Precise control of LISA with Quantitative Feedback Theory Space Flight Track MSc Thesis

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Challenge the future



PRECISE CONTROL OF LISA WITH QUANTITATIVE FEEDBACK THEORY

SPACE FLIGHT TRACK MSC THESIS

by

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in partial fulfillment of the requirements for the degree of

Master of Science in Aerospace Engineering

at the Delft University of Technology, to be defended publicly on 12 March 2019 at 13:30

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PREFACE

When I first started researching LISA I had never been formally introduced to the concept of gravitational waves, or the subject of relativistic physics for that matter – except for some applications in GNSS clock tuning and modeling of perturbations on the orbit of Mercury, as a Master of Science student in Space Engineering. Nevertheless, the breakthrough of LIGO observations made outstanding news in recent years: first, the announcement of the detection of a black hole merger signature in early 2016, followed by the awarding of the Nobel Prize in physics in 2017 to a collaboration of scientists among whom I could just make out the name of Kip Thorne (not for anything else but his involvement in Hollywood back in 2014).

The likeness of fuzzy hair, mustached Albert Einstein – the Father of Relativistic Physics – has been engraved in pop-culture for decades. All over the media, his famous equation $E = MC^2$ is spotted more than often, although, usually, out of the context in which it was conceived. Never an equation has been referenced as much (maybe with the exception of the Pythagorean theorem and the solution of quadratic equations that we are taught in school), and yet, there is an intrinsic unfamiliarity with the work that made the scientist popular as such, while his eccentric personality and philosophical stance made him famous to the World.

Indeed, understanding of his genius requires way more than high school level teachings or basic common sense (which, let's be honest, not everybody possesses either). So, at the dawn of my Master studies I decided to dip my toes in relativistic physics just to get a taste of what it is all about. Without a warning I was overwhelmed by a world of abstractions and foreign mathematics that transcends all I knew: differential geometry, multi-dimensional tensorial calculus, metrics, connections, space-time, field equations and black holes. No wonder why Einstein and the late Stephen Hawking could not just charm the masses with their papers and theories.

The researchers at the Max Planck Institute for Gravitational Physics in Potsdam, where I briefly educated myself on these subjects, could easily be able to put me to shame with their alienating knowledge. Yet, I was welcomed with a warming sense of excitement for LISA! It was then that I realized how important what I was studying was: a whole world of people out there cheer for LISA and have been waiting for its debut with growing anticipation, some even for decades!

In fairness, this thesis has nothing to do with relativistic physics. But every good story must have a beginning somewhere, and this one about LISA might as well start with Einstein and his Field Equations. Besides, it should become clear, after reading this report, what a true fit of engineering we are dealing with. LISA is, after all, a one of a kind space mission, a technological achievement thirty years in the making and bugged by a plethora of obstacles along the way and the technological limitations of our times.

Many thanks to my supervisor, Prof. Pieter N.A.M. Visser for introducing me to the topic of this thesis and guiding me through its implementation, to Dr. Ernst J.O. Schrama for presiding the graduation committee and to Dr. Jian Guo for participating in the process. I would also like to thank the TNO optics team for their interest, especially Dr. Ernst-Jan Buis for attending my graduation. A huge applause goes also to all the people involved – or that have been involved – in LISA and LISA Pathfinder and the physicists that believe in the system: your data will soon come.

A big shout out goes to my friends around – and from all over – the World, especially my fellow peers in Delft, Bremen and Turin for the gift of comradery that greatly relieved the struggles of studying. And, of course, most (but not all) of the numerous flatmates with whom I shared the ever sacred cooking ground and the dining table.

Last, but not least, I need to acknowledge the support both financial and psychological of my relatives back home, with special regards to my parents, grandparents and siblings.

Francesco Lupi Delft, February 2019

ABSTRACT

The Laser Interferometer Space Antenna (LISA) is a European mission for the detection of gravitational waves in space set to be launched in 2034. The mission will see the deployment of 3 spacecraft in heliocentric orbit keeping a triangular formation with side length of 2.5 million km. Laser beams are exchanged between the spacecraft by means of suitably mounted telescopes (2 per spacecraft), with the objective of synthesizing a very-large baseline interferometer. The interferometric measurements are taken between free-floating testmasses placed inside the spacecraft.

Due to the nature of the scientific objectives, the mission requirements on spacecraft-spacecraft pointing precision are exceptionally strict. Moreover, the formation needs to operate in almost perfect free-fall, therefore the solar radiation pressure needs to be continuously compensated for by the on-board thrusters. Gravitational wave signals are measured in the frequency bandwidth of 20 μ Hz to 1 Hz, requiring the vibrations in that domain to also be eliminated both for the attitude and the displacement. The task is made possible by the gravitational reference system, a complex device that keeps the test-masses from touching the walls of the spacecraft by applying on the latter an external force through μ Newton thrusters. This mode of operation is called Drag-Free and Attitude Control System (DFACS).

In this thesis we attempt to study and design a DFACS for LISA using a technique called Quantitative Feedback Theory (QFT). The design process starts from the definition of the orbits, the goal orientation of the spacecraft, the sizing of the solar radiation pressure induced disturbances and the derivation of the dynamics of the 19 degrees of freedom to be controlled. Using QFT, the design process is carried out on the DFACS using separation of the dynamics.

As a result, analytical equations for the calculation of the LISA commands are derived and the methods to design a control system compliant to the scientific requirements imposed on the sensitivity are shown.

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ABBREVIATIONS

- ASD Amplitude Spectral Density. 66, 67, 69, 82, 83, 87, 90–92, 95, 104, 106, 112, 142
- CDF Concurrent Design Facility. 19, 20, 26, 27, 29
- **CoM** Center of Mass. 20, 26, 27, 45, 48, 50, 120, 121, 123

CoP Center of Pressure. 50

- CoT Center of Thrust. 20, 21
- CR3BP Circular Restricted 3-Body Problem. 15, 16, 18
- DC Direct Component. 79, 81, 89, 92, 95, 98, 110–112
- DFACS Drag-Free and Attitude Control System. 3, 4, 19, 20, 77, 117
- **DoF** Degrees of Freedom. 4, 5, 45, 53, 60, 61, 68, 69, 71, 75–78, 89–91, 95–100, 108, 110, 112
- ESTEC European Space Research and Technology Centre. 19
- FEEP Field Emission Electric Propulsion. 28, 52, 65
- **GRS** Gravitational Reference System. 3–5, 45, 60, 61, 63, 64, 68, 77, 78, 96, 107
- GW Gravitational Waves. 1-4, 60
- JILA Joint Institute for Laboratory Astrophysics. 1
- KAGRA Kamioka Gravitational Wave Detector. 1
- LAGOS Laser Antenna for Gravitational-radiation Observation in Space. 1
- LIGO Laser Interferometer Gravitational wave Observatory. 1, 2, 4
- LISA Laser Interferometer Space Antenna. 1–5, 7, 9, 18–22, 26, 31, 43, 45, 47, 48, 60, 68, 69, 71, 75–78, 84
- LTP LISA Test Package. 2
- NDI Non-linear Dynamics Inversion. 80
- PSD Power Spectral Density. 67, 139–145
- **QFT** Quantitative Feedback Theory. 71, 74, 75, 79, 83, 106
- **S/C** Spacecraft. 1–5, 7–9, 12, 13, 16, 18–21, 26–29, 31–45, 47–52, 55, 60–64, 67–69, 76–80, 88, 92, 96, 99, 107, 108, 115
- **SAGITTARIUS** Spaceborne Astronomical Gravitational-wave Interferometer To Test Aspects of Relativity and Investigate Unknown Sources. 1, 2
- **SISO** Single-Input-Single-Output. 80–82
- **SRP** Solar Radiation Pressure. 5, 45, 47–52, 64, 69, 79–81, 83, 88, 89, 92, 93, 95, 97, 98, 110, 112
- SSB Solar System Barycenter. 10, 11, 18, 48
- TM Test-Mass. 1, 3, 4, 45, 60, 61, 63, 67–69, 76–78, 95, 96, 99, 107, 108

NOMENCLATURE

MATH SYMBOLS

- $\vec{0}$ Adimensional physical vector
- 0 Array of zeros
- $1^{n \times n}$ = diag(1, 1, ..., 1)¹ unit $n \times n$ matrix $n \times 1$

$$\mathbb{I}^{n \times n} = \operatorname{diag}(1, 1, ..., 1)^{-1} \operatorname{unit} n \times n \operatorname{matrix}$$

$$\mathbb{I}_{i}^{n \times 1} \quad \text{Unit} n \times 1 \text{ array where } \mathbb{I}_{j \neq i} = 0 \text{ and } \mathbb{I}_{j=i} = 1 \text{ for } i, j \leq n$$

- Set of complex numbers \mathbb{C}
- $\hat{i},\,\hat{j},\,\hat{k}$ Fundamental directives $\hat{i} = (1, 0, 0)^{\tau}$, $\hat{j} = (0, 1, 0)^{\tau}$, $\hat{k} = (0, 0, 1)^{\tau}$
- Set of natural numbers \mathbb{N}
- $\in \mathbb{R}^{3 \times 1}$ Physical vector \vec{x}
- $\|\vec{x}\|$ Magnitude of vector
- Absolute value of $x \in \mathbb{R}$ / magnitude of $x \in \mathbb{C}$ |x|
- $\frac{x}{\vec{x}^{\times 2}}$ Phase of $x \in \mathbb{C}$
- Skew-symmetric matrix of \vec{x} , such that $\vec{x} \times \vec{y} = \vec{x} \times \vec{y}$ (vectorial product) for any vector \vec{y}
- $\in \mathbb{R}^{3 \times 1}$ Unit vector, directive $||\hat{x}|| = 1$ â
- $\in \mathbb{R}^{n \times 1}$ column vector, array, $n \times 1$ matrix x
- \mathbb{R} Set of real numbers
- $\mathbb{R}^{n \times m}$ Set of real $n \times m$ matrices
- $x \overrightarrow{\Box}$ Vector as seen in reference frame "x"
- \Box^{τ} Transpose operation
- Ċ Time derivative
- Ö Double time derivative

CONVERSIONS

1 rad (radian)	= $(180/\pi)^{\circ}$ (degrees)	f- (femto-)	$\times 10^{-12}$
1 year	= 31536000 s	n- (nano-)	$\times 10^{-9}$
1 day	= 86400 s	μ - (micro-)	$\times 10^{-6}$
1 h (hour)	= 3600 s	m- (milli-)	$\times 10^{-3}$
1 min (minute)	= 60 s	k- (kilo-)	$\times 10^3$
1 AU (astron. unit) [1]	$= 1.495978707 \cdot 10^{11} \text{ m}$	M- (mega-)	$\times 19^{6}$
[dB] (decibels)	as in 20log ₁₀	G- (giga-)	$\times 10^9$

1.	$\begin{pmatrix} a_{1,1} \\ 0 \end{pmatrix}$	0 $a_{2,2}$	 	0 0
¹ diag $(a_{1,1}, a_{2,2},, a_{n,n})$	$) = \left(\begin{array}{c} \vdots \\ 0 \end{array} \right)$: 0	·	$\begin{bmatrix} \vdots \\ a_{n,n} \end{bmatrix}$
${}^{2}\vec{x}^{\times} = \begin{pmatrix} 0 & -x_{3} & x_{3} \\ x_{3} & 0 & -x_{2} \\ -x_{2} & x_{1} \end{pmatrix}$	$\begin{pmatrix} x_2 \\ x_1 \\ 0 \end{pmatrix}$			

CONSTANTS

μ_{moon}	$4902.800238 \mathrm{km^3/s^2}$	[2]
μ_e	$398600.4418 \mathrm{km^3/s^2}$	[2]
μ_s	$132712440023.30999 \mathrm{km^3/s^2}$	[2]
g_0	9.81 m/s ²	
m_{dry}	1493.3 kg	
m_{prop_0}	239.6 kg	
ω _e	$1.990983675 \cdot 10^{-7} \text{ rad/s}$	
с	2.99792458·10 ⁸ m/s	
W_{\odot}	$1360.8 \mathrm{W/m^2}$	[<mark>3</mark>]
P_{\odot}	$4.5391 \mu \text{N/m}^2$	[<mark>3</mark>]
I_{sp}	340 s	
	μ_{moon} μ_e μ_s g_0 m_{dry} m_{prop_0} ω_e c W_{\odot} P_{\odot} I_{sp}	$\begin{array}{cccc} \mu_{moon} & 4902.800238 \ {\rm km}^3/{\rm s}^2 \\ \mu_e & 398600.4418 \ {\rm km}^3/{\rm s}^2 \\ \mu_s & 132712440023.30999 \ {\rm km}^3/{\rm s}^2 \\ g_0 & 9.81 \ {\rm m/s}^2 \\ m_{dry} & 1493.3 \ {\rm kg} \\ m_{prop_0} & 239.6 \ {\rm kg} \\ \omega_e & 1.990983675 \cdot 10^{-7} \ {\rm rad/s} \\ c & 2.99792458 \cdot 10^8 \ {\rm m/s} \\ W_{\odot} & 1360.8 \ {\rm W/m}^2 \\ P_{\odot} & 4.5391 \ \mu{\rm N/m}^2 \\ I_{sp} & 340 \ {\rm s} \end{array}$

VARIABLES

- α breath angle (α_i b.a. relative to S/C_i, i = 1,2,3) / in Chapter 6: phase parameter for derivation of *S* and *T*-boundaries
- $\overline{\alpha}$ exposure to sun-light $-\hat{z}\cdot\hat{s}$
- β angle between \hat{z}_p and and \hat{k} in inertial reference frame
- γ nominal Earth trailing angle = 20°
- $\delta \phi$ deviation of real value of ϕ from nominal
- $\delta \boldsymbol{\theta} \qquad (\delta \theta_1, \delta \theta_2, \delta \theta_3)^{\tau}$ Euler angle error in control loop
- $\delta \vec{f}_t$ random thrust jitter
- $\delta \vec{l}_t$ random torque jitter
- $\delta \ddot{x}_{df}$ acceleration noise on the drag-free DoF
- δr DC error limit on the drag-free DoF
- δx DC error on the interferometric directions
- δz DC error on the z-direction
- $\Delta \alpha$ max. divergence of breath angles (limit on the mobility of telescopes)
- Δx used to represent the bin size
- ε ε_i , i = 1, 2, ..., 12, jitter factor, random gaussian value with standard deviation $3\sigma = 0.01$
- θ in Chapter 2: true anomaly (orbital element) / in Chapters 5 8: θ_i , i = 1, 2, 3, Euler angle, component of θ
- $\boldsymbol{\theta}$ Euler angles $(\theta_1, \theta_2, \theta_3)$, $\boldsymbol{\theta}_c$ commands on the E.a. (ideal E.a.), $\boldsymbol{\theta}_m$ measured E.a.
- $\Theta \qquad \text{Lagrange transform of } \boldsymbol{\theta}$
- $\boldsymbol{\lambda} \in \mathbb{R}^{18 \times 1}$ Lagrange multipliers
- μ mass-ratio parameter in circular restricted 3-body problem / gravitational parameter: μ_s Sun g.p., μ_e Earth g.p. / mean, or mathematical expectancy (also E[]) $\mu(S_{xx})$ a. of S_{xx}
- σ standard deviation (square root of variance) $σ_{asd}$ s.d. of zero-mean ASD, $σ_x$ s.d. of random variable *x*, $σ_{d1}$ s.d. of random dynamics disturbance, $σ_{d2}$ s.d. of random readout disturbance
- $\hat{\tau}$ $\hat{\tau}_i$, i = 1, 2, ..., 12 direction of thrust of single thruster
- φ placeholder value for attitude kinematics and dynamics of S/C, (φ_i , i = 1, 2, of TM_i) $\in \mathbb{R}^{3 \times 1}$
- ϕ Inclination of formation plane w.r.t. Ecliptic
- $\overline{\phi}$ complementary angle of ϕ
- ω in Chapter 2: argument of perigee (orbital element) / in Chapters 6–8 angular frequency [rad/s] i.e. $ω = 2πf / ω_o$ numerical fundamental harmonic of low-frequency disturbances / $ω_e$ sidereal motion
- $\vec{\omega}$ angular velocity vector ($\vec{\omega}$ = a. acceleration vector) / without subscript: S/C body a.v.v. w.r.t. inertial reference frame in body-fixed reference frame / $\vec{\omega}_i$ in Chapter 4: a.v.v. of S/C_i i = 1,2,3 / $\vec{\omega}_{i/p}$ a.v.v. of S/C_i i = 1,2,3 w.r.t. formation reference frame / $\vec{\omega}_i$ in Chapters 5–8: a.v.v. of TM_i, i = 1,2 / $\vec{\omega}_{0i}$ a.v.v. of GRS reference frame of TM_i, i = 1,2, w.r.t. S/C body-fixed reference frame.
- Ω longitude of ascending node (orbital element)
- *a* in Chapter 2: semi-major axis (orbital element) / in Chapter 3: short side of trapezoid base, a_1 s.s. of large base, a_2 s.s. of small base, a(z) s.s. of constant z section of trapezoid

- \vec{a} accelerations [m/s²], \vec{a}_g gravitational accelerations, \vec{a}_t ideal thrust acceleration for drag-free operations
- *A* $\in \mathbb{R}^{6\times 12}$ thruster configuration matrix, $A_{-} \in \mathbb{R}^{12\times 6}$ pseudoinverse of *A* with all positive components and $AA_{-} = -\mathbb{1}^{6\times 6}$, $A_{+} \in \mathbb{R}^{12\times 6}$ pseudoinverse of *A* with all positive components and $AA_{+} = \mathbb{1}^{6\times 6}$
- ASD_{\Box} imposed ASD limit on the DFACS DoF errors in measurement bandwidth: ASD_{θ} limit on the attitude errors, ASD_x limit on the displacement along the interferometric directions, ASD_r limit on the drag-free displacement in bandwidth, ASD_z limit on the *z*-direction, ASD_{acc} limit on the accelerations along the interferometric directions
- b long side of trapezoid base, b_1 l.s. of large base, b_2 l.s. of small base, b(z) l.s. of constant z section of trapezoid
- *c* (constant) speed of light in vacuum
- c_n n^{th} harmonic magnitude in Fourier series
- C (without subscript) used to represent any number $C \in \mathbb{C}$
- C_{\Box} (with subscript) direction cosine matrix $\in \mathbb{R}^{3\times3}$: $C_{n/i}$ d.c.m. from inertial to body-fixed reference frame of S/C_i, i = 1,2,3, $(C_{i/n} = C_{n/i}^{\tau})$, $C_{n/p}$ d.c.m. from inertial to formation plane reference frame, $C_{p/i}$ d.c.m. from formation plane to S/C_i, i = 1,2,3, body reference frame, $C_i(\theta_i)$, i = 1,2,3, intermediate d.c.m. associated to the relative fundamental axis of rotation, $C_{n/s}$ d.c.m. from inertial to S/C reference frame, $C_{s/i}$ d.c.m. from S/C body to TM_i, i = 1,2 GRS reference frame / C_{df} selection and transformation matrix from S/C body to drag-free DoF components generic directive $(\hat{x}, \hat{y}, \hat{z})$
- \vec{d} acceleration disturbance: \vec{d}_0 acceleration disturbance on S/C in S/C body reference frame, \vec{d}_i acceleration disturbance on TM_i i = 1, 2 in GRS reference frame

$$d \in \mathbb{R}^{3 \times 1}$$
 time-domain dynamics disturbance in control loop: d_{θ} dynamics disturbance on Euler angles, d_{df} dynamics disturbance on drag-free DoF

- $D \in \mathbb{C}$ frequency-domain disturbance in control loop: D_1 dynamics (control) disturbance, D_2 measurement disturbance
- **D** frequency-domain representation of **d**
- *e* eccentricity (orbital element)
- *E* eccentric anomaly (orbital element)
- *f* frequency [Hz], f_{samp} sampling frequency (usually $f_{samp} = 2$ Hz)
- $f(\mathbf{x})$ target optimization function to minimize
- \vec{f} force \vec{f}_t thrust force, \vec{f}_{d0} disturbance force, \vec{f}_{srp} SRP force, \vec{f}_{spec} force component due to specularly reflected sun-light, $\vec{f}_{blocked}$ force component due to blocked sun-light, \vec{f}_{diff} force component due to diffusely reflected sun-light \vec{f}_i electrostatic suspension force acting on TM_i, i = 1, 2
- $f \in \mathbb{R}^{6\times 1}$ force-torque vector $f = (\vec{f}_t^{\tau}, \vec{l}_t^{\tau})^{\tau}$, f_- array containing all the negative components of f and zeros, f_+ array containing all the positive components of f and zeros
- *F* in Chapters 5-8: upper bandwidth frequency F = 1 Hz.
- $F(\mathbf{x}, \boldsymbol{\lambda})$ in Chapter 3: augmented function
- g_0 gravitational acceleration at Earth's surface
- $\boldsymbol{g} \in \mathbb{R}^{18 \times 1}$ optimization constraints
- $G \in \mathbb{C}$ frequency domain control transfer function
- $G_{ro} \in \mathbb{C}$ frequency domain control transfer function of the drag-free control system for z_1 with rolloff in parallel
- *h* in Chapter 2: magnitude of orbital momentum per unit mass / in Chapter 3: S/C height
- h_{cm} distance of S/C CoM from solar panel (lower base)
- *H* in Chapter 3: system of equations for the solution of the optimization problem, H^* reduced form of *H*
- H_{\Box} in Chapters 7–8: linearizing matrix, H_{θ} l.m. of attitude dynamics system, H_{df} l.m. of drag-free DoF dynamics system, H_{idf} , i = 1,2 selection matrix of drag-free DoF from TM_i relative DoF
- *i* (unless found as subscript) in Chapter 2: inclination (orbital element) / in the Appendices: imaginary unit

- î $= (1,0,0)^{\tau}$ fundamental *x*-direction I
 - $\in \mathbb{R}^{3\times 3}$ inertia matrix: I_{sc} S/C body i.m. w.r.t. CoM, I_{dry} i.m. of the dry mass w.r.t. S/C CoM, I_{prop} i.m. of propellant mass w.r.t. S/C CoM, I_i i.m. of TM_i, i = 1, 2
- specific impulse Isp
- (unless found as subscript) imaginary unit
-] ĵ $= (0, 1, 0)^{\tau}$ fundamental y-direction
- k (unless found as subscript) gain of the transfer function
- ĥ $= (0, 0, 1)^{\tau}$ fundamental *z*-direction
- K $\in \mathbb{R}^{3 \times 3}$ stiffness matrix of the virtual spring-mass system used to model the interaction between the S/C and the TMs: K_i stiffness matrix of the displacement dynamics of TM_i, $i = 1, 2, K_{ai}$ stiffness matrix of the rotational dynamics of the TM_i , i = 1, 2
- 1 nominal arm-length
- ī external torque: \vec{l}_t thruster torque (in S/C reference frame), \vec{l}_{d0} torque noise acting on S/C, \vec{l}_{srp} SRP induced torque, \vec{l}_i electrostatic suspension torque acting on TM_i, $i = 1, 2, \vec{l}_{di}$ excess disturbance torque acting on the TM_i , i = 1, 2, in its enclosure
- L $\in \mathbb{C}$ open loop transfer function of the linear feedback control loop
- mass: m_{sc} S/C mass, m_{drv} dry-mass, m_{prop} propellant mass (\dot{m}_{prop} mass-rate), m_i , mass of TM_i, т i = 1, 2 (= 1.9 kg in simulation)
- Min Chapter 2: mean anomaly (orbital element) / in Chapter 8: M_i matrix used to evaluate the dynamics disturbances on TM_i , i = 1, 2
- (unless otherwise specified) natural number $n \in \mathbb{N}$ п
- in Chapter 5: normal direction (n_i of surface component $i \in \mathbb{N}$) ĥ
- \hat{n}_{ij} (for *i*, *j* = 1,2,3, $i \neq j$) direction of S/C_{*i*} from S/C_{*i*} in inertial reference frame
- (unless otherwise specified) natural number $N \in \mathbb{N}$ Ν
- $N(\boldsymbol{\theta})$ Euler angles kinematics transformation matrix, $N(\theta)^*$ inverse of $N(\theta)$
- semi-latum rectum р
- $i = 1, 2, \dots$ pole of transfer function p_i
- generic vector used in examples \vec{p}
- Р $\in \mathbb{C}$ frequency domain system plant (transfer function)
- P_{\odot} solar radiation pressure at 1 AU from Sun
- solar radiation pressure P_{sr}
- displacement vector: \vec{r}_{cm} position of formation reference point (incenter) in inertial reference \vec{r}_{\Box} frame, \vec{r}_{ij} , $i, j = 1, 2, 3, i \neq j$ displacement of S/C_j w.r.t. S/C_i in inertial reference frame / \vec{r}_{cp} position of CoP w.r.t. S/C CoM, \vec{r}_{t_i} position of CoT of thruster cluster $i = 1, 2, 3, 4, \vec{r}_i$, displacement of TM_i, i = 1,2 in GRS reference frame from rest position, \vec{r}_{0i} rest position of TM_i, i = 1,2 in S/C body reference frame, \vec{r}_{hi} fixed rotation point of telescope i = 1, 2 in S/C body reference frame
- $= (r_{23}, r_{31}, r_{12})^{T}$ concatenation of inter-S/C distances for the calculation of the incenter Cartesian \boldsymbol{r}_a position
- $\ddot{\vec{r}}_{i_{grs}}$ TM_i , i = 1, 2, acceleration that need to be controlled by the electrostatic suspension system
- (for *i*, *j* = 1,2,3, *i* \neq *j*) distance between S/C_{*i*} and S/C_{*i*} (\dot{r}_{ij} drift speed, \ddot{r}_{ij} drift acceleration) r_{ij}
- (without subscript) in Chapter 4: matrix obtained by concatenating the inertial Cartesian positions R of the S/C = $(\vec{R}_1, \vec{R}_2, \vec{R}_3)$
- S/C displacement in inertial reference frame w.r.t. SSB (\vec{R}_i relative to S/C_i in Chapter 4, \vec{R} velocity Ŕ of S/C, \vec{R} acceleration of S/C)
- in Chapter 2: R_{i_0} initial radial distance of S/C_i from Sun / in Chapter 5: reflection coefficients $\in \mathbb{R}$: R_{\Box} R_{spec} specular reflectivity, R_{diff} diffusive reflectivity, R_{abs} absorptivity
- = $j\omega$ complex frequency (Laplace transform parameter) s
- ŝ direction of Sun from S/C CoM
- (chapter 5) S_i ith surface component area, S_{-z} surface of solar panels S_{\Box}
- sensitivity function S = 1/(1 + L). In Chapter 6: S_x sensitivity of output to dynamics disturbances, S_u sensitivity of control to dynamics disturbances / in Chapter 7: $|S_{\theta}|$ upper limit on magnitude of output sensitivity for the attitude / in Chapter 8: $|S_x|$ upper limit on the output sensitivity for the x-directions, $|S_u|$ upper limit on the acceleration sensitivity for the x-directions, $|S_z|$ upper limit on the output sensitivity for z_1

- *t* time parameter
- *T* sensitivity function T = L/(1 + L). In Chapter 6: T_r the closed loop transfer function of a feedback control system, T_x sensitivity of output to readout noise, T_u sensitivity of control to readout noise / in Chapter 7: $|T_{\theta}|$ upper limit on the output sensitivity to readout noise in the attitude control system / in Chapter 8: $|T_x|$ upper limit on the output sensitivity to readout noise along the *x*-directions, $|T_x|$ upper limit on the acceleration sensitivity to readout noise along the *x*-directions, $|T_z|$ upper limit on the output sensitivity to readout noise along the *x*-directions, $|T_z|$ upper limit on the output sensitivity to readout noise along the *x*-directions, $|T_z|$ upper limit on the output sensitivity to readout noise for z_1
- $T \in \mathbb{R}^{12 \times 1}$ array of thrust force magnitudes
- *u* time-domain control variable in the linear feedback control system
- *u* time-domain control variable in the linear feedback control system, u_{θ} c.v. of the attitude (\vec{l}_t) , u_{df} control variable of the drag-free DoF (\vec{f}_t)
- *U* frequency domain representation of *u*
- \boldsymbol{v} time-domain virtual control variable: \boldsymbol{v}_{θ} v.c.v. of the attitude ($\ddot{\boldsymbol{\theta}}$), \boldsymbol{v}_{df} v.c.v. of the drag-free DoF ($\ddot{\boldsymbol{x}}_{df}$)
- V frequency-domain representation of $v: V_{\theta}$ Laplace transform of v_{θ}, V_{df} Laplace transform of v_{df} W_{\odot} solar constant
- *x* used to identify the first direction component on a 3-dimensional reference frame / random variable (depending on context)
- \hat{x} x-directive of the S/C reference frame (in Chapter 4 \hat{x}_i refers to S/C_i)
- *x* in Chapter 3: $\in \mathbb{R}^{12 \times 1}$ optimization variable, column of A_+ or A_-
- $\mathbf{x}_{df} = (x_1, x_2, z_1)^{\tau} \text{ drag-free DoF}$
- $\ddot{\tilde{x}}_{df}$ accelerations to be eliminated by the thrusters in the drag-free control
- \hat{x}_e x-directive of the Sun-Earth Hill's reference frame
- \ddot{x}_{grs} accelerations to be eliminated by the electrostatic suspension system in the drag-free DoF
- \hat{x}_h x-directive of the Sun-formation Hill's reference frame
- x_i i = 1, 2 x-direction displacement of TM_i
- \hat{x}_i in Chapter 4: *x*-directive of S/C_i, *i* = 1,2,3 body-fixed reference frame w.r.t. inertial reference frame / in Chapter 8: *x*-directive of TM_i, *i* = 1,2, reference frame w.r.t. S/C body
- x_p x-position of S/C on formation planar reference frame
- \hat{x}_p x-directive of the formation planar reference frame

 $x_{sim} \in \mathbb{R}^{n \times 1}$ simulation values (observations) of x_p (*n* is the number of observations)

- X x-component of \vec{R} or \vec{R}_{\Box} / in Chapter 6: the output state of the control system
- $ilde{X}_{df}$ frequency domain representation of linearized drag-free DoF dynamics
- y used to identify the second direction component on a 3-dimensional reference frame
- \hat{y} y-directive of the S/C reference frame (in Chapter 4 \hat{y}_i refers to S/C_i)
- \hat{y}_h y-directive of the Sun-formation Hill's reference frame
- y_i i = 1, 2 *y*-direction displacement of TM_i
- \hat{y}_i y-directive of S/C_i, i = 1, 2, 3, body-fixed reference frame w.r.t. inertial reference frame
- \hat{y}_p y-directive of the formation planar reference frame
- z used to identify the third direction component on a 3-dimensional reference frame
- \hat{z} z-directive of the S/C reference frame (in Chapter 4 \hat{z}_i refers to S/C_i)
- \hat{z}_e z-directive of the Sun-Earth Hill's reference frame
- \hat{z}_h z-directive of the Sun-formation Hill's reference frame
- z_i i = 1, 2 z-direction displacement of TM_i
- \hat{z}_i in Chapter 4: *z*-directive of S/C_i, i = 1,2,3, body-fixed reference frame w.r.t. inertial reference frame / in Chapter 8: *z*-directive of TM_i, i = 1,2, reference frame w.r.t. S/C body
- z_p z-position of S/C on formation planar reference frame
- \hat{z}_p z-directive of the formation planar reference frame
- $z_{sim} \in \mathbb{R}^{n \times 1}$ simulation values (observations) of z_p (*n* is the number of observations)

REFERENCE FRAME SUPERSCRIPTS

- a $\vec{\Box}$ in Chapter 5 it represents the reference frame associated to rotation via θ_{2}
- ${}^{b}\vec{\Box}$ in Chapter 5 it represents the reference frame associated to rotation via θ_{3}
- h relative to Hill's reference frame (in Chapter 2 it also signifies the initial orientation of the Hill's reference frame, but inertial)
- n $\overrightarrow{\Box}$ relative to inertial reference frame ("n" is for Newtonian)
- $p \square$ relative to formation planar reference frame
- $s \square$ relative to S/C body-fixed reference frame

1

INTRODUCTION

1.1. SCIENTIFIC AND HISTORICAL MOTIVATIONS

Gravitational Waves (GW) were introduced for the first time in 1917 by A. Einstein as a solution in vacuum of his field equations [4, 5]. In their most grounded definition, GW are space-time curvature propagating at the speed of light. Although any distribution of mass possessing momentum can be a source, notable emissions spawn from rapidly moving stellar masses, such as binary systems of closely co-orbiting neutron stars and black-holes. Such systems are predicted to exist in large quantities at extremely long distances from our Solar System [6].

In 1975, R. A. Hulse and J. H. Taylor proposed GW as a mechanism for the evolution of binary systems during the investigation of pulsars through radio signals [7]. Direct observation of GW would not occur until 2016, almost a century after their theoretical inception, when the Laser Interferometer Gravitational wave Observatory (LIGO) detected the signature due to the merging of two black-holes more than a billion light-years away [8]. The challenge of detecting GW lies in the extremely weak power they carry when reaching Earth. Their signature is space-time strain affecting the time a light beam takes to travel the distance between two Test-Masses (TMs) [9].

The two LIGO observatories and the other Earth based gravitational wave detectors still in phase of commissioning, such as Virgo, Kamioka Gravitational Wave Detector (KAGRA) and GEO600, are based on long range laser interferometry, with arm lengths of 300 m to 4000 m. This range limits the sensitivity to high frequencies (\geq 50 Hz). Moreover, human and natural activities on Earth add drastically to the background noise [10]. As measure of the required sensitivity, consider that the detection of the first GW signal was based on the measurement of a space-time strain smaller than 10^{-21} [8]. By comparison this was as large as the diameter of a proton over the 4 km interferometric range. Figure 1.1 shows how the signal appeared in time.

The limit on the detection bandwidth excludes most of the GW sources in existence. At LIGO's range, the most prominent, if not the only, detectable event is the last few seconds of a merger. At this stage, neutron stars and black holes co-orbit each other several times per second up until they collide due to dissipating energy through emission of gravitational radiation. It has been predicted, however, that the largest portion of binary systems have orbital frequencies at mHz ranges [6]. In order to detect such events, the interferometric range must be increased to millions of km, much larger than anything that could be built on Earth, whose average radius is 6376 km [10].

Space-based GW observatories have been proposed to provide such ranges. In addition, space offers a suitably undisturbed environment in which TMs (i.e. bodies with small enough mass not to generate significant curvature), may be able to experience gravitational-only accelerations. This condition, referred to as *free-falling* or *drag-free* motion, is an imposed constraint for the mathematical derivation of linear GW [9]. In practice, drag-free motion of the TMs is beneficial in order to eliminate spurious accelerations that might contaminate the GW signature [10].

The concept of a space-based GW observatory is more than 30 years old, starting back in 1981 at the Joint Institute for Laboratory Astrophysics (JILA) in Colorado. The following years saw the conceptual design of the Laser Antenna for Gravitational-radiation Observation in Space (LAGOS), a mission comprising three Space-craft (S/C) in heliocentric orbit and having many elements similar to the titular Laser Interferometer Space Antenna (LISA) of this report. The LISA and the Spaceborne Astronomical Gravitational-wave Interferometer



Figure 1.1: The chirp signal detected by LIGO observatories in Hanford (left) and Livingston (right) within ~10 ms of each other, signaling the first ever direct observation of gravitational waves from the merger of two black holes. At its peak, the space-time strain was ~ 10^{-21} . The event becomes visible on the periodogram for only about 0.1 s, starting at ~50 Hz. (Credits: Abbott B. P. [8])

To Test Aspects of Relativity and Investigate Unknown Sources (SAGITTARIUS) missions were proposed at ESA together in 1993 by two separate international teams backed respectively by the historical coordinator of the LISA project K. Danzmann, Max- Planck-Institut für Quantenoptik, and R. W. Hellings, Jet Propulsion Laboratories (JPL). Back then, LISA was a four-S/C heliocentric concept with a baseline of $5 \cdot 10^6$ km, while SAGIT-TARIUS was a six-S/C geocentric mission with a baseline of 10^6 km [11]. The operation of such observatories is based on long-range laser interferometric measurements among pairs of S/C. Similarly to ground-based detectors, the exchanged lasers over the baselines allow to synthesize a Michelson's-like interferometer. The modern, long-lasting configuration of LISA, adopted back in 1996, calls for 3 S/C in heliocentric orbits in an equilateral triangle formation [12].

Formerly a joint NASA-ESA effort, development was supported by both administrations up until the retirement of the *LISA Pathfinder* S/C, a precursor mission launched in 2015 to test many of the enabling low technology readiness level aspects [13]. NASA formally withdrew from the project in 2011, and, since then, LISA underwent several redesigns, such as downsizing the arm-length from $5 \cdot 10^6$ to km $2.5 \cdot 10^6$ km [14].

LISA Pathfinder was fitted with much of the unprecedented technology set to be used on LISA, most notably, the LISA Test Package (LTP) included two TMs, a laser interferometer to precisely measure their relative displacement, a contactless suspension mechanism that allows the TM to float within the S/C, and μ *Newton thrusters* capable of high-performance, low energy thrust forces to enable drag-free motion [15].

LISA was announced once again in early 2017 following a call for proposals by ESA for the L3 (third Largeclass mission) slot of 2034 [14]. The success of LISA Pathfinder and the first confirmed detection of GW by LIGO in the previous year granted LISA approval and secured the L-class mission 1 billion Euro funding in June 2017 [16].

1.2. OVERVIEW OF THE LISA MISSION

LISA is the first mission of its kind. It will deploy three S/C in Earth-trailing heliocentric orbits, keeping an almost constant relative orientation at the vertexes of an equilateral triangle with side length of $2.5 \cdot 10^6$ km [14]. Each S/C is fitted with two telescopes aiming at the respective companions, sending to and receiving from them a laser beam (Figure 1.2). Each S/C contains two TMs, each paired to a companion on the next S/C.



Figure 1.2: The basic functioning principle of LISA. The three S/C exchange lasers through properly fitted telescopes over 2.5 million km to measure the differential displacement of TMs (grey squares) belonging to two different S/C through interferometry. Credits: O. Jennrich [17]

The sent and received laser beam is bounced off the TM to obtain a reading of their differential displacements at the ends of an interferometric arm. The TMs are gold-platinum alloy cubes weighting about 1.9 kg. The combined interferometric measurements of the six links (two sent and two received on each arm) are later processed to identify the GW signatures [12, 17–19]. The reader is referred to the bibliography for the full description of the LISA system.

The triangular geometry is achieved through a *cartwheel orbit*: the formation plane is constantly inclined by 60° w.r.t. the ecliptic while the S/C perform an apparent circular motion with constant angular velocity around a fictional reference point. The reference point orbits the Sun at a nominal trailing distance from Earth of 20°. Figure 1.3 shows how the conceptual design of such a formation is achieved. The formation



Figure 1.3: Depiction of LISA orbital design. The trailing angle of $19^{\circ}-23^{\circ}$ shown here takes into consideration the drifting effect due to the presence of Earth. Credits: K. Danzmann et. al. [14]

performs a revolution every year, with the S/C orbiting the reference point at the same rate [14, 20].

The characteristic frequency-sensitivity curve, shown in Figure 1.4 must be attained by removing nongravitational perturbations along the interferometric arms. This is performed through a Drag-Free and Attitude Control System (DFACS). The DFACS is aided by the Gravitational Reference System (GRS), a mechanism that reads the displacement of the TMs w.r.t. the S/C and performs other operations such as keeping the TM from touching the walls. The TMs, shielded by the outside forces, abide only the gravitational accelerations.



Figure 1.4: The LISA strain sensitivity curve and the encompassed detectable GW sources. This curve drives the control requirements in LISA. GW150914 is the first gravitational wave event observed by LIGO. Credits: K. Danzmann et. al. [14]

By adjusting the position of the S/C w.r.t. the TMs the DFACS is able to eliminate the non-gravitational accelerations, keeping the S/C in free-fall during the science operations [14, 17].

Actuation of the DFACS is provided through a set of μ Newton thrusters, i.e., high-performance, low-thrust actuators working in the range ~0.1÷100 μ N. These values are consistent with the size of the external perturbations, mainly the solar-radiation pressure and the thruster noise. Meanwhile, the GRS needs to shelter the TM from unwanted interaction with the S/C environment, mainly the thermal noise, the electrostatic and self-gravity forces, and centripetal and Coriolis' accelerations [14, 17].

Both the required attitude and the TMs displacement must be achieved under extremely precise levels of accuracy inside the measurement bandwidth, which lies between 0.1 mHz and 0.1 Hz (with a goal of 20 μ Hz to 1 Hz). The refresh frequency of the control hardware is limited to 10 Hz [14]. As such, the DFACS gain must be designed carefully against the perturbations.

1.3. THESIS OBJECTIVES

LISA sensitivity requirements are extremely demanding: the laser beam cone width is just 200 μ rad in diameter [21], requiring precise pointing capabilities in the spectral range of 10 nrad per frequency bin [14]. Spurious accelerations of the TMs along the interferometer arms need not to exceed spectral densities of 3 fm/s² per frequency bin in the bandwidth, while displacement must not exceed 1 pm per frequency bin. Outside the bandwidth, limits are also imposed on the low frequency amplitudes, respectively to 10 nrad and 5 nm [14].

In total, 19 Degrees of Freedom (DoF) per S/C are to be controlled [22]: the six displacement and six orientation DoF of the two TMs, the six displacements and attitude DoF of the S/C and the breath angle between the two telescopes [22]. While the free-floating TMs need to be kept from hitting the walls of the GRS by an *electrostatic suspension system*, some DoF must be allowed to follow their gravitational paths in order to keep the S/C on a drag-free trajectory, using the on-board thrusters. The thrusters are also employed to generate attitude control torque [22]. Assuming that the necessary hardware is fully developed and up to the task, i.e. no improvements are necessary on the technology readiness level of all the involved components, the control algorithm is left to be designed.

Ultimately, this thesis deals with the design of the feedback control loop, thus begging the research question:

What is a suitable algorithm for the drag-free and attitude control system that allows the LISA S/C to perform within all requirements during science operations?

The word *suitable* refers to the capability of the algorithm to, alone, reduce the noise of the DoF to levels that satisfy the mission requirements.

Before the design process we need to understand the size of the perturbations acting on the S/C. This requires a focused analysis of the LISA system. In particular, the following subquestions are to be answered:

- 1. What is the nominal *state* of the formation and how does it relate to the state of each individual S/C?
 - What are the orbits of the LISA formation?
 - How is the S/C orientation defined?
 - What is the expected behavior of the S/C in relation to the other two?
- 2. What are the perturbations acting on each degree of freedom and what are their components?
 - What is the size and orientation of the solar radiation pressure force and torque?
 - What are other spurious accelerations?
 - What are the dynamics of each degree of freedom?
 - What is the sensitivity of each degree of freedom to the perturbations?
- 3. What are the limitations of the control system?

Question 1 encompasses the need to understand the formation dynamics, e.g. LISA's orbital behavior, the nominal orientation of the S/C in order to keep the telescopes aligned to the next two, and the associated dynamical parameters (angular velocities and accelerations).

Question 2 covers the sizing of the perturbing forces, from the constant to low-frequency solar radiation pressure, to the spurious accelerations acting in the measurement band. This step requires the dynamics of each DoF to be known.

Question 3 is about the control system itself, the limitations on the performance imposed by the mission requirements and the implications from controlling the degrees of freedom. For example, the limit on the onboard frequency of 10 Hz is expected to influence the overall performance at high frequencies as experienced on LISA Pathfinder [23].

1.4. METHODOLOGY

This thesis is an individual study on a mission in current status of development. The lack of empirical data and the evolution of the project poses some limits on the extent of this research. Here is the methodology used to tackle the problems.

Working up from the basic informations: The inherent qualities of the mission are the starting point of this study. These are:

- The mission objectives;
- The shape of the orbit;
- The description of the payload;
- The model of known disturbances (i.e. Solar Radiation Pressure (SRP));
- The control objectives, dictated by the science requirements.

The core function of each S/C as floating laboratories at the corners of a triangular interferometer, and whose motion in space is well defined by the formation design can tell us a great deal about its expected behavior, both in its translational and in its rotational state. We start by obtaining information about the motion, i.e. by implementing an orbital simulation, and then try to determine the orientation and the angular velocities and accelerations associated to a well controlled system. These can be used as references for attitude control and dynamics.

The properties of the payload, namely the optical assemblies and the GRS, are used to derive the dynamics of all the DoF included in the system.

The model of the SRP force can be applied to determine the ideal forces required by the drag-free control of the S/C. Finally, the control objectives tell us what is expected of the control system.

Filling up the unknowns: More than seldom we are faced with a lack of values that are necessary for a complete definition of the system. In particular we can only infer about the following:

• Mass and inertias of the S/C;

- The specific impulse of the thrusters;
- The thruster configuration;
- The vibrational noise;
- The precision of the measurement system.

An extrapolation of the above from literature and rationalized guesswork is necessary, therefore, to finalize the model.

Validation and verification: publicly available literature provide data and equations that can be used as benchmarks to validate the model. When these are not available, we need to get creative and make tests from independent calculations. We need to make sure, in particular, that the derived equations are physically sound and provide proof of their veracity.

The verification process is more straightforward, as it needs to ensure that the already validated models are integrated accordingly in the code/simulation, and the results are mutually compatible.

Expected results: conceptually, this thesis can be divided into two parts: the first part is about the definition of the model, its expected behavior, and the derivation of the dynamics; the second part deals with the design of the control system, culminating with a set of simulations aimed at confirming its feasibility. The simulations are performed in Matlab and Simulink[®] [24], being the tool of choice for many engineering applications.

2

ORBITAL MODEL

2.1. INTRODUCTION TO THE ORBITAL SIMULATION

Nominally, the three LISA S/C are placed at the vertices of an equilateral triangle with a side length of $l = 2.5 \cdot 10^6$ km as introduced in Section 1.2 (see Figure 1.3). The formation is bound to what is commonly referred to as a *cartwheel orbit* (Figure 2.1): the S/C (slave) follows an apparent circular motion about a fictional point



Figure 2.1: Apparent motion of a *slave* satellite in a cartwheel orbit around a *master* satellite in the latter's Hill's reference frame. $\overline{\phi}$ is the "complementary angle" of the plane inclination $\phi = 60^\circ$, $h\overline{\omega}$ is the apparent angular motion of the slave satellite (superscript h is for the Hill's reference frame coordinate system) and $h\overline{r}$ is its velocity.

(master) orbiting the Sun, on a plane inclined by 60° w.r.t. the ecliptic from the local Sun direction. This fictional point is the center of the formation, or, in the case of LISA, the geometrical barycenter of the equilateral triangle. The formation orbits alongside Earth with a ~ 20° delay [14].

This motion necessarily rules some aspects of the LISA dynamics: the three S/C, locked onto each-other along the lines-of-sight of the telescopes, need to be oriented accordingly, while the breath-angles between the telescopes need to be tuned to match the angular drift between two consecutive lines-of-sight, thus imposing a few formation-attitude relations. Moreover, the solar radiation pressure is dependent on the position of the S/C w.r.t. the Sun [25].

In this chapter we focus on the modeling of the orbital behavior of the three LISA S/C. Although some first-order approximative models of the orbit exist [26], it is more interesting for the purposes of this thesis, to achieve an exact, inertial definition.

2.2. THE EXACT ORBITS

The *cartwheel orbit* is a solution of the *Clohessy-Whiltshire* equations in the *Hill's reference frame*. On first order approximation, the slave satellite orbits the master with a circular orbit on a plane inclined by 60° about the *y*-axis (direction of motion). The co-orbital and orbital periods are equivalent [20].

For LISA, the master satellite is replaced by a fictitious reference point in circular orbit around the Sun at the same semi-major axis as Earth. The initial Kepler parameters for the exact LISA orbits were derived

by S.V. Dhurandhar et. al. [27]. The equations are provided in the following paragraphs. Let us define the nominal arm-length l, the semi-major axis a, the orbital inclination i, the eccentricity e and the angle of the formation-plane w.r.t. the ecliptic ϕ as shown in Figure 2.2.



Figure 2.2: Orbit of the LISA S/C around the Sun and the shape-related orbital parameters. $a \approx 1$ AU is the semi-major axis, $l \approx 2.5 \cdot 10^6$ km is the arm-length, $\gamma \approx 20^\circ$ is the Earth-trailing angle, $\phi \approx 60^\circ$ is the inclination of the formation plane w.r.t. the Ecliptic.

According to Dhurandhar, the eccentricity of the Sun centered orbit, e, is calculated as

$$e = \sqrt{1 + \frac{l^2}{3a^2} + \frac{2l}{\sqrt{3}a}\cos\phi} - 1$$
(2.1)

and the inclination is calculated as

$$\sin i = \frac{l\sin\phi}{\sqrt{3}a(1+e)}.\tag{2.2}$$

As per formation requirements, we set the arm-length to $l = 2.5 \cdot 10^6$ km, the orbital period of 1 sidereal year, in order to trail Earth, which corresponds to a semi-major axis of a = 1 AU, and the cartwheel nominal co-orbital plane inclination of $\phi = 60^\circ$. The obtained values are shown in table 2.1.

