Solution of Nonlinear Eigenvalue Problems via Parallel Shooting

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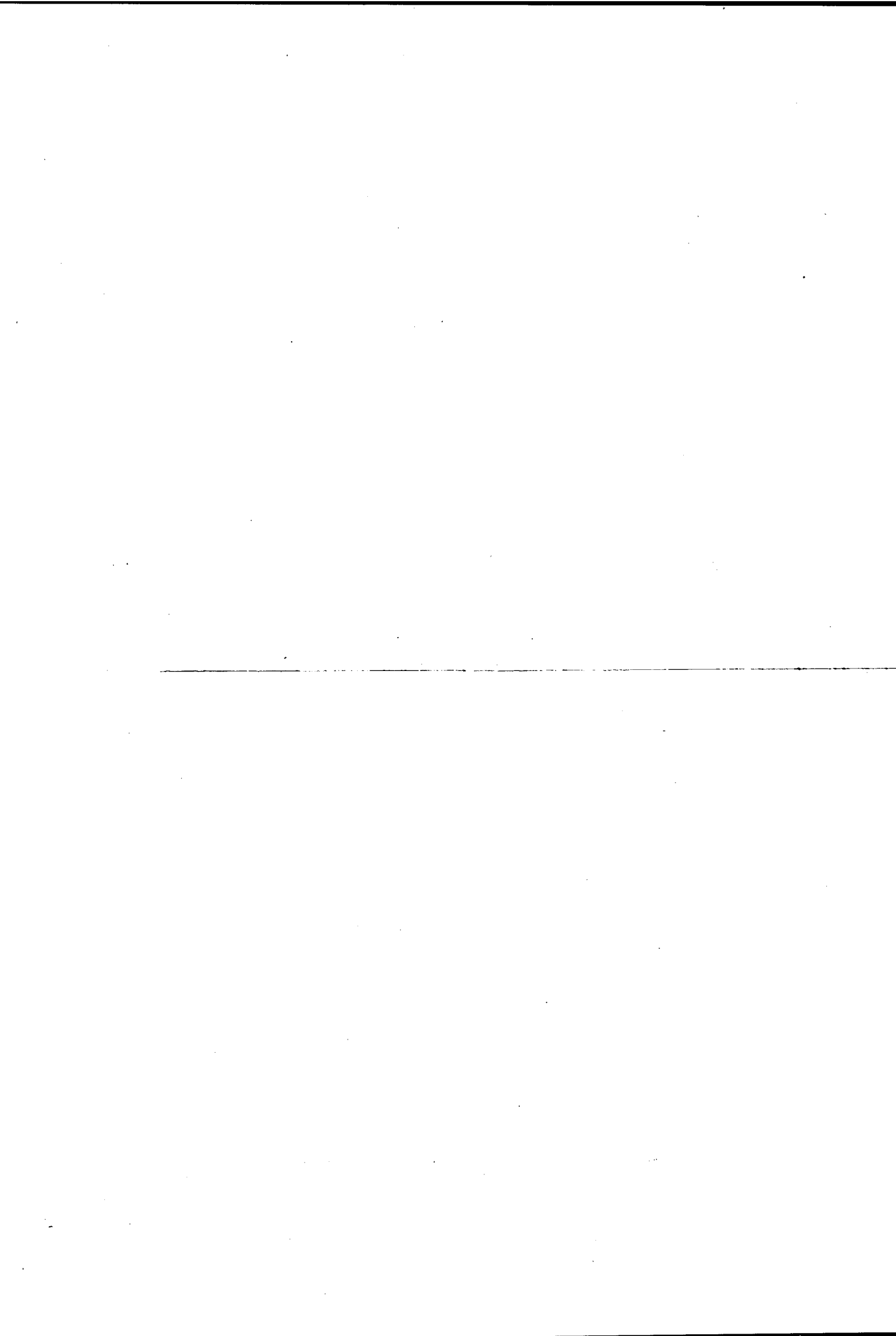# Solution of Nonlinear Eigenvalue Problems via Parallel Shooting

On p.16   Eq.(2.2.14c): '(t/2R){..' should be '(2R/t){..'

On p.16   Eq.(2.2.14d): '(t/2R){..' should be '(2R/t){..'

On p.38   Eq.(A23): ' $\int_0^{2\pi R}$ ..' should be ' $\frac{1}{2} \int_0^{2\pi R}$ ..'

On p.38   Eq.(A23): '= 0' should vanish

On p.49   line 2 from above: '$C_{14} = (4c/\Delta)$..' should be '$C_{14} = (8c/\Delta)$..'

On p.50   line 2 from above: '$\bar{p}_e$' should be '$\bar{p}$'

On p.50   line 6 from above: '$\bar{p}_e$' should be '$\bar{p}$'

On p.53   Eq.(E1a): in '..n^2(..) + ..', ')' should vanish

On p.53   line 2 from below: 'coeffients' should be 'coefficients'

On p.55   Eq.(E5): '.. + D_8(..' should be '.. - D_8(..'

On p.56   Eq.(E6b): '..+$D_{10}\bar{\tau})w'$ ' should be '..+$D_{10}\bar{\tau})w'_R$ '

On p.67   in Fig. 2: 'θ' should be 'θ_k'

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ABSTRACT

Inhomogeneous boundary value problems for ordinary differential equations can be solved by numerical integration of corresponding initial value problems (shooting). Guesses for the unknown boundary values are iteratively adjusted until all prescribed boundary conditions are satisfied. The method of shooting and matching can also be applied to nonlinear eigenvalue problems by treating the eigenvalue as a parameter in this procedure, that is, as an unknown in the (Newton-type) iteration scheme.

In this report a shooting method is described, which simultaneously yields an eigenvalue and the corresponding eigenfunction. A normalization condition is added to the homogeneous boundary value problem to make the eigenfunction unique. This method is applied to the bifurcation buckling of layered anisotropic cylinders under the combined loading of axial compression, radial pressure and torsion. The nonlinear Donnell-type equations formulated in terms of the radial displacement W and an Airy stress function F are used. By Fourier decomposition for the circumferential direction the governing equations are reduced to a set of ordinary differential equations.

Numerical results for a $[30°,0°,-30°]$ laminated glass-epoxy shell are presented, showing the effects of different sets of boundary conditions and nonlinear prebuckling deformations.

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NOTATION

a	first postbuckling coefficient, Eq. (2.2.4)
A	coefficient matrix, Eq. (3.2.2)
A_{ij}	extensional stiffness matrix, App. A
A_{ij}^{*}	semi-inverted extensional stiffness matrix, App. A
$\bar{A}_{ij}^{*}$	nondimensional A_{ij}^{*} ($\bar{A}_{ij}^{*} = Et\, A_{ij}^{*}$)
A_{12}	submatrix of A, Eq. (3.2.3)
b	second postbuckling coefficient, Eq. (2.2.4)
B_{ij}	bending-stretching coupling matrix, App. A
B_{ij}^{*}	semi-inverted bending-stretching coupling matrix, App. A
$\bar{B}_{ij}^{*}$	nondimensional B_{ij}^{*} ($\bar{B}_{ij}^{*} = (2c/t)B_{ij}^{*}$)
B_1-B_{20}	constants in boundary conditions, App. D
B_{11}	coefficient matrix in boundary conditions, Eq. (3.2.5)
BB_1-BB_{35}	constants in boundary conditions, App. D
c	$[3(1-v^2)]^{1/2}$
C	constant in normalization condition (3.2.8)
C_1-C_{31}	constants in Eqs. (2.2.15)
$\tilde{C}_1-\tilde{C}_4$	constants of integration, Eqs. (2.2.10) and (E4)
$\hat{C}_1-\hat{C}_3$	constants in Eq. (3.1.1)
D_{ij}	flexural stiffness matrix, App. A
D_{ij}^{*}	semi-inverted flexural stiffness matrix, App. A
$\bar{D}_{ij}^{*}$	nondimensional D_{ij}^{*} ($\bar{D}_{ij}^{*} = (4c^2/Et^3)D_{ij}^{*}$)
D_1-D_{32}	constants in postbuckling equations, App. E
$\tilde{D}_1-\tilde{D}_8$	constants in Eq. (3.2.4)
E	arbitrarily chosen reference Young's modulus
$E_{11},\ E_{22}$	Young's moduli orthotropic layer, App. A
$\underline{f},\hat{\underline{f}}$	vector functions, Eqs. (3.2.2), (3.2.14)
f_N	function in normalization condition, Eq. (3.2.9)
f_0	prebuckling stress function component
f_1, f_2	buckling stress function components

$f_\alpha, f_\beta, f_\gamma$ postbuckling stress function components

F Airy stress function

$F^{(0)}, F^{(1)}, F^{(2)}$ stress functions for 0th-order, 1st-order, and 2nd-order state, respectively

G_{12} shear modulus orthotropic layer, App. A

h_k thickness of kth layer, App. A

$\hat{J}, \hat{J}'$ Jacobian matrices, Eqs. (3.2.21), (3.2.29)

L shell length

L_{A*}, L_{B*}, L_{D*} linear operators, Eq. (2.2.2)

L_{NL} nonlinear operator, Eq. (2.2.3)

M_x, M_y, M_{xy}, M_{yx} moment resultants

n number of full waves in circumferential direction

N 1) half of the number of intervals in shooting method

     2) number of layers, App. A

N_U, N_V last component of $\hat{\underline{U}}, \hat{\underline{V}}$, Eq. (3.2.13)

N_x, N_y, N_{xy} stress resultants

N_0 applied axial load ($N_0 = - N_x(x=L)$)

p applied external pressure

$\bar{p}$ nondimensional external pressure ($\bar{p} = (cR^2/Et^2)p$)

$\bar{p}_c$ critical nondimensional external pressure $\bar{p}$

q axial load eccentricity (measured from shell midsurface)

$\bar{q}$ nondimensional load eccentricity ($\bar{q} = (4cR/t^2)q$)

Q_{ij} stiffness matrix orthotropic layer with respect to lamina principal axes, App. A

$\bar{Q}_{ij}$ stiffness matrix orthotropic layer with respect to shell reference axes, App. A

Q_x, Q_y transverse shear stress resultants

R radius of shell

$\underline{s}, \underline{t}$ initial guess vectors, Eq. (3.2.17)

$\underline{s}_0, \underline{s}_2$ initial condition vectors, Eqs. (3.2.11), (3.2.12)

$\underline{s}, \hat{\underline{s}}$ initial guess vectors, Eq. (3.2.17)

t shell wall-thickness

T_0 applied torque ($T_0 = N_{xy}(x=L)$)

u	axial displacement
$\underline{U}, \underline{V}$	solution vectors, Eq. (3.2.13)
$\hat{\underline{U}}, \hat{\underline{V}}$	solution vectors, Eqs. (3.2.11), (3.2.12)
v	circumferential displacement
w_0	prebuckling radial displacement component
w_1, w_2	buckling radial displacement components
$w_\alpha, w_\beta, w_\gamma$	postbuckling radial displacement components
W	radial displacement (positive inward)
$W(\bar{x})$	solution matrix of variational system, Eq. (3.2.31)
$\bar{W}$	initial radial imperfection
$\underline{W}_i, \hat{\underline{W}}_i$	solution vectors of variational equations, Eq. (3.2.22) (forward integration)
W_c	Poisson's expansion under combined loading
W_v, W_p, W_t	Poisson's expansions, Eq. (2.2.7)
$W^{(0)}, W^{(1)}, W^{(2)}$	radial displacements for 0th-order, 1st-order and 2nd-order state, respectively
W_{ij}	submatrices of $W(\bar{x})$
x, y	axial coordinate, circumferential coordinate
$\bar{x}$	nondimensional axial coordinate ($\bar{x} = x/R$)
$x(0), x(2)$	starting points in shooting method
$x(1)$	matching point in shooting method
$\underline{Y} = \{\underline{Y}_1^T, \underline{Y}_2^T\}^T$	vector of dependent variables, buckling state
$\underline{Y}_0$	vector of dependent variables, prebuckling state
z	radial coordinate
$\bar{z}$	modified Batdorf parameter ($\bar{Z} = L^2/Rt$)
$\underline{Z}_i, \hat{\underline{Z}}_i$	solution vectors of variational equations, Eq. (3.2.22) (backward integration)
$\underline{\gamma}$	vector of inhomogeneous terms in matching condition (3.2.15)
$\Delta, \Delta_2, \Delta_B$	constants, App. D
$\Delta(\ )$	correction for ( ) in Newton's method
$\varepsilon_x, \varepsilon_y, \gamma_{xy}$	strain components

$\varepsilon_1, \varepsilon_2, \gamma_{12}$	strain components orthotropic layer, App. A
θ	nondimensional circumferential coordinate ($\theta = y/R$)
θ_k	orientation of kth layer, App. A
$\kappa_x, \kappa_y, \kappa_{xy}$	curvature changes and twist, respectively
λ	nondimensional axial load ($\lambda = (cR/Et^2)N_0$)
λ_c	critical nondimensional axial load λ
Λ	load (eigenvalue) parameter
Λ_c	bifurcation buckling load
ν	arbitrarily chosen reference Poisson's ratio
ν_{12}, ν_{21}	Poisson's ratios orthotropic layer, App. A
ξ	perturbation parameter, Eq. (2.2.4)
$\sigma_x, \sigma_y, \tau_{xy}$	in-plane stresses
$\sigma_1, \sigma_2, \tau_{12}$	in-plane stresses orthotropic layer, App. A
$\bar{\tau}$	nondimensional torque ($\bar{\tau} = (cR/Et^2)T_0$)
$\bar{\tau}_c$	critical nondimensional torque $\bar{\tau}$
$\underline{\Phi}, \hat{\underline{\Phi}}$	vectors of matching conditions, Eqs. (3.2.15), (3.2.16)
ω	frequency parameter

$(\ ),$	partial differentiation with respect to coordinate following the comma
$(\ )'$	differentiation with respect to $\bar{x}$
$(\ )^\nu$	$(\ )$ evaluated in the νth iteration step

1. INTRODUCTION

1.1 Methods for vibration and buckling analysis of shells

Free vibration and buckling problems of circular cylindrical shells can be reduced to homogeneous boundary value problems (eigenvalue problems) for ordinary differential equations. The eigenvalue parameters are the square frequency and the loading, respectively. The solution can be based on the numerical integration of corresponding initial value problems.

Kalnins [1] presented a method for the free vibration of shells of revolution which consists of systematically evaluating a characteristic determinant. The eigenvalue parameter is increased in small steps. For each trial value of the eigenfrequency ω one has to obtain the solution vectors of the homogeneous system of differential equations to compute the characteristic determinant. When a sign change occurs, an eigenvalue has been found. This method will be referred to as determinant plotting. It has been employed by Booton [2] in the bifurcation buckling problem of anisotropic cylinders. The eigenvalue can be determined accurately by inverse interpolation.

Another approach was presented by Cohen [3]. This method is a generalization of the well-known Stodola method for beams and will be referred to as the mode iteration method. Substituting an estimate of the eigenmode into the governing equations yields an inhomogeneous boundary value problem giving an improved eigenmode. The corresponding eigenvalue is computed from the Rayleigh quotient. In each iteration step, one has to compute a particular solution vector. The complementary solution vectors have to be calculated only in the first step. It can be proved that this method is convergent for the lowest eigenvalue ω_1^2. The rate of convergence depends on the ratio ω_2^2/ω_1^2, where ω_2^2 is the next smallest eigenvalue. One can speed up the rate of convergence by the method of eigenvalue shifting. Each new reference value for the eigenvalue correction requires the computation of the complementary solution vectors. It is possible to compute higher eigenvalues by orthogonalization with respect to lower modes and/or eigenvalue shifting. The mode iteration method was used by Cohen [4] to compute the buckling load of shells of revolution including a nonlinear prebuckling state. Because of the nonlinear dependence of the

prebuckling state on the load parameter, in general it is necessary to approach the critical eigenvalue by a sequence of linearized problems. Geometrically, this method consists of examining the stability of fictitious equilibrium states on the tangent to the nonlinear load-deformation curve at an assumed load Λ below the critical load. For loads near Λ, the corresponding fictitious states are good approximations to the neighbouring nonlinear states. Consequently, as Λ is increased towards the critical load, the fictitious critical loads approximate with increasing precision the actual critical load. For each Λ, the method of successive approximations can be used to obtain the fictitious critical load. This procedure has also been employed by Arbocz and Hol [5] in the imperfection sensitivity analysis of anisotropic cylindrical shells.

1.2 Shooting for eigenvalues

In the common shooting method (see e.g. Keller [6], Hall and Watt [7]), an inhomogeneous boundary value problem is converted into a sequence of initial value problems which are solved by numerical integration. Guesses for the unknown boundary values are iteratively adjusted until all prescribed boundary conditions are satisfied. In this way, the boundary value problem has been reduced to the solution of a system of (nonlinear) equations for the unknown boundary values. Thus in general, the shooting procedure consists of two steps:
1) numerical integration of corresponding initial value problems with initial guesses for the unknowns,
2) solution of a linear algebraic system for a correction of the unknowns. These two steps can be repeated in an iterative procedure until convergence has been achieved.
To avoid the problems caused by a rapid growth of the initial value solutions, one often has to employ parallel shooting. In this modification, the growth of the solutions is controlled by dividing the range of integration into a number of smaller intervals.

It can be difficult to estimate the unknown boundary values. This problem may be overcome by perturbing a simpler problem in stages into the original problem (a continuation method).

The method of shooting and matching can also be applied to (nonlinear) eigenvalue problems, i.e., to problems in which a coefficient in the differential equation or boundary conditions has to be determined such that a (nontrivial) solution exists. The eigenvalue can be treated as a parameter in the shooting procedure; in other words, the eigenvalue is an unknown in the (Newton-type) iteration scheme. We will distinguish between two approaches.

The first approach is an obvious extension of the determinant plotting method. Thurston [8] presented a Newton-type root-finding procedure for lambda matrices, i.e., for matrices of which the determinant has to equal zero. The problem is reduced to a sequence of linear algebraic eigenvalue problems. The only eigenvalues (i.e., eigenvalue corrections) of interest are those which are small in absolute value. This method can be started when a sign change of the determinant has been detected.

Another approach is given by Keller [6], who described a general shooting method which simultaneously yields an eigenvalue and the corresponding eigenfunction. Apart from the unknown boundary values one has the eigenvalue as a parameter in the Newton-type iteration scheme. For standard eigenvalue problems the eigenfunction is determined to within a multiplicative constant. The eigenfunction can be made unique by adjoining some kind of normalization condition. We then also have an additional equation for the eigenvalue. One can also fix one of the inhomogeneous conditions at a boundary and in this way normalize the solution. The method is applicable to general (nonlinear) eigenvalue problems. The Newton-type iteration procedure converges rapidly (quadratically), but only close to the root. Further, one has to supply an initial guess which is close enough to the desired root, otherwise the iterations may not converge to this root. In this report, this method will be applied to the bifurcation buckling of anisotropic cylinders under the combined loading of axial compression, radial pressure and torsion.

1.3 Bifurcation buckling of anisotropic cylinders

Cylindrical shells are frequently used in structural applications. The buckling behaviour of these thin-walled structures is therefore a widely studied subject.

The two fundamental concepts in buckling analysis are bifurcation and collapse at a limit point, see Fig. 1. A bifurcation (eigenvalue) analysis can be used as an approximation of the failure load and mode, and is the first step in the initial-postbuckling analysis introduced by Koiter [9].

The introduction of fibre reinforced composite materials has led to new possibilities in the design of lightweight structures. By varying the fibre orientations and laminate stacking sequence the designer can try to find optimal structural configurations.

The bifurcation buckling problem of anisotropic cylinders including the effects of exact boundary conditions and nonlinear prebuckling deformations has been investigated by Booton [2] and Arbocz and Hol [5] using Donnell-type equations. By means of separation of variables the buckling problem can be reduced to a homogeneous boundary value problem with one independent variable. In both references methods are employed using numerical integration techniques for initial value problems (shooting methods). Booton uses determinant plotting, while Arbocz and Hol solve the eigenvalue problem by the mode iteration method.

In this report, the general shooting method described by Keller will be used to solve the bifurcation buckling problem. The numerical solution procedure is discussed in Chapter 3. It has to be noted, that the mode iteration method is a more efficient method to solve this eigenvalue problem. First, the method converges more rapidly, and secondly we have a guaranteed convergence to the lowest roots of the linearized problems.

The equations governing bifurcation buckling are derived in Chapter 2. Results for a [30°,0°,-30°] laminated glass-epoxy shell are presented in Chapter 4, showing the effects of various boundary conditions and nonlinear prebuckling deformations.

2. EQUATIONS GOVERNING THE BIFURCATION BUCKLING OF ANISOTROPIC CIRCULAR CYLINDRICAL SHELLS

2.1 Introduction

In this chapter, the equations governing the initial-postbuckling behaviour of anisotropic cylinders under combined loading will be derived. The nonlinear equations used in this analysis are based on a Donnell-type thin shell theory, in which the geometric nonlinearity is limited to moderate rotations (see e.g. Sanders [10]). The basic assumptions of this theory are (see Fig. 2):

- the shell is thin, i.e., $t/R \ll 1$, $t/L \ll 1$, where t is the shell thickness, R the shell radius and L the shell length,
- strains are small (of order ε where $\varepsilon \ll 1$),
- displacements u, v are infinitesimal, W is of the order of the shell thickness,
- flexural rotations of shell elements are moderately small ($W,_x^2$ and $W,_y^2$ of order ε),
- the Kirchhoff assumptions:
    - the transverse normal stress is small compared to the other normal stress components and may be neglected,
    - normals to the undeformed middle surface remain straight and normal to the deformed middle surface and suffer no extension.

It has to be noted, however, that for shells consisting of composite laminae it may be necessary to include the effect of transverse shear deformation, because the transverse shear stiffness is usually very small as compared to in-plane stiffness.

Further, the analysis is based on the assumptions made by Donnell [11]. The term containing the transverse shear stress resultant Q_y (Fig. 3) is neglected in the force equilibrium equation in y-direction, or, equivalently, the displacement term v is neglected in the curvature expressions. Under these assumptions quadratic terms can be omitted in the potential energy expression (Appendix A), i.e., the applied loads can be treated as dead loads. The Donnell stability equations are accurate if the displacement components are rapidly varying functions of the circumferential coordinate. In other words,

the equations are in general not valid for a small number of circumferential waves. The Donnell equations have the advantage that we can reduce the number of dependent variables to two by introducing an Airy stress function.

