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FLUCTUATIONS OF ERGODIC AVERAGES FOR ACTIONS OF GROUPS OF POLYNOMIAL GROWTH.

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Abstract. It was shown by S. Kalikow and B. Weiss that, given a measure-preserving action of $\mathbb{Z}^d$ on a probability space X and a nonnegative measurable function f on X, the probability that the sequence of ergodic averages

$$\frac{1}{(2k+1)^d} \sum_{g \in [-k,\ldots,k]^d} f(g \cdot x)$$

has at least n fluctuations across an interval (α, β) can be bounded from above by $c_1 c_2^n$ for some universal constants $c_1 \in \mathbb{R}$ and $c_2 \in (0,1)$, which depend only on d, α, β. The purpose of this article is to generalize this result to measure-preserving actions of groups of polynomial growth. As the main tool we develop a generalization of effective Vitali covering theorem for groups of polynomial growth.

1. Introduction

Given an integer $n \in \mathbb{Z}_{\geq 0}$ and some numbers $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$, a sequence of real numbers $(a_i)_{i=1}^k$ is said to **fluctuate at least** n **times** across the interval (α, β) if there are indexes $1 \leq i_0 < i_1 < \cdots < i_n \leq k$ such that

1) if j is odd, then $a_{i_j} < \alpha$;
2) if j is even, then $a_{i_j} > \beta$.

In this case it is clear that for every even j we have

$$a_{i_j} > \beta \quad \text{and} \quad a_{i_{j+1}} < \alpha,$$

i.e., $(a_i)_{i=1}^k$ has at least $\lceil \frac{n}{2} \rceil$ **downcrossings** from β to α and at least $\lfloor \frac{n}{2} \rfloor$ **up-crossings** from α to β. If $(a_i)_{i \geq 1}$ is an infinite sequence of real numbers, we use the same terminology and say that $(a_i)_{i \geq 1}$ fluctuates at least n times across the interval (α, β) if some initial segment $(a_i)_{i=1}^k$ of the sequence fluctuates at least n times across (α, β). We denote the sets of all real-valued sequences having at least n fluctuations across an interval (α, β) by $\mathcal{F}^n_{(\alpha,\beta)}$, and it will be clear from the context if we are talking about finite or infinite sequences.

The main result of this article is the following theorem, which generalizes the results in [KW99] about fluctuations of averages of nonnegative functions.

Theorem. *Let Γ be a group of polynomial growth and let $(\alpha, \beta) \subset \mathbb{R}_{>0}$ be some nonempty interval. Then there are some constants $c_1, c_2 \in \mathbb{R}_{>0}$ with $c_2 < 1$, which depend only on Γ, α and β, such that the following assertion holds.*

For any probability space $\mathrm{X} = (X, \mathcal{B}, \mu)$, any measure-preserving action of Γ on X and any measurable $f \geq 0$ on X we have

$$\mu(\{x : (\mathbb{E}_{g \in \mathrm{B}(k)} f(g \cdot x))_{k \geq 1} \in \mathcal{F}^N_{(\alpha,\beta)}\}) < c_1 c_2^N$$

for all $N \geq 1$.

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The paper is structured as follows. We provide some background on groups of polynomial growth in Section 2.1, discuss some special properties of averages on groups of polynomial growth and a transference principle in Section 2.2 and prove effective Vitali covering theorem in Section 2.3. The main theorem of this paper is Theorem 3.1, which is proved in Section 3.

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2. Preliminaries

2.1. **Groups of Polynomial Growth.** Let Γ be a finitely generated group and $\{\gamma_1, \ldots, \gamma_k\}$ be a fixed generating set. Each element $\gamma \in \Gamma$ can be represented as a product $\gamma_{i_1}^{p_1} \gamma_{i_2}^{p_2} \ldots \gamma_{i_l}^{p_l}$ for some indexes $i_1, i_2, \ldots, i_l \in 1, \ldots, k$ and some integers $p_1, p_2, \ldots, p_l \in \mathbb{Z}$. We define the **norm** of an element $\gamma \in \Gamma$ by

$$\|\gamma\| := \inf\{\sum_{i=1}^{l} |p_i| : \gamma = \gamma_{i_1}^{p_1} \gamma_{i_2}^{p_2} \ldots \gamma_{i_l}^{p_l}\},$$

where the infinum is taken over all representations of γ as a product of the generating elements. The norm $\|\cdot\|$ on Γ, in general, does depend on the generating set. However, it is easy to show [CSC10, Corollary 6.4.2] that two different generating sets produce equivalent norms. We will always say what generating set is used in the definition of a norm, but we will omit an explicit reference to the generating set later on. For every $n \in \mathbb{R}_{\geq 0}$ let

$$\mathrm{B}(n) := \{\gamma \in \Gamma : \|\gamma\| \leq n\}$$

be the closed ball of radius n.

The norm $\|\cdot\|$ yields a right invariant metric on Γ defined by

$$d_R(x, y) := \|xy^{-1}\| \quad (x, y \in \Gamma),$$

and a left invariant metric on Γ defined by

$$d_L(x, y) := \|x^{-1}y\| \quad (x, y \in \Gamma),$$

which we call the **word metrics**. The right invariance of d_R means that the right multiplication

$$R_g : \Gamma \to \Gamma, \quad x \mapsto xg \quad (x \in \Gamma)$$

is an isometry for every $g \in \Gamma$ with respect to d_R. Similarly, the left invariance of d_L means that the left multiplications are isometries with respect to d_L. We let $d := d_R$ and view Γ as a metric space with the metric d. For $x \in \Gamma$, $r \in \mathbb{R}_{\geq 0}$ let

$$\mathrm{B}(x, r) := \{y \in \Gamma : d(x, y) \leq r\}$$

be the closed ball of radius r with center x. Using the right invariance of the metric d, it is easy to see that

$$|\mathrm{B}(x, r)| = |\mathrm{B}(y, r)| \qquad \text{for all } x, y \in \Gamma.$$

Let $\mathrm{e} \in \Gamma$ be the neutral element. It is clear that

$$\mathrm{B}(n) = \{\gamma : d_R(\mathrm{e}, \gamma) \leq n\} = \{\gamma : d_L(\mathrm{e}, \gamma) \leq n\},$$

i.e., the ball $\mathrm{B}(n)$ is precisely the ball $\mathrm{B}(\mathrm{e}, n)$ with respect to the left and the right word metric.

