.

Further, $[\omega,1]_K \subset A \Rightarrow [\omega]_{K'} \subset A^1$, where $K' = K\setminus\{1\}$. Hence, because $L \cap K' = \emptyset$, $\omega \in A^1 \square B^{01}$.

ii) Analogous to (i)(by A-B symmetry or 0-1 symmetry)

iii)Follows from (i), (ii) and definition 5.1.

(5.4) Definition. For the pair $A,B \subset \{0,1\}^n$ we define the pair $A^*, {}^*B \subset \{0,1\}^{n+1}$ by:

$A^* = \{(\omega_1\dots, \omega_{n+1}) \in \{0,1\}^{n+1} \mid (\omega_1,\dots,\omega_n) \in A\}$.

${}^*B = \{\omega_1,\dots, \omega_{n+1}) \in \{0,1\}^{n+1} \mid (\omega_1,\dots,\omega_{n-1},\omega_{n+1}) \in B\}$.

Further, if $\mu = \mu_1 x\dots x\mu_n$, where μ is a probability measuren on $\{0,1\}$, then is $\overset{*}{\mu}$ the probability measure $\mu_1 x\dots x\mu_n x\mu_n$ on $\{0,1\}^{n+1}$.

Note that $A^* = [A\ *]$ and ${}^*B = \pi([B\ *])$(where π is the map which exchanges the last two coordinates of each element of B.

Also,

(5.5) (i) $\overset{*}{\mu}(A^*) = \mu(A)$; (ii) $\overset{*}{\mu}({}^*B) = \mu(B)$.

Roughly speaking, A^* and ${}^*B$ are obtained by "making A and B independent in the last coordinate by splitting this coordinate". Analogous operations can be defined for the coordinates 1,..., n-1. Intuitively one would expect that, after applying one of these split operations, the probability of $A\square B$ always increases. This would imply that Conjecture 2.6 is true, because then, after successively "splitting" all coordinates 1,...,n we would have, for the "new" A and B (denoted by $A^\bullet, {}^\bullet B$):

$A^\bullet \perp {}^\bullet B$ hence $\mu(A \square B) \leq \dot{\mu}(A^\bullet\square {}^\bullet B) = \dot{\mu}(A^\bullet)\, \dot{\mu}({}^\bullet B) = \mu(A)\mu(B)$.

However, R. Ahlswede showed a counterexample and afterwards we have observed that it goes wrong very often. It appears that the probability of $A\square B$ does increase if $(A\square B)^{01} = A^{01}\square B^{01}$, which (as we shall show) holds for case a of our theorem. We shall prove that a weaker condition is also sufficient, which we use to prove case b of the theorem.

(5.6) Lemma. Let $A,B\subset\{0,1\}^n$. Then $A^*\square {}^*B = [A^0\square B^0\ 0\ 0] \cup [A^1\square B^0\ 1\ 0]$

$\cup [A^0\square B^1\ 0\ 1] \cup [A^1\square B^1\ 1\ 1]$.

Proof Let $r,s \in\{0,1\}$;

We show that for all $\omega\in\{0,1\}^{n-1}$: $(\omega,r,s)\in A^*\square {}^*B \iff \omega\in A^r\square B^s$.

$(\omega,r,s)\in A^*\square {}^*B \iff \exists K,L\subset(1,\dots n+1)$ s.t. $K\cap L=\emptyset$,

$[(\omega\ r\ s)]_K \subset A^*, [(\omega\ r\ s)]_L \subset {}^*B$. However, bacause $(n+1) \notin \mathrm{supp}(A^*)$ and $n \notin \mathrm{supp}({}^*B)$, the last statement is equivalent to

$\exists K' \subset \{1,\dots n-1\}$, $L'\subset \{1,\dots, n-1\}$ s.t. $K' \cap L' = \emptyset, [\omega]_{K'} \subset A^r$ and $[\omega]_{L'} \subset B^s \iff \omega \in A^r\square B^s$.

(5.7) Lemma Let $A,B \subset \{0,1\}^n$.

(i) If $(A\square B)^{01} = (A^1\square B^{01} \cap A^0\square B^{01}) \cup (B^1\square A^{01} \cap B^0\square A^{01})$,

then $\overset{*}{\mu}(A^*\square {}^*B) \geq \mu(A\square B)$.

(ii) If $(A\square B)^{01} = A^{01}\square B^{01}$ then $\overset{*}{\mu}(A^*\square {}^*B) \geq \mu(A\square B)$.

Proof See lemma 5.3(iii). We only have to prove (i) because (ii) is weaker.

Suppose the condition in i holds. Let $\overline{\omega} = (\omega_1,\dots,\omega_{n-1})$ be given. We show that the conditional probability of $A^*\Box^*B$ is always at least the conditional probability of $A\Box B$: We have four cases (a,b,c,d). First let $p = \mu\{\omega|\omega_n=1\}$

a) $\tilde{\omega} \notin ((A\Box B)^1 \cup (A\Box B)^0)$. Then the conditional probability of $A\Box B=0$.

b) $\tilde{\omega} \in (A\Box B)^1 \setminus (A\Box B)^0$. Then $(\tilde{\omega},\omega_n) \in A\Box B$ iff $\omega_n=1$, which has probability p. On the other hand, by lemma 5.3, $\tilde{\omega} \in A^1\Box B^{01}$ or $\tilde{\omega} \in B^1\Box A^{01}$. If $\tilde{\omega} \in A^1\Box B^{01}$ then it is sufficient, in order to have $(\tilde{\omega},\omega_n,\omega_{n+1}) \in A^*\Box^*B$, that $\omega_n=1$, which has probability p. Analogously, if $\tilde{\omega} \in B^1\Box A^{01}$ it is sufficient that $\omega_{n+1}=1$, which also has probability p.

c) $\tilde{\omega} \in (A\Box B)^0 \setminus (A\Box B)^1$. This case is analogous to case b.

d) $\tilde{\omega} \in (A\Box B)^1 \cap (A\Box B)^0$.