Table 2.1: Eccentricity and inclination of the LISA orbits

 Eccentricity
 e 0.004858926162390

 Inclination
 i 0.476438823626267°

For what concerns the orientation of the three orbits let us consider the coordinate system oriented as the initial Hill's reference frame of the formation barycenter. We need to define the arguments of perihelion ω_i and the longitudes of the ascending node Ω_i for each S/C_i, i = 1,2,3. In order to keep the co-orbital plane inclined in the direction of the Sun, with the orientation shown in Figure 2.2, the perihelion must occur at the lowest Cartesian *z*-coordinate. This means that the orbital plane is inclined about the perpendicular to the semi-major axis, setting the argument of perihelion at $\omega_i = 270^\circ$ for all the S/C. The *z*-axis motion is periodical, with a phase difference between S/C of 120°, therefore, the three Ω_i must also be phased accordingly. The actual values depend on the initial true anomalies θ_i . Figure 2.3 displays the initial configuration used in the context of this thesis.

According to Figure 2.2, we start with S/C_1 at perihelion, laying directly beneath the geometrical barycenter, therefore, with a longitude of ascending node $\Omega_1 = 90^\circ$. The other two S/C are assigned ascending nodes at $\Omega_2 = 330^\circ$ and $\Omega_2 = 210^\circ$. The choice of labeling reflects the co-orbital direction of motion, which seems to rotate with an apparent clock-wise direction from the perspective of the inertial *z*-axis [26].

Finally, according to Durhandhar, the initial mean anomalies M_{i_0} must satisfy the following relation [27]:

$$\Omega_i + \omega_i + M_{i_0} = 2k\pi, \, k \in \mathbb{N} \tag{2.3}$$

with $M_{1_0} = 0$ since S/C₁ starts at perihelion.

The initial true anomalies θ_{i_0} , i = 1, 2, 3, are calculated from M_{i_0} using Relations (2.4)-(2.5) [28]:

$$M = E - e\sin E \tag{2.4}$$



Figure 2.3: The orientation-related orbital parameters and initial true anomalies of the three LISA S/C. $\Omega_1 = 90^\circ$, $\Omega_2 = 330^\circ$ and $\Omega_3 = 210^\circ$ are the longitudes of the ascending nodes of the respective S/C. $\theta_1 = 0$ (not shown), $\theta_2 \simeq 120.4807^\circ$ and $\theta_3 \simeq 239.5193^\circ$ are the initial true anomalies of the respective S/C. The elements are defined w.r.t. the initial Hill's reference frame of the formation's reference point, with the *x*-axis x_h laying on the Ecliptic, with origin at the Sun and pointing $\gamma = 20^\circ$ from the initial Earth's position.

$$\tan\frac{\theta_{i_0}}{2} = \sqrt{\frac{1+e}{1-e}} \tan\frac{E_{i_0}}{2}$$
(2.5)

where *E* is the eccentric anomaly. Equation (2.4) must be resolved in closed form using a numerical method. Table 2.2 summarizes the values of the longitudes of ascending nodes and the respective anomalies.

Table 2.2: Values of initial longitude of ascending nodes, mean and true anomalies of the LISA S/C

		S/C_1	S/C ₂	S/C ₃
Argument of perihelion	ω	270°	270°	270°
Longitude of Ascending Node	Ω	90°	330°	210°
Initial Mean anomaly	M_0	0°	120°	240°
Initial true anomaly	$ heta_0$	0°	120.4807302235256°	239.5192697764743°

2.3. INERTIAL CARTESIAN COORDINATES

The derived orbits in the previous section are Sun-centered. In order to simulate the precise Newtonian motion, we need to translate the orbits to an inertial reference frame. The Cartesian orbital elements are derived in the current reference frame using the following procedure [28]. First, we calculate the semilatum-rectum p, common to all the S/C, as:

$$p = a\left(1 - e^2\right) \tag{2.6}$$

next, we calculate the initial distance from the Sun R_{i_0} of each S/C_i, using:

$$R_{i_0} = \frac{p}{1 + e \cos \theta_{i_0}}$$
(2.7)

The angular momentum per unit mass w.r.t. the Sun is calculated using *p* and the gravitational parameter of the Sun μ_s and it is the same for all S/C:

$$h = \sqrt{\mu_s p}; \tag{2.8}$$

finally, the initial Cartesian displacements ${}^{h}\vec{R}_{i_{0}}$ and velocities ${}^{h}\vec{R}_{i_{0}}$ are calculated using, respectively Equations (2.9) and (2.10) [28]:

$${}^{h}\vec{R}_{i_{0}} = R_{i_{0}} \begin{pmatrix} \cos\Omega_{i}\cos\left(\omega + \theta_{i_{0}}\right) - \sin\Omega_{i}\sin\left(\omega + \theta_{i_{0}}\cos i\right) \\ \sin\Omega_{i}\cos\left(\omega + \theta_{i_{0}}\right) + \cos\Omega_{i}\sin\left(\omega + \theta_{i_{0}}\cos i\right) \\ \sin i\sin\left(\omega + \theta_{i_{0}}\right) \end{pmatrix}$$
(2.9)

$${}^{h}\vec{R}_{i_{0}} = \frac{he}{rp}\sin\theta_{i_{0}}{}^{h}\vec{R}_{i_{0}} + \frac{h}{R_{i_{0}}} \begin{pmatrix} -\cos\Omega_{i}\sin\left(\omega + \theta_{i_{0}}\right) - \sin\Omega_{i}\cos\left(\omega + \theta_{i_{0}}\right)\cos i \\ -\sin\Omega_{i}\sin\left(\omega + \theta_{i_{0}}\right) + \cos\Omega_{i}\cos\left(\omega + \theta_{i_{0}}\right)\cos i \\ \sin i\cos\left(\omega + \theta_{i_{0}}\right) \end{pmatrix}.$$
(2.10)

The values are shown in Table 2.3.

Table 2.3: Initial Cartesian orbital elements in the original reference frame $({}^{h}\vec{R}_{i_{0}})$ and ${}^{h}\vec{R}_{i_{0}})$

	S/C1	S/C_2	S/C_3
${}^{h}x_{0}$ [m]	148865838782.051	149957358979.316	149957358979.316
${}^{h}y_{0}$ [m]	0.000	1255953982.388	-1255953982.388
${}^{h}z_{0}$ [m]	-1237911422.101	632538841.627	632538841.627
$h\dot{x}_0 [\mathrm{m/s}]$	0.000	-125.013	125.013
$h\dot{y}_0 [\mathrm{m/s}]$	29929.767	29710.865	29710.865
${}^{h}\dot{z}_{0}$ [m/s]	0.000	213.444	-213.444

The ^{*h*} superscript refers to the current reference frame. In order to get rid of it and translate to the inertial Cartesian elements, we need the positions and velocities of Earth and Sun at the starting epoch, respectively \vec{R}_{e_0} , \vec{R}_{s_0} , \vec{R}_{e_0} and \vec{R}_{s_0} . Let us identify a Sun-centered reference frame whose *x*-axis points towards the initial position of Earth and the *z*-axis is parallel to its orbital angular momentum. The *x*- and *z*-directives of such reference frame are, respectively,

$$\hat{x}_e = \frac{\vec{R}_{e_0} - \vec{R}_{s_0}}{\|\vec{R}_{e_0} - \vec{R}_{s_0}\|}$$
(2.11)

and

$$\hat{z}_{e} = \frac{\hat{x}_{e} \times \left(\vec{R}_{e_{0}} - \vec{R}_{s_{0}}\right)}{\|\hat{x}_{e} \times \left(\vec{R}_{e_{0}} - \vec{R}_{s_{0}}\right)\|}$$
(2.12)

In the Ecliptic J2000 reference frame, Earth's orbital plane is almost congruent to the Ecliptic [28, 29], hence the *z*-directive can be approximated as $\hat{z}_e \simeq [0,0,1]^{\tau}$.

Next, mission profile requires the formation to trail Earth at $\gamma = 20^{\circ}$. In order to satisfy this condition we need to perform a change of reference frame through a rotation, first about the *z*-axis by γ and then transforming from the Earth's Hill's-like reference frame to the inertial orientation. The total direction-cosine matrix from the original reference frame to the inertial reference frame $C_{h/n}$ is derived as:

$$C_{h/n} = \begin{pmatrix} \hat{x}_e, & \hat{z}_e \times \hat{x}_e, & \hat{z}_e \end{pmatrix} \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0\\ -\sin(\gamma) & \cos(\gamma) & 0\\ 0 & 0 & 1 \end{pmatrix}.$$
 (2.13)

Section 4.3 will discourse more in depth about the method used here. Multiplication by $C_{h/n}$ and addition of the Sun's Cartesian components results in the initial state of each S/C_i in Solar System Barycenter (SSB) centered inertial reference frame, respectively position \vec{R}_{i_0} and velocity \vec{R}_{i_0} :

$$\vec{R}_{i_0} = C_{h/n}{}^h \vec{R}_{i_0} + \vec{R}_{s_0}; \tag{2.14}$$

$$\vec{R}_{i_0} = C_{h/n}{}^h \vec{R}_{i_0} + \vec{R}_{s_0}.$$
(2.15)

Figure 2.4 shows the initial osculating orbits in Sun-centered Ecliptic J2000 orientation, when the initial epoch is set to January 01, 2035, 00:00:00.0h. The initial Earth and Sun states, \vec{R}_{e_0} , \vec{R}_{s_0} , \vec{R}_{e_0} and \vec{R}_{s_0} , shown in Table 2.4, are retrieved from the *SPICE* ephemeris database [30].



Figure 2.4: The three LISA S/C orbits visualized in Sun-centered Ecliptic J2000 orientation. (The z-axis aspect-ratio is 20 times larger than the other two). Initial epoch is January 01, 2035, 00:00:00.0h.

Table 2.4: Earth and Sun ephemeris at epoch January 01, 2035, 00:00:00.0 (Solar System Barycenter centered Ecliptic J2000 \vec{R}_{e_0} , \vec{R}_{s_0} , \vec{R}_{s_0} and $\dot{\vec{R}}_{s_0}$).

	Earth	Sun
<i>X</i> ₀ [m]	-25944943557.455	-675579147.265
Y_0 [m]	144176545713.140	-741669709.405
<i>Z</i> ₀ [m]	8075690.480	19849850.823
$\dot{X}_0 \ [m/s]$	-29823.527	6.804
\dot{Y}_0 [m/s]	-5230.438	-11.237
\dot{Z}_0 [m/s]	-0.545	-0.160

2.4. SIMULATION RESULTS

The simulation was performed using a Cowell's propagator [28], starting at epoch January 01, 2035, 00:00:00.0h. The gravities of Sun, Earth and Moon are included, with the planetary ephemerides extracted using the CSPICE toolkit embedded in TUDAT (TU Delft Astrodynamics Toolbox) and based on the SPICE kernel DE 421. [2, 30, 31]. A 4th order Runge-Kutta integrator with a constant step size of 1000 s is used. Table 2.5 summarizes the simulation properties. On first iteration, TUDAT was employed for the orbit propagation. Later on, the whole simulation was transferred to Simulink.

Table 2.5: Orbit simulation properties.

Name	Value
Reference frame	Ecliptic J2000
Origin	Solar System Barycenter
Initial epoch (t_0)	1104494400.0 [s] (J2000)
Propagator	Cowell's
Massive bodies	Sun, Earth, Moon
Planetary ephemerides	DE 421
Integrator	4 th order Runge-Kutta
Step-size	1000 [s]

Two simulations have been performed:

- Sun-centered, i.e. setting $\vec{R}_{s_0} = \vec{R}_{s_0} = \vec{0}$ in Equations (2.14) and (2.15), without third body gravitational accelerations for the validation of the model;
- SSB-centered, with third body action of Earth and Moon.

Figure 2.5 shows the resulting LISA orbits in the Hill's reference frame of the geometrical barycenter for the unperturbed case. The nominal circular motion with an inclination of 60° degrees on the Hill's reference frame is satisfied.



Figure 2.5: LISA orbits in the Sun-geometrical barycenter Hill's reference frame. The red dot identifies the starting position of the respective S/C. Left to right S/C_1 , S/C_2 and S/C_3 . The simulation is performed without third body gravitational influence.

However, using a rotational transformation of the reference frame around the *y*-axis by $-120^{\circ 1}$, similarly to Bik et. al. [26], and plotting the *z*-axis displacement of the three S/C about this new reference system (Figure 2.6) we are able to visualize the second and higher order discrepancies from a nominal cartwheel



Figure 2.6: Displacement z_p of the unperturbed orbits on the *z*-axis of a reference frame obtained by rotating the Hill's reference frame around the *y*-axis by -120° . The peaks are about ~ -6770 km and ~ 8400 km about the xy-plane.

orbit: the S/C cross the reference formation plane periodically with overshoots on the planar *z*-axis of ~-6770 km and ~8400 km. Since the periodicity of these overshoots is equivalent to the orbit's, the formation plane's tilt is different than the design $\phi \neq 60^{\circ}$.

Since the formation plane is rotated about the *y*-axis, the error on the tilt $\delta \phi$ satisfies a linear relation between the planar x_p and z_p component:

$$z_p = \tan(\delta\phi) x_p. \tag{2.16}$$

Let us call the simulated observations for x_p and z_p , respectively, \mathbf{x}_{sim} and \mathbf{z}_{sim} , with \mathbf{x}_{sim} , $\mathbf{z}_{sim} \in \mathbb{R}^{n \times 1}$, where n is the number of observations. The solution for $\delta \phi$, using a linear least squares algorithm is [32]:

$$\tan(\delta\phi) = \left(\boldsymbol{x}_{sim}^{\mathsf{T}}\boldsymbol{x}_{sim}\right)^{-1}\boldsymbol{x}_{sim}^{\mathsf{T}}\boldsymbol{z}_{sim}$$
(2.17)
¹The rotation matrix used is $C_{h/p}(-120^\circ) = \begin{pmatrix} \cos(-120^\circ) & 0 & \sin(-120^\circ) \\ 0 & 1 & 0 \\ -\sin(-120^\circ) & 0 & \cos(-120^\circ) \end{pmatrix}.$

which results in $\delta \phi \simeq -0.301988653^\circ$, suggesting that the actual tilt is

$$\phi = 60 + \delta \phi \simeq 59.698346358^{\circ}$$

Performing the transformation again, this time using a rotation about the *y*-axis of $-120+\delta\phi$, and plotting the *z*-axis displacement, Figure 2.7 is obtained.



Figure 2.7: *z*-axis displacement of the unperturbed orbits on a reference frame obtained rotating the Hill's reference frame by \sim -120.30199 around the *y*-axis. The amplitudes are about \sim ±817.3 km for a period of 1/2 year.

The resulting displacement on the *z*-axis has an overshoot of amplitude $\sim \pm 817.3$ km, and periodicity of a half orbit. This kind of behavior is inferred the appropriate name of "wobble", as it results in the formation plane to slightly precess at the same rate as the orbital motion in the Hill's reference frame of the geometrical barycenter.

Another second order effect is the variation in time of the orbital motion. Due to the asymmetry of this variation, the arm lengths, or distances between S/C, see a periodical difference from the nominal value of $l = 2.5 \cdot 10^6$ km. The values of the difference for one orbital period are plotted in figure 2.8.



Figure 2.8: Arm length excursion from the nominal value $l = 2.5 \cdot 10^6$ km for the unperturbed orbits. The value $r_{ij} = ||\vec{R}_j - \vec{R}_i||$, i, j = 1, 2, 3, $i \neq j$ is the distance between S/C_i and S/C_j, also referred to as arm-length. They vary about ~23920 km and ~-4779 km from l.

During one orbit the S/C drift from one another between \sim 23920 km and \sim -4779 km equivalent to a total percentage variation of 1.1480%. These values can be minimized by optimizing the initial orbital parameters [33].

The inclusion of Earth and Moon gravity in the model results in "wobble" and arm-length variations as in Figures 2.9 and 2.10.

The data, plotted for 10 years, show how the errors accumulate over time. The wobble is almost non existent from $t \approx 5.5$ years, leaving place to a much more prominent gradual tilting of the formation plane,



Figure 2.9: 10 years *z*-axis displacement of the orbits on a reference frame obtained rotating the Hill's reference frame by \sim -120.30199 around the *y*-axis. Earth and Moon third body gravities are included.



Figure 2.10: Arm length error from the nominal value $l = 2.5 \cdot 10^6$ km for 10 years with Earth and Moon gravities included in the simulation. The value $r_{ij} = \|\vec{R}_j - \vec{R}_i\|$, $i, j = 1, 2, 3, i \neq j$ is the distance between S/C_i and S/C_j, also referred to as arm-length.

characterized by a yearly period of *z*-axis displacement. The arm-length variations start growing dramatically after about ~ 5 years with an increasingly asymmetrical trend and a maximum percentage drift of ~8.7096% for the S/C_1 - S/C_2 arm during the first 10 years.

2.5. ORBITAL DRIFT

Plotting the position of the formation's geometrical barycenter in the Hill's reference frame of Earth for the unperturbed and perturbed cases, respectively, Figures 2.11 and 2.12 are obtained.

In the unperturbed simulation, the geometrical barycenter follows a constant circular orbit around the Sun, which appears as a *planar cartwheel* orbit from an Earth-centered point of view [20]. This orbit is characterized by an elliptical motion with a negative rotation (Figure 2.11). This motion arises when one of the two co-orbiting bodies has non-zero eccentricity (Earth, in this case).

When Earth and Moon gravities are applied, the formation starts drifting away (Figure 2.12). This is due to Earth-Moon system's pull in the direction of motion, causing the semi-major axes of the S/C to rise and, therefore, lengthen their orbital period.

The magnitude of this effect can be verified using the Circular Restricted 3-Body Problem (CR3BP) [34]. Using a mass ratio parameter of

$$\mu = \frac{\mu_e}{\mu_s + \mu_e} \simeq 3.003480642 \cdot 10^{-6} \tag{2.18}$$



Figure 2.11: Unperturbed orbit of the formation's geometrical barycenter in the Earth-Sun Hill's reference frame (xy-plane projection). The grid shows the distance in million km (Gm) and the phase from the Sun-Earth line of sight (x-axis). The underlying reference orbit is a 1 AU circle centered at Sun. The simulation starts at 01/01/2035 00:00:00.0.



Figure 2.12: Perturbed orbit (Earth and Moon third body gravities included) of the formation's geometrical barycenter in the Earth-Sun Hill's reference frame (xy-plane projection) for 10 years. The grid shows the distance in million km (Gm) and the phase from the Sun-Earth line of sight (x-axis). The underlying reference orbit is a 1 AU circle centered at Sun. The simulation starts at 01/01/2035 00:00:00.0.

where μ_e and μ_s are, respectively, the Earth's and Sun's gravitational parameters; and an initial displacement of

$$X_0 = \cos(-20^\circ) - \mu;$$

$$Y_0 = \sin(-20^\circ)$$
(2.19)

with initial zero velocity, the resulting orbit in the Sun-Earth co-rotating reference frame is a *horseshoe* orbit with a period of ~470 years. Figure 2.13 shows the CR3BP solution with the parametrized units of distance and time scaled respectively by 1 AU and $1/(2\pi)$ years: during the first 10 years the formation will drift from Earth from a distance of ~52 $\cdot 10^6$ km to ~70 $\cdot 10^6$ km, which is consistent with the true orbit of Figure 2.12.

The additional loops due to the eccentricity of Earth's orbit and the nominal circular cartwheel orbit of the formation about its barycenter add to the total distance from Earth. Figure 2.14 shows the magnitude of drift for each individual satellite: S/C_3 is the first to cross the recommended communication range limit of $65 \cdot 10^6$ km [14] at about $t \simeq 6.15$ years.



Figure 2.13: Solution of the circular restricted 3-body problem for a starting position of -20° on the Earth orbit and zero initial velocity. **left**: the complete orbit; **right**: the drift for 10 years.



Figure 2.14: Distance from Earth of the three S/C in millions of km (Gm) for 10 years.

2.6. VALIDATION OF ORBITAL MODEL

G. Li et. al. [33] provide a benchmark for the simulator: Figure 2.15 shows the arm-length variations obtained by setting the nominal arm-length and the formation plane tilt, respectively, to

$$l = 5 \cdot 10^{6}$$
 km;
 $\phi = 60.4776^{\circ}$

in Equations (2.1) and (2.2). The model does not include third-body gravitational perturbations. The simulation uses different labeling and displacement than the one used in this thesis, with S/C_1 at aphelion and S/C_2 and S/C_3 inverted. Table 2.6 shows the parameters used for orbital orientation and initial position.

Figure 2.16 shows the simulation results using these values: the shape and magnitude of the plot, on visual inspection, bears no difference from the one in Figure 2.15, therefore proving the validity of the model.

The magnitude of the variation does not change if the values in Table 2.2 are used, instead. As shown in Figure 2.17 the shape of the plot is just phase-shifted w.r.t. Figure 2.16.


Figure 2.15: Validation data: arm-length variations from the nominal value of $r_0 = l = 5 \cdot 10^6$ km [33]

Table 2.6: Values of initial longitude of ascending nodes and mean anomalies of the LISA S/C used in G. Li et. al. [33]

		S/C_1	S/C_2	S/C_3
Argument of parihalian		270°	270°	270°
Argument of permenon	ω	270	270	270
Longitude of Ascending Node	Ω	270°	30°	150°
Initial Mean anomaly	M_0	180°	60°	300°



Figure 2.16: Reproduction of G. Li et. al. plot of figure 2.15 ($r_0 = l = 5 \cdot 10^6$ km).



Figure 2.17: Results using the conditions of Table 2.2, $r_0 = l = 5 \cdot 10^6$ km and $\phi = 60.4776^\circ$.

2.7. Recommendations on the orbital model

The choice of initial orbital parameters and epoch are not optimized for best formation performance. Requirements, for example, call for a maximum distance from Earth of $65 \cdot 10^6$ km, and a cap on the inter-S/C drift speed of <15 m/s in order to relax the Doppler shift on the interferometric measurements [14]. Efforts to minimize these quantities yielded success in the past [33, 35], although application of the same parameters is no longer possible after the arm-lengths were redefined from 5 to 2.5 million km, thus requiring a new optimization process (which is beyond the scope of this thesis). The CR3BP can be invoked in order to optimize the formation drift from Earth by twitching the initial semi-major axis: with a smaller semi-major axis, the formation would initially drift towards Earth and later start drifting away as the gravitational interaction of Earth builds up. This conclusively comes at the cost of inter-S/C drift rate and arm-length variations as shown by Y. Xian et al.[35]. Arm-length, in particular, is also responsible for breath angle variations, required to be kept below 1.5°, due to the triangular geometry of the formation.

Some purposes might benefit from adding additional gravitational perturbations, such as Jupiter and Venus. These where deliberately left out of the simulation to avoid computational complexity.

We notice that the formation retains most of its original shape during the first 4 years of operation, as shown in Figure 2.10. This period coincides with the nominal mission lifetime (although the goal is 10 years) [14]. In the next chapters **we are going to study the formation during the initial 4 years**. We assume that repositioning is performed after this first period.

2.8. CONCLUSIONS ON THE ORBITAL MODEL

In this chapter the motion of the LISA formation has been modeled. The initial orbital parameters have been calculated according to Durandhar et al. [33], by plugging in the required parameter of arm-length $l = 2.5 \cdot 10^6$ km and semi-major axis of a = 1 AU. The initial epoch and configuration were freely chosen to begin at January 1, 2035, 00:00:00h with S/C₁ at perihelion. The center of the formation is positioned at 20° clockwise from the polar angular coordinate of Earth on the Ecliptic. The initial orbital parameters are expressed in SSB-centered Cartesian inertial Ecliptic J2000 coordinates.

The simulation uses a classical Cowell's propagator with Sun, Earth and Moon point-mass gravity included. The integrator used is a 4th order Runge-Kutta.

Preliminary runs including only Sun's gravity confirm that the three S/C revolve around the geometrical barycenter of the formation on nearly-circular relative orbits, inclined by 60° w.r.t. the ecliptic. Closer inspection reveal an actual mean plane inclination of ~59.59834635757° and a periodical tri-phase distance from this plane of the three S/C with an amplitude of \pm 817.3 km and a frequency twice as large as the orbital motion, thus defining a small wobble.

With the inclusion of third-body gravity, the formation drifts away from Earth. The drift is consistent with the CR3BP analysis of similar initial conditions, reaching an average distance of $\sim 70 \cdot 10^6$ km after 10 years. Validation is performed by confronting the results of inter-S/C distance with the ones obtained by G. Li et al [33]. In the next chapters the first 4 years of simulation are studied.

3

THE LISA SPACECRAFT

3.1. INTRODUCTION TO THE LISA SPACECRAFT

In this chapter, the physical parameters of the S/C are quantified. The need for these figures is justified by their presence in the dynamics equations, presented in the following chapters.

All the publications on LISA before 2017 assume a cylindrical S/C design with a maximum diameter of 4.2 m. Just two years before the redaction of this thesis, a new configuration for the LISA S/C was proposed by the Concurrent Design Facility (CDF) Team at the European Space Research and Technology Centre (ESTEC) [21], wildly departing from the original cylindrical geometry, considered a standard for almost 20 years [11, 14, 17].

Originally, each S/C needed to be fitted with a transfer module that would detach at orbit achieved. The satellites, planned to be lifted on an Ariane 64 rocket (still in development during the writing of this thesis), were planned to be stuck on top of each other inside the payload bay in launch configuration. The new design is derived from the Swarm mission S/C: similarly to LISA, Swarm [36], launched in 2013, required also the launch of three satellites so the S/C were designed to fit on a single launch. The adoption of the same concept in LISA allows the three S/C to share a single, chemical propulsion transfer module. Final orbit insertion is performed using on-board thrusters [21].

This new geometry and the figures presented by the CDF Team are used for the estimation of mass, inertia and thruster configuration, the latter used to estimate the efforts necessary to operate the DFACS.

3.2. GEOMETRY

The proposed geometry of the LISA S/C is shown in Figure 3.1. This is a relatively new concept based on the Swarm launch configuration that allows the three satellites to share a single orbital transfer stage [21].

Notably, the two telescopes are used to send and receive the laser beams to and from the other two S/C. They are, therefore, slanted by an angle $\alpha \simeq 60^{\circ}$ to satisfy the almost equilateral triangular shape of the formation. An actuation system is also integrated to allow a margin of $\Delta \alpha = \pm 1.5^{\circ}$ [14].

The body-fixed reference system chosen for such a configuration is depicted in Figure 3.2: the *x*-axis bisects the angle subtended by the telescope's lines-of-sight, and the *z*-axis points in the opposite direction as the solar panel. This convention implies that both the telescopes are articulated, in order to facilitate the modeling of the dynamics. This is not necessarily the case, since early system definitions call for just one of the telescopes to be capable of steering [37].

The orientation of the *z*-axis is related to the co-orbital motion of the formation. A similar reference frame was adopted by Bik et. al. [26]. The positive rotation direction around the *z*-axis (counter-clockwise) was also used to label the S/C in an orderly fashion from 1 to 3 in Chapter 2.



Figure 3.1: The new LISA spacecraft model proposed by the CDF Team in 2017 allows the three S/C o share one launch on the Ariane 64 and one chemical propulsion tranfer stage [21]. The shape resembles a trapezoid on a $4.75 \times 3 \text{ m}^2$ solar panel.



Figure 3.2: Definition of the body-fixed reference frames and their nominal orientations w.r.t. the formation. The *x*-axes bisect the respective breath angles α_i , i = 1, 2, 3 between the telescopes and the *z*-axes are orthogonal to the solar panels. The *y*-axes complete the orthogonal three-axis reference frame with the right-hand rule of thumb.

3.3. THRUSTER CONFIGURATION

The thrusters on a LISA S/C are primarily used as a reaction control system: they provide both the force and the torque for the control of the DFACS [14, 17, 22, 37]. Recently, two concepts have been proposed: one that would use electric propulsion for reaction control and chemical propulsion from a separate stage for orbit insertion, called EP (Electric Propulsion) and one that would use electric propulsion only for both control and orbit insertion, called EP+. The two configurations are very dissimilar, and require different study approaches. Since the EP+ configuration is a relatively new concept based on low technology readiness level components [21], we decide to concentrate on the EP configuration. Figure 3.3 shows the estimated thruster configuration [21]. The arrows represent the orientations of the nozzles w.r.t. the body-fixed reference frame. The 12 thrusters are clustered at about the four positive *z*-axis corners. Each cluster *i* = 1,2,3,4 shares one Center of Thrust (CoT) \vec{r}_{t_i} . Their displacements w.r.t. the Center of Mass (CoM) are estimated as:



Figure 3.3: Direction of plumes and CoT positions of the 12 reaction control thrusters on LISA in the S/C body-fixed reference frame [21].

$$\vec{r}_{t_1} = \begin{pmatrix} 0.75\\ 1.6\\ 0.6 \end{pmatrix}; \quad \vec{r}_{t_2} = \begin{pmatrix} -0.75\\ 1.6\\ 0.6 \end{pmatrix}; \quad \vec{r}_{t_3} = \begin{pmatrix} -0.75\\ -1.6\\ 0.6 \end{pmatrix}; \quad \vec{r}_{t_4} = \begin{pmatrix} 0.75\\ -1.6\\ 0.6 \end{pmatrix} \quad [m].$$

The directions of thrust $\hat{\tau}_j$ for each thruster j = 1, 2, ..., 12, are opposite to the respective directions of plumes shown in Figure 3.3. They are estimated as:

$$\begin{aligned} \hat{\tau}_{1} &= \begin{pmatrix} 0\\ 0\\ -1 \end{pmatrix}; \quad \hat{\tau}_{2} = \begin{pmatrix} -\sqrt{3}/2\\ 0\\ 0.5 \end{pmatrix}; \quad \hat{\tau}_{3} = \begin{pmatrix} 0\\ -\sqrt{3}/2\\ 0.5 \end{pmatrix}; \\ \hat{\tau}_{4} &= \begin{pmatrix} 0\\ 0\\ -1 \end{pmatrix}; \quad \hat{\tau}_{5} = \begin{pmatrix} \sqrt{3}/2\\ 0\\ 0.5 \end{pmatrix}; \quad \hat{\tau}_{6} = \begin{pmatrix} 0\\ -\sqrt{3}/2\\ 0.5 \end{pmatrix}; \\ \hat{\tau}_{7} &= \begin{pmatrix} 0\\ 0\\ -1 \end{pmatrix}; \quad \hat{\tau}_{8} = \begin{pmatrix} \sqrt{3}/2\\ 0\\ 0.5 \end{pmatrix}; \quad \hat{\tau}_{9} = \begin{pmatrix} 0\\ \sqrt{3}/2\\ 0.5 \end{pmatrix}; \\ \hat{\tau}_{10} &= \begin{pmatrix} 0\\ 0\\ -1 \end{pmatrix}; \quad \hat{\tau}_{11} = \begin{pmatrix} -\sqrt{3}/2\\ 0\\ 0.5 \end{pmatrix}; \quad \hat{\tau}_{12} = \begin{pmatrix} 0\\ \sqrt{3}/2\\ 0.5 \end{pmatrix}. \end{aligned}$$

It is important to mention that the values for \vec{r}_{t_i} and $\hat{\tau}_i$ are not found in literature. Instead, they were estimated by closely matching the configuration of Figure 3.3, the only publicly available piece of information about the LISA thruster configuration.

3.4. CALCULATION OF THE THRUSTER COMMANDS

The thrust configuration matrix A is defined such that

$$\boldsymbol{f} = \begin{pmatrix} \vec{f}_t \\ \vec{l}_t \end{pmatrix} = A\boldsymbol{T}$$
(3.1)

where $\mathbf{f} \in \mathbb{R}^{6\times 1}$ is the *thrust-torque* vector, obtained by concatenating the thrust force vector \vec{f}_t and the torque vector \vec{l}_t applied to the S/C body, and $\mathbf{T} \in \mathbb{R}^{n\times 1}$ is an array containing the thrust provided by each of the *n* thrusters available on board (*n* = 12) [38].

Using the definition of thrust and torque provided by the thrusters, respectively as

$$\vec{f}_t = \sum_{i=1}^{12} \hat{\tau}_i T_i \tag{3.2}$$

and

$$\vec{l}_t = \sum_{i=1}^{3} \vec{r}_{t_1} \times \hat{\tau}_i T_i + \sum_{i=4}^{6} \vec{r}_{t_2} \times \hat{\tau}_i T_i + \sum_{i=7}^{9} \vec{r}_{t_3} \times \hat{\tau}_i T_i + \sum_{i=10}^{12} \vec{r}_{t_4} \times \hat{\tau}_i T_i$$
(3.3)

where T_i , i = 1, 2, ..., 12, are the components of **T**, the thruster configuration matrix A of (3.1) is derived as:

$$A = \begin{pmatrix} \begin{bmatrix} \hat{\tau}_1 & \hat{\tau}_2 & \hat{\tau}_3 \end{bmatrix} & \begin{bmatrix} \hat{\tau}_4 & \hat{\tau}_5 & \hat{\tau}_6 \end{bmatrix} & \begin{bmatrix} \hat{\tau}_7 & \hat{\tau}_8 & \hat{\tau}_9 \end{bmatrix} & \begin{bmatrix} \hat{\tau}_{10} & \hat{\tau}_{11} & \hat{\tau}_{12} \end{bmatrix} \\ \vec{r}_{t1}^{\times} \begin{bmatrix} \hat{\tau}_1 & \hat{\tau}_2 & \hat{\tau}_3 \end{bmatrix} & \vec{r}_{t2}^{\times} \begin{bmatrix} \hat{\tau}_4 & \hat{\tau}_5 & \hat{\tau}_6 \end{bmatrix} & \vec{r}_{t3}^{\times} \begin{bmatrix} \hat{\tau}_7 & \hat{\tau}_8 & \hat{\tau}_9 \end{bmatrix} & \vec{r}_{t4}^{\times} \begin{bmatrix} \hat{\tau}_{10} & \hat{\tau}_{11} & \hat{\tau}_{12} \end{bmatrix} \end{pmatrix}$$
(3.4)

where $\vec{r}_{t_i}^{\times}$, j = 1, 2, 3, 4, is a skew symmetric matrix such that $\vec{r}_{t_i} \times \vec{p} \equiv \vec{r}_{t_i}^{\times} \vec{p}$ (\vec{p} is any vector).

In practical applications, T is the unknown. The problem is now finding a proper inverse relation for Equation (3.1) since A is not a square matrix for most applications such as LISA, where $A \in \mathbb{R}^{6 \times 12}$. One could attempt solving for T via a Moore-Penrose pseudoinverse, i.e. [38]

$$\boldsymbol{T} = \left(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathsf{T}}\boldsymbol{f}.\tag{3.5}$$

The problem with the above solution is that T is not guaranteed to have all positive terms, which is necessary as modern reaction control thrusters such as the ones employed on LISA are single-nozzled and only capable of mono-directional thrust-force.

Although there are many methods to solve the problem, the so called *Least square thruster dispatching method* proposed by D. Ferting and S. Wu was selected [38].

In this method, two inverse matrices for A, respectively A_{-} and A_{+} are defined, such that,

$$AA_{+} = \mathbb{1}^{6 \times 6} \tag{3.6}$$

and

$$4A_{-} = -\mathbb{1}^{6 \times 6} \tag{3.7}$$

where $\mathbb{1}^{6 \times 6}$ is the 6-dimensional unit matrix, and

$$A_{+i,i} \ge 0 \text{ and } A_{-i,i} \ge 0, \ \forall i \in \{1, 2, ..., n\}, \forall j \in \{1, 2, ..., 6\}.$$
 (3.8)

where n = 12 is the number of thrusters or the number of columns of A. Basically A_+ is a positive right-inverse of A and A_- is a positive right-inverse of -A.

The thrust-torque vector f is separated into positive and negative terms, thus defining two complementary vectors f_+ and f_- such that

$$\begin{cases} f_{+_i} = f_i, \text{ and } f_{-_i} = 0 & \text{if } f_i \ge 0 \\ f_{-_i} = f_i, \text{ and } f_{+_i} = 0 & \text{if } f_i < 0 \end{cases} \quad \forall i \in \{1, 2, ..., 6\}$$
(3.9)

The thruster force vector **T** is calculated as

$$T = A_{+}f_{+} - A_{-}f_{-} \tag{3.10}$$

This forces $T_i \ge 0$, $\forall i \in \{1, 2, ..., n\}$, where *n* is the size of *T*, since A_- is positive and f_- is negative.

In order to minimize T for a given f, A_+ and A_- must be minimized as well. A least square algorithm is employed. For what concerns A_+ , each i^{th} column, represented by the vector x, is calculated according to the following constrained minimization pattern:

minimize
$$f(\mathbf{x}) = \mathbf{x}^{\mathsf{T}} \mathbf{x}$$

subject to $A\mathbf{x} = \mathbb{I}_{i}^{6 \times 1}$
 $x_{j} \ge 0$ for $j = 1, 2, ..., n$

$$(3.11)$$

where $\mathbb{1}_{i}^{6\times 1}$ is a unit column 6-dimensional vector whose components $\mathbb{1}_{k} = 0 \ \forall k \neq i$ and $\mathbb{1}_{i} = 1$.

The paper [38] does not dive into the minimization algorithm. The one used in this thesis is an analytical *Khun-Tucker* optimization method [39], described in the following paragraphs.

The case study is the LISA configuration, with n = 12 thrusters. This means that $x \in \mathbb{R}^{12 \times 1}$. Following from Equation (3.11) a set of 18 constraints g(x) are defined:

$$\boldsymbol{g}(\boldsymbol{x}): \begin{cases} g_{1,\dots,6} = A\boldsymbol{x} = \mathbb{I}_{i}^{6\times 1} \\ g_{7} = x_{1} \ge 0 \\ g_{8} = x_{2} \ge 0 \\ \vdots \\ g_{18} = x_{12} \ge 0 \end{cases}$$
(3.12)

An augmented function $F(\mathbf{x}, \boldsymbol{\lambda})$ is defined as

$$F(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}^{\mathsf{T}} \boldsymbol{g}(\mathbf{x})$$
(3.13)

where $\lambda \in \mathbb{R}^{18 \times 1}$ are the *Lagrange multipliers*. Together x and λ are the unknowns of the problem.

The necessary condition for a minimum is to find \boldsymbol{x} such that

$$\partial_{j}F = \frac{\partial F(\mathbf{x}, \lambda)}{\partial x_{j}} = 2x_{j} + \sum_{k=1}^{6} A_{k,j}\lambda_{k} + \lambda_{j+6} = 0 \quad \forall j = 1, 2, ..., 12.$$
(3.14)

The sufficient condition must also hold:

$$\frac{\partial^2 F(\mathbf{x}, \boldsymbol{\lambda})}{\partial x_i^2} = 2 > 0 \tag{3.15}$$

which is always true, since $f(\mathbf{x})$ is a sum of squares.

The catch of the Kuhn-Tucker method is that we need to evaluate all the g(x) and all the $\partial F(x, \lambda)$ for both $g_{7,...,18} = x_{1,2,...,12} > 0$ or $g_{7,...,18} = x_{1,2,...,12} = 0$, i.e., by looking whether the x_j fits within boundaries or by setting directly $g_{j+6} = x_j = 0$ for j being any of 1,2,...,12. Since there are 2 conditions, we have to look into $N = 2^{12} = 4096$ cases (actually $N = 2^{12} - 1 = 4095$, since x = 0 cannot be a solution of $g_{1,2,...,6}(x)$). This calls for a direct and efficient algorithm so that a machine could perform the calculations automatically.

First of all, one should notice how $g_{7,8,...,18}(\mathbf{x})$ do not need to be solved for, since they are either inequalities or force the solution $x_j = 0$, for $j \in \{1, 2, ..., 12\}$. Combining $g_{1,2,...,6}(\mathbf{x})$ from Equation (3.12) and the necessary condition $\partial F(\mathbf{x}, \boldsymbol{\lambda})$ from Equation (3.14) the following system is obtained:

$$\begin{pmatrix} g_{1,2,\dots,6}(\boldsymbol{x}) \\ \boldsymbol{\partial} F(\boldsymbol{x},\boldsymbol{\lambda}) \end{pmatrix} = \begin{pmatrix} A & \mathbf{0}^{6\times 6} & \mathbf{0}^{6\times 12} \\ 2\mathbf{1}^{12\times 12} & A^{\tau} & \mathbf{1}^{12\times 12} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{pmatrix} = H\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbb{I}_{i}^{6\times 1} \\ \mathbf{0}^{12\times 1} \end{pmatrix}.$$
(3.16)

Now, for each of the N = 4095 cases in which either $x_k = 0$, i.e., lying on the boundaries of the inequality constraints $g_{7,8,...,18}$, or x_k is an unknown to be localized at $x_k > 0$, we need to set x accordingly. It is advisable to follow a binary pattern, such as described in Table 3.1.

Table 3.1: Binary pattern to evaluate all the boundaries of the inequality constraints. The x_k are set to either 0 or *unknown*.

CASE	x_1	x_2		x_{11}	x_{12}
1	0	0		0	unkn.
2	0	0		unkn.	0
3	0	0		unkn.	unkn.
÷	÷	÷	·	:	÷
4092	unkn.	unkn.		0	0
4093	unkn.	unkn.		0	unkn.
4094	unkn.	unkn.		unkn.	0
4095	unkn.	unkn.		unkn.	unkn.

The Lagrange multipliers have the property that $\lambda_k = 0$ if $g_k > 0$. Therefore, for each j such that x_j is an unknown, $\lambda_{j+6} = 0$ and vice-versa. Because of that, balance arises between the number of unknown x_k and λ_k such that their sum is always equal to 18. Hence, we can reduce the vector $(\mathbf{x}^{\tau}, \boldsymbol{\lambda}^{\tau})^{\tau}$ and the matrix H in Equation (3.16) with the following method:

• eliminate all the columns *j*, such that $x_j = 0$ from *H*, by shifting all the l > j columns to the left by 1 position;

- eliminate all the columns k > 18, $k \neq (j + 18)$, from *H* by shifting all the l > k columns to the left by 1 position;
- eliminate all the x_j elements from **x** by shifting all the $x_{l>j}$ elements up by 1 position;
- eliminate all the λ_k elements from λ for k > 6, $k \neq (j + 6)$, by shifting all the $\lambda_{l>k}$ elements up by 1 position.

From *H*, the square $H^* \in \mathbb{R}^{18 \times 18}$ matrix is obtained, and from $(\mathbf{x}^{\tau}, \boldsymbol{\lambda}^{\tau})^{\tau}$, a properly sized vector $(\mathbf{x}^{*\tau}, \boldsymbol{\lambda}^{*\tau})^{\tau} \in \mathbb{R}^{18 \times 1}$ remains, such that

$$\begin{pmatrix} \boldsymbol{x}^*\\ \boldsymbol{\lambda}^* \end{pmatrix} = H^{*-1} \begin{pmatrix} \mathbb{I}_i^{6\times 1}\\ \mathbf{0}^{12\times 1} \end{pmatrix}$$
(3.17)

If H^* is full rank, the inverse matrix operation is straightforward. We obtain, after N = 4095 operations, N solutions for \mathbf{x} . Only the few of these solutions that satisfy $g_{7...18} \ge 0$ are evaluated, and the \mathbf{x}_m that satisfy

$$f(\boldsymbol{x}_m) = \min f(\boldsymbol{x}) \tag{3.18}$$

is selected as the i^{th} column of A_+ .

It is worth to mention that the Kuhn-Tucker optimization algorithm requires that $\lambda_j < 0$ for $g_j = 0$, but this condition has been relaxed, and, instead, the minimum is found by means of comparison [39].

For A_{-} the method is the same, except, the optimization constraints are modified to:

a . .

minimize
$$f(\mathbf{x}) = \mathbf{x}^{t} \mathbf{x}$$

subject to $A\mathbf{x} = -\mathbb{I}_{i}^{6 \times 1}$
 $x_{j} \ge 0$ for $j = 1, 2, ..., 12$

$$(3.19)$$

The evaluated matrices have been calculated to be, respectively,

	(0	0	0	0	0.41025641026	0.105575187830
	0	0.1875	0.25	0.240384615385	0.153846153846	0.306617300824
	0.4	0	0.25	0.0721153846154	0	0
	0.488675134595	0	0	0	0	0.105575187830
	0.577350269190	0.1875	0.25	0.240384615385	0.153846153846	0
1	0	0	0.25	0.0721153846154	0.512820512821	0.115683450494
$A_+ \simeq$	0.488675134595	0.382425134595	0	0.192307692308	0	0.105575187830
	0.577350269190	0	0.25	0	0.153846153846	0.306617300824
	0	0.577350269190	0.25	0.0721153846154	0.512820512821	0
	0	0.382425134595	0	0.192307692308	0.410256410256	0.105575187830
	0	0	0.25	0	0.153846153846	0
	0.4	0.577350269190	0.25	0.0721153846154	0	0.115683450494
and						
	(0.488675134595	0.382425134595	0.25	0.192307692308	0	0.105575187830
	0.577350269190	0	0	0	0.153846153846	0
	0	0.577350269190	0	0.0721153846154	0.512820512821	0.115683450494
	0	0.382425134595	0.25	0.192307692308	0.410256410256	0.105575187830
	0	0	0	0	0.153846153846	0.306617300824
1~~	0.4	0.577350269190	0	0.0721153846154	0	0
$A_{-} \simeq$	0	0	0.25	0	0.410256410256	0.105575187830
	0	0.1875	0	0.240384615385	0.153846153846	0
	0.4	0	0	0.0721153846154	0	0.115683450494
	0.488675134595	0	0.25	0	0	0.105575187830
	0.577350269190	0.1875	0	0.240384615385	0.153846153846	0.306617300824
	0	0	0	0.0721153846154	0.512820512821	0

Columns 1, 2 and 3 are unitless, while columns 4, 5 and 6 have units of $[m^{-1}]$.

Some solutions of Equation (3.10) for notable cases in which single components of f are set to 1 or -1 (N or Nm) and the all the others to 0, are reported in Figure 3.4. Four other cases are shown:

- Case 1: $f_{t_1} = 1$ N, $l_{t_2} = 1$ Nm and all the other components set to 0;
- Case 2: $f_{t_3} = -1$ N, $l_{t_1} = 1$ Nm and all the other components set to 0;
- Case 3: all components of *f* set to 1;

+1 N x-thrust			0.40	0.49	0.58		0.49	0.58				0.40	2.93
-1 N x-thrust	0.49	0.58				0.40			0.40	0.49	0.58		2.93
+1 N y-thrust		0.19			0.19		0.38		0.58	0.38		0.58	2.29
-1 N y-thrust	0.38		0.58	0.38		0.58		0.19			0.19		2.29
+1 N z-thrust		0.25	0.25		0.25	0.25		0.25	0.25		0.25	0.25	2.00
-1 N z-thrust	0.25			0.25			0.25			0.25			1.00
+1 Nm x-torque		0.24	0.07		0.24	0.07	0.19		0.07	0.19		0.07	1.15
-1 Nm x-torque	0.19		0.07	0.19		0.07		0.24	0.07		0.24	0.07	1.15
+1 Nm y-torque	0.41	0.15			0.15	0.51		0.15	0.51	0.41	0.15		2.46
-1 Nm y-torque		0.15	0.51	0.41	0.15		0.41	0.15			0.15	0.51	2.46
+1 Nm z-torque	0.11	0.31		0.11		0.12	0.11	0.31		0.11		0.12	1.27
-1 Nm z-torque	0.11		0.12	0.11	0.31		0.11		0.12	0.11	0.31		1.27
Case 1	0.41	0.15	0.40	0.49	0.73	0.51	0.49	0.73	0.51	0.41	0.15	0.40	5.39
Case 2	0.25	0.24	0.07	0.25	0.24	0.07	0.44		0.07	0.44		0.07	2.15
Case 3	0.52	1.14	0.72	0.59	1.41	0.95	1.17	1.29	1.41	1.09	0.40	1.42	12.11
Case 4	1.42	0.73	1.28	1.34	0.46	1.05	0.77	0.58	0.59	0.84	1.47	0.58	11.11
	1	2	3	4	5	6	7	8	9	10	11	12	sum
Thruster													

• Case 4: all components of *f* set to -1.

Figure 3.4: Characteristic throttle for single direction thrust and torques and 4 other cases. The values are in [N]. Case 1: $f_{t1} = 1$ N, $l_{t2} = 1$ Nm; Case 2: $f_{t3} = -1$ N, $l_{t1} = 1$ Nm; Case 3: all thrust components set to 1 N and all torque components set to 1 Nm; Case 4: all thrust components set to -1 N and all torque components set to -1 Nm.

Notice how the least expensive maneuver is a thrust in the *-z*-direction, thanks to the alignments of thrusters 1, 4, 7 and 10. A thrust in the *+z*-direction, on the other hand, will require a double effort due to the other thrusters being misaligned by 30° w.r.t. the *z*-axis. Another interesting fact is that a combination of maneuvers will have a total effort as their sum if they were performed independently: the line of Case 1 is obtained by adding lines 1 and 9; Case 2 is the sum of lines 6 and 7; Case 3 is obtained by summing all the positive thrust and torque maneuvers and Case 4 all the negative ones.