The constitutive equations for a layered anisotropic shell are used. The layers are assumed to be orthotropic and their principal axes can be oriented in arbitrary directions.

The basic equations

- the nonlinear strain-displacement relations,

- the equilibrium equations (derived by application of the stationary potential energy criterion),

- the constitutive equations for a layered anisotropic shell,

can be found in Appendix A.

A perturbation procedure will be used to derive the equations governing the initial-postbuckling behaviour. By means of separation of variables the buckling problem (1st-order state) is reduced to a two-point homogeneous boundary value problem in terms of one independent variable. Values of the load parameter are sought, for which a nontrivial solution exists. The lowest eigenvalue is the one of interest.

The solution procedure for the postbuckling problem (2nd-order state) is described by Arbocz and Hol [5].

2.2 Derivation of the governing equations

Assuming that the radial displacement W is positive inward (see Fig. 2) and introducing an Airy stress function F as $N_x = F,_{yy}$, $N_y = F,_{xx}$ and $N_{xy} = -F,_{xy}$, then the Donnell-type nonlinear perfect shell equations for a general anisotropic material can be written as

$$L_{A*}(F) - L_{B*}(W) = -\frac{1}{R} W,_{xx} - \frac{1}{2} L_{NL}(W,W)$$

$$(2.2.1)$$

$$L_{B*}(F) + L_{D*}(W) = \frac{1}{R} F,_{xx} + L_{NL}(F,W) + p$$

where p is the external pressure and where the linear operators are defined by

$$L_{A^*}(\ ) = A_{22}^*(\ )_{,xxxx} - 2A_{26}^*(\ )_{,xxxy} + (2A_{12}^*+A_{66}^*)(\ )_{,xxyy}$$

$$- 2A_{16}^*(\ )_{,xyyy} + A_{11}^*(\ )_{,yyyy} \qquad (2.2.2a)$$

$$L_{B^*}(\ ) = B_{21}^*(\ )_{,xxxx} + (2B_{26}^*-B_{61}^*)(\ )_{,xxxy} + (B_{11}^*+B_{22}^*-2B_{66}^*)(\ )_{,xxyy}$$

$$+ (2B_{16}^*-B_{62}^*)(\ )_{,xyyy} + B_{12}^*(\ )_{,yyyy} \qquad (2.2.2b)$$

$$L_{D^*}(\ ) = D_{11}^*(\ )_{,xxxx} + 2(D_{16}^*+D_{61}^*)(\ )_{,xxxy} + (D_{12}^*+D_{21}^*+4D_{66}^*)(\ )_{,xxyy}$$

$$+ 2(D_{26}^*+D_{62}^*)(\ )_{,xyyy} + D_{22}^*(\ )_{,yyyy} \qquad (2.2.2c)$$

and the nonlinear operator by

$$L_{NL}(S,T) = S_{,xx}T_{,yy} - 2S_{,xy}T_{,xy} + S_{,yy}T_{,xx} \qquad (2.2.3)$$

The stiffness parameters A_{ij}^*, B_{ij}^* and D_{ij}^* are defined in Appendix A.

Assuming a unique buckling mode we have the following perturbation expansion valid in the neigbourhood of the bifurcation point,

$$\Lambda/\Lambda_c = 1 + a\xi + b\xi^2 + \ldots$$

$$W = W^{(0)} + \xi W^{(1)} + \xi^2 W^{(2)} + \ldots \qquad (2.2.4)$$

$$F = F^{(0)} + \xi F^{(1)} + \xi^2 F^{(2)} + \ldots$$

where Λ_c is the bifurcation buckling load, and where $W^{(1)}$ will be normalized with respect to the shell thickness t and $W^{(2)}$ is orthogonal to $W^{(1)}$ in some appropriate sense; ξ is a measure of the displacement amplitude after buckling.

A formal substitution of this expansion into the nonlinear governing equations (2.2.1) yields a sequence of equations for the functions appearing in the expansions.

Equations governing the 0th-order state (prebuckling problem)

The set of governing equations for $W^{(0)}$ and $F^{(0)}$ is

$$L_{A\star}(F^{(0)}) - L_{B\star}(W^{(0)}) = -\frac{1}{R} W,_{xx}^{(0)} - \frac{1}{2} L_{NL}(W^{(0)}, W^{(0)})$$

$$L_{B\star}(F^{(0)}) + L_{D\star}(W^{(0)}) = \frac{1}{R} F,_{xx}^{(0)} + L_{NL}(F^{(0)}, W^{(0)}) + p \qquad (2.2.5)$$

The axisymmetric prebuckling state can be represented as

$$W^{(0)} = t(W_v + W_p + W_t) + tw_0(x)$$

$$F^{(0)} = (Et^2/cR)\{-\frac{1}{2} \lambda y^2 - \frac{1}{2} \bar{p}x^2 - \bar{\tau}xy + R^2 f_0(x)\} \qquad (2.2.6)$$

where the Poisson's expansions

$$W_v = (\bar{A}_{12}^\star/c)\lambda \quad ; \quad W_p = (\bar{A}_{22}^\star/c)\bar{p} \quad ; \quad W_t = -(\bar{A}_{26}^\star/c)\bar{\tau} \qquad (2.2.7)$$

are evaluated by enforcing the periodicity condition (Appendix B), and

$$\lambda = (cR/Et^2)N_0 \quad ; \quad \bar{p} = (cR^2/Et^2)p \quad ; \quad \bar{\tau} = (cR/Et^2)T_0 \qquad (2.2.8)$$

where $N_0 = -N_x(x=L)$ is the applied axial compression, p the applied external pressure, and $T_0 = N_{xy}(x=L)$ the applied counter-clockwise torque. The nondimensional stiffness parameters $\bar{A}_{ij}^\star$, $\bar{B}_{ij}^\star$ and $\bar{D}_{ij}^\star$ are defined in Appendix A. Introduction into the governing equations and rearrangement gives

$$\bar{A}_{22}^\star f_0^{iv} - (t/2R)\bar{B}_{21}^\star w_0^{iv} = -cw_0'' \qquad (2.2.9a)$$

$$2\bar{B}_{21}^\star f_0^{iv} + (t/R)\bar{D}_{11}^\star w_0^{iv} = (4cR/t)f_0'' - 4c\lambda w_0'' \qquad (2.2.9b)$$

where $( )' = d( )/d\bar{x}$ and $\bar{x} = x/R$. Integrating (2.2.9a) twice yields

$$f_0'' = (t/2R)(\bar{B}_{21}^*/\bar{A}_{22}^*)w_0'' - (c/\bar{A}_{22}^*)w_0 + \tilde{C}_1\bar{x} + \tilde{C}_2 \qquad (2.2.10)$$

where the constants of integration $\tilde{C}_1$ and $\tilde{C}_2$ are equal to zero because of the periodicity condition (Appendix B). The equations governing the prebuckling state can now be combined to one equation

$$(\bar{D}_{11}^* + \bar{B}_{21}^{*2}/\bar{A}_{22}^*)w_0^{iv} + (4cR/t)(\lambda - \bar{B}_{21}^*/\bar{A}_{22}^*)w_0'' + (4c^2R^2/t^2)w_0/\bar{A}_{22}^* = 0$$

$$(2.2.11)$$

Equations governing the 1st-order state (buckling problem)

The equations governing the first order state are given by

$$L_{A*}(F^{(1)}) - L_{B*}(W^{(1)}) = -\frac{1}{R}W_{,xx}^{(1)} - (W_{,xx}^{(0)}W_{,yy}^{(1)} + W_{,xx}^{(1)}W_{,yy}^{(0)} - 2W_{,xy}^{(0)}W_{,xy}^{(1)})$$

$$(2.2.12)$$

$$L_{B*}(F^{(1)}) + L_{D*}(W^{(1)}) = \frac{1}{R}F_{,xx}^{(1)} + F_{,xx}^{(0)}W_{,yy}^{(1)} - 2F_{,xy}^{(0)}W_{,xy}^{(1)} + F_{,yy}^{(0)}W_{,xx}^{(1)}$$

$$+ F_{,xx}^{(1)}W_{,yy}^{(0)} - 2F_{,xy}^{(1)}W_{,xy}^{(0)} + F_{,yy}^{(1)}W_{,xx}^{(0)}$$

The stability equations admit separable solutions of the form

$$W^{(1)} = tw_1(x)\cos n\theta + tw_2(x)\sin n\theta$$

$$(2.2.13)$$

$$F^{(1)} = (ERt^2/c)f_1(x)\cos n\theta + (ERt^2/c)f_2(x)\sin n\theta$$

where $\theta = y/R$. Introduction into (2.2.12), regrouping and equating coefficients of like trigonometric terms, gives the following set of four 4th-order differential equations in w_1, w_2, f_1 and f_2

$$\bar{A}_{22}^{*}f_{1}^{iv} - (2\bar{A}_{12}^{*}+\bar{A}_{66}^{*})n^{2}f_{1}'' + \bar{A}_{11}^{*}n^{4}f_{1} - 2\bar{A}_{26}^{*}nf_{2}'' + 2\bar{A}_{16}^{*}n^{3}f_{2}'$$

$$- (t/2R)\{\bar{B}_{21}^{*}w_{1}^{iv} - (\bar{B}_{11}^{*}+\bar{B}_{22}^{*}-2\bar{B}_{66}^{*})n^{2}w_{1}'' + \bar{B}_{12}^{*}n^{4}w_{1} + (2\bar{B}_{26}^{*}-\bar{B}_{61}^{*})nw_{2}''$$

$$- (2\bar{B}_{16}^{*}-\bar{B}_{62}^{*})n^{3}w_{2}'\} + cw_{1}'' - (ct/R)n^{2}w_{0}''w_{1} = 0 \qquad (2.2.14a)$$

$$\bar{A}_{22}^{*}f_{2}^{iv} - (2\bar{A}_{12}^{*}+\bar{A}_{66}^{*})n^{2}f_{2}'' + \bar{A}_{11}^{*}n^{4}f_{2} + 2\bar{A}_{26}^{*}nf_{1}'' - 2\bar{A}_{16}^{*}n^{3}f_{1}'$$

$$- (t/2R)\{\bar{B}_{21}^{*}w_{2}^{iv} - (\bar{B}_{11}^{*}+\bar{B}_{22}^{*}-2\bar{B}_{66}^{*})n^{2}w_{2}'' + \bar{B}_{12}^{*}n^{4}w_{2} - (2\bar{B}_{26}^{*}-\bar{B}_{61}^{*})nw_{1}''$$

$$+ (2\bar{B}_{16}^{*}-\bar{B}_{62}^{*})n^{3}w_{1}'\} + cw_{2}'' - (ct/R)n^{2}w_{0}''w_{2} = 0 \qquad (2.2.14b)$$

$$(t/2R)\{\bar{B}_{21}^{*}f_{1}^{iv} - (\bar{B}_{11}^{*}+\bar{B}_{22}^{*}-2\bar{B}_{66}^{*})n^{2}f_{1}'' + \bar{B}_{12}^{*}n^{4}f_{1} + (2\bar{B}_{26}^{*}-\bar{B}_{61}^{*})nf_{2}''$$

$$- (2\bar{B}_{16}^{*}-\bar{B}_{62}^{*})n^{3}f_{2}'\} + \bar{D}_{11}^{*}w_{1}^{iv} - 2(\bar{D}_{12}^{*}+2\bar{D}_{66}^{*})n^{2}w_{1}'' + \bar{D}_{22}^{*}n^{4}w_{1}$$

$$+ 4\bar{D}_{16}^{*}nw_{2}'' - 4\bar{D}_{26}^{*}n^{3}w_{2}' - (4cR^{2}/t^{2})f_{1}''$$

$$+ (4cR/t)\{\lambda w_{1}'' - \bar{p}n^{2}w_{1} - 2n\bar{\tau}w_{2}' + n^{2}(f_{0}''w_{1}+w_{0}''f_{1})\} = 0 \qquad (2.2.14c)$$

$$(t/2R)\{\bar{B}_{21}^{*}f_{2}^{iv} - (\bar{B}_{11}^{*}+\bar{B}_{22}^{*}-2\bar{B}_{66}^{*})n^{2}f_{2}'' + \bar{B}_{12}^{*}n^{4}f_{2} - (2\bar{B}_{26}^{*}-\bar{B}_{61}^{*})nf_{1}''$$

$$+ (2\bar{B}_{16}^{*}-\bar{B}_{62}^{*})n^{3}f_{1}'\} + \bar{D}_{11}^{*}w_{2}^{iv} - 2(\bar{D}_{12}^{*}+2\bar{D}_{66}^{*})n^{2}w_{2}'' + \bar{D}_{22}^{*}n^{4}w_{2}$$

$$- 4\bar{D}_{16}^{*}nw_{1}'' + 4\bar{D}_{26}^{*}n^{3}w_{1}' - (4cR^{2}/t^{2})f_{2}''$$

$$+ (4cR/t)\{\lambda w_{2}'' - \bar{p}n^{2}w_{2} + 2n\bar{\tau}w_{1}' + n^{2}(f_{0}''w_{2}+w_{0}''f_{2})\} = 0 \qquad (2.2.14d)$$

To be able to use the shooting method, the w_{1}^{iv} term is eliminated from (2.2.14a) and the f_{1}^{iv} term from (2.2.14c). Similarly, the w_{2}^{iv} term is

eliminated from (2.2.14b) and the f_2^{iv} term from (2.2.14d). This finally results in the following equations

$$f_1^{iv} = C_{17}f_1'' - C_{18}f_1 + C_{19}f_2''' + C_{20}f_2' + C_{21}w_1'' + C_{22}w_1 + C_{23}w_2''' + C_{24}w_2'$$
$$+ C_{26}w_0''w_1 + C_{28}\bar{p}w_1 - C_{28}f_0''w_1 + C_{30}\bar{\tau}w_2' - C_{31}\lambda w_1'' - C_{28}w_0''f_1 \qquad (2.2.15a)$$

$$f_2^{iv} = C_{17}f_2'' - C_{18}f_2 - C_{19}f_1''' - C_{20}f_1' + C_{21}w_2'' + C_{22}w_2 - C_{23}w_1''' - C_{24}w_1'$$
$$+ C_{26}w_0''w_2 + C_{28}\bar{p}w_2 - C_{28}f_0''w_2 - C_{30}\bar{\tau}w_1' - C_{31}\lambda w_2'' - C_{28}w_0''f_2 \qquad (2.2.15b)$$

$$w_1^{iv} = C_1 f_1'' + C_2 f_1 - C_3 f_2''' + C_4 f_2' + C_5 w_1'' - C_6 w_1 - C_7 w_2''' + C_8 w_2'$$
$$- C_{10}w_0''w_1 + C_{12}\bar{p}w_1 - C_{12}f_0''w_1 + C_{14}\bar{\tau}w_2' - C_{15}\lambda w_1'' - C_{12}w_0''f_1 \qquad (2.2.15c)$$

$$w_2^{iv} = C_1 f_2'' + C_2 f_2 + C_3 f_1''' - C_4 f_1' + C_5 w_2'' - C_6 w_2 + C_7 w_1''' - C_8 w_1'$$
$$- C_{10}w_0''w_2 + C_{12}\bar{p}w_2 - C_{12}f_0''w_2 - C_{14}\bar{\tau}w_1' - C_{15}\lambda w_2'' - C_{12}w_0''f_2 \qquad (2.2.15d)$$

The constants $C_1 - C_{31}$ are listed in Appendix D.

This set of homogeneous differential equations with variable coefficients together with the appropriate boundary conditions listed in Appendix C form an eigenvalue problem which is solved numerically. The equations governing the 2nd-order state (postbuckling problem) are given in Appendix E.

3. NUMERICAL ANALYSIS

In this chapter, the eigenvalue problem describing the bifurcation buckling of anisotropic shells is solved via the shooting method presented by Keller [6]. By adjoining a normalization condition for the eigenfunction the eigenvalue parameter, i.e., the variable part of the applied loading, can be treated as an unknown in the shooting procedure.

3.1 Solution of the prebuckling state

The equation governing the prebuckling state is

$$w_0^{iv} = (\hat{C}_1 - \hat{C}_2\lambda)w_0'' - \hat{C}_3 w_0 \qquad\qquad (3.1.1)$$

where $\hat{C}_1$, $\hat{C}_2$ and $\hat{C}_3$ are constants, listed in Appendix D. Two sets of boundary conditions (simply supported and clamped) are given in Appendix C. The problem is solved by employing parallel shooting over 2N intervals (see Fig. 4). The prebuckling state is nonlinear in the sense that the radial displacement is a nonlinear function of the applied load. It must be noted that, although Eq. (3.1.1) describes a nonlinear equilibrium path, for a fixed value of the load parameter the differential equation is linear and a closed form solution can be obtained (Arbocz and Hol [5]).
Eq. (3.1.1) is the governing nonlinear bending equation. A linear bending equation is obtained by omission of the nonlinear term containing $\lambda w_0''$ and the linear membrane prebuckling state corresponds to $w_0 = 0$.

3.2 Solution of the buckling state

The equations governing the buckling state are given by Eqs. (2.2.15). Following Booton [2], we introduce the 16-dimensional vector variable $\underline{Y}$ as

$$Y_1 = w_1 \qquad Y_5 = w_1'' \qquad Y_9 = w_2 \qquad Y_{13} = w_2''$$

$$Y_2 = f_1 \qquad Y_6 = f_1'' \qquad Y_{10} = f_2 \qquad Y_{14} = f_2''$$

$$(3.2.1)$$

$$Y_3 = w_2' \qquad Y_7 = w_2''' \qquad Y_{11} = -w_1' \qquad Y_{15} = -w_1'''$$

$$Y_4 = f_2' \qquad Y_8 = f_2''' \qquad Y_{12} = -f_1' \qquad Y_{16} = -f_1'''$$

The applied loading, consisting of a combination of axial compression, radial pressure and torsion, is divided into a fixed and a variable part. The variable part is characterized by a nondimensional load parameter Λ. The governing equations can be written in matrix notation as follows

$$\frac{d}{d\bar{x}} \underline{Y} = \underline{f}(\bar{x}, \underline{Y}_0, \underline{Y}; \Lambda) = A(\bar{x}, \underline{Y}_0; \Lambda) \underline{Y} \qquad (3.2.2)$$

where

$$A = \begin{bmatrix} O & A_{12} \\ -A_{12} & O \end{bmatrix} \qquad (3.2.3)$$

and

$$A_{12} = \begin{bmatrix} & & -1 & & & & & \\ & & & -1 & & & O & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & O & & & & -1 & \\ & & & & & & & -1 \\ -\tilde{D}_1 & \tilde{D}_4 & \tilde{D}_2 & C_4 & \tilde{D}_3 & C_1 & -C_7 & -C_3 \\ \tilde{D}_5 & -\tilde{D}_8 & \tilde{D}_6 & C_{20} & \tilde{D}_7 & C_{17} & C_{23} & C_{19} \end{bmatrix} \qquad (3.2.4)$$

In (3.2.2) $\underline{Y}_0$ is the solution of the prebuckling state, and the constants $\tilde{D}_1$ - $\tilde{D}_8$ in (3.2.4) are listed in Appendix D. The boundary conditions at $\bar{x} = 0$ can be written in the following form

$$B_{11}\underline{Y}_1(0) = \underline{0}$$

$$B_{11}\underline{Y}_2(0) = \underline{0}$$

(3.2.5)

and if the boundary conditions are assumed to be symmetric with respect to the midlength of the cylinder

$$B_{11}\underline{Y}_1(L/R) = \underline{0}$$

$$B_{11}\underline{Y}_2(L/R) = \underline{0}$$

(3.2.6)

where B_{11} is a 4 x 8 coefficient matrix and $\underline{Y} = \{\underline{Y}_1^T, \underline{Y}_2^T\}^T$. The following sets of boundary conditions (simply supported and clamped) are considered

SS-1:	$N_x = - N_0$	;	$N_{xy} = T_0$	;	$W = 0$	;	$M_x = - N_0 q$
SS-2:	$u = 0$	;	$N_{xy} = T_0$	;	$W = 0$	;	$M_x = - N_0 q$
SS-3:	$N_x = - N_0$	;	$v = 0$	;	$W = 0$	;	$M_x = - N_0 q$
SS-4:	$u = 0$	;	$v = 0$	;	$W = 0$	;	$M_x = - N_0 q$

(3.2.7)

C-1:	$N_x = - N_0$	;	$N_{xy} = T_0$	;	$W = 0$	;	$W,_x = 0$
C-2:	$u = 0$	;	$N_{xy} = T_0$	;	$W = 0$	;	$W,_x = 0$
C-3:	$N_x = - N_0$	;	$v = 0$	;	$W = 0$	;	$W,_x = 0$
C-4:	$u = 0$	;	$v = 0$	;	$W = 0$	;	$W,_x = 0$

where q is the axial load eccentricity, measured from the shell midsurface (positive inward). The reduced boundary conditions can be found in Appendix C. For the 1st-order state (buckling problem), the boundary conditions (3.2.7) become homogeneous.

To make the eigenfunctions unique, to within a sign (for simple eigenvalues), we normalize them by

$$\int_0^{L/R} Y_1^2 \, d\bar{x} = C \tag{3.2.8}$$

where C is a (positive) constant. This condition can be written as a differential equation as follows

$$\frac{d}{d\bar{x}} Y_{17} = Y_1^2 = f_N(\bar{x}, \underline{Y}; \Lambda) \qquad 0 \le \bar{x} \le L/R \tag{3.2.9}$$

with the boundary conditions

$$Y_{17}(0) = 0$$
$$\tag{3.2.10}$$
$$Y_{17}(L/R) = C$$

By adding the normalization condition to the homogeneous boundary value problem the eigenvalue can be treated as one of the unknown parameters in the shooting method.

For simplicity, we will consider parallel shooting over 2 intervals. Hence let us introduce the following 2 associated initial value problems:

Forward integration:

$$0 \le \bar{x} \le x(1)$$

$$\frac{d}{d\bar{x}} \underline{\hat{U}} = \underline{\hat{f}}(\bar{x}, \underline{\hat{U}}; \Lambda)$$

$$\tag{3.2.11}$$

$$\underline{\hat{U}}(\bar{x} = 0) = \{\underline{s}_0^T, 0\}^T$$

where $\underline{s}_0$ contains 8 independent parameters s_i in $\underline{s}$.