Hence $\tilde{\omega} \in (A\Box B)^{01}$. So the conditional probability of $A\Box B$ equals 1. We have to show that also the conditional probability of $A^*\Box^*B$ equals 1: By the condition in (i),

$$\tilde{\omega} \in A^1\Box B^{01} \cap A^0\Box B^{01} \text{ or } \tilde{\omega} \in B^1\Box A^{01} \cap B^0\Box A^{01}.$$

Suppose the first holds (By 0-1 symmetry the reasoning is analogous if the latter holds).

$$\tilde{\omega} \in A^1\Box B^{01} \implies \text{(see lemma 5.6) } [\tilde{\omega}\ 1\ *] \subset A^*\Box^*B.$$

$$\tilde{\omega} \in A^0\Box B^{01} \implies [\tilde{\omega}\ 0\ *] \subset A^*\Box^*B.$$

Hence $[\tilde{\omega}\ *\ *] \subset A^*\Box^*B$, so that, indeed, the conditional probability of $A^*\Box^*B$ equals 1.

(5.8) <u>Proof for theorem 4.2, case a.</u>

We show that, if $n \geq 2$ $A,B \subset \{0,1\}^n$ and A,B have the property mentioned in case a, then the condition in lemma 5.7(ii) holds. This is sufficient because also A^* and $^*B$ fall under case a, so we can successively split all coordinates. (More formally the proof can be completed by induction on the number of "unsplit" coordinates).

So, let $A,B \subset \{0,1\}^n$, $A=D\cap E$, $B=F\cap G$, where D and F are increasing, and E and G decreasing subsets of $\{0,1\}^n$. It is easily seen that

$\omega \in A\Box B$ if and only if there are mutually disjoint $K,K', L,L' \subset \{1,\ldots,n\}$, such that $\omega\equiv 1$ on $K\cup L$ and $\omega\equiv 0$ on $K' \cup L'$ and $[\omega]_K \subset D$, $[\omega]_{K'} \subset E$, $[\omega_L] \subset F$ and $[\omega]_{L'} \subset G$. Suppose $\omega \in (A\Box B)^{01}$. We shall show that this implies $\omega \in A^{01}\Box B^{01}$, so that $(A\Box B)^{01} \subset A^{01}\Box B^{01}$. The reverse inclusion is trivial. $\omega \in (A\Box B)^{01} \Rightarrow (\omega,0) \in A\Box B$ and $(\omega,1) \in A\Box B$. $(\omega,0) \in A\Box B$ $\Rightarrow \exists K,L \subset \{1,\ldots,n-1\}$ s.t. $K\cap L=\emptyset$, $\omega\equiv 1$ on $K\cup L$, and $[(\omega,0)]_K \subset D$, $[(\omega,0)]_L \subset F$. $(\omega,1) \in A\Box B \Rightarrow \exists K',L' \subset \{1,\ldots,n-1\}$ s.t. $K' \cap L'=\emptyset$, $\omega\equiv 0$ on $K'\cup L'$, $[\omega,1]_{K'} \subset E$, $[\omega,1]_{L'} \subset G$. Fix such K,L,K',L'.

We have:

$\omega \equiv 1$ on $K \cup L$ and $\omega \equiv 0$ on $K'\cup L' \Rightarrow (K \cup L) \cap (K'\cup L') = \emptyset$.

Hence K,L,K' and L' are mutually disjoint subsets of $\{1,\ldots,n-1\}$ and it is easily seen that $[\omega]_{K\cup K'} \subset A^{01}$ and $[\omega]_{L\cup L'} \subset B^{01}$.

(5.9) <u>Proof for theorem 4.2 case b.</u>

We show that if $A,B \subset \{0,1\}^n$ are permutation invariant, then condition(i) in lemma 5.7 holds. The proof can then be completed by induction on n (using lemma 3.12, noting that $\{n,n+1\} \not\subset \mathrm{supp}(A^*) \cap \mathrm{supp}(^*B)$ and that, in the notation of the lemma, for all $i,j \in \{0,1\}$ the pair $A^*(i,j)$, $^*B(i,j)$ also falls under theorem 4.2, case b). Suppose:

(5.10) $\omega \in (A^1\Box B^{01} \cap B^0\Box A^{01}) \setminus (B^1\Box A^{01} \cup A^0\Box B^{01})$

$\omega \in A^1\Box B^{01} \Rightarrow \exists K,L \subset \{1,\ldots,n\}$, s.t. $[\omega]_K \subset A^1$, $[\omega]_L \subset B^{01}$, and $K \cap L = \emptyset$. Fix such K,L.

We have $[[\omega]_K\ 1] \subset A$, $[[\omega]_L\ *] \subset B$.

Assume: $\exists i \in L\ \omega_i = 1$. Fix such an i.

Define $K'=K \cup \{i\}$, $L' = L\setminus\{i\}$. Obviously $K' \cap L' = \emptyset$.

By the permutation invariance of A and B we have

$[[\omega]_{K'}\ *] \subset A$, $[[\omega]_{L'},1] \subset B \implies [\omega]_{K'} \subset A^{01}$ and $[[\omega]_{L'} \subset B']$ $\implies \omega \in B^1\Box A^{01}$. This is in contradiction to (5.10).

Hence the assumption "$\exists i \in L\ \omega_i = 1$" is false, so that: $\omega \equiv 0$ on L. Analogously, because ω is in $B^0\Box A^{01}$ but not in $A^0\Box B^{01}$, we get that for certain disjoint $\tilde{K},\tilde{L} \subset \{1,\ldots,n-1\}$: $[\omega]_{\tilde{K}} \subset A^{01}$, $[\omega]_{\tilde{L}} \subset B^0$ and $\omega \equiv 1$ on $\tilde{K}$.