3.5. VERIFICATION OF THRUSTER COMMANDS

Equations (3.6) and (3.7) are easily verified for the calculated values of A_+ and A_- . One can further verify that the calculated T is consistent by means of residual check, for example, by randomly generating values for f with $-1 \le f_i \le 1$ and calculating the errors $\delta \vec{f}_t$ and $\delta \vec{l}_t$ as:

$$\begin{pmatrix} \delta \vec{f}_t \\ \delta \vec{l}_t \end{pmatrix} = \boldsymbol{f} - A\boldsymbol{T} = \boldsymbol{f} - A \left(A_+ \boldsymbol{f}_+ - A_- \boldsymbol{f}_- \right)$$
(3.20)

Figure 3.5 shows the probability distributions for a sample size of n = 10000 generated using the command rand() in Matlab (equally distributed random variable). The error spread, mostly within $|\delta f_i| < 1 \cdot 10^{-15}$, is consistent with 0 and the floating point precision in Matlab.



Figure 3.5: Probability distributions for δf_t and δl_t with equally distributed randomly generated $|f_{t_i}| \le 1$ N, $|l_{t_i}| \le 1$ Nm. Sample size n = 10000, bin size $\Delta x = 10^{-16}$ Nm. σ is the standard deviation.

3.6. ESTIMATION OF MASS AND INERTIA

The mass budget for LISA, as estimated by the CDF team, is reported in Table 3.2 [21].

Table 3.2: Mass budget for LISA estimated by the CDF team [21]

Mass budget		
Dry mass		1244.4 kg
+ system margin (20%)	m_{drv}	1493.3 kg
Control propellant mass	m_{prop}	239.6 kg
TOTAL	m_{sc}	1732.9 kg

The derivation of the inertia matrix I_{sc} and position of the CoM within the geometry is explained hereon. Two volumes, V_{dry} and V_{prop} are assigned, respectively, to the dry mass m_{dry} and the propellant mass m_{prop} . These are shown in Figure 3.6. The dry mass is considered uniformly distributed within a trapezoidal shape



Figure 3.6: Modelling of the LISA S/C. *Left*: The dry mass is equally distributed in a trapezoidal solid enveloping the S/C shape presented in figure 3.1. *Right*: the fuel ($m_p r op = 239.6$ kg) is contained in four spherical canisters of radius 0.3665 m [21].

spanning the two bases of the S/C: the largest base, of sides $a_1 = 3$ m and $b_1 = 4.75$ m is the solar panel, while the smallest base, of sides $a_2 = 1.6$ m and $b_2 = 3$ m is the opposite side of the S/C. The height h = 1.1 m is retained from the S/C blueprint of Figure 3.1. The payload is considered part of the dry mass. The propellant

mass is distributed within four canisters of spherical shape, of radius 0.3665 m at an estimated distance (at center) of x = 0.8 m, y = 1.5 m and z = 0 from the S/C CoM. The shape of the canisters was chosen among three possibilities presented by the CDF Team [21], the spherical option being the simplest to model (the other two are orientable capsule shaped canisters, but the nominal orientation is not provided). Propellant slushing is not modeled, instead it is assumed, very roughly, that the propellant fills up the whole volume, changing density as it is depleted.

The main assumption is that the *x*- *y*- and *z*-axes are oriented in the directions of the main inertial axes and centered at the CoM of the S/C, therefore canceling the non-diagonal components of the inertia matrix [40]. Since the cutout surface of the trapezoid is a rectangle at each *z*, the *x*- and *y*-axes are the perpendicular bisectors of, respectively, the local long base b(z) and the local short base a(z). The *xy*-plane must necessarily bisect the volume into two equivalent parts for the CoM to be included. The distance h_{cm} of the CoM from the largest base is found by solving the following equation:

$$\int_{0}^{h_{cm}} a(z')b(z')dz' = \frac{V_{dry}}{2}$$
(3.21)

where z' is the distance from the largest base, parametrized as $z' = z + h_{cm}$. *a* and *b* can be calculated as:

$$a(z') = a_1 - (a_1 - a_2)\frac{z'}{h};$$
(3.22)

$$b(z') = b_1 - (b_1 - b_2) \frac{z'}{h}.$$
(3.23)

For what concerns the volume V_{dry} , it can be calculated by means of integration:

$$V_{dry} = \int_0^h a(z')b(z')dz' = a_1b_1h - [(a_1 - a_2)b_1 + (b_1 - b_2)a_1]\frac{h}{2} + (a_1 - a_2)(b_1 - b_2)\frac{h}{3}$$
(3.24)

and the values obtained are

$$V_{dry} = 10.0283333333333333 \text{ [m}^3\text{]},$$

 $h_{cm} = 0.413394480446386 \text{ [m]}.$

Wolfram Mathematica^{\mathbb{R}} was used to verify the results (see Appendix A).

The knowledge of h_{cm} allows now to calculate the inertia matrix of the dry-mass volume. The formula for the inertia of a volume V of uniformly distributed mass m is calculated as [25]:

$$I = \frac{m}{V} \begin{pmatrix} \int_{V} (y^{2} + z^{2}) dx dy dz & -\int_{V} (xy) dx dy dz & -\int_{V} (xz) dx dy dz \\ -\int_{V} (xy) dx dy dz & \int_{V} (x^{2} + z^{2}) dx dy dz & -\int_{V} (yz) dx dy dz \\ -\int_{V} (xz) dx dy dz & -\int_{V} (yz) dx dy dz & \int_{V} (x^{2} + y^{2}) dx dy dz \end{pmatrix}.$$
(3.25)

Using Wolfram Mathematica, the dry inertia is calculated numerically as:

$$I_{dry} = \frac{m_{dry}}{V_{dry}} \begin{pmatrix} 14.7007247086586 & 0 & 0\\ 0 & 5.98076341435653 & 0\\ 0 & 0 & 18.7692679166667 \end{pmatrix} [kg·m2]$$
(3.26)

where the non-diagonal terms result as 0 because of the choice of aligning the body-fixed axed with the main axes of inertia (see Appendix A).

The same operation is performed for the inertia due to the propellant. Remembering Figure 3.6-*right*, the propellant is stored in four spherical canisters of radius 0.3655 m placed symmetrically w.r.t. all the axes, with centers lying on the *xy*-plane. The inertia matrix is solved, numerically, as

$$I_{prop} = \frac{m_{prop}}{V_{prop}} \begin{pmatrix} 1.9002113473697346 & 0 & 0\\ 0 & 0.572216430204282 & 0\\ 0 & 0 & 2.42810994527123 \end{pmatrix} [\text{kg} \cdot \text{m}^2]$$
(3.27)

where

$$V_{prop} = 4 \cdot \frac{4}{3} \pi \cdot 0.3665^3 \ [\text{m}^3]$$

is the total volume of the four canisters (spheres) (see Appendix A).

Finally, the total inertia of the S/C is calculated by summation

$$I_{sc} = I_{dry} + I_{prop}. (3.28)$$

While I_{dry} is always constant, I_{prop} varies in time as the thrusters are in use, as modeled in the next section. The two extreme cases, identified in Table 3.3 can be asserted: the initial mass and inertia (full tank, $m_{prop} = 239.6$ kg) and the end-of-life mass and inertia (empty tank, $m_{prop} = 0$).

Table 3.3: Associated minimum (empty tanks and lowest possible mass) and maximum (full tanks and largest possible mass) total mass and inertia matrices

Parameter	Symbol	Empty tank					Full tank
Mass	<i>m_{sc}</i> [kg]			1493.3			1732.9
Inertia	I_{sc} [kg m ²]	diag(2189.057,	890.584,	2794.896)	diag(2741.030,	1056.802,	3500.213)

3.7. THRUST AND MASS DEPLETION

One would expect the propellant mass m_{prop} to be depleted by the on-board thrusters with the following relation [28]:

$$\dot{m}_{prop} = -\frac{1}{g_0 I_{sp}} \sum_{i=1}^{12} T_i \tag{3.29}$$

where I_{sp} is the specific impulse of the thrusters and g_0 is Earth's gravitational acceleration at surface. We need therefore to find a suitable value for I_{sp} .

The Field Emission Electric Propulsion (FEEP) μ Newton thrusters studied by ESA and the Colloid μ Newton thrusters studied by NASA both have specific impulses in the extremely efficient ranges of $I_{sp} = 10000$ s [41, 42].

This number is too high, and we explain why: in Section 5.6 we will show that the average total thrust required by all thrusters is $\overline{T}_{tot} \approx 2.0050 \cdot 10^{-4}$ N. Setting the value of the specific impulse to $I_{sp} = 340$ s, which is much lower than the efficiency offered by the market, we obtain an average mass rate of $\overline{m}_{prop} \approx -6.0112 \cdot 10^{-8}$ kg/s. For an initial 4 years of science operations this would amount to a consumption of just

$$\Delta m_{prop} = \overline{\dot{m}}_{prop} \cdot 4$$
 years $\simeq -7.5828$ kg

which, considering a total capacity of 239.6 kg, it is not enough to significantly affect the dynamics of the S/C to study its sensitivity to mass changes.

Figure 3.7 shows the m_{prop} trend for the first 4 years of operation (starting at full tank).





For consistency, and in order to avoid trivial solutions, we stick to a conservative value of

$I_{sp} = 340 \text{ s}$

and consider the tank full at start of science operations. These values are purely arbitrary. The estimation of stored propellant performed by the CDF team takes into consideration the use of the same for orbital injection and control [21], hence the high value w.r.t. the required mass calculated here.

3.8. RECOMMENDATIONS ON LISA SPACECRAFT MODELING

The least square thruster dispatching method presented in Section 3.4 is not optimized. D. Bindel et al. [38] present a few methods to minimize the overall thruster efforts. These methods require either linear programming or numerical optimization to be performed at each simulation step, while the method presented here has the advantage of requiring the computation of matrices A_+ and A_- only once, as their value is not dependent on time.

The calculation of I_{dry} does not take into consideration time-variability of the telescope pointing direction: the bulk of the instruments rotate as the breath angle changes size, thus changing the inertia due to shifting of mass. An analysis of this model has not been performed due to time constraints, despite some preliminary information about the size and mass of the system is available [14]. Besides, we assume that the limited mobility of the device, with a maximum breath angle divergence of $\Delta \alpha = \pm 1.5^{\circ}$, will not affect the overall model performance.

3.9. CONCLUSIONS ON LISA SPACECRAFT MODELING

In summary, the body-fixed reference frame, the thruster configuration matrix, mass and inertia of the spacecraft were defined in this chapter. Most of the information was derived from a model presented by the CDF team [21].

The *x*-axis is defined so that it bisects the angle between the telescopes, assuming that both the telescopes are articulated. The *z*-axis points away and orthogonally from the solar panels.

12 thrusters clustered at 4 corners of the S/C define the thruster configuration matrix. A linear relationship between thruster efforts and their total force and torque was presented, using a Kuhn-Tucker optimization algorithm in order to find a least-squares solution for constant coefficient matrices. The method is not optimized, but it does not require any active algorithm to perform during simulation. A check of conformity between the relation and its inverse has been performed successfully.

Spacecraft inertia has been estimated by modeling the dry mass volume as a trapezoid and the separated propellant volume as four spheres.

An arbitrary value of specific impulse of $I_{sp} = 340$ s is used as a parameter in this thesis.

4

ATTITUDE-FORMATION DEPENDENCIES

4.1. INTRODUCTION TO FORMATION DEPENDENCIES

The motion of the formation in inertial coordinates was described in Chapter 2. The three LISA S/C need to be oriented with respect to each other as shown in Section 3.2. Their attitude is, therefore, strictly dependent on the formation.

In this chapter, a mathematical relation of the attitude requirements w.r.t. the formation is defined. In order to provide preliminary information about the control requirements, an analytical solution of the angular velocities and accelerations experienced by the S/C, while locked onto the moving formation, is derived.

The data provided in this chapter is necessary in order to calculate the attitude commands and to quantify the minimum control input provided by the thrusters.

4.2. FORMATION DEPENDENCIES ON THE BODY-FIXED REFERENCE FRAMES

In Section 3.2 we introduced the body-fixed reference frame (Figure 3.2). The *x*-axis is the bisector of the angles α subtended between the lines-of-sight of the telescopes, while the *z*-axis is orthogonal to the solar panels. In Figure 4.1 we represent the same concept in the inertial reference frame.



Figure 4.1: The required orientations of the three body-fixed reference frames (x_i, y_i, z_i) in the inertial reference frame (X, Y, Z).

For a perfectly controlled S/C w.r.t. the formation, a few properties can be derived

- the lines-of-sight of the telescopes are also the displacement vectors, w.r.t. to S/C, of the other two;
- due to the properties of the triangle, all the lines-of-sight lie on the same plane: the formation plane, or *x*-*y*-plane of the body-fixed reference frames;

- the *z*-axes are orthogonal to the formation plane;
- the *x*-axes all point towards the *incenter* of the formation, i.e., the intersection of the three angle bisectors.

In Euclidean geometry, the incenter \vec{r}_{cm} is the center of the inscribed circle in a triangle (Figure 4.2). In



Figure 4.2: The *incenter* C_{IN} of a triangle is the center of the circle inscribed in it. It is obtained by intersecting the three angle bisectors. The latter property makes it interesting in the framework of this thesis.

the framework of this paper we are mainly interested in the fact that it is the intersection point of the three angle bisectors. Its Cartesian position can be calculated as

$$\vec{r}_{cm} = \frac{r_{23}\vec{R}_1 + r_{31}\vec{R}_2 + r_{12}\vec{R}_3}{r_{12} + r_{23} + r_{31}} \tag{4.1}$$

where \vec{R}_i , $i \in \{1, 2, 3\}$ is the displacement, in the inertial reference frame, of S/C_i, and r_{ij} , $i, j \in \{1, 2, 3\}$, $i \neq j$, is the arm length, or distance, between S/C_i and S/C_j, defined as

$$r_{ij} = \|\vec{r}_{ij}\| = \|\vec{R}_j - \vec{R}_i\| \tag{4.2}$$

The \hat{x}_i directive is then defined as the direction of \vec{r}_{cm} w.r.t. S/C_i, $i \in \{1, 2, 3\}$, as

$$\hat{x}_{i} = \frac{\vec{r}_{cm} - \vec{R}_{i}}{\|\vec{r}_{cm} - \vec{R}_{i}\|}.$$
(4.3)

Since the telescopes lie on the S/C body-fixed xy-plane and their lines-of-sight need to lie on the formationplane, the body-fixed z-axis must be orthogonal to the formation plane, therefore it is unique and shared among the S/C. There are many ways to define its directive \hat{z}_i , the most convenient being the following:

• First, we define the unit direction from S/C_i to S/C_j , $i, j \in \{1, 2, 3\}$, $i \neq j$, as

$$\hat{n}_{ij} = \frac{\vec{r}_{ij}}{\|\vec{r}_{ij}\|}$$
(4.4)

where $\vec{r}_{ij} = \vec{R}_j - \vec{R}_i$ is defined as the displacement of S/C_j from S/C_i, i.e. one of the interferometer arms;

• then, we obtain \hat{z}_i as

$$\hat{z}_1 \equiv \hat{z}_2 \equiv \hat{z}_3 = \frac{\hat{n}_{ij} \times \hat{n}_{ik}}{\sin \alpha_i} \tag{4.5}$$

where α_i , i.e. the angle subtended by the lines of sight of the telescopes in S/C_i, can be obtained via the relation

$$\cos \alpha_i = \hat{n}_{ij} \cdot \hat{n}_{ik}. \tag{4.6}$$

The sequence $\{i, j, k\}$ used here must be any *even* permutation of $\{1, 2, 3\}$ (Table 4.1). Any other permutation would lead to the *z*-axis to be oriented in the opposite direction as the intended one.

In this framework, the S/C are labeled in a rotation-wise sequence according the cartwheel formation apparent motion. This sets the *z*-axis oriented about -30° w.r.t. the ecliptic, towards the opposite direction as the Sun.

Table 4.1: Even permutations of {1,2,3}

The \hat{y}_i directives follow the right-hand rule of thumb:

$$\hat{y}_i = \hat{z}_i \times \hat{x}_i. \tag{4.7}$$

A rendition of the motion of the three axes in the Ecliptic J2000 full-sky geometry is depicted in Figure 4.3. The plot is derived from the simulation data of the first year obtained in Chapter 2. The *z*-axes are almost constantly pointing towards a -30° latitude, as it would be expected from a cartwheel formation plane tilted by 60° w.r.t. the Hill's reference frame. The *x*- and *y*-axes draw yearly *analemmas*, symmetrical w.r.t. the equator with an amplitude of ~ \pm 60° and a maximum width of ~40° in longitude. The three S/C *x*- and *y*-directions have a 120° longitudinal phase between each others.



Figure 4.3: Required path of the three body axes (in order from *left* to *right*) on the celestial sphere, in Ecliptic J2000 coordinates (The inertial X-axis points towards the Vernal Equinox (Υ), and the X-Y plane lies on the ecliptic). *x*- and *y*-body axes draw *analemmas* in the sky phased about 120° from one another. The *z*-axes follow the same path at a constant -30° latitude.

We can verify whether the *x*-directives of Equation (4.3) pointed towards the incenter \vec{r}_{cm} of Equation (4.1) actually bisect the angles by confronting the dot-products $\hat{x}_i \cdot \hat{n}_{ij}$ and $\hat{x}_i \cdot \hat{n}_{ik}$ for any permutation of $\{i, j, k\}$ in Table 4.1: these dot products result in the cosines of the angles between the *x*-directives and the S/C-S/C lines-of-sight on either sides. By definition, these two angles, and therefore their cosines, should be equal. Figure 4.4 shows the differences between the three couples of dot-products for LISA during the first 4 years of operations, as simulated in Chapter 2. The results are small, $< 2 \cdot 10^{-14}$ (compared to the nominal $\cos(30^\circ) = 0.5$) and consistent with floating precision error.



Figure 4.4: Verification of \hat{x}_i , i = 1,2,3, as bisectors of the breath angles subtended by the S/C-S/C laser-beams. The plot shows the differences between the cosines of the two angles at either sides of the \hat{x}_i directives (in the legend $-\hat{n}_{ij} = \hat{n}_{ji}$).

4.3. DIRECTION-COSINE MATRIX

Once the directives have been obtained, the required attitude of S/C_i in the inertial reference frame is fully defined. The direction-cosine matrix $C_{n/i}$ which allows to transform the vectorial quantities from inertial to body-fixed reference frame, is defined as [25]

$$C_{n/i} = \begin{pmatrix} \hat{x}_i^{\mathsf{T}} \\ \hat{y}_i^{\mathsf{T}} \\ \hat{z}_i^{\mathsf{T}} \end{pmatrix}$$
(4.8)

where the superscript τ represents the transpose operation.

This definition reflects the claim according to which a vector ${}^{n}\vec{p}$ in inertial reference frame, (superscript n is for *Newtonian*) can be decomposed in the body-fixed Cartesian coordinates as a sum of the products between the directives and their respective projection of \vec{p} , i.e.

$${}^{n}\vec{p} = {}^{n}\vec{p}\cdot\hat{x}_{i}\hat{x}_{i} + {}^{n}\vec{p}\cdot\hat{y}_{i}\hat{y}_{i} + {}^{n}\vec{p}\cdot\hat{z}_{i}\hat{z}_{i}.$$
(4.9)

In body-fixed reference frame, the directives are the fundamental Euler directions, i.e.

$$\vec{p} = {}^{n}\vec{p}\cdot\hat{x}_{i}\hat{i} + {}^{n}\vec{p}\cdot\hat{y}_{i}\hat{j} + {}^{n}\vec{p}\cdot\hat{z}_{i}\hat{k} = \begin{pmatrix} {}^{n}\vec{p}\cdot\hat{x}_{i} \\ {}^{n}\vec{p}\cdot\hat{y}_{i} \\ {}^{n}\vec{p}\cdot\hat{z}_{i} \end{pmatrix}.$$
(4.10)

In matrix operation $\vec{a} \cdot \vec{b} = \vec{a}^{T} \vec{b} = \vec{b}^{T} \vec{a}$, hence

$$\vec{p} = \begin{pmatrix} \hat{x}_i^{\mathsf{T}} \\ \hat{y}_i^{\mathsf{T}} \\ \hat{z}_i^{\mathsf{T}} \end{pmatrix}^n \vec{p} = C_{n/i}{}^n \vec{p}.$$
(4.11)

One can also demonstrate how \hat{x}_i , \hat{y}_i and \hat{z}_i become the fundamental Euler directions \hat{i} , \hat{j} and \hat{k} via the direction-cosine matrix transformation. For example, for what concerns \hat{x}_i , the following happens

$$C_{n/i}\hat{x}_i = \begin{pmatrix} \hat{x}_i^{\mathsf{T}} \hat{x}_i \\ \hat{y}_i^{\mathsf{T}} \hat{x}_i \\ \hat{z}_i^{\mathsf{T}} \hat{x}_i \end{pmatrix}$$
(4.12)

and since, by definition, \hat{x}_i , \hat{y}_i and \hat{z}_i are mutually orthogonal, Equation (4.12) results in

$$C_{n/i}\hat{x}_{i} = \begin{pmatrix} 1\\0\\0 \end{pmatrix} = \hat{i}.$$
 (4.13)

The analogous results can be obtained for \hat{y}_i and \hat{z}_i . By means of Equations (4.12) and (4.13), the orthonormality of $C_{n/i}$ can be demonstrated. In fact, the transpose of $C_{n/i}$ is

$$C_{n/i}^{\tau} = \begin{pmatrix} \hat{x}_i & \hat{y}_i & \hat{z}_i \end{pmatrix}$$
(4.14)

and multiplying, one obtains

$$C_{n/i}C_{n/i}^{\mathsf{T}} = \begin{pmatrix} \hat{x}_{i}^{\mathsf{T}} \\ \hat{y}_{i}^{\mathsf{T}} \\ \hat{z}_{i}^{\mathsf{T}} \end{pmatrix} \begin{pmatrix} \hat{x}_{i} & \hat{y}_{i} & \hat{z}_{i} \end{pmatrix} = \begin{pmatrix} \hat{x}_{i}^{\mathsf{T}} \hat{x}_{i} & \hat{x}_{i}^{\mathsf{T}} \hat{y}_{i} & \hat{x}_{i}^{\mathsf{T}} \hat{z}_{i} \\ \hat{y}_{i}^{\mathsf{T}} \hat{x}_{i} & \hat{y}_{i}^{\mathsf{T}} \hat{y}_{i} & \hat{y}_{i}^{\mathsf{T}} \hat{z}_{i} \\ \hat{z}_{i}^{\mathsf{T}} \hat{x}_{i} & \hat{z}_{i}^{\mathsf{T}} \hat{y}_{i} & \hat{z}_{i}^{\mathsf{T}} \hat{z}_{i} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}^{3 \times 3}$$
(4.15)

which demonstrates $C_{i/n} = C_{n/i}^{-1} = C_{n/i}^{\tau}$ and vice-versa.

4.4. THE NOMINAL ANGULAR VELOCITY VECTOR

It is relevant, in order to study and verify the attitude evolution of each S/C, to also obtain the value of the nominal angular velocity vector, i.e., the one required to maintain the nominal attitude at each instant.

By definition, the angular velocity vector $\vec{\omega}_i$ of the body-fixed reference frame in the inertial reference frame, is the one that satisfies, in the inertial reference frame [40]

$$\hat{d}_i = {}^n \vec{\omega}_i \times \hat{d}_i \tag{4.16}$$

for $\hat{d}_i \in {\hat{x}_i, \hat{y}_i, \hat{z}_i}$ being either of the three body-fixed directives as defined in the inertial reference frame. *Poisson's kinematics* allow for an inverse relation of Equation (4.16) [40]. The relation reads as

$$\omega_{i}^{\times} = \begin{pmatrix} 0 & -\omega_{iz} & \omega_{iy} \\ \omega_{iz} & 0 & -\omega_{ix} \\ -\omega_{iy} & \omega_{ix} & 0 \end{pmatrix} = C_{n/i} \dot{C}_{n/i}^{\ \tau} = \begin{pmatrix} \hat{x}_{i}^{\tau} \\ \hat{y}_{i}^{\tau} \\ \hat{z}_{i}^{\tau} \end{pmatrix} (\dot{x}_{i} \quad \dot{y}_{i} \quad \dot{z}_{i})$$
(4.17)

which implies that the angular velocity vector in body-fixed reference frame can be written as

$$\vec{\omega}_{i} = \begin{pmatrix} \omega_{i}^{\times}{}_{(3,2)} \\ \omega_{i}^{\times}{}_{(1,3)} \\ \omega_{i}^{\times}{}_{(2,1)} \end{pmatrix} = \hat{z}_{i} \cdot \dot{\hat{y}}_{i} \, \hat{i} + \hat{x}_{i} \cdot \dot{\hat{z}}_{i} \, \hat{j} + \hat{y}_{i} \cdot \dot{\hat{x}}_{i} \, \hat{k}.$$

$$(4.18)$$

where

$$\hat{i} = \begin{pmatrix} 1\\0\\0 \end{pmatrix}; \quad \hat{j} = \begin{pmatrix} 0\\1\\0 \end{pmatrix}; \quad \hat{k} = \begin{pmatrix} 0\\0\\1 \end{pmatrix}$$

are the three fundamental Cartesian directives.

We are now supposed to retrieve an analytical solution for the time derivatives \dot{x}_i , \dot{y}_i and \dot{z}_i . In simulation, and, up to some extent, practice, the velocities of each S/C in their orbits are known at each instant. These are \vec{R}_1 , \vec{R}_2 and \vec{R}_3 , therefore it would be logical to find equations that relate the latter to the time derivatives of the directives.

The following method is used to determine $\vec{\omega}_i$. Let us start by defining the drift rate of S/C_i from S/C_j, defined as the time derivative of the norm of $\vec{r}_{ij} = \vec{R}_j - \vec{R}_i$:

$$\dot{r}_{ij} = \vec{r}_{ij} \cdot \hat{n}_{ij} \tag{4.19}$$

where $\dot{\vec{r}}_{ij}$ is the time derivative of \vec{r}_{ij} , retrieved as

$$\dot{\vec{r}}_{ij} = \vec{R}_j - \vec{R}_i \tag{4.20}$$

by assuming that \vec{R}_1 , \vec{R}_2 and \vec{R}_3 are known at each instant. \hat{n}_{ij} is defined in Equation (4.4). Its time derivative is obtained as following:

$$\dot{\hat{n}}_{ij} = \frac{\dot{\vec{r}}_{ij} - \dot{r}_{ij}\hat{n}_{ij}}{\|\vec{r}_{ij}\|}.$$
(4.21)

We are now capable of deriving the time derivative of breath angle α_i , defined in Equation (4.6):

$$\dot{\alpha}_i = -\frac{\dot{\hat{n}}_{ij} \cdot \hat{n}_{ik} + \hat{n}_{ij} \cdot \dot{\hat{n}}_{ik}}{\sin \alpha_i}.$$
(4.22)

We now need to define a convenient reference frame which lies on the formation plane, but rather than being associated to a single S/C, it is instead dependent on the formation itself and somehow related to the inertial reference frame. In this way we are defining a new system of coordinates in which the motion of the S/C lies on a constant plane, ideally, such as in this case, on the shared *x*-*y*-plane of the body-fixed reference frames. We call this new reference frame *planar* and use the subscript $_p$ to label the quantities associated with it and the left superscript p to define the quantities measured in its coordinates.

Such a reference frame has the *z*-axis orthogonal to the formation plane, and we have already defined z_1 , z_2 and z_3 as such, therefore we have the respective directive defined as

$$\hat{z}_p \equiv \hat{z}_1 \equiv \hat{z}_2 \equiv \hat{z}_3 \tag{4.23}$$

calculated as in Equation (4.5).

Its time derivative can be obtained by differentiation of Equation (4.5) w.r.t. time, thus obtaining:

$$\dot{\hat{z}}_{p} \equiv \dot{\hat{z}}_{1} \equiv \dot{\hat{z}}_{2} \equiv \dot{\hat{z}}_{3} = \frac{\dot{\hat{n}}_{ij} \times \hat{n}_{ik} + \hat{n}_{ij} \times \hat{n}_{ik} - \dot{\alpha}_{i} \cos \alpha_{i} \hat{z}_{p}}{\sin \alpha_{i}}.$$
(4.24)

It is imperative that the triplet $\{i, j, k\}$ be an even permutation of $\{1, 2, 3\}$ as already mentioned. See Table 4.1 to check the allowable permutations.

Since we want the new reference frame to be dependent on the inertial reference frame, we arbitrarily define the *x*-axis parallel to the projection of the inertial negative *z*-axis – which has directive $-\hat{k}$ in inertial coordinates – onto the formation plane. The relation obtained is therefore the following for the \hat{x}_p directive:

$$\hat{x}_p = \frac{\cos\beta \hat{z}_p - \hat{k}}{\sin\beta} \tag{4.25}$$

where β is the angle between the two *z*-axes (inertial and planar), calculated as

$$\cos\beta = \hat{k} \cdot \hat{z}_p. \tag{4.26}$$

We now proceed to calculate first the time-derivative of β , by differentiation of Equation (4.26):

$$\dot{\beta} = -\frac{\dot{k} \cdot \dot{\hat{z}}_p}{\sin \beta} \tag{4.27}$$

and then the time derivative of \hat{x}_p :

$$\dot{\hat{x}}_p = \frac{\cos\beta\dot{\hat{z}}_p - \dot{\beta}\left(\cos\beta\hat{x}_p + \sin\beta\hat{z}_p\right)}{\sin\beta}.$$
(4.28)

The remaining directive is calculated, as usual, with the right-hand rule of thumb, i.e.

$$\hat{y}_p = \hat{z}_p \times \hat{x}_p \tag{4.29}$$

whose time derivative is, therefore:

$$\dot{\hat{y}}_p = \hat{z}_p \times \dot{\hat{x}}_p + \dot{\hat{z}}_p \times \hat{x}_p. \tag{4.30}$$

Now that the directives of the planar reference frame and their respective time-derivatives have been calculated we have the tools to define a direction-cosine matrix $C_{n/p}$ for the transformation of vectorial quantities from inertial to planar reference frame, calculated, analogously to Equation (4.8) as:

$$C_{n/p} = \begin{pmatrix} \hat{x}_p^{\mathsf{T}} \\ \hat{y}_p^{\mathsf{T}} \\ \hat{z}_p^{\mathsf{T}} \end{pmatrix}$$
(4.31)

and the angular velocity vector of the planar reference frame in the inertial reference frame, calculated, analogously to Equation (4.18) as:

$$\vec{\omega}_p = \hat{z}_p \cdot \dot{\hat{y}}_p \hat{i} + \hat{x}_p \cdot \dot{\hat{z}}_p \hat{j} + \hat{y}_p \cdot \dot{\hat{x}}_p \hat{k}.$$

$$(4.32)$$

These quantities allow to relegate the orbits of the three S/C on the formation plane. For convenience we center the planar reference frame at the incenter of the formation whose coordinates in the inertial reference frame \vec{r}_{cm} were calculated in Equation (4.1). It is important to specify that \vec{r}_{cm} can be any point on the plane of the formation. For example we could use the geometrical barycenter and the results for this derivation would not change, except that you would need to calculate its time derivative accordingly. Transformation from inertial to planar reference frame of the displacement of S/C_i \vec{R}_i is calculated by means of shifting and rotation, using $C_{n/p}$ defined in Equation (4.31), as:

$${}^{p}\vec{R}_{i} = C_{n/p}\left(\vec{R}_{i} - \vec{r}_{cm}\right) \tag{4.33}$$

and the velocity \vec{R}_i is transformed as:

$${}^{p}\vec{R}_{i} = C_{n/p}\left(\vec{R}_{i} - \vec{r}_{cm}\right) - \vec{\omega}_{p} \times {}^{p}\vec{R}_{i}$$

$$\tag{4.34}$$

where the $\dot{\vec{r}}_{cm}$ is obtained by time differentiation of Equation (4.1).

It is convenient to reduce Equation (4.1) in matrix terms for sake of compactness. For this purpose let us introduce the matrix R, defined as the horizontal concatenation of the three \vec{R}_i , $i \in \{1, 2, 3\}$,

$$R = \begin{pmatrix} \vec{R}_1 & \vec{R}_2 & \vec{R}_3 \end{pmatrix}.$$
(4.35)

and the *arm-length* vector \mathbf{r}_a , defined as the vertical concatenation of the three $r_{ij} = \|\vec{r}_{ij}\|$, $i, j \in \{1, 2, 3\}$, $i \neq j$, defined in Equation (4.2) and representing the respective opposite sides of the triangle as the three \vec{R}_i ,

$$\boldsymbol{r}_{a} = \begin{pmatrix} r_{23} \\ r_{31} \\ r_{12} \end{pmatrix}.$$
(4.36)

Equation (4.1) can be rewritten in these terms as:

$$\vec{r}_{cm} = \frac{R\boldsymbol{r}_a}{\sum_j r_{a_j}} \tag{4.37}$$

and its time derivative as:

$$\dot{\vec{r}}_{cm} = \frac{\dot{R}\boldsymbol{r}_a + R\dot{\boldsymbol{r}}_a - \sum_j \dot{r}_{a_j} \vec{r}_{cm}}{\sum_j r_{a_j}}$$
(4.38)

where you can use Equation (4.19) to calculate \dot{r}_a . Notice how r_a does not represent a 3-dimensional physical quantity but rather the matrix obtained by arranging three scalars, therefore the bold notation is used rather than the arrow.

In summary, we have now obtained a convenient reference frame for which the motion of all the S/C is relegated to the *x*-*y*-plane, i.e. the *formation plane*. ${}^{p}\vec{R}_{i}$ and ${}^{p}\vec{R}_{i}$ should, therefore, have zero *z* component at each instant. Moreover, this new planar reference frame is centered at the geometrical barycenter of the formation, i.e.,

$${}^{p}\vec{r}_{cm} = {}^{p}\dot{\vec{r}}_{cm} = \vec{0}.$$
(4.39)

Following from Equation (4.3) the directive of the body fixed reference frame in planar reference frame coordinates ${}^{p}\hat{x}_{i}$ is calculated as

$${}^{p}\hat{x}_{i} = -\frac{{}^{p}\vec{R}_{i}}{\|{}^{p}\vec{R}_{i}\|}.$$
(4.40)

The *z*-axis is shared between the planar and the three S/C body-fixed reference frames. This is ${}^{p}\hat{z}_{p} = \hat{k}$ in planar coordinates, therefore, it follows that also for each body-fixed reference frame the same applies:

$${}^p\hat{z}_i = {}^p\hat{z}_p\hat{k} \tag{4.41}$$

for $i \in \{1, 2, 3\}$.

Finally, the *y* directive is calculated with the right-hand rule of thumb:

$${}^{p}\hat{y}_{i} = \hat{k} \times {}^{p}\hat{x}_{i}. \tag{4.42}$$

As for Equation (4.8), transformation from planar to body-fixed reference frame of S/C_i is defined by the direction-cosine matrix $C_{p/i}$:

$$C_{p/i} = \begin{pmatrix} p \hat{x}_i^{\mathsf{T}} \\ p \hat{y}_i^{\mathsf{T}} \\ \hat{k}^{\mathsf{T}} \end{pmatrix}.$$
(4.43)

The body-fixed reference frame of each S/C_i is, by all accounts, a Hill's reference frame: the *z*-axis is parallel to the apparent angular momentum vector w.r.t. \vec{r}_{cm} since all the motion in the *z*-direction is canceled out, therefore the rotation of the *x*-axis must coincide with the angular motion of the S/C in the planar reference frame, in order to keep the *x*-direction fixed on the origin (which physically represents incenter of the formation).

As such, the angular velocity of the body-fixed reference frame is calculated as

$${}^{p}\vec{\omega}_{i/p} = \frac{{}^{p}\vec{R}_{i} \times {}^{p}\hat{x}_{i}}{\|{}^{p}\vec{R}_{i}\|}$$
(4.44)

and, since ${}^{p}\vec{R}_{i_{z}} = {}^{p}\vec{R}_{i_{z}} = 0$, it should only have a *z*-component at each instant.

The nominal angular velocity vector of the body-fixed w.r.t. the inertial reference frame is obtained via addition of $\vec{\omega}_p$ and ${}^p\vec{\omega}_{i/p}$ and subsequent transformation into body-fixed coordinates:

$$\vec{\omega}_i = C_{p/i} \left(\vec{\omega}_p + {}^p \vec{\omega}_{i/p} \right) \tag{4.45}$$

Let us emphasize that this vector does not represent the actual S/C rotation, instead it identifies the angular rate that a S/C *should have*, based on the motion of the formation, in order to keep the correct orientation w.r.t. the other two at any given instant.

Using the data from the simulation of Chapter 2 for the first 4 years, the values of $\vec{\omega}_i$ calculated through Equation (4.45) are shown in Figure 4.5.

The *x*- and *y*-components have a yearly trend with amplitudes of $|\omega_1|, |\omega_2| \le 1.716 \cdot 10^{-7}$ rad/s, while the *z*-component has a small fluctuation around a non-zero value, $\omega_3 \simeq 9.851 \cdot 10^{-8} \pm 0.144 \cdot 10^{-8}$ rad/s, with a periodicity of half a year, consistent with the arm-length variations in Chapter 2.



Figure 4.5: "Natural" angular velocity vector \vec{w}_i , i = 1,2,3, of the three S/C w.r.t. inertial reference frame in body fixed coordinates for 4 years, calculated from Equation (4.45). From *top* to *bottom*, respectively, the angular velocities of S/C₁, S/C₂ and S/C₃ are shown in order. The components ($\omega_1, \omega_2, \omega_3$) refer, respectively, to the rotation about x, y and z in the body-fixed reference frame.

4.5. THE NOMINAL ANGULAR ACCELERATION VECTOR

Analogously to and following the previous section, it is possible to calculate the angular acceleration vector $\dot{\omega}_i$ of S/C_i, $i \in \{1,2,3\}$, if, additionally, the accelerations of the three S/C, \ddot{R}_1 , \ddot{R}_2 and \ddot{R}_3 are known at any instant. The process unfolds in a similar fashion as for ω_i , calculated in the previous section. The following calculations must be performed in sequence:

• time derivative of Equation (4.19), or drift acceleration between S/C_i and S/C_j , $i, j \in \{1, 2, 3\}$:

$$\ddot{r}_{ij} = \vec{\dot{r}}_{ij} \cdot \hat{n}_{ij} + \vec{\dot{r}}_{ij} \cdot \dot{\dot{n}}_{ij}$$

$$(4.46)$$

where $\ddot{\vec{r}}_{ij}$, $i, j \in \{1, 2, 3\}$, $i \neq j$ is the time derivative of Equation (4.20) or,

$$\ddot{\vec{r}}_{ij} = \vec{\vec{R}}_j - \vec{\vec{R}}_i;$$
 (4.47)

• time-derivative of Equation (4.21):

$$\ddot{n}_{ij} = \frac{\ddot{r}_{ij} - \ddot{r}_{ij}\hat{n}_{ij} - 2\dot{r}_{ij}\hat{n}_{ij}}{r_{ij}}$$
(4.48)

• time-derivative of Equation (4.22):

$$\ddot{\alpha_i} = -\frac{\ddot{n}_{ij} \cdot \hat{n}_{ik} + \hat{n}_{ij} \cdot \ddot{n}_{ik} + 2\dot{n}_{ij} \cdot \dot{n}_{ik} + \dot{\alpha}_i^2 \cos \alpha_i}{\sin \alpha_i}; \qquad (4.49)$$

• time-derivative of Equation (4.24):

$$\ddot{z}_p = \frac{\ddot{n}_{ij} \times \hat{n}_{ik} + \hat{n}_{ij} \times \ddot{n}_{ik} + 2\dot{n}_{ij} \times \dot{n}_{ik} + \left(-\ddot{\alpha}_i \cos \alpha_i + \dot{\alpha}_i^2 \sin \alpha_i\right) \hat{z}_p - 2\dot{\alpha}_i \cos \alpha_i \dot{\hat{z}}_p}{\sin \alpha_i};$$
(4.50)

• time-derivative of Equation (4.27):

$$\ddot{\beta} = -\frac{\hat{k} \cdot \ddot{z}_p + \dot{\beta}^2 \cos\beta}{\sin\beta},\tag{4.51}$$

• time-derivative of Equation (4.28):

$$\ddot{\hat{x}}_p = \frac{\cos\beta\ddot{\hat{z}}_p - \ddot{\beta}\left(\cos\beta\hat{x}_p + \sin\beta\hat{z}_p\right) - \dot{\beta}^2\left(\cos\beta\hat{z}_p - \sin\beta\hat{x}_p\right) - 2\dot{\beta}\left(\cos\beta\hat{x}_p + \sin\beta\hat{z}_p\right)}{\sin\beta};\tag{4.52}$$

• time-derivative of Equation (4.30):

$$\ddot{\hat{y}}_p = \ddot{\hat{z}}_p \times \hat{x}_p + \hat{z}_p \times \ddot{\hat{x}}_p + 2\dot{\hat{z}}_p \times \dot{\hat{x}}_p;$$
(4.53)

• time-derivative of Equation (4.32):

$$\vec{\omega}_p = \left(\dot{z}_p \cdot \dot{y}_p + \hat{z}_p \cdot \ddot{y}_p\right)\hat{i} + \left(\dot{x}_p \cdot \dot{z}_p + \hat{x}_p \cdot \ddot{z}_p\right)\hat{j} + \left(\dot{y}_p \cdot \dot{x}_p + \hat{y}_p \cdot \ddot{x}_p\right)\hat{k}; \tag{4.54}$$

• time-derivative of Equation (4.34):

$${}^{p}\vec{\vec{R}}_{i} = C_{n/p}\left(\vec{\vec{R}}_{i} - \vec{\vec{r}}_{cm}\right) - \vec{\omega}_{p} \times \vec{\omega}_{p} \times {}^{p}\vec{R}_{i} - 2\vec{\omega}_{p} \times {}^{p}\vec{\vec{R}}_{i} - \dot{\vec{\omega}}_{p} \times {}^{p}\vec{R}_{i};$$

$$(4.55)$$

• time-derivative of Equation (4.38):

$$\ddot{\vec{r}}_{cm} = \frac{\ddot{R}\boldsymbol{r}_a + R\ddot{\boldsymbol{r}}_a + 2\dot{R}\dot{\boldsymbol{r}}_a - \sum_j \left(\ddot{r}_{a_j}\vec{r}_{cm} + 2\dot{r}_{a_j}\vec{r}_{cm} \right)}{\sum_j r_{a_j}};$$
(4.56)

• time-derivative of Equation (4.44):

$${}^{p}\vec{\omega}_{i/p} = \frac{2\left({}^{p}\hat{x}_{i} \cdot {}^{p}\vec{R}_{i}\right)\vec{\omega}_{i/p} - {}^{p}\hat{x}_{i} \times {}^{p}\vec{R}_{i}}{\|{}^{p}\vec{R}_{i}\|};$$
(4.57)

• time-derivative of Equation (4.43):

$$\dot{C}_{p/i} = \begin{pmatrix} p \, \dot{x}_i^{\tau} \\ p \, \dot{y}_i^{\tau} \\ \vec{0}^{\tau} \end{pmatrix} = \begin{pmatrix} \left(p \, \vec{\omega}_{i/p} \times p \, \hat{x}_i \right)^{\tau} \\ \left(p \, \vec{\omega}_{i/p} \times p \, \hat{y}_i \right)^{\tau} \\ \vec{0}^{\tau}; \end{pmatrix}.$$
(4.58)

Finally we obtain the angular acceleration vector of body-fixed reference frame w.r.t. inertial reference frame in body-fixed coordinates with a time-differentiation of Equation (4.16):

$$\dot{\vec{\omega}}_{i} = C_{p/i} \left({}^{p} \dot{\vec{\omega}}_{i/p} + \dot{\vec{\omega}}_{p} \right) + \dot{C}_{i/p} \left({}^{p} \vec{\omega}_{i/p} + \vec{\omega}_{p} \right).$$
(4.59)

In practice it is almost impossible to measure the acceleration of a body in a gravity field using inertial sensors. Figure 4.6 shows the angular acceleration vector obtained from Equation (4.59) and the simulation data of Chapter 2.



Figure 4.6: "Natural" angular acceleration vector \vec{w}_i , i = 1, 2, 3, of the three S/C w.r.t. inertial reference frame in body fixed coordinates for 4 years, calculated from Equation (4.59). From *top* to *bottom*, respectively, the angular velocities of S/C₁, S/C₂ and S/C₃ are shown in order. The components ($\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\omega}_3$) refer, respectively, to the angular acceleration about x, y and z in the body-fixed reference frame. This graph can be considered as the time-derivative of Figure 4.5.

For the three S/C, the *x*- and *y*-components have an amplitude about 0 of $\dot{\omega}_1$, $\dot{\omega}_2 \simeq \pm 3.413 \cdot 10^{-14} \text{ rad/s}^2$ and the *z*-component has a much smaller amplitude of $\dot{\omega}_3 \simeq \pm 5.772 \cdot 10^{-16} \text{ rad/s}^2$ with double the frequency.

4.6. VALIDATION OF EQUATIONS

A validation scenario is presented in Appendix B: for 3 particles, whose motion is described by finely tuned analytical equations, the values for $\vec{\omega}_i \, \dot{\vec{\omega}}_i$, as well as \vec{R}_i , \vec{R}_i and $\ddot{\vec{R}}_i$ have been derived independently of each other. The shapes of the orbits are presented in Figure 4.7. The values are retrieved on n = 10001 points along



Figure 4.7: The shape of the validation orbits. Time $0 \le t < 2\pi$ is the only parameter of the analytical equations presented in Appendix B that govern the motion of the three particles (S/C₁, S/C₂, S/C₃).

the periodical path of the particles.

The values $\vec{\omega}'_i$ and $\dot{\vec{\omega}}'_i$ are calculated, respectively, with Equations (4.45) and (4.59) as a function of \vec{R}_i , $\dot{\vec{R}}_i$ and $\ddot{\vec{R}}_i$. The probability density distribution of the residuals

$$\delta \vec{\omega}_i = |\vec{\omega}_i - \vec{\omega}_i'| \tag{4.60}$$

and

$$\delta \vec{\omega}_i = |\vec{\omega}_i - \vec{\omega}_i'| \tag{4.61}$$

are shown, respectively, in Figures 4.8 and 4.9.

The results show a small discrepancy with a standard deviation of $4 \div 7 \cdot 10^{-16}$ rad/s for $\vec{\omega}_i$ and $11 \div 23 \cdot 10^{-16}$ rad/s² for $\vec{\omega}_i$ which is consistent with 0, considering an order of magnitude 10^0 for both (See Table B.1).

For more information about the validation process, see Appendix B.



Figure 4.8: Validation results of Equation (4.45). Top to bottom: the three body-fixed angular velocity vectors of the respective S/C. Left to right: the three components of each vector. σ is the standard deviation. The bin resolution is $\Delta x = 2 \cdot 10^{-16}$ rad/s. Results are shown for a population size n = 10001.



Figure 4.9: Validation results of Equation (4.59). Top to bottom: the three body-fixed angular acceleration vectors of the respective S/C. Left to right: the three components of each vector. σ is the standard deviation. The bin resolution is $\Delta x = 4 \cdot 10^{-16} \text{ rad/s}^2$. Results are shown for a population size n = 10001.

4.7. RESULTS AND VERIFICATION

The natural orientation of the body axes and the associated angular velocities and accelerations, based on the orbital simulation of Chapter 2 for a 4 years period have been respectively shown in Figures 4.3, 4.5 and 4.6. These values are representative of a perfect attitude control.

Verification of $\dot{\vec{\omega}}_i$ can be performed by numerically integrating $\dot{\vec{\omega}}_i$ over time and confront the results with $\vec{\omega}_i$. Verification of $\vec{\omega}_i$ can be performed by numerically integrating over time $\dot{\vec{d}}_i$, calculated as

$$\hat{d}_i = C_{n/i}^{\tau} \vec{\omega}_i \times \hat{d}_i \tag{4.62}$$

where \hat{d}_i is any of \hat{x}_i , \hat{y}_i and \hat{z}_i and confronting the results with the respective \hat{d}_i . $C_{n/i}$ is the transformation matrix from body-fixed reference frame of S/C_i to inertial reference frame from Equation (4.8).

You can find the verifications in Appendix C, respectively Section C.1 and C.2. After 4 years, and using a Runge-Kutta 4th order integrator with a step size of 1000 s, the difference in \hat{d}_i keeps at orders of magnitudes of 10^{-13} , while $\vec{\omega}_i$ is precise to the 10^{-19} power, or about 7 orders of magnitudes smaller than its value, thus verifying the results.

