Delft University of Technology

MSc Programme Applied Mathematics

Master's Thesis

Multilinear Fourier multipliers of a locally compact group

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May 17, 2024

Abstract

In [18], Haagerup proves several important results about the weak amenability of locally compact groups. Among these, is the result that a lattice in a second-countable, unimodular, locally compact group is weakly amenable if and only if the surrounding group itself is weakly amenable. A key ingredient in his proof is a method of using (linear) completely bounded Fourier multipliers on the lattice to construct (linear) completely bounded Fourier multipliers on the surrounding group. We use a similar approach to construct multilinear completely bounded Fourier multipliers on the group from multilinear completely bounded Fourier multipliers on the lattice. Our construction is both bounded in the norm of completely bounded Fourier multipliers and preserves uniform convergence on compact sets for bounded nets. We also prove an equivalent characterization of weak amenability where the Fourier algebra is replaced by the space of continuous and compactly supported n-linear Fourier multiplier symbols.

Contents

1 Introduction

In this thesis we study multilinear Fourier multipliers on locally compact groups. As we will see, these can be viewed as a generalization of the more widely known (linear) classical Fourier multipliers on $\mathbb{R}$ or $\mathbb{Z}$.

Linear Fourier multipliers on locally compact groups have been studied longer than their multilinear counterparts. A general goal for this thesis was to attempt to generalize known results for linear Fourier multipliers to multilinear Fourier multipliers.

An important paper in the field of linear Fourier multipliers on locally compact groups was written by Uffe Haagerup back in 1986, but remained unpublished for a long time. An updated version of the paper was published posthumously in 2016 [18]. Haagerup's paper contains several important results involving a property that certain locally compact groups possess that came to be known as "weak amenability". As we shall see, this property can be formulated in terms of linear Fourier multipliers.

In this thesis, we have worked on generalizing some of the results in Haagerup's paper. We have generalized a construction in [18], where Fourier multipliers on a lattice are used to construct Fourier multipliers on the surrounding group. We have also attempted to generalize the definition of weak amenability in terms of multilinear Fourier multipliers. Our proposed generalization turned out to instead be an equivalent characterization of weak amenability.

1.1 Locally compact groups

A locally compact group G is a group equipped with a sufficiently well-behaved topology (i.e. a specification of which subsets of G are open, see Definition 3.2) that is connected to the group structure on G. More precisely, we want that the group multiplication $(x, y) \mapsto xy$ and inversion $x \mapsto x^{-1}$ are continuous functions with respect to the chosen topology. Under these conditions, G is called a topological group (Definition 3.13). The additional requirements imposed on the topology to call G a locally compact group are as follows:

1. For every two distinct points $x, y \in G$, there exist disjoint open neighbourhoods U_x of x and U_y of y.

2. For each point $x \in X$ and each open neighbourhood U of x, there exists a compact neighbourhood K_x of x with $K_x \subseteq U$.

The first of the above requirements is called the Hausdorff condition (Definition 3.15) and is frequently imposed on topologies to ensure that certain desired properties hold (Remark 3.12). Intuitively it can be seen as a way to ensure that distinct points are sufficiently separated from each other and it is one of several different conditions called "separation axioms" that can be imposed on topologies. The second condition is known as local compactness and is often imposed in conjunction with the Hausdorff condition. Local compactness intuitively ensures that locally, i.e. sufficiently close to a point, one can work within a compact space.

One nice thing about locally compact groups is that they can always be equipped with a Haar measure (Definition 3.20) μ. μ is a Radon measure (see Definition 3.19), which means that the measure of a measurable set can be approximated by the measure of larger open sets. Moreover, the measure of an open set can be approximated by the measure of smaller compact sets. A Haar measure differs from general Radon measures by having the additional property that $\mu(xE) = \mu(E)$ for any $x \in G$ and measurable $E \subseteq G$. So, multiplying a set from the left with an element of G does not change the measure of the set. A locally compact group always has a Haar measure and any two Haar measures are positive scalar multiples of each other (Theorem 3.2). For this reason, the choice of Haar measure is usually not very important. In some cases a locally compact group has a Haar measure for which the measure of a set also does not change when it is multiplied from the right with an element of G. Such a locally compact group is called unimodular (Definition 3.23).

Two well-known examples of locally compact groups are the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$ with addition as the group operation. Both of these groups are unimodular and the corresponding Haar measures are the Lebesgue measure and the counting measure, respectively. Lie groups form an important class of locally compact groups. Just like for other measure spaces (such as $\mathbb{R}$ and $\mathbb{Z}$), one can define L^p-spaces in the usual way. On locally compact groups equipped with a Haar measure μ, one can also define convolution:

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y) = \int_G f(xy)g(y^{-1})d\mu(y). \tag{1}$$

This generalizes convolution on $\mathbb{R}$:

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy \tag{2}$$

and $\mathbb{Z}$:

$$(f * g)(n) = \sum_{m\in\mathbb{Z}} f(m)g(n-m). \tag{3}$$

While convolution on a locally compact group is in many ways similar to convolution on $\mathbb{R}$ or $\mathbb{Z}$, there are some differences. A notable difference is that while convolution on $\mathbb{R}$ and $\mathbb{Z}$ is commutative, the same is not true for a general locally compact group G. [17] gives a good introduction to locally compact groups.