It is important to understand how fast the balls $\mathrm{B}(n)$ in the group Γ grow as $n \to \infty$. The **growth function** $\gamma : \mathbb{N} \to \mathbb{N}$ is defined by

$$\gamma(n) := |\mathrm{B}(n)| \quad (n \in \mathbb{N}).$$

We say that the group Γ is of **polynomial growth** if there are constants $C, d > 0$ such that for all $n \geq 1$ we have

$$\gamma(n) \leq C(n^d + 1).$$

Example 2.1. Consider the group $\mathbb{Z}^d$ for $d \in \mathbb{N}$ and let $\gamma_1, \dots, \gamma_d \in \mathbb{Z}^d$ be the standard basis elements of $\mathbb{Z}^d$. That is, γ_i is defined by

$$\gamma_i(j) := \delta_i^j \quad (j = 1, \dots, d)$$

for all $i = 1, \dots, d$. We consider the generating set given by elements $\sum_{k \in I} (-1)^{\varepsilon_k} \gamma_k$ for all subsets $I \subseteq [1, d]$ and all functions $\varepsilon_\cdot \in \{0,1\}^I$. Then it is easy to see by induction on dimension that $\mathrm{B}(n) = [-n, \dots, n]^d$, hence

$$|\mathrm{B}(n)| = (2n+1)^d \quad \text{for all } n \in \mathbb{N}$$

with respect to this generating set, i.e., $\mathbb{Z}^d$ is a group of polynomial growth.

Let $d \in \mathbb{Z}_{\geq 0}$. We say that the group Γ has **polynomial growth of degree** d if there is a constant $C > 0$ such that

$$\frac{1}{C} n^d \leq \gamma(n) \leq C n^d \quad \text{for all } n \in \mathbb{N}.$$

It was shown in [Bas72] that, if Γ is a finitely generated nilpotent group, then Γ has polynomial growth of some degree $d \in \mathbb{Z}_{\geq 0}$. Furthermore, one can show [CSC10, Proposition 6.6.6] that if Γ is a group and $\Gamma' \leq \Gamma$ is a finite index, finitely generated nilpotent subgroup, having polynomial growth of degree $d \in \mathbb{Z}_{\geq 0}$, then the group Γ has polynomial growth of degree d as well. A surprising fact is that the converse is true as well. Namely, it was proved in [Gro81] that, if Γ is a group of polynomial growth, then there is a finite index, finitely generated nilpotent subgroup $\Gamma' \leq \Gamma$. It follows that if Γ is a group of polynomial growth with the growth function γ, then there is a constant $C > 0$ and an integer $d \in \mathbb{Z}_{\geq 0}$, called the **degree of polynomial growth**, such that

$$\frac{1}{C} n^d \leq \gamma(n) \leq C n^d \quad \text{for all } n \in \mathbb{N}.$$

An even stronger result was obtained in [Pan83], where it is shown that, if Γ is a group of polynomial growth of degree $d \in \mathbb{Z}_{\geq 0}$, then the limit

$$c_\Gamma := \lim_{n \to \infty} \frac{\gamma(n)}{n^d} \tag{2.1}$$

exists. As a consequence, one can show that groups of polynomial growth are amenable.

Proposition 2.2. *Let Γ be a group of polynomial growth. Then $(\mathrm{B}(n))_{n \geq 1}$ is a Følner sequence in Γ.*

Proof. We want to show that for every $g \in \Gamma$

$$\lim_{n \to \infty} \frac{|g\mathrm{B}(n) \triangle \mathrm{B}(n)|}{|\mathrm{B}(n)|} = 0.$$

Let $m := d(g, e) \in \mathbb{Z}_{\geq 0}$. Then $g\mathrm{B}(n) \subseteq \mathrm{B}(n+m)$, hence

$$\frac{|g\mathrm{B}(n) \triangle \mathrm{B}(n)|}{|\mathrm{B}(n)|} \leq \frac{|\mathrm{B}(n+m)| - |\mathrm{B}(n)|}{|\mathrm{B}(n)|} \to 0,$$

where we use the existence of the limit in Equation (2.1). □

It will be useful later to have a special notion for the points which are 'close enough' to the boundary of a ball in Γ. Let $W := \mathrm{B}(y, s)$ be some ball in Γ. For a given $r \in \mathbb{R}_{>0}$ the r**-interior** of W is defined as

$$\mathrm{int}_r(W) := \mathrm{B}(y, (1 - 5/r)s).$$

The r**-boundary** of W is defined as

$$\partial_r(W) := W \setminus \mathrm{int}_r(W).$$

If a set $\mathcal{C}$ is a disjoint collection of balls in Γ, we define the r-interior and the r-boundary of $\mathcal{C}$ as

$$\operatorname{int}_r(\mathcal{C}) := \bigsqcup_{W\in\mathcal{C}} \operatorname{int}_r(W)$$

and

$$\partial_r(\mathcal{C}) := \bigsqcup_{W\in\mathcal{C}} \partial_r(W)$$

respectively. It will be essential to know that the r-boundary becomes small (respectively, the r-interior becomes large) for large enough balls and large enough r. More precisely, we state the following lemma, whose proof follows from the result of Pansu (see Equation (2.1)).

Lemma 2.3. *Let Γ be a group of polynomial growth and $\delta \in (0,1)$ be some constant. Then there exist constants $n_0, r_0 \in \mathbb{N}$, depending only on Γ and δ, such that the following holds. If $\mathcal{C}$ is a finite collection of disjoint balls with radii greater than n_0, then for all $r > r_0$*

$$|\operatorname{int}_r(\mathcal{C})| > (1-\delta)\left|\bigsqcup_{W\in\mathcal{C}} W\right|$$

and

$$|\partial_r(\mathcal{C})| < \delta\left|\bigsqcup_{W\in\mathcal{C}} W\right|.$$

2.2. **Averages on Groups of Polynomial Growth and a Transference Principle.** We collect some useful results about averages on groups of polynomial growth in this subsection. At the end of the subsection we will discuss a transference principle, which will become essential later in Section 3. We start with a preliminary lemma, whose proof is straightforward.

Lemma 2.4. *Let f be a nonnegative function on a group of polynomial growth Γ. Let $\{B_1, \dots, B_k\}$ be some disjoint balls in Γ such that*

$$\mathbb{E}_{g\in B_i} f(g) > \beta \quad \text{for each } i = 1, \dots, k.$$

Let B be a ball in Γ, containing all B_i's, such that

$$\mathbb{E}_{g\in B} f(g) < \alpha.$$

Then

$$\frac{\sum_{i=1}^{k} |B_i|}{|B|} < \frac{\alpha}{\beta}.$$

We refine this result as follows.