Backward integration:
_____

$$x(1) \leq \bar{x} \leq x(2)$$

$$\frac{d}{d\bar{x}} \hat{\underline{V}} = \hat{\underline{f}}(\bar{x}, \hat{\underline{V}}; \Lambda)$$

(3.2.12)

$$\hat{\underline{V}}(\bar{x} = x(2)) = \{\underline{s}_2^T, 0\}^T$$

where $\underline{s}_2$ contains 8 independent parameters t_i in $\underline{t}$.

In the above equations is

$$\hat{\underline{U}} = \{\underline{U}^T, U_{17}\}^T = \{\underline{U}^T, N_U\}^T$$

(3.2.13)

$$\hat{\underline{V}} = \{\underline{V}^T, V_{17}\}^T = \{\underline{V}^T, N_V\}^T$$

and

$$\hat{\underline{f}} = \{\underline{f}^T, f_N\}^T$$

(3.2.14)

Introducing a new vector function $\hat{\underline{\Phi}}$ the solutions must satisfy the matching condition and normalization condition

$$\hat{\underline{\Phi}}(\hat{\underline{S}}) = \begin{bmatrix} \underline{U}[\bar{x}=x(1)] - \underline{V}[\bar{x}=x(1)] \\ \\ N_U[\bar{x}=x(1)] - N_V[\bar{x}=x(1)] \end{bmatrix} - \underline{\gamma} = 0$$

(3.2.15)

where

$$\hat{\underline{\Phi}} = \{\underline{\Phi}^T, \Phi_{17}\}^T,$$

(3.2.16)

$$\hat{\underline{S}} = \{\underline{S}^T, \Lambda\}^T = \{\underline{s}^T, \underline{t}^T, \Lambda\}^T$$

(3.2.17)

and

$$\underline{\gamma} = \{\underline{0}^T, \ C\}^T \tag{3.2.18}$$

If we have initial guesses which are sufficiently close to the eigenvalue and eigenfunction, we can use Newton's method to solve (3.2.15). We then have the following iteration scheme

$$\hat{\underline{S}}^{v+1} = \hat{\underline{S}}^v + \Delta\hat{\underline{S}}^v \qquad v = 0,1,\ldots \tag{3.2.19}$$

where $\Delta\hat{\underline{S}}^v$ is the solution of the 17th-order linear system

$$\frac{\partial\hat{\underline{\Phi}}}{\partial\hat{\underline{S}}}(\hat{\underline{S}}^v) \ \Delta\hat{\underline{S}}^v = - \ \hat{\underline{\Phi}}(\hat{\underline{S}}^v) \tag{3.2.20}$$

In order to solve for the components of the Jacobian

$$\hat{J} = \frac{\partial\hat{\underline{\Phi}}}{\partial\hat{\underline{S}}} = \begin{bmatrix} \dfrac{\partial\Phi_1}{\partial S_1} & \cdots & \dfrac{\partial\Phi_1}{\partial S_{16}} & \vline & \dfrac{\partial\Phi_1}{\partial\Lambda} \\ \cdots & & \cdots & \vline & \cdots \\ \cdots & & \cdots & \vline & \cdots \\ \dfrac{\partial\Phi_{16}}{\partial S_1} & \cdots & \dfrac{\partial\Phi_{16}}{\partial S_{16}} & \vline & \dfrac{\partial\Phi_{16}}{\partial\Lambda} \\ \hdashline \dfrac{\partial\Phi_{17}}{\partial S_1} & \cdots & \dfrac{\partial\Phi_{17}}{\partial S_{16}} & \vline & \dfrac{\partial\Phi_{17}}{\partial\Lambda} \end{bmatrix} \tag{3.2.21}$$

let us introduce the following new vectors

$$\hat{\underline{W}}_i = \{\underline{W}_i^T, \ W_{i_{17}}\}^T = \frac{\partial\hat{\underline{U}}}{\partial S_i} \qquad\qquad i = 1,2,\ldots,8$$

$$\tag{3.2.22}$$

$$\hat{\underline{Z}}_i = \{\underline{Z}_i^T, \ Z_{i_{17}}\}^T = \frac{\partial\hat{\underline{V}}}{\partial S_i} \qquad\qquad i = 9,10,\ldots,16$$

These vectors are found by solving the following variational equations, obtained by differentiating (3.2.11) and (3.2.12) with respect to the parameters S_i:

forward:

$$0 \leq \bar{x} \leq x(1) \qquad \frac{d}{d\bar{x}} \hat{\underline{W}}_i = \frac{\partial}{\partial \hat{\underline{U}}} \hat{\underline{f}}(\bar{x}, \hat{\underline{U}}; \Lambda) \hat{\underline{W}}_i \qquad i = 1, 2, \ldots, 8 \qquad (3.2.23)$$

backward:

$$x(1) \leq \bar{x} \leq x(2) \qquad \frac{d}{d\bar{x}} \hat{\underline{z}}_i = \frac{\partial}{\partial \hat{\underline{V}}} \hat{\underline{f}}(\bar{x}, \hat{\underline{V}}; \Lambda) \hat{\underline{z}}_i \qquad i = 9, 10, \ldots, 16 \qquad (3.2.24)$$

with initial conditions

$$\hat{\underline{W}}_1(0) = \{(\partial \underline{s}_0/\partial s_1)^T, 0\}^T$$

$$\hat{\underline{W}}_2(0) = \{(\partial \underline{s}_0/\partial s_2)^T, 0\}^T$$

$$\ldots$$

$$\ldots \qquad (3.2.25)$$

$$\hat{\underline{W}}_8(0) = \{(\partial \underline{s}_0/\partial s_8)^T, 0\}^T$$

and

$$\hat{\underline{z}}_9(L/R) = \{(\partial \underline{s}_2/\partial t_1)^T, 0\}^T$$

$$\ldots \qquad (3.2.26)$$

$$\ldots$$

$$\hat{\underline{z}}_{16}(L/R) = \{(\partial \underline{s}_2/\partial t_8)^T, 0\}^T$$

In addition we must solve the following inhomogeneous variational equations with homogeneous boundary conditions, obtained by differentiating (3.2.11) and

(3.2.12) with respect to the eigenvalue parameter Λ:

forward:

$$0 \le \bar{x} \le x(1) \qquad \frac{d}{d\bar{x}} \frac{\partial \hat{\underline{U}}}{\partial \Lambda} = \frac{\partial}{\partial \hat{\underline{U}}} \hat{\underline{f}}(\bar{x}, \underline{Y}_0, \hat{\underline{U}}; \Lambda) \frac{\partial \hat{\underline{U}}}{\partial \Lambda} + \frac{\partial}{\partial \Lambda} \hat{\underline{f}}(\bar{x}, \underline{Y}_0, \hat{\underline{U}}; \Lambda) \qquad (3.2.27)$$

backward:

$$x(1) \le \bar{x} \le x(2) \qquad \frac{d}{d\bar{x}} \frac{\partial \hat{\underline{V}}}{\partial \Lambda} = \frac{\partial}{\partial \hat{\underline{V}}} \hat{\underline{f}}(\bar{x}, \underline{Y}_0, \hat{\underline{V}}; \Lambda) \frac{\partial \hat{\underline{V}}}{\partial \Lambda} + \frac{\partial}{\partial \Lambda} \hat{\underline{f}}(\bar{x}, \underline{Y}_0, \hat{\underline{V}}; \Lambda) \qquad (3.2.28)$$

The components of the matrix $\hat{J}' = \partial\hat{\underline{f}}/\partial\hat{\underline{U}} = \partial\hat{\underline{f}}/\partial\hat{\underline{V}}$ can be calculated directly:

$$\hat{J}' = \left[\begin{array}{ccccc|c} & & & & & 0 \\ & & & & & . \\ & & & & & . \\ & & A & & & . \\ & & & & & . \\ & & & & & . \\ & & & & & . \\ & & & & & 0 \\ \hline 2U_1 & 0 & \cdots & & 0 & 0 \end{array}\right] \qquad (3.2.29)$$

The Jacobian matrix $\hat{J} = \dfrac{\partial \hat{\Phi}}{\partial \hat{\underline{s}}}$ has the following form

$$
\hat{J} = \left[
\begin{array}{ccc|cc}
\dfrac{\partial U}{\partial \underline{s}}[x(1)] & O & & \dfrac{\partial U}{\partial \Lambda}[x(1)] & -\dfrac{\partial V}{\partial \Lambda}[x(1)] \\[3mm]
O & -\dfrac{\partial V}{\partial \underline{t}}[x(1)] & & \dfrac{\partial U}{\partial \Lambda}[x(1)] & -\dfrac{\partial V}{\partial \Lambda}[x(1)] \\[3mm]
\hline
\dfrac{\partial N_U}{\partial \underline{s}}[x(1)] & -\dfrac{\partial N_V}{\partial \underline{t}}[x(1)] & & \dfrac{\partial N_U}{\partial \Lambda}[x(1)] & -\dfrac{\partial N_V}{\partial \Lambda}[x(1)]
\end{array}
\right]
\qquad (3.2.30)
$$

Booton formally shows that if the boundary conditions at $x = 0$ and $x = L$ are identical, w_1 and f_1 are even, and w_2 and f_2 are odd functions of x (or vice versa) with respect to the midlength of the cylinder. Therefore the eigenvalues can be obtained from a homogeneous boundary value problem defined over only one half of the cylinder length.

Booton also indicates another reduction of the problem. We can show (see Appendix F) that in the present approach the variational equations

$$W' = A \, W \qquad (3.2.31)$$

where $W = [\underline{W}_1, \underline{W}_2, \ldots, \underline{W}_{16}]$, with the corresponding initial conditions, have the following property

$$
W(\bar{x}) = \left[
\begin{array}{cc}
W_{11}(\bar{x}) & -W_{21}(\bar{x}) \\[3mm]
W_{21}(\bar{x}) & W_{11}(\bar{x})
\end{array}
\right]
\qquad (3.2.32)
$$

3.3 Description of the computer program

The numerical computations were carried out by means of a Fortran program on a SUN 4/280 computer. Fig. 5 shows the flow diagram of the program. To start the iteration, an initial guess for the eigenvalue and the eigenmode must be

obtained from a simpler analysis. For this purpose, Khot's method [12] (Galerkin procedure for a one-term deflection function approximately satisfying simply supported boundary conditions) or the method presented by Rong et al. [13] (exact solution of the differential equations for a membrane prebuckling state) can be used.

The main loop is the Newton iteration to solve the eigenvalue problem. In each iteration step, first the prebuckling state and the prebuckling state differentiated with respect to the load parameter are solved. The converged solutions are used in the integration of the buckling state. In an early stage of the iteration process a damping factor can be used for the corrections to guarantee convergence to the desired root.

The solution of the initial value problems was done by the library subroutine DEQ from Caltech's Willis Booth Computer Center, which uses an Adams-Moulton predictor-corrector scheme. Starting values are obtained by the method of Runge-Kutta-Gill. The program includes an option with variable interval size and uses automatic truncation error control.

4. NUMERICAL RESULTS

In this chapter, results of the buckling analysis for anisotropic shells under axisymmetric loadings will be presented. The anisotropic shell investigated in the calculations is a [30°,0°,-30°] laminated glass-epoxy shell first used by Booton [2] and later by Arbocz and Hol [5]. Its geometric and material data are given in Table 1.

The buckling load has been computed for two shells with different lengths, namely,

1) a relatively short shell (L/R = 0.707, $\bar{Z} = L^2/Rt = 50$), and

2) a shell of moderate length (L/R = 2, $\bar{Z} = 400$),

and for three different load cases:

1) axial compression ($\lambda = (cR/Et^2)N_0$),

2) hydrostatic pressure, i.e., a uniform pressure applied to the lateral surface as well as to the ends of the cylinder ($\bar{p} = (cR^2/Et^2)p$ and $\lambda = \frac{1}{2}\bar{p}$), and

3) torsion ($\bar{\tau} = (cR/Et^2)T_0$): counter-clockwise, corresponding to a positive sign, and clockwise, corresponding to a negative sign.

The influence of the eight different sets of boundary conditions (3.2.7) is investigated. In particular, the following effects are examined:

1) the effect of rotational constraint ($W,_x = 0$ versus $W,_{xx} = 0$) and

2) the effect of axial constraint (u = 0 versus $N_x = 0$).

The effect of prebuckling deformations on the buckling load (and mode) is illustrated for the SS-3 and C-4 boundary conditions by a comparison with results of calculations using a membrane prebuckling analysis.

The boundary conditions are symmetric with respect to the shell midlength. The prebuckling shapes and mode shapes are therefore plotted for $0 \leq \bar{x} \leq L/2R = \bar{x}(max)$. In the figures for the buckling modes, the amplitudes are normalized by the wall thickness t.

4.1 Axial compression

It is seen from Table 2 that the buckling load of simply supported shells is drastically reduced for the weak boundary support $N_{xy} = 0$ (v is free). This has been reported by several investigators (see e.g. Almroth [14]). The buckling mode is characterized by a small number of circumferential waves (n = 0 or n = 1). Boundary conditions in which v is free, however, are not likely to be encountered in practical applications.

For $\bar{Z} = 50$, the edge constraint in axial direction u = 0 raises the buckling load by about 5% as compared to the corresponding cases in which u is free. The effect of clamping ($W,_x = 0$) is seen to be predominant, giving increases by more dan 10%. For $\bar{Z} = 400$, the increases are less than 3% for u = 0 and less than 4% for $W,_x = 0$. Buckling modes for SS-3 and C-4 boundary conditions are shown in Figs. 6 and 7.

The effect of prebuckling deformations (Table 6) is seen to be important for the short shell ($\bar{Z} = 50$). The buckling load is lowered by 13% in the SS-3 case, and by 5% for the C-4 case, as compared to the analysis with membrane prebuckling. For $\bar{Z} = 400$, the decreases are less pronounced (4% for SS-3 and negligibly small for C-4 boundary conditions). The buckling modes for a nonlinear and for a membrane prebuckling analysis are depicted in Figs. 8 to 11. For SS-3 boundary conditions, the prebuckling rotations are seen to have a pronounced influence on the buckling mode.

4.2 Hydrostatic pressure

For the hydrostatic pressure case (Table 3), it is seen that for $\bar{Z} = 50$ both the rotational constraint $W,_x = 0$ and the axial constraint u = 0 increase the bifurcation buckling load considerably (by about 10 to 25%). For $\bar{Z} = 400$ the axial restraint has a pronounced influence, raising the buckling load by about 30%. The increase for clamping ($W,_x = 0$) is less than 10%. Buckling modes for SS-3 and C-4 boundary conditions are shown in Figs. 12 and 13. The modes exhibit very little skewedness. The Poisson's expansion $W_v + W_p$ in the prebuckling radial displacement is denoted by W_c.

The effect of prebuckling deformations (Table 7) on the buckling load is seen to be small for the shell of moderate length ($\bar{Z}$ = 400), giving a decrease less than 0.5% for the SS-3 case, and a 1% increase for the C-4 case. For the short shell ($\bar{Z}$ = 50) the buckling load is lowered by 2% for the SS-3 boundary conditions, and by 6% for the C-4 boundary conditions, when a rigorous prebuckling analysis is used.

4.3 Torsion

An interesting phenomenon for torsion is that the buckling load under torsion depends on the sense of the applied torque because of the shearing-bending coupling terms in the constitutive equations (Booton [2]). It is seen that the buckling load under a negative (clockwise) torque is lower than for a positive (counter-clockwise) torque (Tables 4 and 5). For $\bar{Z}$ = 50, the critical load is decreased by about 15 to 20% when a clockwise torque is applied. The critical torque is less than 5% lower for $\bar{Z}$ = 400.

It is seen that for $\bar{Z}$ = 50 the effect of edge clamping is predominant. The rotational constraint $W,_x$ = 0 raises the buckling load by about 10 to 20%, while the axial constraint u = 0 gives an increase of less than 10%. For the longer shell ($\bar{Z}$ = 400) the effect of the axial constraint u = 0 is predominant; the increase due to this constraint is less than 10%. Buckling modes for SS-3 and C-4 boundary conditions are shown in Figs. 14 to 17.

For anisotropic shells under torsion, prebuckling deformations occur if the Poisson's expansion W_t is not equal to zero (see Appendix C). For the Booton shell A_{26}^{*} = 0, and consequently, if the edges are clamped, w_0 = 0. For simply supported shells, however, prebuckling deformations occur, because B_2 in the boundary condition for the prebuckling rotation does not vanish (B_{61}^{*} is not equal to zero). These deformations are about an order of magnitude smaller than for the other external loads considered. The effect of prebuckling rotations on the buckling load (Table 8) is therefore seen to be small (a decrease of 2% for $\bar{Z}$ = 50).

5. CONCLUDING REMARKS

A generalization of the common shooting method has been presented, which can be used to solve nonlinear eigenvalue problems. The eigenvalue is treated as a parameter in the shooting procedure, i.e., as one of the unknowns in the Newton-type iteration scheme.

This idea has been applied to the bifurcation buckling of anisotropic cylindrical shells under the combined loading of axial compression, radial pressure and torsion.

Results for a [30°,0°,-30°] laminated anisotropic shell have been presented, showing the effects of different boundary conditions and nonlinear prebuckling deformations.

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APPENDIX A : Basic equations

Constitutive equations

Using the sign convention shown in Fig. 2, with W positive inward the numbering of the layers begins at the outer surface. The angle of rotation θ_k (k = 1,2,...,N) of the individual layers is defined with respect to the x-axis of the shell. The shell reference surface coincides with the midsurface of the laminate. If the position of the kth lamina is defined by $h_{k-1} < z < h_k$ the total thickness of the laminate is

$$t = \sum_{k=1}^{N} (h_k - h_{k-1}) \tag{A1}$$

We assume that each lamina may be considered as a homogeneous orthotropic medium in a plane stress state. The stress-strain relations for the kth lamina can then be written as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}_k \tag{A2}$$

where

$$Q_{11} = E_{11}/(1-\nu_{12}\nu_{21})$$

$$Q_{22} = E_{22}/(1-\nu_{12}\nu_{21})$$

$$Q_{12} = \nu_{21}E_{11}/(1-\nu_{12}\nu_{21}) = \nu_{12}E_{22}/(1-\nu_{12}\nu_{21}) \tag{A3}$$

$$Q_{66} = G_{12}$$

The relations refer to the lamina principal axes $(1,2)$. Transformation to the shell wall reference axes (x,y) gives

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_k \tag{A4}
$$

where

$$\bar{Q}_{11} = Q_{11}C^4 + 2(Q_{12}+2Q_{66})C^2S^2 + Q_{22}S^4$$

$$\bar{Q}_{12} = (Q_{11}+Q_{22}-4Q_{66})C^2S^2 + Q_{12}(C^4+S^4)$$

$$\bar{Q}_{22} = Q_{11}S^4 + 2(Q_{12}+2Q_{66})C^2S^2 + Q_{22}C^4 \tag{A5}$$

$$\bar{Q}_{66} = (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})C^2S^2 + Q_{66}(C^4+S^4)$$

$$\bar{Q}_{16} = (Q_{11}-Q_{12}-2Q_{66})C^3S + (Q_{12}-Q_{22}+2Q_{66})CS^3$$

$$\bar{Q}_{26} = (Q_{11}-Q_{12}-2Q_{66})CS^3 + (Q_{12}-Q_{22}+2Q_{66})C^3S$$

and

$$C = \cos\theta_k \, , \quad S = \sin\theta_k$$

The stress resultants (see Fig. 3) acting at the shell midsurface are given by

$$
\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k dz \tag{A6}
$$

and the moment resultants (see Fig. 3) acting on the shell midsurface are defined as

$$\begin{bmatrix} M_x \\ M_y \\ \dfrac{M_{xy}+M_{yx}}{2} \end{bmatrix} = \sum_{k=1}^{N} \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k z\,dz \qquad (A7)$$

According to the Kirchhoff-Love hypothesis for a thin shell, the strain at any layer can be written in terms of the strain and curvature of the midsurface as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_k = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} + z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \qquad (A8)$$

Substituting these expressions into Eq. (A4) and introducing the resulting relations into Eq. (A6) and (A7), followed by carrying out the indicated integrations gives the following constitutive equations

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$(A9)$$

$$\begin{bmatrix} M_x \\ M_y \\ \dfrac{M_{xy}+M_{yx}}{2} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$(A10)$$

where

$$A_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k \ (h_k - h_{k-1}) \tag{A11}$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{N} (\bar{Q}_{ij})_k \ (h_k^2 - h_{k-1}^2) \tag{A12}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} (\bar{Q}_{ij})_k \ (h_k^3 - h_{k-1}^3) \tag{A13}$$

for $i, j = 1, 2, 6$.

The constitutive equations (A9) and (A10) can be written in matrix form as

$$\{N\} = A \{\varepsilon\} + B \{\kappa\} \tag{A14}$$

$$\{M\} = B \{\varepsilon\} + D \{\kappa\} \tag{A15}$$

and after partial inversion as

$$\{\varepsilon\} = A^* \{N\} + B^* \{\kappa\} \tag{A16}$$

$$\{M\} = C^* \{N\} + D^* \{\kappa\} \tag{A17}$$

where

$$A^* = A^{-1}$$

$$B^* = -A^{-1}B$$

$$C^* = BA^{-1} = -B^{*T} \tag{A18}$$

$$D^* = D - BA^{-1}B$$

The stiffness parameters are nondimensionalized as follows

$$\bar{A}_{ij} = (1/Et)A_{ij} \quad ; \quad \bar{B}_{ij} = (2c/Et^2)B_{ij} \quad ; \quad \bar{D}_{ij} = (4c^2/Et^3)D_{ij} \quad (A19)$$

and

$$\bar{A}_{ij}^{*} = Et \, A_{ij}^{*} \quad ; \quad \bar{B}_{ij}^{*} = (2c/t)B_{ij}^{*} \quad ; \quad \bar{D}_{ij}^{*} = (4c^2/Et^3)D_{ij}^{*} \quad (A20)$$

where

$$c^2 = 3(1-v^2) \quad (A21)$$

and the quantities E and v are arbitrarily chosen reference values.

Strain-displacement relations

The Donnell-type strain-displacement relations are

$$\varepsilon_x = u,_x + \frac{1}{2} W,_x^2 + W,_x \bar{W},_x$$

$$\varepsilon_y = v,_y - \frac{W}{R} + \frac{1}{2} W,_y^2 + W,_y \bar{W},_y \quad (A22a)$$

$$\gamma_{xy} = u,_y + v,_x + W,_x W,_y + W,_x \bar{W},_y + \bar{W},_x W,_y$$

$$\kappa_x = - W,_{xx}$$

$$\kappa_y = - W,_{yy} \quad (A22b)$$

$$\kappa_{xy} = - 2W,_{xy}$$

where $\bar{W}$ is an initial radial imperfection.

