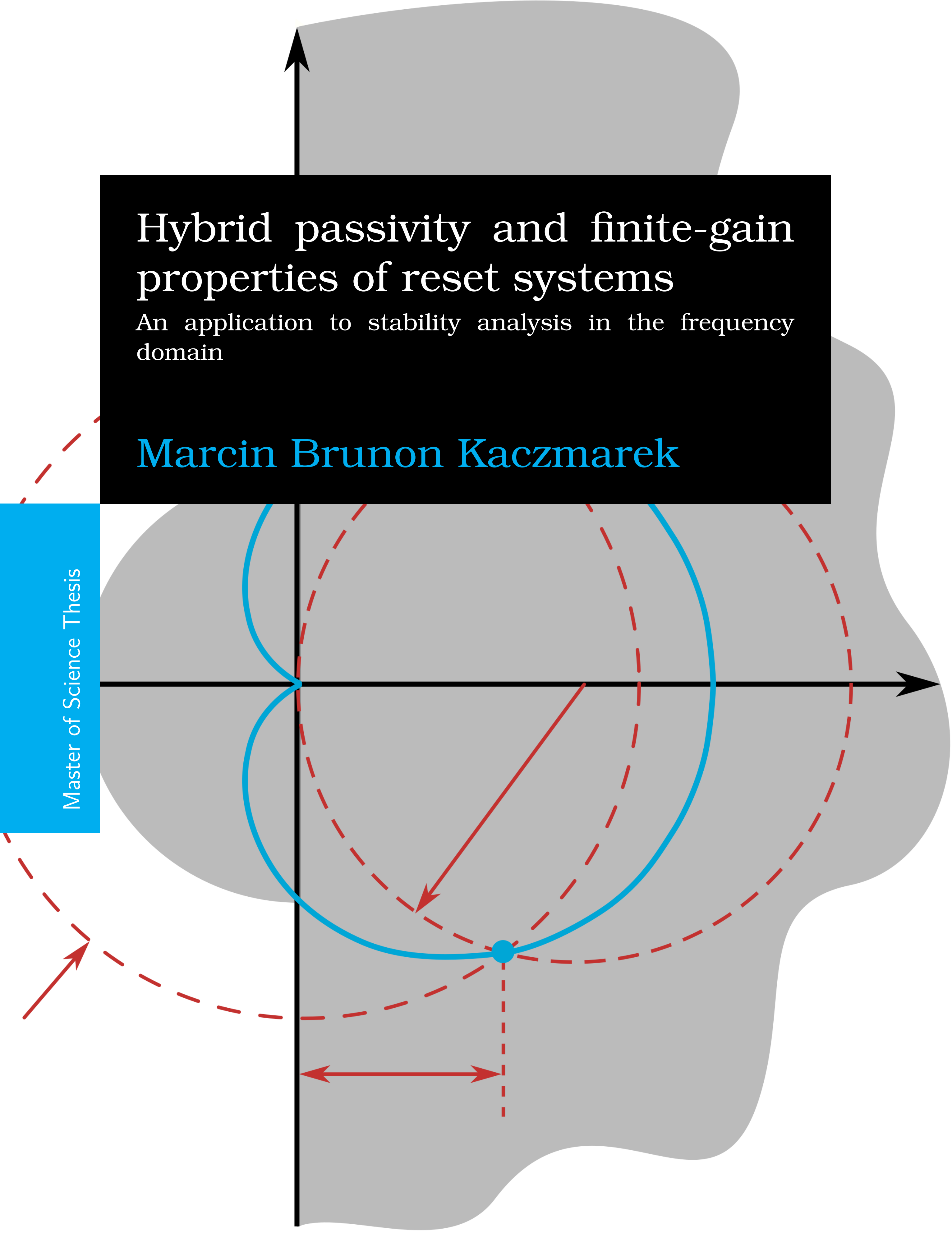
Hybrid passivity and finite-gain properties of reset systems

An application to stability analysis in the frequency domain

Marcin Brunon Kaczmarek

Master of Science Thesis



Delft University of Technology

Delft Center for Systems and Control
Precision and Microsystems Engineering

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Master of Science Thesis

For the individual double degree of Master of Science in Systems and Control and Mechanical Engineering at Delft University of Technology

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June 24, 2020

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Abstract

Reset control is a "simple" nonlinear control strategy that has the potential of being widely adopted and improving the performance of systems traditionally controlled with PIDs. Lack of suitable methods for proving stability, that are in line with the current industrial practice, hampers the wider acceptance of reset control. In this thesis, novel sufficient conditions for stability of reset control systems, that can be evaluated using measured frequency response function of a system to be controlled, are derived using the hybrid passivity and finite-gain framework. A method for analysing the hybrid passivity and finite-gain parameters of reset systems, that can be extended to other classes of nonlinear systems, is developed. Additionally, a variant of the “Constant in Gain Lead in Phase” reset element, that facilitates the use of the proposed method for the stability analysis, is introduced. Stability of several precision positioning systems with reset controllers, designed for different objectives, is studied to demonstrate the applicability of the proposed hybrid passivity and finite-gain approach for the stability analysis of reset control systems. Guidelines for design of reset systems such that their stability can be concluded using the hybrid passivity and finite gain method are shown. This thesis presents a new view on the stability of reset systems and addresses the need for frequency-domain tools for stability analysis of nonlinear control systems in precision mechatronics applications.

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Acknowledgements

This thesis is the final chapter of my Master studies at TU Delft. The last three years have been an exciting adventure during which I explored Systems and Control and Precision Engineering. My experience would not be as rich without the help and contribution of many people.

First, I would like to thank Hassan HosseinNia for guiding me through my studies and giving me the opportunity to contribute to the development of nonlinear control for precision mechatronics. Furthermore, I would like to thank Erik Steur. Meetings with you thought me to think and write more rigorously, and your feedback allowed me to improve this work. I want to thank Niranjan Saikumar for all the discussions that helped to shape this project and broaden my view on high-performance mechatronics. I also want to thank all the friends here in Delft who made my journey not only much more joyful but also more successful.

W szczególności, chciałbym podziękować mojej rodzinie i przyjaciołom. Dziękuję za wsparcie każdego z moich wyborów i motywację do działania. Dziękuję również za czas spędzany wspólnie zarówno osobiście, jak i online.

Delft, University of Technology — Marcin Brunon Kaczmarek
June 24, 2020

Chapter 1

Introduction

In this introductory chapter, the motivations behind this research are explained. The problems addressed in this project and the main contributions made are presented. The chapter is concluded with the outline of this report.

1-1 Research motivations

Moore's law is the observation that the number of transistors in a dense integrated circuit doubles about every two years [1]. The law has been generalized, and it is assumed that the exponential growth should also characterize the development of a wider range of technologies [2]. Chasing these expectations leads to ever-increasing requirements for precision and speed of production machines and scientific instruments enabling this progress.

An increase of speed can be achieved with the development of improved mechanical structures of machines and new actuators. The behaviour of the system can be closely predicted using new modelling techniques what leads to better feedforward control. Finally, robustness against disturbances and uncertainties in the system can be achieved with feedback control techniques based on the measurement of errors appearing in the system. In the past few decades, relatively little has changed in the feedback laws controlling the motion systems, while the other areas witnessed significant improvement.

The vast majority of industrial control systems is based on the PID controllers [3], which generate an input signal for the system that is proportional to the error between the desired and measured behaviour, its integral and derivative. It is a common practice in the high-tech industry to design the controllers using the frequency-domain techniques. These techniques can be used to illustrate the interplay between all the elements of the system and to precisely predict the performance.

The main drawback of the PIDs are the limitations like the magnitude-phase relationship and the Bode sensitivity integral, also known as the waterbed effect [4] [5]. These limitations are inherent to all linear control systems with the relative degree larger than one, and lead to tradeoffs that prevent the improvement of some performance indicators without deteriorating the others.

To overcome these limitations, nonlinear controllers have to be used. Many control strategies that can provide better performance than currently used linear systems exist, for example, model predictive control [6], sliding mode control[7] or impulsive control [8]. However, they are not widely adopted by industry because of design procedures that are complicated and not matching with current practice.

There is a need for "simple" nonlinear controllers for linear plants, that could alleviate tradeoffs typical for linear controllers, while being easy to implement and tune. This need has been addressed through i.a. development of variable gain integral controllers [9], split-path nonlinear integrators [10] [11] or reset controllers, which were first introduced by Clegg in 1958 [12]. These approaches can be combined to even further improve performance of a system [13].

This project is focused on reset control. A reset element is a linear time-invariant system whose states, or a subset of states, reset to values defined by a reset law if its input, state or output satisfy a specified condition [14]. It has been proven that reset systems can overcome limitations of linear controllers [15] [16]. Examples of applications of reset in various fields like process control or networked systems can be found in textbooks [14] [17] [18]. Moreover, reset elements have been successfully applied to control precision positioning systems [13] [19] [20] [21] [22] [23] [24].

One of the factors hampering the wider adoption of reset control is the lack of techniques for assessing the stability of reset systems that are in line with the current industrial practice. Stability of linear controllers is checked using the same frequency-domain tools that are used to predict the performance of the systems. The stability conditions can be evaluated using measured frequency response data. In the case of the reset systems, the stability analysis requires a parametric model of the plant[1] [14] [17]. This complicates the design procedure.

1-2 Problem statement

The goal of this thesis is to develop conditions for assessing the stability of reset control systems that can be evaluated using the measured frequency response of the system to be controlled. The method of choice is to represent the control system as a feedback interconnection of the reset element and a linear block, consisting of the plant and remaining components of the controller. This approach should simplify the process since well-established techniques can be used to analyse the linear components and the analysis of reset elements is simpler than that of the closed-loop reset systems. In the next step, the stability of the feedback system can be concluded using appropriate tools.

The stability of precision mechatronic systems can rarely be assured using the passivity or finite-gain techniques [26]. To accommodate this problem, the *hybrid passivity and finite-gain* approach is used. Note, that the term *hybrid* does not refer here to the dynamics of the system but to the blending of the passivity and finite-gain concepts. The idea of blending the passivity and finite-gain properties have been introduced in [27][28] for linear systems,

[1] The only known stability conditions for reset systems, that can be evaluated using the measured FRF of the plant, have been developed for specific classes of reset systems in [19] and [25]

extended to nonlinear systems in [29] and further developed in [30]. One of the motivations for the development of the hybrid approach was to create a framework for the analysis of systems that lose passivity because of high-frequency dynamics. It is done by using the fact that many systems that lose passivity have low gain at high frequencies. The hybrid passivity and finite-gain framework is closely connected to properties of systems on finite-frequency intervals and the generalized Kalman–Yakubovich–Popov lemma [31][32][33][34].

It has been shown that the hybrid approach can be successfully used for the design of controllers for vibrating structures [35][36], which are robust against passivity violations due to not collocated sensors and actuators [37]. It has been also used for the stability analysis of scheduled [36] and switched controllers [38] which indicates that it may be extended to other classes of nonlinear controllers. However, it has not been applied to the analysis of reset systems.

The research motivations lead to the formulation of questions that this work answers:

How can the hybrid passivity and finite-gain framework be extended to reset systems?
In the case of LTI systems, the frequency-domain description is used to conclude the hybrid passivity and finite-gain. This approach does not apply to nonlinear elements. Furthermore, the methods used for concluding the hybrid passivity and finite gain of scheduled and switched systems cannot be directly extended for the reset systems. In [36], a particular way of scheduling is assumed and in [38], state jumps are not present in the considered dynamics. Therefore, new tools have to be developed.

How can a reset control system be designed such that its stability can be concluded using the hybrid passivity and small-gain theorem?
The *hybrid passivity and small-gain theorem* [30] provides only a sufficient condition for the stability of feedback systems. It is, therefore, possible that this condition can be satisfied only by a specific class of reset control systems. Formulating guidelines for the design of controllers suited to the proposed method for the stability analysis would be a valuable insight.

1-3 Main contributions

The following are the main contributions made in this project.

Development of a method for analysis of the hybrid passivity and finite-gain parameters of reset systems.
In this project, it is shown that the hybrid passivity and finite-gain defined in [30] can not be easily applied to systems producing harmonics of the input signal. To extend the hybrid passivity and small-gain theorem to reset systems, a new definition of the finite-frequency gain is proposed. Subsequently, a method for assessing the finite-frequency gain of a system, based on its steady-state response is proposed. The method can be applied not only to reset systems but also to other classes of reset elements like the *hybrid integrator-gain systems* (HIGS) [39].

Development of a novel *Constant in gain Lead in phase element in a parallel configuration* (CgLpP)
The new CgLpP element is introduced to demonstrate an example reset control system whose stability can be concluded using the hybrid passivity and small-gain theorem. The controller can be tuned using techniques in line with the industrial practice. The *describing function* analysis indicates that the element relaxes the inherent limitations of LTI controllers.

1-4 Thesis outline

The structure of the report is as follows. The background knowledge, fundamental to this research, is presented in Chapter 2. The reset systems are introduced and an approach to represent them in the frequency domain is demonstrated. Next, the input-output approach for the stability analysis of systems is presented. Finally, the hybrid passivity and finite-gain approach is explained.

In Chapter 3 the hybrid passivity and finite-gain of reset systems are developed. We first consider the passivity and gain of reset systems at finite intervals separately. Subsequently, a modified hybrid passivity and finite-gain property and a corresponding stability theorem are introduced. The developed methods are used to analyse the hybrid properties of selected reset elements.

In Chapter 4 the developed theory is applied to design reset controllers for a precision mechatronic system. After considering the structure of the controllers, the use of the proposed method for the stability analysis is presented on an example. Finally, a detailed analysis of a precision positioning system with a reset controller tuned for different objectives is presented.

This project is concluded in Chapter 5, where the answers to the research questions posed above are summarized. Recommendations for further research and open questions are also presented.

Chapter 2

Background knowledge

In this chapter, the theoretical concepts, which are fundamental for this work, are introduced. The first subsection presents the nomenclature used in this report. Subsequently, the class of reset systems is introduced in Section 2-2. The frequency response functions and describing functions are demonstrated in the context of the reset systems in Section 2-3. Finally, the dissipative system framework is used to describe the blending of the passivity and finite-gain properties in Sections 2-4 and 2-5.

2-1 Nomenclature

The $\mathcal{L}_2$-space is a space of square integrable functions defined by

$$\mathcal{L}_2 = \left\{ v : \mathbb{R}^+ \to \mathbb{R}^m | \int_0^\infty v^\top(t)v(t)\mathrm{d}t < \infty \right\}, \tag{2-1}$$

where $\mathbb{R}^+ = [0, \infty)$, v is an arbitrary vector function of time and $v^\top$ is its transpose. The $\mathcal{L}_2$-space is a Hilbert space, where the inner product $\langle\cdot,\cdot\rangle$ defines the norm

$$\begin{aligned} \langle w, v\rangle &= \int_0^\infty w^\top(t)v(t)\mathrm{d}t, \\ \|v\|_2 &= \sqrt{\langle v, v\rangle}, \end{aligned}$$

where $v \in \mathcal{L}_2$, $w \in \mathcal{L}_2$.

The truncation of a vector function is defined as

$$v_T(t) = \begin{cases} v(t), & 0 \le t \le T, \\ 0, & t > T \end{cases}.$$

The extended $\mathcal{L}_2$-space is defined as $\mathcal{L}_{2,e} = \{v : \mathbb{R}^+ \to \mathbb{R}^m | v_T \in \mathcal{L}_2, 0 \le T < \infty\}$, thus $\mathcal{L}_2 \subset \mathcal{L}_{2,e}$.

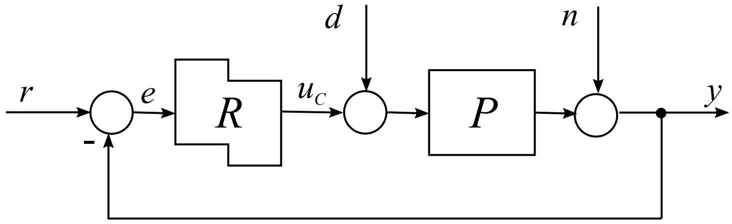


Figure 2-1: Reset control system (RCS). Reset controller R is applied to an LTI plant P. The "broken" box is often used to represent reset elements in block diagrams. Adapted from [14]

The truncated inner product of two arbitrary signals is

$$\langle w, v \rangle_T = \int_0^T w^\top(t) v(t) \mathrm{d}t = \langle w_T, v_T \rangle.$$

The truncated $\mathcal{L}_2$-norm is

$$\|v\|_{2,T} = \sqrt{\langle v, v \rangle_T}.$$

Given an operator G, its operator adjoint $G^\sim$ is defined using an inner product

$$\langle w, Gv \rangle = \langle G^\sim w, v \rangle.$$

2-2 Reset control systems

In general, a reset control system is a hybrid dynamical system, created as a feedback interconnection of a continuous-time plant and a controller with a reset mechanism. The reset causes an instantaneous change of controllers state whenever some condition is met. In this thesis, only the systems with linear base dynamics, which are reset whenever a certain signal crosses zero, are considered. Here, this class of reset systems is briefly presented. More details and information about other classes of reset systems can be found in the monographs [14] [17] [18].

2-2-1 Reset control system definition

The structure of a *Reset Control System* (RCS) is presented in Figure 2-1. An LTI plant P is described by:

$$P : \begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p u(t) \\ y(t) = C_p x_p(t) \end{cases}, \tag{2-2}$$

where $x_p \in \mathbb{R}^{n_p}$, $u \in \mathbb{R}^1$, $y \in \mathbb{R}^1$ and A_p, B_p and C_p are matrices of appropriate dimension.

A reset controller R with a zero-crossing reset condition is described by:

$$R : \begin{cases} \dot{x}_r(t) = A_r x_r(t) + B_r e(t), & \text{if } e \neq 0 \\ x_r(t^+) = A_{\rho_r} x_r(t), & \text{if } e = 0 \ , \\ u = C_r x_r(t) + D_r e(t) \end{cases} \tag{2-3}$$

where $x_r \in \mathbb{R}^{n_r}$ is the controller state, $e = r - y$ is the tracking error, r is the reference input and A_r, B_r, A_{ρ_r}, C_r, D_r are constant matrices of appropriate dimension.

The *linear reset law* $x_r(t^+) = A_{\rho_r} x_r(t)$ describes the change of state that occurs when the *reset condition* $e = 0$ is satisfied. Alternative reset laws and conditions that can be found in literature are not considered in this work. Further in this document the time indices are omitted and $x^+ = x(t^+) = \lim_{\epsilon \to 0^+} x(t + \epsilon)$ is used for clarity.

The linear system described with (A_r, B_r, C_r, D_r) is referred to by the term *Base Linear System* (BLS) and describes dynamics of the system in absence of reset.

A closed-loop description of the reset control system, obtained by combining the plant (2-2) and the reset controller Eq. (2-3), is given by

$$\begin{cases} \dot{x} = Ax + Br, & \text{if } e \neq 0 \\ x^+ = A_\rho x, & \text{if } e = 0 \\ y = Cx, \end{cases} \tag{2-4}$$

with $x = [x_p^T, x_r^T]^T$ and

$$A = \begin{bmatrix} A_p - B_p D_r C_p & B_p C_c \\ -B_r C_p & A_r \end{bmatrix}, B = \begin{bmatrix} B_p D_r \\ B_r \end{bmatrix}, \quad C = \begin{bmatrix} C_p & 0 \end{bmatrix}, A_\rho = \begin{bmatrix} I_{n_p \times n_p} & 0 \\ 0 & A_{\rho_r} \end{bmatrix}. \tag{2-5}$$

The equilibrium point of reset system x_{eq} is defined in [17, Ch. 1]. x_{eq} should be the equilibrium of both the BLS and of the reset mapping. In the case of constant reference r we have

$$\begin{cases} Ax_{eq} + Br = 0, \\ Cx_{eq} - r = 0, \\ A_\rho x_{eq} - x_{eq} = 0 \end{cases} . \tag{2-6}$$

Since reset systems are a special case of hybrid systems, pathological behaviours like beating, deadlock and Zeno behaviour may occur [8].

In practice, existence and uniqueness of the solution are assured by time-regularization [40]. Time-regularization is a modification of reset system, such that reset instants happen only if a minimum time between resets $\Delta_m > 0$ has lapsed. Then, the reset system is described by

$$\begin{cases} \dot{\Delta}(t) = 1, & \dot{x}(t) = Ax(t) + Bu(t), & \text{if } (e \neq) 0 \vee (\Delta \leq \Delta_m) \\ \Delta(t^+) = 0, & x(t^+) = A_\rho x(t), & \text{if } (e = 0) \wedge (\Delta > \Delta_m) \\ & y(t) = Cx(t) + Bu(t). \end{cases} \tag{2-7}$$

More details about solutions of reset systems can be found in [14] and [17].

