

Document Version

Final published version

Licence

CC BY

Citation (APA)

Vojković, T., Boskos, D., & Buchner, A. J. (2026). Parametric reduced-order modeling and closed-loop control of tandem-cylinder wakes. *Physics of Fluids*, 38(9), Article 094107. <https://doi.org/10.1063/5.0336278>

Important note

To cite this publication, please use the final published version (if applicable).
Please check the document version above.

Copyright

In case the licence states "Dutch Copyright Act (Article 25fa)", this publication was made available Green Open Access via the TU Delft Institutional Repository pursuant to Dutch Copyright Act (Article 25fa, the Taverne amendment). This provision does not affect copyright ownership.
Unless copyright is transferred by contract or statute, it remains with the copyright holder.

Sharing and reuse

Other than for strictly personal use, it is not permitted to download, forward or distribute the text or part of it, without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license such as Creative Commons.

Takedown policy

Please contact us and provide details if you believe this document breaches copyrights.
We will remove access to the work immediately and investigate your claim.

RESEARCH ARTICLE | SEPTEMBER 03 2026

Parametric reduced-order modeling and closed-loop control of tandem-cylinder wakes

Tea Vojković  ; Dimitris Boskos  ; Abel-John Buchner 



Physics of Fluids 38, 094107 (2026)

<https://doi.org/10.1063/5.0336278>



Articles You May Be Interested In

Variational study of the two-temperature model parameters based on finite volume method for unraveling ultrafast reaction dynamics on Pd(100) surface

J. Appl. Phys. (October 2026)



Physics of Fluids

Special Topics Open
for Submissions

[Learn More](#)



Parametric reduced-order modeling and closed-loop control of tandem-cylinder wakes

Cite as: Phys. Fluids **38**, 094107 (2026); doi: [10.1063/5.0336278](https://doi.org/10.1063/5.0336278)

Submitted: 30 March 2026 · Accepted: 28 July 2026 ·

Published Online: 3 September 2026



View Online



Export Citation



CrossMark

Tea Vojković,^{1,2,a)} Dimitris Boskos,² and Abel-John Buchner¹

AFFILIATIONS

¹Laboratory for Aero and Hydrodynamics, Delft University of Technology, Delft, The Netherlands

²Delft Center for Systems and Control, Delft University of Technology, Delft, The Netherlands

^{a)}Author to whom correspondence should be addressed: t.vojkovice@tudelft.nl

ABSTRACT

The flow around two circular cylinders arranged in a tandem exhibits complex wake interactions that lead to amplified unsteady loads, particularly in the co-shedding regime where a fully developed wake forms in the gap between the cylinders. Although various control strategies have been proposed to mitigate these effects, most prior studies have focused primarily on load alleviation. Complete suppression of vortex shedding, both in the gap region and in the wake of the second cylinder, has so far only been achieved using open-loop approaches. In this work, we propose a closed-loop control framework for suppressing vortex shedding in tandem-cylinder flows in the co-shedding regime. Focusing on low Reynolds numbers and sufficiently large spacings, we derive a parametric reduced-order model using a global weakly nonlinear analysis of the incompressible Navier–Stokes equations. The model is generalized to account for time-dependent forcing and facilitates the real-time prediction of the flow evolution. Using this model, we design a model predictive controller and apply it to the full-order system via velocity measurements and volumetric forcing. The approach is demonstrated for a cylinder spacing of eight diameters. Vortex shedding is fully suppressed in both the gap region and the downstream wake for Reynolds numbers $Re = 50, 60$, and 70 , while a significant reduction in flow unsteadiness is achieved at $Re = 80$. We further show that effective control is possible with limited sensing: suppression is achieved using a single measurement point for $Re = 50$ and two-point measurements for $Re = 60$ and 70 .

© 2026 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>). <https://doi.org/10.1063/5.0336278>

I. INTRODUCTION

Multiple cylindrical structures immersed in a flow are common in engineering applications, ranging from large-scale systems such as oil pipelines, offshore platform columns, and bridge piers to smaller-scale devices including heat exchangers. Although many engineering applications of bluff-body wakes occur at high Reynolds numbers, practical interest is not restricted to this regime. Low- and moderate-Reynolds-number wake dynamics are relevant, for example, in microfluidic lab-on-chip mixing,¹ pin-array heat exchangers,² and precise particle or mRNA delivery in microfluidic devices.³ Depending on the application, these structures may be arranged in side-by-side, staggered, or tandem configurations.

The flow around two equal-diameter circular cylinders arranged in tandem is a benchmark problem for studying wake interaction in multi-body flows. Owing to its practical relevance and rich flow physics, this configuration has been widely investigated experimentally and numerically. A comprehensive review of these studies is provided by Sumner.⁴

Based on experimental studies, Zdravkovich⁵ and later Xu and Zhou⁶ and Zhou and Yiu⁷ classified the flow around tandem cylinders into three regimes according to the spacing ratio $\gamma = L/D$. Here, L denotes the center-to-center spacing between the cylinders and D their diameter. The reported regime boundaries differ primarily depending on the Reynolds-number ranges considered.

The following regimes are identified: (i) The extended-body regime occurs at small spacings, typically $1 < \gamma < 1.2 - 1.8$ in the experiments of Zdravkovich,⁵ or $1 < \gamma < 2$ in the study of Zhou and Yiu,⁷ and is characterized by the two cylinders behaving as a single bluff body; the shear layers separated from the upstream cylinder enclose the downstream cylinder without reattachment and roll up to form a Kármán vortex street behind it, while the gap flow remains largely stagnant,⁸ with occasional cavity-type oscillations.⁹ (ii) The reattachment regime is observed at intermediate spacings, approximately $1.2 - 1.8 < \gamma < 3.4 - 3.8$ in Zdravkovich⁵ or $2 < \gamma < 5$ in Zhou and Yiu,⁷ where the increased separation allows the upstream shear layers to reattach intermittently onto the downstream cylinder,

generating unsteady gap vortices; within this regime, Zhou and Yiu⁷ further noted that for $2 < \gamma < 3$ reattachment occurs predominantly on the downstream side of the second cylinder, whereas for $3 < \gamma < 5$ it occurs more frequently on the upstream side. (iii) The co-shedding regime occurs at larger spacings, reported as $\gamma > 3.4 - 3.8$ in Zdravkovich⁵ and $\gamma > 5$ in Zhou and Yiu,⁷ where the downstream cylinder lies outside the vortex formation region of the upstream cylinder, permitting the development of a Kármán vortex street in the gap. In this regime, both cylinders shed vortices at the same frequency.⁴ A qualitatively similar three-pattern classification has been reported in Carmo *et al.*¹⁰ at low Reynolds numbers, consisting of: (i) symmetric vortices in the gap at small spacings (observed at $\gamma = 1.5$); (ii) alternating gap vortices at intermediate spacings ($2 < \gamma < 3.7 - 4.6$, depending on Reynolds number); and (iii) fully developed wake in the gap at larger spacings ($\gamma > 3.2 - 4$).

In the co-shedding regime, vortices that have shed from the upstream cylinder periodically impinge on the downstream cylinder, leading to larger fluctuating fluid-dynamic forces compared to those experienced by an isolated cylinder.^{10,11} In this work, the force perpendicular to the direction of the free stream (transverse force) is referred to as lift, while the force parallel to it (streamwise force) is referred to as drag. Depending on the application context, such amplified unsteady loads may result in reduced operational efficiency, flow-induced vibration and fatigue, or increased maintenance costs. Consequently, the suppression or mitigation of vortex shedding in tandem-cylinder configurations is of considerable practical importance.

In general, the suppression or mitigation of flow instabilities such as vortex shedding may be achieved using either passive or active flow-control techniques. Passive control strategies typically rely on geometric modifications. The flow over tandem cylinders at $\gamma = 3.7$ and $Re = 1.66 \times 10^5$ was effectively controlled, for example, using porous coating in the combined numerical and experimental study of Liu *et al.*¹² Vortex shedding in the gap was suppressed, while unsteadiness in the wake of the downstream cylinder persisted but was mitigated. In that study, achieving this performance required a parametric investigation: the coating thickness and porosity were treated as design parameters and systematically varied to identify configurations that provide effective and robust suppression.

Active flow control introduces external forcing into the flow and may be implemented using either open-loop or closed-loop strategies. Active control approaches can be based on forced dynamical models of the flow, in which the control input is determined from model-based predictions of the flow response to actuation, or on model-free formulations. Model-free approaches typically rely on trial-and-error tuning or extensive parametric studies, whereas suitable forced dynamical models can provide insight into the flow response to actuation and reduce reliance on exhaustive parameter searches. Active control of tandem-cylinder wakes has been implemented using plasma actuators, synthetic jets, and prescribed kinematics of the cylinders.

Open-loop control of tandem-cylinder wakes has been investigated using plasma actuators in experimental studies by Kozlov and Thomas,¹³ Latrobe *et al.*,¹⁴ and in numerical work by Eltaweel *et al.*¹⁵ Kozlov and Thomas¹³ and Eltaweel *et al.*¹⁵ considered $\gamma = 4$ at $Re = 22\,000 - 172\,000$, while Latrobe *et al.*¹⁴ examined $\gamma = 3, 4$, and 5 at $Re = 4700$. In all cases, plasma actuators were applied to the upstream cylinder, leading to the attenuation of vortex shedding from

the upstream body. This, in turn, resulted in a more benign wake interaction with the downstream cylinder and a reduction in the magnitude of fluctuating fluid forces acting on it. However, vortex shedding was not fully suppressed, particularly in the wake of the downstream cylinder; rather, the controlled tandem configuration behaved similarly to that of an isolated cylinder. The actuation parameters in these studies were selected based on experimental observations and parametric exploration, rather than being derived from a forced dynamical model of the flow. At lower Reynolds numbers, Liu *et al.*¹¹ numerically employed a constant-velocity jet for open-loop control at $Re = 75 - 200$, considering gap spacings $\gamma = 4$ and 6 . In this study, vortex shedding in the gap region was suppressed, and shedding in the wake of the downstream cylinder was strongly attenuated. The control strategy was not based on a forced dynamical model of the flow; instead, multiple jet velocities were tested to identify effective operating conditions. Furthermore, this open-loop strategy relied on constant actuation inputs, and sustaining such suppression may therefore incur high energy costs or lack robustness to changes in flow conditions.

Closed-loop control, in contrast, relies on flow measurements to determine the control action, making it potentially more energy efficient and capable of adapting to changing flow conditions. A model-free closed-loop control strategy was experimentally applied by Wolfe and Ziada¹⁶ to the flow around tandem cylinders at a spacing ratio $\gamma = 4.5$, where synthetic-jet actuation was employed on the upstream cylinder and the feedback was relying on lift force measurements from the downstream cylinder. This approach achieved a significant reduction in the fluctuating load acting on the downstream cylinder.

Achieving real-time predictive closed-loop control generally requires reduced-order models that enable the rapid simulation of the flow evolution. In this context, a deep-learning-based approach was proposed by Xie *et al.*¹⁷ for a wall-confined tandem-cylinder configuration at Reynolds number $Re = 100$ and spacing $\gamma = 4.5$. In this numerical study, rotation of the downstream cylinder was used as the actuation mechanism, while the state was measured through an array of 45 velocity sensors distributed in the flow field. Using this framework, up to a 75% reduction in the lift force fluctuations on the downstream cylinder was achieved. However, this control strategy primarily alleviated unsteady dynamics in the gap region and did not suppress vortex shedding in the wake downstream of the second cylinder.

To the best of our knowledge, existing studies have either alleviated unsteady loading without suppressing vortex shedding in the wake of the downstream cylinder, or nearly achieved such a suppression only through open-loop control, which typically lacks robustness guarantees. The present work, therefore, focuses on model-based closed-loop control aimed at suppressing vortex shedding in the co-shedding regime, both in the gap region and in the wake downstream of the second cylinder. Low-Reynolds-number tandem-cylinder wakes provide a useful setting in which to develop and test such a model-based closed-loop control strategy. Although the flow remains comparatively simple in terms of stability structure, the interaction between the two wakes is already non-trivial and poses a meaningful control problem.

To this end, we construct a low-dimensional model that enables fast prediction of the flow evolution over tandem cylinders and is suitable for real-time closed-loop control. Our analysis is restricted to

Reynolds numbers for which the flow remains two-dimensional, i.e., below the onset of three-dimensional instabilities. Furthermore, we focus on sufficiently large spacings for which the primary two-dimensional instability of the tandem-cylinder wake arises through a supercritical Hopf bifurcation ($\gamma > 5.5$) (see Mizushima and Suehiro¹⁸) while the downstream cylinder is still subjected to significantly higher unsteady loading than an isolated cylinder. As this bifurcation scenario is analogous to that of an isolated cylinder, we employ a global weakly nonlinear analysis of the incompressible Navier–Stokes equations following the framework introduced by Sipp and Lebedev.¹⁹ To render the resulting model applicable to closed-loop control, we generalize the formulation originally developed in Sipp²⁰ for open-loop harmonic forcing with constant amplitude to the case of slowly time-varying forcing amplitudes.

This framework yields a two-dimensional reduced-order model that (i) captures the nonlinear forced dynamics, (ii) accounts for time-dependent forcing, (iii) explicitly depends on the Reynolds number and is valid over a finite parameter range, and (iv) provides mappings between the full- and reduced-order state and input spaces, enabling the implementation of feedback control in the full-order system. In addition, (v) the model is computationally efficient, as its coefficients are obtained from the solution of a set of linear equations, avoiding the need for long time-resolved simulations typically required by data-driven approaches, and (vi) it is physically interpretable, providing modal decompositions and insight into the underlying flow dynamics.

Based on this reduced-order model, we design a model predictive controller (MPC) that is consistently mapped to a stabilizing controller for the full-order system, which is considered here in a purely numerical setting. The controller is formulated as an output-feedback law using pointwise velocity measurements, and actuation is introduced in the form of volumetric forcing. Following the MPC paradigm, an optimization problem is solved recursively to determine a sequence of forcing amplitudes over a finite prediction horizon, where the future flow evolution is predicted using the reduced-order model as a surrogate for the full dynamics.

Although the present study focuses on tandem-cylinder flows, the proposed methodology is applicable more generally to incompressible flows that lose stability through a supercritical Hopf bifurcation. We demonstrate the approach for tandem cylinders with a spacing ratio $\gamma = 8$ at Reynolds numbers $Re = 50, 60, 70$, and 80 . Successful suppression of wake oscillations, both in the gap and behind the downstream cylinder, is achieved for $Re = 50, 60$, and 70 using localized forcing, with single-point velocity measurements for $Re = 50$ and two-point measurements for $Re = 60$ and 70 . For $Re = 80$, a significant alleviation of the unsteady dynamics is obtained using full-field velocity measurements.

The paper is organized as follows. Section II introduces the tandem-cylinder configuration, governing equations, computational domain, and unforced flow dynamics, and then formulates the closed-loop control problem. Section III presents the generalization of the weakly nonlinear analysis and the derivation of the parametric forced Stuart–Landau reduced-order model. Section IV uses this model to design the output-feedback closed-loop controller. Section V describes the numerical implementation, validates the reduced-order model against DNS, and presents the closed-loop control results for the tandem-cylinder flow at $Re = 50, 60, 70$, and 80 . Finally, conclusions are drawn in Sec. VI.

II. FLOW OVER TWO CYLINDERS IN TANDEM CONFIGURATION

We first introduce the physical configuration, governing equations, and numerical domain used throughout the paper. The purpose of this section is to define the full-order flow problem and to summarize the unforced dynamics that motivate the reduced-order modeling and control strategy developed in Secs. III and IV.

A. Geometry and governing equations

We consider a two-dimensional incompressible flow with free-stream velocity u_∞ past two circular cylinders of equal diameter D , arranged in tandem. The center-to-center spacing between the cylinders is denoted by L , and the corresponding spacing ratio is defined as $\gamma = L/D$. In the numerical demonstrations presented in this work, we focus on the large-spacing case $\gamma = 8$, which lies in the co-shedding regime. The flow is assumed to be subjected to a time-varying volumetric forcing term $\mathbf{f}(t)$.

1. Governing equations

The flow is governed by the incompressible Navier–Stokes equations, written here in non-dimensional form as

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla \mathbf{u} \cdot \mathbf{u} - \nabla p + \frac{1}{Re} \Delta \mathbf{u} + \mathbf{f}(t), \quad (1a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (1b)$$

where $\mathbf{u} = (u, v)$ denotes the velocity field, p is the pressure, and Re is the Reynolds number based on the characteristic velocity u_∞ and length scale D .

For later convenience, the governing equations (1) are written in the compact operator form

$$\mathcal{E} \frac{\partial \mathbf{U}}{\partial t} = \mathcal{N}(\mathbf{U}, \mathbf{f}, Re), \quad (2)$$

where the state vector $\mathbf{U}$ is defined as

$$\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

The nonlinear Navier–Stokes operator $\mathcal{N}$ is given by

$$\mathcal{N}(\mathbf{U}, \mathbf{f}, Re) = \begin{pmatrix} -\nabla \mathbf{u} \cdot \mathbf{u} - \nabla p + \frac{1}{Re} \Delta \mathbf{u} + \mathbf{f}(t) \\ \nabla \cdot \mathbf{u} \end{pmatrix}.$$

Here, $\mathcal{E} = \mathcal{P}\mathcal{P}^T$, where $\mathcal{P}$ is the prolongation operator mapping the velocity field $\mathbf{u}$ to $(\mathbf{u}, 0)$, $\mathcal{P}^T$ is the corresponding restriction operator mapping $(\mathbf{u}, p)$ to $\mathbf{u}$, and $\mathcal{I}$ denotes the identity operator. In this formulation, $\mathbf{f}(t)$ denotes a force density added to the momentum equation over the fluid domain. Such a term can be used to model practical actuation mechanisms whose effect on the fluid is represented as a distributed force. For example, dielectric-barrier-discharge plasma actuators are commonly modeled through a near-wall electrohydrodynamic force density term.²¹ In conducting fluids, electromagnetic actuation provides another example, where the interaction between electric currents and magnetic fields generates a Lorentz force acting on the fluid.²² The same mathematical form can also be

obtained for boundary actuation after a lifting procedure. For blowing/suction or wall motion, the velocity field can be decomposed into a prescribed lifting field satisfying the non-homogeneous boundary condition and a remaining perturbation satisfying homogeneous boundary conditions. After substitution into the governing equations, the perturbation problem has homogeneous boundary conditions and is driven by equivalent source terms in the momentum equation. These terms can be absorbed into $\mathbf{f}(t)$ (see, for example, Barbagallo *et al.*²³) The volumetric forcing formulation used here should therefore be understood as a general control input representation that can model actuator-specific distributed forcing directly, or boundary actuation after lifting.

The steady (base) flow $\mathbf{U}_b$ is defined as an equilibrium solution of the unforced system, satisfying

$$\mathcal{N}(\mathbf{U}_b, \mathbf{0}, Re) = \mathbf{0}. \quad (3)$$

In addition to the governing equations (2), we introduce an observation model for partial measurements of the flow field. We assume that only limited flow information is available, as is typical in experiments, where measurements are obtained from a finite number of sensors, and denote these measurements by $\mathbf{M}$. These measurements are related to the full flow state through

$$\mathbf{M} = \mathcal{H}(\mathbf{U}), \quad (4)$$

where $\mathcal{H}$ denotes the observation operator.

2. Computational domain and mesh

The equations above define the continuous forced flow problem. We now specify the computational domain, boundary conditions, and discretization used for the tandem-cylinder configuration considered in the numerical part of the study.

In this work, the cylinders have diameter $D = 1$ and are separated by the spacing ratio $\gamma = 8$. The computational domain is shown in Fig. 1, where the origin of the Cartesian coordinate system $\mathbf{x} = (x, y)$ is placed at the center of the upstream cylinder. The domain extends to $x_{-\infty} = -60$ in the upstream direction, $x_{+\infty} = 200$ in the downstream direction, and to $\pm y_{\infty}$ in the transverse direction, with $y_{\infty} = 30$. We apply Dirichlet boundary conditions $\mathbf{u} = (1, 0)$ on the inlet Γ_{inlet} , no-slip boundary conditions $\mathbf{u} = (0, 0)$ on the cylinder boundaries Γ_{c1} and Γ_{c2} , the standard freestream boundary condition $(p - Re^{-1}\partial u/\partial x = 0, \partial v/\partial x = 0)$ at the outlet Γ_{outlet} , and the symmetric boundary conditions $(\partial u/\partial y = 0, v = 0)$ on the upper and lower boundary, Γ_{upper} and Γ_{lower} .