1.2 Classical Fourier multipliers

Readers familiar with Fourier analysis on $\mathbb{R}$ or $\mathbb{Z}$ might have certain expectations when it comes to the term "Fourier multipliers". The way Fourier multipliers are defined in this thesis (see Definition 4.10) might seem very different at first glance. Nevertheless, the more abstract Fourier multipliers we consider in this thesis can be viewed as a generalization of the more widely known "classical" Fourier multipliers. Following Vergara [29], we illustrate the connection between classical and abstract Fourier multipliers by considering Fourier analysis on $\mathbb{Z}$.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the torus, which is yet another example of a (unimodular) locally compact group. Given a function $f \in L^1(\mathbb{T})$, one can define a function $\hat{f} : \mathbb{Z} \to \mathbb{C}$ given by

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta. \tag{4}$$

This defines the Fourier transform $\mathcal{F} : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$, $f \mapsto \hat{f}$, where $c_0(\mathbb{Z})$ is the space of all complex-valued sequences on $\mathbb{Z}$ that converge to 0 at $\pm\infty$. Let $A(\mathbb{Z}) = \mathcal{F}(L^1(\mathbb{T})) \subseteq c_0(\mathbb{Z})$. The Fourier transform is injective, so there must be an inverse map $\mathcal{F}^{-1} : A(\mathbb{Z}) \to L^1(\mathbb{T})$.

The Fourier transform restricted to $L^1(\mathbb{T}) \cap L^2(\mathbb{T})$ maps into $l^2(\mathbb{Z})$. This map extends to an (for appropriately chosen Haar measures) isometric, i.e. norm-preserving, isomorphism $L^2(\mathbb{T}) \to l^2(\mathbb{Z})$, which is also called the Fourier transform and written as $\mathcal{F} : f \mapsto \hat{f}$.

For $g, h \in L^2(\mathbb{T})$ we have that $gh \in L^1(\mathbb{T})$ with $\|gh\|_1 \leq \|g\|_2\|h\|_2$ and

$$\mathcal{F}(gh) = \hat{g} * \hat{h}. \tag{5}$$

Moreover, any $f \in L^1(\mathbb{T})$ can be written as $f = gh$ for certain $g, h \in L^2(\mathbb{T})$ with $\|f\|_1 = \|g\|_2\|h\|_2$. It follows that

$$A(\mathbb{Z}) = \{u * v : u, v \in l^2(\mathbb{Z})\}. \tag{6}$$

$A(\mathbb{Z})$ can be equipped with the norm $\|a\|_{A(\mathbb{Z})} = \inf \|u\|_2\|v\|_2$, where the infimum is taken over all $u, v \in l^2(\mathbb{Z})$ such that $a = u * v$. It follows that

$$\|f\|_1 = \|\mathcal{F}(f)\|_{A(\mathbb{Z})} \tag{7}$$

for all $f \in L^1(\mathbb{T})$.

An $L^1(\mathbb{T})$ to $L^1(\mathbb{T})$ Fourier multiplier operator (in the classical sense) is a bounded linear map $L^1(\mathbb{T}) \to L^1(\mathbb{T})$ of the form $f \mapsto \mathcal{F}^{-1}(\phi\mathcal{F}(f))$, where $\phi : \mathbb{Z} \to \mathbb{C}$ is the corresponding Fourier multiplier symbol. Note that ϕ is a Fourier multiplier symbol if and only if $a \mapsto \phi a$ is a bounded (in $\|\cdot\|_{A(\mathbb{Z})}$) linear map $A(\mathbb{Z}) \to A(\mathbb{Z})$.

So, we can characterize $A(\mathbb{Z})$, its norm $\|\cdot\|_{A(\mathbb{Z})}$, and the $L^1(\mathbb{T})$-Fourier multipliers in terms of the group $\mathbb{Z}$, without reference to the group $\mathbb{T}$. The idea behind the upcoming more abstract definition of (linear)

Fourier multipliers in the context of locally compact groups, is to use these characterizations as definitions, but with $\mathbb{Z}$ replaced by an arbitrary locally compact group G.

The above argument can be repeated for general abelian locally compact groups. Every abelian locally compact group G has an abelian dual group $\hat{G}$ and a Fourier transform $\mathcal{F} : L^1(G) \to C_0(\hat{G})$, where $C_0(\hat{G})$ is the space of continuous functions on $\hat{G}$ that "vanish at ∞" (see Definition 3.17). If $G = \mathbb{T}$, then $\hat{\mathbb{T}}$ can be identified with $\mathbb{Z}$. Our treatment of Fourier analysis on $\mathbb{Z}$ remains valid if $\mathbb{T}$ is replaced by an abelian locally compact group G and $\mathbb{Z}$ by the dual group $\hat{G}$. $L^1(\mathbb{R})$-Fourier transforms can also be treated in this manner, where we note that $\hat{\mathbb{R}}$ can be identified with $\mathbb{R}$ itself. Because we do not assume our locally compact groups to be abelian, we will not be using the dual group or the associated Fourier transform in this thesis. We refer the interested reader to chapter 4 in [17] for the relevant definitions and a treatment of abelian locally compact groups.