The results obtained in this chapter, although in part crucial for the advancement of this thesis, can be considered stand-alone, since they build up the current model of the LISA behavior found precedent publications. Although more complicated than others, these equations are the most precise (possibly achievable) method for retrieving the expected rotational velocities and accelerations. Compared to e.g. the orientation model used by Bik et al. [26], we were able to overcome the limits imposed by the first-order approximations and apply purely analytical solutions to an actual orbital model, hence relegating all the uncertainties to the orbital propagator. The significance of this cannot be overstated when considering a system as LISA, where high-end precision is one of the top requirements, and we can justify the core principle of this chapter by referring to the three-phase "wobble" of the plane found in Chapter 2: although very small (~800 km over an arm of 2.5 million km), the precision on the orientation of the LISA S/C of 5 nrad per axis [14] can be largely affected.

Moreover, these equations can be used for any 3-S/C formation where mutual orientation is a mission requirement.

4.8. CONCLUSIONS ON FORMATION DEPENDENCIES

In this chapter we have derived an analytical solution for the attitude commands, i.e. the expected orientations of the body-fixed axes based on the S/C displacements in the inertial reference frames. We assumed that the *x*-axes are pointed towards a common point, represented by the displacement vector \vec{r}_{cm} in the inertial reference frame. Although these equations work for any \vec{r}_{cm} placed on the formation plane, in this analysis we used the incenter of the triangular formation, which satisfies the assumption that the *x*-axes bisect the respective breath angles α_i at the corners of the triangle.

We have further derived the analytical relations of the associated angular velocities $\vec{\omega}_i$ and angular accelerations $\dot{\vec{\omega}}_i$ as function of inertial velocities \vec{R}_i and accelerations \vec{R}_i , based on Poisson's kinematics.

The results show periodical rotations about the axes with components in the range of 10^{-7} rad/s and associated accelerations in the range of 10^{-14} rad/s² for the *x* and *y* components and 10^{-16} rad/s² for the *z*-axis, overall very small.

The relations have been validated using a complementary model (Appendix B) and verified for mutual compatibility.

5

SYSTEM DYNAMICS

5.1. INTRODUCTION TO SYSTEM DYNAMICS

In Chapters 2 and 4 we have studied the S/C as point masses abiding the laws of gravitation. This is just the ideal model, assuming that the S/C follows a perfectly controlled drag-free path, as by requirements. In summary, the orbits and the orientations calculated so far are merely the solution of the control problem.

The system of a single S/C is characterized by 19 DoF [22, 37]. So far we have introduced the S/C displacement in the inertial reference frame \vec{R} , the S/C attitude $\theta \in \mathbb{R}^3$, introduced as the body-axes orientations, and the breath angle α between the lines-of-sight of the telescopes, briefly defined in Equation (4.6). Dragfree control and fine pointing is achieved indirectly by the GRSs, containing the TMs (two per S/C), whose displacements \vec{r}_i (i = 1, 2) and orientation $\theta_i \in \mathbb{R}^3$, add up to the other 12 DoF.

Control is provided by an equal number of forces and torques: the thrusters provide, respectively, thrustforce \vec{f}_t and thrust-torque \vec{l}_t , while an electrostatic suspension system in the GRS provide a force \vec{f}_i and a torque \vec{l}_i to the respective TM_i, i = 1, 2, to keep the TM from getting in contact with the S/C. The steering mechanism of the telescopes provides an additional torque to control α .

In the meanwhile, the SRP and the thruster induced vibrations act as disturbances [14] that need to be compensated.

In this chapter, the dynamics of the various DoF are modeled. These are the equations that are later implemented in the system simulation. Due to the embryonic state of the LISA mission, very little information is provided in the publicly available literature, hence, most of the equations need to be derived.

5.2. Spacecraft translational dynamics

The first DoF that we are going to analyze is the displacement, in the inertial reference frame, of the S/C CoM \vec{R} .

In Chapter 2 we have basically defined the path of such point based solely on Newtonian gravitational dynamics. In reality, the S/C is subject to other forces, namely, the thrust force provided by the reaction control system \vec{f}_t and an external disturbance, that we call \vec{f}_{d0} . Using the placeholder \vec{a}_g for the gravitational accelerations, the dynamics of the CoM are derived as:

$$\ddot{\vec{R}} = \vec{a}_g + \frac{1}{m_{sc}}\vec{f}_t + \frac{1}{m_{sc}}\vec{f}_{d0}$$
(5.1)

where m_{sc} is the spacecraft mass, calculated in Chapter 3. Note that, since we are considering a single S/C, we drop the 1, 2 and 3 subscripts used in Chapters 2 and 4 to identify one of the three satellites in the formation.

5.3. Spacecraft rotational dynamics

In Chapter 4 we derived the angular velocities and accelerations that a LISA S/C *should have* in order to keep a perfect orientation w.r.t. the other two. LISA uses the on-board thrusters to adjust its orientation, providing a torque \vec{l}_t , against an external disturbance \vec{l}_{d0} .

The S/C angular accelerations dynamics $\dot{\vec{\omega}}$ are given, according to Euler's formula as [25]:

$$\dot{\vec{\omega}} = I_{sc}^{-1} \left(\vec{l}_t - \vec{\omega} \times (I_{sc}\vec{\omega}) + \vec{l}_{d0} \right)$$
(5.2)

where I_{sc} is the inertia matrix of the S/C calculated in Chapter 3.

The results obtained in Chapter 4 are to be interpreted as the solution of Equation (5.2), taking into account $\vec{\omega}$ calculated in the same chapter. Note that we also drop the 1, 2 and 3 subscripts as we are considering a single S/C.

5.4. PERTURBATION DUE TO SOLAR RADIATION PRESSURE

The SRP is the most prominent environmental non-gravitational disturbance acting on the LISA S/C [17]. The resulting force \vec{f}_{srp} can be modeled as following [25].

The S/C exterior can be broken down into *n* surface components. Each $i \in 1, 2, ..., n$ component is characterized by a normal direction \hat{n}_i pointing away from the enclosed volume, a surface area S_i , and three coefficients: *absorptivity* R_{abs_i} , *specular reflectivity* R_{spec_i} and *diffuse reflectivity* R_{diff_i} , such that $R_{abs_i} + R_{spec_i} + R_{diff_i} = 1$. Considering the direction to the Sun \hat{s} of the S/C, and the S/C at S/C position P_{sr} , each surface component receives a net force \vec{f}_{srp_i} of:

$$\vec{f}_{srp_i} = -P_{sr}S_i \left[2\left(\frac{R_{diff_i}}{3} + R_{spec_i}\hat{n}_i \cdot \hat{s}\right)\hat{n}_i + \left(1 - R_{spec_i}\right)\hat{s} \right] \max(\hat{n}_i \cdot \hat{s}, 0)$$
(5.3)

where $\max(\hat{n}_i \cdot \hat{s}, 0)$ allows to discriminate between the exposed surfaces $\hat{n}_i \cdot \hat{s} > 0$ and the shadowed ones $\hat{n}_i \cdot \hat{s} \le 0$.

For the modeling of the LISA exposed surface we are going to refer to the LISA Pathfinder properties. The requirements for LISA Pathfinder [43] on the solar panels specify an absorptivity of

$$R_{abs} = 0.14$$

and consider the reflected light to be completely diffusive, i.e.

$$R_{spec} = 0$$

and

$$R_{diff} = 1 - R_{abs}.\tag{5.4}$$

According to the model presented in Chapter 3, the exposed surface is the rectangular solar array on the -z surface, with normal $\hat{n} = -\hat{z}$. By defining the Sun direction w.r.t. S/C as:

$$\hat{s} = \frac{\dot{R}_s - \dot{R}}{\|\vec{R}_s - \vec{R}\|}$$
(5.5)

we can rewrite Equation (5.3) as

$$\vec{f}_{srp} = P_{\odot} \left(\frac{1 \, [\text{AU}]}{\|\vec{R} - \vec{R}_s\|} \right)^2 S_{-z} \left[\frac{2}{3} \left(1 - R_{abs} \right) \hat{z} - \hat{s} \right] \max\left(-\hat{z} \cdot \hat{s}, 0 \right)$$
(5.6)

where

$$P_{\odot} = \frac{W_{\odot}}{c} \tag{5.7}$$

is the solar radiation pressure at 1 AU [3] calculated by means of the standard solar constant

$$W_{\odot} = 1361 \, \text{W/m^2}$$

and the speed of light in vacuum

In order to understand the implications of Equations (5.3) and (5.6), let us take a look at Figure 5.1. The component $\vec{f}_{blocked}$ is always present, and it is proportional to the surface facing the Sun. With $R_{spec} > 0$ the component \vec{f}_{spec} appears. The direction of \vec{f}_{spec} is symmetrical to $\vec{f}_{blocked}$ w.r.t. \hat{z} , but due to $R_{spec} \le 1$, $\|\vec{f}_{spec}\| \le \|\vec{f}_{blocked}\|$. Therefore, part $\vec{f}_{blocked}$ combines with \vec{f}_{spec} in the direction of \hat{z} . When $R_{spec} = 1$ we have the extreme case in which $\|\vec{f}_{spec}\| = \|\vec{f}_{blocked}\|$. This is what happens to a perfect solar sail. The force



Figure 5.1: Decomposition of the solar radiation pressure force \vec{f}_{srp} into the three components \vec{f}_{spec} , \vec{f}_{diff} and $\vec{f}_{blocked}$ acting on a surface element.

 \vec{f}_{srp} is then completely directed in the direction of \hat{z} . With $\overline{\alpha} = -\hat{z} \cdot \hat{s}$, the force at 1 AU acting on a surface of 1 m² is [3]:

$$f_{srn}(1 \text{ AU}) = 2P_{\odot} \cos^2 \overline{\alpha} = 9.126 \cos^2 \overline{\alpha} \ [\mu \text{N}].$$
(5.8)

This is the case for an absorptivity of $R_{abs} = 0$. If instead of being specular, the reflectivity is diffuse, the component \vec{f}_{diff} acts automatically in the direction of \hat{z} with 2/3 of the efficiency: to demonstrate it let us take into consideration the extreme case for $\hat{z} = -\hat{n}_i = -\hat{s}$, or $\cos \overline{\alpha} = 1$. The term inside the square brackets of Equation (5.3) has a magnitude of:

$$\begin{cases} 2 & \text{for } R_{spec} = 1 \text{ and } R_{diff} = 0; \\ \frac{2}{3} + 1 = \frac{5}{3} < 2 & \text{for } R_{diff} = 1 \text{ and } R_{spec} = 0. \end{cases}$$

showing the difference in efficiency between the two reflectivities.

Since LISA does not have a specular reflectivity and we are working with the absorptivity defined, Equation (5.6) is used in the simulation. Let us validate it by setting

$$\hat{s} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\hat{z} = \begin{pmatrix} \cos \overline{\alpha} \\ \sin \overline{\alpha} \\ 0 \end{pmatrix}$$

or, alternatively,

$$\hat{z} = \begin{pmatrix} \cos \overline{\alpha} \\ 0 \\ \sin \overline{\alpha} \end{pmatrix}$$

in order to obtain $\overline{\alpha} = -\hat{z} \cdot \hat{s}$. Let us then set $S_{-z} = 1 \text{ m}^2$ and the distance from the Sun, $\|\vec{R} - \vec{R}_s\| = 1$ AU. If we set $R_{abs} = 1$, no reflectivity is provided, hence, the magnitude of the SRP induced force should only be

$$P_{\odot}\cos\overline{\alpha} = \frac{9.126}{2}\cos\overline{\alpha} \; [\mu N] \tag{5.9}$$

and directed in the direction of $-\hat{s}$.

Figure 5.2 shows the results for varying values of $\overline{\alpha}$. The maximum at 1 AU, 9.126/2 μ N [3] is indicated by the dotted line. At $\overline{\alpha} < 90^{\circ}$ and $\overline{\alpha} > 90^{\circ}$ the force is $f_{srp} = 0$, because of the term max $(-\hat{z} \cdot \hat{s}, 0)$ used in Equation (5.6), which tells us when the surface is pointing away from the Sun. The curve is calculated also for the position of the S/C at nominal aphelion and perihelion. Due to the very small eccentricity of the orbits (See Chapter 2) the curves do not bear much difference.

By setting the exposed surface to the nominal value



Figure 5.2: Results of 5.6 for $R_{abs} = 1$, $S_{-z} = 1$ m² and for $||\vec{R} - \vec{R}_s|| = 1$ AU and at nominal perihelion and aphelion of a LISA S/C. At 1 AU, because of the total lack of reflectivity imposed by the value of R_{abs} we would expect the maximum f_{srp} value for an exposure $\overline{\alpha} = \cos^{-1}(-\hat{z} \cdot \hat{s}) = 90^{\circ}$ as half the one obtained by a perfect 1 m² solar sail of 9.123 μ N [3].



Figure 5.3: Results of 5.6 for $R_{abs} = 0.14$, $S_{-z} = 3.4.75 \text{ m}^2$ and for $\|\vec{R} - \vec{R}_s\| = 1 \text{ AU}$ and at nominal perihelion and aphelion of a LISA S/C. At an exposure of $\overline{\alpha} = -30^\circ$ (nominal for a LISA S/C) the SRP force is $f_{srp} \simeq 85.367 \mu$ N, with small fluctuation at aphelion and perihelion.

$$S_{-z} = a_1 \cdot b_1 = 3.00 \cdot 4.75 \ [m^2]$$

where a_1 and b_1 are the short and long bases of the rectangular solar panel (See section 3.2), and the absorptivity $R_{abs} = 0.14$ as for LISA Pathfinder, the obtained curve is shown in Figure 5.3. With a nominal angle of 30° between \hat{z} and $-\hat{s}$, the nominal solar radiation pressure would exert about $\|\vec{f}_{srp}\| \approx 85.367 \,\mu\text{N}$ on the S/C at 1 AU with small fluctuations due to varying orientation and radial distance from Sun due to the formation dynamics.

In the simulation, \vec{R} and \vec{R}_s are, respectively, the displacements of the S/C CoM and the Sun w.r.t. SSB in the inertial reference frame. Using the simulation data from Chapter 2 and a transformation to body-fixed reference frame through $C_{n/i}$ (Equation (4.8)), the SRP force on the three S/C for a period of 4 years is reported in Figure 5.4.

If we analyze the trend for S/C₁, the *x*-component has an average amplitude of ~28.170522141 μ N and the *y*-component has an average amplitude of ~28.181685265 μ N over the 4 years period.



Figure 5.4: SRP force in body-fixed reference frame of (*top* to *bottom* respectively) S/C_1 , S/C_2 and S/C_3 for 4 years. The components (f_{srp1} , f_{srp2} and f_{srp3}) refer respectively to x, y and z in body-fixed reference frame. The series was determined using the orbital and orientation data calculated in Chapters 2 and 4.

The *z*-component has an average value of ~80.162226950 μ N over the 4 years, with an harmonic component with average amplitude of ±1.531167180 μ N.

While all the components show an annual periodical trend, we can assert that for the *z*-axis this is due to the slight orbital eccentricity, while for the other two this is due mostly to the S/C formation rotation, although a small harmonic due to the eccentricity is also present. Since the harmonics are also phased to the respective orientation the small difference between x and y are expected.

The average magnitude over the 4 years is $\|\vec{f}_{srp}\| \approx 84.915657959 \ \mu$ N, or just a bit smaller than the one estimated for a Sun distance of 1 AU, due to the drifting effect of the Earth/Moon third body gravity influence (which raises the semi-major axis over the 4 years) and it oscillates with an additive max/min difference of +1.495618381 μ N and -1.412570332 μ N.

5.5. SOLAR RADIATION PRESSURE INDUCED TORQUE

In Chapter 3 we defined a model for the S/C which places the CoM at a distance of about 41.3 cm from the -z surface. Meanwhile, the rectangular surface is considered symmetrical in the *x*- and *y*-direction w.r.t. the CoM. Hence we can consider the Center of Pressure (CoP) at a position of

$$\vec{r}_{cp} = \begin{pmatrix} 0 \\ 0 \\ -0.413394480 \end{pmatrix}$$
 m

in the body-fixed reference frame. With the SRP force applied at \vec{r}_{cp} , a SRP induced torque \vec{l}_{srp} acts on the S/C, as [25]:

$$\vec{l}_{srp} = \vec{r}_{cp} \times \vec{f}_{srp}. \tag{5.10}$$

Figure 5.5 shows the values obtained from the results of the previous section. In S/C₁ the torque is 0 for the *z*-component because the *x* and *y* arms are non-existent due to the CoP lying directly underneath the CoM, and it oscillates roughly with amplitude of ~11.65015314 μ Nm for the *x*-component and ~11.64553836 μ Nm for the *y*-component. You can obtain the same results by taking the amplitudes of the *x* and *y* components of the SRP force \vec{f}_{srp} and multiplying them by the *z* component of \vec{r}_{cp} . The *z*-component of \vec{f}_{srp} has no influence on the torque.


Figure 5.5: SRP torque in body-fixed reference frame of (*top* to *bottom* respectively) S/C_1 , S/C_2 and S/C_3 for 4 years. The components (l_{srp1} , l_{srp2} , l_{srp3}) refer respectively to x, y and z in body-fixed reference frame. The series was calculated using the results from Section 5.4.

5.6. ESTIMATION OF THRUST EFFORTS

Operating the S/C in drag-free means that Equation (5.1) should result in

$$\vec{R} = \vec{a}_g \tag{5.11}$$

i.e. the motion of the S/C should be governed only by the gravitational accelerations \vec{a}_g . We saw this in Chapter 2, where the ideal S/C orbits were calculated using only the gravity of the celestial bodies in the Solar System. In order to satisfy Equation (5.11) the thrust force \vec{f}_t and the disturbances \vec{f}_{d0} in Equation (5.1) need to cancel out, or

$$f_t = -f_{d0}.$$
 (5.12)

Moreover, the ideal orientation presented in Chapter 4 requires the angular accelerations $\vec{\omega}$ of Equation (4.59) to always be satisfied. The solution for the control torque \vec{l}_t can be derived from Equation (5.2) as

$$\vec{l}_t = I_{sc}\dot{\vec{\omega}} + \vec{\omega} \times (I_{sc}\vec{\omega}) - \vec{l}_{d0}.$$
(5.13)

The ideal control torque is obtained by substituting the values of $\vec{\omega}$ and $\dot{\vec{\omega}}$ calculated in Chapter 4, with the time-dependent S/C inertia matrix I_{sc} calculated in Chapter 3.

If we consider the SRP as the only disturbance acting on the S/C we can substitute

$$\vec{f}_{d0} = \vec{f}_{srp} \tag{5.14}$$

and

$$l_{d0} = l_{srp} \tag{5.15}$$

into Equations (5.12) and (5.13) and obtain the necessary values of the thrust and torque to attain the desirable control. Since $\|\vec{\omega}\| \ll 1$ and $\|\vec{\omega}\| \ll 1$ we expect $\vec{l}_t \simeq -\vec{l}_{srp}$, nevertheless, it is important to clarify that a small torque necessary to modify the S/C attitude is also present.

In Chapter 3 we derived the thruster configuration of the reaction control system and we presented the relation (3.10) to calculate the throttle of each of the 12 thrusters T given an applied net thrust \vec{f}_t and torque \vec{l}_t . The solutions are shown in Figure 5.6, for a 4 year operation.

Notice how thrusters T_1 , T_4 , T_7 and T_{10} , i.e. the thrusters oriented in the *z*-direction, need to provide the most thrust, up to ~42.9264 μ N, due to the SRP force being more active in that direction.

In general, the thrust requirements calculated here are larger than predicted: the colloidal μ Newton thrusters tested on board of LISA Pathfinder have a maximum thrust capability of 35 μ N [44], while the FEEP μ Newton thrusters studied at ESA have a range of 0-100 μ N [41].

There are several ways to improve the performance of the thrusters, thus obtaining lower thrust requirements. Three are readily identifiable:

- Use a larger absorptivity coefficient R_{abs} , so to lower the SRP force acting on the S/C solar panel.
- Optimize the thruster configuration: the fact that thrusters T_1 , T_4 , T_7 and T_{10} provide an effort much greater than the other eight means that the configuration is not optimized to distribute equally the work.
- Use an active thrust effort estimation algorithm, as opposed to the one used here, which is based on a linear relation with constant coefficient through the whole simulation [38].



Figure 5.6: Thrust provided by each thruster T_i , i = 1, 2, 3..., 12, required to generate net thrust force $f_t = -f_{srp}$ and torque $l_t = -l_{srp} + \vec{\omega} \times I_{sc}\vec{\omega}$, with $\vec{\omega}$ and $\vec{\omega}$ being the angular velocities and accelerations for nominal orientation. From *top* to *bottom* the respective time series for S/C₁, S/C₂ and S/C₃ are shown, for a simulation period of 4 years. These values are the minimum requirement

5.7. ATTITUDE USING EULER ANGLES

The S/C attitude is defined here by means of Euler angles. Although quaternions would be a safer choice for their lack of singularities, we want to minimize the number of DoF.

The Euler angles $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^{\mathsf{T}}$ are defined here with a 3-2-1 rotation sequence, as shown in Figure 5.7. Each angle is associated to a positive rotation around its respective axis, i.e., *x*-axis for θ_1 , *y*-axis for θ_2 and *z*-axis for θ_3 , which are determined by the respective direction-cosine matrices, defined as following [40]:

$$C_{1}(\theta_{1}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{1} & \sin\theta_{1} \\ 0 & -\sin\theta_{1} & \cos\theta_{1} \end{pmatrix};$$
 (5.16)



Figure 5.7: Definition of the Tait-Bryton Euler angles as 3-2-1 rotation of a body-fixed (x,y,z) reference frame in inertial (X,Y,Z) reference frame. In sequence: rotation about inertial *z*-directive ($\hat{z}_n \equiv \hat{z}_b$) by θ_3 ; rotation about intermediate *y*-directive ($\hat{y}_b \equiv \hat{y}_a$) by θ_2 ; rotation about body-fixed *x*-directive ($\hat{x}_a \equiv \hat{x}$) by θ_1 .

$$C_{2}(\theta_{2}) = \begin{pmatrix} \cos\theta_{2} & 0 & -\sin\theta_{2} \\ 0 & 1 & 0 \\ \sin\theta_{2} & 0 & \cos\theta_{2} \end{pmatrix};$$
 (5.17)

$$C_3(\theta_3) = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0\\ -\sin\theta_3 & \cos\theta_3 & 0\\ 0 & 0 & 1 \end{pmatrix}.$$
 (5.18)

The above can be demonstrated by recalling Equation (4.8), in which we define a direction cosine matrix as $C = (\hat{x}, \hat{y}, \hat{z})^{\tau}$.

As shown in figure 5.7, the passage from the inertial reference frame, denoted here with the left superscript n, to the body-fixed reference frame, is obtained by a set of sequential rotations: a vector $n\vec{p}$ in the inertial reference frame becomes $^{b}\vec{p}$ in the first intermediate reference frame, obtained by rotating the inertial reference frame around the *z*-axis by θ_3 :

$${}^{b}\vec{p} = C_{3}(\theta_{3})^{n}\vec{p};$$
 (5.19)

subsequently, ${}^{b}\vec{p}$ becomes ${}^{a}\vec{p}$ in the second intermediate reference obtained by rotating the first around the *y*-axis by θ_2 :

$${}^{a}\vec{p} = C_{2}(\theta_{2}){}^{b}\vec{p} = C_{2}(\theta_{2})C_{3}(\theta_{3}){}^{n}\vec{p};$$
(5.20)

finally ${}^{a}\vec{p}$ becomes \vec{p} in the body-fixed reference frame, which is the second intermediate reference frame rotated around the *x*-axis by θ_1

$$\vec{p} = C_1(\theta_1)^a \vec{p} = C_1(\theta_1) C_2(\theta_2)^b \vec{p} = C_1(\theta_1) C_2(\theta_2) C_3(\theta_3)^n \vec{p} = C_{n/s}(\theta)^n \vec{p}.$$
(5.21)

Direct rotation is achieved via $C_{n/s}(\theta)$ which is a product of the three direction-cosine matrices defined for each angle. This $C_{n/s} \equiv C_{n/i}$ is the same defined in Equation (4.8).

One can write $C_{n/s}(\theta)$ in explicit form, by expanding Equation (5.21) in terms of θ_1 , θ_2 and θ_3 :

$$C_{n/s}(\boldsymbol{\theta}) = \begin{pmatrix} \cos\theta_2\cos\theta_3 & \cos\theta_2\sin\theta_3 & -\sin\theta_2\\ \sin\theta_1\sin\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 & \sin\theta_1\sin\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \sin\theta_1\cos\theta_2\\ \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_3 & \cos\theta_1\sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 & \cos\theta_1\cos\theta_2 \end{pmatrix}.$$
(5.22)

From Equation (5.22) the inverse relation to calculate θ is readily derived:

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} \operatorname{atan2}\left(C_{n/s_{2,3}}, C_{n/s_{3,3}}\right) \\ \operatorname{atan2}\left(-C_{n/s_{1,3}}, \sqrt{C_{n/s_{1,1}}^2 + C_{n/s_{1,2}}^2}\right) \\ \operatorname{atan2}\left(C_{n/s_{1,2}}, C_{n/s_{1,1}}\right) \end{pmatrix} = \begin{pmatrix} \operatorname{atan2}\left(y_3, z_3\right) \\ \operatorname{atan2}\left(-x_3, \sqrt{x_1^2 + x_2^2}\right) \\ \operatorname{atan2}\left(x_2, x_1\right) \end{pmatrix}.$$
(5.23)

where x_i , i = 1, 2, 3, are the components of the *x*-directive \hat{x} and y_3 and z_3 are the inertial *z*-components of the *y*- and *z*- directives \hat{y} and \hat{z} . The mutual relations between Equations (5.21), (5.22) and (5.23) are easily verifiable for any random value of $\boldsymbol{\theta}$.

Notice how each rotation axis is unchanged between two consecutive frames, hence $\hat{x} \equiv \hat{x}_a$, $\hat{y}_b \equiv \hat{y}_a$ and $\hat{z}_a \equiv \hat{z}_n$, where the subscript $_n$ refers to the inertial reference frame ("n" is for Newtonian).

Recalling the S/C body orientation calculated in Chapter 4, specifically Equations (4.3), (4.5), (4.7) and the ideal direction-cosine matrix $C_{n/s}$ calculated with Equation (4.8) and using Equation (5.23) we can calculate the Euler angles θ associated with the ideal orientation of the S/C. The values, for 4 years, are shown in figure 5.8.

For the three S/C the three Euler angles have non-sinusoidal yearly periodical motions. θ_1 and θ_2 have both amplitudes $\pm \pi/3$ rad, respectively around $\pm \pi$ rad and 0 rad. The period of θ_3 is twice as short, with smaller amplitudes and continuous components de-phased by 120° between the three S/C.

The results have been verified by confronting $C_{n/s}$ calculated with Equation (5.22) and $C_{n/i}$ from Equation (4.8) for the three S/C [25].

We need to point out that in the framework of this thesis we are going to define θ_i , i = 1, 2, 3, in the interval $-\pi < \theta_i \le \pi$. You can see, for θ_1 in Figure 5.8, when crossing the boundaries a discontinuity happens. This drawback of the Euler angles needs to be fixed by an active algorithm when integrated in the simulation.



Figure 5.8: "Natural" Euler angles $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^{T}$ calculated with Equation (5.23) equivalent to the nominal orientation calculated in Chapter 4. From *top* to *bottom*, the Euler angles of, respectively, S/C₁, S/C₂ and S/C₃ are shown.

5.8. KINEMATICS AND DYNAMICS OF THE EULER ANGLES

Since the three $\boldsymbol{\theta}$ define a positive rotation around their respective directives, their time-derivatives characterize three angular velocity vectors: $\vec{\omega}_a = \dot{\theta}_1 \hat{x}$, $\vec{\omega}_{a/b} = \dot{\theta}_2 \hat{y}_a$ and $\vec{\omega}_{b/n} = \dot{\theta}_2 \hat{z}_b$. The rotation of the body-fixed reference frame w.r.t. the inertial reference frame can be obtained by adding the three angular velocity vectors together:

$$\vec{\omega} = \dot{\theta}_1 \hat{x} + \dot{\theta}_2 \hat{y}_a + \dot{\theta}_3 \hat{z}_b. \tag{5.24}$$

Let us recall that, in their respective reference frame, the directives are the three fundamental unit vectors, i.e., $\hat{x} = \hat{i}$, ${}^{a}\hat{y}_{a} = \hat{j}$ and ${}^{b}\hat{z}_{b} = \hat{k}$, which can all be transformed to body-fixed reference frame by means of Equations (5.21)–(5.20), i.e.

$$\vec{\omega} = \dot{\theta}_1 \hat{i} + \dot{\theta}_2 C_1(\theta_1) \hat{j} + \dot{\theta}_3 C_1(\theta_1) C_2(\theta_2) \hat{k}.$$
(5.25)

The above can be rewritten as:

$$\vec{\omega} = N^*(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \tag{5.26}$$

where $N^*(\theta)$ can be deduced mathematically as the horizontal concatenation of \hat{i} , the 2^{nd} column of $C_1(\theta_1)$ and the 3^{rd} column of $C_1(\theta_1)C_2(\theta_1)$:

$$N^{*}(\boldsymbol{\theta}) = \begin{pmatrix} 1 & 0 & -\sin\theta_{2} \\ 0 & \cos\theta_{1} & \sin\theta_{1}\cos\theta_{2} \\ 0 & -\sin\theta_{1} & \cos\theta_{1}\cos\theta_{2} \end{pmatrix}.$$
 (5.27)

Usually the inverse relation is more useful, i.e., being able to calculate $\dot{\theta}$ from $\vec{\omega}$, therefore, matrix $N(\theta) = N^*(\theta)^{-1}$ is used instead:

$$N(\boldsymbol{\theta}) = \frac{1}{\cos\theta_2} \begin{pmatrix} \cos\theta_2 & \sin\theta_1 \sin\theta_2 & \cos\theta_1 \sin\theta_2 \\ 0 & \cos\theta_1 \cos\theta_2 & -\sin\theta_1 \cos\theta_2 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix};$$
(5.28)

$$\dot{\boldsymbol{\theta}} = N(\boldsymbol{\theta})\vec{\omega}.\tag{5.29}$$

Again, using the values for $\vec{\omega}$ obtained in Chapter 4 and the results of the previous section for θ , we are able to calculate analytically $\dot{\theta}$, whose solutions are shown in Figure 5.9.

Moreover we can derive a sort of acceleration for the Euler angles, as

$$\hat{\boldsymbol{\theta}} = \dot{N}(\boldsymbol{\theta})\vec{\omega} + N(\boldsymbol{\theta})\vec{\omega}$$
(5.30)

where

$$\dot{N}(\boldsymbol{\theta}) = \frac{1}{\cos\theta_2} \begin{bmatrix} \dot{\theta}_2 \sin\theta_2 N(\boldsymbol{\theta}) + \begin{pmatrix} 0 & \dot{\theta}_1 \cos\theta_1 \sin\theta_2 + \dot{\theta}_2 \sin\theta_1 \cos\theta_2 & -\dot{\theta}_1 \sin\theta_1 \sin\theta_2 + \dot{\theta}_2 \cos\theta_1 \cos\theta_2 \\ 0 & -\dot{\theta}_1 \sin\theta_1 \cos\theta_2 - \dot{\theta}_2 \cos\theta_1 \sin\theta_2 & -\dot{\theta}_1 \cos\theta_1 \cos\theta_2 + \dot{\theta}_2 \sin\theta_1 \sin\theta_2 \\ 0 & \dot{\theta}_1 \cos\theta_1 & -\dot{\theta}_1 \sin\theta_1 & -\dot{\theta}_1 \sin\theta_1 \end{pmatrix} \end{bmatrix}.$$
(5.31)

Using the solution for $\dot{\vec{\omega}}$ derived in Chapter 4, the results are shown in Figure 5.10.

The verification of Equations (5.29) and (5.30), through the consistency of θ , $\dot{\theta}$ and $\ddot{\theta}$ via numerical integration (Runge-Kutta 4th order with time step of 1000 s) can be found in Appendix C, Section C.3.



Figure 5.9: "Natural" time derivative of the Euler angles $\dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$ calculated through Equation (5.29), according to the nominal angular velocities $\vec{\omega}$ from Chapter 4. From *top* to *bottom*, the Euler angle speeds of, respectively, S/C₁, S/C₂ and S/C₃ are shown.



Figure 5.10: "Natural" double time derivative ("acceleration") of the Euler angles $\ddot{\theta} = (\ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}_3)$ calculated through Equation (5.29), according to the nominal angular velocities $\vec{\omega}$ and accelerations $\dot{\vec{\omega}}$ from Chapter 4. From *top* to *bottom*, the Euler angle speeds of, respectively, S/C₁, S/C₂ and S/C₃ are shown.

5.9. THE GRAVITATIONAL REFERENCE SYSTEM DYNAMICS

Let us now discuss the mechanism that allows the drag-free operation of the S/C, as well as the readout of GW signals: the GRS [14, 17].

The received laser beams at each telescope are sent via an optical bench to a TM, located within a vacuum assembly referred to, here, as *cage*. The TM and the walls of its enclosure are separated by just ~2 mm of empty space. Interferometric sensors measure precisely the displacement of the TM within the assembly.

The TM is shielded from external disturbances within the cage, hence abiding only the gravitational accelerations. They are, therefore, used as reference points that the S/C is required to follow in order to keep a drag-free trajectory.

This concept is simple enough, but in reality there are a few limitations. Figure 5.11 presents the strawman concept of the GRS-Optical bench-telescope assembly envisioned on LISA [14]. The telescope and the GRS are rigidly connected. The whole assembly is allowed to rotate about a pivot point in order to satisfy the breath angle α requirements dictated by the evolution of the formation in time (see Equations (4.6), (4.22) and (4.49)).



Figure 5.11: The strawman design of the optical assembly [14]. The telescope has an aperture of 32 cm. The optical assembly defined by the telescope, the GRS and the optical bench rotate around a common pivot (hinge). We assume the both the optical assemblies are articulated.

Two optical assemblies and two TMs are present in one S/C. Due to their number, not all the DoF can be controlled in drag-free, for the rest, the GRS employs an *electrostatic suspension system* to keep the TM from touching the walls and control its orientation [14, 17].

Using Figure 5.11 and the definitions on the body-fixed reference frames determined in Chapter 3 a model for the dynamics of the GRS is derived. Figure 5.12 shows how the various DoF interact.

As always, the S/C reference frame is defined so that the *x*-axis bisects the breath angle α . Two new reference frames are then defined, with their *x*-axis pointing in the directions of the lines-of-sight of the telescopes. The *z*-axes are co-directional with the body-fixed *z*-axis. Their origins \vec{r}_{01} and \vec{r}_{02} define the rest positions of the TMs, whose displacements w.r.t. these points are, respectively \vec{r}_1 and \vec{r}_2 . By convention TM₁ is placed in the -*y* quadrant and TM₂ is on the respective symmetrical spot w.r.t. the S/C body-fixed *x*-axis. The reference frame rotates with the telescope around the *z*-axis about the respective pivot points \vec{r}_{h1} and \vec{r}_{h2} . We call these new reference frames *optical reference frames*.

We now derive the dynamics of the TM in their respective optical orientation. These equations are not provided anywhere in literature. Let us begin by defining the motion of TM_i in the inertial reference frame. As we said, TM_i experiences gravitational accelerations \vec{a}_g of the same magnitude as the ones acting on the S/C (see Equation (5.1)), plus a small difference \vec{d}_i but they are completely shielded by the external environmental disturbance. \vec{d}_i is made to include all the possible alternative disturbances that a TM may experience in its enclosure. The electrostatic suspension system acts with a force \vec{f}_i on TM_i. Moreover, the S/C self-gravity and electrostatic interactions create a virtual spring-mass system with stiffness matrix K_i [22, 23, 45]. The total acceleration experienced by TM_i in the inertial reference frame is, therefore:

$${}^{n}\ddot{\vec{r}}_{i} = {}^{n}\vec{a}_{g} + {}^{n}\vec{d}_{i} + \frac{1}{m_{i}}{}^{n}\vec{f}_{i} - {}^{n}\left(\frac{K_{i}}{m_{i}}\vec{r}_{i}\right)$$
(5.32)

where m_i is the TM mass, whose nominal value is [45]

$$m_i \simeq 1.9$$
 kg.

In the S/C reference frame, defined by the superscript ^{*s*}, we need to account for the S/C angular velocity $\vec{\omega}$ and acceleration $\vec{\omega}$, plus reference frame transformation from inertial to body-fixed, defined by the direction-cosine matrix $C_{n/s}$:



Figure 5.12: Schematics used to derive the dynamics of the TM relative DoF. *left*: In S/C body frame (x,y,z), assuming both optical assemblies to be articulated, the pivot points are positioned at \vec{r}_{hi} (fixed) and the rest-positions of the TM are positioned at \vec{r}_{0i} (rotating around \vec{r}_{hi}) from the S/C center of mass. *right*: the *x*-axes of the GRS reference frames are collinear to the respective telescope lines-of-sight (interferometric arms), the displacements of TM_i w.r.t. their rest positions are defined by \vec{r}_i . The geometry is symmetrical w.r.t. the S/C body *x*-axis, and the rotations around the hinges are equal and opposite and proportional to $\dot{\alpha}/2$. It is assumed that the *xy*-planes of all the reference frames are co-planar.

$${}^{s}\vec{\vec{r}}_{i} = C_{n/s} \left({}^{n}\vec{\vec{r}}_{i} - {}^{n}\vec{\vec{R}} \right) - \vec{\omega} \times \vec{\omega} \times {}^{s}\vec{\vec{r}}_{i} - 2\vec{\omega} \times {}^{s}\vec{\vec{r}}_{i} - \vec{\omega} \times {}^{s}\vec{\vec{r}}_{i}.$$

$$(5.33)$$

Finally, we need to shift to the optical assembly reference frame of TM_{*i*}, whose origin is at \vec{r}_{0i} . A transformation matrix $C_{s/i}$ is defined, such that $\vec{r}_i = C_{s/i} {}^s \vec{r}_i$. For both TM₁ and TM₂, this can be calculated, from Figure 5.12 as, respectively,

$$C_{s/1} = \begin{pmatrix} \cos \alpha/2 & -\sin \alpha/2 & 0\\ \sin \alpha/2 & \cos \alpha/2 & 0\\ 0 & 0 & 1 \end{pmatrix}$$
(5.34)

and

$$C_{s/2} = \begin{pmatrix} \cos \alpha/2 & \sin \alpha/2 & 0\\ -\sin \alpha/2 & \cos \alpha/2 & 0\\ 0 & 0 & 1 \end{pmatrix}.$$
 (5.35)

Moreover, the rotation due to the telescope steering mechanism $\vec{\omega}_{0i}$, i = 1, 2, is accounted for, as, respectively

$$\vec{\omega}_{01} = -\frac{\dot{\alpha}}{2}\hat{k}$$

$$\vec{\omega}_{02} = \frac{\dot{\alpha}}{2}\hat{k}.$$
(5.36)

where $\hat{k} = [0, 0, 1]^{\tau}$. We can write the acceleration of TM_i in the optical assembly reference frame as:

$$\vec{\vec{r}}_{i} = C_{s/i} \left({}^{s} \vec{\vec{r}}_{i} - \vec{\vec{r}}_{0i} \right) - \vec{\omega}_{0i} \times \vec{\omega}_{0i} \times \vec{\vec{r}}_{i} - 2\vec{\omega}_{0i} \times \vec{\vec{r}}_{i} - \vec{\omega}_{0i} \times \vec{\vec{r}}_{i}.$$
(5.37)

We assume that S/C and the optical reference frames share the same *z*-axis and, since $\vec{\omega}_{0i}$ does only have *z*-components, as for Equation (5.36), ${}^{s}\vec{\omega}_{0i} = \vec{\omega}_{0i}$. The s superscript is not necessary for $\vec{\omega}$ and \vec{r}_{0i} in Equations (5.33) and (5.37), as we intend to observe these quantities in the S/C reference frame and nowhere else.

For the values of displacement and velocity in the S/C reference frame, we can determine the following relations to substitute into Equation (5.33):

$${}^{s}\vec{r}_{i} = C_{s/i}^{-1}\vec{r}_{i} + \vec{r}_{0i}; ag{5.38}$$

and

$$\dot{\vec{r}}_i = C_{s/i}^{-1} \left(\dot{\vec{r}}_i + \vec{\omega}_{0i} \times \vec{r}_i \right) + \dot{\vec{r}}_{0i}.$$
(5.39)

Let us recall Equations (5.1) and (5.32): the term ${}^{n}\vec{r}_{i} - {}^{n}\vec{R}$ in Equation (5.33) can be expanded as:

$${}^{n}\ddot{\vec{r}}_{i} - {}^{n}\ddot{\vec{R}} = \frac{1}{m_{i}}{}^{n}\vec{f}_{i} + {}^{n}\vec{d}_{i} - {}^{n}\left(\frac{K_{i}}{m_{i}}\vec{r}_{i}\right) - \frac{1}{m_{sc}}{}^{n}\vec{f}_{t} - {}^{n}\vec{d}_{0}$$
(5.40)

where $\vec{d}_0 = \vec{f}_{d0} / m_{sc}$ is the disturbance acting on the S/C in acceleration form. Performing the transformation in S/C body-fixed reference frame, the first term of Equation (5.33) becomes

$$C_{n/s} \left({}^{n} \ddot{\vec{r}}_{i} - {}^{n} \ddot{\vec{R}} \right) = \frac{1}{m_{i}} {}^{s} \vec{f}_{i} + {}^{s} \vec{d}_{i} - {}^{s} \left(\frac{K_{i}}{m_{i}} \vec{r}_{i} \right) - \frac{1}{m_{sc}} \vec{f}_{t} - \vec{d}_{0}.$$
(5.41)

The ^{*s*} superscript disappears for \vec{f}_t and \vec{d}_0 , as these are observables in the S/C body-fixed reference frame. Transformation into optical reference frame bears

$$C_{s/i}C_{n/s}\left({}^{n}\ddot{\vec{r}}_{i}-{}^{n}\ddot{\vec{R}}\right) = \frac{1}{m_{i}}\vec{f}_{i}+\vec{d}_{i}-\frac{K_{i}}{m_{i}}\vec{r}_{i}+C_{s/i}\left(-\frac{1}{m_{sc}}\vec{f}_{t}-\vec{d}_{0}\right),$$
(5.42)

which appears when expanding the first term of Equation (5.37), specifically $C_{s/i}{}^s \vec{r}_i$. Combining Equations (5.33), (5.37) and (5.42), the following relation is obtained.

$$\ddot{\vec{r}}_{i} = \frac{1}{m_{i}}\vec{f}_{i} + \vec{d}_{i} - \frac{K_{i}}{m_{i}}\vec{r}_{i} + C_{s/i}\left(-\frac{1}{m_{sc}}\vec{f}_{i} - \vec{d}_{0}\right) + C_{s/i}\left[-\ddot{\vec{r}}_{0i} - \vec{\omega} \times \vec{\omega} \times \left(C_{s/i}^{-1}\vec{r}_{i} + \vec{r}_{0i}\right) - 2\vec{\omega} \times \left\{C_{s/i}^{-1}\left(\dot{\vec{r}}_{i} + \vec{\omega}_{0i} \times \vec{r}_{i}\right) + \dot{\vec{r}}_{0i}\right\} - \dot{\vec{\omega}} \times \left(C_{s/i}^{-1}\vec{r}_{i} + \vec{r}_{0i}\right)\right] - \vec{\omega}_{0i} \times \vec{\omega}_{0i} \times \vec{r}_{i} - 2\vec{\omega}_{0i} \times \vec{r}_{i} - \dot{\vec{\omega}}_{0i} \times \vec{r}_{i}.$$

$$(5.43)$$

Further derivation can be performed by expanding \vec{r}_{0i} and \vec{r}_{0i} , i.e., the velocity and acceleration of the optical reference frames origins in the S/C reference frame, dictated by their motion around the hinges:

$$\vec{r}_{0i} = \vec{\omega}_{0i} \times (\vec{r}_{0i} - \vec{r}_{hi});
\vec{r}_{0i} = \vec{\omega}_{0i} \times (\vec{r}_{0i} - \vec{r}_{hi}) + \vec{\omega}_{0i} \times \vec{\omega}_{0i} \times (\vec{r}_{0i} - \vec{r}_{hi}).$$
(5.44)

where $\vec{\omega}_{0i}$ is given by Equation (5.36) for i = 1, 2. Substitution into Equation (5.43) results in the final form for TM_i dynamics:

$$\ddot{\vec{r}}_{i} = \frac{1}{m_{i}}\vec{f}_{i} + \vec{d}_{i} - \frac{K_{i}}{m_{i}}\vec{r}_{i} + C_{s/i}\left(-\frac{1}{m_{sc}}\vec{f}_{t} - \vec{d}_{0}\right) + C_{s/i}\left[-\dot{\vec{\omega}}_{0i} \times (\vec{r}_{0i} - \vec{r}_{hi}) - \vec{\omega}_{0i} \times \vec{\omega}_{0i} \times (\vec{r}_{0i} - \vec{r}_{hi})\right] - \vec{\omega} \times \left(C_{s/i}^{-1}\vec{r}_{i} + \vec{r}_{0i}\right) - 2\vec{\omega} \times \left\{C_{s/i}^{-1}\left(\vec{r}_{i} + \vec{\omega}_{0i} \times \vec{r}_{i}\right) + \vec{\omega}_{0i} \times (\vec{r}_{0i} - \vec{r}_{hi})\right\} - \vec{\omega} \times \left(C_{s/i}^{-1}\vec{r}_{i} + \vec{r}_{0i}\right)\right] - \vec{\omega}_{0i} \times \vec{\omega}_{0i} \times \vec{r}_{i} - \vec{\omega}_{0i} \times \vec{r}_{i}.$$

$$(5.45)$$

The positions of the hinges in the S/C reference frame have been loosely estimated by graphical evaluation of the CAD model presented by the CDF team [21] as:

$$\vec{r}_{h1} = \begin{pmatrix} 0.1074 \\ -0.3216 \\ 0 \end{pmatrix}$$
 m; $\vec{r}_{h2} = \begin{pmatrix} 0.1074 \\ 0.3216 \\ 0 \end{pmatrix}$ m

while the origins of the optical reference frames have been arbitrarily placed at 25 cm from the hinges along the lines-of-sight of the telescope, as suggested in Figure 5.11, or:

$$\vec{r}_{0i} = \vec{r}_{hi} + C_{s/i}^{-1} \begin{pmatrix} -0.25\\0\\0 \end{pmatrix} \text{m.}$$
(5.46)

The values for \vec{r}_{0i} , i = 1, 2, are not constant, as the direction-cosine-matrices $C_{s/i}$ depend on the orientation of the telescopes.

5.10. TEST MASS ORIENTATION DYNAMICS

The angular acceleration vector of TM_i, considered as a rigid body, is derived from the Euler formula and the sum of angular acceleration vectors as:

$$\dot{\vec{\omega}}_{i} = I_{i}^{-1} \vec{l}_{i} - I_{i}^{-1} K_{\varphi i} \boldsymbol{\varphi}_{i} - I_{i}^{-1} \left[(C_{s/i} \vec{\omega} + \vec{\omega}_{0i} + \vec{\omega}_{i}) \times \left\{ I_{i} \times (C_{s/i} \vec{\omega} + \vec{\omega}_{0i} + \vec{\omega}_{i}) \right\} \right] - \dot{\vec{\omega}}_{0i} - C_{s/i} \dot{\vec{\omega}} + I_{i}^{-1} \vec{l}_{di}.$$
(5.47)

In this equation \vec{w}_i is the angular velocity of the TM_i, I_i is its inertia matrix, \vec{l}_i is the torque provided by electrostatic suspension system, $K_{\varphi i}$, as for Equation (5.32), is a stiffness matrix factor that models the interaction between the S/C and TM rotation, creating a torque linearly dependent on the TM_i attitude φ_i . In this case φ_i represent neither the Euler angles nor quaternions, but simply a placeholder value such that [23]

$$\dot{\boldsymbol{\varphi}}_i = \vec{\omega}_i. \tag{5.48}$$

This linear relation is only suitable for $|\varphi_{i_j}| \ll 1$, j = 1, 2, 3. For such case φ_i can be considered Euler angles. Naturally, a disturbance torque \vec{l}_{di} is present as well.

5.11. TELESCOPE STEERING MECHANISM

At last, we are going to talk about the dynamics of the breath angle α . In Chapter 4 we have derived the formation dependent nominal breath angles α_i , i = 1, 2, 3, and their divergence speed $\dot{\alpha}_i$ and accelerations $\ddot{\alpha}_i$, respectively in Equations (4.6), (4.22) and (4.49), as intermediary steps. In this chapter we drop the *i*, and concentrate on the S/C-specific breath angle.

A case-point should be drawn on the breath angle dynamics, as we have not defined a model for the telescope-GRS mass-inertia, therefore we cannot draw any conclusions on the torque required for its control.

We decide, therefore, to leave out the telescope steering mechanism from the control loop and use the quantities α , $\dot{\alpha}$ and $\ddot{\alpha}$ as derived for the perfectly controlled case, when required. Figures 5.13, 5.14 and 5.15 show, respectively, the three quantities for a 4 year operation.