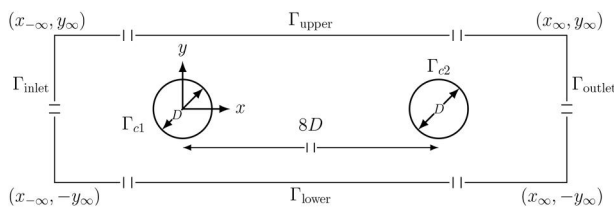


FIG. 1. Schematic defining the computational domain. The streamwise and transverse coordinates, $x_{-\infty}/x_{\infty}$ and $-y_{\infty}/y_{\infty}$, determine, respectively, the location of the inlet/outlet and lower/upper boundaries. The primary flow direction is from left to right.

The incompressible Navier–Stokes equations are spatially discretized via the finite element method (FEM) using the FreeFEM++ software.²⁴ We choose Taylor–Hood elements P_2 for velocity and P_1 for the pressure.

Different meshes were used to check the convergence of the parameters of the weakly nonlinear analysis in Appendix C. In the rest of this paper, we use the mesh denoted with M_2 to present our results.

The mesh is unstructured and the grid point distribution is determined by the automated mesh adaptation method AdaptMesh in FreeFEM++, which relies on the Delaunay–Voronoi algorithm. The mesh resolution is adapted to the base flow and the structure of the direct and adjoint global modes. The mesh has a total of $N_t = 359\,028$ triangles and $N_{\text{dof}} = 1\,619\,133$ degrees of freedom.

B. Hopf bifurcation

In the absence of forcing, i.e., for $\mathbf{f}(t) = \mathbf{0}$, and at sufficiently low Reynolds numbers, the flow past two circular cylinders arranged in tandem is steady and symmetric, corresponding to the base flow $\mathbf{U}_b$ given by Eq. (3). As the Reynolds number increases beyond a critical value Re_c , the steady base flow becomes globally unstable through a Hopf bifurcation and self-sustained vortex shedding develops. The resulting unsteady flow remains two-dimensional in the spatial sense until the onset of a three-dimensional instability. The critical Reynolds number Re_c and the type of Hopf bifurcation (supercritical or subcritical) depend on the spacing ratio γ between the cylinders.

Mizushima and Suehiro¹⁸ reported a supercritical Hopf bifurcation for small ($\gamma < 3.4$) and large ($\gamma > 5.5$) spacing ratios, and a subcritical Hopf bifurcation for intermediate gaps ($3.4 < \gamma < 5.5$). In the present work, we restrict our attention to the supercritical Hopf regime.

In the case of supercritical Hopf bifurcation, the steady base flow $\mathbf{U}_b$ loses stability when a pair of complex-conjugate eigenvalues (λ, λ^*) of the generalized eigenvalue problem

$$\mathcal{L}_b \hat{\mathbf{U}} = \lambda \mathcal{E} \hat{\mathbf{U}} \quad (5)$$

crosses the imaginary axis as the Reynolds number increases, while all remaining eigenvalues remain in the stable half-plane. Here, $\mathcal{L}_b$ denotes the Navier–Stokes operator linearized about $\mathbf{U}_b$. At $Re = Re_c$, the leading eigenvalues satisfy

$$\lambda_0 = i\omega_0, \quad (6)$$

with $\omega_0 > 0$ denoting the bifurcation frequency and the corresponding eigenvector pair is referred to here as the critical global mode.

For Re slightly above Re_c , the unforced dynamics lie on a two-dimensional invariant manifold which connects the unstable equilibrium (the base flow $\mathbf{U}_b$) and an attracting limit cycle²⁵ associated with periodic vortex shedding.

For large-spacing ratios $\gamma > 5.5$, the leading global instability is two-dimensional in the spatial sense and is characterized by vortex shedding both in the inter-cylinder gap and in the wake of the downstream cylinder, corresponding to the co-shedding regime.¹⁰ In our computations for the tandem-cylinder configuration with spacing ratio $\gamma = 8$, a secondary Hopf instability associated with an additional two-dimensional unstable eigenvalue pair is observed only around $Re \simeq 180$, above the range considered here. This is also the range

where the flow becomes three-dimensional through a Mode-A instability, as reported by Carmo *et al.*¹⁰ This differs from the isolated-cylinder wake, for which a second Hopf bifurcation associated with a second unstable eigenvalue pair with two-dimensional spatial structure is known to occur earlier, around $Re \simeq 110$ Verma and Mittal,²⁶ before the onset of the three-dimensional Mode-A instability.²⁷

C. Unforced flow simulation

We now examine the unforced dynamics for the specific test case $\gamma = 8$ and $Re = 50, 60, 70$, and 80 . This baseline simulation characterizes the oscillatory wake that the controller is designed to suppress and provides physical motivation for the control objective.

Figures 2(a), 2(c), 2(e), and 2(g) show the base flow for $Re = 50, 60, 70$, and 80 , respectively, which is steady and symmetric. We can see that the region of the base flow with a negative x -component

u_b , which represents the steady recirculation region, increases in length behind both cylinders with increasing Re .

The DNS results reveal the unsteady nature of the flow above the critical Reynolds number. Figures 2(b), 2(d), 2(f), and 2(h) show instantaneous cross-flow velocity v' at the limit cycle for $Re = 50, 60, 70$, and 80 , respectively. The alternating regions of positive and negative cross-flow velocity v' show the oscillatory nature of the flow, which is physically associated with the von Kármán vortex street. The instantaneous vorticity of the flow at the limit cycle is shown in Fig. 10. From this figure, we observe that vortex shedding occurs both in the gap between the cylinders and in the wake. For the lower Reynolds numbers $Re = 50$ and $Re = 60$, a standard von Kármán vortex street is observed behind both the upstream and downstream cylinders. However, at $Re = 70$ and $Re = 80$, a change in this pattern can be seen. Clockwise and counterclockwise vortex streets about the wake centerline are visible downstream of the second

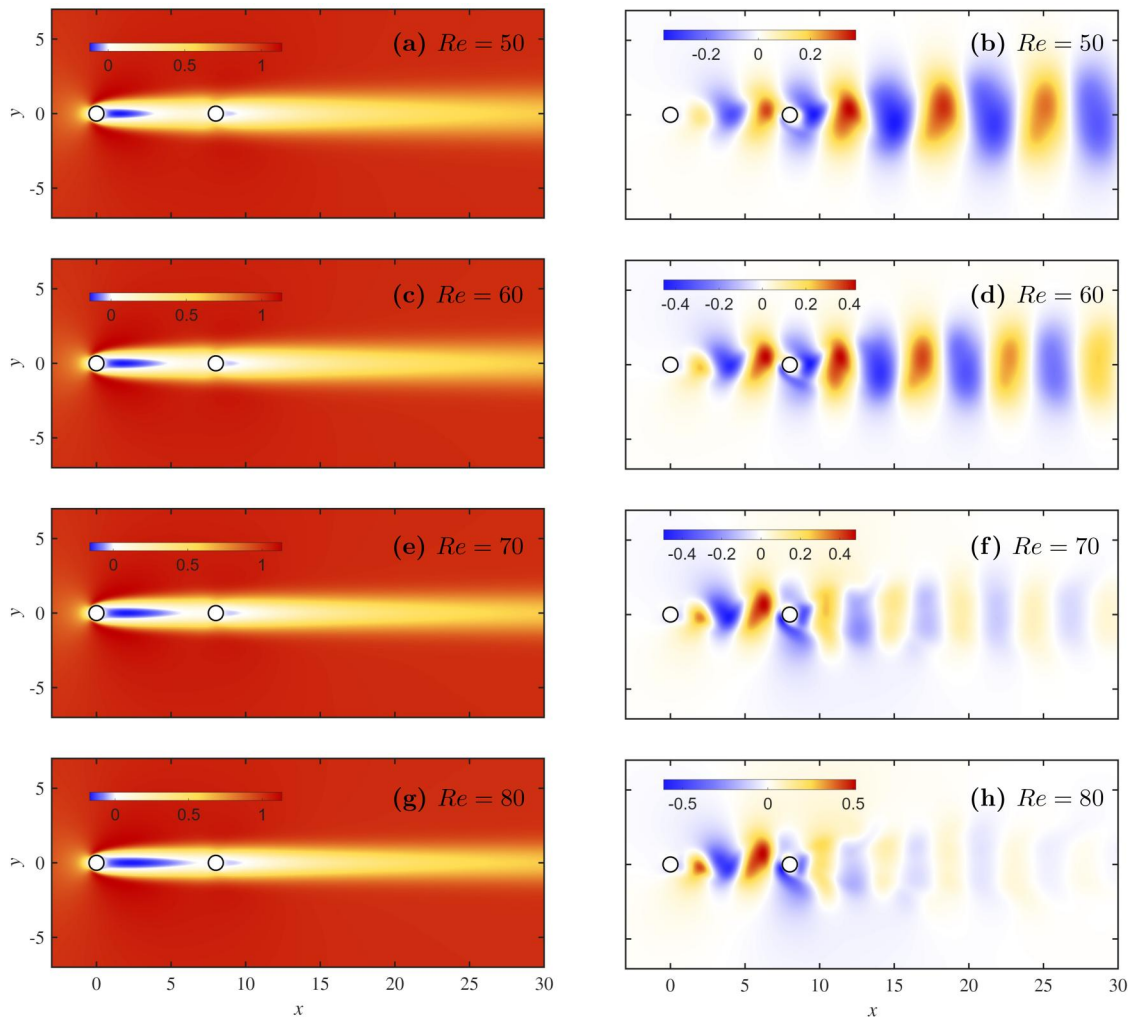


FIG. 2. Velocity fields illustrating the steady base flow and the oscillatory flow on the limit cycle at different Reynolds numbers. Shown are the x -component u_b of the base flow at $Re = 50$ (a), $Re = 60$ (c), $Re = 70$ (e), and $Re = 80$ (g), and snapshots of the y -component v' of the velocity field oscillating on the limit cycle at $Re = 50$ (b), $Re = 60$ (d), $Re = 70$ (f), and $Re = 80$ (h).

cylinder, with a shear layer developing between them. A similar behavior has been reported by Wang *et al.*,²⁸ who also observed the formation of secondary vortices further downstream, where the two vortex rows merge into a single street and the shedding frequency changes.

This suggests a significant change in the flow structure with increasing Reynolds number, particularly downstream of the second cylinder. This change is also visible in the cross-flow velocity fluctuations v' shown in Figs. 2(b), 2(d), 2(f), and 2(h).

Figure 3 shows the hydrodynamic force coefficients acting on the upstream cylinder (UC) and downstream cylinder (DC) at the limit cycle as functions of the Reynolds number. These forces are decomposed into mean and fluctuating components. Here, we denote the mean lift and drag coefficients by $\bar{C}_L$ and $\bar{C}_D$, respectively, and the root mean square of their fluctuating components by C'_L and C'_D .

Note that the mean lift forces $\bar{C}_L$ acting on both cylinders are zero due to the symmetry of the mean flow. However, the vortex shedding induces fluctuating forces on both cylinders. The RMS of the lift and drag coefficients, C'_L and C'_D , acting on the upstream cylinder are similar to those in the single-cylinder case. However, the fluctuating loads on the downstream cylinder are much higher, as shown in Figs. 3(a) and 3(c), since it is immersed in the unsteady wake of the upstream cylinder.

The unsteadiness of the flow also induces higher mean drag coefficients compared to the corresponding base-flow drag C_{D0} for both cylinders at all Reynolds numbers [see Fig. 3(b)]. However, the mean drag coefficient $\bar{C}_D$ is lower on the downstream cylinder because it is located in the wake of the upstream cylinder, where the mean velocity is lower than the freestream velocity.

Overall, the unsteadiness of the flow generates significant fluctuating loads and increases the mean drag on the cylinders, with

particularly strong unsteady loading on the downstream cylinder due to the tandem-cylinder configuration. Therefore, suppressing vortex shedding and stabilizing the flow both in the gap and in the wake is of great interest, as it can minimize its adverse effects.

In a supercritical Hopf bifurcation, oscillations emerge as the Reynolds number exceeds the critical value Re_c , and the flow subsequently settles onto a limit cycle whose amplitude slowly increases with increasing Reynolds number. Consistent with this behavior, C'_l increases slowly with Re in the vicinity of the critical point. To verify the nature of the bifurcation, direct numerical simulations (DNS) were performed for $Re = 43.5$ – 45 with increments of 0.05 . The Reynolds number was first increased from $Re = 43.5$ to $Re = 45$, and then decreased over the same range to check for possible hysteresis in C'_l , as indicated by the arrows in Fig. 3(a). A nonzero value of C'_l was first observed at $Re = 44.5$, and no hysteretic behavior was detected during the decrease in Re , which confirms that the bifurcation is supercritical.

The unforced simulations show that, in the selected co-shedding regime, vortex shedding generates significant oscillations both in the gap and in the wake of the downstream cylinder, together with increased fluctuating loads. This motivates the control problem considered in the remainder of the paper: stabilizing the unstable steady base flow and suppressing the self-sustained oscillations using model-based feedback.

D. Problem formulation

The objective of this work is to suppress the two-dimensional global instability of the flow past tandem cylinders by designing a model-based closed-loop control law. We consider large-spacing ratios for which the instability arises through a supercritical Hopf bifurcation, and Reynolds numbers sufficiently low for the flow to remain two-dimensional.

Control is applied through a time-varying volumetric forcing $\mathbf{f}(t)$ in the incompressible Navier–Stokes equations (2). We assume that only partial measurements $\mathbf{M}$ of the flow state are available through the observation model (4). The goal is therefore to construct an output-feedback law,

$$\mathbf{f} \equiv \mathbf{K}_u(\mathbf{M}), \quad (7)$$

that drives the system toward the steady base flow $\mathbf{U}_b$ from initial conditions spanning the relevant state space, from the vicinity of the equilibrium to the oscillating limit-cycle.

To build such a controller in a model-based manner, a fast evaluation of the system dynamics is required. However, the numerical approximation of Eq. (2) typically relies on a fine spatial discretization, resulting in a high-dimensional nonlinear model. Designing controllers directly for such systems is challenging, in particular for approaches based on online optimization. We therefore introduce a parametric reduced-order model,

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}, \mathbf{q}, Re), \quad (8)$$

with a low-dimensional state $\mathbf{X}$ and input $\mathbf{q}$. The model is complemented by mappings $\mathcal{G}$ and $\mathcal{Q}$, which connect the reduced-order variables to the full-order state and forcing. In particular, $\mathbf{U} \approx \mathcal{G}(\mathbf{X})$ and $\mathbf{f} = \mathcal{Q}(\mathbf{q})$, where $\mathbf{U}$ denotes the solution of Eq. (2) under the forcing $\mathbf{f}$

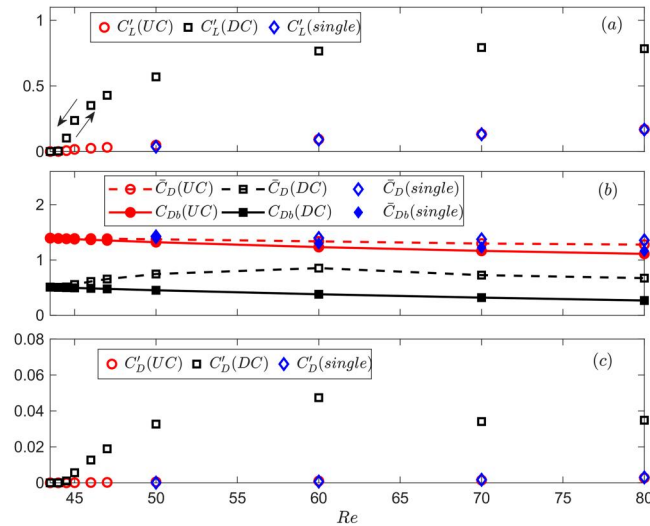


FIG. 3. Hydrodynamic force coefficients for fully developed limit-cycle flow over two cylinders in tandem configuration with spacing $\gamma = 8$, shown separately for the upstream cylinder (UC) and downstream cylinder (DC), and compared with the single-cylinder case as a function of Reynolds number. (a) RMS lift coefficient C'_L . (b) Mean drag coefficient $\bar{C}_D$ and steady drag coefficient C_{D0} . (c) RMS drag coefficient C'_D .

and $\mathbf{X}$ is the corresponding solution of Eq. (8). The reduced-order model is sought to capture the relevant forced nonlinear dynamics across a range of parameter values.

Based on the reduced-order model (8), we design a state-feedback law

$$\mathbf{q} = \mathbf{K}_x(\mathbf{X}, Re). \quad (9)$$

To apply this controller to the full-order system using partial-state measurements, we use an estimator $\mathcal{S}$ to reconstruct the reduced state

$$\tilde{\mathbf{X}} = \mathcal{S}(\mathbf{M}). \quad (10)$$

This way, we obtain the output-feedback law,

$$\mathbf{f} \equiv \mathbf{K}_u(\mathbf{M}) = \mathcal{Q}(\mathbf{K}_x(\tilde{\mathbf{X}}, Re)),$$

for the full-order model. A schematic of the resulting closed-loop controller architecture is shown in Fig. 4.

III. REDUCED-ORDER MODEL

In the following, we outline the methodology for constructing a forced reduced-order model (8) of the incompressible Navier–Stokes equations (2). We build on the weakly nonlinear analysis of Sipp and Lebedev¹⁹ and Sipp²⁰ to obtain a two-dimensional forced Stuart–Landau model. In contrast to the formulation in Sipp,²⁰ where the reduced-order input is assumed to be constant, we allow a time-dependent input $\mathbf{q}(t)$ in Eq. (8). This enables the design of closed-loop feedback controllers at the slow timescale. Although we apply the methodology here to the flow past tandem cylinders, it is applicable to any flow governed by Eq. (2) that loses stability through a supercritical Hopf bifurcation.

A. Weakly nonlinear analysis

Following the classical weakly nonlinear analysis for flows governed by the incompressible Navier–Stokes equations near the onset of a supercritical Hopf bifurcation from Sipp and Lebedev,¹⁹ we consider Reynolds numbers slightly above the critical value Re_c and introduce the small bifurcation parameter

$$\varepsilon := Re_c^{-1} - Re^{-1}, \quad 0 < \varepsilon \ll 1. \quad (11)$$

The analysis is based on a separation of timescales, with a fast time t associated with the oscillatory dynamics of the critical global mode and

a slow time $\tau = \varepsilon t$ governing the modulation of the oscillation amplitude. The flow state $\mathbf{U} = (\mathbf{u}, p)^T$ is expanded asymptotically as

$$\mathbf{U}(t) \approx \mathbf{U}_0 + \sqrt{\varepsilon} \mathbf{U}_1(t, \tau) + \varepsilon \mathbf{U}_2(t, \tau) + \varepsilon \sqrt{\varepsilon} \mathbf{U}_3(t, \tau) + \cdots, \quad (12)$$

about the steady unforced base flow $\mathbf{U}_0 = (\mathbf{u}_0, p_0)^T$ at Re_c . Here, the approximate equality signifies the fact that asymptotic expansions are not necessarily convergent over the whole domain.²⁹

We next introduce a volumetric forcing of the form

$$\mathbf{f}(t, \tau) = E'(\tau) e^{i\omega_f t} \mathbf{f}_E + \text{c.c.}, \quad (13)$$

where $\mathbf{f}_E$ is the complex spatial structure of the forcing, and ω_f is the forcing frequency on the fast timescale. We allow the complex forcing amplitude $E'(\tau)$ to vary on the slow timescale, generalizing the approach form,²⁰ which assumes constant E' . Writing $E'(\tau) = |E'(\tau)| e^{i\phi_E(\tau)}$ shows that both the magnitude of the forcing amplitude $|E'(\tau)|$ and the phase $\phi_E(\tau)$ evolve on the slow timescale τ . Consequently, the instantaneous frequency of the forcing also varies slowly. Substituting the expansion (12) and forcing (13) into the governing equations (2) and collecting terms at successive orders in $\sqrt{\varepsilon}$ yield a hierarchy of linear inhomogeneous problems for $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \dots$, which are solved sequentially.

The forcing amplitude $E'(\tau)$ is assumed to scale with ε as

$$E'(\tau) = \sqrt{\varepsilon}^i E(\tau), \quad (14)$$

where the exponent i determines the order at which the forcing enters this hierarchy. Following Sipp,²⁰ different values of i are chosen for different forcing-frequency classes so that the forcing does not lead to degenerate operator equations at $\mathcal{O}(\sqrt{\varepsilon})$ and $\mathcal{O}(\varepsilon)$.

Here, we proceed with the case of resonant forcing frequencies near ω_0 , since it allows full stabilization of the flow, as explained in the subsection 4 of Appendix A. Other frequency cases are discussed in the subsections 2 and 3 in Appendix A.