Lemma 2.5. *Let $\varepsilon \in (0,1)$. There is $n_0 \in \mathbb{N}$, depending only on the group of polynomial growth Γ and ε, such that the following assertion holds. Given a nonnegative function f on Γ, the condition*

$$\mathbb{E}_{g\in \mathrm{B}(n)} f(g) > \beta \quad \text{and} \quad \mathbb{E}_{g\in \mathrm{B}(m)} f(g) < \alpha \tag{2.2}$$

for some $n_0 \le n < m$ and an interval $(\alpha, \beta) \subset \mathbb{R}_{>0}$ implies that

$$\frac{m}{n} > (1-\varepsilon)\left(\frac{\beta}{\alpha}\right)^{1/d}.$$

Proof. First of all, note that condition (2.2) implies that

$$\frac{|\mathrm{B}(m)|}{|\mathrm{B}(n)|} > \frac{\beta}{\alpha}$$

for *all* indexes $n < m$ (see the previous lemma). Using the result of Pansu (Equation (2.1)), we deduce that there is n_0 depending only on Γ and ε such that for all $n_0 \le n < m$ we have

$$\frac{m^d}{n^d} > (1-\varepsilon)^d \frac{|\mathrm{B}(m)|}{|\mathrm{B}(n)|}.$$

This implies that

$$\frac{m}{n} > (1-\varepsilon)\left(\frac{\beta}{\alpha}\right)^{1/d},$$

and the proof of the lemma is complete. □

Lemma 2.5 has the following straightforward corollary.

Corollary 2.6. *For a constant $\varepsilon \in (0,1)$ and a group of polynomial growth Γ let $n_0 := n_0(\varepsilon)$ be given by Lemma 2.5. Given a measure-preserving action of Γ on a probability space* X*, a nonnegative function f on X and $x \in X$, the condition that the sequence*

$$\left(\mathbb{E}_{g\in \mathrm{B}(i)} f(g\cdot x)\right)_{i=n}^{m}$$

fluctuates at least k times across an interval $(\alpha,\beta) \subset \mathbb{R}_{>0}$ with $n > n_0$ implies that

$$\frac{m}{n} > (1-\varepsilon)^{\lceil \frac{k}{2}\rceil}\left(\frac{\beta}{\alpha}\right)^{\lceil \frac{k}{2}\rceil\cdot \frac{1}{d}}$$

Finally, we will need an adapted version of the 'easy direction' in Calderón's transference principle for groups of polynomial growth. Suppose that a group Γ of polynomial growth acts on a probability space $\mathrm{X} = (X,\mathcal{B},\mu)$ by measure-preserving transformations and that we want to estimate the size of a measurable set E. Fix an integer $m \in \mathbb{Z}_{\ge 0}$. For an integer $L \in \mathbb{N}$ and a point $x \in X$ we define the set

$$B_{L,m,x} := \{g : \ g\cdot x \in E \text{ and } \|g\| \le L-m\} \subseteq \mathrm{B}(L).$$

The lemma below tells us that each universal upper bound on the density of $B_{L,m,x}$ in $\mathrm{B}(L)$ bounds the measure of E from above as well.

Lemma 2.7 (Transference principle)**.** *Suppose that for a given constant $t \in \mathbb{R}_{\ge 0}$ the following holds: there is some $L_0 \in \mathbb{N}$ such that for all $L \ge L_0$ and for μ-almost all $x \in X$ we have*

$$\frac{1}{|\mathrm{B}(L)|}\,|B_{L,m,x}| \le t.$$

Then

$$\mu(E) \le t.$$

Proof. Indeed, since Γ acts on X by measure-preserving transformations, we have

$$\sum_{g\in \mathrm{B}(L)} \int_{\mathrm{X}} \mathbf{1}_E(g\cdot x)d\mu = |\mathrm{B}(L)|\,\mu(E).$$

Then

$$\mu(E) = \int_{\mathrm{X}} \left(\frac{1}{|\mathrm{B}(L)|}\sum_{g\in \mathrm{B}(L)} \mathbf{1}_E(g\cdot x)\right)d\mu \le$$
$$\le \int_{\mathrm{X}} \left(\frac{|B_{L,m,x}| + |\mathrm{B}(L)\setminus \mathrm{B}(L-m)|}{|\mathrm{B}(L)|}\right)d\mu,$$

and the proof is complete since L can be arbitrarily large and Γ is a group of polynomial growth. □

2.3. **Vitali Covering Lemma.** In this section we discuss the generalization of Effective Vitali Covering lemma from [KW99] to groups of polynomial growth. We fix some notation first. Given a number $t \in \mathbb{R}_{\geq 0}$ and a ball $B = \mathrm{B}(x, r) \subseteq \mathrm{X}$ in a metric space X, we denote by $t \cdot B$ the t-enlargement of B, i.e., the ball $\mathrm{B}(x, rt)$. We state the basic finitary Vitali covering lemma first, whose proof is well-known.

Lemma 2.8. *Let $\mathcal{B} := \{B_1, \dots, B_n\}$ be a finite collection of balls in a metric space X. Then there is a finite subset $\{B_{j_1}, \dots, B_{j_m}\} \subseteq \mathcal{B}$ consisting of pairwise disjoint balls such that*

$$\bigcup_{i=1}^{n} B_i \subseteq \bigcup_{l=1}^{m} 3 \cdot B_{j_l}.$$

Infinite version of this lemma is used, for example, in the proof of the standard Vitali covering theorem, which can be generalized to arbitrary doubling measure spaces. However, the standard Vitali covering theorem is not sufficient for our purposes. It was shown in [KW99] that the groups $\mathbb{Z}^d$ for $d \in \mathbb{N}$, which are of course doubling measure spaces when endowed with the counting measure and the word metric, enjoy a particularly useful 'effective' version of the theorem. We prove a generalization of this result to groups of polynomial growth below.