2-2-2 Stability of reset control systems

In the case of reset systems, both base dynamics and reset of states play an important role in establishing stability properties. A reset action can destabilize a stable base system, what makes use of reset for performance improvement not straight forward. Conversely, an unstable base system can be stabilized by the introduction of a reset.

For reset systems with stable base dynamics, stability can be proven without considering when reset actions take place. This fact gives rise to so-called reset-times independent stability conditions. The issue of stability of reset systems with stable base linear dynamics has been addressed in [41]. The main result is:

Theorem 2-2.1. *[41] Let $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continously-differentiable, positive-definite, radially unbounded function such that*

$$\dot{V}(x) = \left[\frac{\partial V}{\partial x}\right] Ax < 0 \qquad \text{if } x \notin \ker(C), \tag{2-8}$$

$$\Delta V(x) = V(x^+) - V(x) \leq 0 \qquad \text{if } x \in \ker(C), \tag{2-9}$$

Then,

1. *there exist a left-continuous function $x(t)$ satisfying Eq. (2-4) for all $t \geq 0$*
2. *the equilibrium point $x_{eq} = 0$ is globally uniformly asymptotically stable.*

The use of a quadratic Lyapunov function $V(x) = x^T Px$ leads to stability conditions that can be tested by LMI solving [17] or with use of the so-called H_β condition derived in [41]. The H_β condition is based on the Kalman-Yakubovich-Popov Lemma, and for systems where only one state is reset, it can be checked visually. The quadratic stability of reset systems has been summarized in [14] by the following proposition.

Proposition 2-2.2. *[14] The following conditions are equivalent:*

1. *The reset system (2-4) with $A_{\rho_r} = 0_{n_r \times n_r}$ is quadratically stable.*
2. *There exists a constant $\beta \in \mathbb{R}^{n_\rho}$ and positive matrix $P > 0$ such that*

$$A^T P + PA < 0 \tag{2-10}$$

$$B_0^T P = C_0 \tag{2-11}$$

3. *(H_β condition) There exist a positive-definite matrix $P_\rho \in \mathbb{R}^{n_\rho \times n_\rho}$ and $\beta \in \mathbb{R}^{n_\rho}$ such that*

$$H_\beta(s) = C_0(sI - A)^{-1} B_0 \tag{2-12}$$

is a strictly positive real (SPR) transfer function, where (A, B_0) is controllable and (A, C_0) is observable.

C_0 and B_0 are defined as

$$C_0 = \begin{bmatrix} \beta C_p & 0_{n_\rho \times (n_c - n_\rho)} & P_\rho \end{bmatrix} \qquad B_0 = \begin{bmatrix} 0_{n_\rho \times n_\rho} \\ 0_{(n_c - n_\rho) \times n_\rho} \\ I_{n_\rho \times n_\rho} \end{bmatrix} \tag{2-13}$$

Guo at al. [17] have shown that if the H_β condition is satisfied, then the RCS is quadratically stable for any reset matrix A_ρ such that

$$A_\rho^T P_\rho A_\rho - P_\rho \leq 0. \tag{2-14}$$

The notions of stability presented above apply only to reset systems with stable base dynamics. Other aspects of stability and conditions applicable to different classes of reset system can be found in monographs [14] [17] [18].

2-3 Frequency domain properties of reset systems

Frequency domain techniques are widely accepted in the engineering community for analysis, modelling and controller design for LTI systems. The Frequency Response Function (FRF) and its representations such that Bode, Nyquist and Nichols plots are standard engineering tools for the design of dynamical systems in the industry. These techniques have been extended for certain classes od nonlinear systems to facilitate the use of nonlinear controllers and simplify the analysis of systems in which nonlinearities can have a significant effect on performance.

A comparative overview of frequency-domain methods for nonlinear systems is presented in [42]. Here, the nonlinear frequency response functions [43] and describing functions [44] [45] [46] are briefly presented in the context of rest systems with the zero-crossing reset condition.

2-3-1 Nonlinear frequency response functions

The nonlinear frequency response functions have been introduced for convergent systems in [43]. First, a class of convergent systems with *uniformly bounded steady-state* is defined to guarantee that for a given input and any initial conditions all solutions converge the same steady-state solution. Next, the steady-state solutions for sinusoidal inputs are described using the nonlinear FRF. Although the nonlinear FRF is not frequency domain function it is closely liked to the frequency-domain concepts. Therefore, is presented here for consistency of the report.

Consider the system

$$\dot{x}(t) \in F(x, w(t)), \tag{2-15}$$

where $x \in \mathbb{R}^n$ is the state and $w : \mathbb{R} \to \mathbb{R}^m$ is the continuous input. $F(x, w)$ is a set-valued mapping $F : \mathbb{R}^{n+m} \to \{\text{subset of } \mathbb{R}^n\}$. It is assumed that for any $(x, w) \in \mathbb{R}^{n+m}$ the set $F(x, w)$ is upper semi-continuous in x, w. The system $\dot{x} = f(x, w(t))$ with a single-valued continuous $f(x, w)$ can be considered as a particular case of (2-15). It is also assumed that the system (2-15) is regular i.e. for any continuous input $w(t)$ and any initial condition $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$ its corresponding solution $x_w(t, t_0, w)$ is right unique.

Definition 2-3.1. *[43] System (2-15) with a given continuous input $w : \mathbb{R} \to \mathbb{R}^m$ is said to be (uniformly, exponentially) convergent if:*

a) all solutions $x_w(t, t_0, x_0)$ are defined for all $t \in [t_0, +\infty)$ and all initial conditions $x_0 \in \mathbb{R}^m$, $t_0 \in \mathbb{R}$

b) there is a solution $\bar{x}_w(t)$ defined and bounded on $\mathbb{R}$

c) the solution $\bar{x}_w(t)$ is (uniformly, exponentially) globally asymptotically stable

System (2-15) is said to be (*uniformly, exponentially*) convergent for a class of inputs $\mathcal{I}$ if it is (*uniformly, exponentially*) convergent for every input $w \in \mathcal{I}$.

Definition 2-3.2. *[43] System (2-15) that is convergent for some class of inputs $\mathcal{I}$ is said to have the uniformly bounded steady-state property if for any $\rho > 0$ there exists $\mathcal{R} > 0$ such that for any input $w \in \mathcal{I}$ the following implication holds:*

$$|w(T)| \leq \rho \quad \forall t \in \mathbb{R} \Rightarrow |\bar{x}_w(t)| \leq \mathcal{R} \quad \forall t \in \mathbb{R}. \tag{2-16}$$

The class of uniformly convergent systems with the UBSS property can be considered as an extension of the class of asymptotically stable LTI systems [43]. An important property of uniformly convergent systems is that if the input applied to the system is constant then the corresponding steady-state solution is also constant and if the input is periodic then the corresponding steady-state solution is also periodic with the same period [47].

After defining convergent systems, the nonlinear FRF can be introduced. Consider an uniformly convergent system

$$\begin{cases} \dot{x} \in F(x, u) \\ y = h(x) \end{cases} \tag{2-17}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}$ and output $y \in \mathbb{R}$.

Theorem 2-3.3. *[43] Suppose system (2-17) is regular and uniformly convergent with the UBSS property for the class of continuous bounded inputs. Then there exists a continuous function $\alpha : \mathbb{R}^3 \to \mathbb{R}^n$ such that for any harmonic excitation of the form $u(t) = a \sin(\omega t)$, the system has a unique periodic solution*

$$\bar{x}_{a,\omega}(t) := \alpha(a \sin(\omega t), a \cos(\omega t), \omega) \tag{2-18}$$

and this solution is UGAS.

The function $\alpha(a \sin(\omega t), a \cos(\omega t), \omega)$ contains all information on the steady-state solution of system (2-17) corresponding to harmonic excitations. For that reason it is called the *state frequency response function.* The function $h(\alpha(a \sin(\omega t), a \cos(\omega t), \omega))$ is called the *output frequency response function.* In general, the frequency response functions cannot be easily found [43]. Examples of nonlinear FRF derived analytically can be found in [43] and [48].

In the case of LTI systems, the Bode magnitude plot is a representation of the gain with which the system amplifies harmonic signals at various frequencies. Similar information about the steady-state response can be represented with the *Nonlinear Bode Plot* introduced in [43]. Consider system (2-17) excited by an harmonic signal $a \sin(\omega t)$. The ratio of the maximal absolute value of the steady-state output and the input amplitude a can be considered as a frequency dependent amplification gain of the convergent system. Formally, the amplification gain $\gamma_{abs}(\omega)$ is defined as

$$\gamma_{abs}(\omega) := \frac{1}{a}\left(\sup_{v_1^2+v_2^2=a^2} |h(\alpha(v_1, v_2, \omega))|\right) \tag{2-19}$$

In general nonlinear case, γ_{abs} depends not only on the frequency but also on the amplitude of the input signal.

An alternative frequency-dependent gain for convergent systems excited by periodic signals has been proposed in [49]. Consider a convergent system excited by periodic input signal $u(t)$ with period T. As mentioned before, the steady-state response of the system $\bar{y}(t)$ is also a periodic signal with the same period T as the input. The gain of the system γ_{rms} can be now defined by writing

$$\int_{t_0}^{t_0+T} \bar{y}^\top(\tau)\bar{y}(\tau)\mathrm{d}\tau \leq \gamma_{rms}^2 \int_{t_0}^{t_0+T} u^\top(\tau)u(\tau)\mathrm{d}\tau \tag{2-20}$$

This inequality can be rewritten as

$$\|\bar{y}\|_{2,T} \leq \gamma_{rms}\|u\|_{2,T} \tag{2-21}$$

This definition of the gain of a nonlinear system links the *rms values* of the periodic exogenous input u and of the corresponding periodic steady-state response $\bar{y}$.

2-3-2 Nonlinear frequency response functions of reset systems

In this subsection, convergence and nonlinear frequency response functions of reset systems are analysed. We focus especially on the *open-loop* reset systems described by Equation (2-3), whose states are reset whenever the input signal e is equal to zero. Convergence of *closed-loop* reset systems, like the one presented in Equation (2-4), whose states are reset based on an internal signal different than the systems input, is a topic of ongoing research [50].

For the open-loop reset system excited with a sinusoidal input

$$e(t) = a\sin(\omega t) \tag{2-22}$$

we have

Proposition 2-3.4. *[45] The reset system (2-3) with input (2-22) has a globally asymptotically stable $2\pi/\omega$-periodic solution if and only if*

$$\rho\left(A_{\rho_r} e^{\frac{\pi}{\omega}A_r}\right) < 1. \tag{2-23}$$

$\rho(\cdot)$ denotes the spectral radius of a matrix.

Proposition 2-3.5. *[45] The reset system (2-3) has a globally asymptotically stable $2\pi/\omega$-periodic solution under sinusoidal input with arbitrary frequency $\omega > 0$ if and only if*

$$\rho\left(A_{\rho_r} e^{A_r\delta}\right) < 1 \quad \forall\delta \in \mathbb{R}^+. \tag{2-24}$$

In fact, Proposition 2-3.5 gives condition for convergence of the open-loop reset system (2-3) for the class of sinusoidal inputs. Note, that the reset matrix A_{ρ_r} has to satisfy $\rho(A_{\rho_r}) \leq 1$, for the condition in Proposition 2-3.5 to be satisfied. At the same time, the condition can be satisfied if A_r is not Hurwitz.

For convenience, define:

$$\Lambda(\omega) = \omega^2 I + A_r^2, \tag{2-25a}$$

$$\Delta(\omega) = I + e^{\frac{\pi}{\omega} A_r}, \tag{2-25b}$$

$$\Delta_r(\omega) = I + A_{\rho_r} e^{\frac{\pi}{\omega} A_r}, \tag{2-25c}$$

$$\Gamma_r(\omega) = \Delta_r^{-1}(\omega) A_{\rho_r} \Delta(\omega) \Lambda^{-1}(\omega). \tag{2-25d}$$

The set of reset time instants $\{t_k\}$ for the reset system (2-4) with input (2-22) is given by

$$t_k = k\pi/\omega, \quad k \in \{0, 1, 2, \dots\}. \tag{2-26}$$

If the conditions of Proposition 2-3.4 are satisfied, the steady-state solution $\bar{y}(t)$ of system (2-3) corresponding to input (2-22) is given by [45]

$$\begin{aligned} \bar{y}(t) &= aC_r e^{A_r t}\theta_k(\omega) - aC_r\Lambda^{-1}(\omega)\left(\omega I \cos(\omega t) + A_r \sin(\omega t)\right) B_r + D_r a \sin(\omega t) \\ \theta_k(\omega) &= (-1)^{k+1} e^{-A_r t_k} \left(\Gamma_r(\omega) - \Lambda^{-1}(\omega)\right) \omega B_r, \end{aligned} \tag{2-27}$$

when $t \in (t_k, t_{k+1}]$.

Function (2-27) is in fact the nonlinear frequency response function of the open-loop reset system (2-3). Note, that the amplitude of $\bar{y}$ is proportional to the amplitude a of the input signal. Knowing $\bar{y}$, the nonlinear Bode plot mentioned earlier can be easily calculated for the open-loop reset system.

2-3-3 Describing functions of reset systems

Describing function (DF) is a quasi-linearization of a nonlinear element subject to a certain input [17, Ch.2]. There exist many types of *Describing Functions* (DF) which are defined for different classes of systems and signals [42]. Typically, definitions of DF are similar to the definition of the FRF for the LTI systems. Here, the DF are presented for open-loop reset systems. DF of the closed-loop reset systems are a topic of ongoing research [50].

Sinusoidal Input Describing Functions (SIDF) are commonly used for analysis and design of reset control systems [14] [17]. They have been used to motivate the use of reset since the very beginning of works on this topic [12]. In [45], an analytical formula for a describing function of a reset element with the zero-crossing reset condition is given. Based on this work, an analytical formula for *Higher-Order Sinusoidal Input Describing Functions* (HOSIDF) of the same kind of reset systems was derived in [46].

Consider the open-loop reset system (2-3) excited with sinusoidal input (2-22). If the conditions of the Proposition 2-3.4 are satisfied, the system has a corresponding globally asymptotically stable solution $\bar{y}$ with the same period as the input signal. The steady state solution

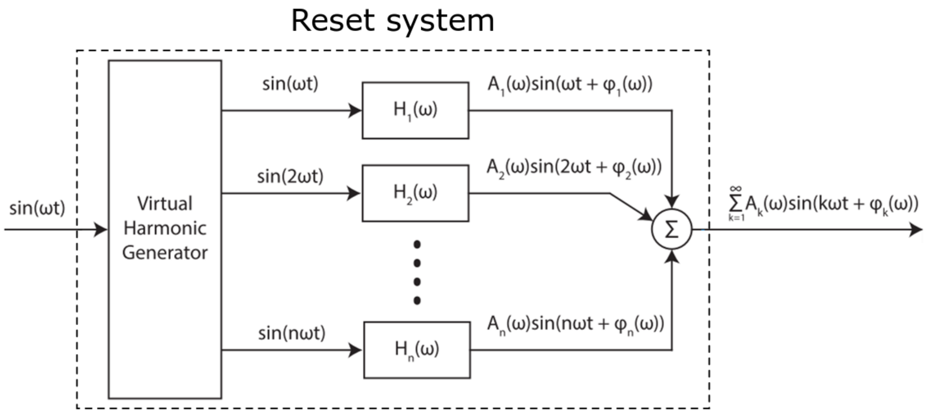


Figure 2-2: Block diagram representation of HOSIDF for a reset system. For reset system the HOSIDF $H_q(\omega)$ is a function of the input frequency only. Adapted from [46]

$\bar{y}$ can be written using the Fourier series as a sum of harmonics of the input signal $e(t)$. The describing function of order q is defined as the complex ratio of the qth harmonic component of the output signal to the virtual qth harmonics signal derived from the excitation signal. SIDF of the system is a HOSIDF of order $q = 1$. Formally, the qth HOSIDF is defined by [46]

$$
\begin{aligned}
H_q(\omega) &= \frac{\bar{Y}_q(\omega)}{E_q(\omega)} \\
\bar{Y}_q(\omega) &= \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \bar{y}(t)e^{-j\omega qt}\mathrm{d}t \\
E_q(\omega) &= \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} a\sin(q\omega t)e^{-j\omega qt}\mathrm{d}t
\end{aligned}
\tag{2-28}
$$

The reset element can be modelled using the virtual harmonic generator as described in [44]. The virtual harmonic generator converts the sinusoidal input $e(t)$ into a harmonic signal $\breve{e}(t) = \sum_{q=1}^{\infty} a\sin(q\omega t)$. The harmonic components are inserted into corresponding describing function subsystems $H_q(\omega)$ which are relating a harmonic component of the non-linear system output to the corresponding harmonic component of the virtual harmonics generator. The response of the system is an infinite sum of outputs generated by H_q. The concept is illustrated for a reset element in Figure 2-2. For resets systems HOSIDF does not depend on the amplitude of the input signal.

Theorem 2-3.6. *[46] For the reset control system (2-3) the Higher-Order Sinusoidal Input Describing Functions (HOSIDF) $H_q(\omega)$, where $\omega \in \mathbb{R}$ is the angular frequency of input excitation and $q \in \mathbb{Z}^+$ is an order of HOSIDF, is given by*

$$
H_q(\omega) = \begin{cases} C_r(j\omega I - A_r)^{-1}(I + j\Theta_D(\omega))B_r + D_r & \textit{for } q = 1 \\ C_r(j\omega q I - A_r)^{-1} j\Theta_D(\omega)B_r & \textit{for odd } q \geq 2 \\ 0 & \textit{for even } q \geq 2 \end{cases} \tag{2-29}
$$

where

$$\begin{aligned}\Theta_D(\omega) &= -\frac{2\omega^2}{\pi}\Delta(\omega)[\Gamma_r(\omega) - \Lambda^{-1}(\omega)]\\ \Lambda(\omega) &= \omega^2 I + A_r^2\\ \Delta(\omega) &= I + e^{\frac{\pi}{\omega}A_r}\\ \Delta_r(\omega) &= I + A_{\rho_r}e^{\frac{\pi}{\omega}A_r}\\ \Gamma_r(\omega) &= \Delta_r^{-1}(\omega)A_{\rho_r}\Delta(\omega)\Lambda^{-1}(\omega)\end{aligned}$$

2-4 $\mathcal{L}_2$-gain, passivity and dissipativity

$\mathcal{L}_2$-gain and passivity are fundamental concepts in control theory. They are commonly used to analyse the stability of feedback interconnections of systems and provide a robustness guarantee [26]. Both concepts can be described using the dissipativity theory, which is based on energy-related considerations [51]. Here, all three notions are presented as a foundation to introduce the blending of passivity and finite-gain in the subsequent section.

2-4-1 Input-output stability and the small gain theorem

In the input-output approach, a model of the system is used, which relates the output directly to the input without focusing on the internal structure that is represented by the state equation. The following basic results are presented as in [52, Ch.5].

Consider a system whose input-output relation is given by

$$y = Hu \tag{2-30}$$

where H is some mapping that specifies $y \in \mathbb{R}^q$ in terms of $u : [0, \infty) \to \mathbb{R}^m$ [52, Ch.5]. The mapping H is defined as a mapping from extended space $\mathcal{L}_{2,e}^m$ to extended space $\mathcal{L}_{2,e}^q$ (see [52, Ch.5])

Definition 2-4.1. *[52] A mapping H: $\mathcal{L}_{2,e}^m \to \mathcal{L}_{2,e}^q$ is $\mathcal{L}_2$ stable if there exist a class $\mathcal{K}$ function α, defined on $[0, \infty)$, and a non negative constant β such that*

$$\|(Hu)_\tau\|_2 \leq \alpha(\|u_\tau\|_2) + \beta \tag{2-31}$$

for all $u \in \mathcal{L}_{2,e}^m$ and $\tau \in [0, \infty)$. It is finite-gain $\mathcal{L}_2$ stable if there exists non negative constants γ and β such that

$$\|(Hu)_\tau\|_2 \leq \gamma\|u_\tau\|_2 + \beta \tag{2-32}$$

for all $u \in \mathcal{L}_{2,e}^m$ and $\tau \in [0, \infty)$.

For an LTI system with rational transfer function $H(s)$, i.e. $y(s) = H(s)u(s)$, where all poles of $H(s)$ have real parts less than zero, the system has finite $\mathcal{L}_2$-gain if the Nyquist plot $H(j\omega)$ is inside the circle of radius γ centred at the origin.

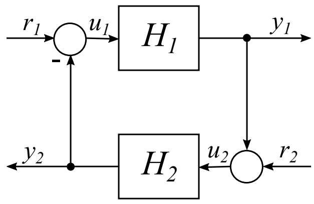


Figure 2-3: Closed-loop system with two external inputs.