B. Resonant forcing

Following the scaling suggested by Fauve³⁰ and Sipp,²⁰ we consider the forcing amplitude

$$E'(\tau) := \sqrt{\varepsilon}^3 E(\tau) \quad (15)$$

when $\omega_f = \omega_0 + \varepsilon\Omega$ for some frequency Ω . Solving the corresponding equations at orders $\sqrt{\varepsilon}^0, \sqrt{\varepsilon}^1, \sqrt{\varepsilon}^2$, and $\sqrt{\varepsilon}^3$, $\mathbf{U}$ is obtained in the form

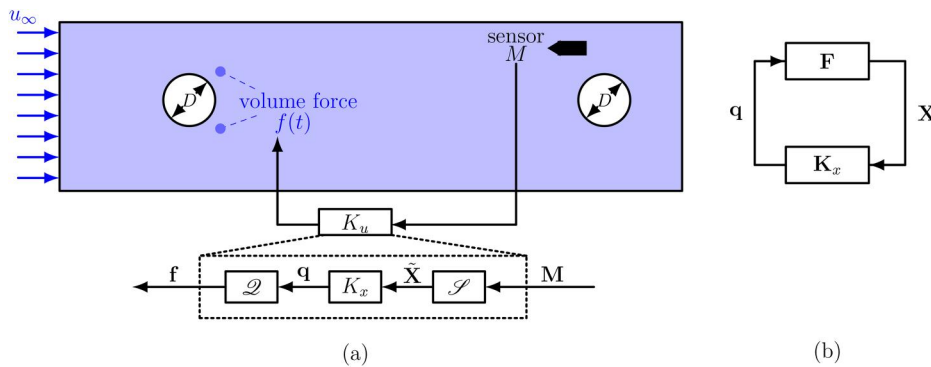


FIG. 4. Schematic of the closed-loop architecture implemented here to control the oscillating flow around tandem cylinders. (a) Output-feedback loop for the forced incompressible Navier–Stokes equations (full-order plant). (b) State-feedback loop for the reduced-order model.

$$\begin{aligned} \mathbf{U} \approx & \mathbf{U}_0 + \sqrt{\varepsilon}(Ae^{i\omega_0 t}\mathbf{U}_1^A + c.c.) \\ & + \varepsilon(\mathbf{U}_2^1 + |A|^2\mathbf{U}_2^{A^2} + (A^2e^{i2\omega_0 t}\mathbf{U}_2^{A^2} + c.c.)) \\ & + \sqrt{\varepsilon}^3(Ae^{i\omega_0 t}\mathbf{U}_3^A + A|A|^2e^{i\omega_0 t}\mathbf{U}_3^{A|A|^2} \\ & + Ee^{i(\omega_0+\varepsilon\Omega)t}\mathbf{U}_3^E + c.c.) + \dots \end{aligned} \quad (16)$$

To ensure that solution (16) remains bounded in time, and hence, retain consistency of the asymptotic expansion (see subsection 1 of Appendix A), the evolution of the complex amplitude A , also referred to as the global mode amplitude, needs to satisfy the Stuart–Landau equation

$$\frac{dA}{dt} = a_0A - a_1A|A|^2 + a_2e^{i\varepsilon\Omega t}E. \quad (17)$$

This can be rewritten with respect to the fast timescale t as

$$\frac{dA}{dt} = \varepsilon a_0A - \varepsilon a_1A|A|^2 + \varepsilon a_2e^{i\varepsilon\Omega t}E. \quad (18)$$

The coefficients a_0 , a_1 , and a_2 are determined by orthogonality conditions that are given in the subsection 1 of Appendix A, where a detailed derivation of the Stuart–Landau equation is also given.

We briefly comment on the physical interpretation of the weakly nonlinear expansion (16) and the Stuart–Landau model (18). In the unforced case ($E = 0$) and for $Re > Re_c$, the Stuart–Landau equation admits two lower-dimensional invariant sets: the unstable equilibrium $A = 0$, and a stable limit cycle with constant magnitude,

$$|A|_{LC} = \sqrt{\frac{\Re(a_0)}{\Re(a_1)}}, \quad (19)$$

and constant phase velocity. This limit-cycle oscillation of the full flow field is captured by expansion (16). As the slowly varying amplitude A in this expansion evolves according to the Stuart–Landau dynamics, the corresponding flow field evolves on a two-dimensional invariant manifold, which is associated with the unstable eigenvalue pair and depends on the parameter $\varepsilon > 0$. This manifold connects the unstable equilibrium, i.e., the base flow, where $A = 0$, with the attracting limit cycle and governs the long-time behavior of the flow near the Hopf onset.

The steady base flow is approximated from Eq. (16) as $\mathbf{U}_b \approx \mathbf{U}_0 + \varepsilon\mathbf{U}_2^1$, where $\mathbf{U}_2^1$ is the base flow correction for $Re > Re_c$. As the flow evolves toward the limit cycle, the mean (time-averaged) flow departs slowly from the base flow through the $\mathcal{O}(\varepsilon|A|^2)$ contribution $\varepsilon|A|^2\mathbf{U}_2^{A^2}$. In analogy with the terminology used in Noack et al.,³¹ $\mathbf{U}_2^{A^2}$ is referred to as the shift mode.

The unsteady component oscillating at the fundamental frequency is given by the terms in Eq. (16) proportional to $e^{i\omega_0 t}$. The linear contribution, $\sqrt{\varepsilon}Ae^{i\omega_0 t}\mathbf{U}_1^A + \sqrt{\varepsilon}^3Ae^{i\omega_0 t}\mathbf{U}_3^A + c.c.$, represents the approximation of the unstable eigenmode of the linearized system about $\mathbf{U}_b$ at the considered Reynolds number: $\mathbf{U}_1^A$ corresponds to the critical global mode, while $\mathbf{U}_3^A$ accounts for its correction for $Re > Re_c$. The nonlinear correction $\sqrt{\varepsilon}^3A|A|^2e^{i\omega_0 t}\mathbf{U}_3^{A|A|^2} + c.c.$ describes the slow, amplitude-dependent deformation of the fundamental mode as the trajectory evolves along the attracting manifold toward the limit cycle. The fundamental frequency is likewise detuned

for $Re > Re_c$: it differs from ω_0 by an $\mathcal{O}(\varepsilon)$ correction near the equilibrium and varies slowly with the amplitude until it approaches a constant value on the limit cycle predicted by the Stuart–Landau phase dynamics.¹⁹ The second-harmonic contribution $\varepsilon A^2e^{i2\omega_0 t}\mathbf{U}_2^{A^2} + c.c.$ oscillates at twice the fundamental frequency. The leading forcing-induced contribution enters (16) at $\mathcal{O}(\sqrt{\varepsilon}^3)$ through the term proportional to E . This term slightly perturbs the flow away from the invariant manifold of the unforced dynamics. However, the dynamical behavior in directions transverse to the manifold is not incorporated through additional state variables in the reduced-order model, which is justified by the fact that the manifold is exponentially attracting. This omission is consistent with the small-amplitude, slowly varying forcing considered here. Hence, any neglected dynamics are not expected to make a leading-order contribution to the slow amplitude dynamics represented by the ROM.

C. Forced Stuart–Landau model

In real coordinates $\mathbf{X} = [\Re(A), \Im(A)]^T$ with input $\mathbf{q} = [\Re(E), \Im(E)]^T$, the Stuart–Landau equation (18) takes the form (8) of a parametric reduced-order model of the incompressible Navier–Stokes equations (2). The model captures the nonlinear forced dynamics, incorporates time-dependent forcing, and depends explicitly on the Reynolds number. Moreover, the weakly nonlinear expansion (16), together with the forcing definitions (13) and (15), provides explicit mappings from the reduced variables to the full-order quantities. In particular, we obtain $\mathcal{G} : \mathbf{X} \mapsto \mathbf{U} \approx \mathcal{G}(\mathbf{X})$ and $\mathcal{Q} : \mathbf{q} \mapsto \mathbf{f} = \mathcal{Q}(\mathbf{q})$, which map the reduced-model state and input spaces to the corresponding spaces of the full-order system (2). The difference between the full state and its reconstruction from the reduced model is captured by the approximation error

$$\mathbf{e} \equiv (\mathbf{e}_u, \mathbf{e}_p) := \mathbf{U} - \mathcal{G}(\mathbf{X}). \quad (20)$$

These mappings enable the construction of the estimator (10) and the deployment of the reduced-order-model controller on the full-order system, as described next.

IV. CLOSED-LOOP CONTROL

In the following, we design an output-feedback law (7) for the Navier–Stokes equations (2), assuming access to the partial-state measurements (4) and using the reduced-order model derived in Sec. III. This is implemented in three successive steps, consistent with the architecture summarized in Sec. II D. First, the reduced state is reconstructed from the available velocity measurements using the estimator (10). Second, the resulting state estimate is supplied to the reduced-model controller (9), which computes the control input $\mathbf{q} = [\Re(E), \Im(E)]^T$ in the reduced-order dynamics (18). These two steps are described in Secs. IV A and IV B. Third, this control input is translated into the full-order forcing $\mathbf{f}(t, \tau)$ via mapping $\mathcal{Q}$. The combination of these three steps yields the output-feedback forcing $\mathbf{f} \equiv \mathbf{K}_u(\mathbf{M})$ applied to the Navier–Stokes equations.

We consider the forcing in Eq. (13) with frequency $\omega_f = \omega_0$ for two reasons. First, for $\omega_f = \omega_0$, the control amplitude E enters the Stuart–Landau equation (18) as an additive input, which is not the case for other frequency classes, except $\omega_f = \omega_0/2$ (see the subsection 4 of Appendix A). This allows the controller to stabilize the model while driving E to zero. When both A and E converge to zero,

the flow approaches the equilibrium base state $\mathbf{U}_b \approx \mathbf{U}_0 + \varepsilon \mathbf{U}_2^1$ according to Eq. (16), and the unsteadiness is suppressed. Second, this forcing case facilitates the characterization of the spatial forcing structure $\mathbf{f}_E$ that maximizes control efficiency, as shown in Sec. IV C.

A. Reduced state estimation from measurements

We reconstruct the complex amplitude A from the available flow measurements $\mathbf{M}$. Since the reduced-order dynamics are two-dimensional and we assume noise-free measurements, we employ an instantaneous (static) reconstruction of the reduced state from measurements. In particular, provided that the measurements contain at least two independent scalar quantities (so that the real and imaginary parts of A can be uniquely determined), a dynamic estimator such as an observer or a Kalman filter is typically not required. In practice, the state reconstructed from measurements, denoted by $\tilde{A}$, generally differs from the reduced-order state A predicted by the surrogate dynamics due to the approximation errors introduced by the model.

We consider two measurement settings: (i) full-field velocity measurements and (ii) pointwise velocity measurements at a finite number of sensor locations. If the whole velocity field is measured, i.e., $\mathbf{M} = (\mathbf{u}, 0)^T$, we use the reduced-to-full state mapping

$$\mathcal{G}(\mathbf{X}) := \mathbf{U}_0 + \sqrt{\varepsilon}(Ae^{i\omega_0 t}\mathbf{U}_1^A + c.c.) + \varepsilon(\mathbf{U}_2^1 + |A|^2\mathbf{U}_2^{A^2} + (A^2e^{i2\omega_0 t}\mathbf{U}_2^{A^2} + c.c.)), \quad (21)$$

which retains the terms up to second order from (16). The corresponding dual mapping $\mathcal{S}$ from the full to the reduced state is obtained by projecting (21) onto the velocity component of the adjoint global mode $\mathbf{u}_1^{A*}$ at Re_c , i.e., by

$$\mathcal{S}(\mathbf{M}) \equiv \tilde{A} := \frac{1}{\sqrt{\varepsilon}} \langle \mathbf{u}_1^{A*}, \mathbf{u} - \mathbf{u}_0 \rangle e^{-i\omega_0 t}, \quad (22)$$

where $\mathbf{u}_1^{A*}$ is normalized such that, $\langle \mathbf{u}_1^{A*}, \mathbf{u}_1^A \rangle = 1$. Owing to the specific symmetry properties in the present test case, all contributions at $\mathcal{O}(\varepsilon)$ vanish under this projection. Hence, we obtain an explicit expression for $\tilde{A}$. A detailed discussion of the symmetry arguments leading to this cancellation is provided in Sec. V. The definition (22) also provides a favorable way of introducing reduced coordinates near the Hopf onset. These coordinates are obtained by projecting onto the critical adjoint mode at Re_c along the subspace annihilated by this mode. To leading order, this subspace corresponds to the stable subspace at the base flow and provides a linear approximation of the stable foliation of the invariant manifold. Thus, even when the DNS state is displaced from the manifold during control transients, the projection assigns it to a nearby manifold point with approximately the same long-time dynamics. This is consistent with the use of projections aligned with stable fibers in invariant-manifold-based model reduction.³²

We next consider point measurements of velocity components at sensor locations $\mathbf{x}_i$. For notational convenience, we first write the measurement relation for the case in which both velocity components are available at each sensor location, and denote these measurements by

$$\mathbf{M} = \left\{ (u(\mathbf{x}_i), v(\mathbf{x}_i))^T \right\}_{i=1}^n.$$

Reconstructing A requires at least two independent scalar measurements. We obtain $\tilde{A}_p$ by using a linearized measurement model that retains only the terms that are linear in A in reconstruction map (21). This approximation is justified when the contributions of $\mathbf{U}_2^{A^2}$ and $\mathbf{U}_2^{A^2}$ are small in the measured components at the selected sensor locations. The real and imaginary parts of resulting reconstruction are given by

$$\begin{aligned} \mathcal{S}(\mathbf{M}) &\equiv \begin{bmatrix} \Re(\tilde{A}_p) \\ \Im(\tilde{A}_p) \end{bmatrix} \\ &:= \frac{1}{2\sqrt{\varepsilon}} e^{-i\omega_0 t} \underbrace{\begin{bmatrix} \Re(u_1^A(\mathbf{x}_1)) & -\Im(u_1^A(\mathbf{x}_1)) \\ \Re(v_1^A(\mathbf{x}_1)) & -\Im(v_1^A(\mathbf{x}_1)) \\ \vdots & \vdots \\ \Re(u_1^A(\mathbf{x}_n)) & -\Im(u_1^A(\mathbf{x}_n)) \\ \Re(v_1^A(\mathbf{x}_n)) & -\Im(v_1^A(\mathbf{x}_n)) \end{bmatrix}}_{\mathbf{G}}^\dagger \\ &\quad \times \begin{bmatrix} u(\mathbf{x}_1) - u_0(\mathbf{x}_1) - \varepsilon u_2^1(\mathbf{x}_1) \\ v(\mathbf{x}_1) - v_0(\mathbf{x}_1) - \varepsilon v_2^1(\mathbf{x}_1) \\ \vdots \\ u(\mathbf{x}_n) - u_0(\mathbf{x}_n) - \varepsilon u_2^1(\mathbf{x}_n) \\ v(\mathbf{x}_n) - v_0(\mathbf{x}_n) - \varepsilon v_2^1(\mathbf{x}_n) \end{bmatrix}. \end{aligned} \quad (23)$$

When only a subset of the velocity components is measured, the corresponding rows of the measurement matrix $\mathbf{G}$ in Eq. (23) are retained. For example, using only transverse-velocity measurements retains only the rows associated with $v_1^A(\mathbf{x}_i)$ and the corresponding entries $v(\mathbf{x}_i) - v_0(\mathbf{x}_i) - \varepsilon v_2^1(\mathbf{x}_i)$.

B. Model predictive control

We design a model predictive controller for the reduced-order model (18), which seeks to optimally steer the amplitude of the global mode A to zero while incorporating suitable constraints for the forcing amplitude E . The latter include soft constraints on the magnitude of the forcing and its rate of change, which are introduced as penalty terms in the objective function of the optimization problem. This enables us to bring the control input E toward zero while naturally complying with our modeling requirement of small temporal gradients of E , to be consistent with the separation of timescales assumed in the derivation of the reduced-order model.

To determine the inputs of the model predictive controller, we consider a discretized version of the forced Stuart–Landau model (18) with sampling period Δt_{MPC} and zero-order hold of the complex input amplitude E . We assume that Δt_{MPC} is sufficiently large compared to the fast timescale, so that potential jumps of E across the sampling times have no significant impact on the validity of the weakly nonlinear analysis.

MPC relies on recursively solving a finite-horizon optimal control problem at each time instant and keeping only the first input of the resulting optimization problem. Considering a horizon of m time steps and using the notation $\mathbf{X}_j$ and $\mathbf{q}_j$ for the state and input of the time-discretized Stuart–Landau model at time $t_j = j\Delta t_{MPC}$, we seek to minimize the quadratic cost function,

$$J(\mathbf{X}_{[j:j+m]}, \mathbf{q}_{[j:j+m-1]}) := \sum_{k=1}^m \|\mathbf{X}_{j+k} - \mathbf{X}^*\|_{\mathbf{Q}}^2 + \sum_{k=0}^{m-1} (\|\mathbf{q}_{j+k}\|_{\mathbf{R}_u}^2 + \|\Delta \mathbf{q}_{j+k}\|_{\mathbf{R}_{\Delta u}}^2), \quad (24)$$

subject to the dynamics of the system. Here, $\mathbf{q}_{[j:j+m-1]} := (\mathbf{q}_j, \dots, \mathbf{q}_{j+m-1})$ denotes the sequence of control actions applied over the horizon and $\mathbf{X}_{[j:j+m]} := (\mathbf{X}_j, \dots, \mathbf{X}_{j+m})$ denotes the corresponding state sequence of the system.

Since the model predictive controller is used for the output feedback of the full-order plant, its initial state satisfies the constraint $\mathbf{X}_j = \tilde{\mathbf{X}}_j$, where the reduced state $\tilde{\mathbf{X}}_j$ is estimated from the measurement $\mathbf{M}(t_j)$ of the flow field at time t_j either via (22) or (23). The future states $\mathbf{X}_{j+1}, \dots, \mathbf{X}_{j+m}$ are predicted by the time-discretized version of the Stuart–Landau model (18) and the inputs are selected in such a way that they minimize the corresponding cost (24). Then the first input $\mathbf{q}_j$ is applied to the system and the same process is repeated recursively. The overall procedure is illustrated schematically in Fig. 5. Each term in Eq. (24) is a quadratic semi-norm of a vector $\mathbf{x}$, denoted by $\|\mathbf{x}\|_{\mathbf{S}}^2 := \mathbf{x}^T \mathbf{S} \mathbf{x}$ for some positive semi-definite matrix $\mathbf{S}$. The cost function J penalizes the deviations of the predicted state $\mathbf{X}_{j+k}$ from the reference state $\mathbf{X}^* = [0, 0]$ as well as the magnitude of the input $\mathbf{q}_{j+k}$ and its increments $\Delta \mathbf{q}_{j+k} = \mathbf{q}_{j+k} - \mathbf{q}_{j+k-1}$, with the first computed using the previously selected input $\mathbf{q}_{j-1}$ of the MPC recursion. The corresponding weight matrix $\mathbf{Q}$ is positive definite, while $\mathbf{R}_u$ and $\mathbf{R}_{\Delta u}$ are positive semi-definite.

C. Optimal forcing structure

The choice of the forcing structure $\mathbf{f}_E$ directly affects control efficiency through the coefficient a_2 in Eq. (18). In particular, due to the structure of the forced Stuart–Landau dynamics, which are fully actuated with respect to E , the higher the magnitude $|a_2|$, the lower the magnitude of the forcing amplitude $|E|$ required to bring $|A|$ to zero. To make this argument precise, let $t \rightarrow E_x(t)$ be a (potentially optimal) control trajectory that stabilizes the system, which is subject to the forcing structure $\mathbf{f}_{E,x}$, and, hence, has a forcing coefficient $a_{2,x}$. When the system is subject to another forcing structure $\mathbf{f}_{E,\beta}$ with a larger coefficient $a_{2,\beta}$, the input $t \rightarrow E_\beta(t) = a_{2,x}/a_{2,\beta} E_x(t)$ will have a smaller magnitude and yield the exact same state trajectory. This means that we can always achieve the same control objective as in the first case with a smaller control effort. This is also consistent with the

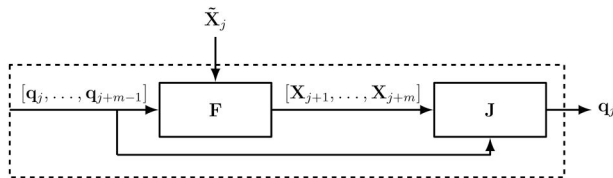


FIG. 5. Schematic of the model-predictive controller. At each time step t_j , the future states $\mathbf{X}_{j+1}, \dots, \mathbf{X}_{j+m}$ are predicted by the surrogate model $\mathbf{F}$ with the initial state $\mathbf{X}_j$ set equal to its estimate $\tilde{\mathbf{X}}_j$ from the flow measurements. The inputs are determined by minimizing the cost function J . The first input $\mathbf{q}_j$ is applied to the system and the same process is repeated at the next time step.