Equilibrium equations

The potential energy is given by

$$\int_0^{2\pi R} \int_0^L \{N_x \varepsilon_x + N_y \varepsilon_y + N_{xy} \gamma_{xy} + M_x \kappa_x + M_y \kappa_y + \frac{M_{xy} + M_{yx}}{2} \kappa_{xy}\} dx dy \qquad \text{(A23)}$$

$$- \int_0^{2\pi R} N_x (x=L) \int_0^L (u,_x - q W,_{xx}) dx dy - \int_0^{2\pi R} \int_0^L p W \, dx dy - \int_0^{2\pi R} N_{xy}(x=L) \int_0^L v,_x dx dy = 0$$

where $N_x(x=L) = -N_0$ is the applied axial compression, p is the applied external pressure, $N_{xy}(x=L) = T_0$ is the applied counter-clockwise torque and q is the load eccentricity (positive inward) of the applied axial compression. We can derive the following equilibrium equations by the stationary potential energy theorem

$$N_{x,x} + N_{xy,y} = 0 \qquad \text{(A24a)}$$

$$N_{xy,x} + N_{y,y} = 0 \qquad \text{(A24b)}$$

$$M_{x,xx} + (M_{xy} + M_{yx}),_{xy} + M_{y,yy} + \frac{1}{R} N_y + N_x(W,_{xx} + \bar{W},_{xx})$$

$$+ 2 N_{xy}(W,_{xy} + \bar{W},_{xy}) + N_y(W,_{yy} + \bar{W},_{yy}) = -p \qquad \text{(A24c)}$$

Introducing an Airy stress function F as $N_x = F,_{yy}$, $N_y = F,_{xx}$ and $N_{xy} = -F,_{xy}$, (A24a) and (A24b) are identically satisfied and (A24c) becomes

$$L_{B*}(F) + L_{D*}(W) = \frac{1}{R} F,_{xx} + L_{NL}(F, W + \bar{W}) + p \qquad \text{(A25)}$$

where the operators L_{B*}, L_{D*} and L_{NL} are defined in the text.
Another equation in W and F is obtained from the compatibility condition

$$\varepsilon_{x,yy} + \varepsilon_{y,xx} - \gamma_{xy,xy} = -\frac{1}{R} W,_{xx} - \frac{1}{2} L_{NL}(W, W + 2\bar{W}) \qquad \text{(A26)}$$

Substitution of $\{\varepsilon\}$ from (A3) gives

$$L_{A\star}(F) - L_{B\star}(W) = -\frac{1}{R} W_{,xx} - \frac{1}{2} L_{NL}(W, W + 2\bar{W})$$

(A27)

where the operators $L_{A\star}$ and $L_{B\star}$ are defined in the text.

APPENDIX B : Periodicity condition

The solution has to satisfy the circumferential periodicity condition

$$\int_0^{2\pi R} v,_y \, dy = 0 \tag{B1}$$

where

$$v,_y = \varepsilon_y + \frac{1}{R} W - \frac{1}{2} W,_y^2 \tag{B2}$$

$$\varepsilon_y = A_{12}^* N_x + A_{22}^* N_y + A_{26}^* N_{xy} + B_{21}^* \kappa_x + B_{22}^* \kappa_y + B_{26}^* \kappa_{xy} \tag{B3}$$

and

$$N_x = F,_{yy} \quad , \quad N_y = F,_{xx} \quad , \quad N_{xy} = -F,_{xy}$$

$$\kappa_x = -W,_{xx} \quad , \quad \kappa_y = -W,_{yy} \quad , \quad \kappa_{xy} = -2W,_{xy} \tag{B4}$$

Substituting for W and F the assumed perturbation expansion yields after regrouping and ordering by powers of ξ

$$v,_y = (t/cR)\{(-\lambda\bar{A}_{12}^* + c\bar{W}_v) + (-\bar{p}\bar{A}_{22}^* + cW_p) + (\bar{\tau}\bar{A}_{26}^* + cW_t) + \bar{A}_{22}^* f_0'' - (t/2R)\bar{B}_{21}^* w_0'' + cw_0\}$$

$$+ (t/cR)\xi\{[\bar{A}_{22}^* f_1'' - \bar{A}_{12}^* n^2 f_1 - \bar{A}_{26}^* n f_2'$$

$$- (t/2R)(\bar{B}_{21}^* w_1'' - \bar{B}_{22}^* n^2 w_1 + 2\bar{B}_{26}^* n w_2') + cw_1]\cos n\theta$$

$$+ [\bar{A}_{22}^* f_2'' - \bar{A}_{12}^* n^2 f_2 + \bar{A}_{26}^* n f_1'$$

$$- (t/2R)(\bar{B}_{21}^* w_2'' - \bar{B}_{22}^* n^2 w_2 - 2\bar{B}_{26}^* n w_1') + cw_2]\sin n\theta\} \tag{B5}$$

$$+ (t/cR)\xi^2\{\bar{A}_{22}^* f_\alpha'' - (t/2R)\bar{B}_{21}^* w_\alpha'' + cw_\alpha - (ct/4R)n^2(w_1^2 + w_2^2) + \ldots$$

$$\ldots + [\bar{A}^*_{22}f''_\beta - \bar{A}^*_{12}4n^2 f_\beta - \bar{A}^*_{26}2nf'_\gamma$$

$$- (t/2R)(\bar{B}^*_{21}w''_\beta - \bar{B}^*_{22}4n^2 w_\beta + 4\bar{B}^*_{26}nw'_\gamma) + cw_\beta + (ct/4R)n^2(w_1^2 - w_2^2)]\cos 2n\theta$$

$$+ [\bar{A}^*_{22}f''_\gamma - \bar{A}^*_{12}4n^2 f_\gamma + \bar{A}^*_{26}2nf'_\beta$$

$$- (t/2R)(\bar{B}^*_{21}w''_\gamma - \bar{B}^*_{22}4n^2 w_\gamma - 4\bar{B}^*_{26}nw'_\beta) + cw_\gamma + (ct/2R)n^2 w_1 w_2]\sin 2n\theta\}$$

$$+ (t/cR)\xi^3\{-(ct/R)n^2[(w_1 w_\beta + w_2 w_\gamma)\cos n\theta \quad - \quad (w_2 w_\beta - w_1 w_\gamma)\sin n\theta$$

$$- (w_1 w_\beta - w_2 w_\gamma)\cos 3n\theta - (w_2 w_\beta + w_1 w_\gamma)\sin 3n\theta]\}$$

$$+ (t/cR)\xi^4\{-(ct/R)n^2[w_\beta^2 + w_\gamma^2 - 2w_\beta w_\gamma \sin 4n\theta - (w_\beta^2 - w_\gamma^2)\cos 4n\theta]\}$$

where $\theta = y/R$. Substituting this expression into equation (B1) and carrying out the y-integration yields

$$\{(-\lambda\bar{A}^*_{12} + cW_v) + (-\bar{p}\bar{A}^*_{22} + cW_p) + (\bar{\tau}\bar{A}^*_{26} + cW_t) + \underline{\bar{A}^*_{22}f''_0 - (t/2R)\bar{B}^*_{21}w''_0 + cw_0}\}$$

$$+ \xi^2\{\underline{\bar{A}^*_{22}f''_\alpha - (t/2R)\bar{B}^*_{21}w''_\alpha + cw_\alpha} - (ct/4R)n^2(w_1^2 + w_2^2)\}$$

$$+ \xi^4\{-(ct/R)n^2(w_\beta^2 + w_\gamma^2)\} = 0 \qquad\qquad (B6)$$

Notice that the underlined terms vanish identically since they are equal to equations (2.2.10) and (E4), respectively, with the constants $\tilde{C}_1 = \tilde{C}_2 = 0$ and $\tilde{C}_3 = \tilde{C}_4 = 0$. If one now lets

$$W_v = \bar{A}^*_{12}\lambda/c \qquad\qquad (B7a)$$

$$W_p = \bar{A}^*_{22}\bar{p}/c \qquad\qquad (B7b)$$

$$W_t = -\bar{A}^*_{26}\bar{\tau}/c \qquad\qquad (B7c)$$

then the periodicity condition (B1) is satisfied up to and including terms of the order ξ^3.

APPENDIX C : Boundary conditions

The reduced boundary conditions are summarized in this appendix. The derivation can be found in the report by Arbocz and Hol [5].

Boundary conditions:

Prebuckling state

simply supported:

$$w_0 = - (W_v + W_p + W_t)$$

$$w_0'' = B_1 \lambda + B_2 \bar{\tau}$$

clamped:

$$w_0 = - (W_v + W_p + W_t)$$

$$w_0' = 0$$

Buckling state

SS-1 : $N_x = - N_0$; $N_{xy} = T_0$; $W = 0$; $M_x = - N_0 q$

$$f_1 = f_2 = 0$$

$$f_1' = f_2' = 0$$

$$w_1 = w_2 = 0$$

$$w_1'' = - B_3 w_2' - B_4 f_1''$$

$$w_2'' = B_3 w_1' - B_4 f_2''$$

SS-2 : $u = 0$; $N_{xy} = T_0$; $W = 0$; $M_x = - N_0 q$

$$f_1^{'''} = \quad BB_{15} f_2^{''} - BB_{16} f_2 + B_{12} w_1^{'''} + BB_{17} w_1^{'}$$

$$f_2^{'''} = - BB_{15} f_1^{''} + BB_{16} f_1 + B_{12} w_2^{'''} + BB_{17} w_2^{'}$$

$$f_1^{'} = f_2^{'} = 0$$

$$w_1 = w_2 = 0$$

$$w_1^{''} = - B_3 w_2^{'} - B_4 f_1^{''} + B_6 f_1$$

$$w_2^{''} = \quad B_3 w_1^{'} - B_4 f_2^{''} + B_6 f_2$$

SS-3 : $N_x = - N_0$; $v = 0$; $W = 0$; $M_x = - N_0 q$

$$f_1 = f_2 = 0$$

$$f_1^{''} = \quad BB_1 f_2^{'} + BB_2 w_2^{'}$$

$$f_2^{''} = - BB_1 f_1^{'} - BB_2 w_1^{'}$$

$$w_1 = w_2 = 0$$

$$w_1^{''} = - BB_{21} w_2^{'} + BB_{22} f_2^{'}$$

$$w_2^{''} = \quad BB_{21} w_1^{'} - BB_{22} f_1^{'}$$

SS-4 : $u = 0$; $v = 0$; $W = 0$; $M_x = - N_0 q$

$$f_1^{'''} = \quad BB_{27} f_1^{'} + BB_{28} w_1^{'} + B_{12} w_1^{'''} - BB_{29} f_2$$

$$f_2^{'''} = \quad BB_{27} f_2^{'} + BB_{28} w_2^{'} + B_{12} w_2^{'''} + BB_{29} f_1$$

$$f_1'' = BB_5 f_1 + BB_1 f_2' + BB_2 w_2'$$

$$f_2'' = BB_5 f_2 - BB_1 f_1' - BB_2 w_1'$$

$$w_1 = w_2 = 0$$

$$w_1'' = - BB_{21} w_2' + BB_{22} f_2' + BB_{25} f_1$$

$$w_2'' = BB_{21} w_1' - BB_{22} f_1' + BB_{25} f_2$$

C-1 : $N_x = - N_0$; $N_{xy} = T_0$; $W = 0$; $W_{,x} = 0$

$$f_1 = f_2 = 0$$

$$f_1' = f_2' = 0$$

$$w_1 = w_2 = 0$$

$$w_1' = w_2' = 0$$

C-2 : $u = 0$; $N_{xy} = T_0$; $W = 0$; $W_{,x} = 0$

$$f_1''' = B_{15} f_2'' - B_{18} f_2 + B_{12} w_1''' + B_{20} w_2''$$

$$f_2''' = - B_{15} f_1'' + B_{18} f_1 + B_{12} w_2''' - B_{20} w_1''$$

$$f_1' = f_2' = 0$$

$$w_1 = w_2 = 0$$

$$w_1' = w_2' = 0$$

C-3 : $N_x = - N_0$; $v = 0$; $W = 0$; $W,_x = 0$

$f_1 = f_2 = 0$

$f_1'' = \quad B_{11}f_2' + B_{12}w_1''$

$f_2'' = - B_{11}f_1' + B_{12}w_2''$

$w_1 = w_2 = 0$

$w_1' = w_2' = 0$

C-4 : $u = 0$; $v = 0$; $W = 0$; $W,_x = 0$

$f_1'''' = BB_{33}f_1' - BB_{34}f_2 + B_{12}w_1''' + BB_{35}w_2''$

$f_2'''' = BB_{33}f_2' + BB_{34}f_1 + B_{12}w_2''' - BB_{35}w_1''$

$f_1'' = B_{10}f_1 + B_{11}f_2' + B_{12}w_1''$

$f_2'' = B_{10}f_2 - B_{11}f_1' + B_{12}w_2''$

$w_1 = w_2 = 0$

$w_1' = w_2' = 0$

Conditions at $x = L/2$:

Prebuckling state

w_0 = symmetric w.r.t. $x = L/2$:

$$w_0' = w_0''' = 0$$

Buckling state

w_1 = symmetric, w_2 = anti-symmetric w.r.t. $x = L/2$:

$$w_2 = f_2 = w_1' = f_1' = w_2'' = f_2'' = w_1''' = f_1''' = 0$$

w_1 = anti-symmetric, w_2 = symmetric w.r.t. $x = L/2$:

$$w_1 = f_1 = w_2' = f_2' = w_1'' = f_1'' = w_2''' = f_2''' = 0$$

The coefficients used in this appendix are listed in Appendix D.

APPENDIX D: Definition of constants

Constants used in the equations:

Prebuckling state (Eq. (3.1.1)):

$$\hat{C}_1 = (4c/\Delta)(R/t)\bar{B}_{21}^*$$

$$\hat{C}_2 = (4c/\Delta)(R/t)\bar{A}_{22}^*$$

$$\hat{C}_3 = (4c^2/\Delta)(R/t)^2$$

Buckling state (Eqs. (2.2.15)):

$$C_1 = (2/\Delta)(R/t)\{[\bar{A}_{22}^*(\bar{B}_{11}^*+\bar{B}_{22}^*-2\bar{B}_{66}^*) - \bar{B}_{21}^*(2\bar{A}_{12}^*+\bar{A}_{66}^*)]n^2 + (2cR/t)\bar{A}_{22}^*\}$$

$$C_2 = (2/\Delta)(R/t)(\bar{B}_{21}^*\bar{A}_{11}^*-\bar{B}_{12}^*\bar{A}_{22}^*)n^4$$

$$C_3 = (2/\Delta)(R/t)\{\bar{A}_{22}^*(2\bar{B}_{26}^*-\bar{B}_{61}^*) + 2\bar{A}_{26}^*\bar{B}_{21}^*\}n$$

$$C_4 = (2/\Delta)(R/t)\{\bar{A}_{22}^*(2\bar{B}_{16}^*-\bar{B}_{62}^*) + 2\bar{A}_{16}^*\bar{B}_{21}^*\}n^3$$

$$C_5 = (1/\Delta)\{[2(\bar{D}_{12}^*+2\bar{D}_{66}^*)\bar{A}_{22}^* + \bar{B}_{21}^*(\bar{B}_{11}^*+\bar{B}_{22}^*-2\bar{B}_{66}^*)]n^2 + (2cR/t)\bar{B}_{21}^*\}$$

$$C_6 = (1/\Delta)(\bar{D}_{22}^*\bar{A}_{22}^*+\bar{B}_{12}^*\bar{B}_{21}^*)n^4$$

$$C_7 = (1/\Delta)\{4\bar{D}_{16}^*\bar{A}_{22}^* + \bar{B}_{21}^*(2\bar{B}_{26}^*-\bar{B}_{61}^*)\}n$$

$$C_8 = (1/\Delta)\{4\bar{D}_{26}^*\bar{A}_{22}^* + \bar{B}_{21}^*(2\bar{B}_{16}^*-\bar{B}_{62}^*)\}n^3$$

$$C_{10} = (2c/\Delta)n^2 \bar{B}_{21}^*$$

$$C_{12} = (4c/\Delta)(R/t)n^2 \, \bar{A}_{22}^*$$

$$C_{14} = (4c/\Delta)(R/t)n\bar{A}_{22}^*$$

$$C_{15} = (4c/\Delta)(R/t)\bar{A}_{22}^*$$

$$C_{17} = (1/\Delta)\{[\bar{D}_{11}^*(2\bar{A}_{12}^*+\bar{A}_{66}^*) + \bar{B}_{21}^*(\bar{B}_{11}^*+\bar{B}_{22}^*-2\bar{B}_{66}^*)]n^2 + (2cR/t)\bar{B}_{21}^*\}$$

$$C_{18} = (1/\Delta)(\bar{D}_{11}^*\bar{A}_{11}^*+\bar{B}_{12}^*\bar{B}_{21}^*)n^4$$

$$C_{19} = (1/\Delta)\{2\bar{D}_{11}^*\bar{A}_{26}^* - \bar{B}_{21}^*(2\bar{B}_{26}^*-\bar{B}_{61}^*)\}n$$

$$C_{20} = (1/\Delta)\{-2\bar{D}_{11}^*\bar{A}_{16}^* + \bar{B}_{21}^*(2\bar{B}_{16}^*-\bar{B}_{62}^*)\}n^3$$

$$C_{21} = (1/\Delta)\{(t/2R)[-\bar{D}_{11}^*(\bar{B}_{11}^*+\bar{B}_{22}^*-2\bar{B}_{66}^*) + 2\bar{B}_{21}^*(\bar{D}_{12}^*+2\bar{D}_{66}^*)]n^2 - c\bar{D}_{11}^*\}$$

$$C_{22} = (t/2R)(1/\Delta)(-\bar{B}_{21}^*\bar{D}_{22}^*+\bar{B}_{12}^*\bar{D}_{11}^*)n^4$$

$$C_{23} = (t/2R)(1/\Delta)\{\bar{D}_{11}^*(2\bar{B}_{26}^*-\bar{B}_{61}^*) - 4\bar{B}_{21}^*\bar{D}_{16}^*\}n$$

$$C_{24} = (t/2R)(1/\Delta)\{4\bar{B}_{21}^*\bar{D}_{26}^* - \bar{D}_{11}^*(2\bar{B}_{16}^*-\bar{B}_{62}^*)\}n^3$$

$$C_{26} = (c/\Delta)(t/R)n^2 \, \bar{D}_{11}^*$$

$$C_{28} = (2c/\Delta)n^2 \, \bar{B}_{21}^*$$

$$C_{30} = (4c/\Delta)n\bar{B}_{21}^*$$

$$C_{31} = (2c/\Delta)\bar{B}_{21}^*$$

where

$$\Delta = \bar{A}_{22}^*\bar{D}_{11}^* + \bar{B}_{21}^{*\,2}$$

Further, the following constants are used in Eq. (3.2.4):

$$\tilde{D}_1 = C_6 + C_{10}w_0'' + C_{12}(f_0'' - \bar{P}_e)$$

$$\tilde{D}_2 = C_8 + C_{14}\bar{\tau}$$

$$\tilde{D}_3 = C_5 - C_{15}\lambda$$

$$\tilde{D}_4 = C_2 - C_{12}w_0''$$

$$\tilde{D}_5 = C_{22} + C_{26}w_0'' - C_{28}(f_0'' - \bar{P}_e)$$

$$\tilde{D}_6 = C_{24} + C_{30}\bar{\tau}$$

$$\tilde{D}_7 = C_{21} - C_{31}\lambda$$

$$\tilde{D}_8 = C_{18} + C_{28}w_0''$$

Constants used in the boundary conditions (Appendix C):

$$B_1 = (1/\Delta_2)\{\bar{q} - (2R/t)(\bar{B}_{21}^*\bar{A}_{12}^*/\bar{A}_{22}^* - \bar{B}_{11}^*)\}$$

$$B_2 = (2/\Delta_2)(R/t)(\bar{B}_{21}^*\bar{A}_{26}^*/\bar{A}_{22}^* - \bar{B}_{61}^*)$$

$$B_3 = 2n\bar{D}_{16}^*/\bar{D}_{11}^*$$

$$B_4 = (2R/t)\bar{B}_{21}^*/\bar{D}_{11}^*$$

$$B_6 = 2n^2(R/t)\bar{B}_{11}^*/\bar{D}_{11}^*$$

$$B_8 = 2n(R/t)\bar{B}_{61}^*/\bar{D}_{11}^*$$

$$B_{10} = n^2 \, \bar{A}_{12}^*/\bar{A}_{22}^*$$

$$B_{11} = n\bar{A}_{26}^*/\bar{A}_{22}^*$$

$$B_{12} = (t/2R)\bar{B}_{21}^*/\bar{A}_{22}^*$$

$$B_{13} = n(t/R)\bar{B}_{26}^*/\bar{A}_{22}^*$$

$$B_{15} = 2n\bar{A}_{26}^*/\bar{A}_{22}^*$$

$$B_{17} = (n^2/\bar{A}_{22}^*)(\bar{A}_{12}^*+\bar{A}_{66}^*)$$

$$B_{18} = n^3 \, \bar{A}_{16}^*/\bar{A}_{22}^*$$

$$B_{19} = (1/\bar{A}_{22}^*)\{(t/2R)n^2(2\bar{B}_{66}^*-\bar{B}_{22}^*) - c\}$$

$$B_{20} = (t/2R)(n/\bar{A}_{22}^*)(2\bar{B}_{26}^*-\bar{B}_{61}^*)$$

$$BB_1 = (1/\Delta_B)(B_{11} + B_8B_{12})$$

$$BB_2 = (1/\Delta_B)(B_{13} - B_3B_{12})$$

$$BB_5 = (1/\Delta_B)(B_{10} + B_6B_{12})$$

$$BB_{15} = B_{15} - B_4B_{20}$$

$$BB_{16} = B_{18} - B_6B_{20}$$

$$BB_{17} = B_{19} + B_3B_{20}$$

$$BB_{21} = B_3 + B_4BB_2$$

$$BB_{22} = B_8 - B_4 BB_1$$

$$BB_{25} = B_6 - B_4 BB_5$$

$$BB_{27} = B_{17} - B_{15} BB_1 - B_{20} BB_{22}$$

$$BB_{28} = B_{19} - B_{15} BB_2 + B_{20} BB_{21}$$

$$BB_{29} = B_{18} - B_{15} BB_5 - B_{20} BB_{25}$$

$$BB_{33} = B_{17} - B_{11} B_{15}$$

$$BB_{34} = B_{18} - B_{10} B_{15}$$

$$BB_{35} = B_{20} + B_{12} B_{15}$$

with

$$\Delta_B = 1 + B_4 B_{12}$$

$$\Delta_2 = \bar{D}_{11}^* + \bar{B}_{21}^{*\,2} / \bar{A}_{22}^*$$

and where

$$\bar{q} = (4cR/t^2) q$$

is the nondimensionalized axial load eccentricity (positive inward).