Hence, since $\omega \equiv 0$ on L and 1 on $\tilde{K}$, L and $\tilde{K}$ are disjoint.

Summarizing: $[\omega]_L \subset B^{01}$, $[\omega]_{\tilde{K}} \subset {}^{01}$, and $L \cap \tilde{K} = \emptyset$, i.e. $\omega \in A^{01} \square B^{01}$.

However, this is in contradiction with (5.10). Therefore, we may conclude that the r.h.s of (5.10) equals $\emptyset$, hence:

(5.11) $(A^1\square B^{01}) \cap (B^0\square A^{01}) \subset (B^1\square A^{01}) \cup (A^0\square B^{01})$. Of course we may replace the r.h.s. of (5.11) by its intersection with the l.h.s, which is: $(B^1\square A^{01} \cup A^0\square B^{01})\cap(A^1\square B^{01}\cap B^0\square A^{01}) = (B^1\square A^{01} \cap A^1\square B^{01}\cap B^0\square A^{01}) \cup (A^0\square B^{01} \cap A^1\square B^{01}\cap B^0\square A^{01})$, which is contained in the r.h.s. of the condition (i)in lemma (5.7). Analogously, by 0-1 symmetry, we can prove that also $A^0 \square B^{01} \cap B^1\square A^{01}$ is contained in the r.h.s. of (5.7,i). The required result now follows from that lemma.

6. Proof of Theorem 4.2, case c.

We first state some definitions and lemmas. A representation of an event A is a set $\{[\omega_i]_{K_i} | i\in I\}$ of cylinders s.t. $\bigcup_{i\in I} [\omega_i]_{K_i} = A$.

(6.1) Definition Let $\mathcal{A} = \{[\omega_i]_{K_i} | i\in I\}$ and $\mathcal{B} = \{[\omega_j]_{K_j} | j\in J\}$ be sets of cylinders. The pair $\mathcal{A}, \mathcal{B}$ is called semi-disjoint if:

$\forall i\in I\ j\in J : K_i\cap L_j = 0$ or $[\omega_i]_{K_i} \cap [\omega_j]_{L_j} = \emptyset$.

A pair of events A,B is called semi-disjoint if there exists a semi-disjoint pair of representations of A and B.

Remark: Note that these pairs of events form exactly case(c) in the theorem. The set of maximal cylinders of an event A is denoted by MR(A). Clearly MR(A) is a representation of A.

(6.2) Lemma: Let A and B be events and let $\mathcal{A}$ be a representation of A. If the property(*) (see below) holds for all ω and for all η, K, then the pair $\mathcal{A}$, MR(BN) is semi-disjoint.

(*): $([\eta]_K \in A,\ \omega\in[\eta]_K \cap B) \rightarrow [\omega]_{K^c} \subset B$.

Proof: Let $[\gamma]_L \in MR(B)$, $[\eta]_K \in \mathcal{A}$ and $[\eta]_K \cap [\gamma]_L \neq \emptyset$.

Then for a suitable σ this intersection can be written as $[\sigma]_{K\cup L}$. (Note that for such σ $[\eta]_K = [\sigma]_K$, $[\gamma]_L = [\sigma]_L$). It is not difficult to see that $[\sigma]_{L\setminus K} = \cup\{[\omega]_{K^c} \mid \omega\in[\sigma]_{K\cup L}\}$ which, by (*), is contained in B. Hence $[\sigma]_{L\setminus K}$ is a cylinder of B. But then, also $[\gamma]_{L\setminus K}$ is a cylinder of B. This is only possible if $L\setminus K = L$. Hence $K \cap L = \emptyset$.

6.3 Lemma The pair of sets D,E is semi-disjoint if and only if the pair MR(D), MR(E) is semi-disjoint.

Proof The "if-part" is trivial so we only have to prove the other direction: *If the pair D,E is semi-disjoint, then by definition there is a semi-disjoint* pair $\mathcal{D}, \mathcal{E}$ of representations of D and E. It is not difficult to show that if we take A=D', $\mathcal{A} = \mathcal{D}$ and B=E, then (*) in lemma 6.2 holds for all ω, η and K. Hence, by that lemma, the pair $\mathcal{D}$, MR(E) is semi-disjoint. Applying lemma 6.2 once more (this time we take A=E, $\mathcal{A}$=MR(E) and B=D) gives the result.

6.4 Lemma Let $\Omega = \prod_{i=1}^{n} \{1,\ldots,k_i\}$ and let $\mu = \prod_{i=1}^{n} \mu_i$, where μ_i is the uniform distribution on $\{1\ldots,k_i\}$ $(i=1,\ldots,n)$. (Hence μ is the uniform distribution on Ω.) *If A,B is a semi-disjoint pair of subsets of Ω, then:* $\mu(A\square B) \leqslant \mu(A)\,\mu(B)$.

Remark By applying lemma 3.4 analogously to the application in the proof of lemma 3.5 the result can be extended to case c of the theorem. It is even sufficient to give a proof for the case that each $k_i=1$; however, the proof for general k_i, which we give, is not more complicated. See also (4.3,c).

Proof of lemma 6.4. The case n=1 is trivial. We shall prove that if the result holds for n-1 (where $n\geqslant 2$), then it also holds for n. The proof consists of five parts, i, ii, iii, iv, v. First, let Ω, A and B be as in the conditions of the lemma. We shall define $\tilde{A}$, $\tilde{B}$, for which we prove in parts i-iii that: $\mu(\tilde{A}) = \mu(A)$, $\mu(\tilde{B} = \mu(B)$ and $\mu(\tilde{A}\cap\tilde{B}) = \mu(A\cap B)$. Further, we show in part iv that the pair $\tilde{A},\tilde{B}$ is also semi-disjoint, so that $\tilde{A}\cap\tilde{B} = \tilde{A}\ \tilde{B}$. Hence, it is sufficient to prove that $\mu(\tilde{A}\square\tilde{B}) \leqslant \mu(\tilde{A})\mu(\tilde{B})$. This will be done in part v by applying the induction hypothesis to $\Omega' = \prod_{i=1}^{n-1} \{1,\ldots,k_i\}$.