MPC cost in Eq. (24), which will be, respectively, lower when the weight matrices $\mathbf{R}_u$ and $\mathbf{R}_{\Delta u}$ are multiples of the identity matrix.

We thus seek a forcing structure $\mathbf{f}_E$ of unit energy $\langle \mathbf{f}_E, \mathbf{f}_E \rangle = 1$ that gives the highest magnitude of $|a_2|$. For the resonant case, where $\omega_f \approx \omega_0$, $|a_2|$ is maximized when the forcing is aligned with the velocity component of the adjoint global mode $\mathbf{u}_1^{A*}$ at Re_c . This follows directly from the definition of the coefficient a_2 [cf. Eq. (A8c) in Appendix A] and is consistent with the findings of Sipp.²⁰

Thus, we obtain an optimal forcing structure $\mathbf{f}_E$ by normalizing $\mathbf{u}_1^{A*}$, i.e., by selecting

$$\mathbf{f}_E := \frac{c \mathbf{u}_1^{A*}}{|c| \sqrt{\langle \mathbf{u}_1^{A*}, \mathbf{u}_1^{A*} \rangle}}, \quad (25)$$

where c is a complex constant.

Similar observations have been made using resolvent analysis for the flow around a single cylinder, where the optimal forcing on the linearized Navier–Stokes equations is obtained from the singular value decomposition (SVD) of the resolvent operator. It has been shown in Symon *et al.*³³ and Jin *et al.*³⁴ that the forcing producing the maximum energy gain occurs at the resonant frequency and has the same spatial structure as the adjoint mode. These results have been leveraged for the control of the single-cylinder wake in Jin *et al.*³⁵ and Ma *et al.*³⁶

Applying a widely spatially distributed forcing is unrealistic in practice. We therefore also consider a more physically realizable forcing structure concentrated on a small domain Ω_s . Again, the structure $\mathbf{f}_E$, which maximizes $|a_2|$, is determined by the restriction of $\mathbf{u}_1^{A*}$ on Ω_s . For a sufficiently small domain, we can assume that $\mathbf{u}_1^{A*}(\mathbf{x})$ is approximately constant on Ω_s . Thus, we may choose any point $\mathbf{x}_i \in \Omega_s$ and consider the forcing structure

$$\mathbf{f}_E := \frac{1}{\sqrt{\text{vol}(\Omega_s)}} \frac{\mathbf{u}_1^{A*}(\mathbf{x}_i)}{\|\mathbf{u}_1^{A*}(\mathbf{x}_i)\|} \chi_{\Omega_s}. \quad (26)$$

Here, χ_{Ω_s} denotes the indicator function of a set $\Omega_s \subset \mathbb{R}^2$, which takes the value 1 on Ω_s and 0 outside. It follows that $|a_2| = \|\mathbf{u}_1^{A*}(\mathbf{x}_i)\| \sqrt{\text{vol}(\Omega_s)}$ for the uniform, normalized $\mathbf{f}_E$ (26), where $\|\mathbf{u}_1^{A*}(\mathbf{x}_i)\| = \sqrt{|u_1^{A*}(\mathbf{x}_i)|^2 + |v_1^{A*}(\mathbf{x}_i)|^2}$ is the norm of $\mathbf{u}_1^{A*}$ evaluated at the spatial location $\mathbf{x}_i$. This indicates that to maximize $|a_2|$, the volume force should be applied at the location $\mathbf{x}_i$, where the norm of the adjoint mode is the highest.

V. NUMERICAL IMPLEMENTATION AND CONTROL RESULTS

In this section, we apply the reduced-order modeling and control framework developed in Secs. III and IV to the tandem-cylinder configuration introduced in Sec. II. Since the geometry, computational domain, mesh, and unforced baseline dynamics have already been described in Sec. II, we focus here on the numerical implementation of the weakly nonlinear analysis and DNS, the validation of the resulting reduced-order model, and the closed-loop control results.

The simulations are performed for the spacing ratio $\gamma = 8$ and Reynolds numbers $Re = 50, 60, 70$, and 80 . This parameter range lies in the two-dimensional co-shedding regime considered in Sec. II, where the unforced flow develops vortex shedding in both the inter-cylinder gap and the wake of the downstream cylinder.

A. Numerical implementation

We first describe the numerical procedures used to compute the coefficients of the weakly nonlinear model and to perform the time-resolved direct numerical simulations. The spatial discretization, computational domain, mesh, and boundary conditions are those specified in Sec. II. The DNS provides the time-resolved flow field used both as the full-order plant in the closed-loop simulations and as reference data for validating the reduced-order model.

1. Numerical implementation of the weakly nonlinear analysis

To obtain the Stuart–Landau model (18), we solve the weakly nonlinear expansion up to order ε , and determine the model coefficients at order $\varepsilon^{3/2}$ from the corresponding compatibility conditions, following the procedure outlined in the subsection 1 of Appendix A.

Solving the equations at the first two orders of the expansion requires the base flow $\mathbf{U}_0$ at Re_c , together with the corresponding marginally stable eigenvalue $\lambda_0 = \sigma_0 \pm i\omega_0$ (with $\sigma_0 \approx 0$) and eigenvector $\mathbf{U}_1^A$. The nonlinear steady Navier–Stokes equations (A1) are solved using a Newton iteration, while the eigenvalue problem (A3) is solved using the shift-and-invert Arnoldi method as implemented in ARPACK.³⁷

Since Re_c is not known *a priori*, this procedure is performed iteratively. Starting from an initial guess for Re_c , we compute the base flow and solve the corresponding eigenvalue problem. The process is repeated until σ_0 is sufficiently small.

To invert the $N_{\text{dof}} \times N_{\text{dof}}$ matrices of the discretized operator equations of the weakly nonlinear analysis, we use the UMFPACK library, which relies on a sparse direct LU solver.³⁸

2. Direct numerical simulation

The direct numerical simulation of the unsteady incompressible Navier–Stokes equations (2) marches the velocity–pressure field $\mathbf{U}$ in time. We consider the perturbation form

$$\varepsilon \frac{\partial \mathbf{U}'}{\partial t} = \mathcal{L} \mathbf{U}' - \mathcal{P}(\nabla \mathbf{u}' \cdot \mathbf{u}' - \mathbf{f}) \quad (27)$$

of the Navier–Stokes equations, where $\mathbf{U}' = (\mathbf{u}', p') = \mathbf{U} - \mathbf{U}_b$ contains the velocity and pressure components of the perturbation around the steady solution $\mathbf{U}_b$ (base flow) at some Re . First, we calculate the base flow $\mathbf{U}_b$ using the iterative Newton method as described in Sec. V A 1, and then we march $\mathbf{U}'$ in time.

For the time discretization of Eq. (27), we use a second-order semi-implicit backward-finite-difference scheme, which gives at time $t_{i+1} = (i+1)\Delta t_{\text{DNS}}$ the field

$$\mathbf{U}'_{i+1} = \left(\frac{3}{2\Delta t_{\text{DNS}}} \varepsilon - \mathcal{L} \right)^{-1} \left(\frac{2}{\Delta t_{\text{DNS}}} \varepsilon \mathbf{U}'_i - \frac{1}{2\Delta t_{\text{DNS}}} \varepsilon \mathbf{U}'_{i-1} - \mathcal{P}(2\nabla \mathbf{u}'_i \cdot \mathbf{u}'_i - \nabla \mathbf{u}'_{i-1} \cdot \mathbf{u}'_{i-1} - 2\mathbf{f}_i + \mathbf{f}_{i-1}) \right). \quad (28)$$

We use the spatial discretization and finite elements from Sec. II A 2 and use different time steps Δt_{DNS} depending on the Reynolds number. For the unforced DNS at $Re \leq 70$, $\Delta t_{\text{DNS}} = 0.05$ is used, while at $Re = 80$, $\Delta t_{\text{DNS}} = 0.025$ is used to keep the time step as large as

possible while ensuring adequate temporal resolution of the flow dynamics and maintaining numerical stability.

To validate our simulations, we compare our results with reference data available in the literature in Appendix D.

B. Reduced-order model validation

We next construct the reduced-order model using the weakly nonlinear analysis described in Sec. III and assess its accuracy against DNS. The purpose of this subsection is twofold: first, to present the modal structures and Stuart–Landau coefficients obtained from the weakly nonlinear expansion; and second, to quantify how well the resulting model reproduces the unforced amplitude and frequency dynamics over the Reynolds-number range considered.

1. Modes and model coefficients

Starting from an initial guess around $Re \approx 44.5$, where the first instability is observed in DNS (see Fig. 3), we determine the critical Reynolds number $Re_c = 44.1$, consistent with the value reported in the literature,³⁹ by iteratively solving the first two orders of the weakly nonlinear analysis. The corresponding eigenvalue is $\lambda_0 = -0.0004 + 0.6756i$.

The modes up to order ε in the weakly nonlinear expansion (16) of the flow are shown in Fig. 6. The x -component u_0 of the base flow $\mathbf{U}_0$ at $\varepsilon = 0$ (or $Re_c = 44.1$) is displayed in Fig. 6(a). The x -component u_2^1 of the modification of $\mathbf{U}_0$ corresponding to the unstable equilibrium at $\varepsilon > 0$ is shown in Fig. 6(c). The negative values of u_2^1 in the gap and the wake indicate that the spatial extent of the recirculation region increases with ε (or Re), consistent with the behavior observed in Fig. 2(a).

Figure 6(b) shows the y -component (real part) $\Re(v_1^A)$ of the critical global mode $\mathbf{U}_1^A$. This mode oscillates approximately at the fundamental frequency of the flow and captures the large-scale structures of its unsteady component. The second-harmonic mode $\mathbf{U}_2^A$, which oscillates at twice the fundamental frequency, is characterized by smaller spatial structures, as illustrated in Fig. 6(d), which shows its y -component (real part) $\Re(v_2^A)$.

The x -component $u_1^{A^2}$ of the shift mode, which approximates the difference between the mean flow and the base flow, is plotted in Fig. 6(e). The positive values of $u_1^{A^2}$ in the gap and the wake indicate a decrease in the recirculation bubble length in the mean flow compared to the base flow, consistent with observations for a single cylinder in Sipp and Lebedev.¹⁹

Finally, Fig. 6(f) shows the y -component $\Re(v_1^{A*})$ of the critical adjoint global mode at Re_c . Its largest values are concentrated near the upstream cylinder, indicating that this region is the most effective location for applying volume forcing for flow control.

The modal structure exhibits a clear symmetry with respect to the centerline $y = 0$: modes arising at odd orders in the expansion are antisymmetric, while those at even orders are symmetric. This behavior is consistent with the weakly nonlinear analysis for the single-cylinder wake reported by Sipp and Lebedev,¹⁹ and is also evident from the modal structures shown in Fig. 6.

As a consequence, the $\mathcal{O}(\varepsilon)$ contributions, which are symmetric, vanish when projected onto the adjoint global mode $\mathbf{u}_1^{A*}$, which is antisymmetric. This explains the cancellation stated in Sec. IV A and yields the explicit expression for $\hat{A}$ in Eq. (22).

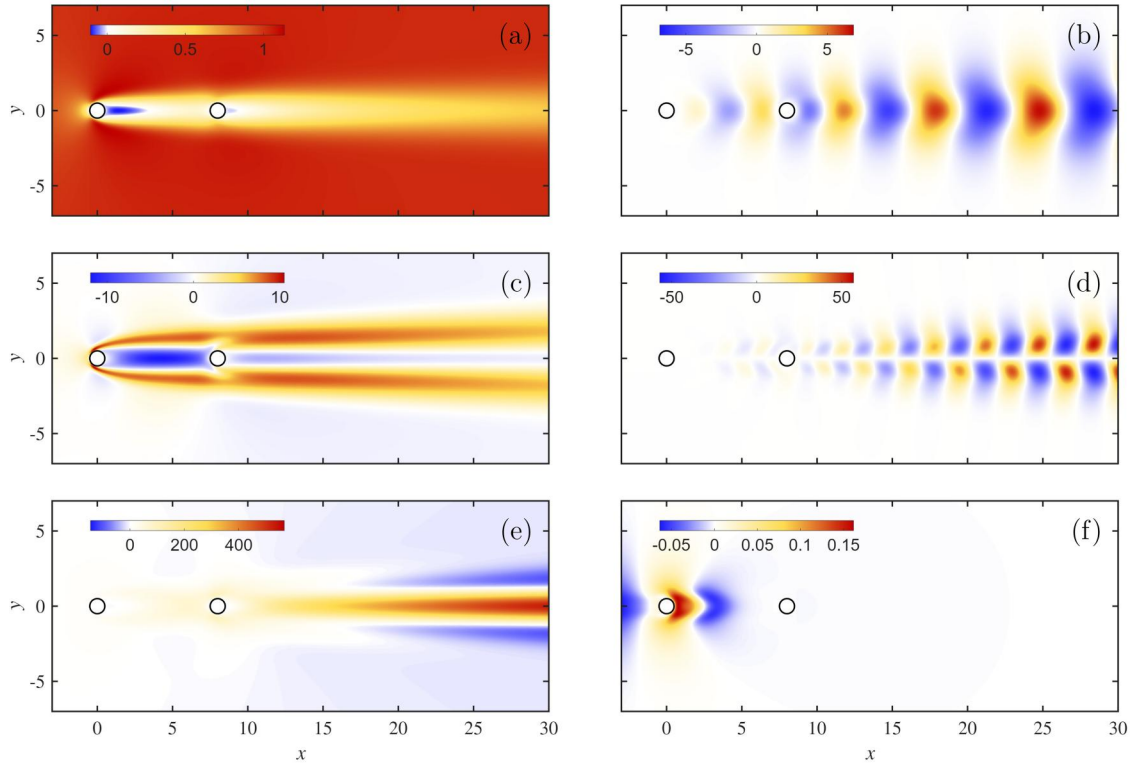


FIG. 6. Spatial structure of the modes appearing in the expansion (16) up to the order ε . (a) The x -component u_0 of the base flow at Re_c . (b) Real part $\Re(v_1^A)$ of the y -component of the critical global mode. (c) The x -component u_2^1 of the base flow modification. (d) Real part $\Re(v_1^{A^2})$ of the y -component of the second-harmonic mode. (e) The x -component u^{A^2} of the shift mode. (f) Real part $\Re(v_1^{A^*})$ of the y -component of the critical adjoint global mode.

The coefficients of the Stuart–Landau model (18) representing this flow are listed in Table I. They are computed from Eq. (A8) in Appendix A. The positive value of $\Re(a_0)$ indicates that the fixed point $A = 0$ is repelling for $\varepsilon > 0$, which is consistent with the DNS observation that the flow undergoes a supercritical Hopf bifurcation. The critical global mode is normalized such that $v_1^A(0.65, 0) = -0.0979 + 0.2661i$, yielding $\Re(a_1) \approx \Re(a_0)$ and therefore a magnitude of limit-cycle amplitude close to unity [see expression (19)]. For the chosen mesh, $|A|_{LC} = 1.0374$. The coefficient a_2 is computed for the forcing structure discussed in Sec. IV C.

2. Comparison with DNS

For the unforced flow, we compare the complex amplitude A predicted by the reduced model with the amplitude $\tilde{A}$ evaluated from

TABLE I. Values of the coefficients of the Stuart–Landau model obtained via weakly nonlinear analysis (WNA). The coefficient a_2 is computed for the localized forcing structure (26), where Ω_s is the union of circular domains Ω_{s_i} of radius 0.07 centered at $x_i = (0.32, \pm 0.59)$ (Fig. 8).

Re_c	ω_0	a_0	a_1	a_2
44.1	0.6756	$7.6571 + 2.9857i$	$7.1153 - 31.2244i$	$0.0368 - 0.0645i$

DNS using Eq. (22), assuming full knowledge of the velocity field over the entire computational domain. The difference

$$e_A \equiv \tilde{A} - A$$

between these two quantities depends on the approximation error $\mathbf{e}_u$ in Eq. (20). In particular,

$$e_A := \frac{1}{\sqrt{\varepsilon}} \langle \mathbf{u}_1^{A*}, \mathbf{e}_u \rangle e^{-i\omega_0 t}. \quad (29)$$

The DNS is initialized close to the base flow with the perturbation velocity $\mathbf{u}'(t=0) = 0.0015 \mathbf{u}_1^A$, which lies in the subspace of the critical global mode. This initialization is used for all Reynolds numbers and corresponds to $\sqrt{\varepsilon}A = 0.0075$. Consequently, the approximation error $\mathbf{e}_u(t=0)$ in Eq. (20) is initially zero.

The evolution of $|\tilde{A}|$ for $Re = 50, 60, 70$, and 80 is shown in Figs. 9(a)–9(d). The model captures the overall trend of the amplitude dynamics and reproduces the qualitative approach to the limit cycle.

The unforced Stuart–Landau model, however, underpredicts the magnitude of the limit-cycle amplitude $|A|_{LC}$. Moreover, the discrepancy between the model and DNS increases with the Reynolds number, as shown in Fig. 7, which displays the variation of $\sqrt{\varepsilon}|A|_{LC}$ and ω_{LC} with Re . The main source of this increasing modeling error is the unsteady flow distortion, evident in the vorticity fields in Figs. 10(c) and 10(d), which is not captured accurately by the weakly nonlinear

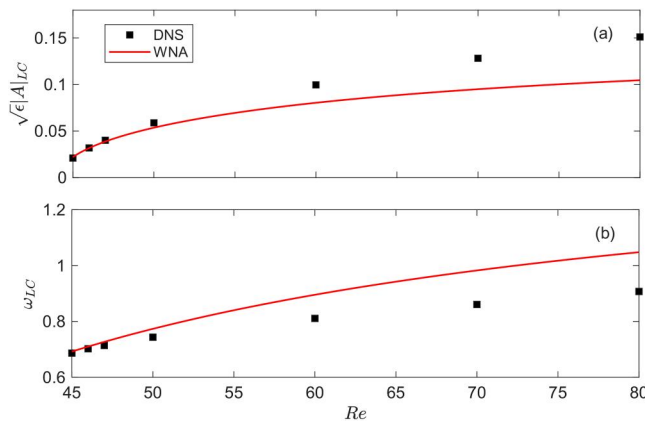


FIG. 7. Prediction of the (scaled) mode amplitude $\sqrt{\epsilon}|A|_{LC}$ and the shedding frequency ω_{LC} at the natural limit cycle vs Re . The values of A_{LC} are estimated as A_{LC} from DNS assuming complete knowledge of the velocity field within the entire computational domain, and obtained from the Stuart–Landau model with coefficients derived from our weakly nonlinear analysis (WNA).

approximation. In particular, for $Re \gtrsim 70$, the flow deviates significantly from its first-order approximation based on the global mode at Re_c , as discussed in Sec. II C.

3. Range of validity and possible model extensions

The comparison of the mode amplitude and shedding frequency between DNS and the weakly nonlinear model in Fig. 7 clarifies the effective range of validity of the model. Since the model is derived from an asymptotic expansion around the primary Hopf bifurcation point, its validity deteriorates with increasing distance from Re_c . As discussed in II B, the instability remains governed by the primary unstable eigenvalue pair up to approximately $Re \simeq 180$, and the three-dimensional Mode-A instability does not occur below this range. Therefore, the deterioration of the weakly nonlinear prediction observed in Fig. 7 is attributed to the local nature of the expansion around the primary Hopf onset, rather than to the appearance of a secondary instability. The weakly nonlinear approximation is therefore not expected to remain accurate all the way up to the Reynolds-number range where secondary instabilities appear. For the present configuration, the model gives reliable predictions close to onset, in particular, for $Re = 50$ and $Re = 60$, while the discrepancy in the limit-cycle amplitude and frequency, as well as in the flow reconstruction, becomes more pronounced at $Re = 70$ and $Re = 80$. The weakly nonlinear model should therefore be interpreted as a local reduced-order model near the primary Hopf bifurcation, which remains useful for control over a finite but limited Reynolds-number interval.

A similar observation regarding model validity applies to variations in the spacing ratio. The third-order Stuart–Landau model derived here is expected to remain appropriate for tandem-cylinder configurations in which the primary instability is governed by a supercritical Hopf bifurcation, such as the small-spacing regime $\gamma < 3.4$ and the large-spacing regime $\gamma > 5.5$ reported by Mizushima and Suehiro.¹⁸ However, the range of validity may differ with the spacing ratio. For a different spacing, the base flow, global modes, adjoint modes, nonlinear coefficients, input/output maps, and

favorable sensor and actuator locations would therefore need to be recomputed and validated. Configurations involving a subcritical Hopf bifurcation, or the unfolding of the supercritical Hopf bifurcation observed at small spacing by Mizushima and Suehiro,¹⁸ would require an extension of the present amplitude equation. In particular, higher-order Stuart–Landau models, derived by introducing additional slow timescales as in Orchini *et al.*,⁴⁰ would be needed to capture such behavior.