Theorem 2.9 (Effective Vitali covering)**.** *Let Γ be a group of polynomial growth of degree d. Let $C \geq 1$ be a constant such that*

$$\frac{1}{C} m^d \leq \gamma(m) \leq C m^d \quad \text{for all } m \in \mathbb{N}$$

and let $c := 3^d C^2$. Let $R, n, r > 2$ be some fixed natural numbers and $X \subseteq \mathrm{B}(R)$ be a subset of the ball $\mathrm{B}(R) \subset \Gamma$. Suppose that to each $p \in X$ there are associated balls $A_1(p), \dots, A_n(p)$ such that the following assertions hold:

(a) $p \in A_i(p) \subseteq \mathrm{B}(R)$ *for $i = 1, \dots, n$;*

(b) *For all $i = 1, \dots, n-1$ the r-enlargement of $A_i(p)$ is contained in $A_{i+1}(p)$.*

Let

$$S_i := \bigcup_{p \in X} A_i(p) \quad (i = 1, \dots, n).$$

There is a disjoint subcollection $\mathcal{C}$ of $\{A_i(p)\}_{p \in X, i=1,\dots,n}$ such that the following conclusions hold:

(a) *The union of $\left(1 + \frac{4}{r-2}\right)$-enlargements of balls in $\mathcal{C}$ together with the the set $S_n \setminus S_1$ covers all but at most $\left(\frac{c-1}{c}\right)^n$ of S_n;*

(b) *The measure of the union of $\left(1 + \frac{4}{r-2}\right)$-enlargements of balls in $\mathcal{C}$ is at least $\left(1 - \left(\frac{c-1}{c}\right)^n\right)$ times the measure of S_1.*

Remark 2.10. Prior to proceeding to the proof of the theorem we make the following remarks. Firstly, we do not require the balls $A_i(p)$ from the theorem to be centered around p. Secondly, the balls of the form $A_i(p)$ for $i = 1, \dots, n$ and $p \in X$ will be called ***i*-th level balls**. An i-th level ball $A_i(p)$ is called **maximal** if it is not contained in any other i-th level ball. It is clear that each S_i is the union of maximal i-level balls as well. It will follow from the proof below that the balls in $\mathcal{C}$ can be chosen to be maximal.

Proof. To simplify the notation, let

$$s := 1 + \frac{4}{r-2}$$

be the scaling factor that is used in the theorem. The main idea of the proof is to cover a positive fraction of S_n by a disjoint union of n-level balls via Lemma

2.8, then cover a positive fraction of what remains in S_{n-1} by a disjoint union of $(n-1)$-level balls and so on. Thus we begin by covering a fraction of S_n by n-level balls. Let $\mathcal{C}_n \subseteq \{A_n(p)\}_{p\in X}$ be the collection of disjoint balls, obtained by applying Lemma 2.8 to the collection of all n-th level *maximal* balls. For every ball $B = \mathrm{B}(p,m) \in \mathcal{C}_n$ we have

$$|3\cdot B| \le C(3m)^d \le C^2 3^d\,|B|\,,$$

hence

$$|S_n| \le \left|\bigcup_{B\in\mathcal{C}_n} 3\cdot B\right| \le \sum_{B\in\mathcal{C}_n} c\,|B|$$

and so

$$\left|\bigsqcup_{B\in\mathcal{C}_n} B\right| \ge \frac{1}{c}\,|S_n|\,.$$

Let $U_n := \bigsqcup_{B\in\mathcal{C}_n} B$. The computation above shows that

$$U_n \text{ covers at least } \frac{1}{c}\text{-fraction of } S_n \tag{2.3}$$

and

$$|S_1| - |U_n| \le |S_1| - \frac{1}{c}\,|S_1| = \frac{c-1}{c}\,|S_1|\,. \tag{2.4}$$

We proceed by restricting to $(n-1)$-level balls. Assume for the moment that the following claim is true.

Claim 1. *If a ball $A_{n-1}(p)$ has a nonempty intersection with U_n, then $A_{n-1}(p)$ is contained in the s-enlargement of the ball in $\mathcal{C}_n$ that it intersects.*

Let

$$\widetilde{\mathcal{C}}_{n-1} := \{A_{n-1}(p) : A_{n-1}(p) \text{ is a maximal } (n-1)-\text{level ball}$$
$$\text{such that } A_{n-1}(p)\cap U_n = \varnothing\}$$

be the collection of all maximal $(n-1)$-level balls disjoint from U_n and let $\widetilde{U}_{n-1}$ be its union. We apply Lemma 2.8 once again to obtain a collection $\mathcal{C}_{n-1} \subseteq \widetilde{\mathcal{C}}_{n-1}$ of pairwise disjoint maximal balls such that

$$\left|\bigsqcup_{B\in\mathcal{C}_{n-1}} B\right| \ge \frac{1}{c}\left|\widetilde{U}_{n-1}\right|.$$

Let $U_{n-1} := \bigsqcup_{B\in\mathcal{C}_{n-1}} B$. In order to show that

$$|S_1| - \left|\bigcup_{B\in\mathcal{C}_n} (s\cdot B)\cup U_{n-1}\right| \le \left(\frac{c-1}{c}\right)^2 |S_1| \tag{2.5}$$

it suffices to prove that

$$\left|\bigcup_{B\in\mathcal{C}_n} (s\cdot B)\cup U_{n-1}\right| \ge |U_n| + \frac{1}{c}\,|S_{n-1}\setminus U_n|\,, \tag{2.6}$$

due to the obvious inequalities

$$|S_{n-1}\setminus U_n| \ge |S_{n-1}| - |U_n| \ge |S_1| - |U_n|\,,$$
$$|U_n| \ge \frac{1}{c}\,|S_1|\,.$$

We decompose the set $S_{n-1}\setminus U_n$ as follows

$$S_{n-1}\setminus U_n = \widetilde{U}_{n-1} \sqcup \left(S_{n-1}\setminus (U_n\cup \widetilde{U}_{n-1})\right).$$

The part $S_{n-1} \setminus (U_n \cup \widetilde{U}_{n-1})$ is covered by the $(n-1)$-level balls intersecting U_n. Hence, if Claim 1 above is true, the set $S_{n-1} \setminus (U_n \cup \widetilde{U}_{n-1})$ is covered by the s-enlargements of balls in $\mathcal{C}_n$. Next, U_{n-1} covers at least $\frac{1}{c}$ fraction of $\widetilde{U}_{n-1}$. It follows that the set $\bigcup\limits_{B\in\mathcal{C}_n} (s \cdot B) \cup U_{n-1}$ covers the set U_n and at least $\frac{1}{c}$-fraction of the set $S_{n-1} \setminus U_n$. Thus we have proved inequalities (2.6) and (2.5). A similar argument shows that

$$\bigcup_{B\in\mathcal{C}_n} (s \cdot B) \cup \bigcup_{B\in\mathcal{C}_{n-1}} (s \cdot B) \cup (S_n \setminus S_{n-1}) \text{ covers all but} \tag{2.7}$$

$$\text{at most } \left(1 - \frac{1}{c}\right)^2 \text{ of } S_n.$$

Comparing Equations (2.7) and (2.5) to the statements (a) and (b) of the theorem, we see that the proof would be complete apart from Claim 1 if n was equal to 2.

So we proceed further to $(n-2)$-level balls and use the following claim.