APPENDIX E: Postbuckling problem

The equations governing the 2nd-order state (postbuckling problem) are

$$L_{A*}(F^{(2)}) - L_{B*}(W^{(2)}) = -\frac{1}{R} W,^{(2)}_{xx} - tw_{0,xx}W,^{(2)}_{yy}$$

$$+ \frac{1}{2}(t/R)^2 n^2 \{w_1 w_{1,xx} + w_{1,x} w_{1,x} + w_2 w_{2,xx} + w_{2,x} w_{2,x})$$

$$+ (w_1 w_{1,xx} - w_{1,x} w_{1,x} - w_2 w_{2,xx} + w_{2,x} w_{2,x}) \cos 2n\theta$$

$$+ (w_1 w_{2,xx} + w_2 w_{1,xx} - 2 w_{1,x} w_{2,x}) \sin 2n\theta \} \qquad (E1a)$$

$$L_{B*}(F^{(2)}) + L_{D*}(W^{(2)}) = \frac{1}{R} F,^{(2)}_{xx} + (ERt^2/c) f_{0,xx} W,^{(2)}_{yy} + tw_{0,xx} F,^{(2)}_{yy}$$

$$- (Et^2/cR)(\lambda W,^{(2)}_{xx} + \bar{p} W,^{(2)}_{yy} - 2\bar{\tau} W,^{(2)}_{xy}) \qquad (E1b)$$

$$- \frac{1}{2}(Et^3/cR) n^2 \{ (w_1 f_{1,xx} + 2 w_{1,x} f_{1,x} + w_{1,xx} f_1 + w_2 f_{2,xx} + 2 w_{2,x} f_{2,x} + w_{2,xx} f_2)$$

$$+ [w_1 f_{1,xx} - 2 w_{1,x} f_{1,x} + w_{1,xx} f_1 - (w_2 f_{2,xx} - 2 w_{2,x} f_{2,x} + w_{2,xx} f_2)] \cos 2n\theta$$

$$+ [w_1 f_{2,xx} - 2 w_{1,x} f_{2,x} + w_{1,xx} f_2 + (w_2 f_{1,xx} - 2 w_{2,x} f_{1,x} + w_{2,xx} f_1)] \sin 2n\theta \}$$

These equations admit separable solutions of the form

$$W^{(2)} = t[w_\alpha(x) + w_\beta(x) \cos 2n\theta + w_\gamma(x) \sin 2n\theta] \qquad (E2a)$$

$$F^{(2)} = (ERt^2/c) [f_\alpha(x) + f_\beta(x) \cos 2n\theta + f_\gamma(x) \sin 2n\theta] \qquad (E2b)$$

Substituting, regrouping and equating coefficients of like trigonometric terms yields the following system of 6 linear inhomogeneous ordinary differential equations with variable coeffients

$$\bar{A}^*_{22} f_\alpha^{iv} - (t/2R) \bar{B}^*_{21} w_\alpha^{iv} + c w_\alpha'' = (ct/2R) n^2 (w_1 w_1'' + w_1' w_1' + w_2 w_2'' + w_2' w_2') \qquad (E3a)$$

$$\bar{A}^*_{22}f^{iv}_\beta - (2\bar{A}^*_{12}+\bar{A}^*_{66})4n^2f''_\beta + \bar{A}^*_{11}16n^4f_\beta - 4\bar{A}^*_{26}nf'''_\gamma + 16\bar{A}^*_{16}n^3f'_\gamma$$

$$- (t/2R)[\bar{B}^*_{21}w^{iv}_\beta - (\bar{B}^*_{11}+\bar{B}^*_{22}-2\bar{B}^*_{66})4n^2w''_\beta + \bar{B}^*_{12}16n^4w_\beta + (2\bar{B}^*_{26}-\bar{B}^*_{61})2nw'''_\gamma$$

$$- (2\bar{B}^*_{16}-\bar{B}^*_{62})8n^3w'_\gamma] + cw''_\beta - (4ct/R)n^2w''_0w_\beta$$

$$= (ct/2R)n^2(w_1w''_1-w'_1w'_1-w_2w''_2+w'_2w'_2) \quad \text{(E3b)}$$

$$\bar{A}^*_{22}f^{iv}_\gamma - (2\bar{A}^*_{12}+\bar{A}^*_{66})4n^2f''_\gamma + \bar{A}^*_{11}16n^4f_\gamma + 4\bar{A}^*_{26}nf'''_\beta - 16\bar{A}^*_{16}n^3f'_\beta$$

$$- (t/2R)[\bar{B}^*_{21}w^{iv}_\gamma - (\bar{B}^*_{11}+\bar{B}^*_{22}-2\bar{B}^*_{66})4n^2w''_\gamma + \bar{B}^*_{12}16n^4w_\gamma - (2\bar{B}^*_{26}-\bar{B}^*_{61})2nw'''_\beta$$

$$+ (2\bar{B}^*_{16}-\bar{B}^*_{62})8n^3w'_\beta] + cw''_\gamma - (4ct/R)n^2w''_0w_\gamma$$

$$= (ct/2R)n^2(w_1w''_2+w_2w''_1-2w'_1w'_2) \quad \text{(E3c)}$$

$$\bar{B}^*_{21}f^{iv}_\alpha + (t/2R)\bar{D}^*_{11}w^{iv}_\alpha - (2cR/t)f''_\alpha + 2c\lambda w''_\alpha \quad \text{(E3d)}$$

$$= - cn^2(w_1f''_1+2w'_1f'_1+w''_1f_1+w_2f''_2+2w'_2f'_2+w''_2f_2)$$

$$\bar{B}^*_{21}f^{iv}_\beta - (\bar{B}^*_{11}+\bar{B}^*_{22}-2\bar{B}^*_{66})4n^2f''_\beta + \bar{B}^*_{12}16n^4f_\beta + (2\bar{B}^*_{26}-\bar{B}^*_{61})2nf'''_\gamma$$

$$- (2\bar{B}^*_{16}-\bar{B}^*_{62})8n^3f'_\gamma + (t/2R)[\bar{D}^*_{11}w^{iv}_\beta - 2(\bar{D}^*_{12}+2\bar{D}^*_{66})4n^2w''_\beta + \bar{D}^*_{22}16n^4w_\beta$$

$$+ 8\bar{D}^*_{16}nw'''_\gamma - 32\bar{D}^*_{26}n^3w'_\gamma] - (2cR/t)f''_\beta + 2c(\lambda w''_\beta-4n^2\bar{p}w_\beta-4n\bar{\tau}w'_\gamma)$$

$$+ 8cn^2(w''_0f_\beta+f''_0w_\beta) = - cn^2(w_1f''_1-2w'_1f'_1+w''_1f_1-w_2f''_2+2w'_2f'_2-w''_2f_2) \quad \text{(E3e)}$$

$$\bar{B}_{21}^{*} f_{\gamma}^{iv} - (\bar{B}_{11}^{*} + \bar{B}_{22}^{*} - 2\bar{B}_{66}^{*}) 4n^2 f_{\gamma}'' + \bar{B}_{12}^{*} 16n^4 f_{\gamma} - (2\bar{B}_{26}^{*} - \bar{B}_{61}^{*}) 2n f_{\beta}'''$$

$$+ (2\bar{B}_{16}^{*} - \bar{B}_{62}^{*}) 8n^3 f_{\beta}' + (t/2R) [\bar{D}_{11}^{*} w_{\gamma}^{iv} - 2(\bar{D}_{12}^{*} + 2\bar{D}_{66}^{*}) 4n^2 w_{\gamma}'' + \bar{D}_{22}^{*} 16n^4 w_{\gamma}$$

$$- 8\bar{D}_{16}^{*} n w_{\beta}''' + 32\bar{D}_{26}^{*} n^3 w_{\beta}'] - (2cR/t) f_{\gamma}'' + 2c(\lambda w_{\gamma}'' - 4n^2 \bar{p} w_{\gamma} + 4n\bar{\tau} w_{\beta}')$$

$$+ 8cn^2 (w_0'' f_{\gamma} + f_0'' w_{\gamma}) = - cn^2 (w_1 f_2'' - 2w_1' f_2' + w_1'' f_2 + w_2 f_1'' - 2w_2' f_1' + w_2'' f_1) \quad (E3f)$$

Equation (E3a) can be integrated twice to yield

$$f_{\alpha}'' = (t/2R) (\bar{B}_{21}^{*} / \bar{A}_{22}^{*}) w_{\alpha}'' - (c/\bar{A}_{22}^{*}) w_{\alpha} + (ct/4R) (n^2 / \bar{A}_{22}^{*}) (w_1^2 + w_2^2) + \tilde{C}_3 \bar{x} + \tilde{C}_4$$

$$(E4)$$

where $\bar{x} = x/R$ and the constants of integration $\tilde{C}_3$ and $\tilde{C}_4$ are identically equal to zero because of the periodicity condition (see Appendix B for details). Eliminating f_{α} between Eqs. (E3a) and (E3d) one obtains

$$w_{\alpha}^{iv} = (D_1 - D_2 \lambda) w_{\alpha}'' - D_3 w_{\alpha} + D_4 (w_1^2 + w_2^2) - D_5 (w_1 w_1'' + w_1' w_1' + w_2 w_2'' + w_2' w_2')$$

$$+ D_8 (w_1 f_1'' + 2w_1' f_1' + w_1'' f_1 + w_2 f_2'' + 2w_2' f_2' + w_2'' f_2) \quad (E5)$$

Further, in order to be able to use the shooting method to solve the governing equations of the 2nd-order state it is necessary, by considering Eqs. (E3b) and (E3e), to eliminate the w_{β}^{iv} term from Eq. (E3b) and the f_{β}^{iv} term from Eq. (E3e). Similarly, by considering Eqs. (E3c) and (E3f) one must eliminate the w_{γ}^{iv} term from Eq. (E3c) and the f_{γ}^{iv} term from Eq. (E3f). Finally, some further regrouping makes it possible to write the resulting equations as

$$f_\beta^{iv} = D_9 f_\beta'' - (D_{10} + D_{17} w_0'') f_\beta + D_{11} f_\gamma''' - D_{12} f_\gamma' - (D_{13} + D_{31} \lambda) w_\beta''$$

$$- (D_{14} - D_{18} w_0'') w_\beta - D_{17} (f_0'' - \bar{p}) w_\beta - D_{15} w_\gamma''' + (D_{16} + D_{19} \bar{\tau}) w_\gamma' \qquad \text{(E6a)}$$

$$+ D_{32} (w_1 w_1'' - w_1' w_1' - w_2 w_2'' + w_2' w_2') - D_5 (w_1 f_1'' - 2 w_1' f_1' + w_1'' f_1 - w_2 f_2'' + 2 w_2' f_2' - w_2'' f_2)$$

$$f_\gamma^{iv} = D_9 f_\gamma'' - (D_{10} + D_{17} w_0'') f_\gamma - D_{11} f_\beta''' + D_{12} f_\beta' - (D_{13} + D_{31} \lambda) w_\gamma''$$

$$- (D_{14} - D_{18} w_0'') w_\gamma - D_{17} (f_0'' - \bar{p}) w_\gamma + D_{15} w_\beta''' - (D_{16} + D_{19} \bar{\tau}) w_\gamma' \qquad \text{(E6b)}$$

$$+ D_{32} (w_1 w_2'' + w_2 w_1'' - 2 w_1' w_2') - D_5 (w_1 f_2'' - 2 w_1' f_2' + w_1'' f_2 + w_2 f_1'' - 2 w_2' f_1' + w_2'' f_1)$$

$$w_\beta^{iv} = - D_{20} f_\beta'' - (D_{21} + D_{22} w_0'') f_\beta + (D_{23} - D_2 \lambda) w_\beta'' - (D_{24} + D_{17} w_0'') w_\beta$$

$$- D_{22} (f_0'' - \bar{p}) w_\beta - D_{25} f_\gamma''' + D_{26} f_\gamma' - D_{27} w_\gamma''' + (D_{28} + D_{29} \bar{\tau}) w_\gamma' \qquad \text{(E6c)}$$

$$- D_5 (w_1 w_1'' - w_1' w_1' - w_2 w_2'' + w_2' w_2') - D_8 (w_1 f_1'' - 2 w_1' f_1' + w_1'' f_1 - w_2 f_2'' + 2 w_2' f_2' - w_2'' f_2)$$

$$w_\gamma^{iv} = - D_{20} f_\gamma'' - (D_{21} + D_{22} w_0'') f_\gamma + (D_{23} - D_2 \lambda) w_\gamma'' - (D_{24} + D_{17} w_0'') w_\gamma$$

$$- D_{22} (f_0'' - \bar{p}) w_\gamma + D_{25} f_\beta''' - D_{26} f_\beta' + D_{27} w_\beta''' - (D_{28} + D_{29} \bar{\tau}) w_\beta' \qquad \text{(E6d)}$$

$$- D_5 (w_1 w_2'' + w_2 w_1'' - 2 w_1' w_2') - D_8 (w_1 f_2'' - 2 w_1' f_2' + w_1'' f_2 + w_2 f_1'' - 2 w_2' f_1' + w_2'' f_1)$$

where f_0'' is given by Eq. (2.2.10), and the constants $D_1 - D_{32}$ are defined as follows:

$$D_1 = (4c/\Delta)(R/t) \bar{B}_{21}^*$$

$$D_2 = (4c/\Delta)(R/t) \bar{A}_{22}^*$$

$$D_3 = (4c^2/\Delta)(R/t)^2$$

$$D_4 = (c^2/\Delta)(R/t)n^2$$

$$D_5 = (cn^2/\Delta)\bar{B}_{21}^*$$

$$D_6 = (t/2R)\bar{B}_{21}^*/\bar{A}_{22}^*$$

$$D_7 = c/\bar{A}_{22}^*$$

$$D_8 = (2cR/t)(n^2/\Delta)\bar{A}_{22}^*$$

$$D_9 = (1/\Delta)\{[\bar{D}_{11}^*(2\bar{A}_{12}^*+\bar{A}_{66}^*) + \bar{B}_{21}^*(\bar{B}_{11}^*+\bar{B}_{22}^*-2\bar{B}_{66}^*)]4n^2 + (2cR/t)\bar{B}_{21}^*\}$$

$$D_{10} = (16n^4/\Delta)(\bar{D}_{11}^*\bar{A}_{11}^*+\bar{B}_{21}^*\bar{B}_{12}^*)$$

$$D_{11} = (2n/\Delta)\{2\bar{D}_{11}^*\bar{A}_{26}^* - \bar{B}_{21}^*(2\bar{B}_{26}^*-\bar{B}_{61}^*)\}$$

$$D_{12} = (8n^3/\Delta)\{2\bar{D}_{11}^*\bar{A}_{16}^* - \bar{B}_{21}^*(2\bar{B}_{16}^*-\bar{B}_{62}^*)\}$$

$$D_{13} = (1/\Delta)\{(t/2R)[\bar{D}_{11}^*(\bar{B}_{11}^*+\bar{B}_{22}^*-2\bar{B}_{66}^*) - 2\bar{B}_{21}^*(\bar{D}_{12}^*+2\bar{D}_{66}^*)]4n^2 + c\bar{D}_{11}^*\}$$

$$D_{14} = (t/2R)(16n^4/\Delta)(\bar{B}_{21}^*\bar{D}_{22}^*-\bar{B}_{12}^*\bar{D}_{11}^*)$$

$$D_{15} = (t/2R)(2n/\Delta)\{4\bar{B}_{21}^*\bar{D}_{16}^* - \bar{D}_{11}^*(2\bar{B}_{26}^*-\bar{B}_{61}^*)\}$$

$$D_{16} = (t/2R)(8n^3/\Delta)\{4\bar{B}_{21}^*\bar{D}_{26}^* - \bar{D}_{11}^*(2\bar{B}_{16}^*-\bar{B}_{62}^*)\}$$

$$D_{17} = (8c/\Delta)n^2\,\bar{B}_{21}^*$$

$$D_{18} = (4ct/R)(n^2/\Delta)\bar{D}_{11}^*$$

$$D_{19} = (8c/\Delta)n\bar{B}_{21}^*$$

$$D_{20} = (2/\Delta)(R/t)\{[\bar{B}_{21}^{*}(2\bar{A}_{12}^{*}+\bar{A}_{66}^{*}) - \bar{A}_{22}^{*}(\bar{B}_{11}^{*}+\bar{B}_{22}^{*}-2B_{66}^{*})]4n^{2} - (2cR/t)\bar{A}_{22}^{*}\}$$

$$D_{21} = (2/\Delta)(R/t)(\bar{A}_{22}^{*}\bar{B}_{12}^{*}-B_{21}^{*}\bar{A}_{11}^{*})16n^{4}$$

$$D_{22} = (4cR/t)(4n^{2}/\Delta)\bar{A}_{22}^{*}$$

$$D_{23} = (1/\Delta)\{[2\bar{A}_{22}^{*}(\bar{D}_{12}^{*}+2\bar{D}_{66}^{*}) + \bar{B}_{21}^{*}(\bar{B}_{11}^{*}+\bar{B}_{22}^{*}-2\bar{B}_{66}^{*})]4n^{2} + (2cR/t)\bar{B}_{21}^{*}\}$$

$$D_{24} = (16n^{4}/\Delta)(\bar{B}_{21}^{*}\bar{B}_{12}^{*}+\bar{A}_{22}^{*}\bar{D}_{22}^{*})$$

$$D_{25} = (2/\Delta)(R/t)\{\bar{A}_{22}^{*}(2\bar{B}_{26}^{*}-\bar{B}_{61}^{*}) + 2\bar{B}_{21}^{*}\bar{A}_{26}^{*}\}2n$$

$$D_{26} = (2/\Delta)(R/t)\{\bar{A}_{22}^{*}(2\bar{B}_{16}^{*}-\bar{B}_{62}^{*}) + 2\bar{B}_{21}^{*}\bar{A}_{16}^{*}\}8n^{3}$$

$$D_{27} = (2n/\Delta)\{4\bar{A}_{22}^{*}\bar{D}_{16}^{*} + \bar{B}_{21}^{*}(2\bar{B}_{26}^{*}-\bar{B}_{61}^{*})\}$$

$$D_{28} = (8n^{3}/\Delta)\{4\bar{A}_{22}^{*}\bar{D}_{26}^{*} + \bar{B}_{21}^{*}(2\bar{B}_{16}^{*}-\bar{B}_{62}^{*})\}$$

$$D_{29} = (8c/\Delta)(R/t)2n\bar{A}_{22}^{*}$$

$$D_{30} = (ct/4R)n^{2}/\bar{A}_{22}^{*}$$

$$D_{31} = (2c/\Delta)\bar{B}_{21}^{*}$$

$$D_{32} = (ct/2R)(n^{2}/\Delta)\bar{D}_{11}^{*}$$

where

$$\Delta = \bar{A}_{22}^{*}\bar{D}_{11}^{*} + \bar{B}_{21}^{*2}$$

This set of inhomogeneous differential equations with variable coefficients together with the appropriate boundary conditions (Arbocz and Hol [5]) form a response problem which is solved numerically.

APPENDIX F: Properties of $W(\bar{x})$

The initial value problem for $W(\bar{x})$ can be written in the following partitioned form

$$
\begin{bmatrix} W'_{11}(\bar{x}) & W'_{12}(\bar{x}) \\ W'_{21}(\bar{x}) & W'_{22}(\bar{x}) \end{bmatrix} = \begin{bmatrix} 0 & A_{12}(\bar{x}) \\ -A_{12}(\bar{x}) & 0 \end{bmatrix} \begin{bmatrix} W_{11}(\bar{x}) & W_{12}(\bar{x}) \\ W_{21}(\bar{x}) & W_{22}(\bar{x}) \end{bmatrix} \tag{F1}
$$

with the initial conditions (if multiple shooting is employed)

$$
\begin{bmatrix} W_{11}[x(2j)] & W_{12}[x(2j)] \\ W_{21}[x(2j)] & W_{22}[x(2j)] \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{F2}
$$

where $x(2j)$ are the starting points for the integration ($j = 1, \ldots, N-1$ if $2N$ is the number of intervals). For edge intervals ($j = 0$ or $j = N$) the analysis is similar.