First some definitions:

(6.5) Definition

$\mathcal{B}^+ = \{C \in MR(B) \mid n \in supp(C)\}$.

$\mathcal{B}^- = \{C \in MR(B) \mid n \notin supp(C)\}$.

$B^+ = \cup$.

$B^- = \cup$.

$\mathcal{A}^-$, $\mathcal{A}^+ A^+$ and A^- are defined analogously.

(6.6) Definition If $V \subset \{1,\ldots,k_n\}$, then $\vec{V} = \{\ell,\ell+1,\ldots,k_n\}$, where $\ell = k_n - |V| + 1$, and $\overleftarrow{V} = \{1,\ldots,|V|\}$.

(6.7) Definition

(a) $\tilde{B}^+ = \bigcup_{\omega' \in \Omega'} \{\overrightarrow{[\omega' \; V(\omega')]}\}$, where $V(\omega') = \{j \mid (\omega',j) \in B^+\}$.

(b) $\tilde{A}^+ = \bigcup_{\omega' \in \Omega'} \{\overleftarrow{[\omega' \; W(\omega)]}\}$, where $W(\omega') = \{j \mid (\omega',j) \in A^+\}$.

(c) $\tilde{B} = B^- \cup \tilde{B}^+$.

(d) $\tilde{A} = A^- \cup \tilde{A}^+$.

We are now ready to start the real work:

(i) It is easy to see, by conditioning on $\omega_1,\ldots,\omega_{n-1}$, that, for all D for which sup (D) $\subset \{1,\ldots,n-1\}$:
$\mu(D \cap B^+) = \mu(D \cap \tilde{B}^+$, and $\mu(D \cap A^+) = \mu(D \cap \tilde{A}^+)$.

(ii) Using (i), we get $\mu(B) = \mu(B^-) + \mu(B^+) - \mu(B^- \cap B^+) =$
$= \mu(B^-) + \mu(\tilde{B}^+) - \mu(B^- \cap \tilde{B}^+) = \mu(\tilde{B})$, and analogously,
$\mu(A) = \mu(\tilde{A})$.

(iii) Application of lemma 6.3 yields $A^+ \cap B^+ = \emptyset$, and by conditioning on $\omega_1,\ldots,\omega_{n-1}$, it follows that also $\tilde{A}^+ \cap \tilde{B}^+ = \emptyset$, so
that $\mu(A \cap B) = \mu((A^- \cup A^+) \cap (B^- \cup B^+)) = \mu(A^- \cap B^-) + \mu(A^- \cap B^+) + \mu(A^+ \cap B^-)$
$- \mu(A^- \cap B^- \cap B^+) - \mu(A^- \cap B^- \cap A^+)$.

Now, by (i), we may, in the last expression, replace B^+ by $\tilde{B}^+$ and A^+ by $\tilde{A}^+$ and then, following the equations backwards, we get $\mu(A \cap B) = \mu(\tilde{A} \cap \tilde{B})$.

(iv) We shall now show that the pair $\tilde{A},\tilde{B}$ is semi-disjoint. First of all, it is clear that $\mathcal{A}^- \cup MR(\tilde{A}^+)$ and $\mathcal{B}^- \cup MR(\tilde{B}^+)$ are representations of $\tilde{A}$ and $\tilde{B}$ respectively. It will turn out that this pair of representations is semi-disjoint: The pair $MR(\tilde{A}^+)$, $MR(\tilde{B}^+)$ is obviously semi-disjoint, because, as we saw in (iii), $\tilde{A}^+ \cap \tilde{B}^+ = \emptyset$.

By lemma 6.3. the pair $\mathcal{A}^-, \mathcal{B}^-$ is also semi-disjoint. It remains to show that also the pairs $\mathcal{A}^-, MR(\tilde{B}^+)$ and $\mathcal{B}^-$, $MR(\tilde{A}^+)$ are semi-disjoint, and, by symmetry, it is sufficient to treat the first pair. This will be done by using lemma 6.2.

Let $[\omega']_K \in \mathcal{A}^-$ and $\omega = (\omega_1,\ldots,\omega_n) \in [\omega']_K \cap \tilde{B}^+$. Obviously $[\omega]_K = [\omega']_K$. Let $K' = \{1,\ldots,n-1\} \setminus K$.

Define the map

$S: \Omega \times \Omega \to \{1,\ldots,k_n\}$ by

$S(\eta,\gamma) = \{j \mid [\eta]_K \cap [\gamma]_{K'} \cap [* \; j] \subset B^+\}$.

Define $\tilde{S}(\eta,\gamma)$ analogously, replacing B^+ by $\tilde{B}^+$. We know, because the pair $\mathcal{A}, \mathcal{B}^+$ is semi-disjoint (lemma 4.3), that for all $\eta \in \Omega$, $S(\omega,\omega) \subset S(\eta,\omega)$.

Further, the definition of $\tilde{B}^+$ implies that

$\tilde{S}(\omega,\omega) = \overrightarrow{S(\omega,\omega)}$, and $\forall \eta \in \Omega$, $\tilde{S}(\eta,\omega) = \overrightarrow{S(\eta,\omega)}$.

So we get:

$\forall \eta \in \Omega \quad \tilde{S}(\omega,\omega) \subset \tilde{S}(\eta,\omega)$, and finally, because $\omega_n \in \tilde{S}(\omega,\omega)$,

$\forall \eta \in \Omega \quad \omega_n \in \tilde{S}(\eta,\omega)$, so that

$\forall \eta \in \Omega \quad [\eta]_K \cap [\omega]_{K^c} \in \tilde{B}^+$, hence $[\omega]_{K^c} \in \tilde{B}^+$, and so, by lemma 6.2,

the pair $\mathcal{A}^-$, $MR(\tilde{B}^+)$ is semi-disjoint.