Claim 2. *If a ball $A_{n-2}(p)$ has a nonempty intersection with $U_n \cup U_{n-1}$, then $A_{n-2}(p)$ is contained in the s-enlargement of the ball in $\mathcal{C}_n \cup \mathcal{C}_{n-1}$ that it intersects.*

We let $\mathcal{C}_{n-2}$ be the collection of all maximal $(n-2)$-level balls disjoint from $U_n \cup U_{n-1}$ and let $\widetilde{U}_{n-2}$ be its union. We apply Lemma 2.8 once again to obtain a collection $\mathcal{C}_{n-2} \subseteq \widetilde{\mathcal{C}}_{n-2}$ of pairwise disjoint balls such that

$$\left| \bigsqcup_{B\in\mathcal{C}_{n-2}} B \right| \geq \frac{1}{c} \left| \widetilde{U}_{n-2} \right|$$

and let $U_{n-2} := \bigsqcup\limits_{B\in\mathcal{C}_{n-2}} B$. Similar arguments show that

$$|S_1| - \left| \bigcup_{B\in\mathcal{C}_n} (s \cdot B) \cup \bigcup_{B\in\mathcal{C}_{n-1}} (s \cdot B) \cup U_{n-2} \right| \leq \left(\frac{c-1}{c}\right)^3 |S_1|$$

and that the union of s-enlargements of balls in $\mathcal{C}_n$, $\mathcal{C}_{n-1}$ and $\mathcal{C}_{n-2}$, together with $S_n \setminus S_{n-2}$, covers all but at most $\left(1 - \frac{1}{c}\right)^3$ of S_n.

It is obvious that one can continue in this way down to the 1-st level balls, using the obvious generalization of Claim 2. This would yield a collection of maximal balls

$$\mathcal{C} := \bigcup_{i=1}^{n} \mathcal{C}_i$$

so that the union of s-enlargements of balls in $\mathcal{C}$ together with $S_n \setminus S_1$ covers all but most $\left(1 - \frac{1}{c}\right)^n$ of S_n and that the measure of the union of these s-enlargements is at least $\left(1 - \left(1 - \frac{1}{c}\right)^n\right)$ times the measure of S_1.

We conclude that the proof is complete once we prove the claims above and their generalizations. For this it suffices to prove the following statement:

Claim 3. *If $1 \leq i < j \leq n$ and $A_j(q)$ is a maximal ball, then for all $p \in X$*

$$A_i(p) \cap A_j(q) \neq \varnothing \Rightarrow A_i(p) \subseteq s \cdot A_j(q).$$

Suppose this is not the case. Let x, y be the centers and r_1, r_2 be the radii of $A_i(p)$ and $A_j(q)$ respectively. Recall that $s = 1 + \frac{4}{r-2}$. Since the s-enlargement of $A_j(q)$ does not contain $A_i(p)$, it follows $\frac{4r_2}{r-2} \leq 2r_1$, hence

$$rr_1 \geq 2r_1 + 2r_2.$$

The intersection of $A_i(p)$ and $A_j(q)$ is nonempty, hence $d \leq r_1 + r_2$. This implies that

$$rr_1 \geq d + r_1 + r_2,$$

so the r-enlargement of the ball $A_i(p)$ contains $A_j(q)$. Since $r \cdot A_i(p) \subseteq A_{i+1}(p)$, we conclude that the ball $A_j(q)$ is not maximal. Contradiction.

□

Corollary 2.11. *Suppose that in addition to all the assumptions of Theorem 2.9 we have*

$$|S_n| \leq (c+1)\,|S_1|\,,$$

where c is the constant defined in Theorem 2.9. Then there is a disjoint subcollection $\mathcal{C}$ of maximal balls such the union of $\left(1+\frac{4}{r-2}\right)$-enlargements of balls in $\mathcal{C}$ covers at least $\left(1-(c+1)\left(\frac{c-1}{c}\right)^n\right)$ of S_1.

Proof. From the proof of Theorem 2.9 it follows that one can find a disjoint collection $\mathcal{C}$ of maximal balls satisfying assertions (a) and (b) of the theorem. The statement of the corollary is an easy consequence of (a). □

As the main application we will use the corollary above in the proof of Theorem 3.1. It will be essential to know that one can ensure that the extra $\left(1+\frac{4}{r-2}\right)$-enlargement does change the size of the union of the balls too much.

Lemma 2.12. *Let Γ be a group of polynomial growth and $\delta \in (0,1)$ be some constant. Then there exist integers $n_0, r_0 > 2$, depending only on Γ and δ, such that the following assertion holds.*

If $\mathcal{C}$ is a finite collection of disjoint balls with radii greater than n_0, then for all $r \geq r_0$ we have

$$\left|\bigsqcup_{W\in\mathcal{C}} W\right| \geq (1-\delta)\left|\bigcup_{W\in\mathcal{C}}\left(1+\frac{4}{r-2}\right)\cdot W\right|.$$

The proof of the lemma follows from the result of Pansu (see Equation (2.1)).

3. Fluctuations of Averages of Nonnegative Functions

The purpose of this section is to prove the following theorem.

Theorem 3.1. *Let Γ be a group of polynomial growth of degree $d \in \mathbb{Z}_{\geq 0}$ and let $(\alpha,\beta) \subset \mathbb{R}_{>0}$ be some nonempty interval. Then there are some constants $c_1, c_2 \in \mathbb{R}_{>0}$ with $c_2 < 1$, which depend only on Γ, α and β, such that the following assertion holds.*

For any probability space $\mathrm{X} = (X, \mathcal{B}, \mu)$, any measure-preserving action of Γ on X and any measurable $f \geq 0$ on X we have

$$\mu(\{x : (\mathbb{E}_{g\in \mathrm{B}(k)} f(g\cdot x))_{k\geq 1} \in \mathcal{F}^N_{(\alpha,\beta)}\}) < c_1 c_2^N$$

for all $N \geq 1$.

To simplify the presentation we use the adjective **universal** to talk about constants determined by Γ and (α,β). When a constant c is determined by $\Gamma, (\alpha,\beta)$ and a parameter δ, we say that c is **δ-universal**. Prior to proceeding to the proof of Theorem 3.1, we make some straightforward observations.

Remark 3.2. It easy to see how one can generalize the theorem above for arbitrary functions bounded from below. If a measurable function f on X is greater than $-m$ for some constant $m \in \mathbb{R}_{\geq 0}$, then

$$\mu(\{x : (\mathbb{E}_{g\in \mathrm{B}(k)} f(g\cdot x))_{k\geq 1} \in \mathcal{F}^N_{(\alpha,\beta)}\}) < \widetilde{c}_1 \widetilde{c}_2^N,$$

where the constants $\widetilde{c}_1, \widetilde{c}_2$ are given by applying Theorem 3.1 to the function $f+m$ and the interval $(\alpha+m, \beta+m)$.