1.3 Abstract Fourier multipliers

Our brief exploration of Fourier analysis on $\mathbb{Z}$ suggests that for a general locally compact group G, we define $A(G)$ as the set of all functions of the form $f * g$, where $f, g \in L^2(G)$ and define $\|a\|_{A(G)}$ as the infimum of $\|f\|_2 \|g\|_2$ taken over all $f, g \in L^2(G)$ such that $a = f * g$. This definition would work in case G is assumed to be unimodular, but in the general case we need to replace g by $\check{g}$, where $\check{g}(x) = g(x^{-1})$. So, we have that

$$A(G) = \{f * \check{g} : f, g \in L^2(G)\}, \tag{8}$$

equipped with the norm $\|a\|_{A(G)} = \inf \|f\|_2 \|g\|_2$, where the infimum is taken over all $f, g \in L^2(G)$ such that $a = f * \check{g}$. Note that in case G is unimodular, $g \mapsto \check{g}$ defines an isometry $L^2(G) \to L^2(G)$. So, if G is unimodular this definition of $A(G)$ agrees with the one we proposed earlier. For $G = \mathbb{Z}$ it is clear that $A(\mathbb{Z})$ is a Banach space, because it is isometrically isomorphic to $L^1(\mathbb{T})$. This turns out to be true for any locally compact group G (Remark 4.9). In fact $A(G)$ is even a Banach algebra (Definition 2.2), i.e. $A(G)$ is a Banach space and $ab \in A(G)$ for $a, b \in A(G)$ and $\|ab\|_{A(G)} \leq \|a\|_{A(G)} \|b\|_{A(G)}$. The study of Fourier algebras dates back to a 1964 paper by Eymard [16]. A detailed treatment can be found in [22], while [29] offers a brief introduction.

Taking inspiration from the $G = \mathbb{Z}$ case, we can define (linear) Fourier multiplier symbols as functions $\phi : G \to \mathbb{C}$ such that $m_\phi : A(G) \to A(G)$, $a \mapsto \phi a$ defines a bounded linear operator. Such functions are automatically continuous and bounded (Remark 4.10). These Fourier multiplier symbols are also called Fourier algebra multipliers and we write $MA(G)$ for the set of Fourier algebra multipliers. Equipped with pointwise operations and the norm $\|\phi\|_{MA(G)} = \|m_\phi\|$, $MA(G)$ is a Banach algebra (Remark 4.10).

There is a second (equivalent) way of defining $MA(G)$, which is less similar to the definition of classical Fourier multipliers, but can be more easily generalized to define multilinear Fourier multipliers. For a locally compact group G and $x \in G$ we can define a linear operator $\lambda(x) \in \mathcal{B}(L^2(G))$ given by $(\lambda(x)f)(y) = f(x^{-1}y)$. The linear span of $\{\lambda(x) : x \in G\}$ can be closed in several different topologies (SOT, WOT and σ-weak operator topology, see Definition 4.1, Definition 4.2 and Definition 4.3). These closures are all the same space: the group von Neumann algebra $VN(G)$ of G (Definition 4.9).

The functions $\phi \in MA(G)$ are such that there exists a (unique) bounded linear map $M_\phi : VN(G) \to VN(G)$ that is continuous in the σ-weak operator topology and such that

$$M_\phi(\lambda(x)) = \phi(x)\lambda(x) \tag{9}$$

for all $x \in G$. Moreover, for a continuous, bounded function $\phi : G \to \mathbb{C}$, the existence of such a map M_ϕ implies that $\phi \in MA(G)$ and $\|M_\phi\| = \|\phi\|_{MA(G)}$ (Theorem 4.21). This results in two equivalent characterizations of $MA(G)$. An important subset of $MA(G)$ consists of those Fourier multiplier symbols ϕ for which M_ϕ is a completely bounded linear map. Complete boundedness of a linear map is a stronger condition than boundedness and has its own associated norm $\|\cdot\|_{CB}$ (see section 2). The completely bounded Fourier multiplier symbols form a subset $M_{cb}A(G) \subseteq MA(G)$. Equipped with pointwise operations and the norm $\|\phi\|_{M_{cb}A(G)} = \|M_\phi\|_{CB}$, this is again a Banach algebra.

1.4 Weak amenability

One important application of $M_{cb}A(G)$ is in the definition of weak amenability (Definition 4.18) of locally compact groups. Weak amenability is an approximation property that states that the function on a locally compact group constantly equal to 1, can be approximated in a specific way by functions in the Fourier algebra of the group. Not every locally compact group has this property. To be precise, a locally compact group G is called weakly amenable if there is a net (ϕ_ι) in $A(G)$ such that $\|\phi_\iota\|_{M_{cb}A(G)} \leq k$ for some constant k and such that ϕ_ι converges to 1 uniformly on compact sets. A net (Definition 3.7) can be thought of as a generalized sequence. Nets are frequently used in topology instead of sequences, when using a sequence does not suffice.

The name weak amenability was first introduced in 1989 by Cowling and Haagerup [7], but the property it refers to was already being studied in the preceding years. Weak amenability has several important applications. We state three of these applications below, but these are certainly not the only results involving weak amenability. We refer to [29] for an introduction to weak amenability and an overview of the literature on this topic.

The first of these applications is the study of operator algebras. For a discrete group Γ, weak amenability of Γ is equivalent to a certain approximation property of the group von Neumann algebra $VN(\Gamma)$ as well as a certain approximation property of the C^*-algebra (Definition 2.2) generated by $\{\lambda(x) : x \in \Gamma\}$. Both of these approximation properties involve the approximation of the identity operator on the respective spaces by maps with finite rank (i.e. their image has finite dimension). Theorem 4.24 gives a more detailed statement of this result, which is proved by Haagerup in Theorem 2.6 of [18].