(v) First define, for $D \subset \Omega$ and $1 \leq i \leq k_n$,

$D^i = \{\omega' \in \Omega' \mid (\omega',i) \in D\}$.

Before we apply the induction hypothesis, we have to show that also the pair $\tilde{A}^i$, $\tilde{B}^i$ is semi-disjoint ($i=1,\ldots,k_n$). This is easily seen by taking the

representations

$= \{C^i \mid C \in MR(\tilde{A})\}$,

$= \{C^i \mid C \in MR \; \tilde{B}\}$.

The induction step is as follows:

Let μ' denote $\mu|\Omega'$, (i.e. μ' is the uniform distribution on Ω').

Let $p = 1/k_n$.

We have:

$$\mu(A)\ \mu(B) - \mu(A \cap B) = \mu(\tilde{A})\ \mu(\tilde{B}) - \mu(\tilde{A} \cap \tilde{B}) =$$

$$= p \sum_{i=1}^{k_n} \mu'(\tilde{A}^i)\, p \sum_{j=1}^{\ell_n} \mu'(\tilde{B}^j) - p \sum_{i=1}^{k_n} \mu'\left((\tilde{A}\cap\tilde{B})^i\right)$$

The last summation equals: $p^2 \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} \mu'(\tilde{A}^i \cap \tilde{B}^i)$

$= p^2 \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} \mu'(\tilde{A}^i \cap \tilde{B}^i)$, which, by the induction hypothesis, is at most $p^2 \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} \mu'(\tilde{A}^i)\ \mu'(\tilde{B}^i)$.

So we have

$$\mu(A)\ \mu(B) - \mu(A \cap B) \geq p^2 \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} \mu'(\tilde{A}^i)\ \mu'(\tilde{B}^j) - \mu'(\tilde{A}^i)\ \mu'(\tilde{B}^i)$$

$= p^2 \sum_{i<j} [\mu'(\tilde{A}^i) - \mu'(\tilde{A}^j)]\ [\mu'(\tilde{B}^j) - \mu'(\tilde{B}^i)]$ which is non-negative because, for all i,j, $i<j$ implies $\tilde{A}^j \subset \tilde{A}^i$ and $\tilde{B}^i \subset \tilde{B}^j$.

7. Proof of Theorem 2.1d

Apparently, there are mutually disjoint $K_1, K_2, \ldots \subset \{1,\ldots,n\}$ such that the support of each maximal cylinder of A is one of the K_i's. By lemma 3.12 (noting that each $A(\omega')$ falls again under case d of the theorem) we may assume that $\bigcup_i K_i = \{1,\ldots,n\}$ and by lemma (3.10) that each K_i consists of consecutive numbers. Further we can reduce the problem to the case that for suitable r_i, s_i:

(7.1) $\Omega = \prod_{i=1}^{n} \{1,\ldots,r_i\}$, $A = \{\omega\in\Omega \mid \exists i\ \omega_i < s_i\}$.

This can be seen as follows: For each i, let

$$\Omega_i = \{\omega' \in \prod_{i\in K_i} S_j \mid [*\ \omega'\ *] \text{ is a maximal cylinder of } A\};$$

take $r_i = |\prod_{j\in K_i} S_j|$, and $s_i = |\Omega_i|$, i.e the number of maximal cylinders of A which have support K_i. Obviously, there exists a 1-1 map $f_i: \{1,\ldots,r_i\} \to \prod_{j\in K_i} S_j$, which maps $\{1,\ldots,s_i\}$ onto Ω_i. Let $\nu_i = (\prod_{j\in K_i} \mu_i) \circ f_i$.

Now define $f = \prod_i f_i$: $\prod_i\{1,\ldots,r_i\} \to \Omega$, and $\nu = \prod_i \nu_i$.

Note that $f^{-1}(A) = \{\omega\in \prod_i \{1,\ldots,r_i\} \mid \exists_j\ \omega_j < s_j\}$, and check that the properties i and ii' in lemma 3.4 hold. Application of that lemma gives that we may indeed restrict ourselves to case (7.1).

Now we apply lemma 3.4 once more to reduce 7.1 to the binary case: Let $\Omega' = \{0,1\}^n \times \prod_{i=1}^{n} \{1,\ldots,s_i\} \times \prod_{i=1}^{n} \{1,\ldots,r_i\}$. Define $g: \Omega' \to \Omega$ by

$$g(x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n) = (g_1(x_1,y_1,z_1),\ g_2(x_2,y_2,z_2),\ldots,g_n(x_n,y_n,z_n)),$$

where, for $1\leq i\leq n$, $g_i(x_i,y_i,z_i) = \begin{cases} y_i & \text{if } x_i=0 \\ z_i & \text{if } x_i=1 \end{cases}$.

Also define, for $1\leq i\leq n$, the probability measures

μ'_{ix}, μ'_{iy} and μ'_{iz} on $\{0,1\}^n$, $\prod_{i=1}^{n}\{1,\ldots,s_i\}$ and $\prod_{i=1}^{n}\{1,\ldots,r_i\}$ respectively by:

$\mu'_{ix}(0) = 1-\mu'_{ix}(1) = \mu\{\omega_i < s_i\}$;

$$\mu'_{iy}(j) = \frac{\mu\{\omega_i=j\}}{\mu\{\omega_i < s_j\}},\ j=1,\ldots,s_i;\quad \mu'_{iz}(j) = \frac{\mu\{\omega_i=j\}}{\mu\{\omega_i > s_j\}},\ j=s_i+1,\ldots,r_i.$$

Finally, define $\mu' = \prod_{i=1}^{n} \mu'_{ix} \times \prod_{i=1}^{n} \mu'_{iy} \times \prod_{i=1}^{n} \mu'_{iz}$.