Remark 3.3. Recall that $\gamma : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is a growth function of a group Γ. Let $C \geq 1$ be a constant such that

$$\frac{1}{C} r^d \leq \gamma(r) \leq C r^d \quad \text{ for all } r \in \mathbb{N}$$

and let $c := 3^d C^2$. Then it suffices to prove Theorem 3.1 only for intervals (α, β) such that

$$\frac{\beta}{\alpha} \leq \frac{c+1}{c}.$$

If the interval does not satisfy this condition, we replace it with a sufficiently small subinterval and apply Theorem 2.9. The importance of this observation will be apparent later.

Remark 3.4. Instead of proving the original assertion of Theorem 3.1, we will prove the following weaker assertion, which is clearly sufficient to deduce Theorem 3.1.

There is a universal integer $\widetilde{N}_0 \in \mathbb{N}$ such that for any probability space $\mathrm{X} = (X, \mathcal{B}, \mu)$, any measure-preserving action of Γ on X and any measurable $f \geq 0$ on X we have

$$\mu(\{x : (\mathbb{E}_{g \in \mathrm{B}(k)} f(g \cdot x))_{k \geq 1} \in \mathcal{F}^N_{(\alpha,\beta)}\}) < c_1 c_2^N$$

for all $N \geq \widetilde{N}_0$.

The upcrossing inequalities given by Theorem 3.1 and Remark 3.2 allow for a short proof of the pointwise ergodic theorem on L^∞ for actions of groups of polynomial growth.

Theorem 3.5. *Let Γ be a group of polynomial growth acting on a probability space $\mathrm{X} = (X, \mathcal{B}, \mu)$ by measure-preserving transformations. Then for every $f \in \mathrm{L}^\infty(\mathrm{X})$ the limit*

$$\lim_{n \to \infty} \mathbb{E}_{g \in \mathrm{B}(n)} f(g \cdot x)$$

exists almost everywhere.

Proof. Let

$$X_0 := \{x \in X : \lim_{n \to \infty} \mathbb{E}_{g \in \mathrm{B}(n)} f(g \cdot x) \text{ does not exist}\}$$

be the set of the points in X where the ergodic averages do not converge. Let $((\alpha_i, \beta_i))_{i \geq 1}$ be a sequence of nonempty intervals such that each nonempty interval $(c, d) \subset \mathbb{R}$ contains some interval (a_i, b_i). Then it is clear that if $x \in X_0$, then there is some interval (a_i, b_i) such that the sequence of averages $\left(\mathbb{E}_{g \in \mathrm{B}(n)} f(g \cdot x)\right)_{n \geq 1}$ fluctuates over (a_i, b_i) infinitely often, i.e.,

$$X_0 \subseteq \{x \in X : \left(\mathbb{E}_{g \in \mathrm{B}(n)} f(g \cdot x)\right)_{n \geq 1} \in \bigcup_{i \geq 1} \bigcap_{k \geq 1} \mathcal{F}^k_{(a_i, b_i)}\}.$$

By Theorem 3.1 and Remark 3.2 we have for every interval (a_i, b_i) that

$$\mu(\{x \in X : \left(\mathbb{E}_{g \in \mathrm{B}(n)} f(g \cdot x)\right)_{n \geq 1} \in \bigcap_{k \geq 1} \mathcal{F}^k_{(a_i, b_i)}\}) = 0,$$

hence $\mu(X_0) = 0$ and the proof is complete. □

We now begin the proof of Theorem 3.1, namely we will prove the assertion in Remark 3.4. Assume from now on that the group Γ of polynomial growth of degree $d \in \mathbb{Z}_{\geq 0}$ and the interval $(\alpha, \beta) \subset \mathbb{R}_{>0}$ are *fixed*.

Given a measure-preserving action of Γ on a probability space $\mathrm{X} = (X, \mathcal{B}, \mu)$, let

$$E_N := \{x : \left(\mathbb{E}_{g \in \mathrm{B}(k)} f(g \cdot x)\right)_{k \geq 1} \in \mathcal{F}^N_{(\alpha,\beta)}\}$$

be the set of all points $x \in X$ where the ergodic averages fluctuate at least $N \geq \widetilde{N}_0$ times across the interval (α, β). Here $\widetilde{N}_0$ is a universal constant, which will be determined later. For $m \geq 1$ define, furthermore, the set

$$E_{N,m} := \{x : \left(\mathbb{E}_{g \in \mathrm{B}(k)} f(g \cdot x)\right)_{k=1}^{m} \in \mathcal{F}_{(\alpha,\beta)}^{N}\}$$

of all points such that the finite sequence $\left(\mathbb{E}_{\mathrm{B}(k)} f(g \cdot x)\right)_{k=1}^{m}$ fluctuates at least N times across (α, β). Then, clearly, $(E_{N,m})_{m \geq 1}$ is a monotone increasing sequence of sets and

$$E_N = \bigcup_{m \geq N} E_{N,m}.$$

We will complete the proof by giving a universal estimate for $\mu(E_{N,m})$ for all $m \geq N$. For that we use the transference principle (Lemma 2.7), i.e., for an integer $L > m$ and a point $x \in X$ we let

$$B_{L,m,x} := \{g : \; g \cdot x \in E_{N,m} \text{ and } \|g\| \leq L - m\}.$$

The goal is to show that the density of the set

$$B_0 := B_{L,m,x} \subset \mathrm{B}(L)$$

can be estimated by $c_1 c_2^N$ for some universal constants c_1, c_2. The main idea is as follows. For every point $z \in B_0$ the sequence of averages

$$k \mapsto \mathbb{E}_{g \in \mathrm{B}(k)} f((gz) \cdot x), \quad k = 1, \ldots, m$$

fluctuates at least N times. Since the word metric $d = d_R$ on Γ is right-invariant, the set $\mathrm{B}(k)z$ is in fact a ball of radius k centered at z for each $k = 1, \ldots, m$. Given a parameter $\delta \in (0, 1 - \sqrt{\alpha/\beta})$, we will pick some of these balls and apply effective Vitali covering theorem (Theorem 2.9) multiple times to replace B_0 by a sequence

$$B_1, B_2, \ldots, B_{\lfloor (N-N_0)/T \rfloor}$$

of subsets of $\mathrm{B}(L)$ for some δ-universal integers $T, N_0 \in \mathbb{N}$ which satisfies the assumption

$$B_{2i+1} \text{ covers at least } (1-\delta) - \text{fraction of } B_{2i} \quad \text{for all indices } i \geq 0 \tag{3.1}$$