Furthermore, in the present approach, the model is constructed from the governing equations and their linearized operators. This is advantageous when such operators can be computed in the available numerical framework, because the model coefficients are obtained by solving steady and linear problems rather than by generating long time-resolved datasets. The resulting model is also physically interpretable, since the reduced variables, spatial modes, and forcing maps are directly connected to the Hopf bifurcation structure. However, this equation-based construction may be less practical when the required operators are not readily accessible or when only experimental data are available. In such cases, data-driven approaches may be beneficial, including feature-based manifold modeling,⁴¹ sparse sensor-based identification of reduced dynamics,⁴² or data-driven invariant-manifold reduced-order models.⁴³

We now discuss two natural extensions for which the control approach proposed here would remain directly applicable. A direct extension is to retain the Stuart–Landau form but identify its coefficients separately at each Reynolds number from DNS or experimental data. This can be done through a relatively inexpensive least squares fit and has been shown in our related work⁴⁴ to improve both unforced and forced amplitude prediction from full-field measurements. Since the reduced amplitude A can also be estimated from pointwise measurements, as done below for feedback control, the same estimated amplitude could in principle be used to fit the Stuart–Landau model. This data-calibrated version would improve the temporal reduced dynamics while preserving the same low-dimensional control structure. However, such a correction would not by itself improve the spatial modes or the state-reconstruction map.

A more substantial extension would be to replace the polynomial expansion around the base flow at Re_c with a data-driven invariant-manifold identification approach constructed at the Reynolds number of interest, as in Cenedese *et al.*⁴³ Such an approach could improve both the reduced dynamics and the state-reconstruction map farther from the bifurcation point, and is part of ongoing research.

C. Closed-loop control results

Having validated the reduced-order model in the unforced setting, we now use it to design and implement the feedback controller described in Sec. IV. The objective is to suppress vortex shedding in both the inter-cylinder gap and the wake of the downstream cylinder, starting from the natural limit-cycle state of the unforced flow.

We consider four Reynolds numbers, namely, $Re = 50, 60, 70$, and 80 . To ensure physical realizability, we employ a spatially localized forcing structure rather than the distributed optimal volume forcing. As shown in Sec. IV C, maximizing the control effectiveness requires choosing a location $\mathbf{x}_i$ where the norm of the adjoint mode, $\|\mathbf{u}_1^{A*}(\mathbf{x}_i)\|$, is large. From Fig. 8, this location is approximately $\mathbf{x}_i = (x_i, y_i) = (0.32, \pm 0.59)$. We therefore employ the localized forcing defined in Eq. (26), with Ω_s taken as the union of the circular

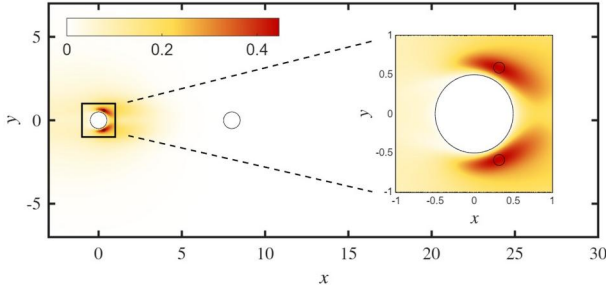


FIG. 8. Spatial distribution of the norm of the adjoint mode $||\mathbf{u}_1^*(\mathbf{x})||$. Circles indicate our choice of localized forcing domain Ω_s , with chosen radius 0.07 and centered on locations $\mathbf{x}_i = (0.32, \pm 0.59)$, where this norm is the highest, thus representing favorable locations for forcing.

domains $\Omega_{s\pm}$ of radius 0.07 centered at $\mathbf{x}_i = (0.32, \pm 0.59)$, indicated by the circles in Fig. 8.

Before applying any control action, we run the unforced DNS for 500 time units for each Reynolds number to ensure that the flow is fully settled on the limit cycle. The controller is then activated at $t = 500$, starting from this limit-cycle state.

The time-discretized sequence of control inputs $\mathbf{q}_j = [\Re(E_j), \Im(E_j)]^T$ is determined by the model predictive control scheme described in Sec. IV B, which minimizes the cost function (24). The control actions are applied to the continuous-time system using a zero-order hold. To solve the minimization problem, predictions of the future states $\mathbf{X}_{j+k} = [\Re(A_{j+k}), \Im(A_{j+k})]^T$ are obtained by numerically integrating the Stuart–Landau model (18).

Following our methodology, the feedback controller is constructed on the slow timescale. Accordingly, the MPC operates with a larger time step than the forced DNS, and we set $\Delta t_{\text{MPC}} = 20 \Delta t_{\text{DNS}}$. Furthermore, the weighting matrix $\mathbf{R}_{\Delta u}$ is used to limit the temporal gradients of E . For $Re = 50$ and $Re = 60$, we use $\Delta t_{\text{DNS}} = 0.05$, as in the unforced simulations, and $\mathbf{R}_{\Delta u} = 0.1$. For $Re = 70$ and $Re = 80$, where wake suppression becomes more challenging, the smaller DNS time steps $\Delta t_{\text{DNS}} = 0.025$ and $\Delta t_{\text{DNS}} = 0.02$ are used together with a reduced control weight $\mathbf{R}_{\Delta u} = 0.08$ to facilitate suppression.

Regarding the remaining weighting matrices, we set $\mathbf{Q} = 1000\mathbf{I}$ for all Reynolds numbers. The control weight is chosen as $\mathbf{R} = 0.0015\mathbf{I}$ for $Re = 50$ and $Re = 60$, and $\mathbf{R} = 0.0008\mathbf{I}$ for $Re = 70$ and $Re = 80$, allowing higher forcing amplitudes for higher Reynolds numbers. The prediction horizon is $m = 20$ time steps Δt_{MPC} .

In the following, we consider two measurement settings: (i) full-field velocity measurements and (ii) pointwise velocity measurements at a finite number of sensor locations. All controller parameters described above are kept identical in both cases.

1. Control with full measurements

We first assume the ideal case of full-domain measurements, i.e., complete knowledge of the velocity field throughout the computational domain. The reduced state $\tilde{\mathbf{X}}_j = [\Re(\tilde{A}_j), \Im(\tilde{A}_j)]^T$ used in the feedback law is then evaluated from the velocity field of the forced DNS solution using Eq. (22). This provides a benchmark for the best achievable performance of the control strategy.

We track the time evolution of the reduced state and reduced input with control applied over the interval $t = 500$ to $t = 1000$. Figure 9 shows their magnitudes, $|\tilde{A}|$ and $|E'|$. Here, we choose to plot forcing amplitude $E' = \varepsilon^{3/2}E$ so that the control amplitudes are directly comparable across Reynolds numbers.

This time interval is sufficient to reduce $|\tilde{A}|$ to very small values, of order 10^{-7} for $Re = 50$ and $Re = 60$, and 10^{-6} for $Re = 70$. The corresponding magnitudes of the forcing amplitude E' are of order 10^{-8} for $Re = 50$ and $Re = 60$, and 10^{-6} for $Re = 70$.

For $Re = 80$, the system is not stabilized to the same extent; instead, it settles into another oscillatory state with a mean $|\tilde{A}|$ of order 10^{-2} , i.e., about two orders of magnitude smaller than the unforced limit-cycle amplitude ($|\tilde{A}|_{LC} \approx 1.5$). The corresponding amplitude of the reduced input is of the same order.

We also introduce the integrated control-effort measure $\mathcal{W}_E = \int_{t_0}^{t_f} |E'(t)| dt$. Since the applied control has the form $E'(t)e^{i\omega_0 t}\mathbf{f}_E + \text{c.c.}$ and the spatial forcing shape $\mathbf{f}_E$ is fixed and normalized to unit norm, this quantity is a good approximation of the time-integrated norm of the control amplitude. The resulting values of $\mathcal{W}_E$ are reported in Table II.

We examine the fluctuation energy

$$K = \frac{1}{2} \langle \mathbf{u}', \mathbf{u}' \rangle$$

of the unsteady velocity $\mathbf{u}' = \mathbf{u} - \mathbf{u}_b$. By definition, $K = 0$ when the flow is at its steady equilibrium. Figure 10 shows the evolution of K for the unforced flow up to $t = 500$ and for the controlled flow over the interval $t = 500$ to $t = 1000$.

For $Re = 50, 60$, and 70 , the energy at $t = 1000$ is significantly reduced, reaching values of order 10^{-8} , 10^{-7} , and 10^{-1} , respectively. For $Re = 70$, the energy continues to decrease under prolonged control (not shown here), reaching values of order 10^{-2} after an additional 100 time units. In contrast, for $Re = 80$, a substantial level of energy remains in the flow.

To further visualize the controlled dynamics, we plot the instantaneous vorticity at $t = 0$, $t = 500$ (corresponding to the limit cycle), and $t = 1000$. At $t = 0$, the vorticity of the base flow is shown; the actual flow differs only by a small perturbation, which is not visually discernible.

For $Re = 50, 60$, and 70 , the control drives the flow close to the steady equilibrium, with no visible difference between vorticity plots at $t = 0$ and $t = 1000$. For $Re = 80$, however, the flow settles into another periodic state, as also observed in Fig. 9(d), with persistent vortex shedding. Compared to the natural limit cycle at $t = 500$, the vortex formation length is increased and the shedding in the gap region is significantly weakened.

Finally, we assess the impact of control on the lift and drag coefficients. Figures 11 and 12 show the time evolution of C_L and C_D for both the unforced and controlled flows. Consistent with the vorticity observations, for $Re = 50, 60$, and 70 , the lift coefficient is reduced to near zero for both cylinders, reaching values of less than 10^{-7} for $Re = 50$, less than 10^{-6} for $Re = 60$, and less than 10^{-4} for $Re = 70$, while the drag coefficient approaches its steady value.

For $Re = 80$, the reduction in the lift coefficient is noticeably less pronounced; however, it is still significantly mitigated compared to the unforced case. For the upstream cylinder, the RMS of lift coefficient is reduced from $C'_L \approx 0.17$ in the unforced limit cycle to

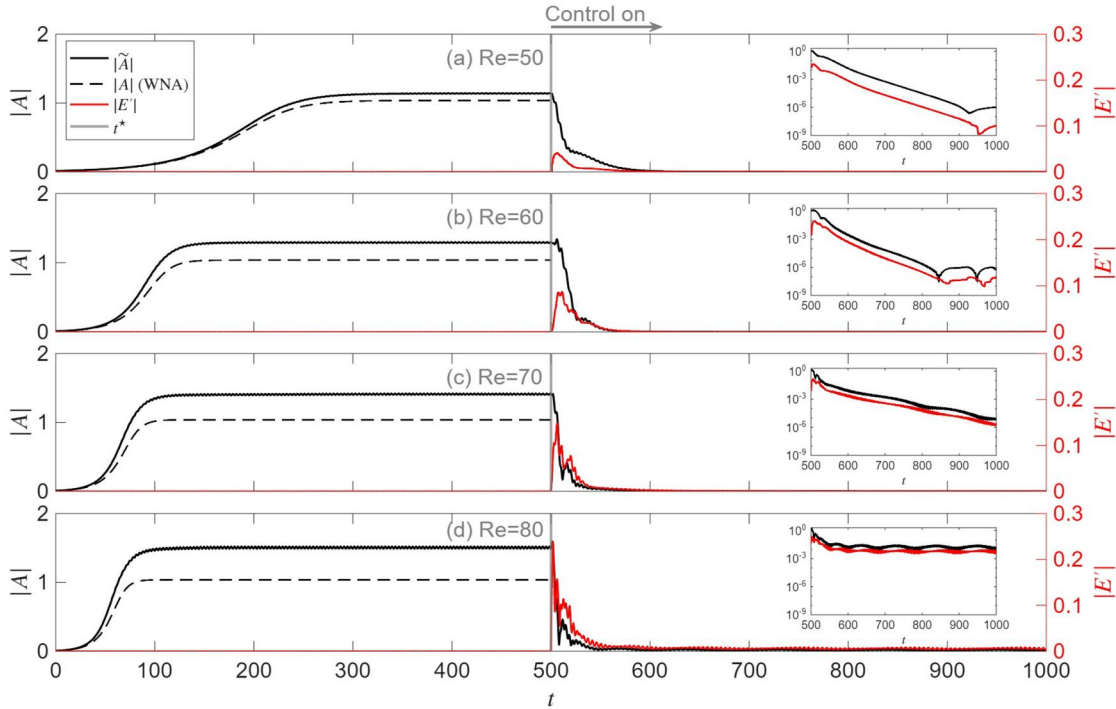


FIG. 9. Time evolution of the global mode amplitude and its suppression. For $t \leq 500$, the unforced evolution is shown, comparing the amplitude obtained from DNS, the estimate $\tilde{A}$ from full velocity measurements (22), and the prediction from the weakly nonlinear analysis (WNA). At $t = 500$, control is activated, and the subsequent decay of the amplitude is achieved using MPC with resonant forcing of amplitude E' and feedback based on $\tilde{A}$. (a) $Re = 50$, (b) $Re = 60$, (c) $Re = 70$, and (d) $Re = 80$.

$C'_L \approx 0.014$ under control. For the downstream cylinder, is C'_L reduced by more than a factor of two, from $C'_L \approx 0.78$ in the unforced case to $C'_L \approx 0.31$.

The drag coefficient of the upstream cylinder closely approaches its steady value ($\bar{C}_D \approx 1.11$), with a controlled mean of approximately $\bar{C}_D \approx 1.13$ (compared to $\bar{C}_D \approx 1.27$ in the unforced case) and small residual oscillations of order 10^{-3} ($C'_D \approx 9 \times 10^{-4}$ compared to 2.7×10^{-3} in the unforced case). For the downstream cylinder, the mean drag is reduced from $\bar{C}_D \approx 0.67$ in the unforced case to $\bar{C}_D \approx 0.45$, approaching an intermediate value between the steady-state value ($\bar{C}_D \approx 0.26$) and the unforced limit cycle. The corresponding drag fluctuations are also reduced ($C'_D \approx 0.02$ compared to 0.035 unforced), although noticeable oscillations persist.

2. Control with point measurements

In Sec. V C 1, we used the reduced state estimated from the DNS velocity field to design the feedback law. However, measuring the velocity field over a large domain, for example using particle image velocimetry (PIV), is often too slow for real-time control. In practical applications, the flow is instead measured using a limited number of spatially localized sensors, which provide only partial information about the velocity field. We therefore consider the use of point measurements of the velocity.

To this end, we seek suitable sensor locations that enable an accurate estimation of the reduced state. The goal is to compute $\tilde{A}_p$ via the mapping (23), such that it closely approximates $\tilde{A}$ obtained

from full-domain measurements using (22), while using as few point measurements $u_i(\mathbf{x}_i)$ and/or $v_i(\mathbf{x}_i)$ as possible. In this work, we restrict ourselves to the minimum number of measurements required by the estimation approach in Eq. (23), namely, two.

In the point-measurement setting, however, the full-field estimate $\tilde{A}$ is not available. The sensor locations are therefore chosen by considering the expected error of the point estimate A_p with respect to the amplitude A predicted by the reduced model. This requires identifying the main sources of error between A_p and A .

For the point reconstruction to recover the amplitude exactly, i.e., $A_p = A$, two conditions would have to be satisfied. First, the reduced-to-full reconstruction map G in Eq. (21), when restricted to the linear terms used in Eq. (23), would have to introduce no error in the reconstruction of the measured velocity components at the selected sensor locations. Second, the corresponding measurement matrix G in Eq. (23) would have to be full rank. In practice, the first condition is not exactly satisfied. We, therefore, assume that the dominant local reconstruction error comes from the leading nonlinear terms omitted in the linear point-measurement relation, in particular, the contributions involving $\mathbf{U}_2^{[A]^2}$ and $\mathbf{U}_2^{A^2}$. These omitted terms, together with the conditioning of G , are used to build the practical sensor-selection indicator defined and illustrated in Appendix B. Based on this criterion, we select the following measurement configurations: $(u_1(3.8, 1.4), v_1(3.8, 1.4))$ for $Re = 50$, $(v_1(2.8, 0), v_2(1.4, 0))$ for $Re = 60$, and $(v_1(2.8, 0), v_2(1.2, 0))$ for $Re = 70$. For $Re = 50$, measuring both velocity components at a single point is advantageous from an application standpoint, since only one sensor location is required. At higher Reynolds numbers,

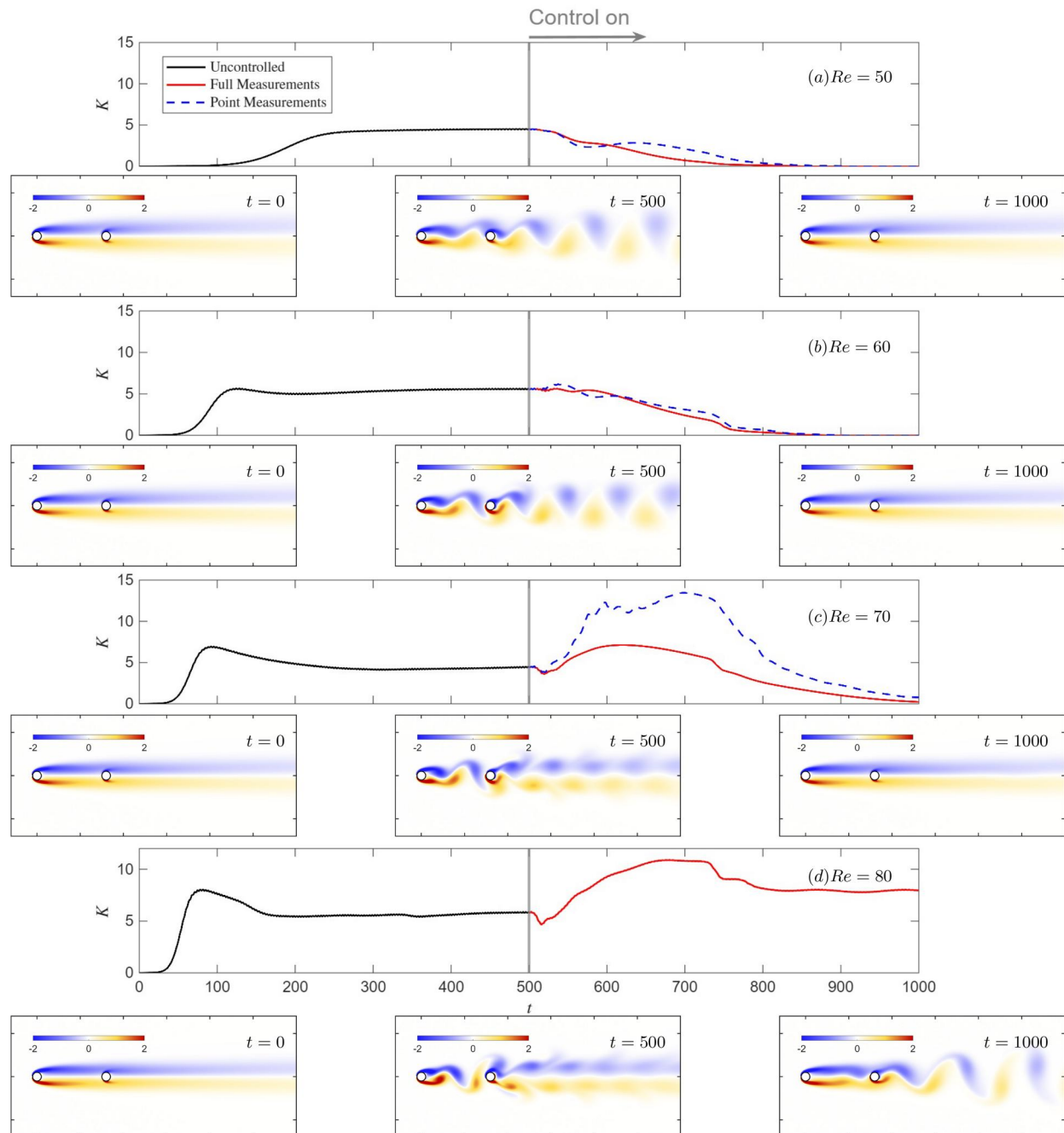


FIG. 10. Evolution of energy $K = 0.5(\mathbf{u}', \mathbf{u}')$ of the unsteady velocity $\mathbf{u}'$ relative to the base flow $\mathbf{u}_b$, as $\mathbf{u}' = \mathbf{u} - \mathbf{u}_b$. The flow evolves to the limit cycle before $t = 500$, at which time control is turned on. The solid red curve represents results obtained by applying control using the full measurements for the feedback, while the dashed blue curve is the result with state estimation via point measurements only. Instantaneous vorticity and streamlines for times $t = 0$, $t = 500$ (when the flow is at the limit cycle), and $t = 1000$ are shown. (a) $Re = 50$, (b) $Re = 60$, (c) $Re = 70$, and (d) $Re = 80$.

however, this single-point configuration does not provide sufficient accuracy. Therefore, for $Re = 60$ and $Re = 70$, we use transverse-velocity measurements at two distinct locations, which gives a more accurate point-measurement estimate of A .