Consider now a feedback interconnection of systems presented in Figure 2-3. Suppose that two systems $H_1 : \mathcal{L}^m_{2,e} \to \mathcal{L}^q_{2,e}$ and $H_2 : \mathcal{L}^q_{2,e} \to \mathcal{L}^m_{2,e}$ are finite gain stable, that is

$$\|y_{1\tau}\|_2 \leq \gamma_1 \|u_{1\tau}\|_2 + \beta_1 \qquad \forall u_1 \in \mathcal{L}^m_{2,e}, \forall \tau \in [0, \infty) \tag{2-33}$$

$$\|y_{2\tau}\|_2 \leq \gamma_2 \|u_{2\tau}\|_2 + \beta_2 \qquad \forall u_2 \in \mathcal{L}^q_{2,e}, \forall \tau \in [0, \infty) \tag{2-34}$$

Assume that the feedback system is well defined, that is for every pair of inputs $r_1 \in \mathcal{L}^m_{2,e}$ and $r_2 \in \mathcal{L}^q_{2,e}$ there exist unique outputs $u_1, y_1 \in \mathcal{L}^m_{2,e}$ and $u_2, y_2 \in \mathcal{L}^q_{2,e}$. The small-gain theorem presented below gives a sufficient condition for finite-gain $\mathcal{L}_2$ stability of the considered feedback connection.

Theorem 2-4.2. *[52] Under the preceding assumptions, the feedback connection is finite gain $\mathcal{L}$ stable if $\gamma_1\gamma_2 < 1$*

2-4-2 Passivity

Passivity techniques are a powerful tool for nonlinear control. They are suitable for analysis of interconnected systems and in the case of linear plants, frequency-domain conditions can be used.

Consider the input-output system H: $\mathcal{L}_{2,e} \to \mathcal{L}_{2,e}$.

Definition 2-4.3. *[26] [51] A system $y = H(u)$ is called passive if there is a constant β such that*

$$\int_0^T y^T(\tau)u(\tau)\mathrm{d}\tau \geq \beta, \forall u \in \mathcal{L}_{2,e} \tag{2-35}$$

If, in addition there are constants $\delta \geq 0$, $\epsilon \geq 0$ such that

$$\int_0^T y^T(\tau)u(\tau)\mathrm{d}\tau \geq \beta + \delta \int_0^T u^T(\tau)u(\tau)\mathrm{d}\tau + \epsilon \int_0^T y^T(\tau)y(\tau)\mathrm{d}\tau, \forall u \in \mathcal{L}_{2,e} \tag{2-36}$$

then the system is, respectively, said to be:

1. *Input Strictly Passive (ISP) if $\delta > 0$*
2. *Output Strictly Passive (OSP) if $\epsilon > 0$*
3. *Very Strictly Passive (VSP) if $\delta > 0$ and $\epsilon > 0$*

The system is said to be pseudo very strictly passive is there are constants $\beta, \delta, \epsilon \in \mathbb{R}$ such that 2-36 holds.

Note, that systems which are OSP or VSP have also a finite $\mathcal{L}_2$-gain.

The passivity of a linear time-invariant system can be conveniently checked using the following frequency domain conditions.

Theorem 2-4.4. *[51] Given a LTI system with a rational transfer function matrix $H(s)$. Assume that all the poles of $H(s)$ have negative real parts. Then the following assertions hold:*

1. *The system is passive if and only if* $\mathrm{Re}(H(j\omega)) \geq 0$ *for all* $\omega \in [-\infty, \infty]$
2. *The system is input strictly passive (ISP) if and only if there exist a* $\delta > 0$ *such that* $\mathrm{Re}(H(j\omega)) \geq \delta > 0$ *for all* $\omega \in [-\infty, \infty]$
3. *The system is output strictly passive if and only if there exist a* $\epsilon > 0$ *such that* $\mathrm{Re}(H(j\omega)) \geq \epsilon |H(j\omega)|^2 > 0$ *for all* $\omega \in [-\infty, \infty]$.
   It is equivalent to $(\mathrm{Re}(H(j\omega)) - \frac{1}{2\epsilon})^2 + (\mathrm{Im}(H(j\omega)))^2 \leq (\frac{1}{2\epsilon})^2$

In a SISO, LTI case, these properties can be checked in a Nyquist diagram: if $H(j\omega)$ is in the closed right half plane then the system is passive, if $H(j\omega)$ is in $\mathrm{Re}\{H(j\omega)\} \geq \delta > 0$ the system is ISP and if $H(j\omega)$ is inside the circle with center $\frac{1}{2\epsilon}$ and radius $\frac{1}{2\epsilon}$ the system is OSP.

The passivity theorem can be used to show the stability of a feedback interconnection of systems presented in Figure 2-3.

Theorem 2-4.5. *[51] Assume that H_1, H_2 are pseudo VSP. The feedback system shown in Figure 2-3 is finite $\mathcal{L}_2$-gain stable if both*

$$\begin{aligned} \epsilon_1 + \delta_2 > 0 \quad & \textit{and} \\ \epsilon_2 + \delta_1 > 0 & \end{aligned} \tag{2-37}$$

are satisfied, where ϵ_i, δ_i may be negative.

2-4-3 Dissipativity

Dissipative system theory was developed in [53], where the concept of passivity was extended to non-mechanical systems. Later, the theory has been adapted for hybrid systems in [8].

Consider a dynamical system represented by the state-space model

$$\begin{cases} \dot{x} = f(x, u) \\ y = g(x, u) \end{cases} \tag{2-38}$$

where $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz, $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(0,0) = 0$, $g(0,0) = 0$, $x \in \mathbb{R}^n$ is the state variable and $u, y \in \mathcal{L}_{2,e}$ are the input and output, respectively.

Definition 2-4.6. *[51] System (2-38) is said to be dissipative with the supply rate $w(u, y)$ if there exist a storage function $V(x) \geq 0$ such that*

$$V(x(T)) \leq V(x(0)) + \int_0^T w(u(t), y(t))\mathrm{d}t, \qquad \forall T \geq 0, \forall u \in \mathcal{L}_{2,e}, \forall x(0) \tag{2-39}$$

If it is imposed that the storage functions satisfy $V(0) = 0$, the dissipativity definition introduced by Hill and Moylan [54] can be used [51].

Definition 2-4.7. *[51] System (2-38) is dissipative with respect to the supply rate $w(u, y)$ if for all admissible u and all $t_1 \geq t_0$ one has*

$$0 \leq \int_{t_0}^{t_1} w(u(t), y(t))\mathrm{d}t \tag{2-40}$$

with $x(t_0) = 0$ and along all trajectories of the system.

The system H is said to be $(\mathbf{Q}, \mathbf{S}, \mathbf{R})$-dissipative when it is dissipative w.r.t. quadratic supply rate

$$w(u, y) = y^\top \mathbf{Q} y + 2y^\top \mathbf{S} u + u^\top \mathbf{R} u \tag{2-41}$$

where $\mathbf{Q}, \mathbf{S}, \mathbf{R}$ are bounded linear operators.

The relationship between dissipative and passive and finite-gain systems is given by the choice of a particular supply rate [51]:

1. The system is passive if it is dissipative with respect to supply rate $w_p(u, y) = u^T y$ (if it is $(0, 1/2, 0)$-dissipative) and $V(0) = 0$
2. The system is ISP if it is dissipative with respect to supply rate $w_i(u, y) = u^T y - \delta u^T u$, for some $\delta > 0$ (if it is $(0, 1/2, -\delta)$-dissipative)
3. The system is OSP if it is dissipative with respect to supply rate $w_o(u, y) = u^T y - \epsilon y^T y$, for some $\epsilon > 0$ (if it is $(-\epsilon, 1/2, 0)$-dissipative)
4. The system is VSP if it is dissipative with respect to supply rate $w_v(u, y) = u^T y - \delta u^T u - \epsilon y^T y$, for some $\epsilon > 0$ and $\delta > 0$ (if it is $(-\epsilon, 1/2, -\delta)$-dissipative)
5. The system has finite gain if it is dissipative with respect to supply rate $w_g(u, y) = \gamma u^T u - \gamma^{-1} y^T y$, for some $\gamma > 0$ (if it is $(-\gamma^{-1}, 0, \gamma)$-dissipative)

The formulation in point 5 above is equivalent to $\|y\|_{2T}^2 \leq \gamma^2 \|u\|_{2T}^2$. It is deliberately chosen for convenience in further derivations.

2-4-4 Passivity of reset systems

The passivity framework can be used for systems with full reset since the system loses all of its memory at every reset time. For systems with a general reset matrix, dissipativity theory has to be used [55]. In both cases, the reset system is assumed to have no Zeno solutions. This can be achieved, for example, by using the time regularization shown in Eq. (2-7).

Theorem 2-4.8. *[55] The reset system (2-3) with $A_\rho = 0_{n\times n}$ is passive, ISP, OSP or VSP if its BLS is passive, ISP, OSP or VSP respectively.*

Conditions for dissipativity of reset system have been derived for the systems with partial reset in [55] but can be also applied for the case of general reset matrices [17, Ch. 3.5].

Theorem 2-4.9. *[55] The reset system (2-3) is passive, ISP, OSP or VSP if its BLS is dissipative with respect to supply rates $w_p(u,y)$, $w_i(u,y)$, $w_o(u,y)$ or $w_v(u,y)$, respectively, and with a storage function $V(x)$ satisfying $V(A_\rho x) \leq V(x)$, $\forall x \in \mathbb{R}^n$.*

2-5 Hybrid passivity and finite-gain

In this section, the idea of blending of the passivity and finite-gain theories is presented. First, an example is given to illustrate the overall concept. Subsequently, the elements of the *hybrid passivity and finite-gain* theory are introduced in two steps. The concept of *finite-frequency properties*, introduced by Iwasaki in [31], is summarized in Subsection 2-5-2. Systems which are passive in a low-frequency range and have finite-gain otherwise can be said to have the *hybrid passivity and finite-gain* property, what is explained in Subsection 2-5-3. Note, that here the term *hybrid* does not refer to the dynamics of the system, but to the blending of the passivity and finite-gain approaches. The idea has been first introduced in [29] and later developed further to the form which is presented in this report in [30]. This framework is extended to the analysis of reset control systems in the main contribution of this thesis.

2-5-1 Illustrative example

To illustrate the general idea of the hybrid passivity and finite-gain, an example adapted from [27] is presented here. The concepts are formally introduced in the following subsections.

The motivation behind the blending of the passivity and small-gain theorems is to provide stability results for feedback interconnections of a class of systems, broader than those that can be analysed with the small-gain or passivity theorems [27].

Consider a negative feedback system consisting of two LTI elements with transfer functions

$$H_1(s) = \frac{5}{(s+1)(s+2)} \qquad\qquad H_2(s) = \frac{5}{(s+0.5)(s+4)}$$

In some frequency range $[0,\Omega]$ both systems are passive (i.e. the real party of each of the transfer functions is positive) and in the frequency range $[\Omega,\infty)$ the product of the amplitudes of the transfer functions is less than one. The later can be checked by evaluating the maximal absolute values of the transfer functions in the high-frequency range, denoted by γ_1, γ_2. In this case, it is not possible that the Nyquist diagram of the cascade H_1H_2 encircles the point -1. Accordingly, the closed-loop system is stable. The example is illustrated in Figure 2-4

Stability of the considered feedback system can not be concluded using the small-gain theorem since both subsystems have $\mathcal{L}_2$ gains greater than 1. The subsystems are also not passive and

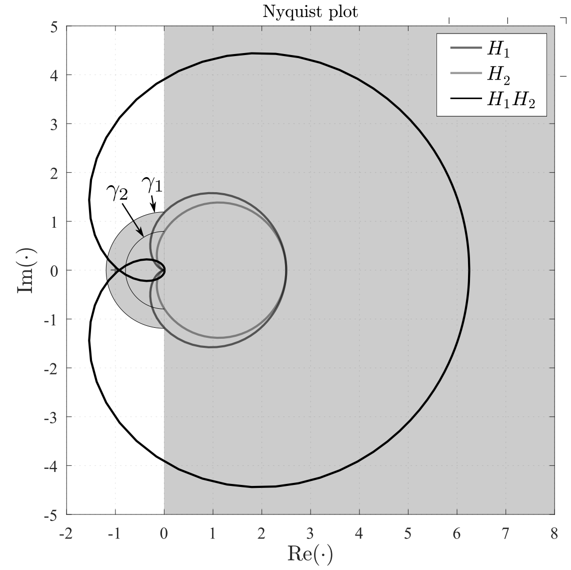


Figure 2-4: Illustration of the hybrid passivity and finite-gain property for LTI systems

multipliers or loop transformations that would enable the use of the passivity theorem can not always be found.

In the case of simple LTI systems, the stability of the closed-loop system can be easily concluded by finding the eigenvalues of the closed-loop system or by plotting the Nyquist plot of the cascade. However, the hybrid passivity approach is beneficial if one of the subsystems is nonlinear [36] [38] or during the synthesis of controllers for flexible structures [56] [37].

2-5-2 Finite-frequency properties

Consider the dissipation inequality from Definition 2-4.6 with the quadratic supply rate given in Equation (2-41). By moving the truncations from the integration limits to the signals u_T, y_T we have

$$V(x(T)) - V(x(0)) \leq \int_0^\infty y_T^\top \mathbf{Q} y_T \mathrm{d}t + 2\int_0^\infty y_T^\top \mathbf{S} u_T \mathrm{d}t + \int_0^\infty u_T^\top \mathbf{R} u_T \mathrm{d}t \tag{2-42}$$

Assuming that the operators $\mathbf{Q}, \mathbf{S}, \mathbf{R}$ are LTI, the inequality (2-42) can be expressed (via the Parseval's theorem) in the frequency domain

$$\begin{aligned} V(x(T)) - V(x(0)) \leq & \frac{1}{2\pi}\int_{-\infty}^{\infty} y_T^{\mathsf{H}}(j\omega)\mathcal{Q}(\omega)y_T(j\omega)\mathrm{d}\omega \\ & + \frac{1}{\pi}\,\mathrm{Re}\left\{\int_{-\infty}^{\infty} y_T^{\mathsf{H}}(j\omega)\mathcal{S}(\omega)u_T(j\omega)\mathrm{d}\omega\right\} + \frac{1}{2\pi}\int_{-\infty}^{\infty} u_T^{\mathsf{H}}(j\omega)\mathcal{R}(\omega)u_T(j\omega)\mathrm{d}\omega \end{aligned} \tag{2-43}$$

To define passivity/finite-gain of a system on a finite frequency interval the integration limits in the Inequality (2-43) can be changed to finite values. For example, system (2-38) is VSP on the frequency interval $[-\omega_H, \omega_H]$ if there exist a storage function V defined in (2-4.6), $\epsilon > 0$, $\delta > 0$ such that

$$\begin{aligned} V(x(T)) - V(x(0)) \leq & -\frac{1}{2\pi}\int_{-\omega_H}^{\omega_H} y_T^{\mathsf{H}}(j\omega)\epsilon y_T(j\omega)\mathrm{d}\omega \\ & + \frac{1}{2\pi}\operatorname{Re}\left\{\int_{-\omega_H}^{\omega_H} y_T^{\mathsf{H}}(j\omega)u_T(j\omega)\mathrm{d}\omega\right\} - \frac{1}{2\pi}\int_{-\omega_H}^{\omega_H} u_T^{\mathsf{H}}(j\omega)\delta u_T(j\omega)\mathrm{d}\omega \end{aligned} \tag{2-44}$$

To obtain an equivalent definition, introduce function $\alpha(\omega) : \mathbb{R} \to \{0, 1\}$ which takes value 1 in the frequency interval where the finite-frequency property is satisfied and 0 elsewhere. The function can be considered as an ideal low-pass filter. α is formally defined by

$$\begin{aligned} \alpha(\omega) &:= \begin{cases} 1, & -\omega_H < \omega < \omega_H \\ 0, & |\omega| \geq \omega_H \end{cases} \\ &= A(-j\omega)A(j\omega) = |A(j\omega)|^2 \end{aligned} \tag{2-45}$$

The transfer function $A(s) \in \mathcal{H}_\infty$ and $A(s)A(-s)$ is the spectral factorisation of the Laplace transform of the inverse Fourier transform of $\alpha(\omega)$. It has been shown in [30], that $A(s)$ can be approximated with a low-pass Butterworth filter.

Now, the finite-frequency VSP is defined using the Inequality (2-43) with new operators $\mathcal{Q}, \mathcal{S}, \mathcal{R}$ given as

$$\mathcal{Q}(\omega) = -\alpha(\omega)\epsilon \tag{2-46a}$$

$$\mathcal{S}(\omega) = \alpha(\omega)\frac{1}{2} \tag{2-46b}$$

$$\mathcal{R}(\omega) = -\alpha(\omega)\delta \tag{2-46c}$$

Definition of a finite-frequency finite-gain system can be obtained in an analogue manner.

The finite-frequency passivity/finite-gain can be described in the time domain, what enables use of this approach for analysis of nonlinear systems. The time domain equivalent of $A(s)$ is the causal convolution operator $\mathcal{A} : \mathcal{L}_2 \to \mathcal{L}_2$.

Define

$$\alpha(\omega) + \beta(\omega) = 1 \Leftrightarrow A(-s)A(s) + B(-s)B(s) = 1 \Leftrightarrow \mathcal{A}^{\sim}\mathcal{A} + \mathcal{B}^{\sim}\mathcal{B} = 1 \tag{2-47}$$

where $\beta(\omega)$ is an ideal high-pass filter with the same corner frequency as the low-pass filter $\alpha(\omega)$. Now, the finite-frequency VSP is described in the time domain using the Inequality (2-42) with $\mathbf{Q}, \mathbf{S}, \mathbf{R}$ operators defined as

$$\mathbf{Q} = -\epsilon\mathcal{A}^{\sim}\mathcal{A}, \tag{2-48a}$$

$$\mathbf{S} = \frac{1}{2}\mathcal{A}^{\sim}\mathcal{A}, \tag{2-48b}$$

$$\mathbf{R} = -\delta\mathcal{A}^{\sim}\mathcal{A}. \tag{2-48c}$$

The finite-frequency finite-gain can be characterized in the time-domain in an analogue manner.

A generalization of the KYP lemma, which can be used to prove the finite-frequency properties of the LTI state-space systems has been introduced in [33]. The authors also provide an alternative time-domain interpretation of the finite-frequency properties [34].

2-5-3 Hybrid passivity and finite gain

A system has the *hybrid passivity and finite-gain* property, introduced in [30], if it is passive in the given low-frequency range and has a finite gain at other frequencies. This property is expressed in the following definition.

Definition 2-5.1. *[30] System (2-38) is hybrid passive and finite-gain if there exists a storage function V defined in Definition 2-4.6 such that the Inequality 2-43 is satisfied with*

$$\mathcal{Q}(\omega) = -\left[\epsilon\alpha(\omega) + \gamma^{-1}(1-\alpha(\omega))\right] \tag{2-49a}$$

$$\mathcal{S}(\omega) = \frac{1}{2}\alpha(\omega) \tag{2-49b}$$

$$\mathcal{R}(\omega) = \left[\gamma(1-\alpha(\omega)) - \delta\alpha(\omega)\right]. \tag{2-49c}$$

Remark 2-5.2. *[30] Alternatively, the hybrid passivity and finite-gain can be described in the time-domain using the Inequality (2-42) and*

$$\mathbf{Q} = -\left[\epsilon\mathcal{A}^{\sim}\mathcal{A} + \gamma^{-1}\mathcal{B}^{\sim}\mathcal{B}\right] \tag{2-50a}$$

$$\mathbf{S} = \frac{1}{2}\mathcal{A}^{\sim}\mathcal{A} \tag{2-50b}$$

$$\mathbf{R} = \left[\gamma\mathcal{B}^{\sim}\mathcal{B} - \delta\mathcal{A}^{\sim}\mathcal{A}\right]. \tag{2-50c}$$

If $V(0) = 0$ and $x(0) = 0$ are imposed, Definition 2-4.7 can be used to characterize dissipativity, what simplifies analysis of systems. In this case, the change of the storage function does not need to be included in the considered inequalities.

In [30] it has been shown that the hybrid passivity/finite-gain property leads to the hybrid passivity and small-gain theorem. The hybrid approach has been used to give sufficient conditions for both the Lyapunov and $\mathcal{L}_2$ stability of nonlinear systems.