Figure 5.13: "Natural" breath angles, required to keep the telescopes locked onto the next S/C calculated using Equation (4.6) for 4 years of simulation.

During the first 4 years of operation the requirement to keep α between $60 \pm 1.5^{\circ}$ is satisfied. In fact, the maximum divergence from 60° is at most ~0.807° for α_3 and the maximum divergence range is ~1.411° for α_2 . Table 5.1 shows the divergence values for all the breath angles.

The divergence speeds are in the order of 10^{-9} rad/s, with maximum value of $|\dot{\alpha}_1|$, $|\dot{\alpha}_2|$ and $|\dot{\alpha}_3|$ of, respectively, 4.149560197 $\cdot 10^{-9}$ rad/s, 4.446549403 $\cdot 10^{-9}$ rad/s and 4.205625000 $\cdot 10^{-9}$ rad/s.

Table 5.1: The maximum divergence from the nominal $\alpha = 60^{\circ}$ and the total range of the breath angles.

	α_1	α_2	α_3
Divergence $ \alpha - 60^{\circ} $	0.751141245°	0.769313356°	0.807311659°
Range $\alpha_{max} - \alpha_{min}$	1.367473566°	1.411040215°	1.375361938°



Figure 5.14: "Natural" breath angle divergence speeds (time-derivative of α_i), required to keep the telescopes locked onto the next S/C calculated using Equation (4.22) for 4 years of simulation.



Figure 5.15: "Natural" breath angle divergence accelerations (double time-derivative of α_i), required to keep the telescopes locked onto the next S/C calculated using Equation (4.22) for 4 years of simulation.

The divergence accelerations are in the order of 10^{-15} rad/s², with maximum value of $|\dot{\alpha}_1|$, $|\ddot{\alpha}_2|$ and $|\ddot{\alpha}_3|$ of, respectively, 1.507598135 $\cdot 10^{-15}$ rad/s², 1.634145201 $\cdot 10^{-15}$ rad/s² and 1.527447759 $\cdot 10^{-15}$ rad/s² The values of the three α_i , i = 1, 2, 3 are verified for

$$\alpha_1 + \alpha_2 + \alpha_3 = 180^\circ$$

because of the nature of the triangle. The values of the three $\dot{\alpha}_i$ and the three $\ddot{\alpha}_i$ are also verified for

$$\dot{\alpha}_1 + \dot{\alpha}_2 + \dot{\alpha}_3 = 0$$

and

$$\ddot{\alpha}_1 + \ddot{\alpha}_2 + \ddot{\alpha}_3 = 0.$$

Another verification is performed on the consistency of α , $\dot{\alpha}$ and $\ddot{\alpha}$ by means of numerical integration (Runge-Kutta 4th order with 1000 s step size). The results are reported in Appendix C, Section C.4.

5.12. NOISE FILTERS

The SRP is not the only disturbance acting on the S/C: several other factors contribute to the deterioration of the dynamics at higher frequencies, including thermal noise, electrostatic and magnetic interactions, atmospheric disturbances due to outgassing inside the GRS and, most prominently, thrust jitter [14, 17, 23]. In this thesis we decide to focus on the latter.

In Chapter 3 we have derived a suitable model for the thrust configuration matrix A. In Section 5.6 we derived the nominal thrust provided by each of the 12 on-board μ Newton thrusters in order to counteract the SRP induced thrust-torque on the S/C. The study case is S/C_1 .

Not much information is provided in literature about the shaping of the noise filters for the thrust jitter, except that it should take into consideration the thrust configuration matrix *A* [23]. Public studies on the characterization of FEEP μ Newton thruster stability struggle on the resolution of the measurement instruments during experimental testing (>1.1 μ N) [41]. The Colloid μ Newton thrusters have been shown to achieve a noise of <0.1 μ N/ $\sqrt{\text{Hz}}$ at 30 μ N [42]. Figure 5.16, extrapolated by J. K. Ziemer and S. M. Merkowitz



Figure 5.16: Characterization of Thrust-Voltage V_N curve of Colloid µNewton thruster for various values of feed current I. From [42]

(2004) [42] shows how the spurious thrust jitters rise in amplitude with the feed current and subsequently with the thrust, but no model is provided by the authors. J. Bik *et al.* (2007) [26] used a frequency-related noise shape to obtain values of thrust-noise with standard deviation $\sigma = 2 \mu N$ at sampling steps of 1 s.

As a preliminary approach, we decide to confer each thruster command a confidence level of $\pm 1\%$ at 3σ (standard deviation). In later iterations this value can be augmented to expand the range of uncertainties.

The noise is simply created by multiplying 12 independent randomly generated factors ε_i , i = 1, 2, ..., 12, with

$$\mu_{\varepsilon} = 0$$
 (mean), $3\sigma_{\varepsilon} = 0.01$ (three standard deviations),

to each thruster effort and calculate the thrust and torque jitter $\delta \vec{f}_t$ and $\delta \vec{l}_t$ by means of the thruster configuration matrix *A* (Section 3.4),

$$\begin{pmatrix} \delta \vec{f}_t \\ \delta \vec{l}_t \end{pmatrix} = A \begin{pmatrix} \varepsilon_1 T_1 \\ \varepsilon_2 T_2 \\ \vdots \\ \varepsilon_{12} T_{12} \end{pmatrix}.$$
 (5.49)



Figure 5.17 shows a sample of ε_i generated using the normrand() function in Matlab. Figure 5.18 shows

Figure 5.17: *left:* a randomly generated ε_i sample (jitter factor) over 4 years (1000 s step size) and *right:* the probability density distribution of the sample with a bin size of $\Delta x = 0.001$. σ is the standard deviation.



Figure 5.18: *left:* the associated thrust noise for a 1% confidence (3σ) in the thruster actuation and *right:* the associated torque noise. The *z*-direction thrust (green) and the torque about te *x*-axis (blue) are more than twice as prominent as the other two respective components. Their zero-mean standard deviation is reported in Table 5.2.

the thrust and torque noise calculated with Equation (5.49) using the values for *T* estimated in Section 5.6. The standard deviations are shown in Table 5.2.

Table 5.2: Standard deviation of thrust noise δf_t and torque noise δl_t due to a 1% confidence level (3σ) in thruster force.

	x	У	Z
$\sigma\left[\delta\vec{f}_t\right] [\mu N]$	0.06421080894	0.08093653289	0.2159621796
$\sigma \left \delta \vec{l}_t \right \left[\mu \text{N m} \right]$	0.3591978114	0.1728324518	0.1193266428

Needing to rely on Equation (5.49) during the simulation is time-consuming. Moreover, we run into the risk of creating an *algebraic loop*, i.e., having an unknown in the loop that Simulink is not able to resolve algebraically [46]. As such, the jitters are simulated independently using the zero-mean standard deviation values of table 5.2. The reconstructed signal used in the dynamics simulator is shown in Figure 5.19.



Figure 5.19: Reproduction of Figure 5.18 by assuming a white noise with the same standard deviations (Table 5.2). *left:* thrust jitter $\delta \vec{f}_t$, *right:* torque jitter $\delta \vec{l}_t$. The period is changed to 10 hours with sampling time of 0.5 s.

Since the requirements in bandwidth are expressed in terms of Amplitude Spectral Density (ASD) (See Appendix E), let us perform a spectral analysis of the noise.

The goal bandwidth is 20 μ Hz< f < 1 Hz, meaning that, by Nyquist's law, the minimum sampling frequency is $f_{sample} = 2$ Hz, which is compatible with the on-board computer frequency of $f_{ref} = 10$ Hz [14], while for the characterization of an entire period the sampling time should be

$$T = \frac{1}{20 \cdot 10^{-6} \text{ Hz}} = 50000 \text{ s} = 13 \text{ h} 53 \text{ m} 20 \text{ s}$$

Since, on the other hand, the required bandwidth is 0.1 mHz < f < 0.1 Hz, we can relax the last requirement and assume a measurement period of T = 10 h. This is also the recommended span, in order to keep the experiment on hold for 14 h every day for reorientation and data upload to Earth [21].

In Appendix E, information about the spectral analysis of a random noise and the definition of its Power Spectral Density (PSD) is exposed. In summary, since the measurement bandwidth has a maximum frequency F = 1 Hz, the one-sided PSD S_{xx} of a zero-mean random signal with standard deviation σ_x is ¹

$$\mu(S_{xx}) = \frac{\sigma_x^2}{F} \tag{5.50}$$

or $\mu(S_{xx}) = \sigma_x^2 \cdot 1/\text{Hz}$ for F = 1 Hz. We assume, therefore that the resulting white noise is *band-limited* between 0 and *F* (or that the power of the noise is 0 for $f \ge F$), as long as we agree, for the rest of the thesis, that the sampling frequency is $f_{samp} = 2F = 2$ Hz [47].

We can expect, therefore, that the ASD, or $S_{xx}^{1/2}$, will have a zero-mean standard deviation of

$$\sigma_{asd} = \sqrt{\mu(S_{xx})} = \sigma_x.$$

Using the periodogram() function in Matlab, the single-sided ASD of the thrust and torque jitters obtained is shown in Figure 5.20. Information about the periodogram() function can be found in Appendix F.



Figure 5.20: Sample amplitude spectral density of Figure 5.19 for 10 hours observations at upper frequency F = 1 Hz. *left:* thrust noise $\delta \vec{f}_t$, *right:* torque noise $\delta \vec{l}_t$.

The on-bandwidth frequency domain control gains designed in Chapters 7 and 8 (introduced in Chapter 6) are based on the ASD of the thruster jitter chosen at 3σ levels and band-limited at 1 Hz.

5.13. LINEARIZATION AND VALIDATION OF THE DYNAMICS

Let us consider all the vectorial quantities in the same reference frame so that no transformation matrices are required (or $C_{s/i} = \mathbb{1}^{3\times 3}$) and let us make the following assumptions:

- the required S/C angular velocities and accelerations are small: $\vec{\omega} \ll 1$, $\vec{\omega} \ll 1$, as calculated in Chapter 4;
- the required angular velocities and accelerations of the telescopes, which are proportional to the breath angle rates (See Equations (5.36)), are also small $\vec{\omega}_{0i} \propto \dot{\alpha}/2 \ll 1$, $\dot{\vec{\omega}}_{0i} \propto \ddot{\alpha}/2 \ll 1$ as shown in Section 5.11;
- the TM displacements and velocities within their assemblies are kept by requirement at $\vec{r}_i \simeq 0$, $\dot{\vec{r}}_i \simeq 0$;
- the angular velocities of the TM are also kept by requirement at $\vec{\omega}_i \simeq 0$.

 $^{{}^{1}\}mu(S_{xx})$ is the symbol used for the mean of S_{xx} . It is calculated through the mathematical expectation $E[S_{xx}]$.

The dynamics equations can be linearized by removing the terms where a multiplication of the above mentioned small values occur.

The translational dynamics of the S/C of Equation (5.1) are already linear. Substituting a placeholder variable φ in Equation (5.2) such that

$$\dot{\boldsymbol{\varphi}} = \vec{\omega},\tag{5.51}$$

the attitude dynamics can be linearized as:

$$\ddot{\boldsymbol{\varphi}} = \vec{\omega} \simeq I_{sc}^{-1} \left(\vec{l}_t + \vec{l}_{d0} \right) \tag{5.52}$$

or simply getting rid of the gyroscopic momentum term.

The linearization of the TM dynamics removes all of the centripetal and Coriolis effects. By substituting Equation (5.52) to $\dot{\vec{\omega}}$, Equation (5.45) becomes:

$$\begin{aligned} \ddot{\vec{r}}_{i} &\simeq \frac{1}{m_{i}} \vec{f}_{i} + \vec{d}_{i} - \frac{K_{i}}{m_{i}} \vec{r}_{i} - \frac{1}{m_{sc}} \vec{f}_{t} - \vec{d}_{0} - \dot{\vec{\omega}} \times \vec{r}_{0i} - \dot{\vec{\omega}}_{0i} \times (\vec{r}_{0i} - \vec{r}_{hi}) \\ &\simeq \frac{1}{m_{i}} \vec{f}_{i} + \vec{d}_{i} - \frac{K_{i}}{m_{i}} \vec{r}_{i} - \frac{1}{m_{sc}} \vec{f}_{t} - \vec{d}_{0} - I_{sc}^{-1} \left(\vec{l}_{t} + \vec{l}_{d0} \right) \times \vec{r}_{0i} - \dot{\vec{\omega}}_{0i} \times (\vec{r}_{0i} - \vec{r}_{hi}). \end{aligned}$$
(5.53)

Linearization of the TM rotational dynamics Equation (5.47) is also performed by substituting the placeholder φ_i and applying Equation (5.52):

$$\ddot{\boldsymbol{\varphi}} = \dot{\vec{\omega}}_i \simeq I_i^{-1} \left(\vec{l}_i + \vec{l}_{di} \right) - I_i^{-1} K_{\varphi i} \boldsymbol{\varphi}_i - I_{sc}^{-1} \left(\vec{l}_i + \vec{l}_{d0} \right) - \dot{\vec{\omega}}_{0i}.$$
(5.54)

While Equation (5.1) is an application of Newtonian dynamics in an inertial reference frame that needs no validation, the linearized dynamics of Equations (5.2), (5.45) and (5.47) can provide a benchmark against all the dynamics equations found in literature:

- Equation (5.52) is used by S.-F. Wu and D. Fertin² [23] to describe the attitude dynamics of LISA Pathfinder.
- Equations (5.53) and (5.54) are also described in the same paper. Since LISA Pathfinder does not have a steering telescope, the $\vec{\omega}_{0i}$ term does not appear there. The disturbance terms do not appear in the time-dependent version but they can be found in the frequency-domain transformation of the same equation. The matrix terms K_i/m_i and $I_i^{-1}K_{\varphi i}$ are substituted by the terms $\omega_{j_i}^2$ (stiffness) of a diagonal matrix, with $j = \{x, y, z, \theta, \eta, \gamma\}$ (representing the matrix components 1,1, 2,2, and 3,3) and i = 1,2 as used in this thesis. M. Armano et al. (2016) [45] uses the same notation.
- A version of Equations (5.53) and (5.54) applied to LISA are provided by P. F. Gath [22]. In this version, K_i/m_i and $I_i^{-1}K_{\varphi i}$ do also appear in the frequency-domain dynamics in the form of matrix Ω^{-3} . In the paper the equations are bound together, with the masses m_{sc} , m_i and inertias I_{sc} , I_i collected in a mass-inertia matrix (called *M* in the paper). The term $\dot{\vec{\omega}}_{0i}$ appears in its scalar form: just the *z*-term is shown (since $\dot{\vec{\omega}}_{0i} = \ddot{\alpha}\hat{k}$).

The non-linear dynamics equations are not provided in publicly available literature. The work of S.-F. Wu and D. Fertin on LISA Pathfinder provides a clearer version than P. F. Gath et al. (or even G. Maghami and T. T. Hyde [37]) and an important starting point in order to retrieve the dynamics of LISA (and understand the process used in P. F. Gath et al.).

A side note should be dedicated to the S/C-TM interaction modeling using the stiffness. As we said, we use the term ω^2 in lieu of K/m or $I^{-1}K$, where K are stiffness matrices of either spring-mass or torsion pendulum models. As predicted and then demonstrated on LISA Pathfinder, $\omega^2 < 0$ is negative for all the DoF where it is applied, meaning that the spring-mass system is Lyapunov unstable [37]. This comes as no surprise, as S/C self-gravity and electrostatic attractions are pull-forces and therefore the TM tends to be knocked off their equilibrium positions for $\vec{r}, \varphi \neq 0$.

 K_i and $K_{\varphi i}$ need to be experimentally estimated after launch, therefore no assumptions can be made on their values. For LISA Pathfinder, $\omega^2 = -525 \pm 30 \times 10^{-9} \text{ rad}^2/\text{s}^2$ was measured in the interferometric measurement direction for both the TM [37].

³Both P. F. Gath and S.-F. Wu and D. Fertin show the term as $R = \frac{F}{s^2 + \omega^2}$, which is the Laplace transform of $\ddot{r} = -\omega^2 r + f$, representing a mass-spring system with an applied force per unit mass f and stiffness ω^2 . The matrix $\Omega = \text{diag}\left(1/(s^2 + \omega_x^2), 1/(s^2 + \omega_y^2), ..., 1/(s^2 + \omega_y^2)\right)$ is applied to a vector containing force and torque.

²This paper inspired to use the notations \vec{l} for the torque and \vec{r}_{0i} for the origin of the GRS axes.

5.14. CONCLUSIONS ON SYSTEM DYNAMICS

In this long chapter we have introduced the dynamics equations of the 19 DoF involved in the LISA system. Spacecraft translational and rotational dynamics have been derived through classical Newtonian and Eulerian physics, with the involvement of thrust and disturbance forces and torques.

The disturbances due to SRP have been modeled by assuming a diffuse reflectivity of the solar panels with an absorptivity coefficient of $R_{abs} = 0.14$ based on LISA Pathfinder specifications. For LISA, with an exposure of -30° , the force shows a magnitude of $f_{srp} \approx 85.367 \ \mu\text{N}$.

The force equation has been validated by showing that an absorptivity set to $R_{abs} = 1$ would exert exactly half of the estimated force than a perfectly reflective solar sail with the same area and facing the Sun with an exposure of 0° [3].

The force and torque due to SRP has been estimated for the S/C shape provided in Chapter 3 and data on orientation and position calculated in Chapter 4. Using the thrust configuration matrix *A* and the relations of Chapter 3, a preliminary study of the thrust efforts has been performed, showing a required capability of \sim 43 μ N for thrusters 1, 4, 7 and 10, to counteract both force and torque, suggesting that the model can be improved.

The attitude representation through Euler angles has been exposed and handy equations to calculate their speed and accelerations have been derived and verified for the current model.

The non-linear dynamics of the TM coupled with the S/C and the rotation of the telescopes has been derived. The angular rotations and accelerations $\dot{\alpha}$ and $\ddot{\alpha}$ associated with the steering mechanism of the telescope are used, as the telescope system has not been modeled.

The noise filters associated to the thruster jitters have been preliminarily identified as white gaussian noise associated to a thrust confidence of 1% at 3 σ , resulting in thrust force noises with standard deviations of $\sigma = (0.0642, 0.0809, 0.2160) \ \mu\text{N}$ and thrust torque noise with standard deviations of $\sigma = (0.3592, 0.1728, 0.1193) \ \mu\text{Nm}$. At a bandwidth of $20 \cdot 10^{-6} \div 1$ Hz, (max frequency F = 1 Hz) the ASD has the same standard deviation of $\sigma_{asd} = \sigma / \sqrt{F} = \sigma \text{ Hz}^{-1/2}$.

Finally, the derived dynamics equations have been linearized, showing a correspondence to the ones provided in literature for both LISA and LISA Pathfinder, thus validating in part the results.

6

ABOUT THE CONTROL SYSTEM

6.1. INTRODUCTION TO THE CONTROL SYSTEM

In the previous chapters we have derived the 19 non-linear DoF dynamics. In this chapter we overview the general strategy to effectively control them so that the science requirements are met during the mission lifetime. To this end, a classic *linear feedback control loop* is introduced, whose purpose is to generate the actuation efforts u(t) given the current state x(t) of a certain degree of freedom and its goal value r(t).

The problem faced in the framework of this thesis is the design of the control block, identified as the frequency dependent gain G(s) (a transfer function). The technique involved in the process must offer a certain degree of advantages as well as being capable of meeting the tight LISA sensitivity requirements.

Dynamic inversion aided by a looping process such as H_{∞} has been already demonstrated to achieve control goals on the linearized version of the LISA dynamics [22]

Quantitative Feedback Theory (QFT) offers a very promising alternative to H_{∞} and it is introduced here as the design technique of choice. It was already applied to the linearized LISA Pathfinder dynamics [23].

6.2. THE LINEAR FEEDBACK CONTROL LOOP

Let us introduce some basic concepts about the control model used in this thesis. The *single-input-single-output* linear feedback control loop schematics are shown in Figure 6.1: the frequency domain reference input R(s) ($s = j\omega$ [rad/s] being the complex frequency) are compared to the frequency domain state variable X(s) and their difference is multiplied by a transfer function *G* to obtain the control variable U(s) [48].





P is a transfer function representing the frequency domain behavior of the dynamics such that

$$X(s) = U(s)P(s). \tag{6.1}$$

Since the frequency domain representation of the dynamics, in many cases, is not linear, this kind of control loop can only be applied to linearizable problems within a certain degree of accuracy.

Two sources of noise are introduced in the system: the control noise D_1 and the measurement noise D_2 .

6.3. The sensitivity functions

Let us recall a few definitions about the frequency domain transfer functions [48]: the product L of G and P

$$L(s) = G(s)P(s) \tag{6.2}$$

is called *open-loop* transfer function, and it represents the relation between the error X(s) - R(s) and the state variable X(s) when no noise is introduced ($D_1 = 0, D_2 = 0$).

The closed loop transfer function T_r :

$$T_r = \frac{X}{R} = \frac{L}{L+1} \tag{6.3}$$

is the relation between X(s) and R(s) when no noise is introduced. The inverse of Equation (6.3) is

$$L = \frac{T_r}{T_r - 1}.\tag{6.4}$$

The *sensitivity functions* allow to quantify the influence of the disturbances on the control loop. These are [22]:

• Influence of control noise on output:

$$S_x = \frac{X}{D_1 P} = \frac{1}{L+1}$$
(6.5)

· Influence of control noise on dynamics:

$$S_u = \frac{U}{D_1} = \frac{1}{L+1}$$
(6.6)

• Influence of measurement noise on output:

$$T_x = -\frac{X}{D_2} = \frac{L}{L+1}$$
(6.7)

· Influence of measurement noise on dynamics:

$$T_u = -\frac{UP}{D_2} = \frac{L}{L+1}$$
(6.8)

and they are calculated by setting the reference commands to R = 0. The output *X* and commands *U* are the errors w.r.t. their ideal values.

By setting a maximum allowable value for their magnitudes $|X|_{max}$ and $|U|_{max}$ in Equations (6.5÷6.8) the *L* function needs to satisfy $|T| \le |T_x|$, $|T| \le |T_u|$, $|S| \le |S_x|$, $|S| \le |S_u|$ for different values of the complex frequency $s = j\omega$ to also satisfy $|X| \le |X_{max}|$ and $|T| \le |T_{max}|$, due to their proportionality.

6.4. S AND T RELATIONS ON THE PHASE-MAGNITUDE PLOT

It is clear now that two types of transfer functions are determined: the closed loop transfer functions T of Equations (6.3),(6.7) and (6.8) and the sensitivity functions S of Equations (6.5) and (6.6).

Inverting the relations we obtain, respectively [48],

$$L = \frac{T}{1 - T} \iff \frac{1}{L} = \frac{1}{T} - 1 \tag{6.9}$$

and

$$L = \frac{1}{S} - 1. \tag{6.10}$$

L, *T* and *S* are complex values. On the real-imaginary plane, a complex value $C \in \mathbb{C}$ is defined as a set of 2-dimensional Cartesian coordinates *x* and *iy*, or, typically when dealing with harmonical phasors, the polar coordinates *magnitude* |C| and *phase* <u>/C</u>. Transcribing relations (6.9) and (6.10) onto the real-imaginary plane, Figure 6.2 is obtained. Note that the magnitude of an inverse complex value |1/C| can be written as



Figure 6.2: Graphical relation between the complex function L = |L|/L and the complex values of constant magnitude (*left*) T = |T|/T = L/(1 + L) and (*right*) S = |S|/S = 1/(1 + L)

1/|C| as demonstrated by the following:

$$\left|\frac{1}{C}\right| = \left|\frac{C^*}{CC^*}\right| = \frac{1}{CC^*}\sqrt{CC^*} = \frac{1}{\sqrt{CC^*}} = \frac{1}{|C|}.$$
(6.11)

The * sign represents the complex conjugate operation. Remember also, that [48]

$$\frac{1}{C} = -\underline{C}.$$
(6.12)

From Figure 6.2 the polar coordinates of the open-loop transfer function *L* can be derived. For the case of Equation (6.9), let us consider a phase variable α such that

$$\alpha = -/\underline{T}.\tag{6.13}$$

The Cartesian coordinates of 1/T are calculated as

$$x_t = \frac{1}{|T|} \cos \alpha, \tag{6.14}$$

$$y_t = \frac{1}{|T|} \sin \alpha; \tag{6.15}$$

and the polar coordinates of *L* can be derived accordingly as:

$$|L| = \frac{1}{\sqrt{(x_t - 1)^2 + y_t^2}} \tag{6.16}$$

and

$$\underline{L} = -\operatorname{atan2}\left(y_t, x_t - 1\right). \tag{6.17}$$

The same can be done for Equation (6.10) by setting α as:

$$\alpha = -\underline{S}.\tag{6.18}$$

Analogously to T, the Cartesian coordinates of 1/S are calculated as

$$x_s = \frac{1}{|S|} \cos \alpha \tag{6.19}$$

and

$$y_s = \frac{1}{|S|} \sin \alpha \tag{6.20}$$

and, using Figure 6.2 as reference, the polar coordinates of Magnitude-phase coordinates of L are derived, respectively, as

$$L| = \sqrt{(x_s - 1)^2 + y_s^2} \tag{6.21}$$

and

$$\underline{L} = \operatorname{atan2}\left(y_s, x_s - 1\right). \tag{6.22}$$

One extremely important result from this derivation is the *Nichols* chart of Figure 6.4: it is obtained by evaluating Equations (6.16) and (6.17) (solid line) for constant values of |T| and Equations (6.21) and (6.22) (dashed line) for constant values of |S| using α as a parameter, for

$$0 \le \alpha < 2\pi$$

and by plotting the curves on a phase-magnitude plane (/L on the x-axis and |L| on the y-axis).



Figure 6.3: Phase-magnitude plot of *L* for constant values of |T| (solid line) and |S| (dashed line). The solid lines are the contours of a Nichols Chart.

The Nichols plot is repeated modularly every 360° , or $|L|(\underline{/L}) = |L|(\underline{/L} + 2\pi)$. Note that, in this context

$$|C| [dB] = 20 \log_{10}(|C|).$$
(6.23)

6.5. DESIGN USING QUANTITATIVE FEEDBACK THEORY

The goal is to design *G*. This can be done indirectly, when *P* is known, by designing *L*. QFT is a design technique for *L* that exploits the Nichols plot relations (6.13-6.22) to tune the response of *L* at various frequency values [48, 49].

In summary, taking into account the contours in Figure 6.4, at each frequency ω_i (angular frequency in rad/s), the complex value $L(j\omega_i)$ needs to be located on the phase-magnitude plane

- Below or outside the constant- $|T_i|$ contour (where $T_i = T(j\omega_i)$)
- Above or outside the constant- $|S_i|$ contour (where $S_i = S(j\omega_i)$).

where $|T_i|$ and $|S_i|$ are the thresholds of a desired output.

The ideal open-loop function is located exactly on one of the two contours at each ω [48]. On the phasemagnitude plane, *L* is a parameter curve as function of ω .

QFT is mostly suitable when the plant *P* is uncertain. The uncertainty extends to *L*, that can assume a whole span of values within a contained area (called *template*) on the phase-magnitude plot. Each point on

the area, on the other hand, must satisfy the above conditions, and that is easily resolvable by graphically identifying the position of the confined area that lies as close as possible to the boundaries (ideally tangentially) while keeping all the points outside (below or above) the respective contours [48].

QFT is too large of a topic to be summarized in just this chapter (it can take the whole size of a book to explain in details all the characteristics and strengths). In the next chapters we are going to explore the use of the technique for the design of the LISA DoF control gains, providing practical examples. The reader is encouraged to get acquainted on external literature [48, 49].

6.6. STABILITY ON THE NICHOLS PLOT

The open loop transfer function (or any transfer function in general) is a ratio between binomial products:

$$L(s) = k \frac{(s-z_1)(s-z_2)(s-z_{...})(s-z_n)}{(s-p_1)(s-p_2)(s-p_{...})(s-p_m)}.$$
(6.24)

 $k \in \mathbb{R}$, k > 0 is the *gain*; z_i and p_j , for i = 1, 2, ..., n and j = 1, 2, ..., m are called, respectively, zeros and poles. They can be either complex or real, as long as for each complex z_i , a complex $z_{l\neq i}$, where $z_l = z_i^*$ exists (complex conjugate).

A few stability issues ensue when defining an open-loop transfer function [48]:

- The real poles of the associated closed loop transfer function *T* have to be negative for the time-domain solution of the control loop to converge.
- For the system to be proper, $m \ge n$ must hold, otherwise the total order of *L*, would be n m > 0, resulting in the time-domain solution to exponentially raise to infinity.
- The singularities for *T* are located at L = -1, or, in polar coordinates |L| = 1 = 0 dB and $\underline{L} = (2q+1)\pi$, $q \in \mathbb{N}$, therefore it is imperative that *L* never assumes this value at any frequency.

The first point in particular is not readily satisfiable by simply looking at the open-loop function: even if all the zeros and poles were negative in *L*, the closed-loop transfer function *T* may still have positive poles. The solution is provided by the shape of *L* on the Nichols-plot: on the phase-magnitude plot *L* appears as a parametric curve of *s*. Depending on weather the function has 0, 1 or two poles at the origin ($p_i = 0$), the $s = j\omega = 0$ point is located at phase /L = 0, $/L = -90^\circ$ or $/L = -180^\circ$. The curve then tends asymptotically to the $l \cdot 90^\circ$ vertical line, between each pole and zero, where *l* is the local order of *L*.

Stability is obtained, for an all negative poles and zeros function, when the vertical line at $\underline{L} = -180^\circ$, above |L| = 0 dB is crossed by *L* an even number of times [50].

The second point simply requires *L* to tend to $|L| \rightarrow r$, where $r \in \mathbb{R}$ for $\omega \rightarrow \infty$. And for the third point, it is simply sufficient to keep the *L* function away from the (-180°, 0 dB) point, by, e.g. adding a stability margin (generally |T| = 3 dB) that *L* must never cross.

Figure 6.4 shows three examples of stable *L* functions with all negative zeros and poles, respectively, L_1 , L_2 and L_3 , where

$$L_1 = 58 \frac{s+1.65}{s^2(s+9.7)}; \quad L_2 = 22 \frac{(s+0.51)^2}{s^2(s+0.05)(s+5.9)}; \quad L_3 = 20 \frac{(s+0.54)^2}{(s+0.05)^3(s+5.6)}.$$
 (6.25)

A stability margin of |T| = 3 dB is added as an example. L_1 and L_2 start at a phase of $\underline{/L} = -180^\circ$ (for $|L| \rightarrow \infty$), meaning that they have two poles at the origin. L_3 starts at $\underline{/L} = 0^\circ$, meaning that is has no poles at the origin. All the functions have an order of -2, as suggested by the asymptotical behavior at $\omega \rightarrow \infty$ with the phase reaching -180° .

 L_1 has only got a stability pole at 1.65 rad/s and a stability zero at 9.7 rad/s to sway away from the |T| = 3 dB margin. At $\omega \rightarrow 0$ the function is already on the right of the (-180°) line (dotted red line), therefore it crosses it 0 times, which is an even number.

 L_2 has a pole added at 0.05 rad/s, which at $\omega \rightarrow 0$ translates to a crossing. A double zero at 0.51 rad/s ensures both a second crossing of the red dotted line and the comply to the stability margin. A pole at 5.9 rad/s returns the function to a -2 order.

 L_3 has a triple pole at 0.05 rad/s, therefore a double zero at 0.54 rad/s must be added to meet the stability boundary requirement and to obtain 2 crosses of the red dotted line. A final pole a 5.6 rad/s confers an order -2.

Another thing to keep in mind is that the magnitude |L| increases or decreases with a trend of $20l/\log_{10}(\omega)$ dB/decade (power of ten) where *l* is the local order of *L* [48].



Figure 6.4: Example of the three stable open-loop transfer functions from Equation (6.25) on the Nichols chart.

6.7. LIMITATIONS OF DESIGN

You can notice that, in their general form, the closed loop functions *T* and the sensitivity functions *S* are not independent of each other [48]:

$$T + S = \frac{L}{1+L} + \frac{1}{1+L} = 1.$$
(6.26)

When defining the boundaries $|T| \le |T_x|$, $|T_u|$ and $|S| \le |S_x|$, $|S_u|$ one must, therefore, be careful to design them so that they always allow a solution for *L*: on the phase-magnitude plot this means that for a certain frequency, the boundaries must allow a point to be outside and below the |T| bound and outside and above the |S| bound. If the two areas do not overlap anywhere, only one of the two boundaries can be satisfied at once.

Apart from the closed-loop transfer function T_r (Equation (6.3)) the *T* functions are dependent on the readout noise D_2 and the *S* functions are dependent on the dynamics noise D_1 . Following from the above reasoning, for a certain value of D_1 , there is a maximum value of D_2 allowable and vice-versa.

Philosophically speaking, one can assert that *a control system can only be as good as the measurements*, or that the precision of the actuations is very much related to the precision of the observations.

6.8. CONTROL OF THE LISA SYSTEM

The LISA system is comprised of 19 DoF [22]:

- Three DoF $\vec{R} = (X, Y, Z)$ of the S/C displacement in the inertial reference frame;
- Six DoF $\vec{r}_1 = (x_1, y_1, z_1)^{\tau}$ and $\vec{r}_2 = (x_2, y_2, z_2)^{\tau}$ of the respective TM displacements within their assemblies;
- Three DoF $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^{\tau}$ of the S/C attitude (Euler angles);
- Six DoF, $\boldsymbol{\varphi}_1 \in \mathbb{R}^{3 \times 1}$ and $\boldsymbol{\varphi}_2 \in \mathbb{R}^{3 \times 1}$ for the TM orientations;
- The angle between the telescopes α also referred to as *breath angle*.

In the dynamics equations of Chapter 5 we have defined the actuation variables used to control the DoF:

• The thruster force \vec{f}_t acts on \vec{R} , \vec{r}_1 and \vec{r}_2 and its purpose is to compensate the non gravitational disturbances and keep the S/C in drag-free;

- The thruster torque \vec{l}_t acts on θ and as a disturbance on \vec{r}_1 , \vec{r}_2 , φ_1 and φ_2 , and its purpose is to steer the S/C and keep its nominal orientation w.r.t. the formation.
- The electrostatic suspension forces $\vec{f_1}$ and $\vec{f_2}$ acting, respectively and separately, on $\vec{r_1}$ and $\vec{r_2}$, needed to keep the TM from touching the walls of the GRS and to compensate for internal disturbances, such as the S/C self-gravity.
- The electrostatic suspension torques \vec{l}_1 and \vec{l}_2 , acting, respectively and separately, on φ_1 and φ_2 , whose purpose is to keep the faces of the TMs aligned to the GRS reference frame.
- Some sort of torque (ignored in this thesis) to steer the telescopes and maintain the breath angle α according to the formation geometry and acting as a disturbance on \vec{r}_1 , \vec{r}_2 , $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$.

The evolution of the Euler angles θ provides information about the disturbance acting on the S/C rotational dynamics, by measuring the difference between the nominal Euler angles θ_c (*c* is for *command*) and the current θ .

The deviation from drag-free behavior of the S/C is provided by the displacement of the TM along the lines-of-sight, i.e., the value of x_1 and x_2 w.r.t. their rest position, and z_1 for the vertical axis [22].

Internal measurements of y_1 , y_2 , z_2 and the TM orientations φ_1 and φ_2 drive the actuation of the GRS electrostatic suspension (\vec{f}_1 , \vec{f}_2 , \vec{l}_1 and \vec{l}_2).

The breath angle α drives the actuation of the telescope steering mechanism.

We are not discussing the complex mechanics involved in the observation of the variables and their translation into DoF measurements, which generally involve a combination of precise interferometry, electrostatic force sensors and classical accelerometers [14, 17, 37]. We assume, on the other hand, that the technology is fully developed to allow the precision associated to the science requirements.

According to the above mentioned sets of variables, four subsystems can be defined:

- The S/C attitude control system, whose state variables (X) are the Euler angles θ and the control variables (U) are the thruster torque l
 *i*_t;
- The drag-free control system, whose state variables are $\mathbf{x}_{df} = (x_1, x_2, z_1)^{\tau}$ and whose control variable is the thruster force \vec{f}_t ;
- The GRS electrostatic suspension system, whose state variables are all the other TM related DoF and the control variables are the electrostatic suspension forces and torques;
- The telescope steering mechanism that control α .

Apart from the telescope steering mechanism, that is only peculiar of LISA, the whole of the other three system comprise what is generally referred to as DFACS.

Recalling Equations (5.52), (5.53) and (5.54), due to the torque \vec{l}_t appearing both as control variable and disturbance, and the electrostatic suspension force \vec{f}_i , i = 1, 2, acting on the drag-free DoF as well, we must face the fact that these three systems cannot be separated from each other [23]. We can, on the other hand, study these systems separately by applying a *separation of dynamics* [22]. This method assumes that all the systems that are not the one being studied, act nominally.

6.9. About the design and simulation

Due to time constraints we are only able to study the attitude control and the drag-free control. The design process is described in the next chapters. **The study is specifically targeted at S/C**₁ (but it can be extended to the other two S/C due to their similarities).

The system is simulated in Simulink[®] [24]. In order to obtain the most from the software while minimizing the computation time, the simulation is run in *rapid accelerator mode* [51]. The ability of Simulink to solve recursive *algebraic loops* [46] is not available in this mode. Separation of the dynamics solves the problems relative to recursion. Specifically, the thruster noise $\delta \vec{f}_t$ and $\delta \vec{l}_t$ defined in Section 5.12 cannot be calculated on the spot from the thrust itself: this would generate a loop in which the value of the actuation variable (\vec{l}_t , \vec{f}_t) is used to calculate the disturbance, which is then fed into the control loop to calculate the value of the actuation variable. As a solution, the thruster noise is introduced as a white noise with the mean standard deviation estimated in Table 5.2. Since the on-board computer works at 10 Hz [14], the simulation is performed with a time-step of 0.1 s. The data is retrieved every 0.5 s to downsize the output variable while still being able to analyze the frequency domain response with a maximum bandwidth of F = 1 Hz.

6.10. CONCLUSIONS ON THE CONTROL SYSTEM

In this Chapter we introduced the basic concepts for the design of a linear control system.

In particular we showed how to calculate the sensitivity functions in frequency domain based on the requirements and the disturbance levels and how to generate boundaries on the Phase-Magnitude plane of the open-loop transfer function in which the sensitivity functions are satisfied.

We then showed how to ensure stability of the system by simply looking at the shape of the open-loop transfer function on the Phase-Magnitude plane.

The control on LISA S/C was explained. The system can be divided into four subsystems: the attitude control, using the thrust torque to control the Euler angles, the drag-free control, using the thrust force to control the displacements of the TM about x_1 , x_2 , and z_1 , the GRS, using the electrostatic suspension system to control displacement along the remaining DoF and orientation of the TM and the telescope steering mechanism, controlling the breath angle.

The simulation of the control is planned only on S/C_1 . Limitations of the simulation process have been exposed, particularly the necessity to separate the dynamics and avoid algebraic loops in Simulink.

7

ATTITUDE CONTROL SYSTEM

7.1. INTRODUCTION TO THE ATTITUDE CONTROL SYSTEM

The S/C attitude, measured with the Euler angles, as reported in Section 5.7, is controlled by the on-board thrusters against the SRP torque (Section 5.5) and the thruster induced torque jitter (Section 5.12). The requirements are specified as [14]:

• Direct Component (DC) error:

$$\delta\theta_i \le 10^{-8} \text{rad} \tag{7.1}$$

for i = 1, 2, 3;

· Bandwidth noise spectrum on all the Euler angles:

$$ASD_{\theta} \le 10^{-8} \frac{\text{rad}}{\sqrt{\text{Hz}}} \sqrt{1 + \left(\frac{3 \text{ mHz}}{f}\right)^4}.$$
(7.2)

In this chapter we are going to design and test the control loop for the attitude θ of S/C₁. The sensitivity functions need to be derived and the open-loop transfer functions are to be determined through the QFT design technique.

We consider DC the disturbance acting outside the measurement band with period T > 10 hours (the measurement time span specified in Section 5.12). This is the SRP torque for the attitude. The noise acting in the bandwidth 20 μ Hz< f < 1 Hz, is the torque due to thrust jitter. Separation of the dynamics is performed.

7.2. LINEARIZED DYNAMICS INVERSION

Let us recall, from Sections 5.3 and 5.8, Euler's equation for the S/C rotational dynamics and the dynamics of the Euler angles, respectively given by Equations (5.2)

$$\dot{\vec{\omega}} = I_{sc}^{-1} \left(\vec{l}_t + \vec{l}_{d0} - \vec{\omega} \times I_{sc} \vec{\omega} \right)$$
(5.2)

where I_{sc} is the inertia matrix, \vec{l}_t is the thruster torque, \vec{l}_{d0} is the disturbance, and Equation (5.30)

$$\ddot{\boldsymbol{\theta}} = \dot{N}(\boldsymbol{\theta})\vec{\omega} + N(\boldsymbol{\theta})\vec{\omega}$$
(5.30)

where $N(\theta)$ is provided in Equation (5.28) and $\dot{N}(\theta)$ is expanded in Equation (5.31). Combining Equations (5.2) and (5.30) the following relation is obtained:

$$\ddot{\boldsymbol{\theta}} = \dot{N}(\boldsymbol{\theta})\vec{\omega} + N(\boldsymbol{\theta})I_{sc}^{-1}\left(\vec{l}_t + \vec{l}_{d0} - \vec{\omega} \times I_{sc}\vec{\omega}\right).$$
(7.3)

The analytical solutions from Sections 4.4 and 5.8 show that $|\omega|, |\dot{\theta}| \ll 1$, which, according to Equation (5.31), translate to $\dot{N}(\theta) \propto \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3 \ll 1$. Hence Equation (7.3) can be linearized by eliminating the second order terms, obtaining the simplified form,

$$\ddot{\boldsymbol{\theta}} \simeq N(\boldsymbol{\theta}) I_{sc}^{-1} \left(\vec{l}_t + \vec{l}_{d0} \right). \tag{7.4}$$

Notice that Equation (7.4) contains a linearizing matrix,

$$H_{\theta} = N(\theta) I_{sc}^{-1}, \tag{7.5}$$

a control term

$$\boldsymbol{u}_{\theta} = \boldsymbol{l}_t \tag{7.6}$$

and a disturbance term

$$\boldsymbol{d}_{\theta} = H_{\theta} \, \vec{l}_{d0}. \tag{7.7}$$

By means of Non-linear Dynamics Inversion (NDI), we can treat the term

$$\boldsymbol{\nu}_{\theta} = H_{\theta} \, \boldsymbol{u}_{\theta} \tag{7.8}$$

as a virtual command, thus releasing the linearized system from uncertainties[48]:

$$\ddot{\boldsymbol{\theta}} = \boldsymbol{v}_{\theta} + \boldsymbol{d}_{\theta}. \tag{7.9}$$

Let us now perform the Laplace transform of Equation (7.9), obtaining (for $\hat{\theta}(t=0) = 0$):

$$s^2 \Theta = V_{\theta} + D_{\theta} \iff \Theta = \frac{1}{s^2} \left(V_{\theta} + D_{\theta} \right).$$
 (7.10)

Here Θ , V_{θ} and D_{θ} are the frequency domain Laplace transforms of, respectively θ , v_{θ} and d_{θ} . *s* is the imaginary frequency variable, defined as

$$s = j\omega \tag{7.11}$$

where ω is the angular frequency in $[\omega] = rad/s$.

Recalling the linear feedback control loop of section 6.2, it becomes clear, after determining that Θ is the state variable of the attitude in frequency domain, that V_{θ} is the control variable U(s), D_{θ} is the dynamics disturbance D_1 and, therefore, the linearized plant is

$$P = \frac{1}{s^2}.$$
 (7.12)

Moreover, the system can be treated and a <u>Single-Input-Single-Output (SISO</u>) control problem, since the plant is a scalar.

7.3. Low Frequency Disturbance

In Section 5.5 we have calculated the SRP induced torque \vec{l}_{srp} on the three S/C with Equation (5.10). This torque acts on the attitude dynamics as a disturbance at low frequency. Using Equation (7.7) and setting $\vec{l}_{srp} = \vec{l}_{d0}$, i.e. assuming that the only disturbance is the SRP induced torque on the ideally oriented S/C, we obtain the induced acceleration on the Euler angles due to the SRP torque:

$$\boldsymbol{d}_{\theta} = H_{\theta} \vec{l}_{srp} = N(\boldsymbol{\theta}) I_{sc}^{-1} \vec{l}_{srp}.$$
(7.13)

The values of d_{θ} for the 4 year period are shown in Figure 7.1.

By analyzing the 4 years sample for S/C_1 , we find that the fundamental harmonic has frequency

$$\omega_o = 1.992384991 \cdot 10^{-7} \text{ rad/s}$$

(subscript *o* is for *orbit*) which is just slightly different from the nominal sidereal motion of [25]

$$\omega_e = 1.990983675 \cdot 10^{-7} \text{ rad/s}$$

due to the second order orbital behavior of the formation.

We can analyze the sinusoidal components by performing a discrete Fourier analysis. See Appendix D for more information. The discrete Fourier series of d_{θ} (SRP disturbance on $\ddot{\theta}$) is shown in Figure 7.2. This is the magnitude of the dynamics disturbance (called D_1 in chapter 6) in low frequency ($f < 20 \mu$ Hz).



Figure 7.1: SRP induced accelerations on the Euler angles d_{θ} on S/C₁ for the 4 years period, calculated by means of Equation (7.13) and the values for \vec{l}_{srp} calculated in Section 5.5.



Figure 7.2: Fourier series of d_{θ} for fundamental frequency $\omega_o \approx 1.9924 \cdot 10^{-7}$. Calculated numerically through the Matlab fft () function [52] (See Appendix D). The magnitude of the harmonic components is shown here.

For our SISO problem we have that Θ_i , i = 1,2,3 is the state variable in frequency domain (called X in Chapter 6). Let us call the sensitivity function of Equation (6.5) $S_{\theta_{lf}}$ in low frequency and notice that the linearized plant of Equation (7.12) holds a magnitude of

$$|P| = \frac{1}{\omega^2}.$$
 (7.14)

The low-frequency, in this thesis, represents the concept of DC w.r.t. the bandwidth. Therefore, in low frequency, we would like to keep the error below 10^{-8} rad as specified by Equation (7.1). Since, as shown in Figure 7.2, more than one component is important to the disturbance, we tentatively impose a margin factor of 4 and combine Equations (6.5), (7.1) and (7.14) to obtain the requirement on the sensitivity function *S* in low-bandwidth:

$$|\Theta_i| \le \frac{\delta \theta_i}{4} \iff |S| = \left|\frac{1}{1+L}\right| \le |S_{\theta_{lf}}| = \frac{\delta \theta_i \cdot \omega^2}{4|D_1|(\omega)} = \frac{10^{-8} \operatorname{rad} \cdot \omega^2}{4|D_1|(\omega)}.$$
(7.15)

The values of $|S_{\theta_{lf}}|$ are used later to calculate the boundaries on phase-magnitude plot for various values of $\omega < 2\pi \cdot 20 \cdot 10^{-6}$ rad/s (because $f = 20 \,\mu\text{Hz}$ is the minimum goal frequency).

The $|D_1|$ component at each frequency is chosen as the maximum among the three $\ddot{\theta}_i$, i = 1, 2, 3 at each evaluated frequency. As shown in Figure 7.2, $\ddot{\theta}_2$ is chosen at ω_o and $\ddot{\theta}_3$ is chosen at all the other $2^k \omega_o$ frequencies, k = 1, 2, 3, ... (every octave).

7.4. MEASUREMENT BANDWIDTH REQUIREMENTS

In Section 5.12 a noise model for the torque $\delta \vec{l}_t$ was tentatively identified, with the values of the standard deviation reported in table 5.2.

In order to add a margin to the design, we consider the disturbance on the torque at 3 standard deviations (3σ) of the ASD of $\delta \vec{l}_t$. The dynamics disturbance on the Euler angles D_{θ} is calculated by picking the largest value obtained by means of dynamics inversion. In order to save time we want to design a SISO control loop that is suitable for the three Euler angles, therefore, a single value D_1 will serve as the design limit of each component of D_{θ} :

$$|D_1| = \max|3H_\theta \boldsymbol{\sigma}[\delta \vec{l}_t]| \tag{7.16}$$

where H_{θ} is the linearizing dynamic inversion matrix calculated from Equation (7.5). The maximum is chosen among the three components and the time-series due to the time dependence of H_{θ} shown in Figure 7.3. D_1



Figure 7.3: The temporal variability of $3H_{\theta}\sigma[\delta \vec{l}_{t}]$ for a 4 years period. The red dot represents the value chosen for $|D_{1}|$ which is the maximum value for the θ_{1} noise in bandwidth.

is evaluated at:

$$|D_1| = 8.649762966 \cdot 10^{-10}$$
 rad s⁻²/ $\sqrt{\text{Hz}}$ at all frequencies.