Control with point measurements is not considered for $Re = 80$, due to the large state-estimation error arising from the altered structure of the global mode at this Reynolds number, which is not adequately captured by the present model.

TABLE II. Integrated control effort for the closed-loop simulations. The control effort is defined as $\mathcal{W}_E = \int_{t_0}^{t_f} |E'(t)| dt$, where $t_0 = 500$ is the time at which control is activated and $t_f = 1000$ is the final time of the controlled simulation. The sustained input amplitude denotes the approximate order of $|E'|$ near the end of the controlled interval.

Case	Measurement type	Re	Sustained $ E' $	$\mathcal{W}_E$
F1	Full velocity field	50	$\mathcal{O}(10^{-8})$	0.9589
F2	Full velocity field	60	$\mathcal{O}(10^{-8})$	1.9582
F3	Full velocity field	70	$\mathcal{O}(10^{-6})$	2.5729
F4	Full velocity field	80	$\mathcal{O}(10^{-2})$	7.1332
P1	Point measurements	50	$\mathcal{O}(10^{-8})$	1.5794
P2	Point measurements	60	$\mathcal{O}(10^{-8})$	2.9979
P3	Point measurements	70	$\mathcal{O}(10^{-5})$	18.4094

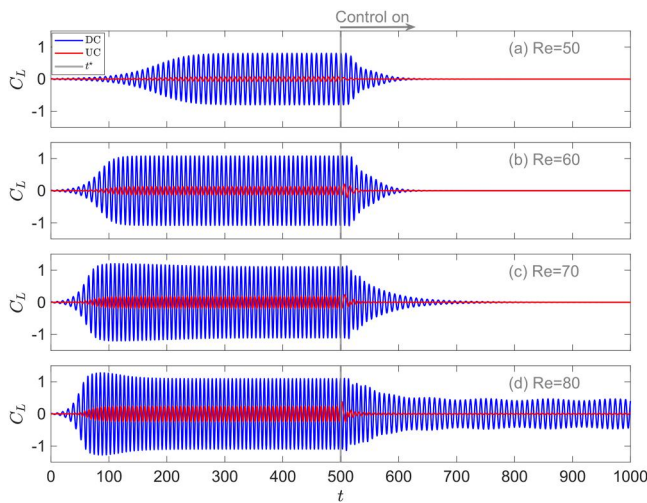


FIG. 11. Time evolution and suppression of lift coefficient C_L acting on the upstream (UC) and downstream (DC) cylinders. The MPC feedback is based on full velocity measurements. (a) $Re = 50$, (b) $Re = 60$, (c) $Re = 70$, and (d) $Re = 80$.

Figure 13 shows the time evolution of the amplitude of the reduced state $|\tilde{A}_p|$, estimated from the selected point measurements, for the unforced flow over the interval $t = 0-500$. An oscillatory discrepancy, compared to the amplitude obtained from full measurements (Fig. 9), is observed in $\tilde{A}_p$ and grows as the flow approaches the limit cycle. This behavior does not reflect the physical flow dynamics, but arises from the increasing approximation error in the state estimation.

The estimation error is significantly larger at $Re = 70$ than at $Re = 50$ and $Re = 60$. Moreover, the error at $Re = 50$ exceeds that at $Re = 60$, since only a single measurement location is used in that case, whereas for higher Reynolds numbers, two measurement points of the v -velocity component are employed, leading to improved accuracy according to the chosen criterion.

The control is activated at $t = 500$ and applied until $t = 1000$. The MPC settings are identical to those used in the full-measurement case.

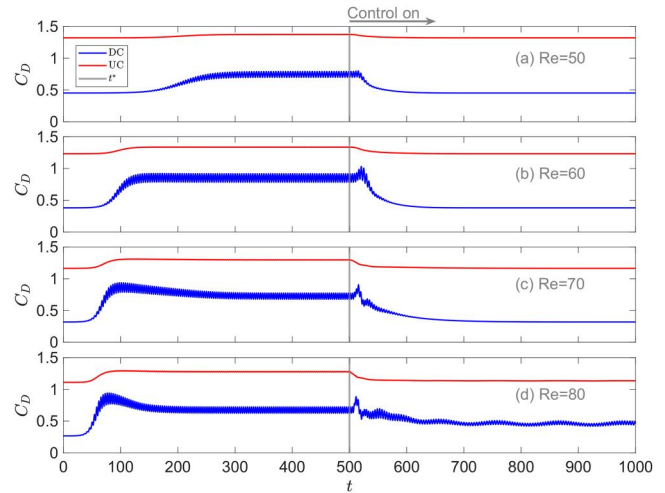


FIG. 12. Time evolution and suppression of drag coefficient C_D acting on the upstream (UC) and downstream (DC) cylinders. The MPC feedback is based on full velocity measurements. (a) $Re = 50$, (b) $Re = 60$, (c) $Re = 70$, and (d) $Re = 80$.

This time interval is sufficient to reduce $|\tilde{A}_p|$ to very small values, of order 10^{-7} for $Re = 50$ and $Re = 60$, with the corresponding amplitudes of the reduced input decreasing to values of order 10^{-8} , similar to in the full-measurement case.

For $Re = 70$, control is more challenging due to the larger modeling errors. The amplitude of the reduced state initially increases but is eventually reduced to approximately 10^{-4} , with the corresponding input converging to values of order 10^{-5} .

The fluctuation energy evolution given in Fig. 10 (dashed lines) confirms that the flow is effectively stabilized and brought to the steady base flow for $Re = 50$ and $Re = 60$. In both cases, the energy decreases steadily, albeit more slowly than in the full-measurement case, reaching values of order 10^{-5} at $t = 1000$. A high level of suppression is also achieved for $Re = 70$; although the energy initially increases, it is subsequently reduced to approximately 0.8 by $t = 1000$ and continues to decay under prolonged control, decreasing by about a factor of two over an additional 100 time units (not shown here). We do not show the vorticity fields again, as the differences between the full-measurement and point-measurement cases are not visually discernible.

Finally, we plot the time evolution of C_L and C_D in Figs. 14 and 15. Consistent with the vorticity observations, for $Re = 50, 60$, and 70 , the lift coefficient is reduced to near zero for both cylinders, reaching values of less than 10^{-6} for $Re = 50$ and $Re = 60$, and less than 10^{-3} for $Re = 70$, while the drag coefficient approaches its steady value.

A natural extension is to combine the static point-measurement reconstruction with a dynamic estimator, such as an extended Kalman filter.⁴⁵ In such a formulation, the Stuart–Landau equation serves as an approximate dynamical prior, while the measurements are used to update the estimated amplitude at each time instant. For parameter values close to criticality, two measurements are sufficient to ensure accurate static state reconstruction. Hence, the main added value of a dynamic estimator is the possibility of reducing the required

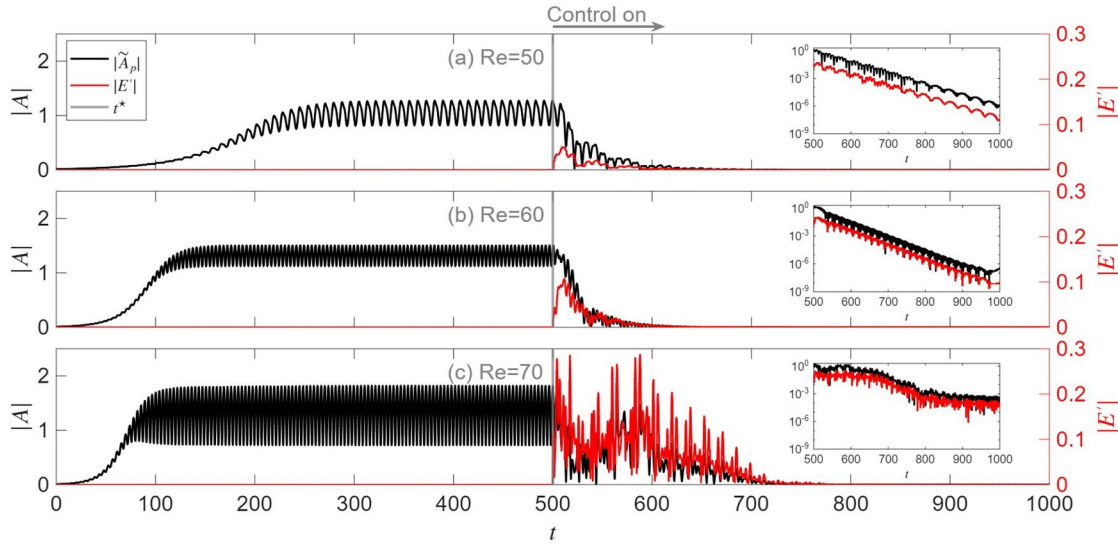


FIG. 13. Time evolution of the global mode amplitude and its suppression. For $t \leq 500$, the estimate $\tilde{A}_p$ obtained from pointwise velocity measurements (23) in the unforced flow is shown. The following measurement configurations are used: $(u_1(3.8, 1.4), v_1(3.8, 1.4))$ for $Re = 50$, $(v_1(2.8, 0), v_2(1.4, 0))$ for $Re = 60$, and $(v_1(2.8, 0), v_2(1.2, 0))$ for $Re = 70$. At $t = 500$, control is activated, and the subsequent decay of the amplitude is achieved using MPC with a resonant forcing of amplitude E' and feedback based on $\tilde{A}_p$. (a) $Re = 50$, (b) $Re = 60$, and (c) $Re = 70$.

number of measurement locations to one, which is not a particularly substantial gain. For higher parameter values, the estimator dynamics are expected to reduce the oscillatory behavior observed in the static estimation (cf. Fig. 13). It is less clear, however, how such a dynamic estimator can be tuned effectively for larger parameter values, since underestimated model mismatch may be significantly amplified by the observer or extended Kalman filter gains. Nevertheless, filtering-based formulations are likely to be the natural choice for experimental applications, as they also provide a systematic way to account for measurement noise and moderate model uncertainty.

VI. CONCLUSION

In this work, we have presented a model-based closed-loop control strategy for the flow over two cylinders in tandem configuration.

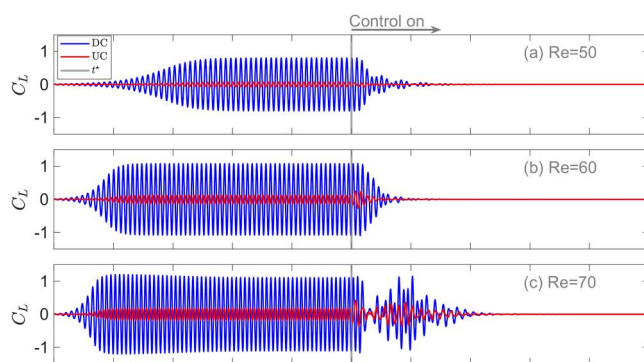


FIG. 14. Time evolution and suppression of lift coefficient C_L acting on the upstream (UC) and downstream (DC) cylinders. The MPC feedback is based on pointwise velocity measurements. (a) $Re = 50$, (b) $Re = 60$, and (c) $Re = 70$.

Building on a weakly nonlinear analysis of the incompressible Navier–Stokes equations, we derived a low-dimensional, forced Stuart–Landau model that captures the essential nonlinear dynamics near the onset of instability, i.e., in the vicinity of the critical Reynolds number associated with a supercritical Hopf bifurcation. The formulation was extended to allow for time-dependent forcing amplitudes, enabling the design of feedback control laws.

Based on this reduced-order model, we designed a model predictive controller (MPC) to determine the control input in the reduced-order dynamics. The resulting control law is then mapped to the full-order system through explicit state and input mappings, yielding an output-feedback formulation. In this framework, the reduced state is estimated from available velocity measurements, and actuation is

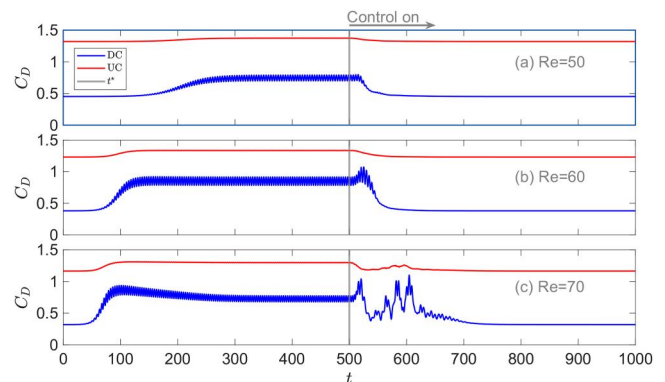


FIG. 15. Time evolution and suppression of drag coefficient C_D acting on the upstream (UC) and downstream (DC) cylinders. The MPC feedback is based on pointwise velocity measurements. (a) $Re = 50$, (b) $Re = 60$, and (c) $Re = 70$.

introduced through localized volumetric forcing to ensure physical realizability. Both full-field and pointwise measurement configurations are considered.

The proposed approach was demonstrated for tandem cylinders at spacing ratio $\gamma = 8$, corresponding to the co-shedding regime, and Reynolds numbers $Re = 50, 60, 70$, and 80 . As expected from the weakly nonlinear expansion, the model accuracy decreases with increasing Reynolds number.

Using full-domain velocity measurements, the closed-loop controller successfully suppresses vortex shedding in both the gap region and the wake of the downstream cylinder for $Re = 50, 60$, and 70 , driving the flow close to the steady base state. For $Re = 80$, the controller significantly mitigates the unsteady dynamics, leading to a reduced-amplitude oscillatory state. The control input required to achieve this suppression also approaches zero as the unsteadiness is suppressed, thereby achieving efficient control over long times.

We further demonstrated that effective control can be achieved using only a small number of pointwise velocity measurements. In particular, suppression is obtained for $Re = 50$ using measurements at a single spatial location, and for $Re = 60$ and $Re = 70$ using two-point measurements, although the performance degrades compared to the full-measurement case as the Reynolds number increases.

The quantitative validity of the present Stuart–Landau model is limited to a finite neighborhood of the Hopf onset. Nevertheless, the underlying control idea can be transferred to other flows in which the relevant unsteadiness is organized by a dominant Hopf-type global instability, provided that an appropriate reduced-order model is constructed and validated for the flow of interest. This may include more complex or turbulent flows in which the coherent instability remains identifiable despite additional fluctuations.^{46,47} For applications involving sparse or noisy measurements, the static amplitude estimate used here could also be replaced by a dynamic estimator, such as an extended Kalman filter, to improve the accuracy of the reduced-state estimate.

Overall, these results demonstrate that the proposed model-based closed-loop control framework can effectively suppress or significantly reduce vortex shedding in tandem-cylinder flows using low-dimensional models, limited sensing, and localized actuation, while acknowledging the limitations associated with increasing Reynolds number and reduced-model accuracy.

SUPPLEMENTARY MATERIAL

See the [supplementary material](#) for additional closed-loop control tests. It examines the effect of the MPC prediction horizon on the controlled response and the sensitivity of the controller to perturbations of the nonlinear Stuart–Landau coefficient.

ACKNOWLEDGMENTS

A.-J. Buchner was supported by the Netherlands Organisation for Scientific Research (NWO), under VENI Project No. 18176.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Dimitris Boskos and Abel-John Buchner contributed equally to this work.

Tea Vojkovic: Conceptualization (equal); Data curation (lead); Formal analysis (equal); Investigation (equal); Methodology (equal); Software (lead); Validation (equal); Visualization (lead); Writing – original draft (lead); Writing – review & editing (equal). **Dimitris Boskos:** Conceptualization (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (supporting); Resources (lead); Supervision (equal); Validation (equal); Visualization (supporting); Writing – review & editing (equal). **Abel-John Buchner:** Conceptualization (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (lead); Resources (supporting); Supervision (equal); Validation (equal); Visualization (supporting); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are openly available in Zenodo at <https://doi.org/10.5281/zenodo.22124859>, Ref. 50.

APPENDIX A: WEAKLY NONLINEAR ANALYSIS

Here, we provide a detailed derivation of the Stuart–Landau equation (17) for resonant forcing frequencies close to ω_0 . We also briefly summarize the corresponding results for other forcing frequencies, which can be obtained analogously.

1. Forcing near the resonant frequency ω_0

Substituting the weakly nonlinear expansion (12) and the forcing (13) into the governing equations (2), together with the scaling (15), and collecting terms at successive orders in $\sqrt{\varepsilon}$ yields a hierarchy of linear inhomogeneous problems for $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \dots$. We solve these equations at each order $\sqrt{\varepsilon}^i$ with $i = 0, 1, 2, 3$, and derive the Stuart–Landau equation (17).

a. Order $\sqrt{\varepsilon}^0$

At zeroth order $O(\sqrt{\varepsilon}^0)$, we obtain the steady incompressible Navier–Stokes equation,

$$\mathcal{N}(\mathbf{U}_0, Re_c) = \begin{pmatrix} -\nabla \mathbf{u}_0 \cdot \mathbf{u}_0 - \nabla p_0 + \frac{1}{Re_c} \Delta \mathbf{u}_0 \\ \nabla \cdot \mathbf{u}_0 \end{pmatrix} = \mathbf{0}, \quad (\text{A1})$$

at the critical Reynolds number Re_c . By construction, this equation is satisfied by the base flow $\mathbf{U}_0$, which is an equilibrium of the unforced system at Re_c .

b. Order $\sqrt{\varepsilon}$

At order $\sqrt{\varepsilon}$, we obtain the homogeneous linear problem

$$\left(\frac{\partial}{\partial t} \mathcal{E} - \mathcal{L}_0 \right) \mathbf{U}_1 = \mathbf{0}, \quad (\text{A2})$$

where

$$\mathcal{L}_0 = \begin{pmatrix} -\nabla(\cdot) \cdot \mathbf{u}_0 - \nabla \mathbf{u}_0 \cdot (\cdot) + \frac{1}{Re_c} \Delta & \nabla \\ \nabla^T & 0 \end{pmatrix}$$

denotes the linearized Navier–Stokes operator about the base flow $\mathbf{U}_0$. To express the general solution of Eq. (A2) in terms of the eigenmodes of the operator pencil,

$$\mathcal{K}_{\lambda_0} = \lambda \mathcal{E} - \mathcal{L}_0, \quad \lambda_0 = \sigma_0 + i\omega_0,$$

we consider the generalized eigenvalue problem

$$\mathcal{K}_{\lambda_0} \hat{\mathbf{U}} = 0. \quad (\text{A3})$$

Since we assume that the first Hopf bifurcation occurs at Re_c , the spectrum of $\mathcal{K}_{\lambda_0}$ contains a single marginally stable complex-conjugate eigenvalue pair $\pm i\omega_0$, with the associated eigenvector $\mathbf{U}_1^A$ (and its complex conjugate) satisfying

$$\mathcal{K}_{i\omega_0} \mathbf{U}_1^A = 0,$$

while all remaining eigenvalues are stable. We, therefore, retain only the contribution along this eigenspace, which governs the long-term dynamics. The resulting leading-order solution can be written as

$$\mathbf{U}_1(t, \tau) = A(\tau) e^{i\omega_0 t} \mathbf{U}_1^A + c.c.,$$

where the complex amplitude $A(\tau)$ evolves on the slow timescale. The mode $\mathbf{U}_1^A$ is referred to as the critical global mode,¹⁹ and the term $A(\tau) e^{i\omega_0 t} \mathbf{U}_1^A$ represents the first harmonic, oscillating at the fundamental frequency ω_0 on the fast timescale.

c. Order $\sqrt{\varepsilon}^2$

At order $\sqrt{\varepsilon}^2$, we obtain the linear inhomogeneous problem

$$\left(\frac{\partial}{\partial t} \mathcal{E} - \mathcal{L} \right) \mathbf{U}_2 = \mathbf{F}_2 + |A(\tau)|^2 \mathbf{F}^{|A|^2} + (A(\tau))^2 e^{i2\omega_0 t} \mathbf{F}^{A^2} + c.c.,$$

where the right-hand side terms are

$$\mathbf{F}^{|A|^2} := \mathcal{P}(-\nabla \mathbf{u}_1^A \cdot \overline{\mathbf{u}_1^A} - \nabla \overline{\mathbf{u}_1^A} \cdot \mathbf{u}_1^A), \quad \mathbf{F}^{A^2} := \mathcal{P}(-\nabla \mathbf{u}_1^A \cdot \mathbf{u}_1^A), \\ \mathbf{F}_2 := \mathcal{P}(-\Delta \mathbf{u}_0).$$

Accordingly, we write $\mathbf{U}_2(t, \tau)$ as a superposition of the responses to each right-hand-side contribution,

$$\mathbf{U}_2(t, \tau) = \mathbf{U}_2^1 + |A(\tau)|^2 \mathbf{U}_2^{|A|^2} + A(\tau)^2 e^{i2\omega_0 t} \mathbf{U}_2^{A^2} + c.c., \quad (\text{A4})$$

where the fields $\mathbf{U}_2^1$, $\mathbf{U}_2^{|A|^2}$, and $\mathbf{U}_2^{A^2}$ are obtained from

$$\mathcal{K}_0 \mathbf{U}_2^1 = \mathbf{F}_2, \quad \mathcal{K}_0 \mathbf{U}_2^{|A|^2} = \mathbf{F}^{|A|^2}, \quad \mathcal{K}_{i2\omega_0} \mathbf{U}_2^{A^2} = \mathbf{F}^{A^2}. \quad (\text{A5})$$

By the assumptions of the Hopf bifurcation theorem (see Theorem 8.25 in Chicone⁴⁸) $\lambda = 0$ and $\lambda = i2\omega_0$ do not belong to the spectrum of the operator pencil $\mathcal{K}_{\lambda_0}$, and, therefore, the problems in Eq. (A5) admit unique solutions.