Theorem 2-5.3. *[30] Consider the negative feedback interconnection (presented in Fig. (2-3)) of two systems which are dissipative w.r.t. $(\mathbf{Q}_i, \mathbf{S}_i, \mathbf{R}_i)$ supply rates defined in Equation (3-25) with the same operators $\mathcal{A}$, $\mathcal{B}$. A sufficient condition for the feedback system with no exogenous inputs $r_1 = r_2 = 0$ to have a globally asymptotically stable equilibrium is*

$$\left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} -\mathbf{Q}_1 - \mathbf{R}_2 & \mathbf{S}_1 - \mathbf{S}_2^{\sim} \\ \mathbf{S}_1^{2} - \mathbf{S}_2 & -\mathbf{R}_1 - \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle_T > 0 \tag{2-51}$$

for all admissible signals y_1, y_2 and all T.

Theorem 2-5.4. *[30] Consider the negative feedback interconnection (presented in Fig. (2-3)) of two systems which are dissipative w.r.t.* $(\mathbf{Q}_i, \mathbf{S}_i, \mathbf{R}_i)$ *supply rates defined in Equation (3-25) with the same operators* $\mathcal{A}$, $\mathcal{B}$. *A sufficient condition for the feedback system to be input-output* $\mathcal{L}_2$*-stable is*

$$\left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} -\mathbf{Q}_1 - \mathbf{R}_2 & \mathbf{S}_1 - \mathbf{S}_2^{\sim} \\ \mathbf{S}_1^{2} - \mathbf{S}_2 & -\mathbf{R}_1 - \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle_T > 0 \tag{2-52}$$

for all admissible signals y_1, y_2 *and all* T.

It has been also shown that the conditions of Theorems 2-5.3 and 2-5.4 are satisfied if the parameters $\epsilon_i, \delta_i, \gamma_i$ corresponding to the subsystems satisfy [30]

$$\epsilon_1 + \delta_2 > 0 \tag{2-53a}$$

$$\epsilon_2 + \delta_1 > 0 \tag{2-53b}$$

$$\gamma_1 \gamma_2 < 1 \tag{2-53c}$$

The difference between the conditions in Equation 2-53 and the standard passivity and finite gain theorems is that the parameters ϵ_i, δ_i and γ_i are defined on separate frequency ranges. This makes the hybrid approach applicable to a wider class of systems.

2-5-4 Finding the hybrid passivity and finite-gain parameters

After defining the hybrid passivity and finite-gain property, we focus on the ways to find whether a given system satisfies the conditions of the definition. The main focus is on the properties of SISO systems.

Analysis of the hybrid passivity and finite gain of the SISO LTI systems is relatively straight forward, which can already be seen in the example presented in Subsection 2-5-1. The corresponding procedure for finding the hybrid parameters is illustrated in Figure 2-5.

For a SISO LTI transfer function $H(j\omega)$ we have [35]

$$\delta = \inf_{-\omega_H < \omega < \omega_H} \operatorname{Re}\{H(j\omega)\}, \quad \kappa = \sup_{-\omega_H < \omega < \omega_H} |H(j\omega)|, \quad \epsilon = \frac{\delta}{2\kappa^2}, \quad \gamma = \sup_{|\omega| \geq \omega_H} |H(j\omega)|. \tag{2-54}$$

The benefit of the formulas above is that the values of estimates can be easily calculated. However, the estimate of ϵ is conservative. To obtain a closer estimate of ϵ, recall that the Nyquist plot of an output strictly passive SISO LTI system is inside a circle with center at $\frac{1}{2\epsilon}$ and radius $\frac{1}{2\epsilon}$. A similar approach is taken here to find the finite-frequency passivity parameters of an LTI system.

Proposition 2-5.5. *Consider a LTI system with a transfer function matrix* $H(s)$. *Assume that all the poles of* $H(s)$ *have negative real parts. The system is output strictly passive on a frequency range* $[-\omega_H, \omega_H]$ *if and only if there exists an* $\epsilon > 0$ *such that*

$$\operatorname{Re}\{H(j\omega)\} \geq \epsilon |H(j\omega)|^2, \quad \forall \omega \in [-\omega_H, \omega_H].$$

This is equivalent to

$$\left(\operatorname{Re}\{H(j\omega)\} - \frac{1}{2\epsilon}\right)^2 + \left(\operatorname{Im}\{H(j\omega)\}\right)^2 \leq \left(\frac{1}{2\epsilon}\right)^2, \quad \forall \omega \in [-\omega_H, \omega_H]. \tag{2-55}$$

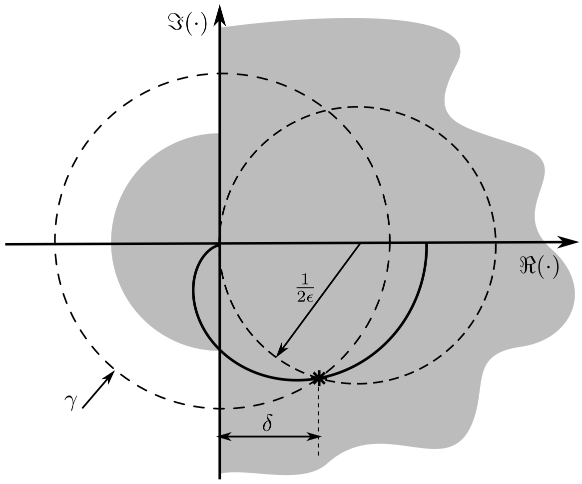


Figure 2-5: Illustration of the procedure for finding the hybrid passivity and finite-gain parameters for SISO LTI systems. The solid black line illustrates the frequency response of a system, and the border frequency ω_H is indicated with the symbol $*$. The choice of the ω_H influences the values of the parameters. Increasing the border frequency leads to a decrease of the high-frequency gain γ and of the low-frequency passivity parameters. The grey zone corresponds to the ω_H such that $\delta = 0$.

This result is given without a proof since it is a straightforward extension of a classical result for output passive systems [51]. In the SISO LTI case, the ϵ can be found by solving the linear optimization problem

$$\min \quad r \qquad \text{s.t.} -2\,\mathrm{Re}\,\{H(j\omega)\}\, r \leq -\left(\mathrm{Re}\,\{H(j\omega)\}^2 + \mathrm{Im}\,\{H(j\omega)\}^2\right), \tag{2-56}$$

with $\omega \in [-\omega_H, \omega_H]$ and $\epsilon = 1/2r$.

In the case of LTI *multiple-input multiple-output* (MIMO) systems, more complicated procedures have to be applied. In [57] and [37] the hybrid passivity and finite-gain parameters of MIMO LTI systems are found using the generalized Kalman–Yakubovich–Popov lemma [33]. In [58] a frequency-domain test to check whether a MIMO LTI systems satisfies the conditions of the "mixed" passivity and finite-gain property, related to the hybrid approach, is developed.

Although the conditions of the definition 2-5.1 can be satisfied by a nonlinear system, a general procedure to check if this is the case for a given system is not available. Only examples of analysis of specific classes of systems can be found in the literature.

In [36], gain scheduling of a number of hybrid passive and finite-gain LTI controllers is considered. It is shown, that use of a particular scheduling scheme yields a resulting controller also hybrid passive and finite-gain. The "mixed" passivity and finite-gain of switched systems is analysed in [38].

None of the approaches mentioned above can be directly applied for the hybrid passivity and finite-gain analysis of reset systems. Since the reset systems are nonlinear, the methods used for LTI systems are not applicable. Moreover, the two solutions proposed for the scheduled and switched systems can not be easily extended, since they rely on specific properties of the considered classes of systems.

2-6 Concluding remarks

This chapter presented the concepts that are fundamental to this research. In reset systems, the state jumps are introduced to alleviate the tradeoffs inherent to linear controllers. The stability of such systems is often shown using the so-called H_β condition. The fact that a parametric model of the system to be controlled is required for the stability analysis is one of the main factors hampering the wider adoption of reset control systems. The behaviour of the reset system can be approximately described in the frequency domain using the nonlinear frequency response functions and the describing functions.

The hybrid passivity and finite-gain approach can be used to guarantee the stability of a feedback interconnection of systems and is an extension of the passivity and the small-gain theorems. Finding whether a given system is hybrid passive and finite-gain is relatively simple in the LTI SISO case. While methods for concluding this property for specific classes of nonlinear systems exist, they cannot be easily extended to reset systems. This issue led to the formulations of the first research question. A solution to this problem is proposed in the next chapter.

Chapter 3

Hybrid passivity and finite-gain property for reset systems

In this chapter, the hybrid passivity and finite-gain properties of reset systems are developed. Analysis of the available methods for concluding whether a given system has the hybrid passivity and finite-gain, conducted in the previous chapter, showed that they cannot be extended to reset systems. To develop a method applicable to reset elements, we consider the finite-frequency passivity and the finite-frequency gain of a system separately. To simplify the problem, only passive systems are analysed. In the next step, the finite-frequency gain of a system is calculated using the nonlinear frequency response function. The adjustments in the definition of the hybrid passivity and finite-gain property, required to simplify the analysis of the reset systems, lead to a modification of hybrid passivity and small-gain theorem for the stability analysis of feedback systems. Finally, the hybrid passivity and gain of selected first-order reset systems are studied.

3-1 Finite-frequency passivity of reset systems

In this section, it is shown that a passive system is also finite-frequency passive. This result applies not only to reset systems but also to general nonlinear systems.

The passivity of a reset system is analysed in [55] using the dissipativity theory. It is shown that a reset system is passive if its base system is dissipative w.r.t. supply rate $w(u, y) = u^\top y$ with a storage function $V(x)$ such that

$$V(A_\rho x) - V(x) \leq 0 \tag{3-1}$$

for every $x \in \mathbb{R}^n$. The condition in Eq. (3-1) means that the storage function should not increase at reset instants.

To show that every passive system is also finite-frequency passive it is required that $V(0) = 0$. In this case, the Definition 2-4.7 according to Hill and Moylan can be used [51]. We have then the following result.

Theorem 3-1.1. *Every system satisfying the conditions of Definition 2-4.7 with the supply rate $w(u,y) = u^\top y$ is finite-frequency passive at any frequency interval $[-\omega_F, \omega_F]$.*

Proof. If a system satisfies the conditions of Definition 2-4.7 with the supply rate $w(u,y) = u^\top y$ it is passive. It holds then that

$$\int_0^T y_r^\top u_r \mathrm{d}t \geq 0 \qquad \Longleftrightarrow \qquad \int_{-\infty}^{\infty} y_{r,T}^\top(-j\omega) u_{r,T}(j\omega) \mathrm{d}\omega \geq 0, \tag{3-2}$$

where, with some abuse of notation, $u_{r,T}(j\omega), y_{r,T}(j\omega)$ denote the Fourier transform of signals $u_{r,T}(t), y_{r,T}(t)$.

Consider the supply rate corresponding to the finite-frequency passivity. We have

$$\mathrm{Re}\left\{\int_{-\infty}^{\infty} y_{r,T}^\top(-j\omega)\alpha(\omega) u_{r,T}(j\omega) \mathrm{d}\omega\right\} \geq \underbrace{\inf_{-\infty<\omega<\infty}\{\alpha(\omega)\}}_{=0} \underbrace{\mathrm{Re}\left\{\int_{-\infty}^{\infty} y_{r,T}^\top(-j\omega) u_{r,T}(j\omega) \mathrm{d}\omega\right\}}_{\geq 0} = 0 \tag{3-3}$$

where $\alpha(\omega)$, $\alpha : \mathbb{R} \to \mathbb{R}$ is an ideal low-pass filter, whose corner frequency ω_H defines the passive frequency range. The relationship in Eq. (3-3) confirms that every passive system with a storage function that satisfies $V(0) = 0$ is finite-frequency passive for any frequency range. □

3-2 Finite-frequency finite-gain of reset systems

A system has the finite-frequency finite-gain property if the relationship

$$\gamma^2 \int_{-\infty}^{\infty} u_{r,T}^\top(-j\omega)\beta(\omega) u_{r,T}(j\omega) \mathrm{d}\omega - \int_{-\infty}^{\infty} y_{r,T}^\top(-j\omega)\beta(\omega) y_{r,T}(j\omega) \mathrm{d}\omega \geq 0 \tag{3-4}$$

holds for any input signal u_r [30] . Function $\beta(\omega)$, $\beta : \mathbb{R} \to \mathbb{R}$ is a high-pass filter, defined in Section 2-5, whose corner frequency ω_H defines the finite-gain frequency range.

The property described by Equation 3-4 does not hold for any system producing output signal consisting of higher harmonics of the input signal. To see that, consider such a system with a sinusoidal input $u_r(t) = \sin(\omega t)$ with $\omega < \omega_H$. In this case, the first integral of (3-4) is equal to zero. Since the output of the system can be described as an infinite sum of higher harmonics of the input signal, the second integral is greater than zero. Condition (3-4) is therefore not satisfied.

A new definition of finite-frequency finite-gain property, more suited to systems producing higher harmonics of the input signal, is proposed.

Definition 3-2.1. *A system has the finite-frequency finite-gain property if*

$$\gamma^2 \int_{-\infty}^{\infty} u_{r,T}^\top(-j\omega) u_{r,T}(j\omega) \mathrm{d}\omega - \int_{-\infty}^{\infty} y_{r,T}^\top(-j\omega)\beta(\omega) y_{r,T}(j\omega) \mathrm{d}\omega \geq 0 \tag{3-5}$$

holds for any input signal u_r.

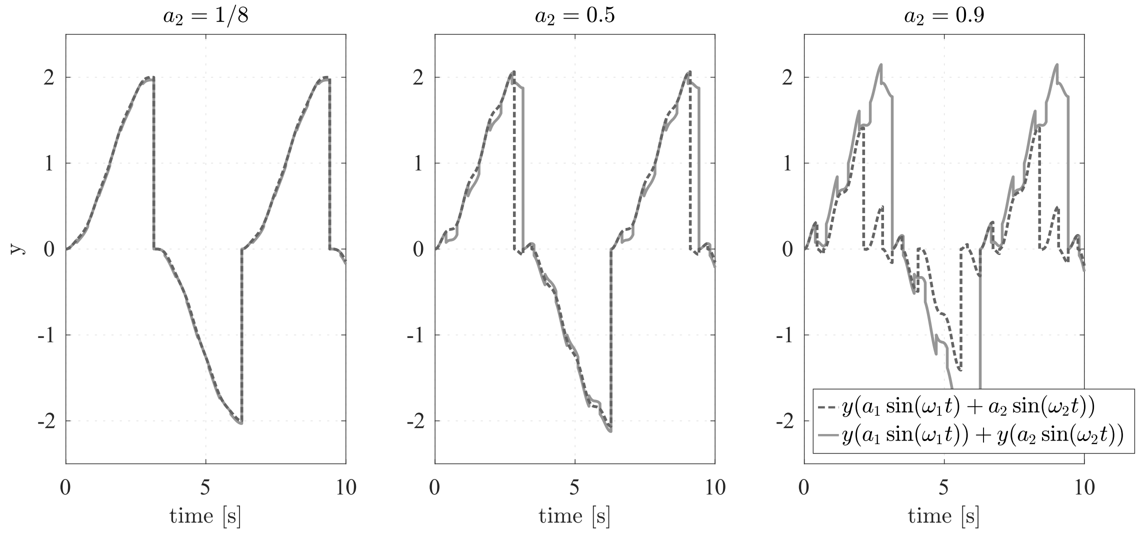


Figure 3-1: Comparison of a response of a reset integrator to a sum of sinusoids with sum of responses to component sinusoids. $\omega_1 = 1$ rad/s, $\omega_2 = 8$ rad/s, $a_1 = 1$, a_2 has been varied to illustrate different behaviours of the system.

In this definition, the "high-pass filter" $\beta(\omega)$ is included only in the output integral and not in the input integral.

For convergent reset systems the finite-frequency gain corresponding to a given border frequency ω_H, described by the property (3-5), can be found using the nonlinear FRF of the system. As described earlier, formulas for the steady-state output of a reset system with sinusoidal excitation given in [45] are the nonlinear FRF of the reset system. It is assumed, that the transient response of the reset controller does not have a significant influence on the stability of the overall system.

The superposition principle does not hold for reset systems with the state-dependent reset. Nevertheless, the following assumption is made

Assumption 3-2.2. *For the reset system excited by an input signal consisting of multiple harmonics, the magnitude of the sum of responses caused by separate harmonics is identical or greater than the magnitude of the response caused by combined harmonics.*

Assumption 3-2.2 is supported by an illustrative example presented in Figure 3-1, where a response of a reset integrator to a sum of sinusoids $a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t)$ with $\omega_1 = 1$ rad/s, $\omega_2 = 8$ rad/s and $a_1 = 1$, is considered. Presence of higher frequency sinusoid does not cause new reset instants when $a_2/a_1 \leq \omega_1/\omega_2$ [59], what can be observed in Figure 3-1 with $a_2 = 1/8$. When few new reset instants are introduced, magnitudes of $y(a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t))$ and $y(a_1 \sin(\omega_1 t)) + y(a_2 \sin(\omega_2 t))$ are comparable, what can be seen in Figure 3-1 with $a_2 = 0.5$. When many new resets are introduced $y(a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t)) \ll y(a_1 \sin(\omega_1 t)) + y(a_2 \sin(\omega_2 t))$.

The following definition is introduced as a step towards finding the high-frequency gain of a convergent system, defined in the Definition 3-2.1.

Definition 3-2.3. *The high-frequency nonlinear frequency response function $\mathcal{B}\bar{y}(t)$, with the border frequency ω_H, of a convergent system is given by*

$$\mathcal{B}\bar{y}(t) = \sum_{n=n_F+1}^{\infty} \bar{Y}_n(\omega)e^{j\omega nt} = \bar{y}(t) - \sum_{n=0}^{n_F} \bar{Y}_n(\omega)e^{j\omega nt}, \tag{3-6}$$

where $n_F = \max n \in \mathbb{Z}_+$ s.t. $n\omega \leq \omega_H$, $\bar{Y}_n(\omega)$ denotes the n-th HOSIDF and $\bar{y}(t)$ is the nonlinear FRF of the system.

To motivate the Definition 3-2.3, consider the steady-state response $\bar{y}(t)$ of the reset system corresponding to a sinusoidal excitation with frequency ω. $\bar{y}(t)$ can be expressed as an infinite sum of the Fourier series components $\bar{Y}_n(\omega)$, $n \in \mathbb{Z}_+$

$$\bar{Y}_n(\omega) = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \bar{y}(t)e^{-j\omega nt}\mathrm{d}t \tag{3-7}$$

$$\bar{y}(t) = \sum_{n=0}^{\infty} \bar{Y}_n(\omega)e^{j\omega nt} \tag{3-8}$$

Analytical formulas for $\bar{Y}_n(\omega)$ have been derived in [46] and are presented in Section 2-3-3 of this report.

To find the finite-frequency gains described by Eq. (3-5) with given ω_H, the right-hand side of Eq. (3-8) is written as

$$\bar{y}(t) = \sum_{n=0}^{n_F} \bar{Y}_n(\omega)e^{j\omega nt} + \sum_{n=n_F+1}^{\infty} \bar{Y}_n(\omega)e^{j\omega nt}, \tag{3-9}$$

where $n_F = \max n \in \mathbb{Z}_+$ s.t. $n\omega \leq \omega_H$. Now, the low- and high-pass filtered steady state response of the convergent reset system can be calculated as

$$\mathcal{A}\bar{y}(t) = \sum_{n=0}^{n_F} \bar{Y}_n(\omega)e^{j\omega nt}, \tag{3-10}$$

$$\mathcal{B}\bar{y}(t) = \sum_{n=n_F+1}^{\infty} \bar{Y}_n(\omega)e^{j\omega nt} = \bar{y}(t) - \mathcal{A}\bar{y}(t) = \bar{y}(t) - \sum_{n=0}^{n_F} \bar{Y}_n(\omega)e^{j\omega nt}, \tag{3-11}$$

where $\mathcal{A}$ and $\mathcal{B}$ are operators used to express the low- and high-pass filters $\alpha(\omega)$ and $\beta(\omega)$ in the time domain, as described in Section 2-5. The low and high-frequency components of the steady state response of a rest integrator are illustrated in Figure 3-2.

The high-frequency nonlinear FRF of a convergent system, introduced in the Definition 3-2.3, is used to find the high-frequency gain of a convergent reset system.

Theorem 3-2.4. *The high-frequency gain of a convergent system, with the border frequency ω_H, defined in the Definition 3-2.1, is given by*

$$\gamma = \sup_{-\infty<\omega<\infty} \gamma_{\mathcal{B}}(\omega), \tag{3-12}$$

where $\gamma_{\mathcal{B}}(\omega)$ is the gain, calculated as described in Equation (3-22) or (3-23), of the high-frequency nonlinear FRF of the system with the same border frequency ω_H.