It is not difficult to see that the properties i and ii of lemma 3.4 hold with $f=g$ and $\nu=\mu'$) and that $g^{-1}(A) =$

$= \bigcup_{1\leq i\leq n} \{x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n)\in\Omega \mid \exists i\ x_i=0\}$. Application of this lemma and of lemma 3.11 (note that $g^{-1}(A)$ does not depend on the $y_i's$ and $z_i's$) reduces the problem to the case that:

(7.2) $\Omega=\{0,1\}^n$, $A=\{\omega\in\Omega \mid \exists i\ \omega_i=0\}$, and for this case we have the following direct proof: If Ω and A are as in (7.2) then, for arbitrary $B \subset \Omega$,

(7.3) $\omega \in A\Box B \Longleftrightarrow \omega \in B,\ \exists i\ \omega_i = 0,\ (\omega_1,\dots,\omega_{i-1},1,\omega_{i+1},\dots,\omega_n) \in B.$

Call an element $\omega \in B$ maximal if there is no $\omega' \neq \omega$ in B with $\omega' \geq \omega$. Denote the set of maximal elements of B by B_{max}. Obviously, by 7.3, $A\Box B \subset B \setminus B_{max}$, hence

(7.4) $\mu(A\Box B) \leq \mu(B) - \mu(B_{max})$.

Further, for each $\omega \in \Omega$:

(7.5) $\mu(\{\omega' | \omega' < \omega\}) = \prod_{\omega_i = 0} \mu(\omega_i) = \mu(\omega) / \prod_{\omega_i = 1} \mu(\omega_i) \leq$

$\leq \mu(\omega)/\mu(1,\dots,1) = \mu(\omega)/(1-\mu(A))$.

Hence

(7.6) $\mu(B) = \mu(\bigcup_{\omega \in B_{max}} \{\omega' | \omega' \in B,\ \omega' \leq \omega\}) \leq \sum_{\omega \in B_{max}} \mu(\{\omega' | \omega' \in B,\ \omega' \leq \omega\}) \leq$

$\sum_{\omega \in B_{max}} \mu(\omega) / (1-\mu(A)) = \mu(B_{max}) / (1-\mu(A))$,

so that

(7.7) $\mu(B_{max}) \geq \mu(B)(1-\mu(A))$.

Combining (7.4) and (7.7) we get

(7.8) $\mu(A\Box B) \leq \mu(B) - \mu(B)(1-\mu(A)) = \mu(A)\mu(B)$.

Acknowledgments

We wish to thank R. Ahlswede and H. Kesten for many helpful discussions.

The first author's research was supported by the Netherlands Foundation for Mathematics SMC with financial aid from the Netherlands Organization for the Advancement of Pure Research (ZWO).

Most of this research has been done between September 1983 and July 1984 when the second author had a research position at Delft University of Technology.

REFERENCES

[1] R. Ahlswede and D.E. Daykin, Inequalities for a pair of maps S×S→S with S a finite set, Math. Z. 165 (1979) 269-289.

[2] J. van den Berg and H. Kesten, Inequalities with applications to percolation and reliability, to appear in J. Appl. Prob. (1985).

[3] C.M. Fortuin, P.W. Kasteleyn and J. Ginibre, Correlation inequalities on some partially ordered sets, Comm. Math. Phys. 22(1971) 89-103.

[4] J.M. Hammersley, Comparison of atom and bond percolation, J. Math. Phys. 2(1961) 728-733.

[5] T.E. Harris, A lower bound for the critical probability in a certain percolation process, Proc. Camb. Phil. Soc. 56(1960) 13-20.

[6] C. Mc.Diarmid, Clutter percolation and random graphs, Math. Progr. Study 13(1980) 17-25.

Dept. of Mathematics
Delft University of Technology
The Netherlands.

Institute of Math. Stochastics
Georg August University Göttingen
Federal Republic of Germany.

ENIGE BIJDRAGEN TOT DE PERCOLATIETHEORIE

en verwante gebieden

SAMENVATTING

In de percolatietheorie bestudeert men de stochastische eigenschappen van netwerken waarvan de knooppunten en / of verbindingen met bepaalde kansen eigenschap A of eigenschap B hebben. De specifieke betekenis van A en B hangt af van de aard van het probleem. Het onderwerp werd voor het eerst in de literatuur behandeld in 1957 door Broadbent en Hammersley, die een wiskundig model probeerden te maken van de verspreiding van een gas of vloeistof door een poreus materiaal.

Percolatiemodellen bleken spoedig van belang te zijn bij de beschrijving van diverse "coöperatieve" verschijnselen, zoals halfgeleiding, betrouwbaarheid van grote communicatienetwerken en de vorming van polymeren. Ook bestaat er een grote mate van analogie met het Ising model voor ferromagneten.

Dit proefschrift bestaat uit een algemene inleiding, gevolgd door zeven artikelen over percolatie en verwante gebieden. Bovendien wordt op ieder artikel afzonderlijk een kort commentaar gegeven.

Het eerste artikel (A) betreft resultaten en vermoedens van Sykes en Essam (1964).

Het tweede artikel (B) houdt zich bezig met de vraag welke kritische waarschijnlijkheden kunnen voorkomen bij de subgrafen van een gegeven graaf, en formuleert een eigenschap van "unieke oneindige componenten", welke gebruikt wordt in artikel (D).

De artikelen (C) en (E) laten tegenvoorbeelden zien op intuitief voor de hand liggende vermoedens van Hammersley en Welsh (1965) en Hammersley (1957).

In artikel (D) (gezamenlijk werk met M.Keane) wordt een verband gelegd tussen twee belangrijke onderwerpen, namelijk de continuïteit van de percolatie-waarschijnlijkheidsfunctie enerzijds en het aantal en de aard van oneindige componenten anderzijds.