A second important application of weak amenability is the study of Lie groups. An important result about weak amenability in the context of Lie groups is that the weak amenability of a connected, simple Lie group depends on its rank. In particular, a connected, simple Lie group is weakly amenable if and only if it has real rank 0 or 1. This result is Theorem 5.1 in [29] and we refer to section 5 in [29] and the references provided there for the relevant definitions. This theorem was not proved all at once and different parts of the problem were solved by different people. The fact that no Lie group with real rank ≥ 2 can be weakly amenable was proved by Haagerup [18] with an additional assumption that was later removed by Dorofaeff [11, 12]. The weak amenability of Lie groups with rank 0 or 1 was proved separately for different Lie groups by Cowling ([6]), De Cannière and Haagerup ([8]), and Cowling and Haagerup ([7]) (with an additional assumption that was later removed by Hansen [19]).

A third important result, especially in the context of this thesis, involving weak amenability is the fact that a lattice Γ in a second-countable (Definition 3.10), unimodular, locally compact group G is weakly amenable if and only if G itself is weakly amenable. Γ being a lattice means that Γ is a closed discrete subgroup of G and the quotient G/Γ has a finite measure that is invariant under left multiplication with elements of G. This result was proved by Haagerup [18] and, together with the results leading up to it, forms the inspiration behind section 5.

1.5 Multilinear Fourier multipliers

We mentioned earlier that the characterization of $MA(G)$ in terms of $VN(G)$ can be generalized to a definition of multilinear Fourier multiplier symbols. For $n \in \mathbb{N}$, an n-linear map is a map that depends on n variables and is linear in each separate variable (so 1-linear maps are just linear maps). The definitions of boundedness and complete boundedness for linear maps can be generalized to n-linear maps (see section 2). If $\phi : G^{\times n} \to \mathbb{C}$ is a (n-variable) bounded and continuous function, such that a (unique) bounded n-linear map $M_\phi : VN(G)^{\times n} \to VN(G)$ exists that is continuous in each variable in the σ-weak operator topology, and such that

$$M_\phi(\lambda(x_1), \ldots, \lambda(x_n)) = \phi(x_1, \ldots, x_n) \prod_{j=1}^{n} \lambda(x_j), \tag{10}$$

then the map M_ϕ is called an n-linear Fourier multiplier and ϕ its symbol. $M^n A(G)$ denotes the space of symbols of n-linear Fourier multipliers equipped with the norm $\|\phi\|_{M^n A(G)} = \|M_\phi\|$. In case M_ϕ is a completely bounded n-linear map, we call it a completely bounded Fourier multiplier. $M^n_{cb}A(G)$ is the space of symbols of completely bounded n-linear Fourier multipliers and is equipped with the norm $\|\phi\|_{M^n_{cb}A(G)} =$

$\|M_\phi\|_{CB}$. This definition generalizes linear Fourier multipliers in the sense that for $n = 1$ the n-linear (completely bounded) Fourier multipliers agree with the (linear) Fourier multipliers we defined earlier.

A generalization of Fourier multipliers has also been introduced by Todorov and Turowska [28]. Their approach involves defining a multidimensional Fourier algebra $A^n(G)$ and using it to define certain multipliers. These multipliers have an equivalent characterization that is very similar (and possibly equivalent) to our definition of multilinear Fourier multipliers. See Remark 4.12 and [28] for more details. A multivariable generalization of the Fourier algebra for discrete groups was previously also introduced by [13]. There also exists a different type of multilinear Fourier multipliers, where the Fourier multipliers act on non-commutative L^p-spaces (see for example [3]), but these differ from the Fourier multipliers that are covered in this thesis.

1.6 Schur multipliers

Fourier multipliers are closely related to Schur multipliers, which form another class of (multi)linear maps. Although the general definition of Schur multipliers (see subsection 4.3) is more involved, the basic idea behind Schur multipliers is relatively straightforward. If $m \in \mathbb{N}$ and $\phi : \{1, \dots, m\}^{\times 2} \to \mathbb{C}$ is a function, we can define a linear map $S_\phi : M_m(\mathbb{C}) \to M_m(\mathbb{C})$ given by

$$S_\phi \left((a_{i,j})_{i,j=1}^m \right) = (\phi(i,j) a_{i,j})_{i,j=1}^m . \tag{11}$$

The map S_ϕ is called a (linear) Schur multiplier with symbol ϕ. More generally, if $n \in \mathbb{N}$, given a function $\phi : \{1, \dots, m\}^{\times(n+1)} \to \mathbb{C}$, we can define an n-linear Schur multiplier $S_\phi : M_m(\mathbb{C})^{\times n} \to M_m(\mathbb{C})$ given by

$$S_\phi \left(\left(\left(a_{i,j}^{(k)} \right)_{i,j=1}^m \right)_{k=1}^n \right) = \sum_{j_1=1}^m \cdots \sum_{j_{n-1}=1}^m \left(\phi(j_0, \dots, j_n) \prod_{k=1}^n a_{j_{k-1},j_k}^{(k)} \right)_{j_0,j_n=1}^m . \tag{12}$$

Note that we can view the matrices in $M_m(\mathbb{C})$ as bounded linear operators on the space $l^2(\{1, \dots, m\})$. The idea behind the more general definition of Schur multipliers (Definition 4.15) is to replace $l^2(\{1, \dots, m\})$ by $L^2(X)$ for a more general measure space X. This results in the replacement of sums by integrals and requires more technicalities. We are mostly interested in the special case where X is a locally compact group G. In this context, for every $\phi \in C_b(G^{\times n})$, we can define a function $\tilde{\phi} \in C_b(G^{\times(n+1)})$ given by

$$\tilde{\phi}(x_0, \dots, x_n) = \phi(x_0 x_1^{-1}, \dots, x_{n-1} x_n^{-1}). \tag{13}$$