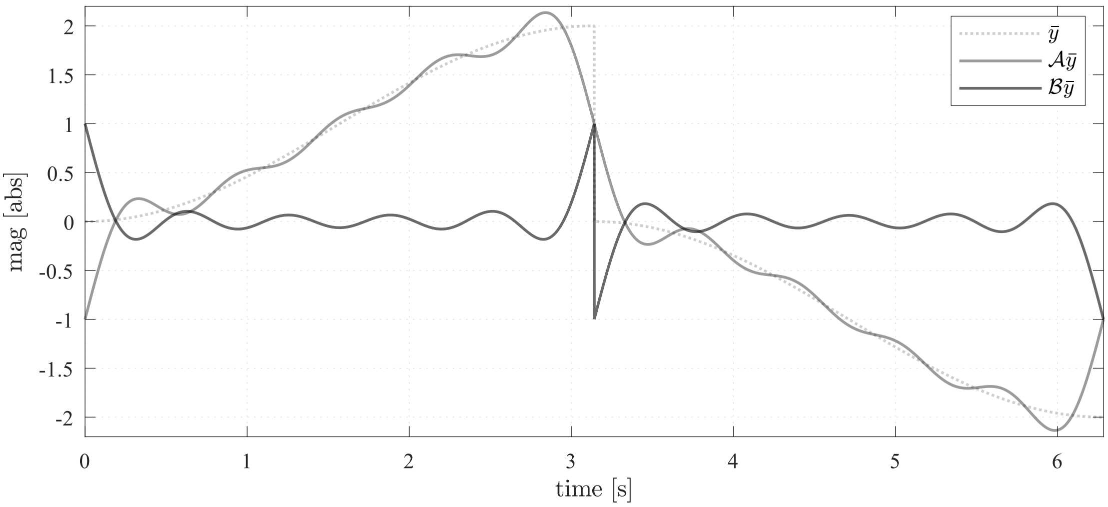


Figure 3-2: Low and high-frequency components of a steady-state response of a reset integrator to input $\sin t$ with $\omega_H = 10\,\mathrm{rad\,s^{-1}}$

Proof. We suppose assumption 3-2.2 is satisfied. Consider a response of a reset system to an arbitrary signal. The input signal $u_{tot}(t)$ can be expressed using the Fourier series as a sum of sinusoids $u_i(t) = a\sin(\omega_i t + \phi_i)$.

$$u_{tot}(t) = \sum_{i=1}^{\infty} u_i(t). \tag{3-13}$$

According to the Assumption 3-2.2, the high frequency component of the steady-state response to the total signal satisfies

$$\mathcal{B}\bar{y}_{tot}(t) \leq \sum_{i=1}^{\infty} \mathcal{B}\bar{y}_i(t), \tag{3-14}$$

where $\bar{y}_i(t)$ denote the steady-state response of the reset system to a component sinusoid $u_i(t)$. We have then

$$|\mathcal{B}\bar{y}_{tot}(t)|^2 \leq |\sum_{i=1}^{\infty} \mathcal{B}\bar{y}_i(t)| \leq \sum_{i=1}^{\infty} |\mathcal{B}\bar{y}_i(t)| \tag{3-15}$$

$$\int_0^{\infty} |\mathcal{B}\bar{y}_{tot}(t)|^2 \mathrm{d}t \leq \sum_{i=1}^{\infty} \int_0^{\infty} |\mathcal{B}\bar{y}_i(t)|^2 \mathrm{d}t \tag{3-16}$$

Using the Parseval's theorem we have

$$\int_{-\infty}^{\infty} \bar{y}_{tot}^{\top}(-j\omega)\beta(\omega)\bar{y}_{tot}(j\omega)\mathrm{d}\omega \leq \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} \bar{y}_i^{\top}(-j\omega)\beta(\omega)\bar{y}_i(j\omega)\mathrm{d}\omega \tag{3-17}$$

where $\bar{y}_{tot}(j\omega)$ and $\bar{y}_i(j\omega)$ denote the Fourier transforms of the total steady-state response and the response to the component sinusoid of the input.

In the second step, note that the high-frequency component of the response $\mathcal{B}\bar{y}(t)$ can be used to find the finite-frequency gain of the considered system. First, consider the high-frequency component $\mathcal{B}\bar{y}(t)$ of the response to the sinusoidal excitation $u_i(t)$. The corresponding non-linear bode plot magnitudes $\gamma_{\mathcal{B},abs}$, $\gamma_{\mathcal{B},rms}$ can be calculated as described in Equations (3-22) and (3-23). We have then

$$\int_0^{\infty} |\mathcal{B}\bar{y}_i(T)|^2 \mathrm{d}t \le \gamma^2_{\mathcal{B},abs}(\omega_i) \int_0^{\infty} |a_i \sin(\omega_i t + \phi_i)|^2 \mathrm{d}t, \tag{3-18}$$

$$\int_0^{\infty} |\mathcal{B}\bar{y}_i(T)|^2 \mathrm{d}t \le \gamma^2_{\mathcal{B},rms}(\omega_i) \int_0^{\infty} |a_i \sin(\omega_i t + \phi_i)|^2 \mathrm{d}t. \tag{3-19}$$

Using the Parseval's theorem, the relationships above can be translated to the frequency domain as

$$\int_{-\infty}^{\infty} \bar{y}_i^\top(-j\omega)\beta(\omega)\bar{y}_i(j\omega)\mathrm{d}\omega \le \gamma^2_{\mathcal{B},abs}(\omega_i) \int_{-\infty}^{\infty} u_i^\top(-j\omega)u_i(j\omega)\mathrm{d}\omega, \tag{3-20}$$

$$\int_{-\infty}^{\infty} \bar{y}_i^\top(-j\omega)\beta(\omega)\bar{y}_i(j\omega)\mathrm{d}\omega \le \gamma^2_{\mathcal{B},rms}(\omega_i) \int_{-\infty}^{\infty} u_i^\top(-j\omega)u_i(j\omega)\mathrm{d}\omega, \tag{3-21}$$

where $u_i(j\omega)$ denotes the Fourier transform of the input signal $u_i(t)$.

When Equation (3-17) is combined with (3-20) we get

$$\begin{aligned}\int_{-\infty}^{\infty} \bar{y}_{tot}^\top(-j\omega)\beta(\omega)\bar{y}_{tot}(j\omega)\mathrm{d}\omega &\le \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} \bar{y}_i^\top(-j\omega)\beta(\omega)\bar{y}_i(j\omega)\mathrm{d}\omega \\ &\le \sum_{i=1}^{\infty} \gamma^2_{\mathcal{B},abs}(\omega_i) \int_{-\infty}^{\infty} u_i^\top(-j\omega)u_i(j\omega)\mathrm{d}\omega \\ &= \big(\sup_{-\infty<\omega<\infty} \gamma_{\mathcal{B},abs}(\omega)\big)^2 \int_{-\infty}^{\infty} u_{tot}^\top(-j\omega)u_{tot}(j\omega)\mathrm{d}\omega\end{aligned} \tag{3-22}$$

Similarly, after combining Equations (3-17) and (3-21) we get

$$\int_{-\infty}^{\infty} \bar{y}_{tot}^\top(-j\omega)\beta(\omega)\bar{y}_{tot}(j\omega)\mathrm{d}\omega \le \big(\sup_{-\infty<\omega<\infty} \gamma_{\mathcal{B},rms}(\omega)\big)^2 \int_{-\infty}^{\infty} u_{tot}^\top(-j\omega)u_{tot}(j\omega)\mathrm{d}\omega \tag{3-23}$$

□

Inequalities (3-22) and (3-23) present two approaches to estimate the finite-frequency gain of reset system based on its nonlinear FRF. The presented method may be easily applied to other classes of convergent systems.

3-3 Stability of systems with the modified hybrid passivity and finite gain definition

In the previous sections, the finite-frequency passivity and gain of reset systems were considered. Based on them, a modified definition of hybrid passivity and finite-gain, suitable for systems creating harmonics of the input signal in their response, is given below

Definition 3-3.1. *The reset system (2-7) is said to be hybrid passive and finite-gain if it satisfies the dissipation inequality (2-43) with*

$$\mathcal{Q}(\omega) = -\left[\epsilon\alpha(\omega) + \gamma^{-1}(1-\alpha(\omega))\right] \tag{3-24a}$$
$$\mathcal{S}(\omega) = \frac{1}{2}\alpha(\omega) \tag{3-24b}$$
$$\mathcal{R}(\omega) = [\gamma - \delta\alpha(\omega)] \tag{3-24c}$$

Remark 3-3.2. *Equivalently, the hybrid passivity and finite-gain can be described in the time-domain using the Inequality (2-43) with*

$$\mathbf{Q} = -\left[\epsilon\mathcal{A}^{\sim}\mathcal{A} + \gamma^{-1}\mathcal{B}^{\sim}\mathcal{B}\right] \tag{3-25a}$$
$$\mathbf{S} = \frac{1}{2}\mathcal{A}^{\sim}\mathcal{A} \tag{3-25b}$$
$$\mathbf{R} = [\gamma - \delta\mathcal{A}^{\sim}\mathcal{A}] \tag{3-25c}$$

Note, that systems producing higher harmonics of the input signals, including the reset systems, may still satisfy the conditions of the Definition 2-5.1. However, this can not be shown by considering the passivity and gain of a system over separate frequency ranges, like it is done for LTI systems. The reason is that the nonlinear systems, unlike the LTI systems, can shift the energy of an input signal to other frequencies. Additionally, note that the Section 3-1 provides tools only to the check finite-frequency passivity with $\delta = \epsilon = 0$.

Consider a negative feedback interconnection of a system that is hybrid passive and finite-gain according to the Definition 2-5.1, with a system satisfying the modified definition of hybrid passivity and finite gain 3-3.1. Such a system can be obtained, for example, by connecting appropriate linear and reset systems.

Since the modified definition 3-3.1 has been used, the hybrid passivity and small-gain theorem (2-5.3, 2-5.4) has to be adjusted.

Theorem 3-3.3. *The negative feedback interconnection of systems $H_1 : \mathcal{L}_{2,e} \to \mathcal{L}_{2,e}$ and $H_2 : \mathcal{L}_{2,e} \to \mathcal{L}_{2,e}$, presented in Figure 2-3, where system H_1 satisfies conditions of Definition 2-5.1 and system H_2 satisfies conditions of Definition 3-3.1, is $\mathcal{L}_2$-stable if*

$$\epsilon_1 + \delta_2 - \gamma_2 > 0 \tag{3-26a}$$
$$\delta_1 + \epsilon_2 > 0 \tag{3-26b}$$
$$\gamma_1\gamma_2 < 1 \tag{3-26c}$$

Proof. As in Theorem 2-5.4, to guarantee the Lyapunov and $\mathcal{L}_2$-stability of the feedback system we require

$$\left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \overbrace{\begin{bmatrix} -\mathbf{Q}_1 - \mathbf{R}_2 & \mathbf{S}_1 - \mathbf{S}_2^{\sim} \\ \mathbf{S}_1^{2} - \mathbf{S}_2 & -\mathbf{R}_1 - \mathbf{Q}_2 \end{bmatrix}}^{\mathbf{M}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle_T > 0 \tag{3-27}$$

Assume that $\mathbf{Q}_1, \mathbf{S}_1, \mathbf{R}_1$ correspond to the subsystem satisfying the definition 2-5.1 and $\mathbf{Q}_2, \mathbf{S}_2, \mathbf{R}_2$ to the subsystem satisfying the definition 3-3.1. Consider the operator $\mathbf{M}$ in the frequency domain. In the considered case, it is given by

$$\mathbf{M}(\omega) = \begin{bmatrix} \epsilon_1\alpha(\omega) + \gamma_1^{-1}\beta(\omega) - \gamma_2 + \delta_2\alpha(\omega) & \frac{1}{2}\alpha(\omega) - \frac{1}{2}\alpha(\omega) \\ \frac{1}{2}\alpha(\omega) - \frac{1}{2}\alpha(\omega) & -\gamma_1\beta(\omega) + \delta_1\alpha(\omega) + \epsilon_2\alpha(\omega) + \gamma_2^{-1}\beta(\omega) \end{bmatrix} \tag{3-28}$$

When $\alpha(\omega) = 1$, $\beta(\omega) = 0$ $\mathbf{M}(\omega)$ is given by

$$\mathbf{M}(\omega)|_{\alpha(\omega)=1} = \begin{bmatrix} \epsilon_1 + \delta_2 - \gamma_2 & 0 \\ 0 & \delta_1 + \epsilon_2 \end{bmatrix} \tag{3-29}$$

When $\alpha(\omega) = 0$, $\beta(\omega) = 1$ $\mathbf{M}(\omega)$ is given by

$$\mathbf{M}(\omega)|_{\alpha(\omega)=0} = \begin{bmatrix} \gamma_1^{-1} - \gamma_2 & 0 \\ 0 & \gamma_2^{-1} - \gamma_1 \end{bmatrix} \tag{3-30}$$

Recall that α and β are defined over $\omega \in (-\infty, \infty)$ such that over the "passive frequency range" $(-\omega_H, \omega_H)$ $\alpha(\omega) = 1$ and $\beta(\omega) = 0$, and overt the $(-\infty, -\omega_H] \cup [\omega_H, \infty)$ $\alpha(\omega) = 0$ and $\beta(\omega) = 1$. Clearly, inequality (3-28) is satisfied if

$$\epsilon_1 + \delta_2 - \gamma_2 > 0 \tag{3-31a}$$
$$\delta_1 + \epsilon_2 > 0 \tag{3-31b}$$
$$\gamma_1\gamma_2 < 1 \tag{3-31c}$$

□

The difference between conditions in (3-26) and the conditions of the hybrid passivity and small-gain theorem presented in (2-53) is presence of $-\gamma_2$ in the first inequality. This is caused by the change of the definition of the finite-frequency gain for reset systems, which resulted in an artificial shortage of input passivity. In view of Section 3-1, this shortage has to be compensated by the excess of output passivity in the ϵ_1, since a way to calculate δ_2 is not available.

3-4 Hybrid properties of first-order reset systems

In this section, the finite-frequency properties of *first-order rest elements* (FORE) are analysed. The FORE are frequently used in reset control systems. They can be applied both as Clegg integrators or reset low-pass filters and be a basis for more complex elements like the CgLp [20]. For this reason, they are a relevant class of elements to study.

After introduction of the FORE, the finite-frequency passivity and gain of reset integrators and low-pass filters are analysed separately. Next, the hybrid passivity and finite-gain is concluded as described in the previous chapter.

3-4-1 First order reset elements

The *First Order Reset Element* (FORE) is a reset element described by Equation (2-3) with $x_r \in \mathbb{R}$, $e \in \mathbb{R}$ and $u \in \mathbb{R}$. The two most commonly used FOREs are reset integrators and reset low-pass filters.

The matrices of the reset integrator in (2-3) are

$$A_r = 0, \qquad B_r = 1, \qquad C_r = 1, \qquad D_r = 0, \qquad A_{\rho_r} = \rho, \tag{3-32}$$

where $\rho \in \mathbb{R}$ is a parameter describing the level of reset. The reset integrator was the first reset element introduced in [12].

The matrices of the first order reset low-pass filter in (2-3) are

$$A_r = -\omega_r, \qquad B_r = \omega_r, \qquad C_r = 1, \qquad D_r = 0, \qquad A_{\rho_r} = \rho, \tag{3-33}$$

where $\omega_r \in \mathbb{R}$ is the corner frequency of the filter. The element has been introduced in [60] and generalized in [45], by considering non full reset $\rho \neq 0$.

The benefits of using reset integrators and low-pass filters can be seen with the DF analysis. At steady state, both elements have the magnitude of the response similar to their linear counterparts while the phase lag they introduce is reduced and depends on the value of the parameter ρ [45].

3-4-2 Finite-frequency passivity of FORE

As has been shown in Section 3-1, a passive system is finite-frequency passive for any frequency range. This fact, in combination with [55], is used here to show finite frequency passivity of reset systems.

According to Theorem 2-4.9, the reset system is passive if its base system is dissipative with respect to supply rate $w = eu$ with a storage function $V(x)$ satisfying $V(A_{\rho_r x}) \leq V(x), \forall x \in \mathbb{R}$.

Consider a time-regularized reset low-pass filter with (3-33) and a quadratic storage function $V(x) = \frac{1}{2\omega_r}x^2$. The dissipativity of the base dynamics can be studied using the *differential dissipation inequality* [26]. We have

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = -x^2 + xe = -u^2 + eu \leq eu \tag{3-34}$$

It can be concluded that the base dynamics of the system are dissipative w.r.t. $w = eu$ for all $\omega_r > 0$. For the storage function we have

$$V(A_{\rho_r x}) = \rho^2 \frac{1}{2\omega_r}x^2 \leq \frac{1}{2\omega_r}x^2 = V(x), \quad \forall x \in \mathbb{R}, \forall \rho \in (-1, 1) \tag{3-35}$$

A similar procedure can be repeated for the reset integrator. It is concluded that time-regularized reset integrators and low-pass filters are finite-frequency passive at any frequency range if $\rho \in (-1, 1)$ and $\omega_r > 0$.

3-4-3 Finite-frequency gain of FORE

The finite-frequency $\mathcal{L}_2$-gain has been defined for reset systems in Section 3-2. Here, the finite-frequency gains of selected FORE are analysed.

A Matlab implementation of the procedure for calculating the gain is presented in Appendinx A. For each frequency of the input sinusoid, the steady-state response of the element is calculated using formula (2-27). Subsequently, the HOSIDF up to the considered corner frequency ω_H are computed, and the low and high-frequency parts of the output signal are obtained, as described in Equations (3-10) and (3-11). Next, the gains $\gamma_{\mathcal{A}}(\omega)$ and $\gamma_{\mathcal{B}}(\omega)$ for the specific frequency can be calculated as in Equations (2-19) and (2-20). This procedure is repeated for a grid of frequencies. The finite-frequency gain of the reset system is the supremum of $\gamma_{\mathcal{B},abs}(\omega)$ or $\gamma_{\mathcal{B},rms}(\omega)$ over the entire frequency range.

The nonlinear bode magnitude plots $\gamma_{\mathcal{A}}(\omega)$, $\gamma_{\mathcal{B}}(\omega)$, calculated using the absolute and rms-values of the signals, are compared for reset integrators and low-pass filters in Figure 3-3. At high frequency range $\omega > \omega_H$, the steady-state response of the system is equal to the high-frequency component $\bar{y} = \mathcal{B}\bar{y}$ and the low-frequency component $\mathcal{A}\bar{y}$ is equal to zero. At low frequencies $\omega < \omega_H$, both components contribute to the steady-state response.

The high-frequency gain defined using the absolute value is significantly higher than to gain calculated using the rms values. This is a result of the nature of the high-frequency signal component, which can be observed in Figure 3-2.

Recall, that in the Theorem 3-2.4, the supremum of the gain of the high-frequency component $\mathcal{B}\bar{y}$ over all frequencies ω is used to find the high-frequency gain of the system. In the case of the reset integrator, the gain $\gamma_{\mathcal{B}}(\omega)$ of the high-frequency component is increasing as the frequency ω decreases, which can be seen in Figures 3-3a and 3-3b. It is therefore concluded that reset integrators do not have a finite high-frequency gain. The high-frequency gain can be found for the reset low-pass filter. In this case, the gain $\gamma_{\mathcal{B}}(\omega)$ has a clear maximum and decreases as frequency ω tends to zero (Figures 3-3c and 3-3d). It is concluded that the reset low-pass filters are hybrid passive and finite gain.

Figure 3-3d presents $\gamma_{\mathcal{B},rms}$ of the reset low-pass filter for different border frequencies ω_H. Using the Theorem 3-2.4, it can be concluded that the value of $\gamma_{\mathcal{B},rms}$ can be underestimated using the values of γ_{rms} at the corresponding ω_H. When the difference between ω_H and ω_R is smaller then approximately 1 order of magnitude, the $\gamma_{\mathcal{B},rms}$ is equal to $\gamma_{rms}(\omega_H)$. This observation is used later in this report to simplify the stability analysis of reset control systems.

3-5 Concluding remarks

In this chapter, the hybrid passivity and finite gain of reset systems were studied. In this way, the first research question: *How can the hybrid passivity and finite-gain framework be extended to reset systems?* was answered. These considerations lead to general observations, that apply to a wide range of nonlinear systems.

The finite frequency passivity and gain of systems were considered separately. It has been shown that every passive system is also finite-frequency passive on any frequency interval.