Het artikel (F) (gezamenlijk werk met H.Kesten) is voortgekomen uit een geïsoleerd percolatieprobleem, dat echter aanleiding gaf tot een aantal ongelijkheden die op een breed gebied van toepassing zijn. Een aantal resultaten in de betrouwbaarheidstheorie en in de percolatietheorie werden verbeterd, van een aantal andere werden eenvoudiger bewijzen gevonden.

Artikel (G) (gezamenlijk werk met U. Fiebig) is gewijd aan een in het vorig artikel geformuleerd combinatorisch vermoeden betreffende het "disjunct optreden van gebeurtenissen". Dit vermoeden heeft betekenis voor de percolatietheorie (zoals voorbeeld 1.2 laat zien) maar heeft ook geheel andere interessante interpretaties (zoals voorbeeld 1.1). Enkele gevallen worden bewezen; één daarvan is een uitbreiding van de FKG-Harris ongelijkheid.

Curriculum Vitae

De schrijver van dit proefschrift is in 1956 geboren te Vianen. In 1974 behaalde hij het diploma Atheneum B aan het Christelijk Lyceum te Gouda, waarna hij Wiskunde ging studeren aan de Rijksuniversiteit Utrecht. In 1977 behaalde hij het kandidaatsexamen (met bijvak Natuurkunde) en in 1980 studeerde hij af in de Toegepaste Wiskunde bij Prof.dr.ir. J.W.Cohen. Van november 1980 tot maart 1982 was hij als dienstplichtig militair gedetacheerd aan het Physisch Laboratorium TNO waar hij deel uitmaakte van de groep Operational Research. Sindsdien werkt hij aan de Technische Hogeschool Delft onder leiding van Prof.dr. M.S.Keane aan een door ZWO gefinancierd project, waaruit dit proefschrift is voortgekomen.

Acknowledgments

Since March 1982 my research has been supported by the Netherlands Foundation for Mathematics (SMC) with financial aid from the Netherlands Organization for the Advancement of Pure Research (ZWO).

In the autumn of 1983 I spent three weeks at the University of Bielefeld to cooperate with R. Ahlswede, and in July 1984 I stayed at Cornell University, where I had instructive discussions with H.Kesten. The first version of the joint paper with U. Fiebig has been written in the beginning of November 1984 when I visited the University of Göttingen.

I thank these universities for their financial support and hospitality.

Some contributions to percolation theory and related fields

Errata and addenda

Page	line	printed	read
contents	10↓	21	22
I5	17↓	*Chayes, Fröhlich*	Fröhlich and Russo
G21	5↑	B'	B^1
G22	3↑	BN	B
G23	11↓	D'	D
G23	12↑	1	2
G24	4↓	U	$\cup\mathcal{B}^+$
G24	5↓	U	$\cup\mathcal{B}^-$
G24	10↓	$\overrightarrow{\omega' V(\omega')}$	$\omega'\overrightarrow{V(\omega')}$
G24	11↓	$\overleftarrow{\omega' W(\omega)}$	$\omega'\overleftarrow{W}(\omega')$
G24	11↑	sup	supp
G25	16↓	4.3	6.3
G25	2↑	=	$\mathcal{A}^i =$
G25	1↑	=	$\mathcal{B}^i =$
G26	6↓	ℓ	k
G26	7↓	$(\tilde{A}^i \cap \tilde{B}^i)$	$((\tilde{A} \cap \tilde{B})^i)$
G27	9↓, 13↓	$1, \ldots, r_i$	$s_i+1, \ldots, r_i$
G27	10↓	z_i	z_1
G28	8↓(2x)	$\mu(\omega_i)$	$\mu_i(\omega_i)$

By a technical failure a part of the text in the first column of page A154 has been omitted. The complete text is as follows:

The following references should be added to the list on p. 22-24:

Kingman, J.F.C. (1973) *Subadditive* ergodic theory, Ann. Probab. 1, 883-909.

Mc Diarmid, C. (1980) Clutter percolation and random graphs, Math. Progr. Study 13, 17-25.

When we call two vertices v_1 and v_2 equivalent if, for all n and p, $S_n(p;v_1) = S_n(p;v_2)$, then we can generalize the above result as follows:

Theorem 1: Let L be a lattice which has only a finite number of classes of equivalent vertices and which possesses a pair of orthogonal symmetry-axes. Then

$$P_c^{(s)}(L) + P_c^{(s)}(L^*) = 1.$$

The proof of Theorem 1 is similar to that of the special case of the simple quadratic lattice (see also Fisher[7]).

3. SOME NONTRIVIAL SECTION-GRAPHS OF S WITH CRITICAL PROBABILITY $P_c^{(s)}(S)$.

In this section it will first be shown that $P_c^{(s)}(S(\frac{1}{2}\pi)) = P_c^{(s)}(S)$, where $S(\frac{1}{2}\pi)$ denotes the quadrant of S with vertex-set $\{(n,m)|n,m\geq 0\}$. Analogously $S^*(\frac{1}{2}\pi)$ will denote the quadrant of S^* with the same vertex-set as $S(\frac{1}{2}\pi)$.

From the matching-property (see Ref. 1) it follows that the vertex $0 = (0,0)$ belongs to an infinite white cluster of $S^*(\frac{1}{2}\pi)$ *if and only if there is no black walk in* $S(\frac{1}{2}\pi)$ *from some* vertex $(n,0)$ to some vertex $(0,m)$, $n,m\geq 0$. It is trivial that the

A154 J. Math. Phys., Vol. 22, No. 1, January 1981

Stellingen bij het proefschrift "Some Contributions to Percolation Theory and related fields" van J. van den Berg, 8 januari 1985.

1. De in [1] beschreven methode voor het simuleren van de golfhoogten onder een zich boven het oceaanoppervlak voortbewegende waarnemer is niet geschikt voor hoge snelheden, omdat de variantie van het gesimuleerde proces dan aanzienlijk ($\pm$ 35%) te laag is.