Let us recall the limit imposed on the attitude jitter ASD_{θ} :

$$ASD_{\theta} = 10^{-8} \frac{\mathrm{rad}}{\sqrt{\mathrm{Hz}}} \sqrt{1 + \left(\frac{3 \mathrm{mHz}}{f}\right)^4}.$$
(7.2)

Again, Θ_i , i = 1,2,3 is the state variable in frequency domain (called *X* in Chapter 6). Let us call the sensitivity function of Equation (6.5) as S_{θ} . Equations (6.5), (7.2) and (7.14) can then be combined to obtain the requirement limit on sensitivity function S_x :

$$|\Theta_i| \le ASD_\theta \iff |S| = \left|\frac{1}{1+L}\right| \le |S_\theta| = \frac{ASD_\theta \cdot \omega^2}{|D_1|}.$$
(7.17)

The readout noise D_2 determines the requirements on the closed loop function for T_x , once again, substituting $|\Theta_i| = ASD_{\theta} = |X|$ in Equation (6.7) as:

$$|T| = \left|\frac{L}{L+1}\right| \le |T_{\theta}| = \frac{ASD_{\theta}}{|D_2|}.$$
(7.18)

We have not defined D_2 , yet, but, since S and T are related by Equation (6.26) the relation

$$S_{\theta} + T_{\theta} = 1 \tag{7.19}$$

must hold true. Using the variable Θ_i in lieu of *X* in Equations (6.5) and (6.7), and substituting T_x and S_x in Equation (7.19) we obtain the relation

$$\frac{\Theta_i}{D_2} = \frac{\Theta_i s^2 - D_1}{D_1} \Longleftrightarrow D_2 = \frac{\Theta_i D_1}{\Theta_i s^2 - D_1}$$
(7.20)

which, in a bandwidth that reaches F = 1 Hz, hence

$$s_{max} = 2\pi j \text{ rad/s}$$

and a goal to keep the disturbances at bay, i.e. making sure that

$$|\Theta_i s^2| \ll |D_1| \tag{7.21}$$

results in a cap for the readout noise of

$$|D_2|_{max} \simeq |\Theta_i| \le ASD_{\theta}. \tag{7.22}$$

This requirement for the read-out noise is applied so that the inequalities $|T| < |T_{\theta}|$ and $|S| < |S_{\theta}|$ can be both satisfied. On the Nichols chart this means that we can always find a point in the phase-magnitude plane that lies outside and above $|T_{\theta}|$ and outside and below $|S_{\theta}|$ (Section 6.7).

For convenience we simulate a white noise, i.e. with a constant amplitude at each frequency with a value of:

$$|D_2| = ASD_{\theta}(f = 1 \text{ Hz}) = 10^{-8} \text{ rad}/\sqrt{\text{Hz}} \text{ at } 3\sigma$$

which is simulated, accordingly to a maximum frequency bandwidth of F = 1 Hz, as a zero-mean random Gaussian variable with standard deviation

$$\sigma_{d2\theta} = \frac{10^{-8} \operatorname{rad}/\sqrt{\operatorname{Hz}}}{3} \sqrt{F} = \frac{10^{-8}}{3} \operatorname{rad}.$$
 (7.23)

Figure 7.4 depicts the noise and sensitivity function magnitudes used in the context of this analysis.



Figure 7.4: *left*: the ASDs of the dynamics and readout noise, respectively D_1 and D_2 and the limit ASD_{θ} on the output. $|D_1|$ is divided by the squared angular frequency ω^2 to match the dimension. *right*: the maximum magnitudes of sensitivity function S_{θ} and closed loop function T_{θ} associated with the noise levels.

From Figure 7.4, we see that $|S_{\theta}| \simeq -47.72804$ dB at $f \to 0$ Hz, and it raises by 40 dB/decade from f = 3 mHz on. $|T_{\theta}|$, on the other hand, lowers by -40 dB/decade until f = 3 mHz, and it settles to $|T_{\theta}| \simeq 0$ dB.

7.5. Shaping the open-loop function

With the limit on |S| and |T| defined we can now use QFT to generate an optimal open-loop function.

For the SRP control, Equation (7.15) is used to calculate the values for $|S_{\theta_{lf}}|$ for a few values of ω , starting from $\omega = \omega_{\rho}$ and then at each octave. The values for the first few frequencies are reported in Table 7.1.

Using Equations (6.21) and (6.22), the boundaries on the phase-magnitude plot of the open-loop function are calculated (Figure 7.5). For $|S| \gg 1$, the value of the magnitude of Equation (6.10) can be approximated to $|L| \simeq 1/|S|$, therefore the boundaries appear as almost constant horizontal lines on the Nichols plot.

For the boundaries in the measurement bandwidth we start at $f = 1 \mu$ Hz and sample $|S_{\theta}|$ and $|T_{\theta}|$ at each octave. Figure 7.6 shows the calculated levels. The 3 mHz double-zero on the S_{θ} function (and double-pole on the $T_{\theta}|$) happens between the 2¹¹ and 2¹² μ Hz bounds. On the left graph *L* is therefore almost constant until 2¹¹ μ Hz, where it has a magnitude of |L| = 46.913739 dB at -180°, and from 2¹² μ Hz on it starts to lower by

	ω $ S_{\theta_{lf}} $ [dB]		
	$\begin{array}{rcrr} \omega_o & -279.00895428 \\ 2\omega_o & -266.50714653 \\ 2^2\omega_o & -249.66762597 \\ 2^3\omega_o & -228.12371726 \\ 2^4\omega_o & -197.54088397 \end{array}$		
300	Low frequency bounds		
280	$2^0 \cdot \omega_o$		
200	$2^1 \cdot \omega_o$		
260	$2^2 \cdot \omega_o$		
<u>م</u> 240	$2^3 \cdot \omega_o$ -		
ت ع 220			
pn in 200	$2^4 \cdot \omega_o$ _		
ißen 180			
doc 160	$2^5 \cdot \omega_o$		
- - - - 140	$2^6\cdot\omega_o$		
Ope	$2^7\cdot\omega_a$		
120	$2^8 \cdot \omega_z$		
100	$-\frac{2^{9}}{2^{9}}$		
80	$2 \omega_0$		
60			
	-180 -150 -120 -90 -60 -30 0		
	Open-Loop phase °		

Table 7.1: Values of $|S_{\theta_{lf}}|$ for different octaves of $\omega_o \simeq 1.9924 \cdot 10^{-7}$ rad/s (fundamental frequency of LISA orbital motion)

Figure 7.5: The low-frequency *S*-boundaries on the phase-magnitude plane of *L*, calculated through Equations (6.21) and (6.22) for $|S_{\theta_{lf}}|$ of Table 7.1. The plot is symmetrical w.r.t. the -180° vertical line.

about 40 dB/decade until the boundaries wind at f = 0.05036 Hz or between $2^{15} \mu$ Hz and from $2^{16} \mu$ Hz. On the right graph *L* is contained within the |T| = 0 dB curve (see Figure 6.3 for reference). A |T| = 3 dB boundary is also added for stability (thick dotted curve).

The open-loop function *L* needs to satisfy these boundaries on the phase-magnitude plot. Let us recall that the approximative plant $P = 1/s^2$ as derived in Section 7.2, therefore,

$$L = \frac{1}{s^2}G\tag{7.24}$$

where *G* is the control transfer function. Since *G* must have a total order of 0 or lower, in order to start *L*(0) at $\underline{/L} = 0$, we would need to add a second-order zero at the origin and an additional two poles at other frequencies, risking to increase the complexity of the function. Moreover, according to Table 7.1, the optimal |L| magnitude needs to fall by >60 dB/decade up to $8\omega_o$ and then >100 dB/decade at $16\omega_o$. At $2^{11} \mu$ Hz, then, it should fall by only 40 dB/decade needing 2 additional zeros to raise the order before then. Since, by this point, it should not cross the -180° phase (for all negative poles and zeros and keep away from the |T| = 3 dB boundary, another zero and another pole must be added. At $f \to \infty$ it should at least have an order of -2. Keeping in mind these considerations and the actual boundaries, *G* is designed, with the resulting *L* shown in Figure 7.7.

The control transfer function has been evaluated as:

$$G = k \frac{(s-z_1)^2(s-z_2)(s-z_3)(s-z_4)}{(s-p_1)^2(s-p_2)(s-p_3)(s-p_4)}$$
(7.25)

with



Figure 7.6: *left:* the S_{θ} boundaries calculated from the values of $|S_{\theta}|$ from Equation (7.17) and Equations (6.21) and (6.22) for each octave $f = 2^{k} \mu$ Hz. The boundaries between $2^{0} \mu$ Hz $< f < 2^{11} \mu$ Hz are not shown because $|S_{\theta}|$ is almost constant in the interval. *right:* the T_{θ} boundaries calculated from the values of $|T_{\theta}|$ from Equation (7.18) and Equations (6.16) and (6.17) until $f = 2^{21} \mu$ Hz (>1 Hz). The boundaries approach closer and closer the |T| = 0 dB curve. For both, the stability boundary |T| = 3 dB is shown as a thick dotted line. The plots are symmetric w.r.t. the -180° vertical line.



Figure 7.7: The graphically evaluated *L* function (solid black line) of Equations (7.24) and (7.25). The bubbles \circ represent the evaluated low-frequency harmonics and the bullets \bullet represent the bandwidth frequencies starting at 2¹¹ µHz. The markers need to stay close to the respective boundaries, above and outside the |*S*| boundaries and below and outside the |*T*| boundaries. The latter intentionally not satisfied, because the looping is inside the blue U curve (|*T*| = 0 dB).

k = 0.5745913347

and the values of the zeros and poles shown in Table 7.2.

In the Figure 7.7, the bubbles (\circ) show the *L* function coordinates at the relative evaluation frequencies of $|S_{\theta_{l_f}}|$, and the bullets (\bullet) show the coordinates starting at $2^{11} \mu$ Hz in order to visualize their position w.r.t. the

Table 7.2: Zeros and poles of the *G* function of Equation (7.25).

Zeros [rad/s]	Poles [rad/s]
$z_{1} = -7.969539963 \cdot 10^{-6} (2^{nd} \text{ order})$ $z_{2} = -1.255891789 \cdot 10^{-4}$ $z_{3} = -1.657159151 \cdot 10^{-4}$ $z_{4} = -0.2206646689$	$p_{1} = -8.766493960 \cdot 10^{-7} (2^{nd} \text{ order})$ $p_{2} = -1.653679542 \cdot 10^{-6}$ $p_{3} = -0.02758308361$ $p_{4} = -1.125001995$

$|S_{\theta}|$ bounds.

One can notice that the *L* function is forced close to the $|S_{\theta}|$ boundaries without satisfying the $|T_{\theta}|$ boundary of 0 dB. The reason is that we prefer to keep the gain *k* low, as it affects the time-step required to run the simulation. The obtained value is low enough to satisfy the minimum 0.1 s required by the on-board computer without lowering it any further (k = 0.5745913347 is much less than actually needed). On the other hand, the overshoot is theoretically ≤ 3 dB (actually $|T|_{max} \approx 2.961175$ dB at $f \approx 0.06025596$ Hz). We have also decided to simulate the read-out noise with a 3σ margin, therefore the final evaluation is left to the numerical simulation.

Figure 7.8 shows the expected behavior of the error output in bandwidth frequency, with the reaction



Figure 7.8: Theoretical limit behavior of linear feedback control output $|\Theta|$ (for reference value set to R = 0) in bandwidth due to the influence of $|D_1|$ (green dash-dot line) and $|D_2|$ (blue solid line), calculated, respectively, through the inverse of Equations (6.5) and (6.7). The red-dashed line represents the design upper limit ASD_{θ} .

to measurement noise overshooting the noise limit ASD_{θ} at $f \simeq 0.06025596$ Hz by $0.4062377 \cdot 10^{-8}$ rad/ $\sqrt{\text{Hz}}$. The combined jitter and read-out noise should result in an error following the higher curve at each frequency.

For verification, a preliminary simulation is performed using an actual linear plant $P = 1/s^2$, introducing zero-mean random Gaussian disturbances d_1 and d_2 and keeping the reference commands r(t) = 0. d_1 is the dynamics noise, with a standard deviation of

$$\sigma_{d1} = 2.883254322 \cdot 10^{-10} \text{ rad/s}^2$$

as by design. d_2 is the readout noise, with a standard deviation of

$$\sigma_{d2} = \frac{10^{-6}}{3}$$
 rad

The simulation is run in Simulink, with a 0.1 s time-step, Runge-Kutta 4th order integrator for 100 intervals of T = 10 hours, and evaluated with a sampling frequency of $f_{samp} = 2$ Hz, obtaining a maximum spectrum frequency of F = 1 Hz, as assumed in the control design process. Figure 7.9 shows a sample and the average over the 100 intervals of the ASD, calculated with the periodogram() function (see Appendix F). The theoretical 3σ level is determined by the 3μ (3 times the average ASD of the error) plot. The simulation shows that the design of *G* is slightly overcompensated, but, overall, it fits the shape of the expected behavior.


Figure 7.9: Results for *G* in bandwidth with a linear feedback control system $(P = 1/s^2)$ and design random noises at 3σ levels (Gaussian). *left*: ASD of a 10 h sample with maximum frequency F = 1 Hz. *right*: average μ over 100 samples and 3μ level (representing approximately a zero-mean 3σ of ASD). The theoretical shape of Figure 7.8 is retained, with an unexpected scaling factor for a better result.

7.6. SIMULATION OF THE ATTITUDE CONTROL

The simulation is carried out in Simulink according to the scheme proposed in Figure 7.10. The attitude dynamics are isolated, and the reference commands θ_c are calculated using Equations (4.8) and (5.23) in the block labeled *Orbit-Attitude dependency*. \vec{R}_1 , \vec{R}_2 and \vec{R}_3 are the instantaneous positions of the respective three S/C, simulated based on the model of Chapter 2. Since we are only commanding S/C₁, the other two S/C are propagated just as point-masses.



Figure 7.10: Schema of the Simulink simulation and control of the Euler angles: the block *Orbit-attitude dependency* uses Equations (4.8) and (5.23) to calculate the reference θ_c commands for the Euler angles, based on the three propagated S/C positions \vec{R}_i . The block *P* propagates the dynamics of the attitude of S/C₁, where *Euler dynamics* and *Euler kinematics* refer, respectively, to Equations (5.2) and (5.29). The *control subsystem* block simulate the control transfer function *G* and the non-linear dynamics inversion.

In the *Control subsystem* block, the control transfer function *G* and the linear dynamics inversion H_{θ} , calculated by Equation (7.5) provide the command torque \vec{l}_t . The inertia I_{sc} is calculated based on the propellant mass left due to thrust consumption. The Euler angles used in the dynamics inversion, $\boldsymbol{\theta}_m$, are the measured one, i.e. $\boldsymbol{\theta} + \boldsymbol{d}_{\theta 2}$, where $\boldsymbol{d}_{\theta 2}$ are the measurement errors. The torque noise \vec{l}_{d0} is calculated as

$$\vec{l}_{d0} = \vec{l}_{srp} + \delta \vec{l}_t \tag{7.26}$$

where \vec{l}_{srp} is the SRP induced torque, calculated by Equation (5.10) and based on the SRP model of Section 5.4, and $\delta \vec{l}_t$ is the thruster torque noise, defined in Section 5.12.

The non-linear plant *P* is the combination of *Euler dynamics* of Equation (5.2) and *Euler kinematics* of Equation (5.29). The real $\boldsymbol{\theta}$ are used for the evaluation of $N(\boldsymbol{\theta})$ in this case. $\boldsymbol{d}_{\theta 2}$ is a Gaussian random variable with standard deviation $\sigma_{d2\theta}$ as in Equation (7.23).

For what concerns the initial conditions, in order to keep the transient to a minimum we start right-on with the ideal attitude value,

$$\boldsymbol{\theta}(t=0) = \boldsymbol{\theta}_c(t=0) \tag{7.27}$$

and the ideal initial spin, i.e. $\vec{\omega}(t = 0)$ calculated from Equation (4.45). Only the initial torque is left as an unknown, starting with

$$\vec{l}_t(t=0) = \vec{0}.$$

The transient will be determined, therefore, only by the response to the disturbances.

The validation of the Simulink model is performed by eliminating all the random variables, i.e. by setting $d_{2\theta} = 0$ and $\vec{l}_{d0} = \vec{l}_{srp}$. Reference $\vec{\omega}'$ and $\dot{\vec{\omega}}'$ are calculated through, respectively, Equations (4.45) and (4.59), and fed into Equation (5.13) to obtain the ideal value of \vec{l}_t . A step-size of 1000 s is used. The propagated θ and the analytically determined, orbit-dependent θ_c are then confronted, with their difference for 4 years shown in Figure 7.11. The difference between the two processes to calculate the current θ (analytical vs propagated)



Figure 7.11: Validation. Difference between the analytically determined θ_c and θ , propagated through the plant block *P* dynamics, based on ideal values of \vec{l}_t , for 4 years and with step size 1000 s.

bears a maximum of 8.489431380 \cdot 10⁻¹² rad for θ_3 over the 4 years period.

7.7. ATTITUDE CONTROL RESULTS IN LOW FREQUENCY

Until now we had no means to verify the validity of the low-frequency boundaries $|S_{\theta_{lf}}|$ of Table 7.1. Let us recall that we need the errors due to SRP torque to lie below $|\delta\theta_i| < 10^{-8}$ rad (10 nrad). The strategy required to limit the components c_n of the Fourier series, whose fundamental harmonic is ω_o , to:

$$c_n < \frac{10^{-8}}{4}$$
 rad = 2.5 nrad.

as specified in Equation (7.15).

The pay-off is visible when we simulate the system without thruster torque jitter $\delta \vec{l}_t = \vec{0}$. Since we are not interested in the bandwidth results and taking advantage of the low gain k, we simulate the control system with a step size of 1 s for 2 years, obtaining the error $\delta \theta = \theta_c - \theta$ shown in Figure 7.12.

 θ_2 overshoots, initially, by $\delta\theta_2 \simeq -6.440873 \ \mu$ rad after ~ 1 h 45 min of operation and it is also the last DoF to reach the goal of $\delta\theta_2 = 10$ nrad after about $t \simeq 7$ days 21 h 56 min 40 s, defined as our *settling time*.

After that, the three quantities keep below

$$|\delta \theta_1| \le 4.569352896 \text{ nrad},$$

 $|\delta \theta_2| \le 3.159098649 \text{ nrad},$
 $|\delta \theta_3| \le 4.660129393 \text{ nrad}.$

with an annual trend for θ_1 and θ_2 and a six month period for θ_3 .

The 2 years data availability allows to analyze the entire period using the discrete Fourier series (See Appendix D). Figure 7.13 shows the components. It is evident that, by design, these are kept below the $c_n < 2.5$



Figure 7.12: Low frequency attitude control loop response $\delta \theta = \theta_c - \theta$. *top*: Transient response for 5 days. *bottom*: 2 years response at nrad level. The requirements are $\delta \theta_i < 10$ nrad (i = 1, 2, 3).

nrad level at all frequencies, with the best *L* function fit at $2\omega_o$, for θ_3 ($\delta\theta_3(2\omega_o) \approx 2.4449937$ nrad). For θ_2 and θ_3 the *L* function does not fit optimally the $S_{\theta_{lf}}$ bounds at all the other frequencies but the limit is still satisfied.



Figure 7.13: Fourier series of Figure 7.12 showing the harmonic components forced below 2.5 nrad at multiples of the fundamental frequency $\omega_o \simeq 1.9924 \cdot 10^{-7}$ rad/s.

In general it is difficult to achieve perfect control at all the frequencies. Knowing this we opted for the factor 4 margin (in fact, this factor would be much larger if all the frequencies between ω_o and $2^{10}\omega_o$ were to be perfectly controlled).

The most important result, of course, is the time-domain series: it shows that DC control is achieved as specified by the constraints using the same controller for all the Euler angles.

7.8. ATTITUDE CONTROL RESULTS IN BANDWIDTH

The full simulation is performed with a time-step of 0.1 s for 35 days. Leaving the settling period of 7 days 21 h 56 min 40 s, we can analyze up to 65 periods of 10 h each. The results are shown in Figures 7.14 and 7.15.



Figure 7.14: ASD of the error $\delta \theta = \theta_c - \theta$ calculated with periodogram() on a random 10 h sample (maximum frequency F = 1 Hz) after the settling time. *top-left*: θ_1 , *top-right*: θ_2 , *bottom*: θ_3 .

The Figures show, respectively, a sample ASD of the error $\delta\theta$ and the average for 98 periods. The results are a close match to the ones obtained in Section 7.5 (Figure 7.9), and all the DoF have almost the same behavior.

According to Figure 7.3 one would expect θ_1 to be the most affected by the torque, but the evaluation might not be completely representative of the action of the torque on the DoF, since it does not take into consideration negative components.



Figure 7.15: Average ASD between 65 consecutive samples of 10 h periods of the error $\delta \theta = \theta_c - \theta$ calculated with periodogram() (maximum frequency F = 1 Hz) after the settling time. *top-left*: θ_1 , *top-right*: θ_2 , *bottom*: θ_3 .

7.9. RECOMMENDATIONS ON THE CONTROL SYSTEM

Overall the system performs as desired. The main issue is, of course, the slow settling time t > 7 days even when the initial error is set to 0 and the initial S/C angular velocity $\vec{\omega}$ is set to match the nominal one. The solution to this inconvenience is to use intermediary controllers with faster roll-offs and coarser precision during what is referred to as an *acquisition phase* [21], and then switch to the slower science mode controller keeping the initial value of \vec{l}_t .

We must make a point about the sensitivity of the measurement device: the precision of $< 10^{-8}$ rad assumed here is nothing achievable by the classic star trackers. Instead, it is assumed that the attitude is measured by means of laser interferometry by an instrument called *point ahead mechanism* [17, 37]. At the time of writing this thesis, the point ahead mechanism has not yet been fully developed.

7.10. CONCLUSIONS ON THE ATTITUDE CONTROL SYSTEM

In this chapter we designed a control system for the attitude control both in low-frequency (compensation of SRP torque) and in bandwidth, based on the linearized dynamics of the Euler angles. Dynamic inversion was used to eliminate plant uncertainties. A single controller is used for the three Euler angles.

The DC margin of 10 nrad was used as a drive requirement for the low frequency control, based on the influence of the SRP torque on the Euler angle dynamics, whose fundamental harmonic is estimated at ω_o (average sidereal motion of the formation for the first 4 years). The boundaries on the *S* function are calculated by allowing a maximum error of 2.5 nrad at each harmonic.

The ASD of the dynamics noise in bandwidth was estimated from the torque noise of Section 5.12 at 3σ for the worst value over the studied 4 year period and assuming a white noise. The readout noise is estimated as the maximum allowable noise ASD_{θ} at 1 Hz. The boundaries for *S* and *T* on the phase-magnitude plane are calculated at every octave from 1 μ Hz. A stability margin of |T| = 3 dB is added.

A control transfer function with 5 zeros and 5 poles has been shaped ignoring the T boundaries due to the

readout noise. The readout noise is simulated as a random variable with standard deviation 3 times smaller than the design noise.

The expected behavior of the system in bandwidth has been verified with dummy linear dynamics. The system seems to perform better than the design.

A simulation using only SRP torque as noise reveals that the design level was correctly estimated, and the low frequency error stays below the 10 nrad requirement. The attitude control system with real dynamics behave as in the verification step, achieving control under the sensitivity curve.

The only drawback of this system is the very slow settling time of >7 days, meaning that this controller cannot be used for the initial acquisition of the attitude.

8

DRAG-FREE CONTROL SYSTEM

8.1. INTRODUCTION TO THE DRAG-FREE CONTROL SYSTEM

The displacement of the TM along the interferometric paths x_1 and x_2 and the *z*-direction of TM₁ displacement z_1 drive the drag-free control, actuated by the on-board thrust \vec{f}_t against the SRP force \vec{f}_{srp} modeled in Section 5.4, and the thruster jitter $\delta \vec{f}_t$, modeled in Section 5.12.

The maximum DC displacement for all the DoF is [14]:

$$\delta x, \delta z \le 5 \cdot 10^{-9} \,\mathrm{m} \tag{8.1}$$

In bandwidth, the displacement noise for the x-axes must keep an ASD below

$$ASD_{x}(f) \le 10 \cdot 10^{-12} \frac{\mathrm{m}}{\sqrt{\mathrm{Hz}}} \cdot \sqrt{1 + \left(\frac{2 \mathrm{mHz}}{f}\right)^{4}}$$

$$(8.2)$$

and the accelerations are to be kept under

$$ASD_{acc}(f) \le 3 \cdot 10^{-15} \frac{\text{m s}^{-2}}{\sqrt{\text{Hz}}} \cdot \sqrt{1 + \left(\frac{0.4 \text{ mHz}}{f}\right)^2} \cdot \sqrt{1 + \left(\frac{f}{8 \text{ mHz}}\right)^4}.$$
(8.3)

Moreover, the displacements on both the x-axes and z_1 need to stay below

$$ASD_r(f) \le 5 \cdot 10^{-9} \frac{\mathrm{m}}{\sqrt{\mathrm{Hz}}}$$

$$(8.4)$$

at every frequency.

In this chapter, the drag-free control transfer function is designed with QFT. As for the attitude, we consider DC all the disturbances with period T > 10 h and separation of the dynamics is performed.

8.2. DRAG-FREE DYNAMIC INVERSION

Let us define the array of drag-free controlled DoF x_{df} as [22]

$$\boldsymbol{x}_{df} = \begin{pmatrix} x_1 \\ x_2 \\ z_1 \end{pmatrix} \tag{8.5}$$

where x_1 and x_2 are the displacements of the TM from their rest positions along the lines-of-sight of the telescopes, while z_1 , is the vertical displacement of TM₁. Selection of the three acceleration components is performed through a set of selection matrices:

$$\ddot{x}_{df} = H_{1df} \vec{\dot{r}}_1 + H_{2df} \vec{\dot{r}}_2 \tag{8.6}$$

where one can derive H_{1df} and H_{2df} from Equation (8.5) respectively as

$$H_{1df} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } H_{2df} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let us recall the dynamics Equation (5.43) for $\ddot{\vec{r}}_i$, i = 1, 2, derived in Section 5.9. The system is dependent on a large number of variables:

- the thrust and external disturbances \vec{f}_t/m_{sc} and $\vec{f}_{d0}/m_{sc} = \vec{d}_0$ acting on the S/C (m_{sc} is the S/C mass);
- the virtual spring-mass term $-K_i \vec{r}_i / m_i$ due to the spurious S/C-TM interactions, with K_i being a stiffness matrix and m_i being the TM mass;
- the S/C angular velocity and rotational acceleration $\vec{\omega}$ and $\vec{\omega}$, responsible for Coriolis, centrifugal and Euler's acceleration, together with TM displacement \vec{r}_i and velocity \vec{r}_i ;
- the telescope steering velocity and acceleration $\vec{\omega}_{0i}$ and $\vec{\omega}_{0i}$, both dependent on the breath angle α kinematics and dynamics, and also responsible for rotational accelerations;
- the electrostatic suspension control force \vec{f}_i and the rest of the spurious accelerations \vec{d}_i .

By definition, the drag-free control system employs the on-board thrusters \vec{f}_t to compensate for the external forces \vec{f}_{d0} , to which the TM are sensible [17].

For analysis purposes, let us assume that the electrostatic suspension system eliminates all the internal disturbances and the non Newtonian accelerations, thus leaving the *complementary drag-free acceleration* $\ddot{\tilde{x}}_{df}$, which is defined as:

$$\ddot{\boldsymbol{x}}_{df} = H_{1df}C_{s/1}\left(-\frac{1}{m_{sc}}\vec{f}_t - \vec{d}_0\right) + H_{2df}C_{s/2}\left(-\frac{1}{m_{sc}}\vec{f}_t - \vec{d}_0\right) = \left(H_{1df}C_{s/1} + H_{2df}C_{s/2}\right)\left(-\frac{1}{m_{sc}}\vec{f}_t - \vec{d}_0\right)$$
(8.7)

where $\vec{d}_0 = \vec{f}_{d0}/m_{sc}$. From Equation (8.7) we can derive a transformation matrix C_{df} that allows to retrieve the control efforts \vec{f}_t to manage the three drag-free DoF \mathbf{x}_{df} ,

$$C_{df} = \left(H_{1df}C_{s/1} + H_{2df}C_{s/2}\right)^{-1} = \begin{pmatrix}\cos\alpha/2 & -\sin\alpha/2 & 0\\\cos\alpha/2 & \sin\alpha/2 & 0\\0 & 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}\frac{1}{2\cos\alpha/2} & \frac{1}{2\cos\alpha/2} & 0\\-\frac{1}{2\sin\alpha/2} & \frac{1}{2\sin\alpha/2} & 0\\0 & 0 & 1\end{pmatrix},$$
(8.8)

recalling the definition of $C_{s/1}$ and $C_{s/2}$, respectively Equations (5.34) and (5.35) from Section 5.9.

Conceptually, C_{df} can be derived graphically by taking into consideration Figure 8.1 explained in the following paragraphs. The basic objective of the drag-free control is to accelerate the S/C-fixed frame together with the inertial *x*-direction accelerations of the two TM and the *z*-direction acceleration of TM₁. Let us call the three accelerations \ddot{x}_{df1} , \ddot{x}_{df2} and \ddot{z}_{df1} . Their vectors, in the body-fixed reference frame of the S/C are oriented according to the respective axes orientations of their GRS reference frames, with the *x*-directions slanted by an angle $\alpha/2$ symmetrically to the body-fixed *x*-axis. The compensating acceleration \vec{a}_t imprinted on the S/C is a vector that can be decomposed into \ddot{x}_{df1} , \ddot{x}_{df2} and \ddot{z}_{df1} along their respective orientations:

$$\begin{pmatrix} \ddot{x}_{df1} \\ \ddot{x}_{df2} \\ \ddot{z}_{df1} \end{pmatrix} = \begin{pmatrix} \hat{x}_1 \cdot \vec{a}_t \\ \hat{x}_2 \cdot \vec{a}_t \\ \hat{z}_1 \cdot \vec{a}_t \end{pmatrix} = \begin{pmatrix} \hat{x}_1^{\mathsf{T}} \vec{a}_t \\ \hat{x}_2^{\mathsf{T}} \vec{a}_t \\ \hat{z}_1^{\mathsf{T}} \vec{a}_t \end{pmatrix} = \begin{pmatrix} \cos \alpha/2 & -\sin \alpha/2 & 0 \\ \cos \alpha/2 & \sin \alpha/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{a}_t.$$
(8.9)

Defining C_{df} as the transformation matrix that allows to derive \vec{a}_t from the three drag-free accelerations, i.e. the inverse relation of Equation (8.9),

$$\vec{a}_t = C_{df} \begin{pmatrix} \ddot{x}_{df1} \\ \ddot{x}_{df2} \\ \ddot{z}_{df1} \end{pmatrix}$$
(8.10)

it can be calculated as

$$C_{df} = \begin{pmatrix} \cos \alpha/2 & -\sin \alpha/2 & 0\\ \cos \alpha/2 & \sin \alpha/2 & 0\\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2\cos \alpha/2} & \frac{1}{2\cos \alpha/2} & 0\\ -\frac{1}{2\sin \alpha/2} & \frac{1}{2\sin \alpha/2} & 0\\ 0 & 0 & 1 \end{pmatrix}.$$
 (8.11)



Figure 8.1: Vectorial identification. \ddot{x}_{df1} and \ddot{x}_{df2} are, respectively, the *x*-accelerations of TM₁ and TM₂ in their respective reference frames. \ddot{z}_{df1} is the *z*-acceleration of TM₁, chosen as third drag-free controlled DoF. \vec{a}_t is the resulting acceleration actuated by the S/C thrusters needed to overcome the TM displacements.

which bears the same result as Equation (8.8).

Going back to Equation (8.7), we can rewrite it as:

$$\ddot{\tilde{\mathbf{x}}}_{df} = -\frac{C_{df}^{-1}}{m_{sc}} \left(\vec{f}_t + \vec{f}_{d0} \right) = H_{df} \left(\vec{f}_t + \vec{f}_{d0} \right).$$
(8.12)

Similarly to the linearized attitude dynamics of Equation (7.4) (Section 7.2), Equation (8.12) contains a dynamics inversion matrix

$$H_{df} = -\frac{C_{df}^{-1}}{m_{sc}},$$
(8.13)

a command variable

$$\boldsymbol{u}_{df} = \boldsymbol{f}_t \tag{8.14}$$

which can be transformed into a virtual command \boldsymbol{v}_{df} by

$$\boldsymbol{v}_{df} = H_{df} f_t, \tag{8.15}$$

and a dynamics disturbance term:

$$\boldsymbol{d}_{df} = H_{df} f_{d0}. \tag{8.16}$$

Let us then rewrite Equation (8.12) by substituting these terms:

$$\ddot{\tilde{\boldsymbol{x}}}_{df} = \boldsymbol{v}_{df} + \boldsymbol{d}_{df} \tag{8.17}$$

By applying the Laplace transform, the above equation, in frequency domain, becomes

$$s^{2}\widetilde{X}_{df} = V_{df} + D_{df} \iff \widetilde{X}_{df} = \frac{1}{s^{2}} \left(V_{df} + D_{df} \right)$$
(8.18)

where \tilde{X}_{df} , V_{df} and D_{df} are, respectively, the Laplace transforms of \tilde{x}_{df} , v_{df} and d_{df} . The linearized feedback control loop plant *P*, described in Section 6.2, is, once again, a double integrator

$$P = \frac{1}{s^2},$$
 (8.19)

similarly to the attitude linearized dynamics of Section 7.2.

8.3. DRAG-FREE DISTURBANCES IN LOW FREQUENCIES

As for the Euler angles in Chapter 7, the low frequencies disturbances on the drag-free DoF are the accelerations due to the SRP force \vec{f}_{srp} , modeled in Section 5.4. Let us set $\vec{f}_t = \vec{0}$ and $\vec{f}_{d0} = \vec{f}_{srp}$ in Equation (8.12): the disturbance obtained on the drag-free DoF is shown in Figure 8.2: the behavior is consistent with the SRP force. The Fourier analysis for the 4 years period results in Figure 8.3. Only the component at $\omega_o = 1.99234991 \cdot 10^{-7}$ rad/s is present for x_1 and x_2 , whose average is

$$\ddot{x}_{1,2}(\omega_o) \simeq 1.622644577 \cdot 10^{-8} \text{ m/s}^2$$



Figure 8.2: SRP induced acceleration on the drag-free DoF (x_1, x_2, z_1) for 4 years, based on $\vec{f}_{d0} = \vec{f}_{srp}$ modeled in Section 5.4 and calculated with Equation (8.12).



Figure 8.3: Fourier series (magnitude) of SRP induced accelerations $\ddot{\tilde{x}}_{df}$ on the drag-free DoF, with fundamental harmonic $\omega_o = 1.9923 \cdot 10^{-7}$ rad/s.

while z_1 has a strong continuous component

$$\ddot{z}_1(\omega = 0) \simeq 4.633356186 \cdot 10^{-8} \text{ m/s}^2$$

and a small component at the fundamental harmonic of

$$\ddot{z}_1(\omega_0) = 8.224090528 \cdot 10^{-10} \text{ m/s}^2$$

The science requirements call for a maximum DC displacement of $\delta x = \delta z \le 5 \cdot 10^{-9}$ m (Equation (8.1)). A safety margin factor of 2 is implemented and, according to Equation (6.5), with $|P| = 1/\omega^2$ like for the Euler angles, the sensitivity functions $S_{x_{lf}}$ and $S_{z_{lf}}$ at $\omega = \omega_o$ are calculated respectively as:

$$|S_{x_{lf}}| = \frac{\omega_o^2 \delta x}{2|D_1|} = \frac{\omega_o^2 \cdot 5 \cdot 10^{-9} \text{ m}}{2|D_1|} = \frac{\omega_o^2}{\ddot{x}_{1,2}(\omega_o)} \cdot 2.5 \cdot 10^{-9}$$
(8.20)

and

$$|S_{z_{lf}}| = \frac{\omega_o^2 \delta z}{2|D_{1_z}|} = \frac{\omega_o^2 \cdot 5 \cdot 10^{-9} \text{ m}}{2|D_{1_z}|} = \frac{\omega_o^2}{\ddot{z}_1(\omega_o)} \cdot 2.5 \cdot 10^{-9}$$
(8.21)

obtaining the following values:

$$|S_{x_{lf}}|(\omega_o) \simeq -284.270737385 \text{ dB};$$

 $|S_{z_{lf}}|(\omega_o) \simeq -258.368027014 \text{ dB}.$

Other harmonic components are not calculated as we assume that the SRP force acts, on visual inspection, with an almost sinusoidal behavior.

8.4. DISTURBANCES IN BANDWIDTH ALONG *x*-AXES

In Section 5.12 the thrust noise $\delta \vec{f}_t$ has been defined as a zero-mean random variable with standard deviation $\sigma \left[\delta \vec{f}_t\right]$ reported in Table 5.2. The acceleration noise $\delta \ddot{x}_{df}$ acting on the drag-free DoF can be estimated using the dynamics inversion matrix H_{df} of Equation (8.13) by a process described by the following equation:

$$\sigma[\delta \ddot{\boldsymbol{x}}_{df}] = \max\left(\left|H_{df}M_{1}\sigma\left[\delta \vec{f}_{t}\right]\right|, \left|H_{df}M_{2}\sigma\left[\delta \vec{f}_{t}\right]\right|\right)$$
(8.22)

where

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
 (8.23)

Basically we need to check the influence that the direction of the *y*-axis noise in S/C-fixed reference frame has on the TMs *x*-accelerations, as the TM *x*-axes are oriented symmetrically w.r.t. the S/C *x*-axis, and pick the maximum value obtained. M_1 favors TM₂, as the direction of the x_2 -axis is in the upper half of the S/C body-reference frame, while M_2 favors TM₁ as the direction of the x_1 -axis lies in the lower half.

For sake of brevity we can set $\alpha = 60^{\circ}$ (nominal breath angle) in Equation (8.8) and use the value of the mass at the end of the 4 years period to calculate Equation (8.13), with the propellant mass being

$$m_{prop}(t = 4 \text{ years}) \simeq 232.0165 \text{ kg}$$

We define the dynamics noise $|D_1|$ at a level of 3σ , i.e. $|D_1| = 3\sigma [\delta \ddot{x}_{df}]$, and calculate the sensitivity functions $|S_x|$, $|S_z|$ and $|S_u|$ based on Equations (6.5) and (6.6) and values for the limits at:

$$ASD_{x}(f) \le 10 \cdot 10^{-12} \frac{\mathrm{m}}{\sqrt{\mathrm{Hz}}} \cdot \sqrt{1 + \left(\frac{2 \mathrm{mHz}}{f}\right)^{4}}$$

$$(8.2)$$

$$ASD_{acc}(f) \le 3 \cdot 10^{-15} \frac{\text{m s}^{-2}}{\sqrt{\text{Hz}}} \cdot \sqrt{1 + \left(\frac{0.4 \text{ mHz}}{f}\right)^2} \cdot \sqrt{1 + \left(\frac{f}{8 \text{ mHz}}\right)^4}$$
(8.3)

while also keeping

$$ASD_x(f) \le ASD_r \le 5 \cdot 10^{-9} \frac{\mathrm{m}}{\sqrt{\mathrm{Hz}}}$$
(8.4)

at all frequencies [14].

For the x_1, x_2 -axes we estimate that

$$|D_1| = 1.670588439 \times 10^{-10} \text{ m s}^{-2}/\sqrt{\text{Hz}}$$

and the $|S_x|$ and $|S_u|$ upper limits are calculated using:

$$|S_x|(f) = \frac{ASD_x(f)\omega^2}{|D_1|}$$
(8.24)

and

$$|S_u|(f) = \frac{ASD_{acc}(f)}{|D_1|}.$$
(8.25)

The readout noise $|D_2|$ is calculated as the maximum white noise allowable at f = 1 Hz, i.e. the upper measurement frequency:

$$|D_2| = \frac{ASD_{acc}(1 \text{ Hz})}{(2\pi \text{ rad/s})^2} = 1.187357718 \cdot 10^{-12} \text{ m/}\sqrt{\text{Hz}}$$
(8.26)

and the sensitivity functions $|T_x|$ and $|T_u|$ are calculated accordingly to Equations (6.7) and (6.8) (setting $|X| = ASD_x$ and $|U| = ASD_{acc}$).

Figure 8.4 shows the values of the various noise and sensitivity levels. The *top-right* picture shows the motive behind Equation (8.26): the chosen value of $|D_2|$ allows the double integration trend of the error to follow ASD_{acc} at infinity, while finally crossing at f = 1 Hz, where the value of $|T_u|(1Hz) = 0$ dB is the minimum allowable for Equation (6.26) to inherently hold true at every frequency in the bandwidth, as shown in the *bottom-left* picture. The crossing between requirements of Equations (8.2) and (8.4) happens at about $f = 8.94 \cdot 10^{-5}$ Hz, where the trends of $|S_x|$ and $|T_x|$ change abruptly.



Figure 8.4: *top-left*: Noises and limits on the displacements in frequency domain. $|D_1|$ is divided by the squared angular frequency ω^2 to match the dimensions. The limit ASD_x is shown for both Equations (8.4) and (8.2). *top-right*: Noises and limits on the accelerations in frequency domain. $|D_2|$ is multiplied by the squared angular frequency ω^2 to match the dimensions. *bottom-left*: the limits on the |T| sensitivities calculated with Equations (6.7) and (6.8) (setting $|X| = ASD_x$ and $|U| = ASD_{acc}$. *bottom-right*: the limits on the *S* sensitivity functions calculated with Equations (8.24) and (8.25).

8.5. DISTURBANCES IN BANDWIDTH ALONG *z*-AXIS

The control of z_1 does not have requirements on the acceleration, therefore only requirement

$$ASD_z \le ASD_r \le 5 \cdot 10^{-9} \frac{\mathrm{m}}{\sqrt{\mathrm{Hz}}}$$
(8.4)

needs to be satisfied. From Equation (8.22) and the 3σ margin imposed on all the DoF, we obtain that the dynamics noise is

$$|D_1| = 3.755175069 \cdot 10^{-10} \text{ m s}^{-2} / \sqrt{\text{Hz}}.$$

The sensitivity function $|S_z|(f)$ is calculated, similarly for $|S_x|(f)$ as:

$$|S_z|(f) = \frac{ASD_z \omega^2}{|D_1|}.$$
(8.27)

In order to obtain a minimum sensitivity of $|T_z| = 0$ dB, with $|T_z|$ being calculated with Equation (6.7), the readout noise is set to:

$$|D_2| = ASD_z. \tag{8.28}$$

Since ASD_z and $|D_2|$ are constant over f, $|T_z| = 0$ dB is also constant.

Figure 8.5 shows the various levels for the noises and the sensitivity functions in the bandwidth frequencies.



Figure 8.5: *left:* the noises and limit on the displacement of z_1 . $|D_1|$ is divided by the squared angular frequency ω^2 to match the dimensions. *right:* the limits on the sensitivity functions $|S_z|$ and $|T_z|$. $|T_z| = 0$ dB is constant because of the choice of $|D_2| = ASD_z$. Required bandwidth and goal bandwidth bounds are shown.

8.6. SHAPING OF *x*-AXES OPEN LOOP TRANSFER FUNCTION

Figure 8.6 shows the boundaries of $|S_x|$, $|S_u|$ and $|T_x|$ on the phase-magnitude plane, for f starting at 1 μ Hz and then at each octave. The boundary for T_x is not shown, as its minimum value is $|T_x|(f \to \infty) \approx 18.508368$ dB, which is much larger than 3 dB at each frequency.

The $|S_x|$ bounds (*top-left*) show a steady 40 dB/decade decline from ~178.55079 dB at 1 μ Hz until the switch at $f \simeq 8.94 \cdot 10^{-5}$ Hz (between 2⁶ and 2⁷ μ Hz), where |L| stays constant until f = 2 mHz ($\simeq 2^{11} \mu$ Hz), and start falling again with the same trend.

For $|S_u|$ (*top-right*), the maximum magnitude |L| happens at octave $f = 2^{11} \mu$ Hz, where it stays approximativaly constant |L| = 94.733922 dB until about $2^{13} \mu$ Hz, where the trend becomes declining at 40 dB/decade.The $|T_u|$ bounds (*bottom*) are meant to match $|T| \simeq 0$ dB, by design, from f = 8 mHz on ($f \simeq 2^{13}$ Hz).

The open-loop transfer function

$$L = PG = \frac{G}{s^2}$$

like for the Euler angles, needs to satisfy $|L| < |S_{x_{lf}}|$ at ω_o and $|L| < |S_x|$, $|S_u|$, $|T_u|$ and the |T| < 3 dB stability margin shown in Figure 8.6. *G*, as a transfer function, also needs to have a total order ≤ 0 , meaning that the order of *L* needs to be ≤ 2 due to the presence of *P* as a double integrator.

After some design, we find an appropriate control transfer function in:

$$G = k \frac{(s - z_1)(s - z_2)^2(s - z_3)}{(s - p_1)^2(s - p_2)(s - p_3)}$$
(8.29)

with

k = 512.331124704

and the values of the zeros and poles shown in Table 8.1.

Table 8.1: Zeros and poles of the *G* function of Equation (8.29).

Zeros [rad/s]			Poles [rad/s]		
$z_1 = z_2 = z_3 =$	$\begin{array}{r} -2.869034387\cdot 10^{-6} \\ -5.026548246\cdot 10^{-6} \\ -9.4247779607 \end{array}$	(2 nd order)	$p_1 = p_2 = p_3 =$	-4.9809625·10 ⁻⁸ -0.03900837094 -28.2743339	(2 nd order)

The shape of *L* on the phase-magnitude plane is shown in figure 8.7. The following steps were followed for the definition of *G*:



Figure 8.6: *top-left:* the S_x boundaries calculated from the values of $|S_x|$ from Equation (8.24) and Equations (6.21) and (6.22) for each octave $f = 2^k \mu Hz$. The boundaries between $2^7 \mu Hz < f < 2^{11} \mu Hz$ are not shown because $|S_x|$ is almost constant in the interval. *top-right:* the S_u boundaries calculated from the values of $|S_u|$ from Equation (8.25) and Equations (6.21) and (6.22) until $f = 2^{20} \mu Hz (\ge 1 Hz)$. The boundaries for $f < 2^{11} \mu Hz$ are not shown because their magnitude is raising until that point. *bottom:* The T_u boundaries calculated with Equation (6.8). The boundaries approach closer and closer the |T| = 0 dB curve getting to |T| > 0 dB at $f = 2^{20} \mu Hz > 1$ Hz. For all the pictures, the stability boundary |T| = 3 dB is shown as a thick dotted line. The plots are symmetric w.r.t. the -180° vertical line. The T_x boundary is not shown, because $|T_x| > 18$ dB is a series of ellipses within the |T| = 3 dB boundary.

- The initial gain to satisfy the ω_o boundary is calculated as $k \simeq \omega_o^2 / |S_{x_{lf}}|$ and it is then updated for every pole and zero added.
- In order to rush to a minimum separation from the boundary at $f = 1 \mu$ Hz, the function crosses the ω_o boundary at 80 dB/decade thanks to the double pole p_1 .
- Between 1 μ Hz and 2⁶ μ Hz *L* is supposed to decrease by 40 dB/decade, but the boundaries between 2⁶ and 2¹³ μ Hz require, instead, a shallower decline at 20 dB/decade in order to stay above the boundary at $f = 2^{13} \mu$ Hz, therefore 3 zeros (z_1 and double zero z_2) are added.



Figure 8.7: The graphically evaluated *L* function (solid black line) of Equations (8.6) and (8.29). The bubble \circ represents the evaluated low-frequency harmonic at ω_o , with the limit of $|S_{x_{lf}}|$ represented by the dark horizontal line. The bullets \bullet represent the bandwidth frequencies starting at $2^{11} \mu$ Hz. The markers need to stay close to the respective boundaries, above and outside the |S| boundaries and below and outside the |T| boundaries. The latter is intentionally not satisfied, because the looping is inside the blue U-shaped curve (|T| = 0 dB). The boundaries in bandwidth are a combination of the S_x , S_u and T_u boundaries: below $f < 2^{12} \mu$ Hz the S_x boundaries are used, above the S_u boundaries are used. T_u and S_u overlap at $f = 2^{16} \mu$ Hz. The stability requirement is raised from $|T| \le 3$ dB to $|T| \le 6$ dB to lower the value of the gain *k*.

- The $|S_u|$ boundaries replace the $|S_x|$ boundaries at $2^{12} \mu$ Hz for $|S_u| < |S_x|$ at every phase.
- from $2^{13} \mu$ Hz *L* is required to lower at 40 dB/decade, therefore another pole p_2 is added.
- at $f = 2^{16} \mu$ Hz the $|T_x|$ boundaries start overlapping with the $|S_u|$ boundaries (now shown as dotted lines in the prohibited area). Nevertheless we decide to violate these boundaries to avoid too large of a gain k.
- Since *L* has a –2 order we need to add a zero and a pole to sway from the singularity out of the stability |T| = 3 dB bound. We find that at f = 1 Hz the closed-loop gain is still |T| < 3 dB and that k > 1000, therefore we raise the stability bound to |T| = 6 dB, obtaining $k \approx 512.3311$ for zero and pole z_3 and p_3 .