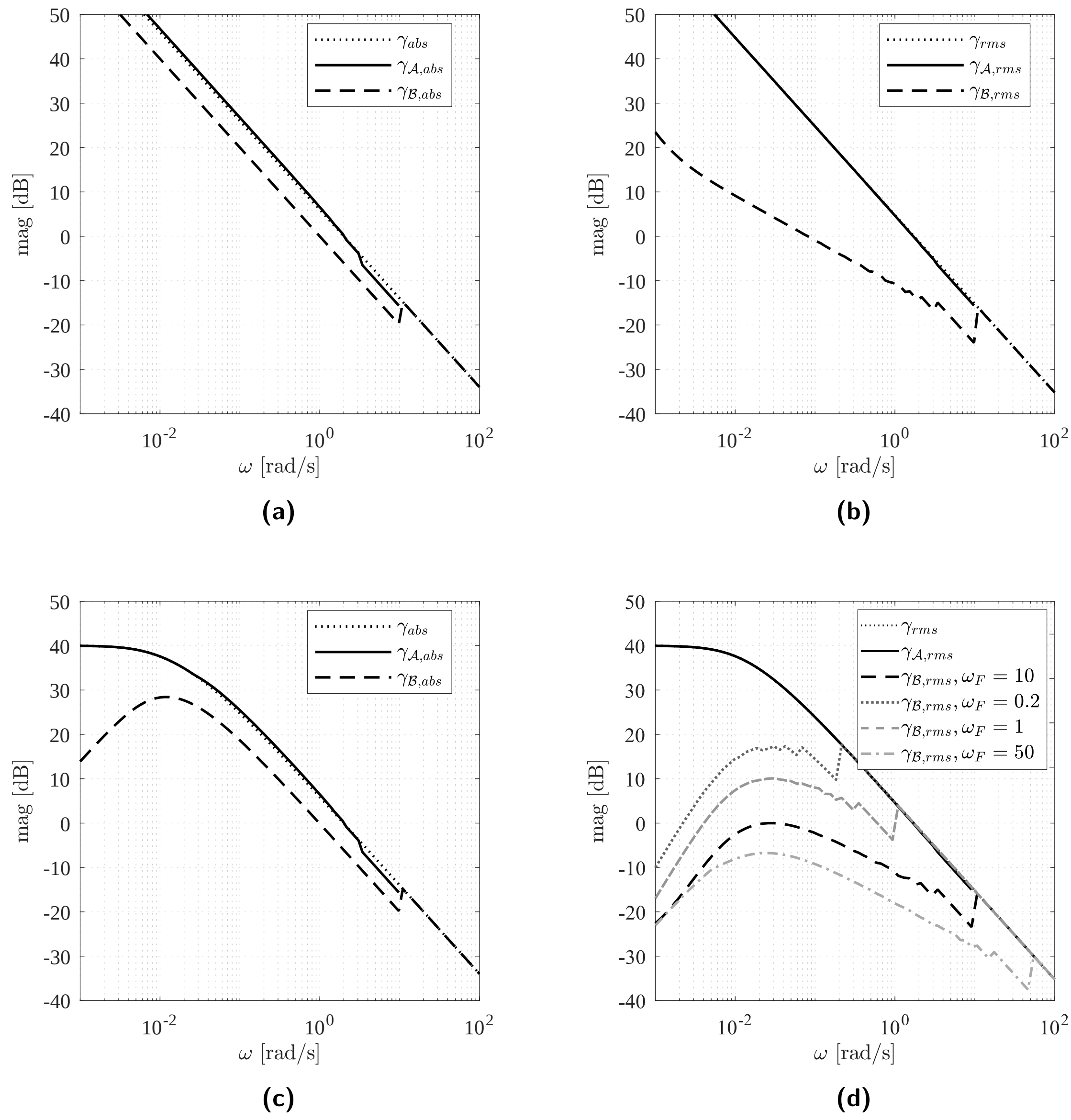


Figure 3-3: Nonlinear Bode magnitude plots of First order reset elements. (a),(b) Reset integrator with $A_r = 0, B_r = 1, C_r = 1, D_r = 0, A_\rho = 0$ and $\omega_H = 10\,\mathrm{rad\,s^{-1}}$,(c),(d) reset low-pass filter with $A_r = -0.01, B_r = 0.01, C_r = 100, D_r = 0, A_\rho = 0$ and $\omega_H = 10\,\mathrm{rad\,s^{-1}}$. Figure (d) presents also $\gamma_{\mathcal{B},rms}$ for other values of ω_H.

Analysis of finite-frequency gains of reset systems revealed that the definition of the finite-frequency passivity used in [30] is not suitable for nonlinear systems whose responses consist of higher harmonics of the input signal. To enable the analysis of hybrid passivity and gain of reset systems, an alternative definition of the finite-frequency passivity of finite gain was proposed. In consequence, a new definition of the hybrid passivity and finite-gain property was coined, and a new variant of the hybrid passivity and small-gain theorem for stability analysis of feedback systems was derived.

It is important to note that systems producing higher harmonics of the input signal may still have the hybrid passivity and finite-gain property defined in [30]. However, showing this is complicated, since the energy of the input signal can be shifted to other frequencies in the response.

The proposed approach has been used to analyse the properties of selected first-order reset elements. It has been concluded that first-order reset low-pass systems satisfy the conditions of the proposed definition of the hybrid passivity and finite-gain.

Chapter 4

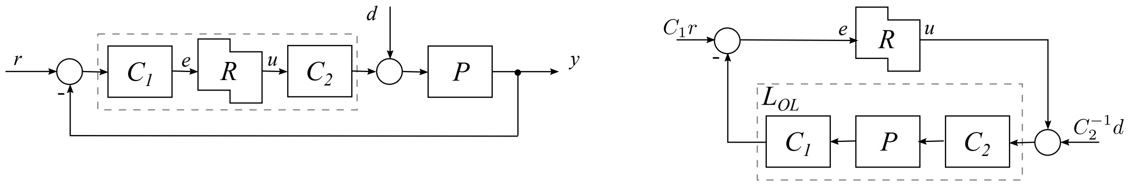
Application of the hybrid passivity and finite gain to stability of reset control systems

In this chapter, the hybrid passivity and finite gain of reset systems, derived in the previous chapter, are applied to the stability analysis of a reset control system. First, a class of control systems suitable for analysis using the proposed method is presented. Next, the stability of a reset control system for a precision positioning stage is studied as an example. The chapter is concluded with a summary regarding the benefits and limitations of the proposed method.

4-1 Reset control systems suitable for the hybrid stability analysis

The hybrid passivity and finite-gain theorem and its variant adjusted for reset systems (introduced in Section 3-3), can be used to analyse feedback interconnections of systems presented in Figure 2-3. In this section, examples of reset control systems that can be analysed using the Theorem 3-3.3 are studied. The scope of the analysis is limited to precision positioning systems, in which plants can be assumed to be LTI. Nevertheless, this framework may be extended to other kinds of systems and possibly nonlinear plants. The reset low-pass filter will be considered as an addition to a PID controller [61], to improve the performance of the system.

Figure 4-1a shows a general configuration of a reset controller [62], where P denotes an LTI plant, C_1, C_2 denote LTI components of the controller and R is a reset element. Since one of the elements is nonlinear, the order of elements in the structure has an influence on the behaviour of the system. If C_1 is bounded and C_2 is invertible, the considered control system can be transformed into the feedback interconnection shown in Figure 4-1b. Here, the reset control system is divided into the linear part $L_{OL} = C_1 P C_2$ and the reset element R. This structure is suitable for the stability analysis using the proposed variant of the hybrid passivity and small-gain theorem.



(a) **(b)**

Figure 4-1: Transformation of a reset control system to a feedback interconnection. (a) Standard configuration, (b) Feedback interconnection suitable for hybrid passivity and small-gain stability analysis.

To conclude the stability of the reset control system in Figure 4-1, it is necessary that the linear subsystem L_{OL} is hybrid passive and finite-gain. The first class of suitable systems are plants with passivity violations due to high-frequency dynamics. Examples of such systems, like flexible structures and manipulators, can be found in literature [56][35][36][57][37].

Another class of systems that satisfy the requirement are systems modelled with the second-order LTI transfer function

$$G(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{4-1}$$

where ω_n is the undamped natural frequency and ζ is the damping ratio. This transfer function can be used as a simple model of a motion stage or of a complementary sensitivity function of a closed-loop control system. We are interested in the overall loop-shape, and the second-order transfer function is chosen only as a convenient example to analyse.

Figure 4-2 shows the Bode plot of the second-order system (4-1) with $\omega_n = 10\,\mathrm{rad\,s^{-1}}$ and different values of the damping ratio ζ. It can be seen that the system is hybrid passive and finite-gain, with ω_n separating the passive a finite-gain frequency regions.

The majority of reset control systems in the precision positioning applications have the structure presented in Figure 4-1 and the plants can be at least roughly approximated using the transfer function (4-1). However, the stability of such systems can rarely be concluded using the hybrid passivity and small-gain theorem. Typically, the bandwidth of the control system ω_c is significantly higher than the first natural frequency of the plant. In consequence, the product of high-frequency gains with the border frequency $\omega_H = \omega_n$ would be higher than 1. If the bandwidth $\omega_c \approx \omega_n$ is chosen, the corner frequency of the reset low-pass ω_R should be significantly lower then ω_n to compensate the magnitude peak of the subsystem L_{OL}.

To achieve greater design freedom, consider the reset control system with the configuration shown in Figure 4-3. In this case, the reset element is applied in parallel to the main control line, and an additional LTI element C_R is added. Similar to the previous configuration, if $C_1 C_R$ is bounded and C_2 is invertible, the control system can be transformed into structure presented in Figure 4-3b. We have then a feedback interconnection of the reset element R with a closed-loop LTI system, represented by the transfer function

$$L_{CL} = \frac{C_1 P C_2}{1 + C_1 P C_2 C_R}. \tag{4-2}$$

Note, that L_{CL} is identical to the complementary sensitivity function of the system with input w and output v in Figure 4-3b.

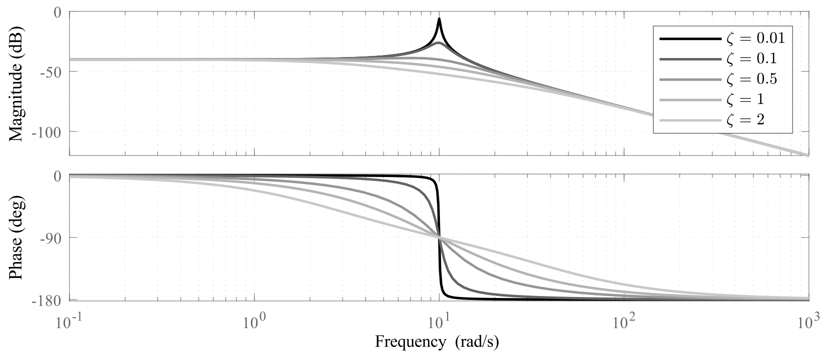


Figure 4-2: Bode plot of the second-order transfer function (4-1) with $\omega_n = 10\,\mathrm{rad\,s^{-1}}$ and different values of ζ.

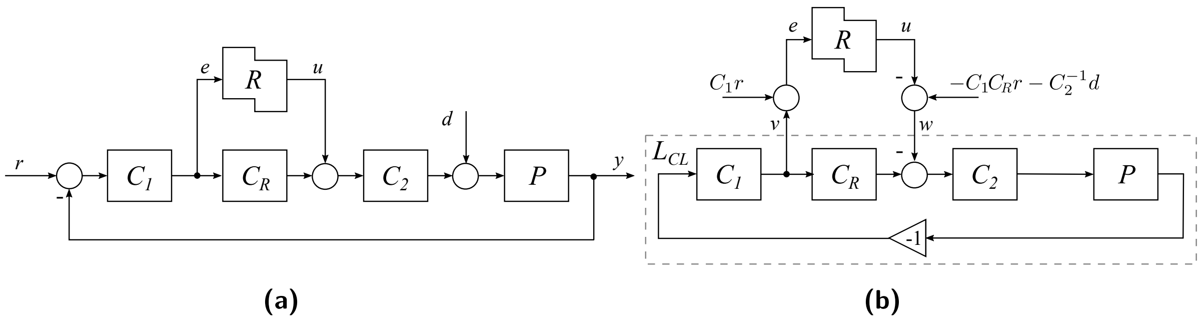


Figure 4-3: Transformation of a reset control system with a parallel configuration to a feedback interconnection. (a) General structure (b) Feedback interconnection suitable for hybrid passivity and small-gain stability analysis.

The dynamics of L_{CL}, consisting of a positioning stage as a plant and a PID-style controller, can be crudely approximated with the second-order transfer function (4-1), with ω_n equal to the bandwidth of the open-loop system $C_1PC_2C_R$. This fact can be used to gain insight into when the stability of reset control system with the structure presented in Figure 4-3 can be concluded using the Theorem 3-3.3. Similar to the case of the structure presented in Figure 4-1, the stability may be concluded using the hybrid passivity and finite-gain theorem if the corner frequency of the reset low-pass system is placed below ω_n. The benefit of configuration presented in Figure 4-3 is that the corner frequency ω_n of the 2nd-order approximation of the L_{CL} depends not only on the dynamics of the plant P but also on the control elements C_1, C_R and C_2. This leads to greater freedom of design. An example of such a system is analysed later in this chapter.

4-2 Illustrative example

To demonstrate the use of the hybrid passivity and finite-gain approach for the stability analysis of reset control systems we consider first a simple feedback interconnection of a second-order transfer function with a reset low-pass filter based on FORE. The structure presented in Figure 4-1 with

$$P = \frac{10000}{s^2 + 200s + 10000}, \qquad C_1 = 1, \qquad C_2 = k_P, \tag{4-3}$$

and R given in Equation (3-33) with $\omega_r = 1\,\mathrm{rad\,s^{-1}}$ is used for simplicity. $k_P \in \mathbb{R}$ denotes the proportional gain which is set to achieve a selected bandwidth ω_c of the open-loop system.

First, the bandwidth $\omega_c = 20\,\mathrm{rad\,s^{-1}}$ is selected. The DF based open-loop Bode plot of the system can be seen in Figure 4-4a. The resulting closed-loop system is quadratically stable, what was confirmed with the H_β condition presented in the Proposition 2-2.2 [41] [17].

To assess the stability of the system using the hybrid passivity and finite-gain approach, we need to find finite-frequency parameters $\epsilon_L, \delta_L, \gamma_L$ corresponding to the linear subsystem and the high-frequency gain of the reset element γ_R.

Figure 4-5a presents the values of terms appearing in the stability conditions for a range of frequencies ω_H. For each frequency ω_H, the parameters of the linear subsystems ϵ_L, δ_L and γ_L are calculated as described in the Section 2-5-4. The value of γ_R is underestimated by the value of the nonlinear FRF of the reset element at ω_H, as described in the Section 3-4-3. The estimated values of the terms appearing in the stability conditions of the Theorem 3-3.3 are shown with the solid lines. It can be seen, that there is a range of ω_H for which the underestimated parameters satisfy all the stability conditions. To conclude the stability $\omega_H = 50\,\mathrm{rad\,s^{-1}}$ is chosen and the value of γ_R is calculated using the Theorem 3-2.4. At $\omega_H = 50\,\mathrm{rad\,s^{-1}}$ we have $\delta_L = 6.470$, $\epsilon_L = 0.056$, $\gamma_L = 10.78$ and $\gamma_R = 0.046$. All the stability conditions are satisfied and the corresponding values are indicated with markers $*$ in the figure.

For the second example, the bandwidth $\omega_c = 300\,\mathrm{rad\,s^{-1}}$ is selected. The DF based open-loop Bode plot of the system can be seen in Figure 4-4b. The resulting closed-loop system is unstable.

The procedure described for the previous example is repeated, and the values of terms appearing in the stability conditions are plotted for a range of ω_H in Figure 4-5b. It can be seen that there does not exist ω_H for which the underestimated terms satisfy the stability conditions. The stability of the system can not be concluded using the hybrid passivity and small-gain theorem. The example illustrates that the proposed correctly does not indicate stability for unstable reset systems.

4-3 Application of the hybrid passivity and finite-gain theorem to the stability analysis of a reset control system for a precision positioning stage

In this section, an example of a reset control system in the parallel configuration introduced in Figure 4-3 is given. First, the overall design of the control system is presented. Next, the stability of several controllers with different parameters is analysed using the hybrid passivity and finite-gain method adjusted for reset systems.

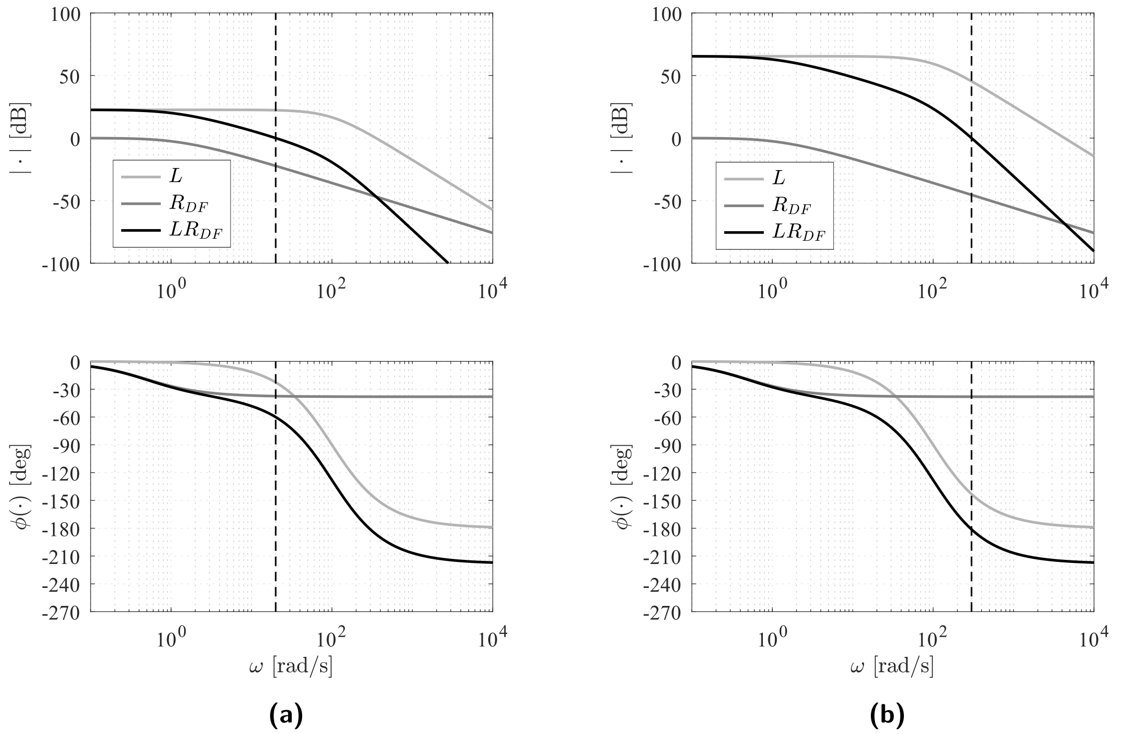
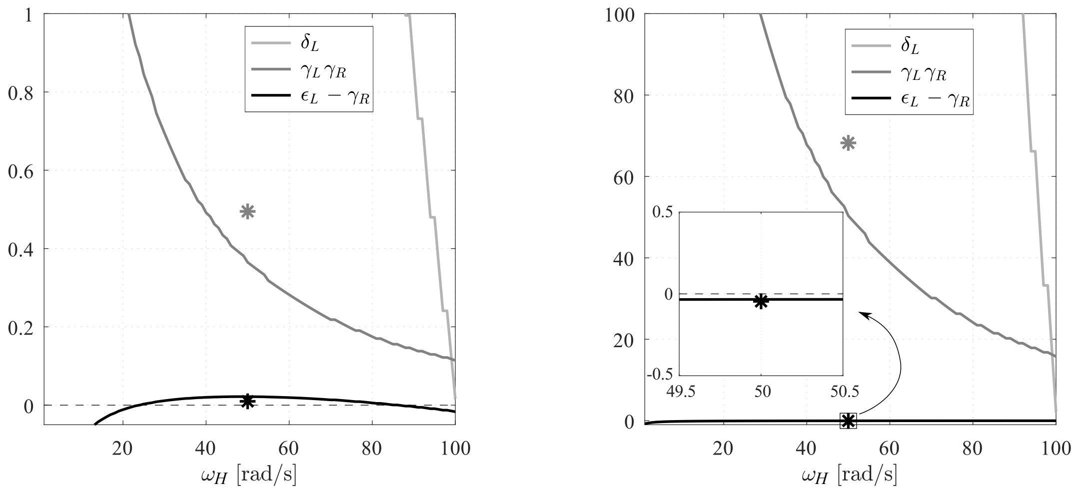


Figure 4-4: DF-based open-loop Bode plot of the feedback interconnection of a second-order LTI plant with a FORE low-pass filter. The proportional gain of the LTI system is set to achieve a selected bandwidth ω_c. (a) $\omega_c = 20\,\mathrm{rad\,s^{-1}}$, (b) $\omega_c = 300\,\mathrm{rad\,s^{-1}}$



(a) **(b)**

Figure 4-5: Hybrid passivity and small-gain stability analysis of (a) stable and (b) unstable reset control systems. The lines represent the approximated values of terms. Exact values, calculated for $\omega_H = 50\,\mathrm{rad\,s^{-1}}$, are indicated with the symbol $*$. Note that the scale of the y-axis is different in both figures.