We now have obtained two uncoupled initial value problems

$$
W'_{11} = A_{12}W_{21}
$$

$$
\tag{F3}
$$

$$
W'_{21} = - A_{12}W_{11}
$$

$$
W_{11}[x(2j)] = I
$$

$$
\tag{F4}
$$

$$
W_{21}[x(2j)] = 0
$$

and

$$
W'_{22} = - A_{12}W_{12}
$$

$$
\tag{F5}
$$

$$
W'_{12} = A_{12}W_{22}
$$

$$W_{22}[x(2j)] = I \tag{F6}$$

$$W_{12}[x(2j)] = O$$

Comparing the two initial value problems, it is seen that

$$W_{11} = W_{22} \tag{F7}$$

and

$$W_{21} = -W_{12} \tag{F8}$$

Consequently, $W(\bar{x})$ can be written as follows:

$$W(\bar{x}) = \begin{bmatrix} W_{11}(\bar{x}) & -W_{21}(\bar{x}) \\ W_{21}(\bar{x}) & W_{11}(\bar{x}) \end{bmatrix} \tag{F9}$$

Therefore only eight solutions of the variational equations are required.

shell geometry:

    radius   R   = 2.67   in.

    thickness t   = 0.0267 in.

    length   L   = 1.88798 in.   or   L   = 5.34 in.

    geometry parameters:

        $R/t = 100$

        $L/R = 0.70711$ or $L/R = 2$

        $\bar{Z} = 50$ or $\bar{Z} = 400$

laminate geometry:

    3 layers:

        layer thickness: $h_1 =$ $h_2 =$ $h_3 = 0.0089$ in.

        layer orientation: $\theta_1 = 30°$ $\theta_2 = 0°$ $\theta_3 = -30°$

layer properties:

    composite material: glass-epoxy

        modulus of elasticity 1-direction $E_{11} = 0.583\ 10^7$ psi

        modulus of elasticity 2-direction $E_{22} = 0.242\ 10^7$ psi

        major Poisson's ratio $\nu_{12} = 0.363$

        shear modulus 12-direction $G_{12} = 0.668\ 10^6$ psi

laminate properties:

$$A^* = (1/Et)\begin{bmatrix} 1.3751 & -0.7582 & 0.0000 \\ -0.7582 & 2.6292 & 0.0000 \\ 0.0000 & 0.0000 & 4.8885 \end{bmatrix} \quad B^* = (t/2c)\begin{bmatrix} 0.0000 & 0.0000 & 0.1785 \\ 0.0000 & 0.0000 & -0.0096 \\ 0.7430 & 0.1965 & 0.0000 \end{bmatrix}$$

$$C^* = (t/2c)\begin{bmatrix} 0.0000 & 0.0000 & -0.7430 \\ 0.0000 & 0.0000 & -0.1965 \\ -0.1785 & 0.0096 & 0.0000 \end{bmatrix} \quad D^* = D\begin{bmatrix} 0.5634 & 0.2214 & 0.0000 \\ 0.2214 & 0.3898 & 0.0000 \\ 0.0000 & 0.0000 & 0.1856 \end{bmatrix}$$

where $c^2 = 3(1-\nu^2)$, $D = Et^3/4c^2$, $\nu = 0.363$, and $E = 0.583\ 10^7$ psi

Table 1: Booton's anisotropic composite shell

	AXIAL COMPRESSION	
B.C.	$\bar{z} = 50$	$\bar{z} = 400$
SS-1	0.23058 (0)	0.23145 (0)
SS-2	0.23910 (1)	0.23498 (1)
SS-3	0.37096 (8)	0.39303 (7)
SS-4	0.39349 (8)	0.40556 (8)
C-1	0.41815 (7)	0.40790 (6)
C-2	0.43689 (8)	0.41178 (6)
C-3	0.41993 (8)	0.40865 (6)
C-4	0.43835 (8)	0.41194 (6)

Table 2: Nondimensional buckling loads λ_c (circumferential wave numbers) under axial compression for different boundary conditions (Booton's shell, $\bar{z} = 50$ and $\bar{z} = 400$)

	HYDROSTATIC PRESSURE	
B.C.	$\bar{z} = 50$	$\bar{z} = 400$
SS-1	0.10382 (9)	0.03646 (6)
SS-2	0.12367 (10)	0.04831 (7)
SS-3	0.10985 (10)	0.03795 (6)
SS-4	0.13360 (11)	0.05046 (7)
C-1	0.12933 (10)	0.04036 (7)
C-2	0.14739 (11)	0.05141 (7)
C-3	0.13019 (10)	0.04042 (7)
C-4	0.14984 (11)	0.05188 (7)

Table 3: Nondimensional buckling loads $\bar{p}_c$ (circumferential wave numbers) under hydrostatic pressure for different boundary conditions (Booton's shell, $\bar{z} = 50$ and $\bar{z} = 400$)

	COUNTER-CLOCKWISE TORSION	
B.C.	$\bar{z} = 50$	$\bar{z} = 400$
SS-1	0.25924 (10)	0.13182 (7)
SS-2	0.27846 (10)	0.14391 (8)
SS-3	0.27989 (11)	0.13918 (8)
SS-4	0.30249 (11)	0.15012 (8)
C-1	0.30858 (11)	0.14062 (8)
C-2	0.32193 (11)	0.14995 (8)
C-3	0.31878 (11)	0.14394 (8)
C-4	0.33205 (11)	0.15287 (8)

Table 4: Nondimensional buckling loads $\bar{\tau}_c$ (circumferential wave numbers) under counter-clockwise torsion for different boundary conditions (Booton's shell, $\bar{Z} = 50$ and $\bar{Z} = 400$)

	CLOCKWISE TORSION	
B.C.	$\bar{z} = 50$	$\bar{z} = 400$
SS-1	-0.21964 (9)	-0.13100 (7)
SS-2	-0.23486 (9)	-0.14038 (8)
SS-3	-0.23849 (10)	-0.13820 (8)
SS-4	-0.24878 (10)	-0.14524 (8)
C-1	-0.25331 (10)	-0.13841 (8)
C-2	-0.26221 (10)	-0.14536 (8)
C-3	-0.25991 (10)	-0.14039 (8)
C-4	-0.27100 (10)	-0.14744 (8)

Table 5: Nondimensional buckling loads $\bar{\tau}_c$ (circumferential wave numbers) under clockwise torsion for different boundary conditions (Booton's shell, $\bar{Z} = 50$ and $\bar{Z} = 400$)

AXIAL COMPRESSION			
$\bar{z}$	B.C.	nonlinear prebuckling	membrane prebuckling
50	SS-3	0.37096 (8)	0.42465 (6)
	C-4	0.43835 (8)	0.46257 (7)
400	SS-3	0.39303 (7)	0.40929 (6)
	C-4	0.41194 (6)	0.41224 (6)

Table 6: Comparison of nondimensional buckling loads λ_c (circumferential wave numbers) under axial compression for nonlinear and membrane prebuckling analysis (Booton's shell, $\bar{z} = 50$ and $\bar{z} = 400$, SS-3 and C-4 boundary conditions)

HYDROSTATIC PRESSURE			
$\bar{z}$	B.C.	nonlinear prebuckling	membrane prebuckling
50	SS-3	0.10985 (10)	0.11710 (10)
	C-4	0.14984 (11)	0.15341 (11)
400	SS-3	0.03795 (6)	0.03808 (6)
	C-4	0.05188 (7)	0.05146 (7)

Table 7: Comparison of nondimensional buckling loads $\bar{p}_c$ (circumferential wave numbers) under hydrostatic pressure for nonlinear and membrane prebuckling analysis (Booton's shell, $\bar{z} = 50$ and $\bar{z} = 400$, SS-3 and C-4 boundary conditions)

COUNTER-CLOCKWISE TORSION			
$\bar{z}$	B.C.	nonlinear prebuckling	membrane prebuckling
50	SS-3	0.27989 (11)	0.28490 (11)
	C-4	0.33205 (11)	0.33205 (11)
400	SS-3	0.13918 (8)	0.13911 (8)
	C-4	0.15287 (8)	0.15287 (8)

Table 8: Comparison of nondimensional buckling loads $\bar{\tau}_c$ (circumferential wave numbers) under counter-clockwise torsion for nonlinear and membrane prebuckling analysis (Booton's shell, $\bar{z} = 50$ and $\bar{z} = 400$, SS-3 and C-4 boundary conditions)
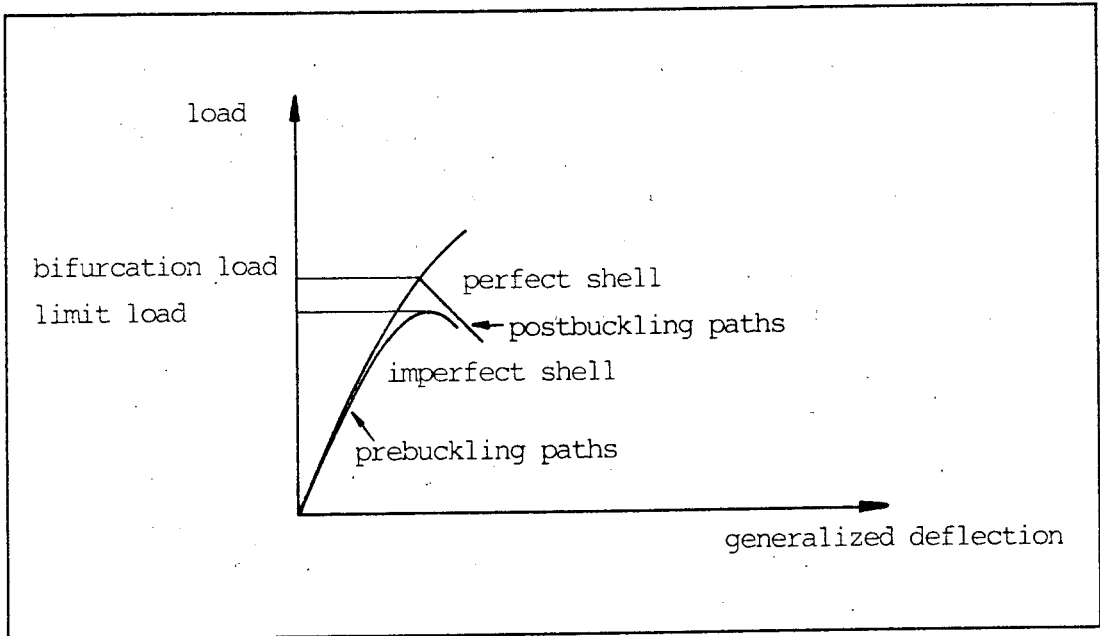
Fig. 1: Fundamental concepts in buckling analysis
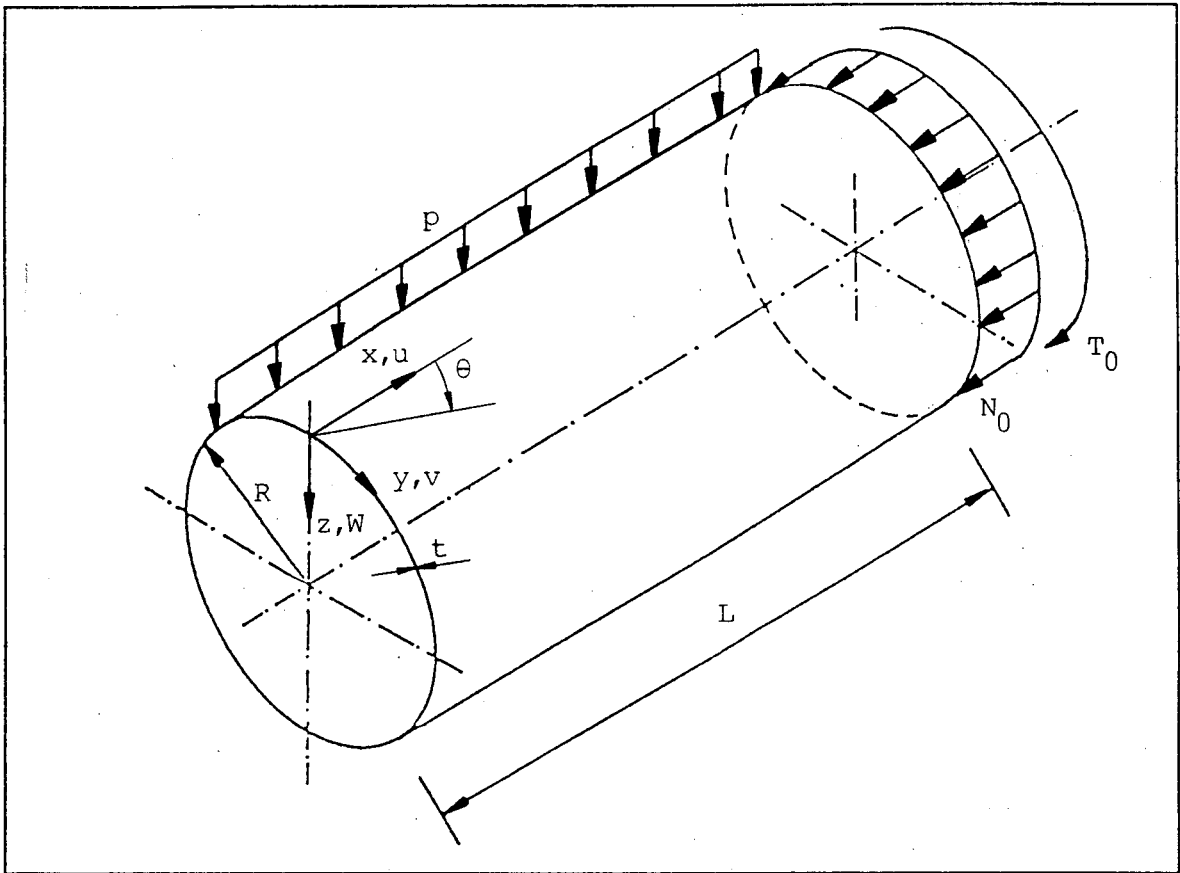
Fig. 2: Cylinder geometry and applied loading
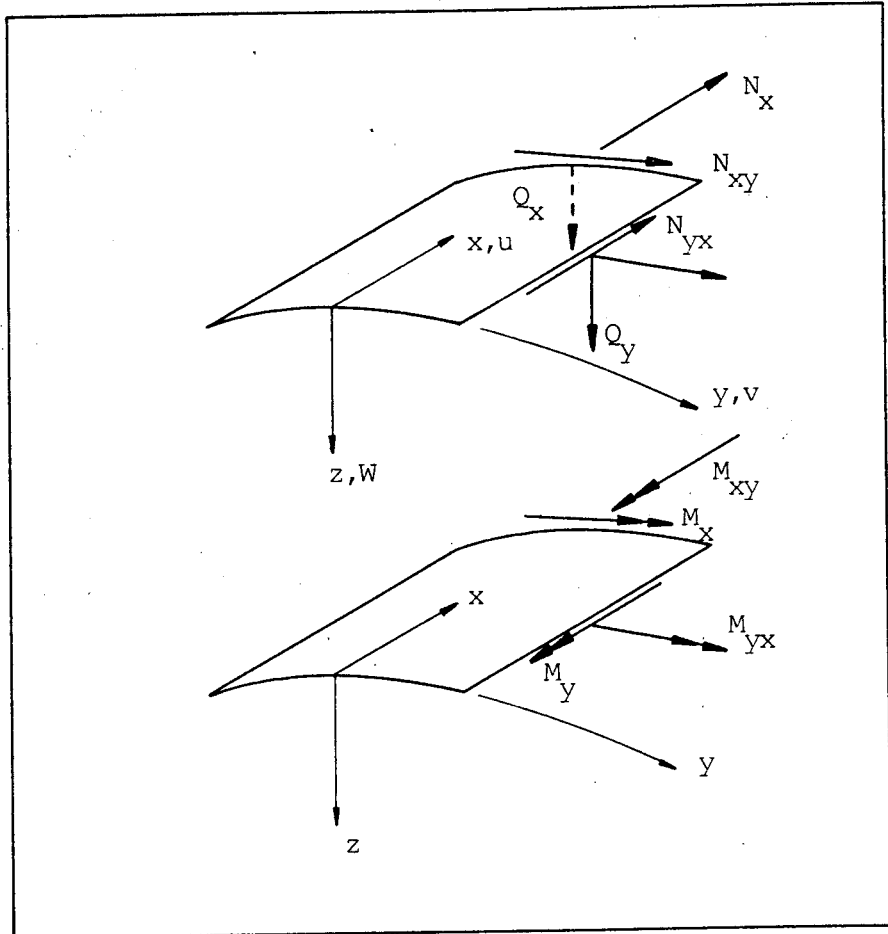
Fig. 3: Stress and moment resultants
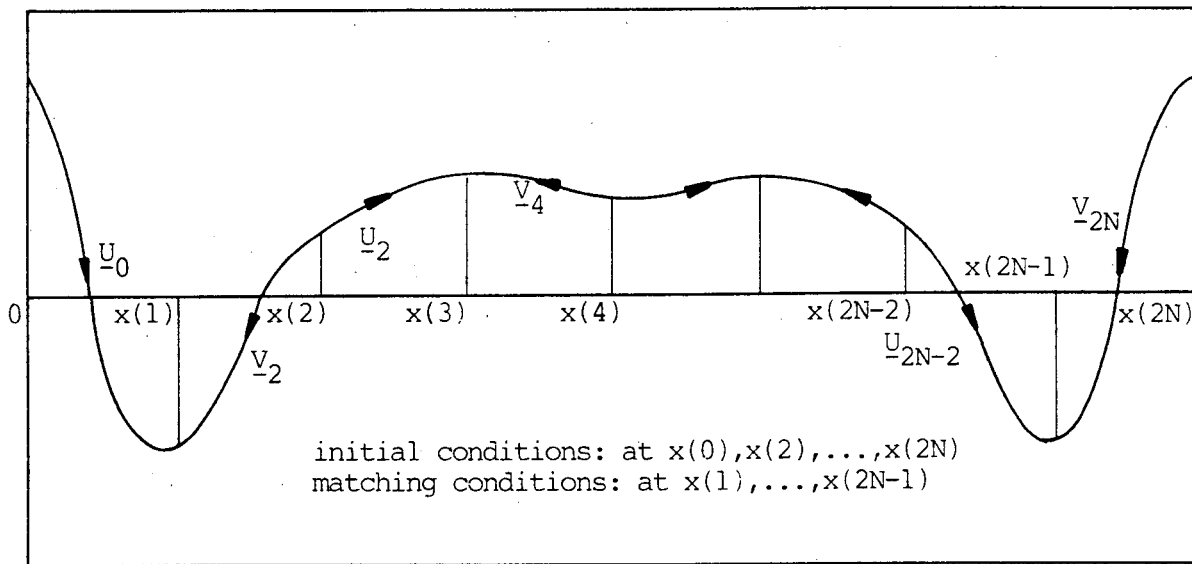
Fig. 4: Parallel shooting over 2N intervals
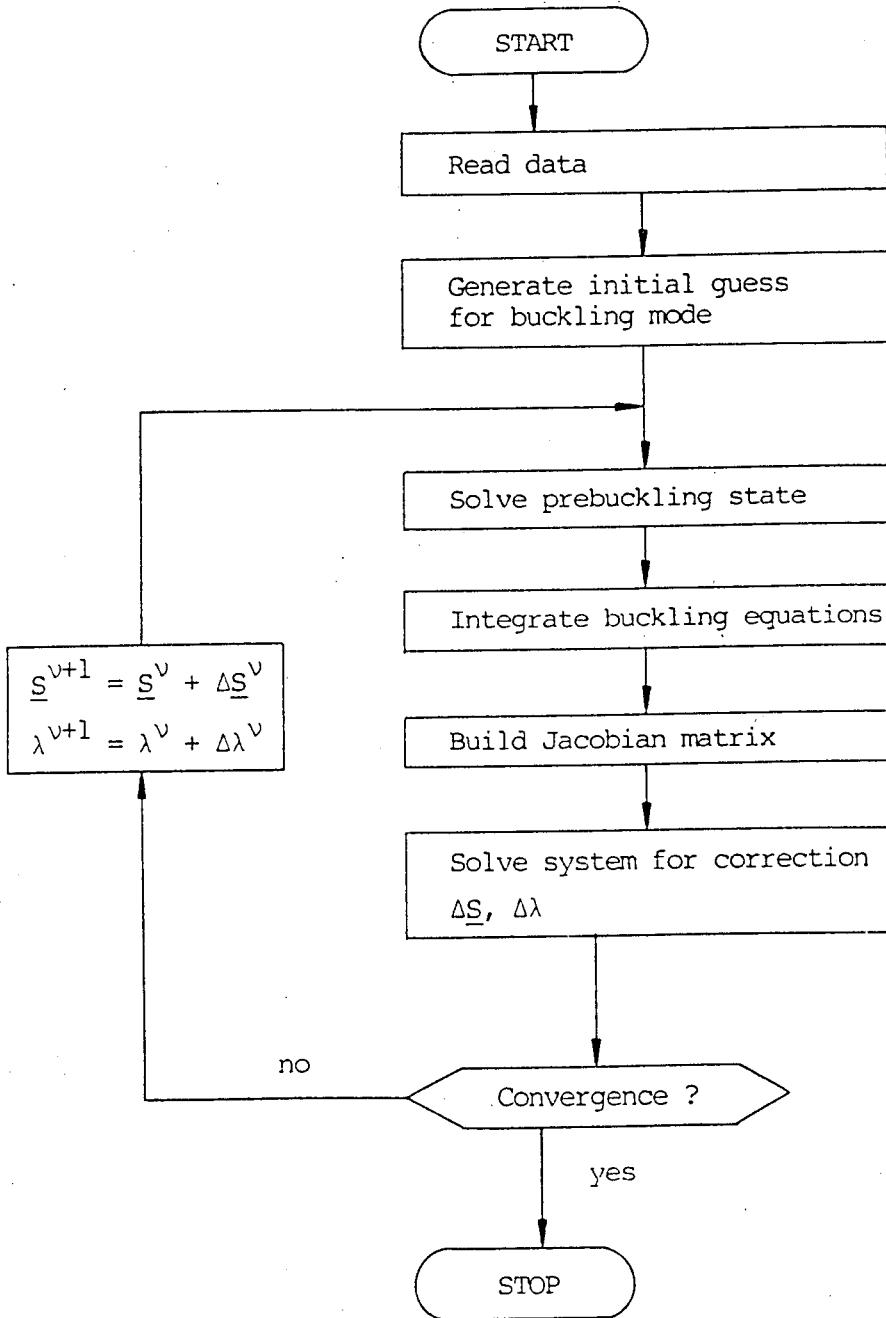
Fig. 5: Flow diagram of the computer program

a) Prebuckling shapes

b) Buckling modes

$\bar{Z} = 50$
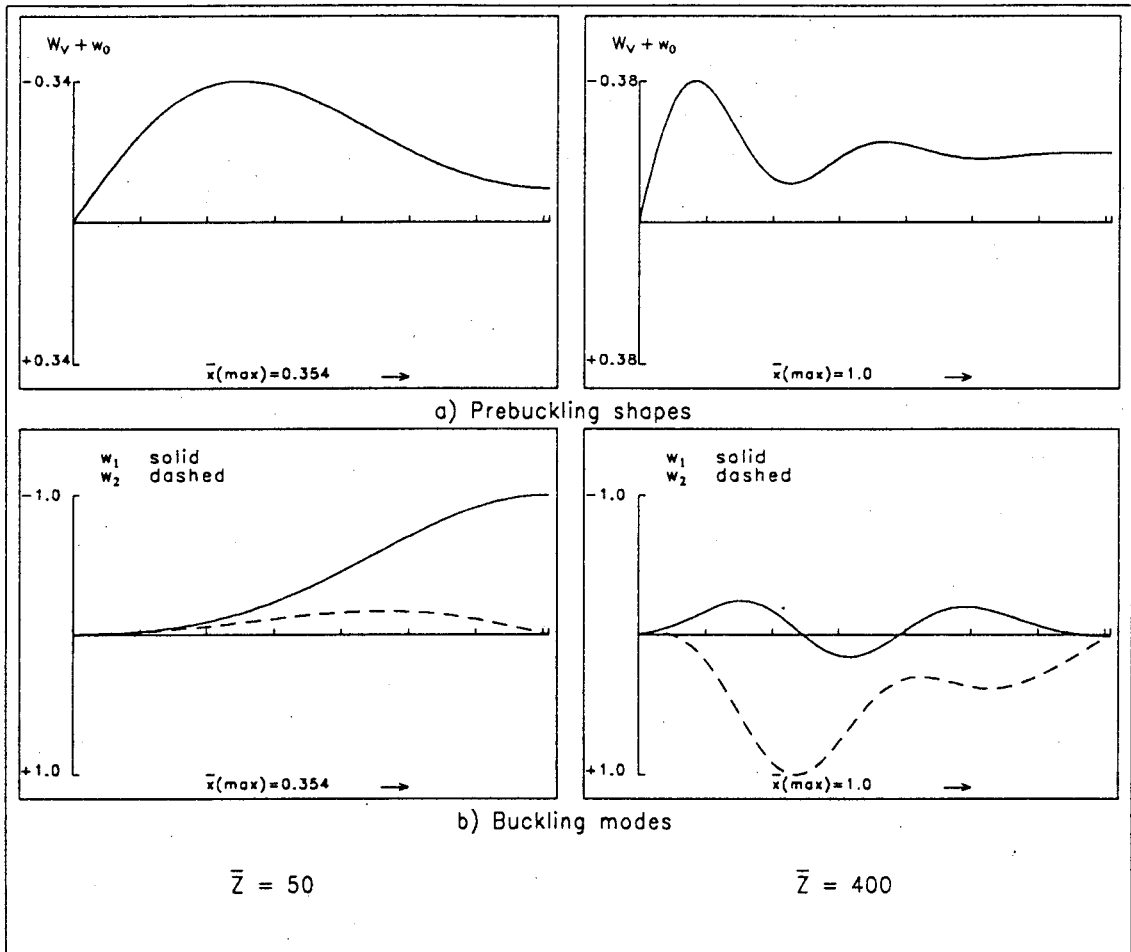
$\bar{Z} = 400$

$\bar{Z} = 50: \lambda_c = 0.37096$ (8) $\bar{Z} = 400: \lambda_c = 0.39303$ (7)