at 'odd' steps and the assumption

$$|B_{2i}| \geq \frac{\beta}{\alpha}(1-\delta)\,|B_{2i-1}| \quad \text{for all indexes } i \geq 1 \tag{3.2}$$

at 'even' steps. Each B_i is, furthermore, a union

$$\bigsqcup_{B \in \mathcal{C}_i} B$$

of some family $\mathcal{C}_i$ of disjoint balls with centers in B_0. If such a sequence of sets $B_1, \ldots, B_{\lfloor (N-N_0)/T \rfloor}$ exists, then

$$|\mathrm{B}(L)| \geq \left|B_{\lfloor (N-N_0)/T \rfloor}\right| \geq \left(\frac{\beta}{\alpha}(1-\delta)^2\right)^{\lfloor \frac{N-N_0}{2T} \rfloor} |B_0|\,,$$

which gives the required exponential bound on the density of B_0 with

$$c_2 := \left(\frac{\alpha}{\beta}(1-\delta)^{-2}\right)^{1/2T}$$

and a suitable δ-universal c_1. To ensure that conditions (3.1) and (3.2) hold, one has to pick sufficiently large δ-universal parameters r and n for the effective Vitali covering theorem. We make it precise at the end of the proof, for now we assume that r, n are 'large enough'.

In order to force the sufficient growth rate of the balls (condition (b) of Theorem 2.9), we employ the following argument. Let $K > 0$ be the smallest integer such that

$$\left(1 - \frac{1 - (\alpha/\beta)^{1/d}}{2}\right)^{\lceil \frac{K}{2} \rceil} \left(\frac{\beta}{\alpha}\right)^{\lceil \frac{K}{2} \rceil \cdot \frac{1}{d}} \geq r.$$

Then, applying Corollary 2.6, we obtain a universal integer $n_0 \in \mathbb{N}$ such that if a sequence

$$(\mathbb{E}_{g \in B(i)} f((gz) \cdot x))_{i=n}^{m} \quad \text{for some } n > n_0, z \in B_0$$

fluctuates at least K times across the interval (α, β), then

$$\frac{m}{n} > \left(1 - \frac{1 - (\alpha/\beta)^{1/d}}{2}\right)^{\lceil \frac{K}{2} \rceil} \left(\frac{\beta}{\alpha}\right)^{\lceil \frac{K}{2} \rceil \cdot \frac{1}{d}} \geq r. \tag{3.3}$$

Let n be large enough for use in effective Vitali covering theorem. We define $T := 2nK$ and let $N_0 \geq n_0$ be sufficiently large (this will be made precise later). The first N_0 fluctuations are skipped to ensure that the balls have large enough radius, and the rest are divided into $\lfloor (N - N_0)/T \rfloor$ groups of T consecutive fluctuations. The i-th group of consecutive fluctuations is used to construct the set B_i for $i = 1, \ldots, \lfloor (N - N_0)/T \rfloor$ as follows. We distinguish between the 'odd' and the 'even' steps.

Odd step: First, let us describe the procedure for odd i's. For each point $z \in B_{i-1}$ we do the following. By induction we assume that $z \in B_{i-1}$ belongs to some unique ball $\mathrm{B}(u, s)$ from $(i-1)$-th step with $u \in B_0$. If $i = 1$, then $z \in B_0$. Let $A_1(z)$ be the $(K+1)$-th ball $\mathrm{B}(u, s_1)$ in the i-th group of fluctuations such that

$$\mathbb{E}_{g \in A_1(z)} f(g \cdot x) > \beta,$$

$A_2(z)$ be the $(2K+1)$-th ball $\mathrm{B}(u, s_2)$ in the i-th group of fluctuations such that

$$\mathbb{E}_{g \in A_2(z)} f(g \cdot x) > \beta$$

and so on up to $A_n(z)$. It is clear that the r-enlargement of $A_j(z)$ is contained in $A_{j+1}(z)$ for all indexes $j < n$ and that the balls defined in this manner are contained in $\mathrm{B}(L)$. Thus the assumptions of Theorem 2.9 are satisfied. There are two further possibilities: either this collection satisfies the additional assumption in Corollary 2.11, i.e.,

$$|S_n| \leq (c+1)\,|S_1| \tag{3.4}$$

or not. If (3.4) holds, then by the virtue of Corollary 2.11 we obtain a disjoint collection $\mathcal{C}$ of maximal balls such that the measure of the union of $\left(1 + \frac{4}{r-2}\right)$-enlargements of balls in $\mathcal{C}$ covers at least $\left(1 - (c+1)\left(\frac{c-1}{c}\right)^n\right)$ of S_1. We let

$$B_i := \bigsqcup_{B \in \mathcal{C}} B$$

and $\mathcal{C}_i := \mathcal{C}$. Condition (3.1) is satisfied if r and n are large enough, and we proceed to the following 'even' step. If, on the contrary,

$$|S_n| > (c+1)\,|S_1|,$$

then we apply the standard Vitali covering lemma to the collection of maximal n-th level balls and obtain a disjoint subcollection $\mathcal{C}$ such that

$$\left|\bigsqcup_{B \in \mathcal{C}} B\right| \geq \frac{1}{c}\,|S_n| > \frac{c+1}{c}\,|S_1| \tag{3.5}$$

We assume without loss of generality that $\frac{\beta}{\alpha} \leq \frac{c+1}{c}$ (see Remark 3.3). We let

$$B_i := B_{i-1},$$
$$B_{i+1} := \bigsqcup_{B\in\mathcal{C}} B$$

and

$$\mathcal{C}_i := \mathcal{C}_{i-1},$$
$$\mathcal{C}_{i+1} := \mathcal{C}.$$

The conditions (3.1), (3.2) are satisfied and we proceed to the next 'odd' step.
Even step: We now describe the procedure for even i's. For each point $z \in B_{i-1}$ we do the following. By induction we assume that $z \in B_{i-1}$ belongs to some unique ball $\mathrm{B}(u,s)$ from $(i-1)$-th step with $u \in B_0$. Let $A_1(z)$ be the $(K+1)$-th ball $\mathrm{B}(u,s_1)$ in the i-th group of fluctuations such that

$$\mathbb{E}_{g\in A_1(z)} f(g\cdot x) < \alpha,$$

$A_2(z)$ be the $(2K+1)$-th ball $\mathrm{B}(u,s_2)$ in the i-th group of fluctuations such that

$$\mathbb{E}_{g\in A_2(z)} f(g\cdot x) < \alpha$$

and so on up to $A_n(z)$. It is clear that the r-enlargement of $A_j(z)$ is contained in $A_{j+1}(z)$ for all indexes $j < n$ and that the balls defined in this manner are contained in $\mathrm{B}(L)$. Thus the assumptions of Theorem 2.9 are satisfied. There are two further possibilities: either this collection satisfies the additional assumption in Corollary 2.11, i.e.,

$$|S_n| \leq (c+1)\,|S_1| \tag{3.6}$$

or not. If

$$|S_n| > (c+1)\,|S_1|\,,$$

then we apply the standard Vitali covering lemma to the collection of maximal n-th level balls and obtain a disjoint subcollection $\mathcal{C}$ such that

$$\left|\bigsqcup_{B\in\mathcal{C}} B\right| \geq \frac{1}{c}\,|S_n| > \frac{c+1}{c}\,|S_1| \tag{3.7}$$