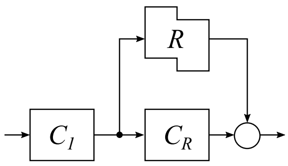


Figure 4-6: Parallel configuration of reset and LTI elements

4-3-1 Phase compensating reset element in parallel configuration

A reset element in the parallel configuration, inspired by the CgLp, is introduced here to demonstrate the use of the hybrid passivity and finite-gain approach for the stability analysis of reset systems. The structure of the developed element resembles that of the $PI + CI$ controller introduced in [14].

Consider the parallel configuration of reset and LTI elements presented in Figure 4-6. R is the first-order reset low-pass filter given by

$$R : \begin{cases} \dot{x}_r(t) = -\omega_R x_r(t) + \omega_R e(t), & \text{if } e \neq 0 \\ x_r(t^+) = A_\rho x_r(t), & \text{if } e = 0 \\ u = x_r(t), \end{cases} \tag{4-4}$$

where $e \in \mathbb{R}$, $u \in \mathbb{R}$, $x \in \mathbb{R}$ are the input, output and state of the reset element, $\omega_R \in \mathbb{R}$ is the corner frequency and $A_\rho \in (-1, 1]$ is the reset parameter of the element.

Table 4-1: Values of the offset parameter α and the phase advantage $\Delta\phi_{\omega_C}$ at ω_c (estimated using DF analysis) of the CgLpP element with $\omega_d = \omega_c/10$, $\omega_t = 1000\omega_c$ and different values of A_ρ

A_ρ	α	$\Delta\phi_{\omega_c}[°]$
0	1,28	25,62
0,2	1,14	19,43
0,4	1,06	13,64
0,6	1,02	8,42
0,8	1,00	3,86

C_1 is a linear lead filter

$$C_1(s) = \frac{s/\omega_d + 1}{s/\omega_t + 1}, \tag{4-5}$$

where $\omega_d, \omega_t \in \mathbb{R}$ are the corner frequencies and C_R is a linear low-pass filter

$$C_R(s) = \frac{1}{s/\omega_d + 1}. \tag{4-6}$$

The DF analysis is used to design a phase compensating reset controller. First, focus on the low-pass elements C_R and R. The frequency response of the element is approximated by adding the FRF of C_R with the first-order DF of R and presented in Figure 4-7a. $\omega_R = \omega_d/\alpha$ has been chosen, where α is a parameter to compensate for the shift of the corner frequency due to the introduction of the reset [20] [62]. Combining the elements $C_R + R$ results in doubling of the magnitude of a linear low-pass filter. The phase lag introduced by combined elements is approximately equal to $-64°$.

The combination of the lead filter C_1 with the low-pass element $C_R + R$ is presented in Figure 4-7b. The lead filter is centred at frequency $\omega_c = 100\,\mathrm{rad\,s^{-1}}$ by choosing $\omega_d = \omega_c/10$ and $\omega_t = 10\omega_c$. The parameter α which shifts the corer frequency of the reset element has been adjusted to minimize the variation of the magnitude of the element at frequencies lower than ω_t. The DF analysis indicates that the element provides a constant gain and phase advantage over a wide range of frequencies. At high frequencies, the element has a low-pass behaviour. The values of the offset parameter α and achieved phase advantage (estimated using the DF analysis) for the introduced CgLp element with different values of A_ρ are presented in Table 4-1.

It can be concluded that the obtained compound element closely resembles the CgLp element [20]. In the remainder of this report, it will be referred to as the *Constant in gain Lead in phase element in a parallel configuration* (CgLpP).

Note, that the series interconnection of C_1 and C_R is equivalent to a linear low-pass filter with the corner frequency at ω_t.

4-3-2 CgLpP in the PID framework

The phase advantage provided by the CgLpP element described in the previous section, corresponding to nearly constant magnitude over a wide frequency range, can be used to

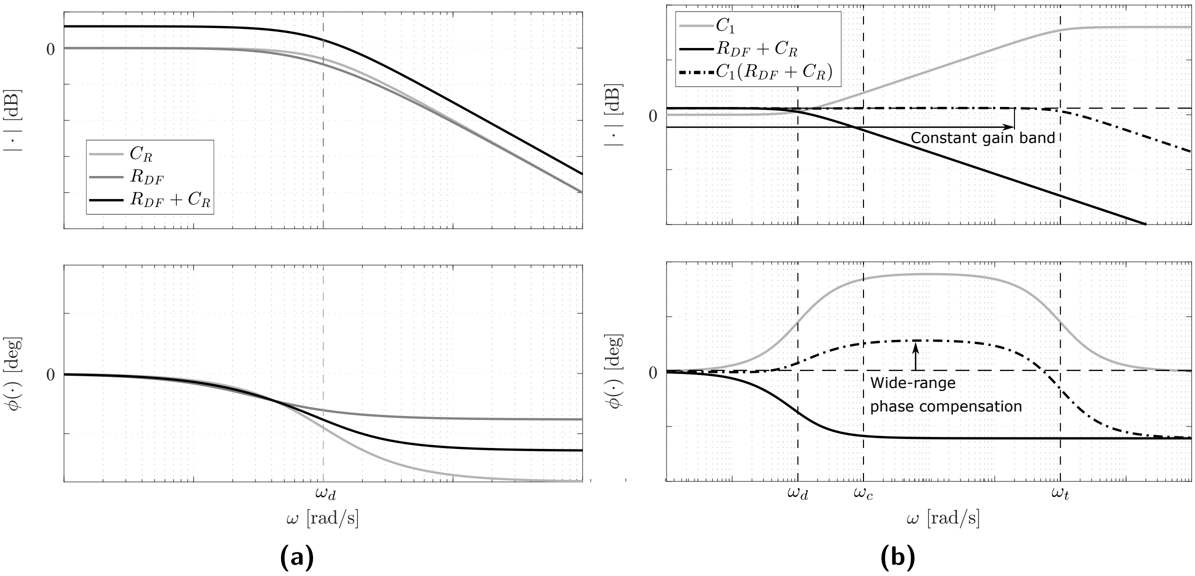


Figure 4-7: Frequency response approximated using the DF analysis. (a) Reset and linear low-pass filter in parallel configuration $\omega_d = 10\,\mathrm{rad\,s^{-1}}$, $\omega_R = \omega_d/1.62$ (b) Lead filter in series with the parallel combination of reset and linear low-pass filters. $\omega_c = 100\,\mathrm{rad\,s^{-1}}$, $\omega_d = \omega_c/10$, $\omega_t = 10\omega_c$, $\omega_R = \omega_d/1.27$, $A_\rho = 0$.

relax the Bode's gain-phase relationship inherent for linear controllers. In this section, the CgLpP element is integrated with a PID controller to improve the performance of the system.

A typical PID controller used in the precision positioning industry consists of a proportional gain K_p and integral, lead and low-pass elements

$$C_I = \frac{s + \omega_I}{s}, \qquad C_D = \frac{s/\omega_D + 1}{s/\omega_T + 1}, \qquad C_F = \frac{1}{s/\omega_F + 1}, \tag{4-7}$$

where ω_I is the frequency at which integrator is terminated, ω_D and ω_T are the beginning and end frequencies of the lead filter and ω_F is the corner frequency of the low-pass filter with $\omega_I < \omega_D < \omega_T < \omega_F$. Additional elements like notch filters and higher-order low-pass filters can be included in the structure. A procedure and rules of thumb for the design of PID controllers for high-tech applications are given in [61]. Once the bandwidth ω_c of the control system is chosen, the following corner frequencies can be set as a starting point for the design:

$$\omega_I = \omega_c/10, \qquad \omega_F = 10\omega_c, \qquad \omega_D = \omega_c/a, \qquad \omega_T = a\omega_c, \tag{4-8}$$

where $a > 1$ is a parameter introduced to adjust the amount of phase lead provided by the element C_D and to ensure that the maximum of added phase overlaps with ω_c.

When reset and linear controllers are integrated into one control system, the arrangement of elements in the structure has a significant influence on the performance. It is suggested in [62], that the best results can be achieved when lead filters are placed before a reset element and lag elements (like integrators and low-pass filters) after. Following this result, the lead filter C_D is implemented as a part of C_1 and elements C_I, C_F as part of C_2 in the structure presented in Figure 4-3.

The phase lead provided by the element C_D influences not only the stability/robustness of the control system but also its tracking and precision performance. This is a consequence of the Bode magnitude-phase relationship. In [20], three different scenarios for overcoming this fundamental problem by integrating a phase-compensating reset element with the PID controller are given. In all three cases an LTI PID controller, satisfying certain requirements regarding the systems bandwidth, phase margin, reference tracking and high-frequency disturbance rejection, is considered as a reference for comparison. In each of the scenarios, selected performance criteria can be improved without compromising the other. While the scenarios have been developed for the standard CgLp element, they can be also realized with the element in the parallel configuration.

- S1. The values of ω_D and ω_T can be fixed to the values obtained for linear PID and CgLp designed to add required additional phase and hence improve stability and robustness without affecting precision, tracking or bandwidth.
- S2. CgLp can be designed first to provide part of the phase resulting in a smaller scale a for derivative action (to obtain same phase margin), which should result in improved tracking and precision without affecting stability and bandwidth.
- S3. CgLp can be designed to provide part of the phase again as in the second case, but instead of improving precision, the closed-loop bandwidth of the system can be increased which thereby increases tracking as well without affecting stability or precision.

Figure 4-8 presents the DF-based frequency responses of controllers consisting of an LTI PID and a CgLpP element, designed following the scenarios mentioned above.

Intermediate options, where stability/robustness, precision and tracking are simultaneously improved can be achieved. [20] presents design procedures, based on the DF analysis, for controllers following the second and the third scenario. The performance of several control systems with the CgLp element is validated experimentally and it is concluded that the DF analysis can be used to predict the performance of the system.

4-3-3 Hybrid stability analysis

To explore the use of the hybrid passivity and finite-gain method for stability analysis, the CgLpP element, introduced in Section 4-3-1, is applied in the three scenarios presented above. A number of stable reset control systems are designed following the design procedures presented in [20]. Their stability is studied using the method introduced in Chapter 3. In this way, it is shown that the proposed method is conservative. Comparison of the results provides insights into the design of reset controllers, such that their stability can be concluded using the hybrid approach. Since this thesis is focused on the stability of reset control systems, the achieved performance is not studied.

One degree of freedom of the planar precision positioning stage presented in Figure 4-9 is used as a plant. The dynamics of the system are modelled using the transfer function [20]:

$$P = \frac{1.429e8}{175.9s^2 + 7738s + 1.361e6}. \tag{4-9}$$

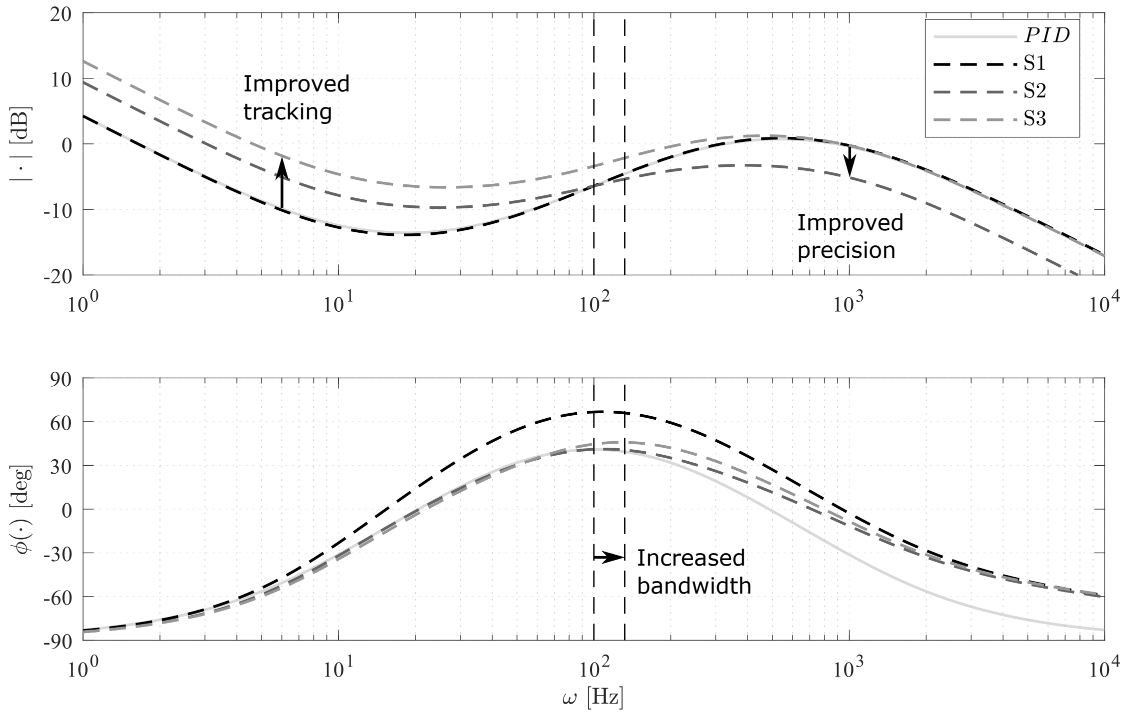


Figure 4-8: DF-based bode plot of reset controllers consisting of a PID and the CgLpP elements designed for an increase of PM ($S1$), improvement of tracking and precision ($S2$) and increase of the bandwidth ($S3$).

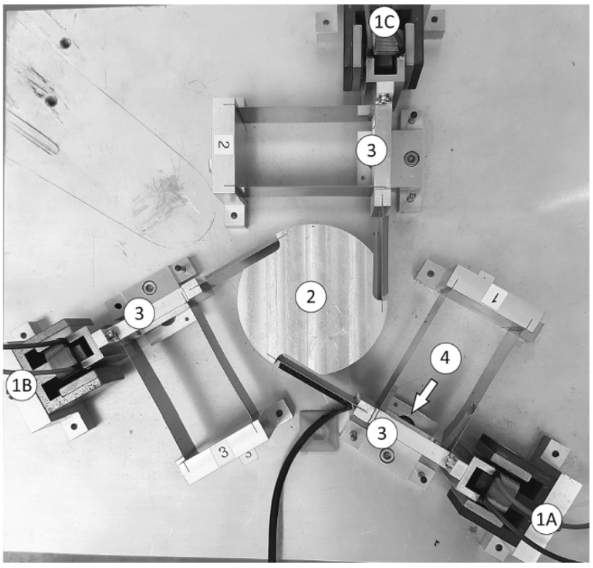


Figure 4-9: Three-DOF planar precision positioning "Spider" stage. The voice-coil actuators (1A,1B,1C) actuate the intermediated bodies (3) connected to the base and the main body (2) with leaf springs. The displacement of bodies (3) is measured by linear encoders (4) and results in a motion of the main body (2). Adapted from [20].

A time delay of $\tau = 0.53$ ms can be added to the transfer function to match the phase of the model at frequency 100 Hz with the phase of $-195°$ reported in [20]. The model of the plant with the time delay is given by $Pe^{-\tau s}$.

Robustness against uncertainties is an important requirement for control systems. An overview of results on the robustness of reset control systems can be found in [17]. In this project, the phase margin at the cross-over frequency, estimated using the DF analysis, is used as a measure of robustness [20] [46].

The plant model without and with the time delay was considered. When the time delay is ignored, the reset control system can be described using the state-space model and its stability can be assessed using the H_β condition [41]. If the time delay is significant, it can be included in the DF-based design procedure but not in the model used for a stability check. While this is a common approach in the design of reset controllers for motion systems [20] [63] [62], it has clear flaws since the influence of time delay on the stability of the system can be substantial [14].

An overview of methods for assessment of the stability of reset systems in the presence of time delay can be found in [14] and [17]. In this project, the stability of reset systems with the time delay is confirmed by approximating the delay with a second-order rational transfer function obtained with the Padé approximation [64] and checking if the H_β condition is satisfied.

To demonstrate the stability analysis, consider the plant (4-9) without the time delay and a reset controller with the CgLpP element, designed following the scenario S1. Stability of the control system is confirmed using the H_β condition [41]. An LTI controller ensures the PM of 25°. A CgLpP element with $A_\rho = 0.2$ is used to provide additional phase lead. A DF-based Bode plot of the system can be seen in Figure 4-10. The open-loop of the complete control system $k_p C_1(R_{DF} + C_R)C_2 P$ has the same magnitude but a higher phase than a system with a PID and the same bandwidth. The open-loop of the linear subsystem $k_p C_1 C_R C_2 P$ has the same phase as the system with the PID but lower bandwidth. This is caused by the increase of the gain due to adding of the reset element in parallel.

Although the stability of the system was confirmed with the H_β condition, it is analysed with the hybrid passivity and finite-gain approach to compare the methods. The control system is divided into the linear and reset parts, as described in Section 4-1. The stability of the reset control system can be concluded using the Theorem 3-3.3 if the conditions

$$\epsilon_L - \gamma_R > 0, \tag{4-10a}$$

$$\delta_L > 0, \tag{4-10b}$$

$$\gamma_L \gamma_R < 0 \tag{4-10c}$$

are satisfied, with $\epsilon_L, \delta_L, \gamma_L$ and γ_R denoting the hybrid passivity parameters and gains of the LTI subsystem L and of the reset subsystem R.

In the first step of the analysis, the hybrid passivity and finite-gain parameters of the subsystems L and R are evaluated for a grid of frequencies ω_H. The high-frequency gain of the reset system is approximated using the RMS amplitude of the nonlinear bode plot at ω_H. The obtained results can be seen in Figure 4-11a. Since $\gamma_{rms}(\omega_H)$ provides an underestimate of $\gamma_{\mathcal{B},rms}$, it can be concluded that if the stability conditions are not satisfied by the approximated parameter, they are also not satisfied with the exact value. The value of ω_H for further analysis is chosen from a range of frequencies for which the stability conditions are satisfied with the approximated high-frequency gain of R. In the second step, the exact gain $\gamma_{\mathcal{B},rms}$ is calculated. The stability can be concluded if the conditions of Theorem 3-3.3 are satisfied with the exact value $\gamma_{\mathcal{B},rms}$.

Tables 4-2 and 4-3 show the hybrid passivity and finite gain parameter for systems with controllers designed following the scenario S1, for a plant without and with the time delay included. The results of the stability analysis of the controllers designed following the second and the third scenario are presented in Tables 4-4 and 4-5. All the considered systems are stable, what was confirmed with the H_β condition [41]. In the cases in which the stability could not be concluded using the Theorem 3-3.3 the values of hybrid passivity and finite-gain parameters have been calculated for an arbitrary ω_H for illustration.

The first observation is that the stability of the control system can be concluded using the hybrid approach only if sufficient phase margin is provided by the linear lead filter, which is not a part of the CgLpP. Figures 4-11b and 4-11c illustrate the frequency responses of the subsystems L and R and provide insights into the design of control systems which stability can be concluded with the hybrid passivity and small-gain method. It is desired to pick $\omega_F >> \omega_R$ such that the high-frequency gain γ_R is low. γ_R can be then used to compensate the high-frequency gain of the linear subsystem γ_L. The finite-frequency passivity parameters δ_L and ϵ_L should be possibly high to compensate for γ_R. This is related to the low magnitude peak of L. The similarity between the PM of the open-loop characteristic of L and the damping ratio of the second-order transfer function [61] indicates, that the high open-loop PM of L facilitates concluding the stability.