Fig. 6: Prebuckling shapes and buckling modes under axial compression for SS-3 boundary conditions (Booton's shell)

a) Prebuckling shapes

b) Buckling modes

$\bar{Z} = 50$
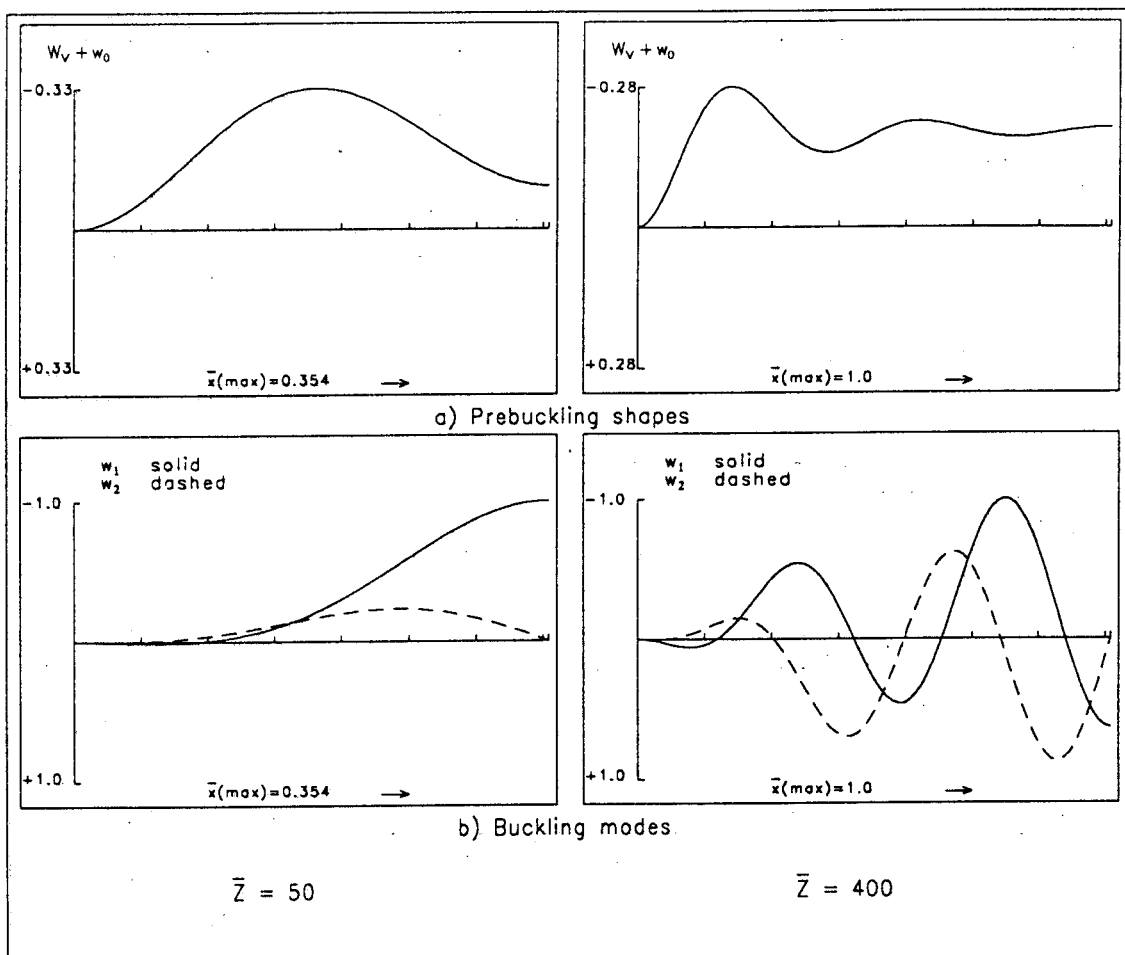
$\bar{Z} = 400$

$\bar{Z} = 50: \lambda_c = 0.43835$ (8) $\bar{Z} = 400: \lambda_c = 0.41194$ (6)

Fig. 7: Prebuckling shapes and buckling modes under axial compression for C-4 boundary conditions (Booton's shell)
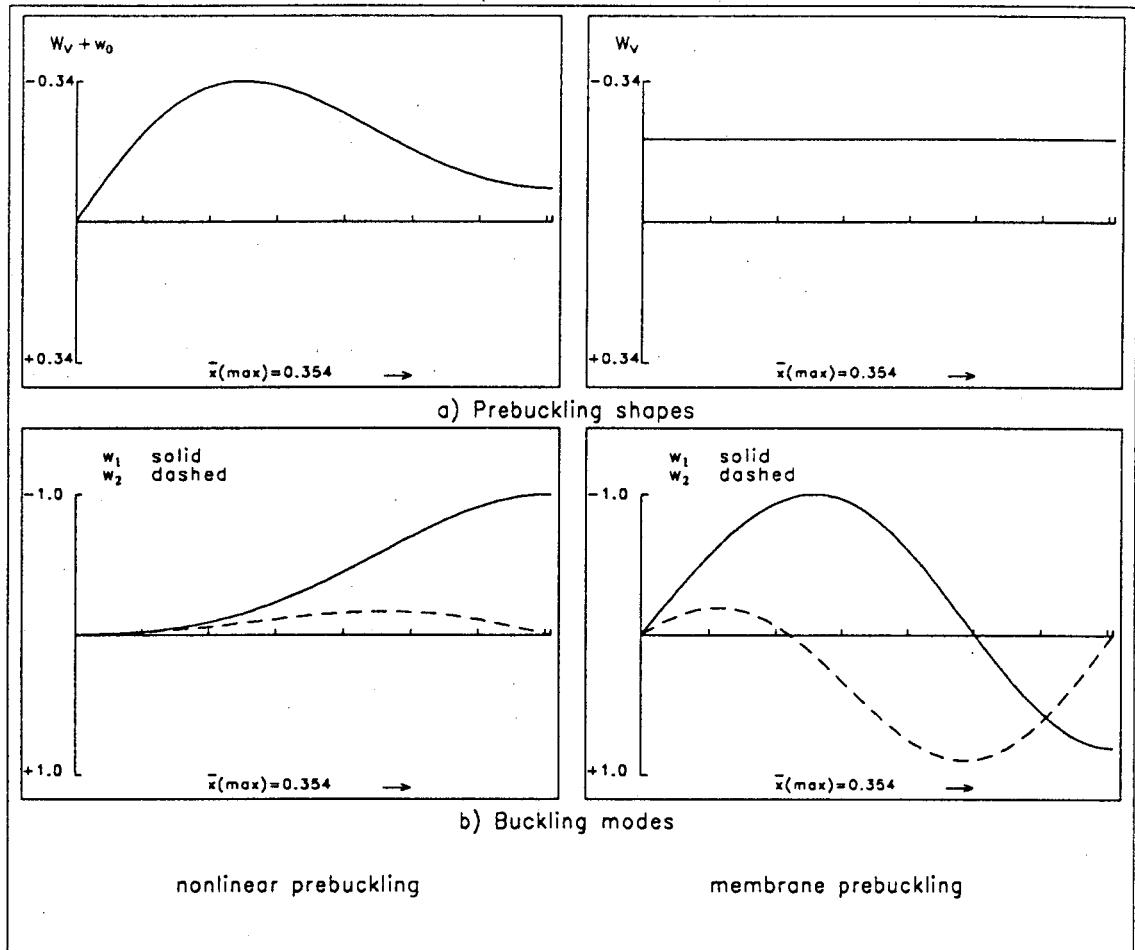
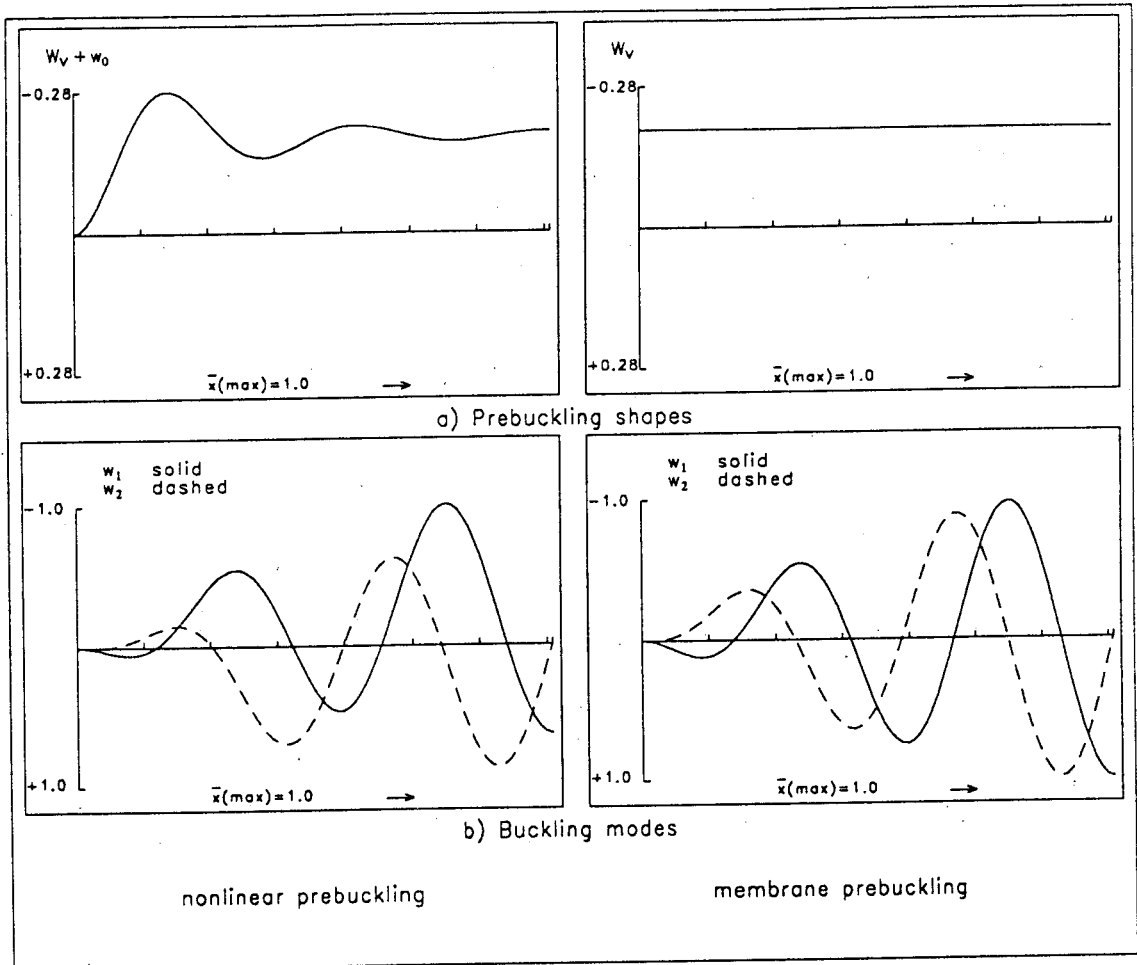nonlinear: λ_c = 0.37096 (8)   membrane: λ_c = 0.42465 (6)

Fig. 8: Comparison of buckling modes under axial compression for nonlinear and membrane prebuckling analysis (Booton's shell, $\bar{Z}$ = 50, SS-3 boundary conditions)

a) Prebuckling shapes

b) Buckling modes

nonlinear prebuckling   membrane prebuckling
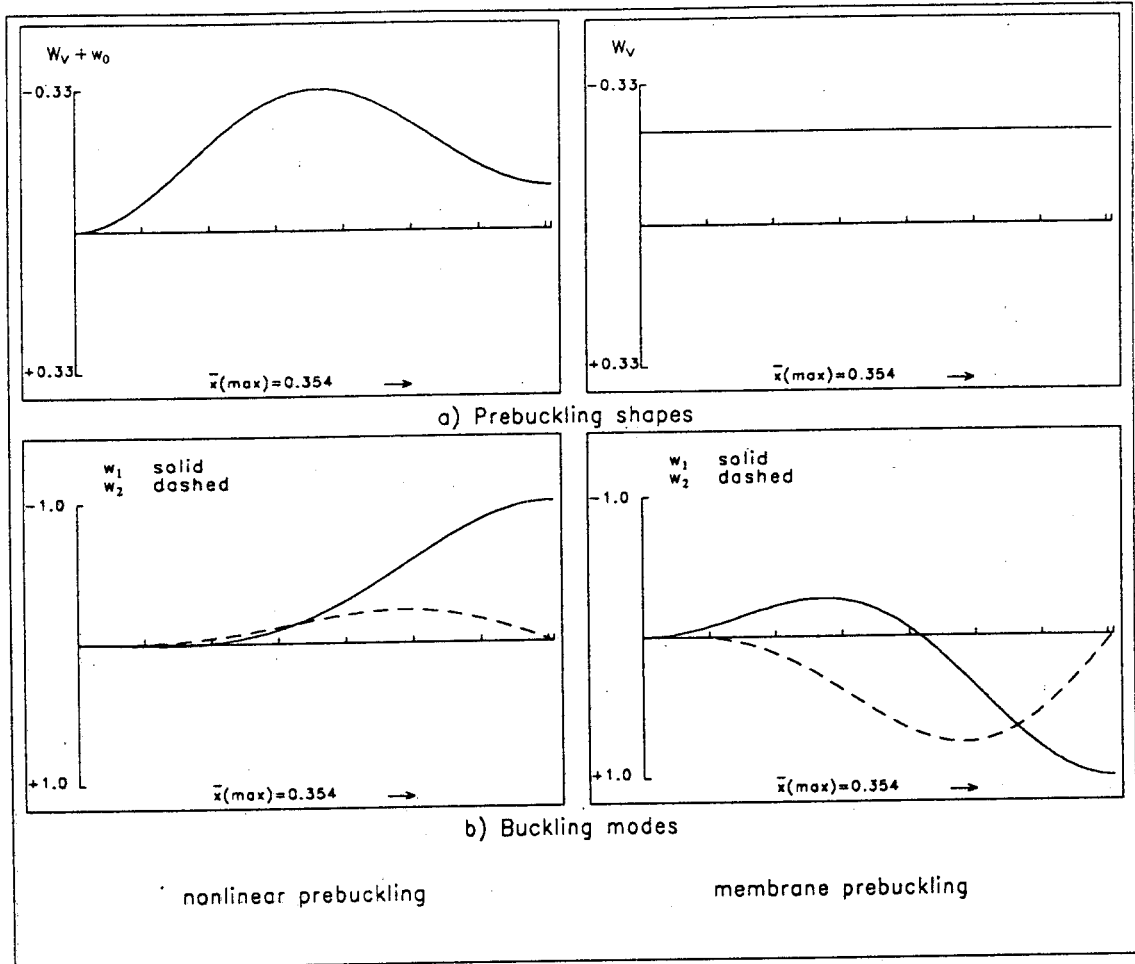
nonlinear: λ_c = 0.43835 (8)   membrane: λ_c = 0.46257 (7)

Fig. 9: Comparison of buckling modes under axial compression for nonlinear and membrane prebuckling analysis (Booton's shell, $\bar{Z} = 50$, C-4 boundary conditions)

a) Prebuckling shapes

b) Buckling modes

nonlinear prebuckling   membrane prebuckling
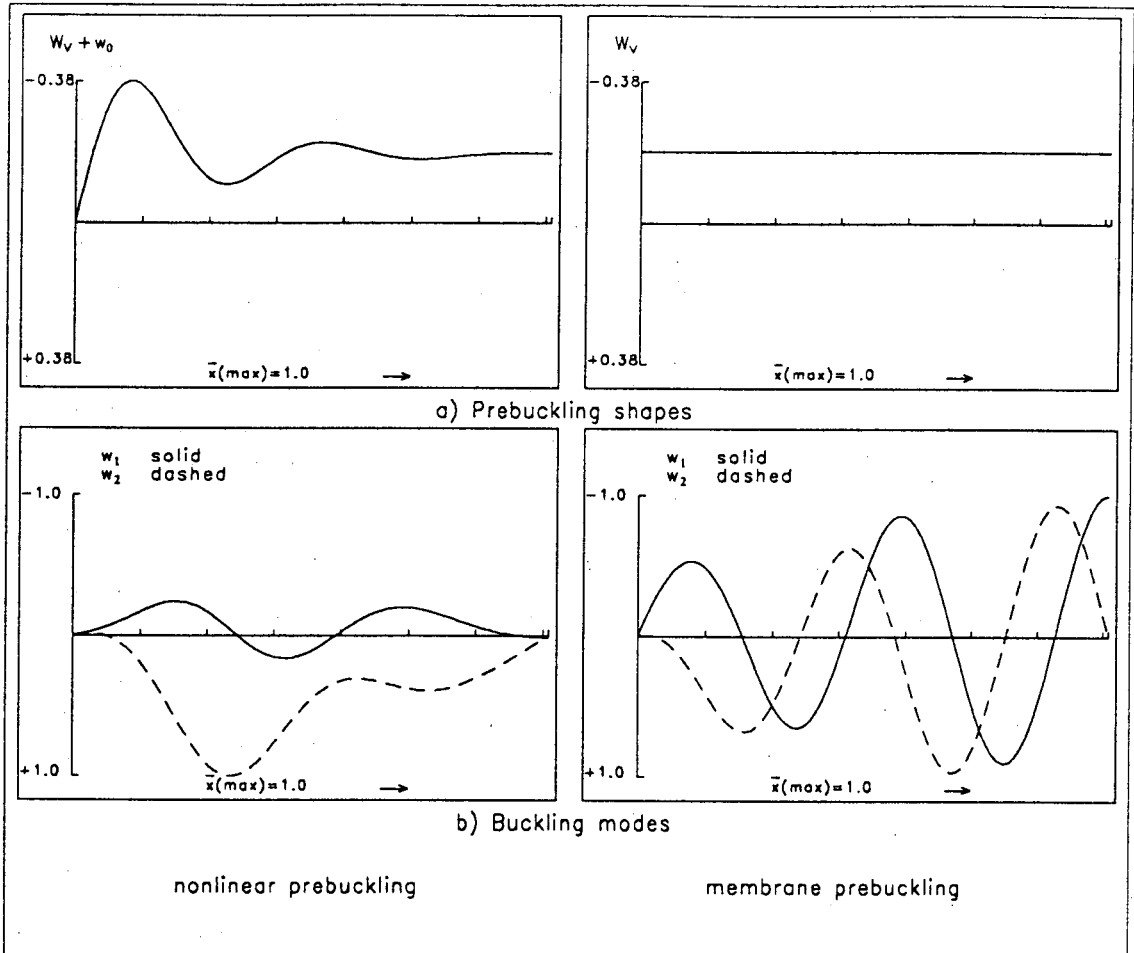
nonlinear: λ_c = 0.39303 (7)   membrane: λ_c = 0.40929 (6)

Fig. 10: Comparison of buckling modes under axial compression for nonlinear and membrane prebuckling analysis (Booton's shell, $\bar{Z}$ = 400, SS-3 boundary conditions)

a) Prebuckling shapes

b) Buckling modes

nonlinear prebuckling   membrane prebuckling

nonlinear: $\lambda_c = 0.41194$ (6)   membrane: $\lambda_c = 0.41224$ (6)