We assume without loss of generality that $\frac{\beta}{\alpha} \leq \frac{c+1}{c}$ (see Remark 3.3). We let

$$B_i := \bigsqcup_{B\in\mathcal{C}} B$$

and proceed to the following 'odd' step. If (3.6) holds, then by the virtue of Corollary 2.11 we obtain a disjoint collection $\mathcal{C}$ of maximal balls such that the measure of the union of $\left(1+\frac{4}{r-2}\right)$-enlargements of balls in $\mathcal{C}$ covers at least $\left(1-(c+1)\left(\frac{c-1}{c}\right)^n\right)$ of S_1. We let

$$B_i := \bigsqcup_{B\in\mathcal{C}} B$$

and $\mathcal{C}_i := \mathcal{C}$. The goal is to prove that condition (3.2) is satisfied. If the balls from $\mathcal{C}_{i-1}$ were completely contained in the balls from $\mathcal{C}_i$, the proof would be completed by applying Lemma 2.4. This, in general, might not be the case, so we argue as follows. First, we prove the following lemma.

Lemma 3.6. *If a ball W_1 from $\mathcal{C}_{i-1}$ intersects $\mathrm{int}_r(W_2)$ for some ball $W_2 \in \mathcal{C}_i$, then $W_1 \subseteq W_2$.*

Proof. Let $W_1 = \mathrm{B}(y_1, s_1)$ and $W_2 = \mathrm{B}(y_2, s_2)$ for some $y_1, y_2 \in B_0$. Since W_1 intersects $\mathrm{int}_r(W_2)$, we have

$$d(y_1, y_2) \leq s_2(1 - 5/r) + s_1.$$

If W_1 is not contained in W_2, then $d(y_1, y_2) > s_2 - s_1$. From these inequalities it follows that

$$s_1 \geq d(y_1, y_2) - s_2(1 - 5/r) > s_2 - s_1 - s_2 + \frac{5s_2}{r},$$

hence $s_2 < \frac{2rs_1}{5}$. We deduce that the r-enlargement of W_1 contains W_2. This is a contradiction since W_2 is maximal and the r-enlargement of W_1 is contained in n-th level ball $A_n(y_1)$. □

From the lemma above it follows that the set B_{i-1} can be decomposed as

$$B_{i-1} = \left(\bigsqcup_{W \in \mathcal{C}'_{i-1}} W \right) \sqcup (\partial_r(\mathcal{C}_i) \cap B_{i-1}) \sqcup (B_{i-1} \setminus B_i),$$

where

$$\mathcal{C}'_{i-1} := \{W \in \mathcal{C}_{i-1} : \ W \cap \mathrm{int}_r(V) \neq \varnothing \text{ for some } V \in \mathcal{C}_i\}.$$

The rest of the argument depends on how much of B_{i-1} is contained in $\partial_r(\mathcal{C}_i)$, so let

$$\Delta := \frac{|\partial_r(\mathcal{C}_i) \cap B_{i-1}|}{|B_{i-1}|}.$$

There are two possibilities. First, suppose that $\Delta > \frac{\delta}{3}$. Then $|B_{i-1}| \leq \frac{|\partial_r(\mathcal{C}_i)|}{\delta/3}$. Let r and the radii of the balls in $\mathcal{C}_i$ be large enough (see Lemma 2.3) so that

$$\frac{|\partial_r(\mathcal{C}_i)|}{|B_i|} < \frac{\alpha}{\beta}\frac{\delta}{3}(1-\delta)^{-1}.$$

It is then easy to see that condition (3.2) is satisfied. Suppose, on the other hand, that $\Delta \leq \frac{\delta}{3}$. Then, if n and r are large enough so that $|B_{i-1} \setminus B_i|$ is small compared to $|B_{i-1}|$, we obtain

$$\begin{aligned} |B_{i-1}| &\leq \frac{\alpha}{\beta} |B_i| + |\partial_r(\mathcal{C}_i) \cap B_{i-1}| + |B_{i-1} \setminus B_i| \leq \\ &\leq \frac{\alpha}{\beta} |B_i| + \frac{\delta}{3} |B_{i-1}| + \frac{\delta}{3} |B_{i-1}| , \end{aligned}$$

which implies that

$$|B_i| \geq \frac{\beta}{\alpha}(1 - \frac{2\delta}{3}) |B_{i-1}| ,$$

i.e., condition (3.2) is satisfied as well. We proceed to the following 'odd' step.

The proof of the theorem is essentially complete. To finish it we only need to say how one can choose the constants N_0, r, n and $\widetilde{N}_0$. Recall that $\delta \in (0, 1 - (\alpha/\beta)^{1/2})$ is an arbitrary parameter. First, the integer $n \in \mathbb{N}$ is chosen so that

$$(c+1)\left(1 - \frac{1}{c}\right)^n \leq 1 - \sqrt{1 - \delta/4}.$$

Next, we choose r as the maximum of

1) the integer r_0 given by Lemma 2.3 with the parameter $\frac{\alpha}{\beta}\frac{\delta}{3}(1-\delta)^{-1}$;
2) the integer r_0 given by Lemma 2.12 with the parameter $1 - \sqrt{1 - \delta/4}$.

The integer $K > 0$ is picked so that condition (3.3) is satisfied. We choose N_0 as the maximum of

1) the integer n_0 given by Lemma 2.3 with the parameter $\frac{\alpha}{\beta}\frac{\delta}{3}(1-\delta)^{-1}$;
2) the integer n_0 given by Lemma 2.12 with the parameter $1 - \sqrt{1 - \delta/4}$;

3) the integer n_0 given by Corollary 2.6 with the parameter $\frac{1-(\alpha/\beta)^{1/d}}{2}$;

Finally, we define $\widetilde{N}_0$ as $\widetilde{N}_0 := N_0 + 4nK + 1$. A straightforward computation shows that this choice of constants satisfies all requirements. We do not assert, however, that this choice yields *optimal* constants c_1 and c_2. □

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