The component $\mathbf{U}_2^1$ represents the $O(\varepsilon)$ correction to $\mathbf{U}_0$ that accounts for the shift of the unstable equilibrium branch for $\varepsilon > 0$. The term $A(\tau)^2 e^{i2\omega_0 t} \mathbf{U}_2^{A^2}$ is the second harmonic, oscillating at twice the fundamental frequency, and stems from the nonlinear

interaction between the first harmonic with itself. The mean-flow component $|A(\tau)|^2 \mathbf{U}_2^{|A|^2}$ results from the interaction of the first harmonic with its complex conjugate; it evolves on the slow timescale and approximates the difference between the mean flow and the steady base flow.¹⁹

d. Order $\sqrt{\varepsilon}^3$

At order $\sqrt{\varepsilon}^3$, we obtain the inhomogeneous linear problem

$$\left(\frac{\partial}{\partial t} \mathcal{E} - \mathcal{L} \right) \mathbf{U}_3 = -E \frac{dA}{d\tau} e^{i\omega_0 t} \mathbf{U}_1^A + A(\tau) e^{i\omega_0 t} \mathbf{F}^A \\ + A(\tau) |A(\tau)|^2 e^{i\omega_0 t} \mathbf{F}^{|A|^2} + E(\tau) e^{i\omega_0 t} \mathcal{P} \mathbf{f}_E \\ + c.c. + \dots \quad (\text{A6})$$

where the right-hand side terms $\mathbf{F}^A$ and $\mathbf{F}^{|A|^2}$ are defined by

$$\mathbf{F}^A := \mathcal{P}(-\nabla \mathbf{u}_1^A \cdot \mathbf{u}_2^1 - \nabla \mathbf{u}_2^1 \cdot \mathbf{u}_1^A - \Delta \mathbf{u}_1^A),$$

$$\mathbf{F}^{|A|^2} := \mathcal{P}(-\nabla \mathbf{u}_1^A \cdot \mathbf{u}_2^{|A|^2} - \nabla \mathbf{u}_2^{|A|^2} \cdot \mathbf{u}_1^A - \nabla \overline{\mathbf{u}_1^A} \cdot \mathbf{u}_2^{A^2} - \nabla \mathbf{u}_2^{A^2} \cdot \overline{\mathbf{u}_1^A}).$$

The term $\mathbf{F}^A$ stems from the viscous diffusion of the first harmonic $A e^{i\omega_0 t} \mathbf{U}_1^A$ and its interaction with the $O(\varepsilon)$ base-flow correction $\mathbf{U}_2^1$. The term $\mathbf{F}^{|A|^2}$ results from interactions between the first harmonic $A(\tau) e^{i\omega_0 t} \mathbf{U}_1^A$, the zeroth harmonic $|A(\tau)|^2 \mathbf{U}_2^{|A|^2}$, and the second-harmonic $A(\tau)^2 e^{i2\omega_0 t} \mathbf{U}_2^{A^2}$. Since the right-hand side of Eq. (A6) contains contributions oscillating at (or close to) the marginal frequency ω_0 , the corresponding particular solution $\mathbf{U}_3(t, \tau)$ may exhibit secular growth in time unless appropriate compatibility conditions are enforced.

e. Stuart–Landau equation

To ensure that the solution remains bounded in time, and hence that the asymptotic expansion remains valid, the resonant terms on the right-hand side of Eq. (A6) must satisfy a compatibility condition. In particular, they must be orthogonal, at all times, to the adjoint eigenvector $\mathbf{U}_1^{A*}$ associated with $\mathcal{K}_{i\omega_0}$. This condition imposes an evolution equation for the global-mode amplitude A , which takes the Stuart–Landau form

$$\frac{dA}{d\tau} = a_0 A - a_1 |A|^2 + a_2 e^{i\varepsilon \Omega t} E, \quad (\text{A7})$$

where the coefficients a_0 , a_1 , and a_2 are obtained from the corresponding compatibility (orthogonality) conditions

$$a_0 := \frac{\langle \mathbf{U}_1^{A*}, \mathbf{F}^A \rangle}{\langle \mathbf{U}_1^{A*}, \mathcal{E} \mathbf{U}_1^A \rangle}, \quad (\text{A8a})$$

$$a_1 := -\frac{\langle \mathbf{U}_1^{A*}, \mathbf{F}^{|A|^2} \rangle}{\langle \mathbf{U}_1^{A*}, \mathcal{E} \mathbf{U}_1^A \rangle}, \quad (\text{A8b})$$

$$a_2 := \frac{\langle \mathbf{U}_1^{A*}, \mathcal{P} \mathbf{f}_E \rangle}{\langle \mathbf{U}_1^{A*}, \mathcal{E} \mathbf{U}_1^A \rangle}. \quad (\text{A8c})$$

Here, the scalar product between two fields $\mathbf{U}_\alpha$ and $\mathbf{U}_\beta$ is defined as

$$\langle \mathbf{U}_\alpha, \mathbf{U}_\beta \rangle := \iint_{\Omega} (\bar{u}_\alpha u_\beta + \bar{v}_\alpha v_\beta + \bar{p}_\alpha p_\beta) dx dy. \quad (\text{A9})$$

Thus, the solution at order $\sqrt{\varepsilon}^2$ takes the form

$$\mathbf{U}_3(t, \tau) = A(\tau) e^{i\omega_0 t} \mathbf{U}_3^A + A(\tau) |A(\tau)|^2 e^{i\omega_0 t} \mathbf{U}_3^{A|A|^2} + E(\tau) e^{i\omega_f t} \mathbf{U}_3^E + c.c. + \dots$$

where we retain only the contributions oscillating at (or close to) the resonant frequency.

2. Other resonant forcing frequencies

We next discuss other resonant forcing cases, namely $\omega_f = 0$, $\omega_f \approx 2\omega_0$, and $\omega_f \approx \omega_0/2$. Each case requires a different forcing scaling in order for the forcing to enter the Stuart–Landau equation at $O(\sqrt{\varepsilon}^3)$, while avoiding degenerate (non-invertible) operators at lower orders.

a. Forcing at $\omega_f = 0$

For $\omega_f = 0$, the forcing amplitude is scaled as

$$E'(\tau) := \varepsilon E(\tau).$$

Solving the resulting equations at orders $\sqrt{\varepsilon}^0$, $\sqrt{\varepsilon}^1$, $\sqrt{\varepsilon}^2$, and $\sqrt{\varepsilon}^3$, yields

$$\begin{aligned} \mathbf{U} \approx & \mathbf{U}_0 + \sqrt{\varepsilon} (A e^{i\omega_0 t} \mathbf{U}_1^A + c.c.) \\ & + \varepsilon (\mathbf{U}_2^1 + |A|^2 \mathbf{U}_2^{A^2} + (A^2 e^{i2\omega_0 t} \mathbf{U}_2^{A^2} + E \mathbf{U}_2^E + c.c.)) \\ & + \sqrt{\varepsilon}^3 (A e^{i\omega_0 t} \mathbf{U}_3^A + A |A|^2 e^{i\omega_0 t} \mathbf{U}_3^{A|A|^2} \\ & + A E e^{i\omega_0 t} \mathbf{U}_3^{AE} + A \bar{E} e^{i\omega_0 t} \mathbf{U}_3^{\bar{A}\bar{E}} + c.c.) + \dots, \end{aligned} \quad (\text{A10})$$

where $\mathbf{U}_0$, $\mathbf{U}_1^A$, $\mathbf{U}_2^1$, $\mathbf{U}_2^{A^2}$, and $\mathbf{U}_2^{AE}$ are identical to those in the subsection 1 of Appendix A, while the forcing response $\mathbf{U}_2^E$ is obtained from

$$\mathcal{K}_0 \mathbf{U}_2^E = \mathcal{P} \mathbf{f}_E.$$

At order $\sqrt{\varepsilon}^3$, the contributions $\mathbf{U}_3^A$, $\mathbf{U}_3^{A|A|^2}$, $\mathbf{U}_3^{AE}$, and $\mathbf{U}_3^{\bar{A}\bar{E}}$ must again satisfy compatibility conditions. These yield the Stuart–Landau equation

$$\frac{dA}{dt} = \varepsilon (a_0 - a_2 E - a_3 \bar{E}) A - \varepsilon a_1 |A|^2. \quad (\text{A11})$$

Since the coefficients a_0 and a_1 depend only on the unforced modes, they are independent of the forcing and are given, as in subsection 1 of Appendix A, by Eqs. (A8a) and (A8b). The forcing-related coefficient is

$$\begin{aligned} a_2 &:= \frac{\langle \mathbf{U}_1^{A*}, \mathcal{P}(\nabla \mathbf{u}_1^A \cdot \mathbf{u}_1^E + \nabla \mathbf{u}_1^E \cdot \mathbf{u}_1^A) \rangle}{\langle \mathbf{U}_1^{A*}, \mathcal{E} \mathbf{U}_1^A \rangle}, \\ a_3 &:= \frac{\langle \mathbf{U}_1^{A*}, \mathcal{P}(\nabla \mathbf{u}_1^A \cdot \bar{\mathbf{u}}_1^E + \nabla \bar{\mathbf{u}}_1^E \cdot \mathbf{u}_1^A) \rangle}{\langle \mathbf{U}_1^{A*}, \mathcal{E} \mathbf{U}_1^A \rangle}. \end{aligned}$$

Equation (A11) shows that, for $\omega_f = 0$, the forcing enters the reduced dynamics bilinearly through the term $(a_2 E + a_3 \bar{E}) A$, in contrast to the purely additive forcing term in Eq. (A7).

b. Forcing near $2\omega_0$

For a forcing frequency near $2\omega_0$, namely, $\omega_f = 2\omega_0 + \varepsilon\Omega$ for some frequency Ω , we scale the forcing amplitude as

$$E'(\tau) := \varepsilon E(\tau).$$

Solving the resulting equations yields an expansion of the form

$$\begin{aligned} \mathbf{U} \approx & \mathbf{U}_0 + \sqrt{\varepsilon} (A e^{i\omega_0 t} \mathbf{U}_1^A + c.c.) \\ & + \varepsilon (\mathbf{U}_2^1 + |A|^2 \mathbf{U}_2^{A^2} \\ & + (A^2 e^{i2\omega_0 t} \mathbf{U}_2^{A^2} + E e^{i(2\omega_0 + \varepsilon\Omega)t} \mathbf{U}_2^E + c.c.)) \\ & + \sqrt{\varepsilon}^3 (A e^{i\omega_0 t} \mathbf{U}_3^A + A |A|^2 e^{i\omega_0 t} \mathbf{U}_3^{A|A|^2} \\ & + \bar{A} E e^{i(\omega_0 + \varepsilon\Omega)t} \mathbf{U}_3^{\bar{A}E} + c.c.) + \dots \end{aligned} \quad (\text{A12})$$

Here, the forcing response is obtained from

$$\mathcal{K}_{i(2\omega_0 + \varepsilon\Omega)} \mathbf{U}_2^E = \mathcal{P} \mathbf{f}_E.$$

The corresponding Stuart–Landau equation is

$$\frac{dA}{dt} = \varepsilon a_0 A - \varepsilon a_1 |A|^2 - \varepsilon a_2 e^{i\varepsilon\Omega t} \bar{A} E$$

with the coefficient

$$a_2 := \frac{\langle \mathbf{U}_1^{A*}, \mathcal{P}(\nabla \bar{\mathbf{u}}_1^A \cdot \mathbf{u}_1^E + \nabla \mathbf{u}_1^E \cdot \bar{\mathbf{u}}_1^A) \rangle}{\langle \mathbf{U}_1^{A*}, \mathcal{E} \mathbf{U}_1^A \rangle}$$

determined by the interaction between the forcing response and the complex conjugate of the global mode.

c. Forcing near $\omega_0/2$

The last resonant forcing case concerns frequencies near $\omega_0/2$. We assume $\omega_f = \omega_0/2 + \varepsilon\Omega$. The appropriate scaling of $E'(\tau)$ is

$$E'(\tau) := \varepsilon^{3/4} E(\tau),$$

and the weakly nonlinear expansion takes the form

$$\begin{aligned} \mathbf{U} \approx & \mathbf{U}_0 + \sqrt{\varepsilon} (A e^{i\omega_0 t} \mathbf{U}_1^A + c.c.) \\ & + \sqrt{\varepsilon}^{3/2} (E e^{i(\omega_0/2 + \varepsilon\Omega)t} \mathbf{U}_2^E + c.c.) \\ & + \varepsilon (\mathbf{U}_2^1 + |A|^2 \mathbf{U}_2^{A^2} + A^2 e^{i2\omega_0 t} \mathbf{U}_2^{A^2}) \\ & + \sqrt{\varepsilon}^3 (A e^{i\omega_0 t} \mathbf{U}_3^A + A |A|^2 e^{i\omega_0 t} \mathbf{U}_3^{A|A|^2} \\ & + E^2 e^{i(\omega_0 + 2\varepsilon\Omega)t} \mathbf{U}_3^{E^2} + c.c.) + \dots, \end{aligned} \quad (\text{A13})$$

where the forcing response $\mathbf{U}_2^E$ is obtained from

$$\mathcal{K}_{i(\omega_0/2 + \varepsilon\Omega)} \mathbf{U}_2^E = \mathcal{P} \mathbf{f}_E.$$

The corresponding Stuart–Landau equation is

$$\frac{dA}{dt} = \varepsilon a_0 A - \varepsilon a_1 |A|^2 - \varepsilon a_2 e^{i2\varepsilon\Omega t} E^2, \quad (\text{A14})$$

where

$$a_2 := \frac{\langle \mathbf{U}_1^{A*}, \mathcal{P}(\nabla \mathbf{u}_1^E \cdot \mathbf{u}_1^E) \rangle}{\langle \mathbf{U}_1^{A*}, E \mathbf{U}_1^A \rangle}$$

stems from the nonlinear self-interaction of the forcing response. In this case, the forcing enters (A14) as a nonlinear additive term, in analogy to Eq. (A7) for forcing near the natural frequency.

3. Non-resonant forcing frequencies

For non-resonant forcing, i.e., for $\omega_f \neq 0$ and $\omega_f \not\approx \omega_0$, $\omega_0/2$, $2\omega_0$, the forcing amplitude scales as

$$E'(\tau) := \sqrt{\varepsilon} E(\tau). \quad (\text{A15})$$

The equations at orders $\sqrt{\varepsilon}^0$, $\sqrt{\varepsilon}^1$, $\sqrt{\varepsilon}^2$, and $\sqrt{\varepsilon}^3$ are solved successively as in the resonant cases, yielding the expansion

$$\begin{aligned} \mathbf{U} \approx & \mathbf{U}_0 + \sqrt{\varepsilon} (A e^{i\omega_0 t} \mathbf{U}_1^A + E e^{i\omega_f t} \mathbf{U}_1^E + c.c.) \\ & + \varepsilon (\mathbf{U}_2^1 + |A|^2 \mathbf{U}_2^{1^2} + (A^2 e^{i2\omega_0 t} \mathbf{U}_2^{A^2} + c.c.)) \\ & + |E|^2 \mathbf{U}_2^{E^2} + (E^2 e^{i2\omega_f t} \mathbf{U}_2^{E^2} + c.c.) \\ & + (A E e^{i(\omega_0 + \omega_f)t} \mathbf{U}_2^{AE} + A \bar{E} e^{i(\omega_0 - \omega_f)t} \mathbf{U}_2^{A\bar{E}} + c.c.) \\ & + \sqrt{\varepsilon}^3 (A e^{i\omega_0 t} \mathbf{U}_3^A + A |A|^2 e^{i\omega_0 t} \mathbf{U}_3^{A|A|^2} \\ & + A |E|^2 e^{i\omega_0 t} \mathbf{U}_3^{A|E|^2} + c.c.) + \dots \end{aligned} \quad (\text{A16})$$

The fields $\mathbf{U}_1^E$, $\mathbf{U}_2^{1^2}$, $\mathbf{U}_2^{E^2}$, $\mathbf{U}_2^{AE}$, and $\mathbf{U}_2^{A\bar{E}}$ are obtained by solving

$$\begin{aligned} \mathcal{K}_{i\omega_f} \mathbf{U}_1^E &= \mathcal{P} \mathbf{f}_E, \\ \mathcal{K}_0 \mathbf{U}_2^{1^2} &= \mathcal{P}(-\nabla \mathbf{u}_1^E \cdot \bar{\mathbf{u}}_1^E - \nabla \bar{\mathbf{u}}_1^E \cdot \mathbf{u}_1^E), \\ \mathcal{K}_{2i\omega_f} \mathbf{U}_2^{E^2} &= \mathcal{P}(-\nabla \mathbf{u}_1^E \cdot \mathbf{u}_1^E), \\ \mathcal{K}_{i(\omega_0 + \omega_f)} \mathbf{U}_2^{AE} &= \mathcal{P}(-\nabla \mathbf{u}_1^A \cdot \mathbf{u}_1^E - \nabla \mathbf{u}_1^E \cdot \mathbf{u}_1^A), \\ \mathcal{K}_{i(\omega_0 - \omega_f)} \mathbf{U}_2^{A\bar{E}} &= \mathcal{P}(-\nabla \mathbf{u}_1^A \cdot \bar{\mathbf{u}}_1^E - \nabla \bar{\mathbf{u}}_1^E \cdot \mathbf{u}_1^A). \end{aligned}$$

As in the previous cases, at order $\sqrt{\varepsilon}^3$, the terms $\mathbf{U}_3^A$, $\mathbf{U}_3^{A|A|^2}$, and $\mathbf{U}_3^{A|E|^2}$ need to satisfy compatibility conditions, which yield the Stuart–Landau equation

$$\frac{dA}{dt} = \varepsilon(a_0 - a_2 |E|^2)A - \varepsilon a_1 A |A|^2$$

with

$$a_2 := \frac{\langle \mathbf{U}_1^{A*}, \mathcal{P}(\nabla \mathbf{u}_1^A \cdot \mathbf{u}_2^{1^2} + \nabla \mathbf{u}_2^{1^2} \cdot \mathbf{u}_1^A + \nabla \mathbf{u}_1^E \cdot \mathbf{u}_2^{A\bar{E}} + \nabla \mathbf{u}_2^{A\bar{E}} \cdot \mathbf{u}_1^E) \rangle}{\langle \mathbf{U}_1^{A*}, E \mathbf{U}_1^A \rangle}.$$

As in the cases of zero-frequency forcing and forcing near $2\omega_0$, the input acts multiplicatively on the linear part of the reduced dynamics.

4. Stabilization capabilities across different frequencies

As suggested by the weakly nonlinear approximations of $\mathbf{U}$ given in Eqs. (16), (A10), (A12), (A13), and (A16), the equilibrium $\mathbf{U}_b \approx \mathbf{U}_0 + \varepsilon \mathbf{U}_2^1$ corresponds to $A = 0$ and $E = 0$.

In the case of zero-frequency forcing, forcing near $2\omega_0$, and non-resonant frequency, the forcing amplitude E cannot converge to zero while maintaining stability of the Stuart–Landau model as it acts multiplicatively on the linear part of its dynamics. This implies the following. First, to stabilize the Stuart–Landau model, we need a persisting control input, which can be energy inefficient. Second, although we might stabilize the Stuart–Landau model, and therefore the global mode and all the other unsteady terms in Eqs. (A10), (A12), and (A16) that arise from its interaction with itself or the forcing response, terms that depend exclusively on the forcing persist. For unsteady forcing with a frequency near $2\omega_0$ or a non-resonant frequency, this implies that the flow will remain unsteady.