Influence of the time delay can be seen by comparing the results obtained for the systems without and with the time delay and with the controllers designed following the scenario S1, presented in Tables 4-2 and 4-3. In many cases, the stability of the system with the time delay

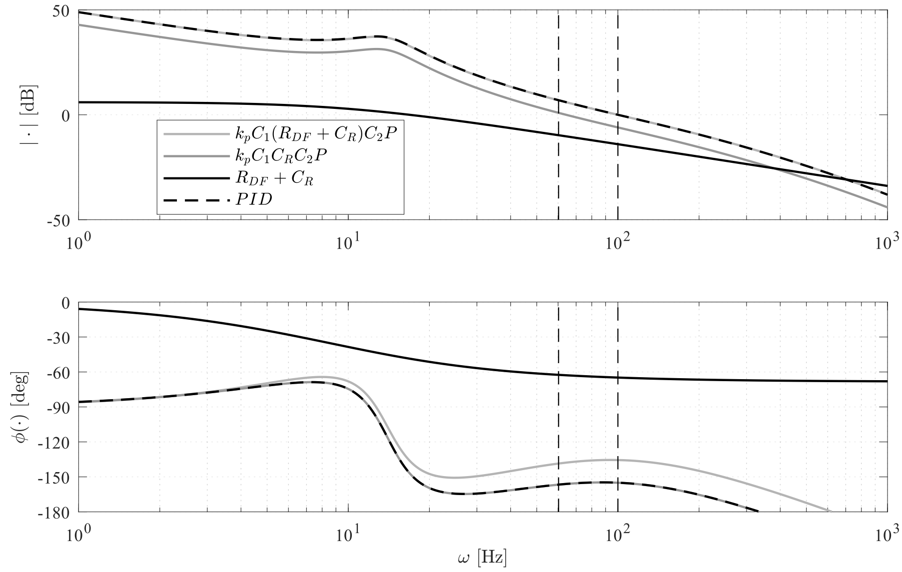
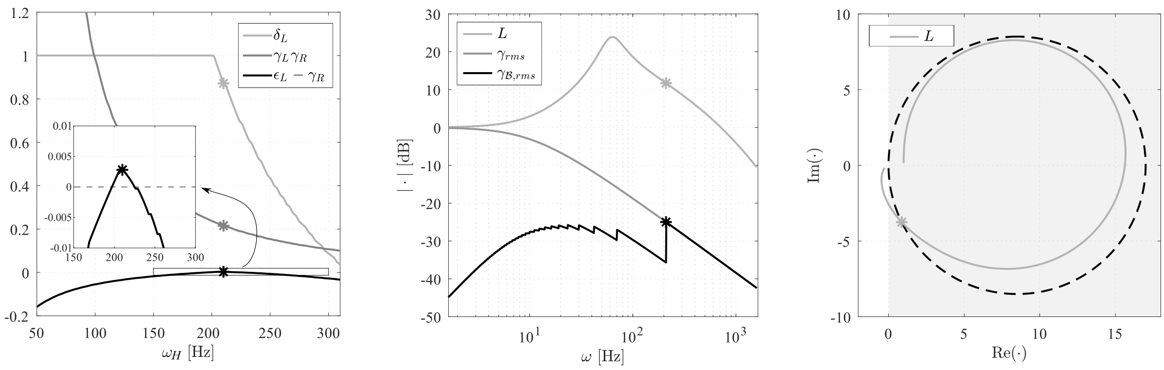


Figure 4-10: DF-based bode plot of a reset control system with a CgLpP element, designed for an increase of the PM. Solid lines represent the open-loop of the complete system and its constituting elements. The dashed line corresponds to an LTI PID controller designed as a reference.



(a) **(b)** **(c)**

Figure 4-11: Illustration of the hybrid passivity and small-gain stability analysis for a reset control system. (a) Estimated and exact values of parameters for a range of ω_H, (b) magnitudes of the response of the linear and reset subsystems, (c) Nyquist plot of the linear subsystem.

can not be concluded using the hybrid approach, even if it is possible for a system without the time delay with the same initial PM and the same amount of phase added by the CgLpP element. This indicates that not only the PM of the open-loop of the subsystem L, but also the overall shape of the phase influence the hybrid passivity and finite-gain parameters of the system.

All the analysed systems were stable, and their stability was assured using the H_β condition. In multiple cases, the stability of these systems could not be concluded using the hybrid passivity and finite-gain approach. This indicates that the proposed method is more conservative than the well-known H_β condition. Nevertheless, the approach introduced in this research does not require the identification of parametric models which is in line with the industrial practice.

4-4 Concluding remarks

The use of the proposed variant of the hybrid passivity and small-gain theorem for the analysis of the stability of reset control systems was demonstrated in this chapter. Two control structures suitable for the analysis were introduced.

Analysis of simple feedback interconnection of a second-order plant and a reset low-pass element was used as an initial example. It confirmed that the proposed method can be successfully used to conclude the stability of a reset system. Moreover, it was demonstrated that the method correctly does not indicate stability for unstable reset systems.

A novel *Constant in gain Lead in phase element in a parallel configuration* (CgLpP) was introduced. The reset element can be used to relax the limitation resulting from the Bode's magnitude-phase relationship for LTI systems. It was demonstrated that the element can be integrated with PID controllers in high-precision mechatronic systems, such that the stability can be shown using the proposed hybrid passivity and small-gain theorem.

The examples presented in this chapter confirm that the proposed stability method can be used for assessing the stability of reset controllers designed in line with the industrial practice. Unlike the other methods for stability analysis of reset systems, this approach does not require a parametric model of the system. In consequence, the condition can be evaluated based on the measured FRF function of the linear elements in the system and effects like time delay can be taken into account.

The proposed conditions are sufficient but not necessary for the stability of a reset control system. Because of that, they are not satisfied by many stable control systems that could be used in precision positioning applications. Analysis of several examples shown, that loop-shape of the linear components of the control system which are not a part of the CgLpP, is important for the fulfilment of the proposed stability conditions. Further research is required to identify other control structures facilitating the use of Theorem 3-3.3 for the stability analysis and to formulate exact design guidelines.

Table 4-2: Hybrid passivity and finite-gain parameters of reset control systems designed for the improvement of stability/robustness. The time delay was not included in the design procedure. A small variance in the parameters of the linear subsystem is caused by adjustments of the proportional gain, necessary to maintain the same bandwidth after including the CgLpP element in the control structure. Terms satisfying and violating the stability conditions are indicated with the green and red colour respectively.

Initial PM	A_ρ	ω_F[Hz]	δ_L	ϵ_L	γ_L	γ_R	$\epsilon_L - \gamma_R$	$\gamma_L\gamma_R$
25	0	210	**0,8729**	0,0588	3,8539	0,0643	**-0,0055**	**0,2477**
	0,2	210	**0,8718**	0,0588	3,8500	0,0550	**0,0038**	**0,2118**
	0,4	210	**0,8719**	0,0588	3,8504	0,0508	**0,0080**	**0,1955**
	0,6	210	**0,8723**	0,0588	3,8520	0,0481	**0,0106**	**0,1855**
	0,8	210	**0,8727**	0,0588	3,8532	0,0469	**0,0119**	**0,1805**
30	0	210	**1,0000**	0,0718	4,0777	0,0643	**0,0076**	**0,2621**
	0,2	210	**1,0000**	0,0719	4,0736	0,0550	**0,0168**	**0,2241**
	0,4	210	**1,0000**	0,0718	4,0740	0,0508	**0,0211**	**0,2069**
	0,6	210	**1,0000**	0,0718	4,0757	0,0481	**0,0237**	**0,1962**
	0,8	210	**1,0000**	0,0718	4,0770	0,0469	**0,0250**	**0,1910**

Table 4-3: Hybrid passivity and finite-gain parameters of reset control systems designed for the improvement of stability/robustness. The time delay $\tau = 0.53\,\text{ms}$ was included in the design procedure. A small variance in the parameters of the linear subsystem is caused by adjustments of the proportional gain, necessary to maintain the same bandwidth after including the CgLpP element in the control structure. Terms violating the stability conditions are underlined.

Initial PM	A_ρ	ω_F[Hz]	δ_L	ϵ_L	γ_L	γ_R	$\epsilon_L - \gamma_R$	$\gamma_L\gamma_R$
30	0	120	**0,9999**	0,0779	7,3238	0,1048	**-0,0269**	**0,7678**
	0,2	120	**0,9999**	0,0780	7,3146	0,0961	**-0,0181**	**0,7031**
	0,4	120	**0,9999**	0,0780	7,3154	0,0895	**-0,0115**	**0,6546**
	0,6	120	**0,9999**	0,0780	7,3192	0,0852	**-0,0072**	**0,6236**
	0,8	120	**0,9999**	0,0779	7,3221	0,0830	**-0,0051**	**0,6081**
35	0	130	**0,9987**	0,0862	6,9127	0,0963	**-0,0101**	**0,6656**
	0,2	120	**0,9987**	0,0941	7,1396	0,0961	**-0,0020**	**0,6863**
	0,4	120	**0,9987**	0,0941	7,1404	0,0895	**0,0046**	**0,6389**
	0,6	120	**0,9987**	0,0941	7,1439	0,0852	**0,0089**	**0,6087**
	0,8	120	**0,9987**	0,0941	7,1466	0,0830	**0,0110**	**0,5935**
40	0	120	**0,9948**	0,1094	6,9365	0,1048	**0,0045**	**0,7272**
	0,2	120	**0,9948**	0,1095	6,9286	0,0961	**0,0133**	**0,6660**
	0,4	120	**0,9948**	0,1095	6,9293	0,0895	**0,0200**	**0,6200**
	0,6	120	**0,9948**	0,1094	6,9325	0,0852	**0,0242**	**0,5907**
	0,8	120	**0,9948**	0,1094	6,9350	0,0830	**0,0264**	**0,5759**

Table 4-4: Hybrid passivity and finite-gain parameters of reset control systems designed for the improvement of tracking and precision. The total PM of $45°$ is required at the bandwidth. PM_L denotes the PM of the open-loop linear subsystems. The time delay $\tau = 0.53\,\mathrm{ms}$ was included in the design procedure. Terms violating the stability conditions are underlined.

A_ρ	$\mathrm{PM}_L[°]$	ω_F[Hz]	δ_L	ϵ_L	γ_L	γ_R	$\epsilon_L - \gamma_R$	$\gamma_L\gamma_R$
0	19,33	120	**1,0000**	0,0414	7,5230	0,1048	**-0,0634**	**0,7887**
0,2	25,51	120	**1,0000**	0,0631	7,4295	0,0961	**-0,0330**	**0,7141**
0,4	31,31	120	**0,9997**	0,0825	7,2719	0,0895	**-0,0070**	**0,6507**
0,6	36,52	120	**0,9979**	0,0990	7,0805	0,0852	**0,0138**	**0,6033**
0,8	41,08	120	**0,9933**	0,1128	6,8830	0,0830	**0,0297**	**0,5716**

Table 4-5: Hybrid passivity and finite-gain parameters of reset control systems designed for the increase of bandwidth and improvement of tracking. Total PM of $45°$ is required at the new bandwidth $\omega_{c,2}$. PM_L denotes the PM of the open-loop linear subsystems at $\omega_{c,2}$. The time delay was included in the design procedure. Terms violating the stability conditions are underlined.

A_ρ	$\omega_{c,2}$[Hz]	$\mathrm{PM}_L[°]$	ω_F[Hz]	δ_L	ϵ_L	γ_L	γ_R	$\epsilon_L - \gamma_R$	$\gamma_L\gamma_R$
0	137	19,33	150	**0,9993**	0,0471	8,3767	0,1124	**-0,0653**	**0,9418**
0,2	129	25,52	150	**0,9988**	0,0617	7,7852	0,0974	**-0,0357**	**0,7586**
0,4	121	31,31	150	**0,9978**	0,0746	7,3133	0,0852	**-0,0106**	**0,6228**
0,6	113	36,52	150	**0,9957**	0,0854	6,9448	0,0759	**0,0095**	**0,5275**
0,8	106	41,08	150	**0,9917**	0,0944	6,6587	0,0694	**0,0250**	**0,4622**

Chapter 5

Conclusion

In this final chapter, the answers to the initial research questions are summarized. Next, open questions and recommendations for future work are presented.

5-1 The answers to the research questions

Reset controllers have been developed to alleviate the tradeoffs inherent to linear systems. Although it has been proven that they can improve the performance of precision mechatronic systems, they are not widely adopted by the industry. One of the reasons is a lack of tools for guaranteeing the stability of rest control systems, that are in line with the current industrial practice. To address this problem, a frequency-domain method for the stability analysis was proposed in this work. The hybrid passivity and finite-gain approach was used as the basis for the method.

The project consisted of two major steps. First, the hybrid passivity and finite-gain properties of selected reset systems were analysed. Subsequently, design of reset controllers whose stability can be guaranteed using the hybrid passivity and finite-gain approach was considered. The proposed method was applied to several examples of reset control systems for a precision motion stage. The initial research questions, stated in the introductory chapter, could be answered with the insights gained from the work.

How can the hybrid passivity and finite-gain framework be extended to reset systems?
A way of extending the hybrid framework to reset systems was presented in Chapter 3. Although a reset system may have the property described in [30], it is unclear how to find the hybrid passivity and finite-gain parameters for a given reset system. A modified definition of the property is required to simplify the finding of the parameters. With the new definition, it is possible to consider the finite-frequency passivity and gain of a system separately, using methods developed in this project. However, the modified definition leads to more strict stability conditions.

How can a reset control system be designed such that its stability can be concluded using the hybrid passivity and small-gain theorem?
In Chapter 4 stability of several examples of reset control systems was analysed using the hybrid passivity and small-gain theorem. From the analysis, it became clear that the structure of a control system should be such that it can be converted into a feedback interconnection of a reset and an LTI subsystem. It was shown that the stability can be concluded using the hybrid approach when the open-loop of the LTI subsystem has a sufficient phase margin. It would be beneficial if the part of the phase-providing reset element assigned to the linear subsystem provided phase advantage even without the reset component. Moreover, the controller is easier to design if the bandwidths of the complete control system and of the open-loop LTI subsystem are similar.

5-2 Recommendations

This research showed that the hybrid passivity and finite-gain approach can be successfully used for the stability analysis of reset control systems. Moreover, the developed methods for the finding of the hybrid parameters of reset elements can be extended to other classes of nonlinear systems. This, in combination with the earlier applications of the hybrid approach for the robust controller design for linear systems, suggests that the hybrid passivity and finite-gain method can be used to unify various tools for guaranteeing the stability of different classes of control systems. Since the presented stability conditions can be evaluated in a way preferred by the industry, i.e. using the measured FRFs of a system, this research may encourage wider use of nonlinear control system.

The "merging" of the passivity and small-gain theorems should provide stability results for feedback interconnections of a class of systems, broader than those dealt with by the small gain and passivity theorems, respectively [29]. In this project, this objective was achieved only partially, since the reset element is required to be passive.

The main point that deserves attention is finding the hybrid passivity and finite-gain parameters of reset systems. The method proposed in this research allows only to find the high-frequency gain of a convergent system. Knowledge of the input and output passivity parameters of a reset system would result in less conservative conditions for the stability of a reset control system. Moreover, the ability to conclude finite-frequency passivity for reset systems that are not passive would broaden the applicability of the hybrid approach.

Another closely related topic is concluding the hybrid passivity and finite-gain of a system simultaneously, without resorting to the separate analysis of low-frequency passivity and high-frequency gain. This could enable a direct use of the Definition 2-5.1 [30] and simplification of the stability conditions.

It would be also beneficial to study if a reset system can be dissipative w.r.t. a specific supply rate when its base linear system is not dissipative w.r.t the same supply rate. Results on the *reset passivation* suggest that this is possible [65]. Possibility of concluding the hybrid passivity and finite-gain for a reset system, without relying on the base dynamics, would result in more freedom in design of reset comptrollers.

The proposed conditions are sufficient but not necessary for the stability of considered reset control systems. Further research is required to characterize reset systems whose stability can be concluded using the approach introduced in this thesis and to develop design guidelines.

Future work could also focus on the development of new reset control structures, for which the stability could be concluded using the hybrid passivity and finite-gain approach. Moreover, the properties of the *Constant in gain Lead in phase element in a parallel configuration* introduced in this project should be closely studied.

Appendix A

Finite-frequency gain of FORE

```
function [NLB, NLBa, NLBb] = betaDF(sys, Ar, freqs, wc)
%betaDF Calculates the finite-frequency version of the nonlinear bode
% magnitude plot of a reset system
%
% SYS is the reset element described in state space
% AR is the amount of reset you want to achieve
% FREQS contains the frequencies the NL bode is calculated
% for, represented in rad/s
% wc is the corner frequency of the ideal low/high pass filters
%
% NLB is the standard nonlinear Bode plot
% NLBa is the low-frequency nonlinear Bode plot
% NLBb is the high-frequency nonlinear Bode plot
%
% M.B. Kaczmarek - TU Delft - 2020

if (~(nargin == 3 && nargout == 1) && ~(nargin == 4 && nargout == 3))
    warning('Wrong number of input/output arguments')
    return
end

A = sys.a; B = sys.b; C = sys.c; D = sys.d;

tlen = 1e4;
flen = numel(freqs);
yss = zeros(flen, tlen); % Steady-state response in the time domain
ylow = zeros(flen, tlen); % Low-pass filtered steady-state response in the time domain
yhigh = zeros(flen, tlen); % High-pass filtered steady-state response in the time domain

NLB = zeros(1,flen);
NLBa = zeros(1,flen); % Low-pass filtred nonlinear Bode plot
```

```
NLBb = zeros(1,flen); % High-pass filtered nonlinear Bode plot


for i=1:flen % For each frequency
    w = freqs(i);
    t = linspace(0,2*pi/w,tlen);% tlen time points per period

    Lambda = w*w*eye(size(A)) + A^2;
    LambdaInv = inv(Lambda);

    Delta = eye(size(A)) + expm(A*pi/w);
    DeltaR = eye(size(A)) + Ar*expm(A*pi/w);

    GammaR = inv(DeltaR)*Ar*Delta*LambdaInv;

    Theta0 = -1*eye(size(A))*[GammaR-LambdaInv]*w*B; % tk = k*pi/w, expm
        (-A*0) = I
    Theta1 = 1*expm(-A*pi/w)*[GammaR-LambdaInv]*w*B;
    ThetaD = (-2*w*w/pi)*Delta*(GammaR-LambdaInv);
    % alpha is omitted, since the response in linear in alpha (Guo2009)
    % Calculate the response in the time domain
    for j = 1:tlen/2
        yss(i,j) = C.'*expm(A*t(j))*Theta0 - C.'*LambdaInv*(w*eye(size(A)
            )*cos(w*t(j)) + A*sin(w*t(j)))*B + D*sin(w*t(j));
    end
    for j = 1:tlen/2
        j2 = tlen/2+j;
        yss(i,j2) = C.'*expm(A*t(j2))*Theta1 - C.'*LambdaInv*(w*eye(size(
            A))*cos(w*t(j2)) + A*sin(w*t(j2)))*B+ D*sin(w*t(j));
    end
    % Calculate harmoincs up to wc
    if (nargin == 4) && (nargout == 3)
        n = 1; % number of harmonics
        while n*w<wc
            if (n==1)
                G = C*inv(1i*w*eye(size(A)) - A)*(eye(size(A)) + 1i*
                    ThetaD)*B;
            else
                G = C*inv(1i*w*n*eye(size(A)) - A)*1i*ThetaD*B;
            end
            % add signals in the time domain
            ylow(i,:)= ylow(i,:) + abs(G)*sin(n*w*t + angle(G));
            n = n+2;
        end
        nF = n-2;% correct for the last increase
        % ABS value
        % NLB(1,i) = max(abs(yss(i,:)));
        % NLBa(1,i) = max(abs(ylow(i,:))); % low-pass filtered
        % NLBb(1,i) = max(abs(yss(i,:)-ylow(i,:))); % high-pass filtered
        %L2 simpsons
        NLBa(1,i) = sqrt(((ylow(i,1)^2/2+sum(ylow(i,2:end-1).^2)+ylow(i,
            end)^2/2)*t(2))*w/pi);
        yhigh(i,:) = yss(i,:)-ylow(i,:);
```

```
        NLBb(1,i) = sqrt(((yhigh(i,1)^2/2+sum(yhigh(i,2:end-1).^2)+yhigh(
            i,end)^2/2)*t(2))*w/pi);
    end
    NLB(1,i) = sqrt(((yss(i,1)^2/2+sum(yss(i,2:end-1).^2)+yss(i,end)^2/2)
        *t(2))*w/pi);

end
end
```

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Glossary

List of Acronyms

CgLp *Constant in gain Lead in phase*

CgLpP *Constant in gain Lead in phase element in a parallel configuration*

BLS *Base Linear System*

FORE *First Order Reset Element*

RCS *Reset Control System*

SPR *strictly positive real*

LMI *Linear Matrix Inequality*

SISO *single-input single-output*

MIMO *multiple-input multiple-output*

DF *Describing Functions*

SIDF *Sinusoidal Input Describing Functions*

HOSIDF *Higher-Order Sinusoidal Input Describing Functions*

FRF Frequency Response Function

VSP *Very Strictly Passive*

ISP *Input Strictly Passive*

OSP *Output Strictly Passive*

LTI *Linear and time invariant*

PM *phase margin*

UBSS *uniformly bounded steady-state*

UGAS *uniformly globally asymptotically stable*