Moreover, by choosing all negative poles and zeros and allowing the function to only cross the $\underline{L} = -180^{\circ}$ line twice (the first being at the origin), the stability conditions for *T* are also satisfied.

In Figure 8.8 the expected behavior of the errors due to the combination of jitter noise and read-out noise is shown, both for the displacement and the accelerations along the *x*-axes: we notice that, by design, ASD_{acc} is perfectly satisfied by the acceleration noises and a small overshoot due to the read-out noise happens around 1 Hz (we know that the limit is violated at around $2^{17} \mu$ Hz = 0.065536 Hz). We also notice that ASD_x is actually 10 times larger than the same limit ASD_{acc} double integrated on the same frequency, therefore all the displacements are expected to be 10 times lower than the limit.

For verification we can simulate the linear control loop with $P = 1/s^2$ and the dynamics and readout noises d_1 and d_2 with a standard deviation

$$\sigma_{d1} = \frac{|D_1|}{3} \\ \sigma_{d2} = \frac{|D_2|}{3}.$$



Figure 8.8: Expected output of the errors for design $L |D_1|$ and $|D_2|$. *left*: displacement errors due to dynamics noise, calculated with manipulation of Equation (6.5), and due to measurement noise, calculated with Equation (6.7). *right*: acceleration errors due to dynamics noise, calculated with manipulation of Equation (6.6), and due to measurement noise, calculated with Equation (6.8).

The simulation is performed in Simulink with a step size of 0.1 s and evaluated with a sampling frequency of $f_{samp} = 2$ Hz using the periodogram() function (see Appendix F). 100 periods of T = 10 hours are simulated. The results for both the displacement and the accelerations are reported in Figure 8.9. The displace-



Figure 8.9: Results for designed *L* in bandwidth with a linear feedback control system ($P = 1/s^2$) and design random noises at 3σ levels (Gaussian) of a random 10 h sample sampled every 0.5 s. *left*: ASD of displacement error F = 1 Hz. *right*: ASD of the acceleration error (averaged over 0.5 s steps).

ment $x(t_i)$ at epoch t_i is measured directly, but the acceleration $\ddot{x}(t_i)$ needs to be averaged over the sampling time: because the sampling frequency f_{samp} is 5 times smaller than the simulation frequency $f_{sim} = 10$ Hz, and the acceleration is proportional to the square of the frequency, the standard deviation of the simulation accelerations $\sigma_{acc}(f_{sim})$ is 25 times larger than the sample accelerations:

$$\sigma_{acc}(f_{sim}) = \frac{f_{sim}^2}{f_{samp}^2} \sigma_{acc}(f_{samp}) = 25\sigma_{acc}(f_{samp});$$
(8.30)

you may find more information about the reasons of Equation (8.30) and the extrapolation procedure in Appendix F, Section F.3. Figure 8.9 shows the results of the verification linear feedback control for a random sample, while Figure 8.10 shows the average over 100 samples.

We notice that the numerical simulation fits almost perfectly the expected behavior. The largest discrepancy happening at f < 0.3 mHz, where the acceleration is above the expected levels, yet below the accepted limits of ASD_{acc} .



Figure 8.10: Average of 100 periods of 10 h of the results for *L* in bandwidth with a linear feedback control system ($P = 1/s^2$) and design random noises at 3σ levels (Gaussian). *left*: average ASD of the displacement noise, *right*: average ASD of the acceleration noise. The theoretical shape of Figure 7.8 is retained, except for the smoother form around the overshoot frequency. The 3μ level is shown as a reference to the 3σ design level of the noises.

8.7. Shaping of z_1 open-loop transfer function

The design of the open-loop transfer function for z_1 is easier, as only the $|S_z|$ and $|T_z|$ boundaries are specified. Figure 8.11 shows the bounds on the phase-magnitude plane of the open-loop transfer function. At $f = 1 \mu Hz$,



Figure 8.11: *left:* the S_z boundaries calculated from the values of $|S_z|$ from Equation (8.27) and Equations (6.21) and (6.22) for each octave $f = 2^k \mu Hz$: the bounds are dependent on the square of the frequency, therefore they lower in magnitude at a -40 dB/decade trend until lower magnitudes. *right:* the T_z boundaries, i.e. $|T_z| = 0$ dB at each frequency. In both the graphs, the stability boundary |T| = 3 dB is shown as a thick dotted line.

 $|S_z| \simeq -185.586008958$ dB and it raises by 40 dB/decade (hence, |L| declines roughly at the same rate). The boundary for T_z is unique, as $|T_z| = 0$ dB at every frequency.

Moreover the low-frequency boundary for $\omega = \omega_o$ is located at $|S_{z_{lf}}|(\omega_o) \simeq -258.368027014$ dB, or roughly $|L| \simeq 1/|S_{z_{lf}}| = 258.368027014$ dB.

As for the x-axes control and the Euler angles control the open-loop function is defined as

$$L = PG = \frac{G}{s^2}.$$
(8.31)

G is designed as

$$G = k \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}$$
(8.32)

where

k = 0.6509071933

and the values of the zeros and poles shown in Table 8.2.

Table 8.2: Zeros and poles of the *G* function of Equation (8.32).

Zeros [rad/s]	Poles [rad/s]		
$z_1 = -8.726646260 \cdot 10^{-7}$	p_1 -2.490481200·10 ⁻⁸		
$z_2 = -0.2058874161$	p_2 -1.536798014		

Figure 8.12 shows the shape of L on the phase-magnitude plot. As for the Euler angles and the x-axis



Figure 8.12: Shape of the *L* function for the z_1 DoF control on the phase-magnitude graph. The bubble \circ represents the evaluated low-frequency harmonic at ω_o , with the limit of $|S_{zlf}|$ represented by the dark horizontal line. The bullets \bullet represent the bandwidth frequencies starting at 1 μ Hz and for each octave in the bandwidth. The function is shaped so that the markers lie just above the dark line and the relative S_z bounds. The U-shaped blue line represents the T_z bound, which is intentionally not satisfied, although a maximum |T| = 3 dB is imposed (thick dotted line).

control the T_z boundaries are ignored, with just a swaying zero-pole to avoid the singularity at -180° phase and 0 dB magnitude.

The first pole p_1 lowers the order of *L* from -2 to -3, in order to rush the 1 µHz marker close to the first $|S_z|$ bound, z_1 re-sets the system to an order of -2 to closely follow the -40 dB/decade trend of the $|S_z|$ boundaries. z_2 and p_2 are then roughly placed to asymptotically touch the |T| = 3 dB stability boundary.

Figure 8.13 shows how the errors are expected to behave on a theoretical linear feedback control loop. We see that with the QFT technique we were able to estimate |L| w.r.t. $|S_z|$ with a factor of 1.1611 (1.2974 dB above the ideal value), due to the expected error associated to the jitter noise (green dot-dashed line in Figure 8.13) being limited to ~4.30626 nm rather than 5 nm. The maximum overshoot for |T|, and the measurement noise influence on the error, happens at $f \simeq 0.04677351413$ Hz.

We can verify this, again, by simulating the linearized control loop by using P = 1/S and randomly generated noises at 3 standard deviations:



Figure 8.13: Theoretical limit behavior of linear feedback control output for z_1 in bandwidth due to the influence of $|D_1|$ (green dash-dot line) and $|D_2|$ (blue solid line), calculated, respectively, through the inverse of Equations (6.5) and (6.7). The red-dashed line represents the design upper limit ASD_z .

$$\sigma_{d1} = \frac{|D_1|}{3}; \sigma_{d2} = \frac{|D_2|}{3}.$$

Figure 8.14 shows the ASD results of the noise for a 10 hour period sample and the average for 100 samples. The expected overshoot at $f \simeq 0.0468$ Hz does not seem to appear, instead, the simulated behavior is more



Figure 8.14: Results for *G* in bandwidth with a linear feedback control system ($P = 1/s^2$) and design random noises at 3σ levels (Gaussian). *left*: ASD of a 10 h sample with maximum frequency F = 1 Hz. *right*: average μ over 100 samples and 3μ level (representing approximately a zero-mean 3σ of ASD). The theoretical shape of Figure 8.13 is retained without the ~ 3 dB overshoot.

similar to the jitter noise response, probably due to the 3σ margin imposed on the readout noise.

8.8. SIMULATION SETUP

The next step is the simulation of the S/C and GRS dynamics with the inclusion of the drag-free control system.

For this part, the attitude dynamics of the S/C are provided by the equations of Chapter 4 and the control of the Euler angles is not simulated.

Since no control is provided on the attitude, the torque noises are switched off $l_{d0} = 0$. Moreover, we assume that the S/C follows a perfectly gravitationally bound motion by switching off \vec{f}_t and \vec{f}_{d0} in Equation (5.1) describing the dynamics of the S/C. This is to avoid the algebraic loop that would ensue from retrieving

the attitude from the S/C displacement: since the SRP depends on the attitude and the controller that generates the accelerations from which the displacement is integrated depends on the SRP, the system cannot be expressed in closed form (moreover Simulink returns an error addressing this issue when trying to simulate the full loop). We also assume that by controlling the TM with the accuracy determined by the requirements the difference between the actual path of the S/C and drag-free orbit would differ by an order of 10^{-8} m (while the inertial positions of the S/C are in the order of 10^{11} m).

For the calculation of the electrostatic suspension forces \vec{f}_i , in the absence of a controller, in Equation (5.43) all the terms, excluding the first

$$\vec{f}_i/m_i$$

and the fourth

$$C_{s/i}(-\vec{f}_t - \vec{f}_{d0})/m_{s0}$$

are grouped into an acceleration component $\ddot{\vec{r}}_{i_{grs}}$ which can be basically defined as

$$\ddot{\vec{r}}_{igrs} = \ddot{\vec{r}}_i - \left[\frac{\vec{f}_i}{m_i} + C_{s/i}\frac{-\vec{f}_t - \vec{f}_{d0}}{m_{sc}}\right],\tag{8.33}$$

with \vec{r}_i referring to Equation (5.43). In the *y*-direction the whole \vec{r}_i as well as the *z*-component of \vec{r}_2 , need both to be controlled by \vec{f}_i . The force is simply calculated by retrieving the in-simulation accelerations. Obviously this step needs to be replaced by a proper control system in later iterations, but for the moment we are largely interested in the control of the drag-free DoF.

Figure 8.15 shows the schematic operation of selection of the electrostatic suspension controlled compo-



Figure 8.15: Simplified simulation model of the GRS control system. The "GRS dynamics" block refers to Equations (5.43) and (8.33).

nents and calculation of \vec{f}_1 and \vec{f}_2 . Random spurious accelerations \vec{f}_{di} in Equation (5.37) are also absent. The TM masses are set to $m_1 = m_2 = 1.9$ kg, as for the ones used on LISA Pathfinder [45].

In order to verify the action of the dummy electrostatic suspension control system, we perform a simulation without external disturbance, $\vec{f}_{d0} = \vec{0}$, and no drag-free control (the net thrust force results in $\vec{f}_t = \vec{0}$). The stiffness matrix is simply set to $K_i = -\text{diag}[10^{-7}, 10^{-7}, 10^{-7}]$ N/m, which is of the same order as the ones measured on LISA Pathfinder [45]. The only forces acting on the TMs are, therefore, the rotational forces and the elastic force due to K_i (which should be 0 for $\vec{r}_i = \vec{0}$). The simulation is performed with a step-size of 100 s for 4 years. The displacement of the TM \vec{r}_i is shown in Figure 8.16: after 4 years the displacement is in the order of 10^{-17} m for all the components (that is, de facto, 0).



Figure 8.16: Verification results for the dummy feedback electrostatic suspension control system. The divergence on the TM displacements shown here is too small to be attributed to the remaining \vec{r}_{igrs} accelerations due to reference frame rotational accelerations. It is however possible that the small accelerations are due to reference frame conversion operation in Matlab.

On the other hand, the accelerations calculated from the dynamics, \ddot{r}_1 and \ddot{r}_2 are non-zero, as shown in Figure 8.17. These are the rotational accelerations acting on the TM due to $\vec{\omega}$ and $\dot{\vec{\omega}}$ (angular velocity and acceleration of the S/C), as provided in Chapter 4, and, in minor part, to $\ddot{\alpha}$. A decomposition of the forces verifies that the accelerations in the *x*- and *y*-directions are due almost completely to the centrifugal accelerations, with minor influence from the Euler forces due to the telescope steering mechanism. The forces along the *z*-axis, instead are due to Euler accelerations only for about ~33.12% and the rest are due to centrifugal accelerations.

On a side note, these results do not account for the coupling with the torques \vec{l}_t and \vec{l}_{d0} (See Equation (5.53)), which are controlled by the attitude control system. Moreover, due to the segregation of $\vec{r}_{i_{grs}}$, i = 1, 2 the system becomes completely linear.



Figure 8.17: The accelerations due to non-inertial reference frame rotation (i.e. $\ddot{\vec{r}}_{igrs}$ from Equation (8.33) for no thruster jitter and no SRP force included) acting on the two TM for a 4 years period.

8.9. Results in low frequency

In this section, the simulation is performed by setting $\vec{f}_{d0} = \vec{f}_{srp}$. Figure 8.18 shows the results of a preliminary



Figure 8.18: Initial results for the control of z_1 against the SRP force induced accelerations. The DC component of the displacement is stuck at ~ $3.544 \cdot 10^{-9}$ m due to the presence of a strong DC component in the disturbance accelerations.

simulation on the z_1 DoF: the displacement error, $-z_1$ does not fall below $5 \cdot 10^{-9}$ m. This is due to the DC component of the SRP force acting in the *z*-direction (Figure 8.2). The annual trend, on the other hand, follows the design $2.5 \cdot 10^{-9}$ m amplitude imposed by $|S_{z_{lf}}|$ (Equation (8.21)).

The control transfer function G for z_1 must be modified to add a *roll-off* filter, i.e. an integrator, in parallel.

The gain of such filter is tentatively chosen as ω_o , as we are looking for a low value:

$$G_{ro} = G + \frac{\omega_o}{s} \tag{8.34}$$

Following the modification, the system is simulated again, including x_1 and x_2 in the loop, at a time-step of 0.1 s for 1 year: when trying to use a higher step size, the simulation fails after a few seconds due to state derivative of x_1 and x_2 approaching infinity. This is due to the large k > 512 gain factor associated with the control transfer function *G* of Equation (8.29) and the resonant frequency being located at $f_r \approx 2.5$ Hz (see Figure 8.8) with a ~6 dB overshoot (by design). Simulation frequency is required to stay at $f_{sim} \gg f_r$ and the sampling is performed at $f_{samp} = 2$ Hz, requiring, therefore f_{sim} to be a multiple of f_{samp} in Simulink in order to avoid an interpolation step.

The initial conditions are set to 0, both for displacement and velocity. Figure 8.19 shows the results of the



Figure 8.19: Results in low frequency for both the *x*-directions and z_1 . *top*: the transient for the first 26 days; *bottom*: the simulated behavior for 1 year at nm scale. The improved version of the control transfer function for z_1 allows its displacement to lower below the required level of $\delta z \le 5$ nm. The imposed limit ≤ 2.5 nm (due to the margin factor of 2) on x_1 and x_2 is satisfied. (showed here are the errors $-x_1$, $-x_2$ and $-z_1$).

simulation (the output is retrieved every 1000 s). The $5 \cdot 10^{-9}$ m displacement mark is reached at about $t \approx 25$ d 23 h 3 min 20 s on the x_2 -axis while the displacement on the *z*-axis reaches the mark at $t \approx 18$ d 1 h 20 min 00 s. The transient shows a maximum peak, at $x_1 \approx 1.766417510 \cdot 10^{-7}$ m for the *x*-axes, and $z_1 = 5.389283204 \cdot 10^{-7}$ m for the *z*-axis.

After the transient, an expected sinusoidal behavior ensues, with a final peak of $|x_1| = 2.131023790$ nm and $|x_2| = 2.254077908$ nm at t = 365 days (the first peak reaches $|x_2| = 2.065391871$ nm).

 z_1 shows an improved behavior, although the DC components is still present around 1 nm and the periodical oscillations appear to shrink. The maximum value after the transient is $|z_1| = 1.303460420$ nm.

8.10. RESULTS IN BANDWIDTH

The final simulation is performed for 35 days with the inclusion of the jitter noise on the dynamics and the readout noise on the control loop. This would correspond to little less than 10 days worth of observations after the initial roll-off.

The sampling is initiated after all the DoF reach $5 \cdot 10^{-9}$ m. In our case, it happens at t = 25 days 22 h 46 min 1.0 s. This value may change, depending on the randomly generated noises. This leaves us with 20 samples of 10 h each.

Figure 8.20 show the results obtained using the periodogram() function for the x-axes displacement error in a random sample.



Figure 8.20: ASD of the x-direction displacements calculated with periodogram() on a random 10 h sample (maximum frequency F = 1 Hz) after the settling time. *left*: x_1 , *right*: x_2 .

Figure 8.21 shows the ASD of the numerically extrapolated accelerations (see Section E3) for the same sample.



Figure 8.21: ASD of the *x*-direction accelerations calculated with periodogram() on a random 10 h sample (maximum frequency F = 1 Hz) after the settling time. *left*: \ddot{x}_1 , *right*: \ddot{x}_2 .

Figure 8.22 shows the periodogram of the z_1 DoF for a random sample. While the shapes obtained for x_1 and x_2 match the theoretical results obtained in 8.6, the z_1 ASD is about 4 times smaller than theoretically designed. Since the same behavior is attributed to the low frequency components we can assume that this is because of the introduction of the roll-off filter for the compensation of the DC component of the SRP force.

The mean ASD for the 20 samples are shown, respectively, in Figures 8.23, 8.24 and 8.25.



Figure 8.22: ASD of the z_1 displacement calculated with periodogram() on a random 10 h sample (maximum frequency F = 1 Hz) after the settling time.



Figure 8.23: Average ASD between 20 consecutive samples of 10 h periods of the *x*-displacement errors calculated with periodogram() (maximum frequency F = 1 Hz) after the settling time. *left*: x_1 , *right*: x_2 .



Figure 8.24: Average ASD between 20 consecutive samples of 10 h periods of the *x*-direction acceleration errors calculated with periodogram() (maximum frequency F = 1 Hz) after the settling time. *left*: \ddot{x}_1 , *right*: \ddot{x}_2 .



Figure 8.25: Average ASD between 20 consecutive samples of 10 h periods of the z_1 displacement error calculated with periodogram() (maximum frequency F = 1 Hz) after the settling time.

8.11. RECOMMENDATIONS ON THE DRAG-FREE CONTROL

There is much to be improved on the control system. With more time at disposal more observation can be performed to discern a more accurate average behavior.

One goal is to achieve the simulation of the system without separation of the dynamics, with control on attitude and electrostatic suspension system. This step requires large computational power.

A good inclusion would be to see how the S/C orbit, and therefore the formation, would change if the rotational accelerations were also counteracted by the thruster force along the *x*-axes.

8.12. CONCLUSION ON THE DRAG-FREE CONTROL SYSTEM

In this Chapter a control system for the drag-free DoF $\mathbf{x}_{df} = (x_1, x_2, z_1)^{T}$ has been defined using the separation of dynamics and NDI. The linearized dynamics can be modeled as a simple multiple SISO problem with a double integrator plant $P = 1/s^2$. The DC requirements were used to derive the sensitivity functions in low frequency, using the SRP induced accelerations on the TM as reference disturbances and adding a margin factor of 2.

In bandwidth, the sensitivity requirements for the controller of x_1 and x_2 are defined separately from the ones for z_1 . The noise is determined by the ASD of the thrust jitters defined in Section 5.12 at 3σ . While for z_1 only the limits on the displacement needs to be evaluated, for the control on the *x*-direction the sensitivity requirements on the accelerations add two more boundaries. The readout noise for the *x*-directions is defined by the limit on the accelerations. The boundaries for *S* and *T* on the phase-magnitude plane are calculated at every octave from 1 μ Hz. A stability margin of |T| = 3 dB is added. During the design process using QFT the stability margin for the *x*-direction controller is raised to |T| = 6 dB to lower the gain factor *k* to ~ 512.33.

Two controllers are defined: a four zero, four pole transfer function for the *x*-direction and a two zero, two pole transfer function for z_1 .

A linear feedback control loop is simulated by adding the associated noises as white noises, simulated with the design values at 3σ . The result of the ASD in bandwidth for both displacement and acceleration requirements is compliant with the expectations and the requirements.

A simulation using only SRP force on the z_1 control loop reveals that a strong DC component does not allow the DoF to comply with the requirements. The problem is fixed by adding a small gain roll-off filter in parallel. In the next simulation, all the DoF are controlled below the required displacement limits.

The only drawback of this system is the very slow settling time of >25 days, meaning that this controller cannot be used for the initial acquisition of the attitude.

Finally, the whole system is simulated using the real dynamics. The results for x_1 and x_2 in bandwidth satisfy the requirements for both displacement and accelerations. The control on z_1 also satisfies the requirements but the output is slightly overcompensated probably due to the added roll-off filter. Separation of the dynamics is forced by simulating a dummy electrostatic suspension system directly feeding back the unwanted accelerations. The dummy feedback system is verified.

9

CONCLUSIONS

9.1. SUMMARY

The objective of this thesis was to design a DFACS algorithm for LISA that would satisfy the strict science requirements dictated by the mission specifications.

A generic orbital model for the three S/C was defined in Chapter 2. The task had the objective to simulate the exact motion of the three S/C in a gravitationally perturbed environment. For sake of brevity the orbits are not optimized, showing a compliance with the requirements during at least the first 4 years of operation. As such, this period was selected for analysis in the later chapters. The model has been adapted in part from S.V. Dhurandhar et al. [27] and validated against G. Li et al. [33]. Earth and Moon gravities have been added to the model and propagation was compared to a simulated Earth-Sun CR3BP with the same initial conditions, showing compliance for the first 10 years of simulation to the formation drift from Earth. As a result, a simulator has been developed using a Runge-Kutta 4th order integrator and a Cowell propagator, thus answering the subquestion "What are the orbits of the LISA formation?". The model is an improvement from first-order approximations used in the past to simulate the Cartwheel orbit [20] behavior of the formation.

In Chapter 3 a physical model for the S/C has been derived using the guidelines provided by a presentation by the CDF team at ESTEC [21]. The S/C mass and inertia have been estimated using a geometrical manipulation and separating the volumes dedicated to dry-mass and propellant storage. The thrust configuration matrix and the relation between necessary thrust-torque and effort of each thruster has been defined using a *Least square thruster dispatching method* by D. Ferting and S. Wu [38]. The full solution, using a Kuhn-Tucker optimization algorithm [39] has been shown. The solution was verified, although little can be said about its validity due to a lack of independent observations, except that it complies with the physical definition of thruster configuration matrix. Propellant mass depletion due to thruster usage has been modelled, using a conservative value for the specific impulse of $I_{sp} = 340$ s.

The orientation of the body-fixed reference frame has been defined with the *x*-axis oriented so to bisect the angle subtended by the lines of sight of the telescopes and the *z*-axis orthogonal and pointing in the opposite direction as the solar panel, thus providing an answer to the question: "How is the S/C orientation defined?". We have assumed that both the telescopes are articulated and capable of symmetrical steering.

In Chapter 4 we tried to answer the subquestion: "What is the expected behavior of the S/C in relation to the other two?". As a solution, an analytical model for the S/C orientation and rotational behavior dependent on the formation state has been derived. First, the generic orientation of the S/C body-fixed directives has been defined according to their position in the inertial reference frame. Then, using simulation data for their Cartesian velocities and accelerations, an analytical relation for their expected rotational velocities and accelerations have been derived. Poisson's kinematics [40] have been exploited for the definition of the angular velocity in body-fixed reference frame. The S/C are assumed to point the body *x*-axes towards a common point, here defined as the "incenter" of the triangle whose vertexes are the S/C themselves. The *z*-axes, orthogonal to the plane swooped by the telescopes lines-of-sights have been assumed to also be perpendicular to the formation plane, and therefore congruent.

In order to validate the equations, a different situation involving three particles in an analytically defined motion in the 3D space has been modelled, allowing to determine positions, velocities, accelerations and angular velocities and accelerations analytically and separately from each other. The derived equations have

been applied successfully to retrieve the exact same angular states from the translational states. When applied to LISA, the equations showed that the angular velocities and accelerations for a perfectly controlled LISA are small, with components in the order of, respectively, $\simeq 10^{-7}$ rad/s and $\simeq 10^{-14}$ rad/s², which are compatible to the sidereal motion ($\simeq 2 \cdot 10^{-7}$ rad/s).

In the process, the equations for the determination of the breath angles and their divergence speeds and accelerations have also been derived as an intermediary step. Once again, the solution considers the telescopes to be able to steer symmetrically w.r.t. the body-fixed *x*-axes.

In Chapter 5 the dynamics and the disturbances acting on the LISA DoF have been modeled. While the S/C translational and rotational dynamics are trivially determined through the Newtonian and Eulerian equations of motion, aided by the estimation of mass and inertia in Chapter 3, the dynamics of the DoF pertaining the TM are way more complicated due to the coupling of several factors, including S/C rotations and steering of the optical assembly, S/C-TM interactions [23], reference frame transformations between S/C body and GRS specific reference frames and the presence of the electrostatic suspension system, providing forces and torques to the TM. Both the displacement and orientation related dynamics for the TM have been derived. Validation of such equations in their natural form is not possible due to a gap of information in the publicly available literature. The linearized equations, on the other hand, show compliance with the ones provided by P.F. Gath [22] and S.-F. Wu and D. Fertin (for LISA Pathfinder) [23] in their expressions.

In the same chapter, the SRP force and torque is modeled according to the LISA Pathfinder specifications (calling for an absorptivity coefficient of 0.14 and diffuse reflectivity of the solar panels [43]) and the S/C geometry defined in Chapter 3. A preliminary study on the thruster efforts required to compensate both the torques and the forces has been carried out by means of the thruster configuration results of Chapter 3. Using the data on the required thruster efforts, the noise filters for the thrust and the torque jitters have been modeled applying a random factor with standard deviation $3\sigma = 0.01$ to the force of each thruster. These filters allow to model the noise in the measurement bandwidth of $20 \cdot 10^{-6}$ Hz< f < 1 Hz. According to this range, we decided to analyze the periodogram of the noise and the results with a maximum bandwidth frequency of F = 1 Hz, which requires to sample the simulation outputs with a frequency of $f_{samp} = 2$ Hz.

Finally, the attitude definition using the Euler angles and the associated kinematics have been presented. The use of Euler angles has been justified in order to limit the number of DoF (3 for the Euler angles, 4 for the quaternions, 12 for the axis orientations).

In summary, Chapter 5 was able to answer the subquestion "What are the perturbations acting on each degree of freedom and what are their components?": size and orientation of the SRP force and torque have been modeled and preliminary data for a perfectly controlled LISA S/C has been calculated, the other spurious accelerations considered are the thrust and torque jitters, preliminary defined by the $\pm 1\%$ confidence at 3σ applied to all the thrusters involved; other forces, in this iteration, are not considered; the full analytical dynamics of each DoF have been derived, except that the telescope steering mechanism has been modeled using only the accelerations of the breath angles for a perfect case due to a lack of investigation on its physical properties; linearization of the dynamics provides in part information on the sensitivity of each DoF on the perturbations. In total 19 DoF have been defined.

Chapter 6 was redacted to present the strategy used for the design of the control system using the QFT technique [48]. The method exploits the phase-magnitude graph of the open-loop transfer function L of a classic SISO linearized feedback control loop to determine its allowed response at each frequency. Methods to calculate the sensitivity functions based on the disturbances (both on dynamics and due to the limitations of the state readout) and to ensure theoretical stability [50] are presented. It was also determined, due to the complementarity of the sensitivity functions, that the dynamics noise and the readout noise cannot be mutually free. The dynamics noise D_1 is associated to the *S* sensitivity functions, while the readout noise D_2 is associated to the *T* sensitivity functions (that have the same relationship as the closed-loop transfer function L). Based on the sensitivity functions, the areas where L is allowed at each frequency can be identified on the phase-magnitude plot.

Due to a lack of physical time only the attitude and drag-free control system is studied, using separation of dynamics to avoid computational complexity and to allow the two systems to be studied separately. The remaining subsystem, that is not studied here, is the GRS electrostatic suspension control. Finally we can assert that the answer to the subquestion "What are the limitations of the control system?" has been provided here.

The final two chapters deal with the design and simulation of the control systems. The control requirements have been defined in K. Danzmann *et al.* (2017) [14].

In Chapter 7 the attitude of S/C_1 is controlled separately from any other DoF. The Euler angle dynamics

have been linearized and treated through NDI to transform the frequency-domain plant to a simple SISO double integrator $P = 1/s^2$ (*s* being the Laplace transform parameter, or complex frequency). The linearized disturbance at low frequency due to the SRP torque is modeled first in time-domain and then transformed numerically to a frequency-domain Fourier series; the limits on the *S* function in low frequency are then determined by forcing each component to fall below the DC requirements of $\delta\theta < 10$ nrad (divided by a safety margin factor of 4).

In bandwidth, the disturbance acting on the Euler angle accelerations have been modeled from the torque jitter in frequency domain. Their estimated ASDs at 3σ are used as dynamics disturbance on the control loop. The relative *S* function limits at various frequency levels have been determined in order to keep the ASD of the error below the science requirements. The maximum readout noise allowable has also been calculated to determine a *T* function limit compatible with the *S* functions (converging to |T| = 0 dB at high frequencies). The limits have been translated to boundaries on the phase-magnitude plot and an open-loop transfer function L = GP, where *G* is a transfer function with 5 poles and 5 zeros, has been designed. The limits on the *T* function are not satisfied, although the closed loop transfer function is limited to |T| = 3 dB for stability reasons. The limits on the *S* functions are closely matched to avoid overdesign. The model has been verified on a linear SISO control loop, showing that the results comply to the imposed limits due to the several margins added (e.g. the readout noise is simulated considering the design level at 3 standard deviations 3σ).

The simulation is performed using a model on Simulink which integrates the true dynamics of the system. The reference commands are generated as calculated in Chapter 4. The validation of the model is performed by checking whether feeding the theoretical torques to the system and propagating the S/C attitude results as analytically determined. A Runge-Kutta 4 integrator is used.

First, a large step size is used, without the jitters, to verify that the control of the SRP torque satisfy the DC requirements. Then, a 35 days, 0.1 s step-size simulation, with jitters and readout noise included, is performed. The ASD of the errors on the Euler angles has been numerically calculated to check the feasibility of the control system. The simulation reveals satisfying results for both the DC and bandwidth requirements.

In Chapter 8 the same procedure is performed. First, the drag-free DoF have been determined. These are the *x*-displacements of the two TM and the *z*-axis displacement of $TM_1 x_{df} = (x_1, x_2, z_1)^{T}$. The NDI is performed, but due to the separation of the dynamics we consider only the concerning control thrust to act on the drag-free control system while assuming that the electrostatic suspension control system of the GRS and the attitude control take care of the rest of the accelerations with 0 error. As a result, the linearized dynamics can be modeled as a simple multiple SISO problem with a double integrator plant $P = 1/s^2$ as for the Euler angles.

As for the Euler angles, the DC requirements on the drag-free DoF are associated to the SRP force. The limits on the S sensitivity function are calculated by dividing the requirements $\delta r \leq 5 \cdot 10^{-9}$ m by a safety margin factor of 2 and evaluated for the amplitude of the signals at $\sim 2 \cdot 10^{-7}$ rad/s. The bandwidth requirements are associated to the ASD of the accelerations due the spurious thrust jitters. Since x_1 , and x_2 have different requirements than z_1 , two separate control transfer functions need to be evaluated. Moreover, the requirements on the x-directions include limits on the accelerations in bandwidth, therefore two S boundaries and two T boundaries need to be determined (one of both for the displacement requirements and the others for the acceleration requirements). The readout noise along the x-directions has been sized to comply with the acceleration requirements. For z_1 only a flat cap at $5 \cdot 10^{-9}$ m/ $\sqrt{\text{Hz}}$ for all frequencies is specified in the requirements. The same value is attributed to the readout noise, hence the T boundary is evaluated at |T| < 0 dB at every frequency. The boundaries are translated onto the phase-magnitude plot of the openloop transfer function and the L = GP function loop is defined obtaining a 4-zero, 4-pole G function for the control of the x-directions and a simpler 2-zero, 2-pole G function for the z_1 control. The stability bound for the *x*-direction was raised from $|T| \le 3$ dB to 6 dB otherwise the required gain factor $k \approx 512.33$ would be much higher and not compatible with the step size of the controller of 0.1 s. A verification step with a linear control loop and noises simulated as design at a 3σ level confirms that the control transfer function works as expected in bandwidth.

The non-linear simulator is set up in Simulink. The unwanted accelerations are directly fed back to emulate the electrostatic suspension system, since we have not defined a control loop for it yet. Moreover, all the S/C related and telescope steering mechanism rotations and angular accelerations are calculated analytically using the relations derived in Chapter 4. The system is tested by verifying that the TM position stays to 0 when no external acceleration is added. The only accelerations are, therefore, Euler, centrifugal and Coriolis accelerations due to the S/C orientation.

The control system for z_1 fails in DC due to a continuous component of the SRP force in that direction,

therefore the transfer function *G* needs to be modified by adding a roll-off filter. We also find that simulating the system at a step size larger than 0.1 s is impossible due to the high gain of the *x*-direction control function. On the other hand, the system performs exactly as expected both in low frequency and in bandwidth, except for a small overcompensation in the *z*-direction, we assume, due to the roll off filter.

9.2. RESULTS

The very first batch of results comes from the analytical equations for the determination of orientation, angular velocity and angular acceleration and breath angle divergence speed and accelerations derived in Chapter 4. The equations offer a very interesting tool for the analysis of LISA-like formations composed of 3 S/C whose orientations are strictly co-dependent. The equations can be easily adapted for the case in which only one telescope is capable of steering [37] by redefining the body-fixed *x*-axis.

Information about the angular rotation and acceleration was used to calculate the required thruster efforts (Figure 5.6). This result is very specific for the case presented here, where the thrusters are configured as defined in Chapter 3 and the SRP acts in the way modeled in Section 5.4. We obtained that the minimum capability for the μ Newton thrusters is ~ 43 μ N. Darkening the exposed surface, i.e. raising its absorptivity, would lower this constraint. It is also shown how the effort required by the thrusters is very asymmetrical, mostly concentrated on the *z*-axis of the S/C.

A minor result is the definition of the free-floating TM dynamics inside the GRS. Its linearization (Equations (5.53)-(5.54)) clearly shows the first-order effects of the coupling between different DoF dynamics in the GRS: the S/C torques and the telescope steering mechanism add an important contribution to the dynamics.

The second set of results comes from Chapters 7 and 8. To begin with, QFT is very interesting for the design of an optimal control system while meeting various criteria at the same time: for the case of the dragfree control along the telescope lines-of-sight we were able to define the open-loop function while satisfying requirements for the displacement and accelerations, against dynamics and readout noise. Moreover using the Nichols plot, we were able to predict with a certain degree of accuracy how the output would react by adding and removing poles and zeros. The drawback is that QFT also relies on a learning curve (in fact, it took a few months to master how to define boundaries and *L* accordingly). Moreover, as in the case of the control of z_1 , I did not foresee the implications of the DC component in the SRP disturbance before the numerical simulation, therefore adding the roll-off filter caused the system to be less optimal and more overcompensated.

A danger from the determination of the control command using QFT coupled with NDI, as it was performed here, is that information about the S/C mass and inertia are essential both during design and during the control. It is not wise, for example, to use a generic value for the mass and inertia: instead of using the mass propagated at the end of the first 4 years of operation (which is very close to the initial mass), as it was done here, it would be more logical to use the value of the dry mass directly, i.e. at the lowest possible S/C mass; this is because the accelerations are more prominent for lower values of mass and inertia, and therefore the design disturbances much higher. The reason why this was not done in this thesis is that we are interested in showing the behavior of the control system during the first part of the mission and without repositioning in between (that would deplete the propellant mass stored on board). With more time at disposal the simulations can be performed for various scenarios involving different values of mass, although the control transfer function would have to be redesigned.

The most interesting results come from the simulation of the control loop under SRP forces and torques: it is shown here that the minimum possible settling time for an ideal starting point is > 7 days for the attitude control and > 25 days for the precise drag-free control. Obviously this means that this control system, although suitable for the science operations, cannot be used alone during acquisition, considering the various disruptions happening in between observations, such as antenna repositioning for downlink and micrometeoroid strikes [14, 21]. Ironically, while the roll-off is not fast enough for the acquisition phase, the control on the bandwidth along the *x*-axes requires a somewhat high proportional gain $k \approx 512.33$, at the expense of simulation time and possible incompatibility with the on-board computer frequency of 10 Hz.

We can conclude by recalling the research question:

What is a suitable algorithm for the drag-free and attitude control system that allows the LISA S/C to perform within all requirements during science operations?

While the system designed here is certainly suitable to control a LISA S/C during the science operations, much more time and efforts are required to finalize a complete working system.

9.3. RECOMMENDATIONS

Here are some steps to be taken from here on. First of all, it would be interesting to simulate both attitude and drag-free control not separated from each other. As Equation (5.53) shows, the torque commands and the torque disturbances are part of the TM dynamics. The non linearized dynamics in Equation (5.45) expand on this by adding also the centrifugal and Coriolis accelerations arising from the coupling of the two systems. As shown by S.-F. Wu and D. Fertin [23] for LISA Pathfinder, even the readout noise on one system is responsible for the degradation of the output of the other (provided that the readout of the attitude on LISA Pathfinder is not as precise as LISA).

An important aspect that we have not discussed here due to time constraints is that the μ Newton thrusters have a discrete resolution. This was also taken into consideration by S.-F. Wu and D. Fertin [23] while designing a control system for LISA Pathfinder.

We should also address the fact that modeling of the bandwidth disturbances as a band-limited white noise based on the 1% confidence level at 3σ of the thruster commands was an arbitrary choice. We have seen in Chapter 8 that, at these levels, the gain required for the noise reduction for the TM accelerations forced the simulation to run at a minimum step size of 0.1 s. Building up from this requires: 1) higher computing power for larger confidence levels or 2) a more realistic noise model based on experimental data.

The control system for the GRS electrostatic suspension system has a very difficult challenge to overcome that is the presence of the stiffness terms K_i and $K_{\varphi i}$: these terms, due to the nature of their presence, create unstable positive poles in the frequency-domain representation of the dynamics [22, 23] and their values need to be estimated during the mission [45].

Another problem that constrained the results for this thesis is the available computer power and the limits imposed on the Simulink system. With more time at disposal more simulations can be performed without needing to rely on the Rapid Accelerator mode [51], and therefore allowing algebraic loops in the system [46].

A

SPACECRAFT MODELING WITH MATHEMATICA

A.1. INERTIA OF THE DRY MASS

The definition of the trapezoidal shape, used to model the inertia of the dry mass can be defined as a subset V_T of the 3-dimensional Euclidean space:

$$V_{T} = V_{A} \cap V_{B} \cap V_{C} \cap V_{D} \cap V_{E}$$
where
$$V_{A} = \{(x', y', z') \in \mathbb{R} | 0 \le z' \le h\}$$

$$V_{B} = \left\{(x', y', z') \in \mathbb{R} | z' \le \frac{2(a_{1} - x')h}{a_{1} - a_{2}}\right\}$$

$$V_{C} = \left\{(x', y', z') \in \mathbb{R} | z' \le \frac{2x'h}{a_{1} - a_{2}}\right\}$$

$$V_{D} = \left\{(x', y', z') \in \mathbb{R} | z' \le \frac{2(b_{1} - y')h}{b_{1} - b_{2}}\right\}$$

$$V_{E} = \left\{(x', y', z') \in \mathbb{R} | z' \le \frac{2y'h}{b_{1} - b_{2}}\right\}$$
(A.1)

Operator \cap represents the intersection between sets. Values $a_1 = 3$ m, $b_1 = 4.5$ m, $a_2 = 1.6$ m, $b_2 = 3$ m and h = 1.1 m are the bases and the height of the trapezoid.

In Mathematica, the set can be defined using the ImplicitRegion[] function as demonstrated in the following lines [53]:

Out[1]:=



Using the Volume [] function, the volume V_{dry} of the trapezoid lying on the *xy*-plane V_T is easily calculated:

In[2] := Vdry = Volume[VT](*Volume*)

Out[2]:= 10.0283

Alternatively, one can use the formula:

$$V_{dry} = a_1 b_1 h - [(a_1 - a_2)b_1 + (b_1 - b_2)a_1]\frac{h}{2} + (a_1 - a_2)(b_1 - b_2)\frac{h}{3}$$
(A.2)

which in Mathematica results in:

$$In[2] := Vdry = a1*b1*h - ((a1 - a2) b1 + (b1 - b2) a1) h/2 + (a1 - a2) (b1 - b2) h/3$$

Out[2]:= 10.0283

The two values differ by 1.77635683940025 $\times 10^{-15}~m^3.$

The position on the *z*-axis of the CoM is h_{cm} . This is the point in which the volume of the trapezoid is bisected by a *z*-orthogonal plane. Using the relation between the bases *a* and *b* of the sectioned rectangle and the position of the plane z',

$$a(z') = a_1 - (a_1 - a_2)\frac{z'}{h}$$
(A.3)

and

$$b(z') = b_1 - (b_1 - b_2)\frac{z'}{h},$$
(A.4)

 h_{cm} has solution:

$$\int_{0}^{h_{cm}} a(z')b(z')dz' = \frac{V_{dry}}{2}$$
(A.5)

being V_{dry} the volume of the trapezoid. This operation is performed in Mathematica using the Solve[] and Integrate[] function as following:
Out[3]:= 0.413394

In order to center the CoM at the origin, it is sufficient to shift V_T by h_{cm} along the *z*-direction by setting $z = z' - h_{cm}$. Mathematically, we define a new set, called V_{lisa} as:

$$V_{lisa} = \{(x, y, z) \in \mathbb{R} | (x, y, z + h_{cm}) \in V_T \}$$
(A.6)

In Mathematica this is simply performed by repeating the ImplicitRegion[] function by substituting z + hcm to z:

```
Out[4]:=
```



We now proceed to calculate the inertial components (per unit density) of the V_{lisa} three-dimensional space. The diagonal components are calculated as:

```
In[5] := Ixx = Integrate[z^2 + y^2, {x, y, z} \[Element] lisa]
```

```
Out[5]:= 14.700725
```

```
In[6] := Iyy = Integrate[z^2 + x^2, {x, y, z} \[Element] lisa]
```

```
Out[6]:= 5.980763
```

```
In[7] := Izz = Integrate[x^2 + y^2, {x, y, z} \[Element] lisa]
```

```
Out[7]:= 18.769267
```

We can demonstrate that the origin of the axes and the CoM coincide by verifying that the non-diagonal components are zero:

In[8] := Ixy = -Integrate[x*y, {x, y, z} \[Element] lisa]

 $Out[8] := 2.694242 \times 10^{-17}$

In[9] := Ixz = -Integrate[x*z, {x, y, z} \[Element] lisa]

 $Out[9] := -2.875846 \times 10^{-17}$

In[10] := Iyz = -Integrate[y*z, {x, y, z} \[Element] lisa]

 $Out[10] := -4.729832 \times 10^{-17}$ The non-diagonal terms calculated by Mathematica have a magnitude in the order of 10^{-17} m⁵ (17 to 18 orders of magnitude lower than the diagonal elements), consistent with 0.

A.2. INERTIA OF THE PROPELLANT TANKS

The canisters containing the propellant are spheres of radius $r_s = 0.3665$ m located at coordinates $x_t = 0.8$ m and $y_t = 1.5$ m in all quadrants, symmetrically w.r.t. the origin of the axes and at z = 0.

Mathematically the 3-dimensional Euclidean space V_S used to represent them is defined as:

$$V_{S} = V_{S1} \cup V_{S2} \cup V_{S3} \cup V_{S4}$$
where
$$V_{S1} = \{(x, y, z) \in \mathbb{R} | (x - x_{t})^{2} + (y - y_{t})^{2} + z^{2} \le r_{s}^{2} \}$$

$$V_{S2} = \{(x, y, z) \in \mathbb{R} | (x + x_{t})^{2} + (y - y_{t})^{2} + z^{2} \le r_{s}^{2} \}$$

$$V_{S3} = \{(x, y, z) \in \mathbb{R} | (x - x_{t})^{2} + (y + y_{t})^{2} + z^{2} \le r_{s}^{2} \}$$

$$V_{S4} = \{(x, y, z) \in \mathbb{R} | (x + x_{t})^{2} + (y + y_{t})^{2} + z^{2} \le r_{s}^{2} \}$$
(A.7)

The \cup operator represents the union of sets. In Mathematica this can be accomplished by using the ImpliciRegion[] function:





The volume V_{prop} of V_S can be calculated using the function Volume[]

Out[2]:= 0.824842

Alternatively the equation for the volume of four spheres with the same radius can be used:

$$V_{prop} = 4 \cdot \frac{4}{3} \pi r_s^3 \tag{A.8}$$

which in Mathematica results in

In[2] := 4*4/3*Pi*rs^3 = Volume[VS]

Out [2] := 0.824842 The two results differ by 3.23188675821484 × 10⁻⁹ m³. The diagonal elements of the inertia matrix (per unit density) are calculated accordingly: In [3] := Ixx = Integrate [$z^2 + y^2$, {x, y, z} \[Element] VS] Out [3] := 1.900211 In [4] := Iyy = Integrate [$z^2 + x^2$, {x, y, z} \[Element] VS] Out [4] := 0.572216 In [5] := Izz = Integrate [$x^2 + y^2$, {x, y, z} \[Element] VS] Out [5] := 2.428110 And we can verify that the origin is congruent to the CoM by calculating the non-diagonal components: In [6] := Ixy = -Integrate [x*y, {x, y, z} \[Element] VS] Out [6] := $-2.873136 \times 10^{-18} + 3.231352 \times 10^{-25}i$ In [7] := Ixz = -Integrate [x*z, {x, y, z} \[Element] VS] Out [7] := 0. In [2] := Iyz = -Integrate [y*z, {x, y, z} \[Element] VS]

Out[8]:= 0.

While the latter two are 0 even to machine precision, the first has a real component in the order of magnitude of 10^{-18} and an imaginary component in the order of magnitude of 10^{-25} . Although it might be recorded as a strange result, the values are so small that they are consistent with 0. The complexity of the result might be due to the method used by Mathematica to calculate the integral.

B

FORMATION DEPENDANCIES VALIDATION

For the validation of the equations (4.45) and (4.59) an analytical case scenario is defined so that, for a threepoint formation, $\vec{\omega}_i$ and $\dot{\vec{\omega}}_i$ can be evaluated independently from \vec{R}_i , $\dot{\vec{R}}_i$ and $\ddot{\vec{R}}_i$. The schema is presented in Figure B.1.



Figure B.1: The various orientation transformations used to derive the analytical orbits of three particles for validation.

The three S/C perform motions around a central point in time t with their respective longitudes being defined as:

$$\lambda_1 = 2t + \frac{\pi}{12}\sin(2t); \tag{B.1}$$

$$\lambda_2 = 2t - \frac{\pi}{12}\sin(2t) + \frac{2\pi}{3};\tag{B.2}$$

$$\lambda_3 = 2t + \frac{4\pi}{3}.\tag{B.3}$$

The time derivatives of the longitudes are:

$$\dot{\lambda}_1 = 2 + \frac{\pi}{6}\cos(2t);$$
 (B.4)

$$\dot{\lambda}_2 = 2 - \frac{\pi}{6} \cos(2t);$$
 (B.5)

$$\dot{\lambda}_3 = 2 \tag{B.6}$$

and

$$\ddot{\lambda}_1 = -\frac{\pi}{3}\sin(2t); \tag{B.7}$$

$$\ddot{\lambda}_2 = \frac{\pi}{3}\sin(2t);\tag{B.8}$$

$$\ddot{\lambda}_3 = 0. \tag{B.9}$$

The points lie on the *x*-*y*-plane of the original reference frame, defined by the letter *p* (as in Planar). The radial distances of S/C_1 and S/C_2 are chosen as constants of value 1. The radial distance of S/C_3 is defined as:

$${}^{p}r_{3} = 1 + 2\left[\cos\frac{2\pi}{3} - \cos\left(\frac{2\pi}{3} + \frac{\pi}{12}\sin(2t)\right)\right]$$
 (B.10)

in order to keep the origin of reference frame p at the geometrical barycenter of the formation.