This transformation is such that ϕ is the symbol of a completely bounded n-linear Fourier multiplier if and only if $\tilde{\phi}$ is the symbol of a completely bounded n-linear Schur multiplier (Theorem 4.14). Using this connection, one can prove that ϕ is a completely bounded Fourier multiplier by proving that $\tilde{\phi}$ is a completely bounded Schur multiplier. We use this approach in section 5. The multilinear Schur multipliers on measure spaces that we use, were introduced in [21].

1.7 Main results

Our main contribution in this thesis is the generalization of several results in section 2 of [18] from linear to multilinear Fourier multipliers. In section 2 of [18], Haagerup proves several results about lattices in second-countable, unimodular, locally compact groups. One of the main results, which we mentioned earlier, is that such a group is weakly amenable if and only if any one of its lattices is weakly amenable. A key ingredient in this prove is a way to construct completely bounded Fourier multipliers on the entire group G, based on completely bounded Fourier multipliers on the lattice Γ (see Lemma 2.1 in [18]).

In the same setting as section 2 of [18], we introduce a similar construction that works for multilinear Fourier multipliers. Our construction consists of two steps. The first is a contractive (i.e. norm-decreasing) linear map $l^\infty(\Gamma^{\times n}) \to L^\infty(G^{\times n})$, $\phi \mapsto \hat{\phi}$ (see Definition 5.1). For $n = 1$ this step agrees with the construction in Lemma 2.1 of [18] and results in a completely bounded Fourier multiplier $\hat{\phi}$ if ϕ is a completely bounded Fourier multiplier. In particular for $n = 1$, $\hat{\phi}$ will be a continuous function for any $\phi \in l^\infty(G)$. The continuity of $\hat{\phi}$ is not apparent for $n \geq 2$, which motivated us to add a second step to the construction.

The second step of the construction is a "pseudo-convolution" $L^1(G^{\times(n+1)}) \times L^\infty(G^{\times n}) \to C_b(G^{\times n})$, $(F, \phi) \mapsto F \tilde{*} \phi$ (see Definition 3.29). This pseudo-convolution is similar (but not identical) to a construction used in the proof of Theorem 4.5 in [4]. We show that $\phi \mapsto F\tilde{*}\hat{\phi}$ preserves uniform convergence on compact sets for bounded nets. Our approach here is similar to the approach taken by Haagerup in [18] as part of proving that weak amenability of a lattice implies weak amenability of the surrounding group. We also show that for an appropriate choice of F, $\phi \mapsto F\tilde{*}\hat{\phi}$ maps completely bounded Fourier multipliers on the lattice Γ to completely bounded Fourier multipliers on the surrounding group G (see Theorem 5.10). This result relies on Schur multiplier theory and the connection between Fourier and Schur multipliers.

One of the original goals for this thesis was to generalize the notion of weak amenability to n-weak amenability, formulated in terms of n-linear Fourier multipliers, and take first steps in trying to generalize known results for weak amenability to n-weak amenability. In particular we wanted to show that a lattice is n-weakly amenable if and only if the surrounding group is n-weakly amenable, by generalizing Haagerup's results in [18]. Our proposed definition for n-weak amenability of a locally compact group G was the existence of a net (ϕ_ι) in $C_c(G^{\times n}) \cap M^n_{cb}A(G)$ that converges to 1 uniformly on compact subsets of $G^{\times n}$ and is bounded in $\|\cdot\|_{M^n_{cb}A(G)}$. Here 1-weak amenability is equivalent to weak amenability (Theorem 4.26). We discovered that the condition of n-weak amenability is equivalent for different n and therefore no different from weak amenability itself (see Theorem 4.26 and Remark 4.15). For this reason all results on weak amenability generalize in a trivial way to our proposed definition of n-weak amenability. The intent to generalize the equivalence between weak amenability of a lattice and weak amenability of the surrounding group explains why most of our results are inspired by Haagerup's proof of this result. Our proof of the equivalence between weak amenability and n-weak amenability relies on different ways of combining Fourier multiplier symbols to obtain new Fourier multiplier symbols. In particular we show (in subsection 4.2) that a product of linear Fourier multiplier symbols (in different variables) results in the symbol of a multilinear Fourier multiplier. We also show that setting all but one variable equal to the identity element in the symbol of a multilinear Fourier multiplier results in a linear Fourier multiplier.

1.8 Further research

The fact that our proposed definition of n-weak amenability turned out to be equivalent to weak amenability, while not intended, gives an equivalent characterization of weak amenability. In particular this characterization does not involve the Fourier algebra. It is possible that this characterization of weak amenability might contribute to the field either by aiding in the derivation of certain properties of weakly amenable groups or by making it easier to prove that a group is weakly amenable.

If another generalization of weak amenability in terms of multilinear Fourier multipliers is found that is not equivalent to weak amenability, a natural direction for further research would be to investigate which results involving weak amenability generalize to the new definition. One such result is the equivalence of weak amenability of a lattice and the surrounding group. We hope that our construction of multilinear Fourier multipliers, based on Haagerup's proof of this equivalence, might play a role in generalizing this result to a generalization of weak amenability. One possible generalization of weak amenability that we think might be worth investigating, is to modify the original definition by replacing $A(G)$ by the multidimensional Fourier algebra $A^n(G)$ as introduced in [28] and linear Fourier multipliers by multilinear ones. As far as we know, this possibility has not yet been explored.