However, for the cases of $\omega_f \approx \omega_0$ and $\omega_f \approx \omega_0/2$, the amplitude E enters the Stuart–Landau equation as an additive term, which allows bringing E to zero while stabilizing the model. Once both A and E converge to zero, the flow is at its equilibrium $\mathbf{U}_b \approx \mathbf{U}_0 + \varepsilon \mathbf{U}_2^1$ according to Eqs. (16) and (A13). Therefore, the unsteadiness of the flow can be fully suppressed.

We choose forcing near the resonant frequency $\omega_f \approx \omega_0$ since this case facilitates characterizing the structure $\mathbf{f}_E$, which optimizes the control efficiency, as shown in Sec. IV C. Finding such a forcing structure for the forcing frequency $\omega_f \approx \omega_0/2$ is a nontrivial task and we leave it for future research.

APPENDIX B: SENSOR LOCATION INDICATOR FOR POINT MEASUREMENTS

As stated in the main text, the point-measurement estimate is obtained from the least squares expression (23), which is derived from the reconstruction map (21) by retaining only the terms linear in A . The nonlinear terms involving $\mathbf{U}_2^{1^2}$ and $\mathbf{U}_2^{A^2}$ are, therefore, omitted from the measured velocity components. These omitted terms provide a practical indicator of the local reconstruction error associated with a given choice of sensor locations and measured velocity components. This local contribution to the measurement error is then amplified in the amplitude estimate through the pseudo-inverse of the measurement matrix $\mathbf{G}$. The sensor-selection indicator, therefore, combines the magnitude of the omitted nonlinear terms in the measured components with the conditioning of the corresponding matrix $\mathbf{G}$.

For a given choice of measured velocity components and sensor locations, we define the indicator

$$\mathcal{J} = \frac{\hat{e}}{\sigma_{\min}(\mathbf{G})}, \quad (\text{B1})$$

where $\hat{e}$ denotes the norm of the omitted nonlinear contributions in the measured velocity components, namely, those involving $\mathbf{U}_2^{1^2}$ and $\mathbf{U}_2^{A^2}$, and $\sigma_{\min}(\mathbf{G})$ is the smallest singular value of the corresponding measurement matrix in Eq. (23). Small values of $\mathcal{J}$ therefore correspond to sensor choices for which the omitted nonlinear terms are small and are not strongly amplified by $\mathbf{G}^\dagger$.

For one of the point-measurement configurations considered here, both velocity components are measured at the same spatial location $\mathbf{x}$. In this case, the corresponding omitted nonlinear contribution is measured by

$$\hat{e}(\mathbf{x}) = \| (u_2^{1^2}(\mathbf{x}), v_2^{1^2}(\mathbf{x}), u_2^{A^2}(\mathbf{x}), v_2^{A^2}(\mathbf{x})) \|. \quad (\text{B2})$$

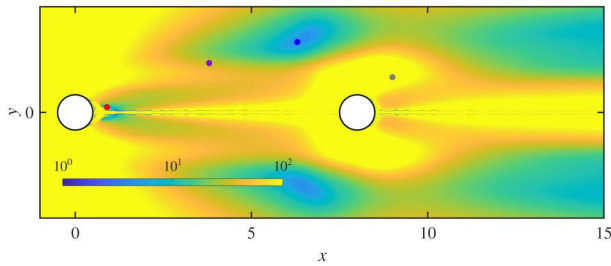


FIG. 16. Spatial distribution of the sensor-location indicator $\mathcal{J}(\mathbf{x})$ for the point-measurement configuration in which both velocity components are measured at a single location. The four indicated locations are $(6.3, 2)$, $(0.9, 0.15)$, $(3.8, 1.4)$, and the unfavorable location $(9, 1)$.

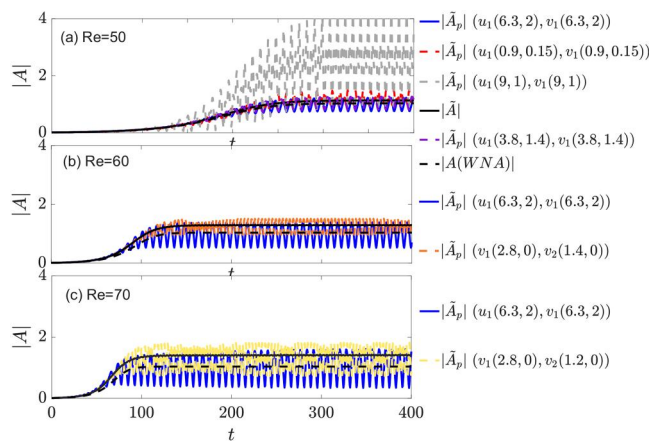


FIG. 17. Comparison of amplitude estimates A_p . (a) $Re = 50$: estimate A_p obtained from one-point measurements of both velocity components at four different sensor locations, compared with the full-field estimate $\tilde{A}$ and the Stuart-Landau prediction A . (b) and (c) Comparison between two-point transverse-velocity measurements and one-point two-component measurements for $Re = 60$ and $Re = 70$.

The associated indicator $\mathcal{J}(\mathbf{x})$ can be evaluated as a function of this single location and visualized directly over the computational domain. This provides a simple way to identify favorable and unfavorable sensor locations. In Fig. 16, we show this map. We then compare the amplitude estimate A_p , obtained from Eq. (23) using measurements of both u - and v -velocity components, at four different locations for $Re = 50$, namely, $(6.3, 2)$, $(0.9, 0.15)$, $(3.8, 1.4)$, and the unfavorable location $(9, 1)$. These locations are indicated

in Fig. 16 using colors consistent with the corresponding curves in Fig. 17(a). The first three locations have relatively small values of $\mathcal{J}(\mathbf{x})$, whereas $(9, 1)$ has a large value. As expected, the latter produces a more oscillatory estimate A_p and a larger discrepancy with respect to A , predicted by the model, as well as $\tilde{A}$ predicted from full-field measurements, which is consistent with the expected effect of the proposed indicator.

We also found that using two measurements of the transverse velocity at distinct spatial locations gives more accurate estimates of A . In this case, only the v -components of the omitted nonlinear terms are retained in $\hat{e}$, and only the corresponding rows are retained in $\mathbf{G}$. The indicator then depends on a pair of sensor locations rather than on a single spatial point, so it cannot be visualized directly as a scalar field over the domain. The same criterion can nevertheless be evaluated for candidate pairs, with the same interpretation: the omitted nonlinear contribution should be small at the selected locations, and the rows of the corresponding measurement matrix should not be nearly linearly dependent. The resulting estimates based on two transverse-velocity measurements are shown in Figs. 17(b) and 17(c) for $Re = 60$ and $Re = 70$, where they are compared with the estimates obtained from one-point measurements of both velocity components. The two-point transverse-velocity configuration provides a more accurate estimate of A_p .

APPENDIX C: MESH CONVERGENCE

We assess the convergence of the coefficients obtained from the weakly nonlinear analysis with respect to the computational mesh. For a consistent comparison, the critical global mode is normalized such that $v_1^A(0.65, 0) = -0.0979 + 0.2661i$ for all meshes. The meshes are unstructured and generated using the adaptive meshing procedure AdaptMesh in FreeFEM++, based on a Delaunay-Voronoi algorithm. In all cases, the mesh is adapted to the base flow as well as to the structure of the direct and adjoint global modes.

We consider three meshes, M1, M2, and M3, defined on the same computational domain (see Sec. II A 2), which differ only in the minimal element size and, consequently, in the number of elements. The results reported in Table III show that mesh M2 provides sufficiently accurate estimates of the weakly nonlinear coefficients while being computationally more efficient than the finer mesh M1. In contrast, further coarsening to mesh M3 leads to more noticeable deviations, particularly in the nonlinear coefficient a_1 .

We also examine the influence of the computational domain extent. The upper and lower boundaries are fixed at $y = \pm 30$, following Sipp and Lebedev,¹⁹ where it was shown for the single-cylinder case that smaller vertical extents introduce slight blockage effects. To assess the sensitivity to the downstream boundary, mesh

TABLE III. Values of the coefficients of the Stuart-Landau model obtained via weakly nonlinear analysis (WNA) obtained with different meshes.

Mesh	Nt	h_{min}	λ_0	a_0	a_1
M1	490 896	0.0047	$-5.9 \times 10^{-5} + 0.6747i$	$7.6888 + 2.8863i$	$7.6883 - 30.1099i$
M2	359 028	0.0528	$-0.0004 + 0.6756i$	$7.6571 + 2.9857i$	$7.1153 - 31.2244i$
M3	253 344	0.0577	$-0.0005 + 0.6756i$	$7.5126 + 3.1444i$	$8.3083 - 36.4366i$
M1bd	537 456	0.0038	$5.9 \times 10^{-5} + 0.6747i$	$7.6896 + 2.8860i$	$7.6457 - 29.9435i$

TABLE IV. Comparison of DNS results with available literature⁴⁹ for $\gamma = 8$ at $Re = 100$. C_l and C_d here refer to RMS value of lift and drag coefficients at the limit cycle, $\bar{C}_d$ is the mean drag coefficient at the limit cycle and St is Strouhal number.

Variable	Present	Literature
C_l (UC)	0.233 2	0.2377
C_l (DC)	0.755 0	0.7760
C_d (UC)	1.255 2	1.2966
C_d (DC)	0.587 6	0.6031
$\bar{C}_d$ (UC)	0.006 7	0.0066
$\bar{C}_d$ (DC)	0.048 0	0.0501
St	0.156 25	0.1529

M1bd uses the same resolution as M1 but extends the outlet to $x_{+\infty} = 250$. As shown in Table III, this modification has a negligible effect on the computed coefficients, indicating that the original domain size is sufficient to capture the relevant flow dynamics and that the parameter estimation is not sensitive to the outlet position.

APPENDIX D: CODE VALIDATION

To validate the DNS carried out in this paper, we compare the hydrodynamic loads experienced by the cylinders, as well as the Strouhal number St of vortex shedding, against available results in the literature. Since reference data are available at $Re = 100$, we perform this validation at this Reynolds number.

Specifically, we compare the root mean square (RMS) lift and drag coefficients, C_l and C_d , together with the mean drag coefficient $\bar{C}_d$, for the fully developed unforced flow at the limit cycle. The results are summarized in Table IV and compared with the numerical data of Sharman *et al.*⁴⁹ for $\gamma = 8$ at $Re = 100$.

We observe good agreement between the present results and those reported in the literature.

In addition, we compare the critical Reynolds number obtained from our analysis, $Re_c = 44.1$, with the value reported in Wang *et al.*,³⁹ further supporting the accuracy of the numerical framework.

REFERENCES

- L. Ding, J. Xiong, Y. Han, Z. Zhu, and J. Ran, "Improvement of mixing performance in laminar micromixers utilizing vortex-induced vibration of two circular cylinders," *Int. Commun. Heat Mass Transfer* **159**, 108285 (2024).
- Z. Liu, C. Zhang, H. Zhao, and N. Guan, "Forced convective heat transfer characteristics in micro/minicylinders group at low Reynolds number," *Nanoscale Microscale Thermophys. Eng.* **16**, 165–180 (2012).
- J. A. Jarrell, B. J. Sytsma, L. H. Wilson, F. L. Pan, K. H. Lau, G. T. Kirby, A. A. Lievano, and R. S. Pawell, "Numerical optimization of microfluidic vortex shedding for genome editing T cells with Cas9," *Sci. Rep.* **11**, 11818 (2021).
- D. Sumner, "Two circular cylinders in cross-flow: A review," *J. Fluids Struct.* **26**, 849–899 (2010).
- M. Zdravkovich, "The effects of interference between circular cylinders in cross flow," *J. Fluids Struct.* **1**, 239–261 (1987).
- G. Xu and Y. Zhou, "Strouhal numbers in the wake of two inline cylinders," *Exp. Fluids* **37**, 248–256 (2004).
- Y. Zhou and M. Yiu, "Flow structure, momentum and heat transport in a two-tandem-cylinder wake," *J. Fluid Mech.* **548**, 17–48 (2006).
- M. Zdravkovich, "Flow induced oscillations of two interfering circular cylinders," *J. Sound Vib.* **101**, 511–521 (1985).
- A. Hetz, M. Dhaubhadel, and D. Telonis, "Vortex shedding over five in-line cylinders," *J. Fluids Struct.* **5**, 243–257 (1991).
- B. S. Carmo, J. R. Meneghini, and S. J. Sherwin, "Possible states in the flow around two circular cylinders in tandem with separations in the vicinity of the drag inversion spacing," *Phys. Fluids* **22**, 054101 (2010).
- Z. Liu, L. Zhou, H. Tang, Z. Wang, F. Zhao, X. Ji, and H. Zhang, "Primary instability, sensitivity and active control of flow past two tandem circular cylinders," *Ocean Eng.* **294**, 116863 (2024).
- H. Liu, M. Azarpeyvand, J. Wei, and Z. Qu, "Tandem cylinder aerodynamic sound control using porous coating," *J. Sound Vib.* **334**, 190–201 (2015).
- A. Kozlov and F. Thomas, "Plasma flow control of cylinders in a tandem configuration," *AIAA J.* **49**, 2183–2193 (2011).
- B. Latrobe, E. G. Ohanu, E. Fernandez, and S. Bhattacharya, "Flow control over tandem cylinders using plasma actuators," *Exp. Therm. Fluid Sci.* **159**, 111274 (2024).
- A. Eltaweel, M. Wang, D. Kim, F. Thomas, and A. Kozlov, "Numerical investigation of tandem-cylinder noise reduction using plasma-based flow control," *J. Fluid Mech.* **756**, 422–451 (2014).
- D. Wolfe and S. Ziada, "Feedback control of vortex shedding from two tandem cylinders," *J. Fluids Struct.* **17**, 579–592 (2003).
- Z. Xie, H. Hu, J. Chen, J. Song, T. Lu, and F. Ren, "Applying reinforcement learning to mitigate wake-induced lift fluctuation of a wall-confined circular cylinder in tandem configuration," *Phys. Fluids* **35**, 053617 (2023).
- J. Mizushima and N. Suehiro, "Instability and transition of flow past two tandem circular cylinders," *Phys. Fluids* **17**, 104107 (2005).
- D. Sipp and A. Lebedev, "Global stability of base and mean flows: A general approach and its applications to cylinder and open cavity flows," *J. Fluid Mech.* **593**, 333–358 (2007).
- D. Sipp, "Open-loop control of cavity oscillations with harmonic forcings," *J. Fluid Mech.* **708**, 439–468 (2012).
- Y. Suzen, G. Huang, J. Jacob, and D. Ashpis, "Numerical simulations of plasma based flow control applications," AIAA Paper No. 2005-4633, 2005.
- T. Weier, G. Gerbeth, G. Mutschke, O. Lielausis, and G. Lammers, "Control of flow separation using electromagnetic forces," *Flow, Turbul. Combust.* **71**, 5–17 (2003).
- A. Barbagallo, D. Sipp, and P. Schmid, "Closed-loop control of an open cavity flow using reduced-order models," *J. Fluid Mech.* **641**, 1–50 (2009).
- F. Hecht, "New development in FreeFEM++," *J. Numer. Math.* **20**, 251–266 (2012).
- Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory* (Springer, 1998).
- A. Verma and S. Mittal, "A new unstable mode in the wake of a circular cylinder," *Phys. Fluids* **23**, 121701 (2011).
- D. Barkley and R. D. Henderson, "Three-dimensional Floquet stability analysis of the wake of a circular cylinder," *J. Fluid Mech.* **322**, 215–241 (1996).
- S.-Y. Wang, F.-B. Tian, L.-B. Jia, X.-Y. Lu, and X.-Z. Yin, "Secondary vortex street in the wake of two tandem circular cylinders at low Reynolds number," *Phys. Rev. E* **81**, 036305 (2010).
- M. van Dyke, *Perturbation Methods in Fluid Mechanics* (Parabolic Press, Stanford, CA, 1975).
- S. Fauve, "Pattern forming instabilities," in *Hydrodynamics and Nonlinear Instabilities* (Cambridge University Press, 2009).
- B. Noack, K. Afanasiev, M. Morzyński, G. Tadmor, and F. Thiele, "A hierarchy of low-dimensional models for the transient and post-transient cylinder wake," *J. Fluid Mech.* **497**, 335–363 (2003).
- H. Buurmeijer, L. Pabon, J. I. Alora, R. S. Kaundinya, G. Haller, and M. Pavone, "Taming high-dimensional dynamics: Learning optimal projections onto spectral submanifolds," in *IEEE 64th Conference on Decision and Control (CDC)* (IEEE, 2025), pp. 373–380.
- S. Symon, K. Rosenberg, S. T. Dawson, and B. J. McKeon, "Non-normality and classification of amplification mechanisms in stability and resolvent analysis," *Phys. Rev. Fluids* **3**, 053902 (2018).
- B. Jin, S. Symon, and S. J. Illingworth, "Energy transfer mechanisms and resolvent analysis in the cylinder wake," *Phys. Rev. Fluids* **6**, 024702 (2021).
- B. Jin, S. Illingworth, and R. Sandberg, "Resolvent-based approach for H_2 -optimal estimation and control: An application to the cylinder flow," *Theor. Comput. Fluid Dyn.* **36**, 491–515 (2022).
- R. Ma, C. Gao, K. Ren, H. Yuan, and W. Zhang, "Suppression of oscillatory fluid forces in cylinder wake: Optimal jet control position designed through resolvent analysis," *Phys. Fluids* **36**, 073622 (2024).

- ³⁷R. Lehoucq, D. Sorensen, and C. Yang, *ARPACK Users' Guide: Solution of Large-Scale Eigenvalue Problems with Implicitly Restarted Arnoldi Methods* (SIAM, 1998).
- ³⁸T. Davis, "Algorithm 832: UMFPACK V4. 3—An unsymmetric-pattern multifrontal method," *ACM Trans. Math. Software* **30**, 196–199 (2004).
- ³⁹J. Wang, X. Shan, and J. Liu, "First instability of the flow past two tandem cylinders with different diameters," *Phys. Fluids* **34**, 074112 (2022).
- ⁴⁰A. Orchini, G. Rigas, and M. P. Juniper, "Weakly nonlinear analysis of thermoacoustic bifurcations in the Rijke tube," *J. Fluid Mech.* **805**, 523–550 (2016).
- ⁴¹N. Deng, Y. Yan, C. Ji, and B. R. Noack, "Feature-based manifold modeling for the quasiperiodic wake dynamics of a pair of side-by-side cylinders," *Phys. Fluids* **36**, 093613 (2024).
- ⁴²J.-C. Loiseau, B. R. Noack, and S. L. Brunton, "Sparse reduced-order modeling: Sensor-based dynamics to full-state estimation," *J. Fluid Mech.* **844**, 459–490 (2018).
- ⁴³M. Cenedese, J. Axås, B. Bäuerlein, K. Avila, and G. Haller, "Data-driven modeling and prediction of non-linearizable dynamics via spectral submanifolds," *Nat. Commun.* **13**, 872 (2022).
- ⁴⁴T. Vojković, D. Boskos, and A.-J. Buchner, "Reduced-order modelling and closed-loop control of the cylinder wake," [arXiv:2410.03330](https://arxiv.org/abs/2410.03330) (2024).
- ⁴⁵R. E. Kalman, "A new approach to linear filtering and prediction problems," *ASME. J. Basic Eng.* **82**(1), 35–45.
- ⁴⁶F. Sartor, C. Mettot, and D. Sipp, "Stability, receptivity, and sensitivity analyses of buffeting transonic flow over a profile," *AIAA J.* **53**, 1980–1993 (2015).
- ⁴⁷D. Busquet, O. Marquet, F. Richez, M. Juniper, and D. Sipp, "Bifurcation scenario for a two-dimensional static airfoil exhibiting trailing edge stall," *J. Fluid Mech.* **928**, A3 (2021).
- ⁴⁸C. C. Chicone, *Ordinary Differential Equations with Applications* (Springer, 2006), Vol. 34.
- ⁴⁹B. Sharman, F.-S. Lien, L. Davidson, and C. Norberg, "Numerical predictions of low Reynolds number flows over two tandem circular cylinders," *Int. J. Numer. Methods Fluids* **47**, 423–447 (2005).
- ⁵⁰T. Vojkovic, D. Boskos, and A.-J. Buchner (2026). "Parametric reduced-order modeling and closed-loop control of tandem-cylinder wakes," Zenodo. <https://doi.org/10.5281/zenodo.22124859>