Its time derivatives are derived analytically as:

$${}^{p}\dot{r}_{3} = \frac{\pi}{3}\sin\left(\frac{2\pi}{3} + \frac{\pi}{12}\sin(2t)\right)\cos(2t)$$
(B.11)

and

$${}^{p}\ddot{r}_{3} = -\frac{\pi^{2}}{18}\cos\left(\frac{2\pi}{3} + \frac{\pi}{12}\sin(2t)\right)\cos^{2}(2t) - \frac{2\pi}{3}\sin\left(\frac{2\pi}{3} + \frac{\pi}{12}\sin(2t)\right)\sin(2t)$$
(B.12)

Using the above equations we can define the orbits of the three S/C in *p*, respectively, as:

$${}^{p}\vec{r}_{1} = \begin{pmatrix} \cos\lambda_{1}\\ \sin\lambda_{1}\\ 0 \end{pmatrix}; \tag{B.13}$$

$${}^{p}\vec{r}_{2} = \begin{pmatrix} \cos\lambda_{2}\\ \sin\lambda_{2}\\ 0 \end{pmatrix}; \tag{B.14}$$

$${}^{p}\vec{r}_{3} = {}^{p}r_{3} \begin{pmatrix} \cos\lambda_{3}\\ \sin\lambda_{3}\\ 0 \end{pmatrix}; \tag{B.15}$$

Their velocities and accelerations in p are, therefore, derived analytically as:

$${}^{p}\dot{\vec{r}}_{1} = \dot{\lambda}_{1} \begin{pmatrix} -\sin\lambda_{1} \\ \cos\lambda_{1} \\ 0 \end{pmatrix}; \tag{B.16}$$

$${}^{p}\dot{\vec{r}}_{2} = \dot{\lambda}_{2} \begin{pmatrix} -\sin\lambda_{2} \\ \cos\lambda_{2} \\ 0 \end{pmatrix}; \tag{B.17}$$

$${}^{p}\dot{\vec{r}}_{3} = {}^{p}\dot{r}_{3} \begin{pmatrix} \cos\lambda_{3} \\ \sin\lambda_{3} \\ 0 \end{pmatrix} + \dot{\lambda}_{3}{}^{p}r_{3} \begin{pmatrix} -\sin\lambda_{3} \\ \cos\lambda_{3} \\ 0 \end{pmatrix} = {}^{p}\dot{r}_{3} \begin{pmatrix} \cos\lambda_{3} \\ \sin\lambda_{3} \\ 0 \end{pmatrix} + 2{}^{p}r_{3} \begin{pmatrix} -\sin\lambda_{3} \\ \cos\lambda_{3} \\ 0 \end{pmatrix}$$
(B.18)

and

$${}^{p}\ddot{\vec{r}}_{1} = \ddot{\lambda}_{1} \begin{pmatrix} -\sin\lambda_{1} \\ \cos\lambda_{1} \\ 0 \end{pmatrix} + \dot{\lambda}_{1}^{2} \begin{pmatrix} -\cos\lambda_{1} \\ -\sin\lambda_{1} \\ 0 \end{pmatrix};$$
(B.19)

$${}^{p}\ddot{\vec{r}}_{2} = \ddot{\lambda}_{2} \begin{pmatrix} -\sin\lambda_{2} \\ \cos\lambda_{2} \\ 0 \end{pmatrix} + \dot{\lambda}_{2}^{2} \begin{pmatrix} -\cos\lambda_{2} \\ -\sin\lambda_{2} \\ 0 \end{pmatrix};$$
(B.20)

$${}^{p}\ddot{\vec{r}}_{3} = {}^{p}\ddot{r}_{3} \begin{pmatrix} \cos\lambda_{3}\\ \sin\lambda_{3}\\ 0 \end{pmatrix} + \dot{\lambda}_{3}{}^{p}\dot{r}_{3} \begin{pmatrix} -\sin\lambda_{3}\\ \cos\lambda_{3}\\ 0 \end{pmatrix} + 2^{p}\dot{r}_{3} \begin{pmatrix} -\sin\lambda_{3}\\ \cos\lambda_{3}\\ 0 \end{pmatrix} + 2\dot{\lambda}_{3}{}^{p}r_{3} \begin{pmatrix} -\cos\lambda_{3}\\ -\sin\lambda_{3}\\ 0 \end{pmatrix} =$$

$$= {}^{p}\ddot{r}_{3} \begin{pmatrix} \cos\lambda_{3}\\ \sin\lambda_{3}\\ 0 \end{pmatrix} + 4^{p}\dot{r}_{3} \begin{pmatrix} -\sin\lambda_{3}\\ \cos\lambda_{3}\\ 0 \end{pmatrix} + 4^{p}r_{3} \begin{pmatrix} -\cos\lambda_{3}\\ -\sin\lambda_{3}\\ 0 \end{pmatrix} .$$
(B.21)

Next, in order to create a non-trivial solution, the *p* reference frame is given a tilt around the *y*-axis by an angle θ_2 defined as:

$$\theta_2 = \frac{2\pi}{3} + \frac{\pi}{12}\sin t \tag{B.22}$$

Its time derivatives are derived analytically as:

$$\dot{\theta}_2 = -\frac{\pi}{12}\cos t \tag{B.23}$$

and

$$\ddot{\theta}_2 = -\frac{\pi}{12}\sin t. \tag{B.24}$$

The direction-cosine matrix for the rotation about the *y*-axis is

$$C_2 = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$$
(B.25)

and its time derivatives are, respectively:

$$\dot{C}_{2} = \dot{\theta}_{2} \begin{pmatrix} -\sin\theta_{2} & 0 & \cos\theta_{2} \\ 0 & 0 & 0 \\ -\cos\theta_{2} & 0 & -\sin\theta_{2} \end{pmatrix}$$
(B.26)

and

$$\ddot{C}_{2} = \ddot{\theta}_{2} \begin{pmatrix} -\sin\theta_{2} & 0 & \cos\theta_{2} \\ 0 & 0 & 0 \\ -\cos\theta_{2} & 0 & -\sin\theta_{2} \end{pmatrix} + \dot{\theta}_{2}^{2} \begin{pmatrix} -\cos\theta_{2} & 0 & -\sin\theta_{2} \\ 0 & 0 & 0 \\ \sin\theta_{2} & 0 & -\cos\theta_{2} \end{pmatrix}.$$
(B.27)

Next another periodical rotation about the *z*-axis is performed, with an angle θ_3 defined as:

$$\theta_3 = \frac{\pi}{12} \sin t. \tag{B.28}$$

Its time derivatives are:

$$\dot{\theta}_3 = \frac{\pi}{12}\cos t \tag{B.29}$$

and

$$\ddot{\theta}_3 = -\frac{\pi}{12}\sin t. \tag{B.30}$$

The direction-cosine matrix for the transformation about the *z*-axis is

$$C_3 = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0\\ \sin\theta_3 & \cos\theta_3 & 0\\ 0 & 0 & 1 \end{pmatrix}$$
(B.31)

and its time derivatives are, respectively:

$$\dot{C}_{3} = \dot{\theta}_{3} \begin{pmatrix} -\sin\theta_{3} & -\cos\theta_{3} & 0\\ \cos\theta_{3} & -\sin\theta_{3} & 0\\ 0 & 0 & 0 \end{pmatrix}$$
(B.32)

and

$$\ddot{C}_{3} = \ddot{\theta}_{3} \begin{pmatrix} -\sin\theta_{3} & -\cos\theta_{3} & 0\\ \cos\theta_{3} & -\sin\theta_{3} & 0\\ 0 & 0 & 0 \end{pmatrix} + \dot{\theta}_{3}^{2} \begin{pmatrix} -\cos\theta_{3} & \sin\theta_{3} & 0\\ -\sin\theta_{3} & -\cos\theta_{3} & 0\\ 0 & 0 & 0 \end{pmatrix}$$
(B.33)

Furthermore, the whole frame is centered at a reference point \vec{r}_0 which moves in a circular motion around the inertial reference frame on the *x*-*y*-plane, with a time-dependent latitude of

$$\lambda_{cm} = t - \frac{\pi}{12}\sin(2t) \tag{B.34}$$

The time-derivatives of the reference point latitude are:

$$\dot{\lambda}_{cm} = 1 - \frac{\pi}{6} \cos(2t) \tag{B.35}$$

and

$$\ddot{\lambda}_{cm} = \frac{\pi}{3}\sin(2t) \tag{B.36}$$

The reference point orbit is therefore defined as

$$\vec{r}_0 = 3 \begin{pmatrix} \cos \lambda_{cm} \\ \sin \lambda_{cm} \\ 0 \end{pmatrix}$$
(B.37)

Its velocity and acceleration are derived, analytically, as:

$$\dot{\vec{r}}_0 = 3\dot{\lambda}_{cm} \begin{pmatrix} -\sin\lambda_{cm} \\ \cos\lambda_{cm} \\ 0 \end{pmatrix}$$
(B.38)

and

$$\ddot{\vec{r}}_0 = 3\ddot{\lambda}_{cm} \begin{pmatrix} -\sin\lambda_{cm} \\ \cos\lambda_{cm} \\ 0 \end{pmatrix} + 3\dot{\lambda}_{cm}^2 \begin{pmatrix} -\cos\lambda_{cm} \\ -\sin\lambda_{cm} \\ 0 \end{pmatrix}$$
(B.39)

Apart from the translation, the formation orientation is defined in the Hill's reference frame of the reference point. The last transformation matrix to the inertial reference frame is, therefore:

$$C_0 = \begin{pmatrix} \cos \lambda_{cm} & -\sin \lambda_{cm} & 0\\ \sin \lambda_{cm} & \cos \lambda_{cm} & 0\\ 0 & 0 & 1 \end{pmatrix}$$
(B.40)

and its time-derivatives are, respectively

$$\dot{C}_0 = \dot{\lambda}_{cm} \begin{pmatrix} -\sin\lambda_{cm} & -\cos\lambda_{cm} & 0\\ \cos\lambda_{cm} & -\sin\lambda_{cm} & 0\\ 0 & 0 & 0 \end{pmatrix}$$
(B.41)

and

$$\ddot{C}_{0} = \ddot{\lambda}_{cm} \begin{pmatrix} -\sin\lambda_{cm} & -\cos\lambda_{cm} & 0\\ \cos\lambda_{cm} & -\sin\lambda_{cm} & 0\\ 0 & 0 & 0 \end{pmatrix} + \dot{\lambda}_{cm}^{2} \begin{pmatrix} -\cos\lambda_{cm} & \sin\lambda_{cm} & 0\\ -\sin\lambda_{cm} & -\cos\lambda_{cm} & 0\\ 0 & 0 & 0 \end{pmatrix}.$$
(B.42)

Let us now calculate, analytically, the positions, velocities and accelerations of the three S/C_i in the inertial reference frame according to the transformations made:

• rotation about the *y*-axis from reference frame *p* to reference frame *b*,

$${}^b\vec{r}_i = C_2{}^p\vec{r}_i,\tag{B.43}$$

and calculation of velocities and accelerations in *b* by time-differentiation:

$${}^{b}\vec{r}_{i} = \dot{C}_{2}{}^{p}\vec{r}_{i} + C_{2}{}^{p}\vec{r}_{i}; \tag{B.44}$$

$${}^{b}\ddot{\vec{r}}_{i} = \ddot{C}_{2}{}^{p}\vec{r}_{i} + 2\dot{C}_{2}{}^{p}\dot{\vec{r}}_{i} + C_{2}{}^{p}\ddot{\vec{r}}_{i}.$$
 (B.45)

• rotation about the *z*-axis from reference frame *b* to *a*,

$${}^a\vec{r}_i = C_3{}^b\vec{r}_i \tag{B.46}$$

and calculation of velocities and accelerations in *a* by time-differentiation:

$${}^{a}\vec{r}_{i} = \dot{C}_{3}{}^{b}\vec{r}_{i} + C_{3}{}^{b}\vec{r}_{i} \tag{B.47}$$

and

$${}^{a}\vec{r}_{i} = \ddot{C}_{3}{}^{b}\vec{r}_{i} + 2\dot{C}_{3}{}^{b}\vec{r}_{i} + C_{3}{}^{b}\vec{r}_{i}$$
(B.48)

• rotation from the Hill's reference frame of, and translation to \vec{r}_0 :

$$\vec{R}_i = C_0{}^a \vec{r}_i + \vec{r}_0 \tag{B.49}$$

and calculation of velocities and accelerations in the inertial reference frame by time-differentiation:

$$\vec{R}_i = \dot{C}_0{}^a \vec{r}_i + C_0{}^a \dot{\vec{r}}_i + \dot{\vec{r}}_0 \tag{B.50}$$

and

$$\vec{R}_{i} = \ddot{C}_{0}{}^{a}\vec{r}_{i} + 2\dot{C}_{0}{}^{a}\vec{r}_{i} + C_{0}{}^{a}\vec{r}_{i} + \vec{r}_{0}.$$
(B.51)

Let us now derive analytically the body-fixed reference frame angular velocities and accelerations. Let us start by defining the rotation of p in the inertial reference frame as a sum of the rotations of, respectively, the Hill's reference frame of \vec{r}_0 , the reference frame a w.r.t. it, and the reference frame b w.r.t. a:

$$\vec{\omega}_p = C_2^{\tau} \left\{ C_3^{\tau} \left[C_0^{\tau} \begin{pmatrix} 0\\0\\\dot{\lambda}_{cm} \end{pmatrix} + \begin{pmatrix} 0\\0\\\dot{\theta}_3 \end{pmatrix} \right] + \begin{pmatrix} 0\\\dot{\theta}_2\\0 \end{pmatrix} \right\} = C_2^{\tau} \begin{pmatrix} 0\\\dot{\theta}_2\\\dot{\lambda}_{cm} + \dot{\theta}_3 \end{pmatrix}.$$
(B.52)

The rightmost simplification can be performed because C_3 and C_0 are rotations about the *z*-axis.

The angular velocity vector of p is derived by time-differentiation as:

$$\dot{\omega}_p = \dot{C}_2^{\tau} \begin{pmatrix} 0\\\dot{\theta}_2\\\dot{\lambda}_{cm} + \dot{\theta}_3 \end{pmatrix} + C_2^{\tau} \begin{pmatrix} 0\\\ddot{\theta}_2\\\ddot{\lambda}_{cm} + \ddot{\theta}_3 \end{pmatrix}.$$
(B.53)

In Chapter 4 we defined the body-fixed reference points pointing towards the incenter of the formation. This relation can only be derived for S/C_3 in the context of this validation, as, by definition, the other two S/C have a symmetrical motion w.r.t. the line passing through S/C_3 and the origin of p.

Transformation from body-fixed reference frame to p for S/C₃ is defined by the direction-cosine matrix about the *z*-axis

$$C_{3/p} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \lambda_3 & -\sin \lambda_3 & 0 \\ \sin \lambda_3 & \cos \lambda_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
(B.54)

whose time-derivative is

$$\dot{C}_{3/p} = \dot{\lambda}_3 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin\lambda_3 & -\cos\lambda_3 & 0 \\ \cos\lambda_3 & -\sin\lambda_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
(B.55)

The change of sign for *x* and *y* are needed to point the *x*-axis towards the barycenter, rather than away from it.

The value of the angular velocity vector of S/C_3 for the validation, is finally obtained as

$$\vec{\omega}_3 = C_{3/p}^{\tau} \left[\vec{\omega}_p + \begin{pmatrix} 0\\0\\\dot{\lambda}_3 \end{pmatrix} \right] = C_{3/p}^{\tau} \left[\vec{\omega}_p + \begin{pmatrix} 0\\0\\2 \end{pmatrix} \right]$$
(B.56)

and the angular acceleration vector is

$$\dot{\vec{\omega}}_{3} = \dot{C}_{3/p}^{\tau} \left[\vec{\omega}_{p} + \begin{pmatrix} 0\\0\\2 \end{pmatrix} \right] + C_{3/p}^{\tau} \dot{\vec{\omega}}_{p}.$$
(B.57)

Let us recall, however, that the incenter is just a middlepoint that we can easily replace with any other coplanar point and we still would be able to validate all the equations apart from Equations (4.38) and (4.56). Therefore, let us substitute it with the barycenter, by replacing Equation (4.1) with

$$\vec{r}_{cm} = \frac{\vec{R}_1 + \vec{R}_2 + \vec{R}_3}{3},\tag{B.58}$$

Equation (4.38) with

$$\vec{r}_{cm} = \frac{\vec{R}_1 + \vec{R}_2 + \vec{R}_3}{3}$$
(B.59)

and Equation (4.56) with

$$\ddot{\vec{r}}_{cm} = \frac{\ddot{\vec{R}}_1 + \ddot{\vec{R}}_2 + \ddot{\vec{R}}_3}{3}.$$
(B.60)

The transformation matrix from body-fixed reference frame of S/C_i to p and its time-derivative are unchanged. They are calculated, respectively, as:

$$C_{i/p}^{*} = \begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \lambda_{i} & -\sin \lambda_{i} & 0\\ \sin \lambda_{i} & \cos \lambda_{i} & 0\\ 0 & 0 & 1 \end{pmatrix}$$
(B.61)

and

$$\dot{C}_{i/p}^{*} = \dot{\lambda}_{i} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin\lambda_{i} & -\cos\lambda_{i} & 0 \\ \cos\lambda_{i} & -\sin\lambda_{i} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
(B.62)

Finally, the angular velocity vector of the geometrical barycenter-dependent body-fixed reference frame w.r.t. the inertial reference frame is calculated by addition and transformation to body-fixed coordinates,

$$\vec{\omega}_i^* = C_{i/p}^{*\tau} \left[\vec{\omega}_p + \begin{pmatrix} 0\\0\\\dot{\lambda}_i \end{pmatrix} \right] \tag{B.63}$$

and the angular acceleration vector is the time-derivative:

$$\dot{\vec{\omega}}_{i}^{*} = \dot{C}_{i/p}^{*\tau} \left[\vec{\omega}_{p} + \begin{pmatrix} 0\\0\\\dot{\lambda}_{i} \end{pmatrix} \right] + C_{i/p}^{*\tau} \left[\dot{\vec{\omega}}_{p} + \begin{pmatrix} 0\\0\\\ddot{\lambda}_{i} \end{pmatrix} \right].$$
(B.64)

You can easily verify that

$$\vec{\omega}_3 \equiv \vec{\omega}_3^* \tag{B.65}$$

since Equations (B.61)–(B.64) are equivalent to Equations (B.54)–(B.57) for i = 3.

The consistency of \vec{R}_i , \vec{R}_i and \vec{R}_i can be verified by integrating the respective derivatives numerically using the Matlab function trapz() and subtracting the results, i.e.

$$\delta \vec{R}_i(t) = \vec{R}_i(t) - \int_0^t \dot{\vec{R}}_i dt$$
(B.66)

$$\delta \dot{\vec{R}}_{i}(t) = \dot{\vec{R}}_{i}(t) - \int_{0}^{t} \ddot{\vec{R}}_{i} dt$$
 (B.67)

The same can be done with $\vec{\omega}_i$ and $\dot{\vec{\omega}}_i$, i.e.

$$\delta \vec{\omega}_i(t) = \vec{\omega}_i(t) - \int_0^t \dot{\vec{\omega}}_i dt$$
(B.68)

The consistency between \vec{R}_i and $\vec{\omega}_i$ can be verified by calculating \hat{x}_i , \hat{y}_i and \hat{z}_i from Equations (4.3), (4.7) and (4.5) and using relation (4.16) to calculate \dot{x}_i , \dot{y}_i , \dot{z}_i for the numerical integration, i.e.

$$\delta \hat{d}_i(t) = \hat{d}_i(t) - \int_0^t C_{n/i}^\tau \omega_i \times \hat{d}_i dt$$
(B.69)

where $C_{n/i}$ is the transformation matrix from inertial to body-fixed reference frame of S/C_i, calculated using Equation (4.8) and \hat{d}_i is any of \hat{x}_i , \hat{y}_i or \hat{z}_i calculated via Equations (4.3) to (4.7).

Figure B.2 shows the result for all the above, calculated between $t_0 = 0$ and $t_{end} = 2\pi$ for a sample size of 201 and 2001. If the respective derivatives are consistent, then the error should scale down with increasing sample size, which is demonstrated in the figure.

Some validation samples are reported in Table B.1.

Table B.1: Validation values for Equations (4.45) and (4.59). ${}^*\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_1 \text{ and } \vec{\omega}_2 \text{ are derived for } \vec{r}_{cm} = (\vec{R}_1 + \vec{R}_2 + \vec{R}_3)/3$. Use Equations (B.58), (B.59) and (B.60) instead of (4.1), (4.38) and (4.56) for the calculations of \vec{r}_{cm} , $\dot{\vec{r}}_{cm}$, $\ddot{\vec{r}}_{cm}$.

t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$
\vec{R}_1	2.50000	2.09728	-0.18301	-2.43624	-2.50000	-2.35840	-0.06699	2.03464
	0.00000	2.34297	3.68301	2.46706	0.00000	-2.43966	-3.25000	-2.28926
	-0.86603	0.19649	0.70711	0.19649	-0.86603	0.24413	0.96593	0.24413
\vec{R}_2	3.25000	3.24383	0.92802	-2.39876	-3.25000	-2.98656	-0.80302	2.88005
	0.86603	1.71269	2.88264	0.68019	-0.86603	-1.36237	-2.65086	-0.81035
	0.43301	0.73333	-0.35355	-0.53683	0.43301	0.91112	-0.48296	-0.66699
\vec{R}_3	3.25000	2.45312	-0.74501	-2.95924	-3.25000	-2.44927	0.87001	2.87954
	-0.86603	0.44434	2.43435	1.35275	0.86603	-0.69797	-3.09914	-1.40039
	0.43301	-0.92982	-0.35355	0.34034	0.43301	-1.15525	-0.48296	0.42285
$\dot{\vec{R}}_1$	-0.22672	-1.17990	-4.18534	-0.89581	-0.22672	1.53138	3.52560	1.64577
	3.58370	2.45363	0.10328	-2.40752	-3.84550	-2.11250	0.28006	1.92544
	0.13090	1.43547	0.00000	-1.43547	-0.13090	1.83815	0.00000	-1.83815
$\dot{\vec{R}}_2$	0.11336	-0.64742	-5.21081	-2.26352	-0.34009	1.06760	5.40405	1.86591
	0.87555	1.76579	-0.40537	-3.43099	-0.74465	-0.97932	-1.00369	3.78064
	1.04185	-0.50937	-1.54538	0.98847	1.17275	-0.42889	-2.11103	1.37744
$\dot{\vec{R}}_3$	0.11336	-2.67268	-4.31624	-1.34067	0.56681	1.90103	4.78274	0.98832
	-0.17164	3.57481	0.30209	-1.95572	0.30254	-4.70241	0.72363	2.08815
	-1.17275	-0.92610	1.54538	0.44700	-1.04185	-1.40926	2.11103	0.46071
	-0.91570	-3.54938	-0.11391	4.74869	-1.47756	4.73927	-0.86288	-2.70411
$\ddot{\vec{R}}_1$	-0.33474	-0.53697	-5.82369	-1.01677	-0.09731	2.32632	3.48195	2.08351
-	5.57468	-1.99499	-1.35620	-1.99499	5.57468	-1.71743	-2.17325	-1.71743
$\ddot{\vec{R}}_2$	0.29048	-3.76100	-2.57534	4.81796	1.51741	2.08681	6.24974	-7.49044
	-1.24843	2.16634	-8.59288	2.91024	1.70188	-1.99166	8.00198	-3.03494
	-1.30828	-2.92309	2.15906	1.85915	-0.63881	-4.05437	3.10965	1.86194
$\ddot{\vec{R}}_2$	-1.41740	-5.19624	2.68925	2.93997	2.00278	5.68054	-5.38686	-2.31207
	1.58316	2.03272	-6.47561	1.76863	-1.60457	-3.99677	9.40825	-2.71067
0	-4.26640	4.91808	-0.80286	0.13584	-4.93588	5.77181	-0.93641	-0.14451
$\vec{\omega}_1^*$	0.63930	-0.41168	-1.07735	-0.33893	0.18585	-0.02013	-1.47168	-0.11052
	-0.26180	-0.82117	-0.00000	0.54966	0.26180	-0.79037	-0.00000	1.12770
1	2.15450	1.22865	0.39905	1.46963	2.41630	1.72942	1.08206	1.60648
$ec{\omega}_2^*$	-0.54638	-0.82117	0.53867	0.56835	0.13380	-0.79037	0.73584	0.65956
	-0.42275	0.41168	0.93301	-0.30655	-0.29185	0.02013	1.27452	-0.92136
	1.10730	1.22865	1.44625	1.46963	1.36910	1.72942	2.12926	1.60648
$ec{\omega}_3$	-0.09293	0.87176	0.53867	-0.44321	-0.31965	0.57311	0.73584	-1.06067
	0.68455	0.28955	-0.93301	-0.46965	0.03005	0.54464	-1.27452	-0.39862
	1.63090	1.22865	0.92265	1.46963	1.89270	1.72942	1.60566	1.60648
$\dot{ec \omega}_1^*$	-0.75731	-1.59596	0.18512	1.13724	0.68877	-2.07336	-0.25288	2.66353
	-1.61334	0.28119	1.32880	-0.17890	-0.46901	-1.08301	2.43459	-0.58688
	-0.16737	-1.77485	0.18512	1.96379	0.04866	-1.31410	-0.06776	1.12651
$\dot{ec \omega}_2^*$	-0.57583	0.28119	2.03526	-1.07433	-0.44493	-1.08301	3.56953	-2.60012
	0.89035	1.59596	-1.38881	-0.41369	-0.22187	2.07336	-1.76887	-0.82351
	-0.16737	0.31954	0.18512	-0.13060	0.04866	0.78029	-0.06776	-0.96788
$\dot{ec \omega}_3$	1.41742	0.92968	-1.73185	-0.12153	0.04606	2.23189	-2.64932	-0.12249
	0.10217	-1.32735	-0.78613	1.14480	0.66363	-0.70029	-1.82158	2.72467
	-0.16737	-0.72765	0.18512	0.91659	0.04866	-0.26690	-0.06776	0.07931



Figure B.2: The difference of various validation values and the numerical integration of their derivatives calculated with Matlab function *trapz*. It shows that the residuals scale down of a factor of 100 for sample size increasing by 10.

C

VERIFICATION OF DERIVATIVE FUNCTIONS

C.1. ANGULAR VELOCITY VS. BODY FIXED DIRECTIVES

Simulated body-fixed reference frame directives \hat{d}_i , where \hat{d} is any of \hat{x} , \hat{y} , \hat{z} and angular velocity vector $\vec{\omega}_i$, relative to S/C_i, i = 1, 2, 3 (Chapter 4) are verified for consistency by showing that the numerical integration of $\hat{d}_i = C_{n/i}^{-1} \vec{\omega}_i \times \hat{d}_i$ does not diverge over time:

$$\delta \hat{d}_i(t_j) = \hat{d}_i(t_j) - \int_{t_0}^{t_j} C_{n/i}^{-1} \vec{\omega}_i(t) \times \hat{d}_i(t) dt - \hat{d}_i(t_0) \simeq \vec{0}$$
(C.1)

Results are shown in Figures C.1, C.2 and C.3



Figure C.1: Verification of results of angular velocities vs. directive consistency for S/C1.



Figure C.2: Verification of results of angular velocities vs. directive consistency for S/C_2 .



Figure C.3: Verification of results of angular velocities vs. directive consistency for S/C₃.

C.2. ANGULAR ACCELERATION VS. VELOCITY VECTORS

In order to verify whether the analytically obtained angular acceleration vectors $\vec{\omega}_i$ are the actual time-derivatives of the angular velocity vectors $\vec{\omega}_i$ (relative to S/C_i, i = 1, 2, 3, in Chapter 4), $\dot{\vec{\omega}}_i$ are integrated numerically over time and compared to $\vec{\omega}_i$ according to the following:

$$\delta \vec{\omega}_i(t_j) = \vec{\omega}_i(t_j) - \int_{t_0}^{t_j} \dot{\vec{\omega}}_i(t) dt - \vec{\omega}_i(t_0)$$
(C.2)

Results are shown in Figure C.4.



Figure C.4: Verification of results of angular accelerations vs. angular velocities for S/C1, S/C2 and S/C3.

C.3. EULER ANGLES

In order to verify whether the Euler angle time-rates $\dot{\theta}$ are the actual time-derivatives of θ the former are numerically integrated over time and compared to the latter as following:

$$\delta \boldsymbol{\theta}(t_j) = \boldsymbol{\theta}(t_j) - \int_{t_0}^{t_j} \dot{\boldsymbol{\theta}}(t) dt - \boldsymbol{\theta}(t_0)$$
(C.3)

The results are shown in Figure C.5.

In order to verify whether the Euler angle accelerations $\ddot{\theta}$ are the actual time-derivatives of $\dot{\theta}$ the former are numerically integrated over time and compared to the latter as following:

$$\delta \dot{\boldsymbol{\theta}}(t_j) = \dot{\boldsymbol{\theta}}(t_j) - \int_{t_0}^{t_j} \ddot{\boldsymbol{\theta}}(t) dt - \dot{\boldsymbol{\theta}}(t_0)$$
(C.4)

The results are shown in Figure C.6.



Figure C.5: Verification of results of Euler angles time-rates vs Euler angles for S/C₁, S/C₂ and S/C₃.



Figure C.6: Verification of results of Euler angle accelerations vs Euler angle time-rates for S/C₁, S/C₂ and S/C₃.

C.4. BREATH ANGLES

In order to verify whether the analytically evaluated breath angle time-rates $\dot{\alpha}_i$ are the actual time-derivatives of α_i for S/C_i, i = 1, 2, 3 the former are numerically integrated and compared to the latter as following:

$$\delta \alpha_i(t_j) = \alpha_i(t_j) - \int_{t_0}^{t_j} \dot{\alpha}_i(t) dt - \alpha_i(t_0).$$
(C.5)

To verify that the analytically evaluated breath angle divergence accelerations $\ddot{\alpha}_i$ are the actual timederivative of the speeds $\dot{\alpha}_i$, the the former are numerically integrated and compared to the latter as following:

$$\delta \dot{\alpha}_{i}(t_{j}) = \dot{\alpha}_{i}(t_{j}) - \int_{t_{0}}^{t_{j}} \ddot{\alpha}_{i}(t) dt - \dot{\alpha}_{i}(t_{0}).$$
(C.6)

The results are shown in Figure C.7



Figure C.7: Verification results of α_i vs. $\dot{\alpha}_i$ vs. $\ddot{\alpha}_i$ for S/C_i, i = 1, 2, 3.

D

DISCRETE FOURIER ANALYSIS WITH *fft*

The Fourier series of a periodical signal x(t) with period $T = 1/f_0$ is:

$$x(t) = a_0 + \sum_{n=1}^{N} \left[a_n \cos\left(2\pi n f_0 t\right) + b_n \sin\left(2\pi n f_0 t\right) \right] = a_0 + \sum_{n=1}^{N} c_n \cos\left(2\pi n f_0 t + \varphi_n\right)$$
(D.1)

 f_0 is the fundamental harmonic. The coefficients a_n , b_n and c_n can be evaluated for a time-series using the Matlab function fft() [52].

We present here an example on how the procedure to extract $c_n = c(f_n)$ from a time series x pans out. First of all let us create a time series as a combination of sinusoids of different frequencies:

```
%Time epochs
t = 0:0.001:2;
%Signal made up of 4 sinusoidal components
x = sin(4*2*pi*t) +...
3*cos(7*2*pi*t) +...
0.5*sin(13*2*pi*t) +...
2*cos(17*2*pi*t);
```

the frequencies in the example are $f_k = 4,7,13,17$, whose common denominator is $f_0 = 1$. Since the timeperiod is $t_{end} = 2$, the fft() function will resolve the coefficients at a frequency rate of $f_{0_{out}} = 1/t_{end} = 0.5$, up to $f_{max} = 1/\Delta t$, where $\Delta t = 0.001$ in the example.

Moreover fft() returns a double sided spectrum which is mirrored about f_{max} , or $X_c(f) = X_c(f + f_{max})$, whose components are

$$X_{c}(f) = \frac{1}{2}(a(f) + b(f)i)$$
(D.2)

therefore, in order to calculate c(f) one must take into account the factor of 2 and use:

$$c(f) = 2\sqrt{a(f)^2 + b(f)^2} = 2\sqrt{X_c(f)X_c(f)^*}$$
(D.3)

as shown in the example.

```
%Use the fft() function and normalize
c = 2*abs(fft(x))/(length(t)-1);
%Calculate the frequencies. The fundamental harmonic is
%f_0 = 1/t(end) = 0.5, f(0) = 0 and f(1) = f_0
f = (0:length(c)-1)/t(end);
```

Figure D.1 shows the results by plotting c (y-axis) vs. f (x-axis):

Table D.1 show how close the numerically evaluated coefficients are to the expected ones.



Figure D.1: *left*: the time-domain *x* series. *right*: the Fourier series magnitude coefficients plotted for each frequency.

Table D.1: Comparison between the expected Fourier harmonic components c at frequency f and the ones calculated numerically through fft().

f[1/time u]	<i>c</i> [sign. u]	<i>c</i> numerical [sign. u]
4	1	1.000414897079195
7	3	3.004416133747166
13	0.5	0.499658499264590
17	2	1.999937124110733

E

SPECTRAL DENSITY OF A RANDOM NOISE

E.1. About the power spectral density

The PSD of a time-continuous function is the distribution of signal power over frequency [47]. *Power*, used in this context, refers to the squared norm of the signal, which might not always correspond to physical power. The term was born in the context of electronics, where the power dissipated by a voltage signal on a resistor would correspond to an actual physical power with units in *watts*.

Given a signal x(t), whose Fourier transform X(f) is

$$X(f) = \lim_{T \to \infty} \int_0^T x(t) e^{-2\pi f i t} dt$$
(E.1)

as a function of frequency f, its PSD S_{xx} is defined as the squared norm of X(f) averaged over an infinite interval of time $T \to \infty$:

$$S_{xx}(f) = \lim_{T \to \infty} \frac{1}{T} E[X(f)X^{*}(f)]$$
(E.2)

where *E*[] is the statistical average (or weighted average, or expected value) and $X^*(f)$ is the complex conjugate of *X*(*f*). The product *X*(*f*)*X*^{*}(*f*) can also be expressed as $|X(f)|^2$ or squared norm.

In order to understand why the PSD is calculated as such, let us work out Equation (E.2) from the autocorrelation function of x(t), $R_{xx}(\tau)$, defined as

$$R_{xx}(\tau) = E[x(t)x(t+\tau)] = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)x(t+\tau)dt.$$
 (E.3)

First of all, if we set $\tau = 0$, the above becomes a zero-lag autocorrelation:

$$R_{xx}(0) = E[x(t)^2] = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)^2 dt$$
(E.4)

which represents the total average *power* carried by the signal for an infinite averaging time (If you remove the 1/*T* term, then the integral returns the total *energy*). Again, the term *power* used here does not necessarily mean physical power (with units in *watts*).

The squared norm of the Fourier transform X(f) is calculated as:

$$X(f)X^{*}(f) = \left(\int_{0}^{T} x(t)e^{-2\pi f i t} dt\right) \left(\int_{0}^{T} x(s)e^{2\pi f i s} ds\right) =$$

= $\int_{0}^{T} \int_{0}^{T} x(t)x(s)e^{-2\pi f i (t-s)} dt ds$ (E.5)

and, by applying *E*[] to both terms one obtains:

$$E[X(f)X^*(f)] = \int_0^T \int_0^T E[x(t)x(s)]e^{-2\pi f i(t-s)}dtds$$
(E.6)

Now, the following procedure is used to calculate this double integral: let us use a change of variable: $s = t + \tau$ and substitute into $E[x(t)x(s)] = E[x(t)x(t + \tau)] = R_{xx}(\tau)$ and Equation (E.6), to obtain

$$E[X(f)X^{*}(f)] = \int_{0}^{T} \int_{0}^{T} R_{xx}(\tau) e^{2\pi f i\tau} dt ds$$
(E.7)

Now, in order to evaluate the above integral let us use another set of variables

$$\begin{aligned} \tau &= s - t \\ \eta &= s + t \end{aligned} \tag{E.8}$$

and calculate the Jacobian J (for the change of variable theorem in the integration):

$$J = \begin{vmatrix} \frac{\partial \tau}{\partial t} & \frac{\partial \tau}{\partial s} \\ \frac{\partial \eta}{\partial t} & \frac{\partial \eta}{\partial s} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$
(E.9)

Let us refer to figure E.1 for the change of integration interval: assuming to integrate first by η , we have



Figure E.1: Change of variable domains in Equation (E.8)

that τ belongs in the interval $-T \le \tau \le T$, and the area inside the *ABCD* square in E.1-*right* is defined by $|\tau| \le \eta \le 2T - |\tau|$. Equation (E.7) is therefore evaluated as [47]:

$$E[X(f)X^{*}(f)] = \int_{-T}^{T} \int_{|\tau|}^{2T-|\tau|} R_{xx}(\tau) e^{-2\pi f i\tau} |J|^{-1} d\eta d\tau =$$

$$= \frac{1}{2} \int_{-T}^{T} \int_{|\tau|}^{2T-|\tau|} R_{xx}(\tau) e^{-2\pi f i\tau} d\eta d\tau =$$

$$= T \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) R_{xx}(\tau) e^{-2\pi f i\tau} d\tau$$
(E.10)

By dividing the above by $T \rightarrow \infty$ one obtains, finally:

$$S_{xx}(f) = \lim_{T \to \infty} \frac{1}{T} E[X(f)X^*(f)] = \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-2\pi f i\tau} d\tau$$
(E.11)

This relationship shows that the PSD is none other than the Fourier transform of the autocorrelation function of x(t).

Using the Wiener-Khinchin theorem the double relationship

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} dS_{xx}(f) e^{2\pi f i\tau} = \int_{-\infty}^{\infty} S_{xx}(f) e^{2\pi f i\tau} df$$
(E.12)

also holds from Equation (E.11). Setting $\tau = 0$ in Equation (E.12) and recalling Equation (E.4), the so-called Parseval's identity is obtained:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)^2 dt = \int_{-\infty}^\infty S_{xx}(f) df$$
(E.13)

showing that $S_{xx}(f)df$ is the infinitesimal power carried by the signal at frequency f, or that

$$\int_{f_1}^{f_2} S_{xx}(f) df$$
(E.14)

is the average power contained between two frequencies f_1 and f_2 .

Another important aspect worth noting is that S(f) = S(-f), therefore the PSD is symmetrical w.r.t. to f = 0, and

$$\int_{-\infty}^{\infty} S_{xx}(f) df = 2 \int_{0}^{\infty} S_{xx}(f) df = \frac{a^2}{2}.$$
 (E.15)

Take into consideration that $S_{xx}(f)$ can be defined both ways as a *double-sided* spectrum for $f = (-\infty, \infty)$ or *single-sided*, for $f = [0, \infty)$. Keep in mind that, according to their definition, factor of 2 is applied.

Looking into an example, the average power of a process (Figure E.2)



Figure E.2: The signal x(t) fog Equation (E.16). The period showed is $T_a = 1/f_a$.

$$x(t) = a\sin(2\pi f_a t) \tag{E.16}$$

is

$$E[a^{2}\sin^{2}(2\pi f_{a}t)] = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} a^{2} \sin^{2}(2\pi f_{a}t) dt = \frac{a^{2}}{2}$$
(E.17)

The PSD of x(t) is, without much of a surprise (Figure E.3),



Figure E.3: The vertical arrows represent the Dirac-delta function at frequencies $-f_a$ and f_a

$$S_{xx}(f) = \frac{a^2}{4} \left(\delta(f - f_a) + \delta(f + f_a) \right)$$
(E.18)

meaning that the average signal power is concentrated at frequencies f_a and $-f_a$ over an infinitesimal interval $(\delta(f - f'))$ being the Dirac delta function) and both the frequencies carry half the total average power, or

$$\int_{-\infty}^{\infty} S_{xx}(f)df = 2\int_{0}^{\infty} S_{xx}(f)df = \frac{a^2}{2}$$
(E.19)

E.2. POWER AND AMPLITUDE SPECTRAL DENSITY OF A RANDOM NOISE

For a random noise x(t) with an average value $\mu_x = E[x(t)] = 0$, the expected value of its square is actually its variance [47]:

$$E[x(t)^{2}] = E[(x(t) - E[x(t)])^{2}] = \sigma_{x}^{2}$$
(E.20)

which means that the net power of a zero-mean random signal is its variance. Observing, therefore, that same average power in a finite frequency bandwidth [0, F], the relation with the expected value of the PSD can be deducted, from equations Equations (E.4) and (E.13), as:

$$\sigma_x^2 = 2\int_0^F S_{xx}(f)df = 2F \cdot \frac{1}{F} \int_0^F S_{xx}(f)df = 2F \cdot E[S_{xx}(f)].$$
(E.21)

If $S_{xx}(f)$ is already representative of a single-sided PSD, the 2 can be dropped in Equation (E.21) and assume that $\sigma_x^2 = F \cdot E[S_{xx}(f)]$. The same result is obtained when integrating to $f \to \infty$ when the PSD is zero for f > F. In that case we talk about a *band-limited* white noise.

The bottom line is that, the larger the observation bandwidth of a given random noise, the smaller the average PSD, as the information about the signal power is contained solely in its variance.

Sometimes the Amplitude Spectral Density (ASD) of a random noise is defined. This is non other than the square root of the PSD or $S_{xx}^{1/2}(f)$. The implication of using this notation is that, instead of defining $E[S_{xx}(f)]$ as an actual average, we are assuming it as the variance of a zero-mean quantity whose absolute value at each frequency is the ASD. This is convenient, as we can define a random noise at different density intervals, such as $\sigma = \sqrt{E[S_{xx}(f)]}$ (~68.27% of all the amplitudes for a white noise), 2σ (~95.45%) or 3σ (~99.73%).

F

THE PERIODOGRAM FUNCTION

F.1. HOW TO USE THE PERIODOGRAM FUNCTION

The Matlab built-in periodogram() function is used to calculate the PSD of a signal [54]. As an example, let us define a sinusoidal

an example, let us define a sinasoldar

 $x_s = \sin(2\pi t)$

(F.1)

with frequency $f_s = 1$ Hz over a period T = 1000 s and let us discretize the observation with a sampling frequency $f_{samp} = 200$ Hz (or every 0.005 s). The periodogram() function calculates the one-sided PSD of the signal with a resolution of $\Delta f = 1/T$ and a range of frequencies $0 \le f \le f_{samp}/2$, where the 2 factor is due to Nyquist's law.

In Matlab this is performed using the following lines:

```
T = 1000; %Observation period
f_samp = 200; %Sampling frequency
t = 0:1/f_samp:T; %Time-series
x_s = sin(2*pi*t);
[Sxx_s, f] = periodogram(x_s,[],length(t)-1,f_samp,'psd','onesided');
```

The second input "[]" requires a discrete window function (that here is left empty). The output is a frequency series such that the integral in Equation (E.14) can be solved numerically using the trapz() function (trapezoidal integration). In this case, by using trapz(f,Sxx_s) the output is expected to be 0.5, i.e. the average power of a sinusoid of amplitude 1.

Because this is a Matlab convention, it is more opportune to transform the output into an histogram, i.e., by transferring the areas under the trapeziums into rectangles with base Δf . This is done by adding the following lines:

```
Sxx_hist = zeros(size(Sxx_s));
%At the edges
Sxx_hist(1) = (3*Sxx_s(1) + Sxx_s(2))/8;
Sxx_hist(end) = (3*Sxx_s(end-1) + Sxx_s(end))/8;
%In the middle
Sxx_hist(2:end-1) = (3*Sxx_s(2:end-1) + Sxx_s(1:end-2)/2 +...
Sxx_s(3:end)/2)/4;
```

The integral is then resolved using sum(Sxx_hist)/T, which also returns 0.5. Figure F.1 shows the output around the frequency f = 1 Hz for both the trapezium and the histogram solutions.



Figure E1: Periodogram of single-harmonic sinusoid of frequency 1 Hz and amplitude 1 calculated with periodogram(). For a continuous f the PSD S_{xxS} would be a Dirac-delta at 1 Hz with value 0.5. The blue line "trapz" represents the discrete PSD curve enveloping the 0.5 area which can be solved numerically using the trapz() function in Matlab. The stems "hist" represent the same PSD as an histogram with bin size 0.001 Hz.

F.2. PERIODOGRAM OF RANDOM SIGNALS

Let us now study a uniformly distributed random signal. Two signals x_1 and x_2 are generated using the normrnd() function in Matlab, both with a mean $\mu = 0$ and standard deviation $\sigma = 1$. x_1 is sampled at $f_{samp1} = 200$ Hz and x_2 is sampled at $f_{samp2} = 20$ Hz, on a period T = 1000 s. The periodogram() function is then used to calculate the PSD of the signals, respectively S_{xx1} and S_{xx2} .

```
T = 1000; %Measurement period [s]
f_samp_1 = 200; %Hz
f_samp_2 = 20; %Hz
t_1 = 0:1/f_samp_1:T; %s
t_2 = 0:1/f_samp_2:T; %s
%Normally distributed random variables with sigma = 1
x_1 = normrnd(0,1,[1,length(t_1)]);
x_2 = normrnd(0,1,[1,length(t_2)]);
%Periodograms
[Sxx_1, f_1] = periodogram(x_1,[],length(t_1)-1,f_samp_1,'psd',...
'onesided');
[Sxx_2, f_2] = periodogram(x_2,[],length(t_2)-1,f_samp_2,'psd',...
'onesided');
```

The PSDs are shown in Figure E2. The two periodograms are evaluated with the same frequency step $\Delta f = 1/T$, but the upper frequency, due to Nyquist's law, is $F_1 = 100$ Hz for x_1 and $F_2 = 10$ Hz for x_2 . Recalling Equation (E.21) (single sided, i.e. no factor of 2 is applied), we expect the average PSDs to be $\overline{S}_{xx1} = \sigma^2/100 = 0.01 \text{ Hz}^{-1}$ and $\overline{S}_{xx2} = \sigma^2/10 = 0.1 \text{ Hz}^{-1}$. In fact it is readily visible that S_{xx1} is about 10 times smaller than S_{xx2} . Using the mean () function we obtain values of 0.0100 and 0.1014 respectively for this specific case, the latter being less accurate, since the sample is 10 times smaller.



Figure E2: *left:* the periodogram estimate S_{xx1} of random signal x_1 , whose maximum analyzable frequency is $F_1 = 100$ Hz. *right:* the periodogram estimate S_{xx2} of random signal x_2 , whose maximum analyzable frequency is $F_2 = 10$ Hz. S_{xx1} is on average 10 times larger than S_{xx2} .

F.3. PERIODOGRAM OF ACCELERATION

We can extrapolate numerically the accelerations in a signal by double integration. The acceleration of a signal x(t) at t_i , $\ddot{x}(t_i)$, can be estimated as:

$$\dot{x}(t_i) = f_{samp} \left(x(t_{i+1}) - x(t_i) \right)$$
(E.2)

$$\ddot{x}(t_i) = f_{samp} \left(\dot{x}(t_{i+1}) - \dot{x}(t_i) \right)$$
(F.3)

The value of $\ddot{x}(t_i)$ is actually the average acceleration between two bins $t_i \le t \le t_{i+1}$. Since the double derivative of a harmonic is proportional to the square of the angular frequency $\omega = 2\pi f$, and the PSD of *x*, S_{xx} , is proportional to the square of its frequency domain harmonics, we expect the PSD of \ddot{x} to be roughly

$$S_{acc}(f) = (2\pi f)^4 \cdot S_{xx}(f)$$
 (F.4)

Because of the averaging process in Equations (E2) and (E3), the signal is affected by overlapping low frequency components, causing spectral leakage at low frequencies. In order to resolve the components, a window function h(t) must be multiplied to x(t). In Matlab there are predefined discrete window functions. We show here the results with the Von Hann function hann(N) [54], where N is the size of the discrete sample $\ddot{x}(t)$.

Taking the random signal x_1 in Section C.4 this is implemented with the following lines in Matlab:

```
%Time derivatives
dx_1 = [f_samp_1*(x_1(2:end) - x_1(1:end-1)),0];
ddx_1 = [f_samp_1*(dx_1(2:end) - dx_1(1:end-1)),0];
%Periodogram without window function
[S_acc, f] = periodogram(ddx_1,[],length(t_1)-1,f_samp_1,'psd',...
'onesided');
%Periodogram with window function
[S_acc_win, f] = periodogram(ddx_1,hann(length(t_1)),length(t_1)-1,...
f_samp_1,'psd','onesided');
```

Figure E3 shows the results for both the unwindowed and windowed periodograms of \ddot{x} . The modified periodogram actually follows the expected trend:

$$S_{acc} \simeq 0.01 (2\pi f)^4$$
 (E.5)

for the average PSD of x_1 being $\overline{S}_{xx1} = 0.01 \text{ 1/Hz}$.



Figure F.3: The periodogram estimate S_{acc} of the average accelerations \ddot{x} of the random signal x_1 . The blue line is calculated without window function ([]), the red line is calculated using a Von Hann window function (hann(length(t_1))). The dotted line shows the expected trend (at 1σ), raising by $(2\pi f)^4$.

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