   [1] R.T. Schmitke (1971), A computer simulation of the performance and dynamics of HMCS Bras d'or (FHE-400), Canadian Aerodynamics and Space Journal, March 1971.

2. Door middel van laboratoriumproeven onderzoekt men in welke mate DNA beschadigd wordt door chemotherapie en bestraling. In [2] wordt een wiskundig model behandeld van twee aanvankelijk onbeschadigde en van elkaar gescheiden DNA strengen *die onderhevig zijn aan: (i) breuk; (ii) vorming van onderlinge verbindingen;* (iii) aanhechting van andere celproteïnen . Een stuk DNA waarmee het laatste heeft plaats gevonden is zo groot geworden dat het niet meer door bepaalde *filters kan. De auteurs berekenen de fractie van* het DNA *materiaal* dat tot dergelijke clusters behoort. Aan het einde van het artikel merken zij onder andere op dat " the calculation of the total length of all the fragments in a given cluster seems to pose a more difficult problem". Dit lijkt mee te vallen: de oplossing komt neer op het inverteren van een Laplace transformatie .

   [2] George H. Weiss and John Rice (1982), A combinatorial problem in pharmacology, J.Math.Biology 14, 195-201.

3. Het in [3] besproken model van een bosbrand is, in tegenstelling tot een bewering van de auteurs, geheel equivalent met gewone (d.w.z. onafhankelijke naaste-buur) lijnpercolatie.

   [3] Gary MacKay and Naeem Jan (1984), Forest fires as critical phenomena, J. Phys. A 17, L757 - L760.

4. Voor simpele symmetrische exclusieprocessen (zie [4]) is bekend dat, gegeven de begintoestand, voor iedere $t \geq 0$ en voor elk tweetal roosterpunten i,j ($i \neq j$) de gebeurtenissen {op tijdstip t bevindt zich een deeltje in i} en {op tijdstip t bevindt zich een deeltje in j} negatief gecorreleerd zijn. In deze stelling is de symmetrie conditie zeer belangrijk. Er is namelijk een voorbeeld van een proces waarvoor nog wel in ieder roosterpunt de ingaande rate gelijk is aan de uitgaande rate maar waarvoor de symmetrie-conditie niet geldt en de uitspraak van de stelling onwaar is.

   [4] T.M. Liggett (1977), The stochastic evolution of infinite systems of interacting particles; in: Ecole d'été de probabilités de Saint-Flour VI-1976, LNM 598, Springer-Verlag.

5. Zij $d > 0$, $\ell, v, g \geq d$ en B de balk $[0,\ell] \times [0,v] \times [0,g]$. Laten L, V en G meetbare deelverzamelingen zijn van respectievelijk het linker zijvlak, het voorvlak en het grondvlak van B. Laat verder C_G de cylinder zijn in B die loodrecht op het grondvlak staat en als basis G heeft (dus $C_G = G \times [0,g]$. Definieer op analoge wijze C_L en C_V. Stel nu dat van elk van deze cylinders het gedeelte op afstand $\leq d$ van de basis gekleurd wordt (Dus van C_G wordt de deelverzameling $G \times [0,d]$ gekleurd enz.). Dan geldt dat de fractie van $C_L \cup C_V \cup C_G$ die gekleurd is maximaal is als L het gehele linker zijvlak, V het gehele voorvlak en G het gehele grondvlak is.

6. Beschouw een groot net met vierkante mazen van lengte 1. Stel dat ten gevolge van slijtage breukvorming optreedt overeenkomstig een homogeen Poisson proces. Hierdoor valt het net (mogelijk) in fragmenten uiteen. De kans dat twee buurknooppunten tot verschillende fragmenten behoren is kleiner dan de overeenkomstige kans voor twee knooppunten met onderlinge afstand 2.

7. Laat $\mathcal{F}$ een partitie van Z^d zijn in eindige klassen. Beschouw naaste-buur puntpercolatie modellen waarbij tot verschillende klassen behorende punten onafhankelijk zijn, terwijl binnen iedere klasse afzonderlijk de punten open of gesloten zijn overeenkomstig een SNBU verdeling. Wanneer het antwoord op probleem 1.11 van artikel F in dit proefschrift bevestigend is, dan voldoen al deze modellen aan ongelijkheid 3.18. Voor modellen die bovendien voldoende periodiciteit bezitten zou dan tevens, in het kritische gebied, het volgende analogon van corollarium 3.15 gelden: $P[B_n] \geq c(d)\, n^{-(d-1)/2}$.

8. Zij $a \in [0,1]$, $A = [a,1]$, $x,y \in [0,1]$, $x \geq y$. Voor diffusieproblemen met reflecterende punten 0,1 geldt $P_x\{B(t) \in A\} \geq P_y\{B(t) \in A\}$, $0 \leq t < \infty$.

9. Wie twee reizen als bijrijder op een vrachtauto maakt, een van New York naar Los Angeles en een van Stockholm naar Rome, behoeft geen econoom te zijn om in te zien dat de EEG geen wereldrol van betekenis kan spelen zolang haar inwendige handelsobstakels niet verder geslecht worden.

10. In verband met de steunverlening van de overheid aan noodlijdende bedrijven wordt vaak de kritiek geuit dat bescherming van deze, meestal verouderde, bedrijven weggegooid geld is en ten koste gaat van levensvatbare en vernieuwende bedrijven. Wanneer men nu personen beschouwt als (dienstverlenende) ondernemingen die hun dienstenpakket (lees: kennis, vaardigheid en inzet) voortdurend dienen aan te passen aan omstandigheden en ontwikkelingen, dan zou een gelijksoortige kritiek op zijn plaats kunnen zijn.

11. Bij het oogsten van biezen wordt men geconfronteerd met diverse lastige percolatieproblemen.