There are two more questions about multilinear Fourier multipliers that we have encountered while working on this thesis, which we think might be worth investigating. Firstly, we have shown that symbols of multilinear Fourier multipliers can be obtained as products of symbols of linear Fourier multipliers. By taking linear combinations of these, even more symbols of multilinear Fourier multipliers can be obtained. The question is whether all symbols of multilinear Fourier multipliers can be constructed in this manner and which counterexamples can be found if this is not the case. The second question is whether the map $\phi \mapsto \hat{\phi}$ maps $l^\infty(\Gamma^{\times n})$ into $C_b(G^{\times n})$. It does when $n = 1$, but the proof in [18] to show this, does not seem to easily generalize to $n \geq 2$. We suspect that $\hat{\phi}$ is not always continuous when $n \geq 2$, but we have not done any explicit calculations to confirm this by finding counterexamples. Answering this question might be especially relevant if further attempts are made to generalize weak amenability and Haagerup's result about the weak amenability of lattices.

1.9 Thesis structure

The remainder of this thesis is structured as follows. In section 2 we define (complete) boundedness for multilinear maps and prove several results about them. We give special attention to ways of constructing multilinear maps from existing multilinear maps. These constructions are later applied to Fourier multipliers. Our separate treatment of multilinear maps allows us to prove some more technical and notationally complicated results long before needing to apply them to Fourier multipliers. This streamlines our treatment of Fourier multipliers, while still covering some of the technical details.

In section 3 we cover most of the general theory on locally compact groups that we will need. Our treatment of locally compact groups mostly follows chapter 2 in [17], to which we refer for most proofs. We also briefly introduce some definitions and results from general topology that we apply to locally compact groups. In subsection 3.3 we prove several needed results that are not proved in [17] or the other literature we have encountered. In subsection 3.5, we define the pseudo-convolution that forms the second step in our construction of Fourier multipliers and prove several results about this pseudo-convolution. We have decided to cover the pseudo-convolution before introducing Fourier multipliers and lattices, because the pseudo-convolution does not rely on the latter two notions and could have applications that do not involve Fourier multipliers.

In section 4 we treat multilinear Fourier multipliers and related topics such as the group von Neumann algebra, the Fourier algebra, weak amenability and Schur multipliers. Unlike in the introduction, in subsection 4.2 we define Fourier multipliers in terms of the group von Neumann algebra and make the connection with the Fourier algebra afterwards in subsection 4.4. This allows us to immediately define multilinear Fourier multipliers. This approach may be considered less intuitive than the approach taken in the introduction, hence the approach taken in the introduction is included to complement section 4. Both results taken from the literature (usually without proof) and some results of our own are contained in section 4.

In section 5, we work entirely in the setting of section 2 of [18]. After explaining the setting and introducing some notation, mostly following [18], the remainder of section 5 consists of our own results, which are inspired by parts of section 2 in [18]. A large part of section 5 is centered around the map $\phi \mapsto \hat{\phi}$ and the subsequent construction of multilinear Fourier multipliers.

All vector spaces we consider in this thesis are assumed to be complex, i.e. their scalar field is equal to $\mathbb{C}$, the field of complex numbers. We also note that $\mathbb{N}$ will denote the natural numbers excluding 0.

2 Multilinear maps

In this section we introduce multilinear maps and cover some definitions and results involving multilinear maps. The main reason for devoting a section to multilinear maps, is that Fourier multipliers are multilinear maps. By treating multilinear maps in general in this section, we can shorten the proofs of some results in section 4. Because these topics are most relevant for our application of multilinear maps to Fourier multipliers, special attention is given to boundedness and complete boundedness of multilinear maps as well as methods to obtain new multilinear maps from existing ones. In particular we will explore how multilinear maps can be obtained by fixing variables and as "compositions" of multilinear maps. These constructions and how they preserve (complete) boundedness will be used to construct Fourier multipliers.

2.1 Bounded multilinear maps

In this subsection we introduce (bounded) multilinear maps and cover some of their basic properties and constructions.

Definition 2.1. Multilinear Map
Let $n \in \mathbb{N}$ and $A_1, \ldots, A_n, B$ vector spaces. Let $\bigtimes_{j=1}^n A_j$ denote the n-ary Cartesian product of $A_1, \ldots, A_n$. We call a map $T : \bigtimes_{j=1}^n A_j \to B$ multilinear (more specifically n-linear) if it is linear in each component. Explicitly this means that T must satisfy

$$T(x_1, \ldots, cx_j + dy_j, \ldots, x_n) = cT(x_1, \ldots, x_j, \ldots, x_n) + dT(x_1, \ldots, y_j, \ldots, x_n) \tag{14}$$

for all $j \in \{1, \ldots, n\}$, $c, d \in \mathbb{C}$ and $x_i, y_i \in A_i$ for all $i \in \{1, \ldots, n\}$. In other words, this means that for all $j \in \{1, \ldots, n\}$ and all $x_i \in \{1, \ldots, j-1, j+1, \ldots, n\}$, the map $A_j \to B$, $x_j \mapsto T(x_1, \ldots, x_j, \ldots, x_n)$ should be a linear map. Note that if $T : \bigtimes_{j=1}^n A_j \to B$ is multilinear, then $T(x_1, \ldots, x_n) = 0$ if $x_j = 0$ for at least one $j \in \{1, \ldots, n\}$. If $A_1, \ldots, A_n, B$ are normed vector spaces, we call an n-linear map $T : \bigtimes_{j=1}^n A_j \to B$ bounded if

$$\sup\{\|T(x_1, \ldots, x_n)\| : \|x_j\| \leq 1 \, \forall j \in \{1, \ldots, n\}\} < \infty. \tag{15}$$

We denote the above quantity by $\|T\|$.