Fig. 11: Comparison of buckling modes under axial compression for nonlinear and membrane prebuckling analysis (Booton's shell, $\bar{Z} = 400$, C-4 boundary conditions)

a) Prebuckling shapes

b) Buckling modes

$\bar{Z} = 50$
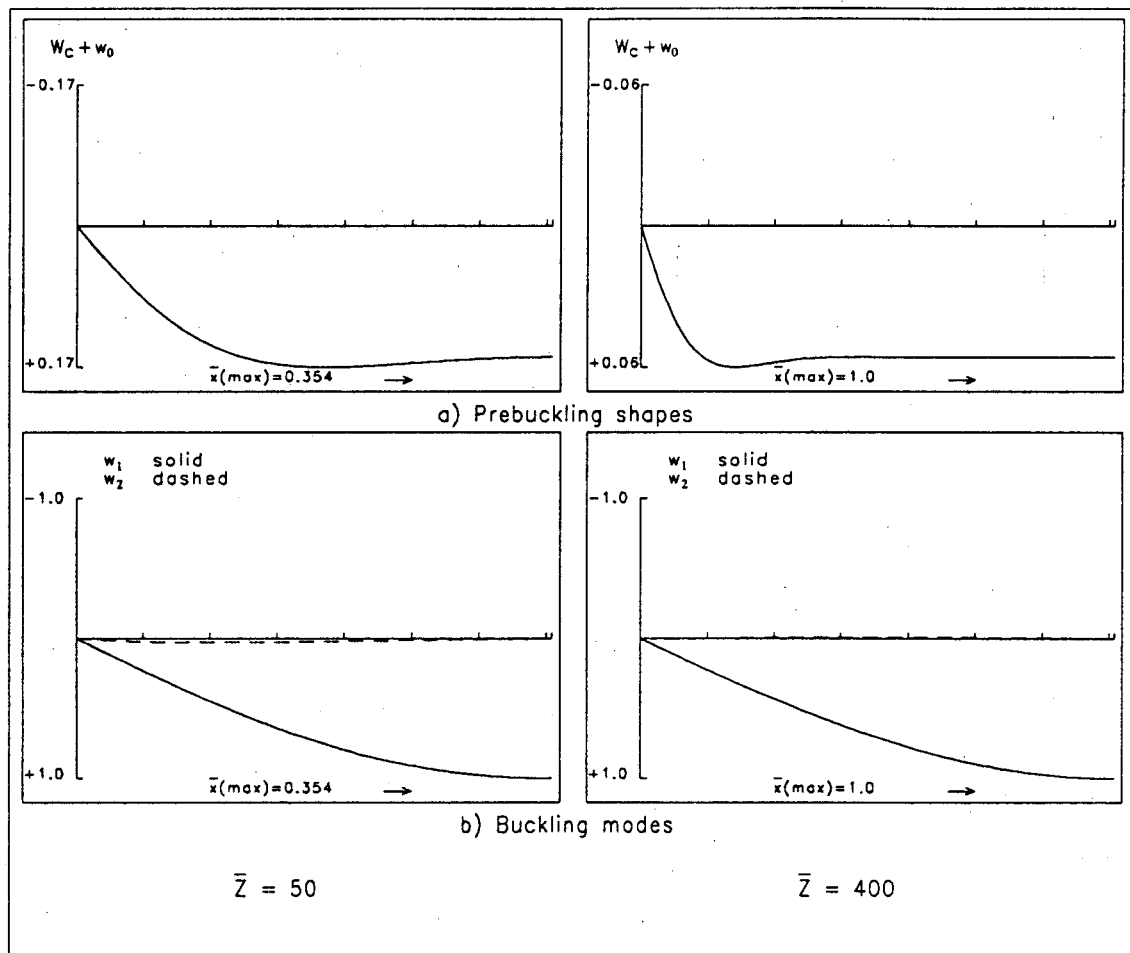
$\bar{Z} = 400$

$\bar{z} = 50: \bar{p}_c = 0.10985 \ (10)$ $\bar{z} = 400: \bar{p}_c = 0.03795 \ (6)$

Fig. 12: Prebuckling shapes and buckling modes under hydrostatic pressure for SS-3 boundary conditions (Booton's shell)

a) Prebuckling shapes

b) Buckling modes

$\bar{Z} = 50$
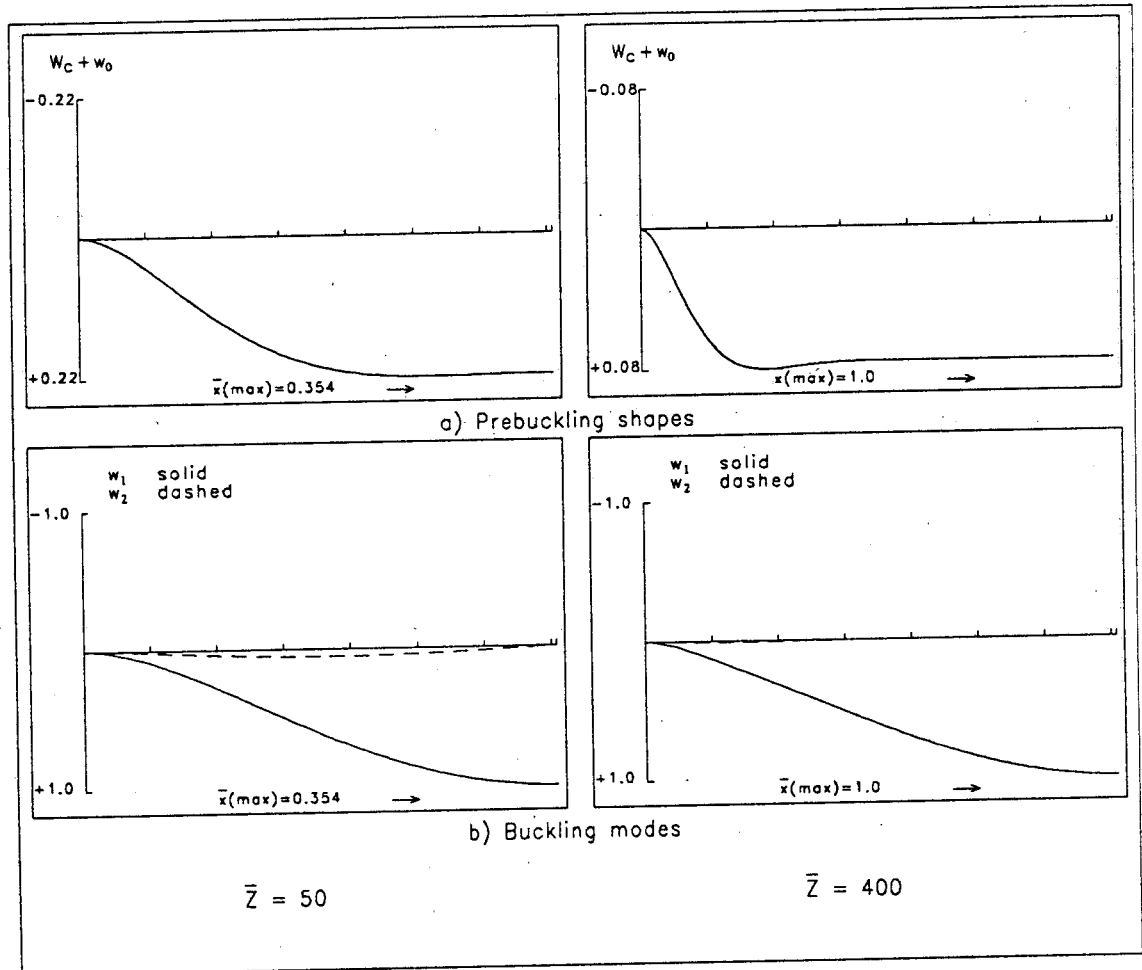
$\bar{Z} = 400$

$\bar{z} = 50: \bar{p}_c = 0.14984$ (11) $\bar{z} = 400: \bar{p}_c = 0.05188$ (7)

Fig. 13: Prebuckling shapes and buckling modes under hydrostatic pressure for C-4 boundary conditions (Booton's shell)

a) Prebuckling shapes

b) Buckling modes

$\bar{Z} = 50$
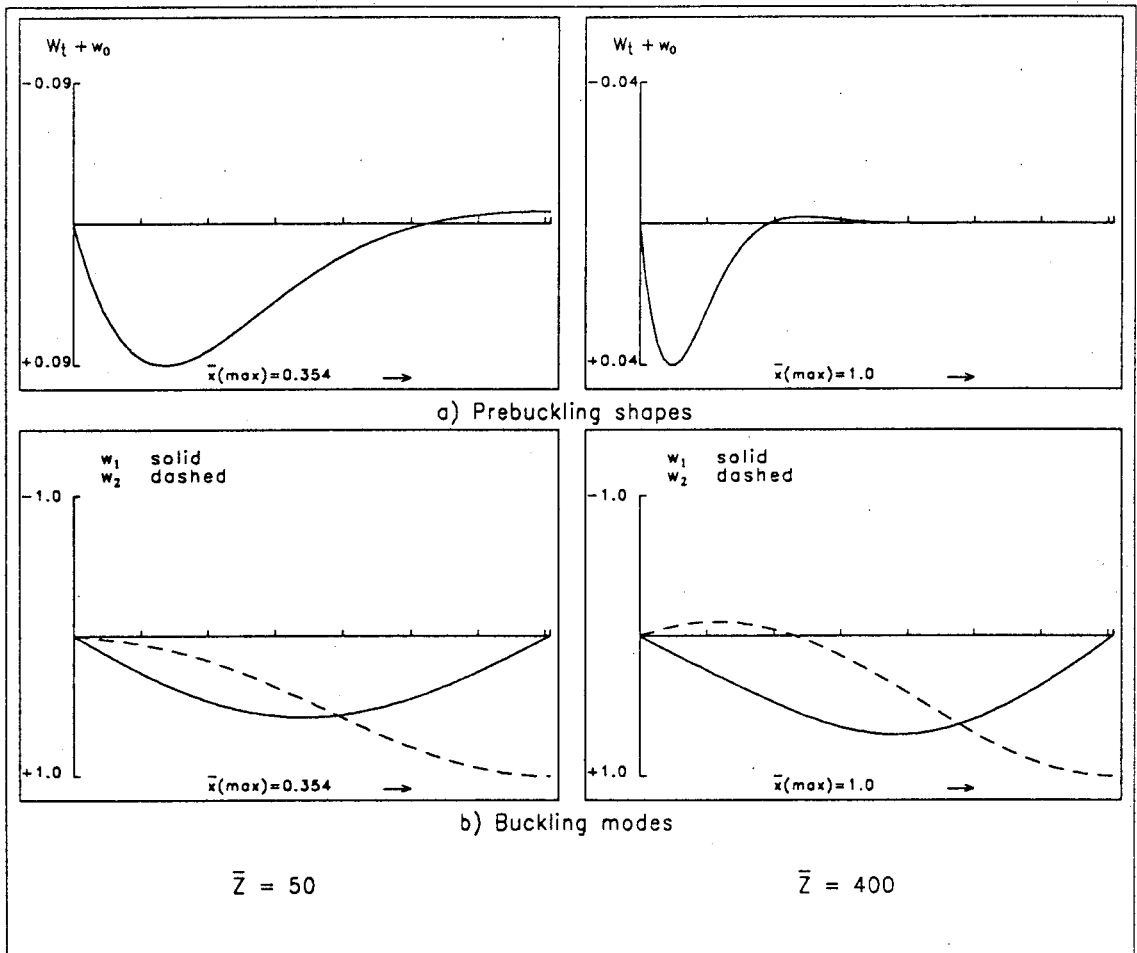
$\bar{Z} = 400$

$\bar{Z} = 50: \bar{\tau}_c = 0.27989$ (11) $\bar{Z} = 400: \bar{\tau}_c = 0.13918$ (8)
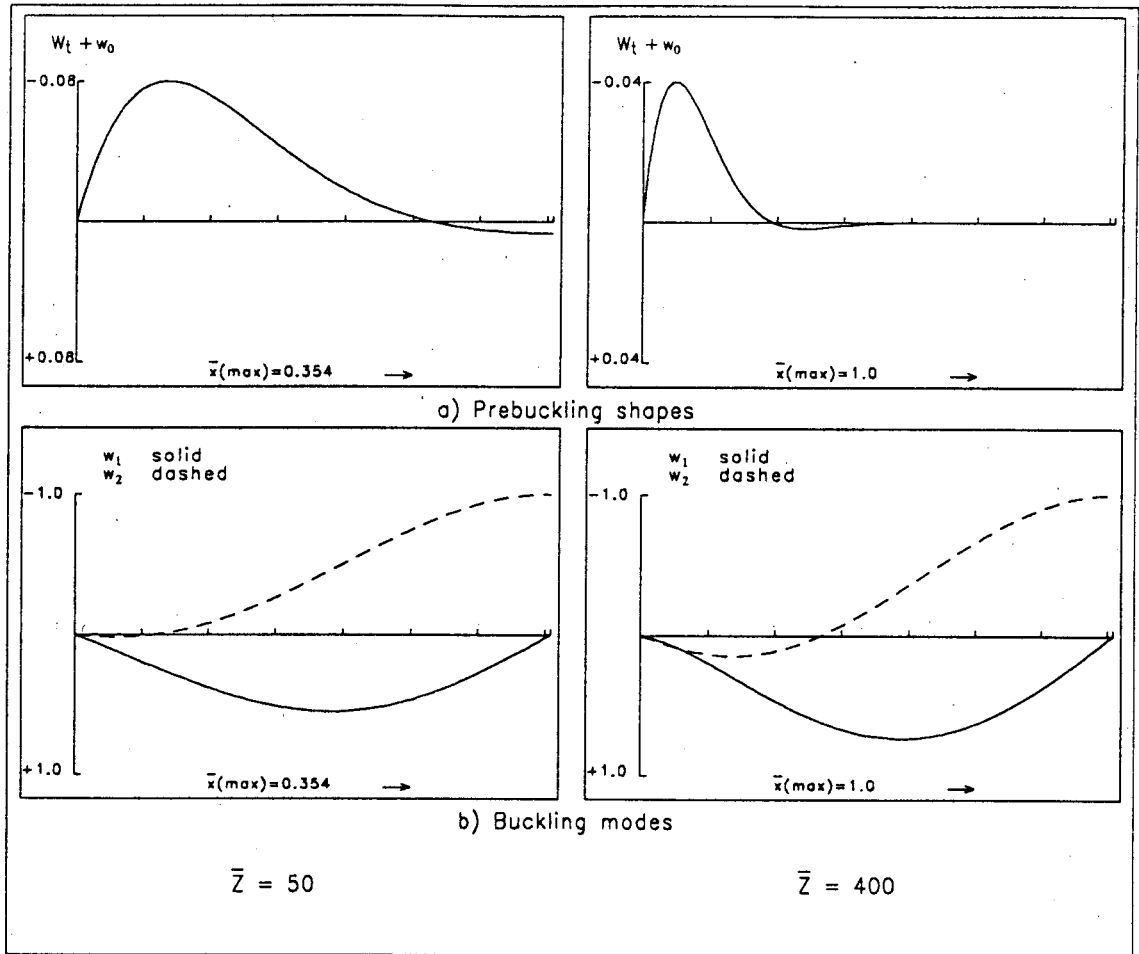
Fig. 14: Prebuckling shapes and buckling modes under counter-clockwise torsion for SS-3 boundary conditions (Booton's shell)

a) Prebuckling shapes

b) Buckling modes

$\bar{Z} = 50$
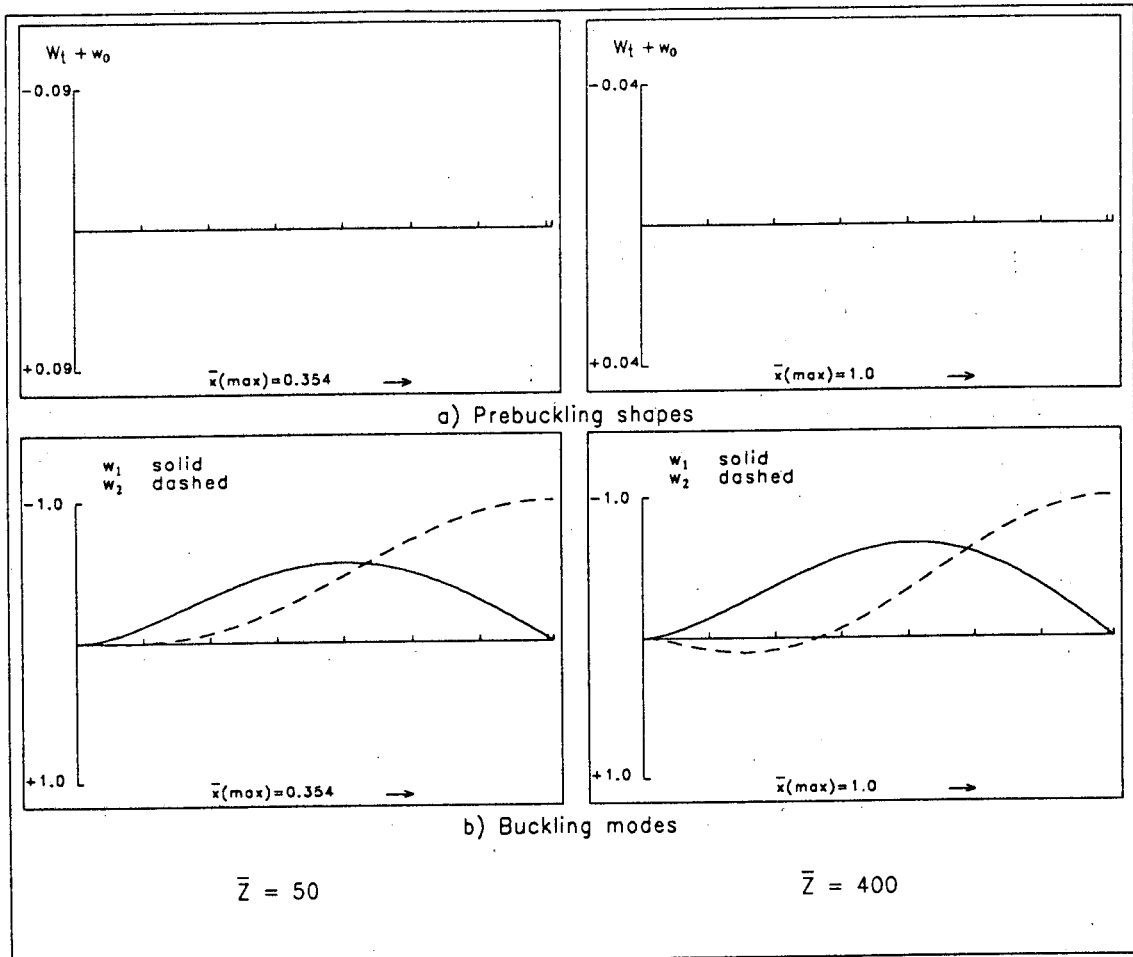
$\bar{Z} = 400$

$\bar{Z} = 50: \bar{\tau}_c = 0.33205$ (11) $\bar{Z} = 400: \bar{\tau}_c = 0.15287$ (8)

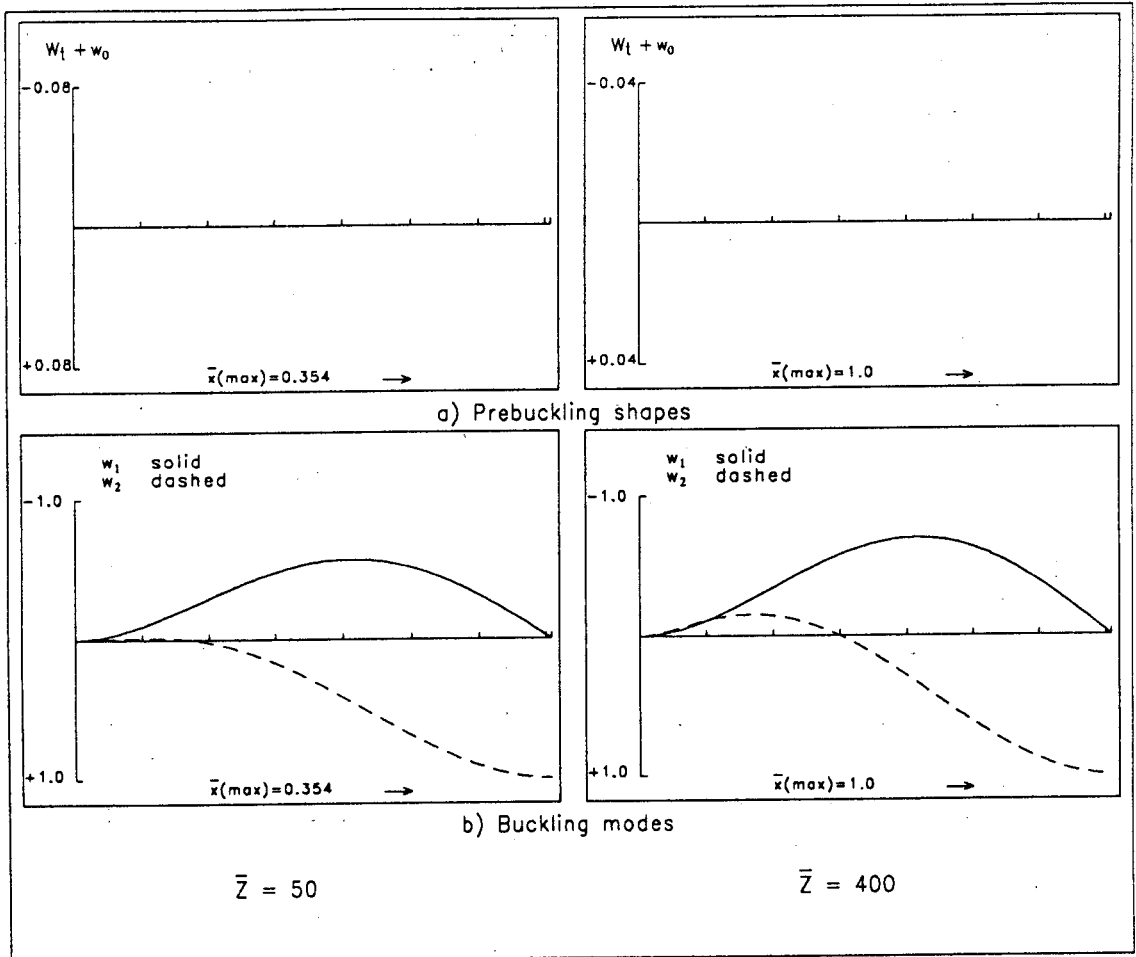Fig. 15: Prebuckling shapes and buckling modes under counter-clockwise torsion for C-4 boundary conditions (Booton's shell)

a) Prebuckling shapes

b) Buckling modes

$\bar{Z} = 50$ $\bar{Z} = 400$

$\bar{Z} = 50: \ \bar{\tau}_c = -0.23849 \ (10)$ $\bar{Z} = 400: \ \bar{\tau}_c = -0.13820 \ (8)$

Fig. 16: Prebuckling shapes and buckling modes under clockwise torsion for SS-3 boundary conditions (Booton's shell)

a) Prebuckling shapes

b) Buckling modes

$\bar{Z} = 50$

$\bar{Z} = 400$

$\bar{Z} = 50: \ \bar{\tau}_c = -0.27100 \ (10)$ $\bar{Z} = 400: \ \bar{\tau}_c = -0.14744 \ (8)$

Fig. 17: Prebuckling shapes and buckling modes under clockwise torsion for C-4 boundary conditions (Booton's shell)