Note that the above definition reduces to the definition of a (bounded) linear map in case $n = 1$. A number of basic results about (bounded) linear maps generalize to the multilinear case, as can be seen from some of the remarks below.

Remark 2.1. There are several alternative characterizations of the quantity $\|T\|$ for a multilinear map $T : \bigtimes_{j=1}^n A_j \to B$. First of all we have that

$$\|T\| = \sup\{\|T(x_1, \ldots, x_n)\| : \|x_j\| < 1 \, \forall j \in \{1, \ldots, n\}\}, \tag{16}$$

$$\|T\| = \sup\{\|T(x_1, \ldots, x_n)\| : \|x_j\| = 1 \forall j \in \{1, \ldots, n\}\} \tag{17}$$

and

$$\|T\| = \sup\left\{\frac{\|T(x_1, \ldots, x_n)\|}{\prod_{j=1}^n \|x_j\|} : x_j \neq 0 \, \forall j \in \{1, \ldots, n\}\right\}. \tag{18}$$

The equivalence of these characterizations follows from the definitions of multilinearity and the supremum with the proof being very similar to the linear case. Here we note that the last two of these characterizations are only valid if none of the vector spaces $A_1, \ldots, A_n$ is equal to $\{0\}$. The last of the above characterizations implies that

$$\|T(x_1, \ldots, x_n)\| \leq \|T\| \prod_{j=1}^n \|x_j\| \tag{19}$$

for all $(x_1, \dots, x_n) \in \bigtimes_{j=1}^n A_j$. This is also true if any of the vector spaces $A_1, \dots, A_n$ is equal to $\{0\}$. In fact, $\|T\|$ is always the infimum of all $C \geq 0$ that satisfy

$$\|T(x_1, \dots, x_n)\| \leq C \prod_{j=1}^{n} \|x_j\| \tag{20}$$

with the convention that $\inf(\emptyset) = \infty$. This is another equivalent characterization of $\|T\|$. As such, boundedness of T follows from the existence of a constant $C \geq 0$ that satisfies the above inequality and any such constant C gives an upper bound on $\|T\|$.

Remark 2.2. The set of all multilinear maps $\bigtimes_{j=1}^n A_j \to B$ forms a linear subspace, which we will denote as $\mathcal{L}^0(A_1, \dots, A_n; B)$, of the vector space of all functions $\bigtimes_{j=1}^n A_j \to B$. If $A_1, \dots, A_n, B$ are normed spaces, then the set of bounded multilinear maps $\bigtimes_{j=1}^n A_j \to B$, which we will denote as $\mathcal{L}(A_1, \dots, A_n; B)$, is a further linear subspace and $\|\cdot\|$ defines a norm on $\mathcal{L}(A_1, \dots, A_n; B)$. Note that

$$\|T(x_1, \dots, x_n) + S(x_1, \dots, x_n)\| \leq \|T(x_1, \dots, x_n)\| + \|S(x_1, \dots, x_n)\| \tag{21}$$

and

$$\|cT(x_1, \dots, x_n)\| = |c|\|T(x_1, \dots, x_n)\| \tag{22}$$

for all $S, T \in \mathcal{L}(A_1, \dots, A_n; B)$, $c \in \mathbb{C}$ and $(x_1, \dots, x_n) \in \bigtimes_{j=1}^n A_j$. By using any one of the characterizations of $\|\cdot\|$ on $\mathcal{L}(A_1, \dots, A_n; B)$ and taking the appropriate supremum it follows that $\|S + T\| \leq \|S\| + \|T\|$ and $\|cT\| = |c|\|T\|$. This shows that $\mathcal{L}(A_1, \dots, A_n; B)$ is indeed a vector space and $\|\cdot\|$ is a seminorm (i.e. it satisfies all defining properties of a norm except that $\|T\| = 0$ is not required to imply that $T = 0$) on $\mathcal{L}(A_1, \dots, A_n; B)$. If $T \in \mathcal{L}(A_1, \dots, A_n; B)$ is such that $\|T\| = 0$, then $\|T(x_1, \dots, x_n)\| = 0$ and therefore $T(x_1, \dots, x_n) = 0$ for all $(x_1, \dots, x_n) \in \bigtimes_{j=1}^n A_j$. So $T = 0$ whenever $\|T\| = 0$, hence $\|\cdot\|$ is indeed a norm on $\mathcal{L}(A_1, \dots, A_n; B)$.

Remark 2.3. If $A_1, \dots, A_n, B$ are normed vector spaces and $(T_m)_{m \in \mathbb{N}}$ is a sequence in $\mathcal{L}(A_1, \dots, A_n; B)$ that converges in norm to $T \in \mathcal{L}(A_1, \dots, A_n; B)$, then T_m also converges to T pointwise. Similarly if $(T_m)_{m \in \mathbb{N}}$ is a Cauchy sequence, then $(T_m(x_1, \dots, x_n))_{m \in \mathbb{N}}$ is Cauchy for all $x_1 \in A_1, \dots, x_n \in A_n$. Both these results follow from the inequality

$$\|S(x_1, \dots, x_n) - T(x_1, \dots, x_n)\| \leq \|S - T\| \prod_{j=1}^{n} \|x_j\|. \tag{23}$$

Remark 2.4. A well-known fact is that a composition of two (bounded) linear maps will again be a (bounded) linear map. A similar result holds for multilinear maps, but is a bit more involved in terms of notation. The idea is that one can take several multilinear maps and use their output as the variables for another multilinear map. The precise statement of the result is as follows. Let $N \in \mathbb{N}$ and for $n \in \{1, \dots, N\}$ let $M_n \in \mathbb{N}$. For $n \in \{1, \dots, N\}$ and $m_n \in \{1, \dots, M_n\}$ let $A_{m_n,n}$ be a vector space. Also for $n \in \{1, \dots, N\}$, let B_n be a vector space and let C be a vector space. For $n \in \{1, \dots, N\}$ let $S_n \in \mathcal{L}(A_{1,n}, \dots, A_{M_n,n}; B_n)$ and let $T \in \mathcal{L}(B_1, \dots, B_N; C)$. We write $\times_{n=1}^N S_n$ for the map $\bigtimes_{n=1}^N \bigtimes_{m_n=1}^{M_n} A_{m_n,n} \to \bigtimes_{n=1}^N B_n$ given by $((x_{m_n,n})_{m_n=1}^{M_n})_{n=1}^N \mapsto (S_n(x_{1,n}, \dots, x_{M_n,n}))_{n=1}^N$. Then it follows from the multilinearity of $S_1, \dots, S_N, T$ that $T \circ \times_{n=1}^N S_n \in \mathcal{L}^0(A_{1,1}, \dots, A_{M_1,1}, \dots, A_{1,N}, \dots, A_{M_N,N}; C)$. Moreover, if all the vector spaces $A_{m_n,n}, B_n, C$ are normed and the multilinear maps $S_1, \dots, S_N, T$ are bounded, then $T \circ \times_{n=1}^N S_n \in \mathcal{L}(A_{1,1}, \dots, A_{M_1,1}, \dots, A_{1,N}, \dots, A_{M_N,N}; C)$. To see this, let $x_{m_n,n} \in A_{m_n,n}$ for $n \in \{1, \dots, N\}$ and $m_n \in \{1, \dots, M_n\}$. Using boundedness of $S_1, \dots, S_N, T$ we see that

$$
\begin{aligned}
&\left\|T(\times_{n=1}^{N} S_n(((x_{m_n,n})_{m_n=1}^{M_n})_{n=1}^{N}))\right\| = \|T(S_1(x_{1,1},\ldots,x_{M_1,1}),\ldots,S_N(x_{1,N},\ldots,x_{M_N,N}))\| \\
&\le \|T\| \prod_{n=1}^{N} \|S_n(x_{1,n},\ldots,x_{M_n,n})\| \le \|T\| \prod_{n=1}^{N} \|S_n\| \prod_{m_n=1}^{M_n} \|x_{m_n,n}\| \\
&= \|T\| \left(\prod_{n=1}^{N} \|S_n\|\right) \prod_{n=1}^{N} \prod_{m_n=1}^{M_n} \|x_{m_n,n}\|.
\end{aligned} \tag{24}
$$

This shows that $T \circ \times_{n=1}^{N} S_n$ is indeed bounded and we have

$$
\left\|T \circ \times_{n=1}^{N} S_n\right\| \le \|T\| \prod_{n=1}^{N} \|S_n\|. \tag{25}
$$

Note that if $N = 1$ and $M_1 = 1$, the above construction reduces to the composition of two (bounded) linear maps.

We will now consider an example where the above construction can be applied to construct (bounded) multilinear maps. Suppose B is an algebra (see Definition 2.2). Then $(x_1,\ldots,x_N) \mapsto \prod_{n=1}^{N} x_n$ defines an N-linear map $B^{\times N} \to B$. If B is also a normed algebra (algebra with a submultiplicative norm), then this multilinear map is bounded (with norm at most 1). Suppose that for each $n \in \{1,\ldots,N\}$ there is an $M_n \in \mathbb{N}$ and for $m_n \in \{1,\ldots,M_n\}$ we have a vector space $A_{m_n,n}$. Also suppose that we have multilinear maps $S_n \in \mathcal{L}^0(A_{1,n},\ldots,A_{M_n,n};B)$ for each $n \in \{1,\ldots,N\}$. Note that we are in the same situation as in Remark 2.4 but now with $B_n = B$, $C = B$ and T given by $T(x_1,\ldots,x_N) = \prod_{n=1}^{N} x_n$. It follows that

$$
((x_{m_n,n})_{m_n=1}^{M_n})_{n=1}^{N} \mapsto \prod_{n=1}^{N} S_n(x_{1,n},\ldots,x_{M_n,n}) \tag{26}
$$

defines a multilinear map $\bigtimes_{n=1}^{N} \bigtimes_{m_n=1}^{M_n} A_{m_n,n} \to B$. If we assume that $A_{m_n,n}$ and B are normed and all S_n are bounded, then this will be a bounded multilinear map with norm bounded from above by $\prod_{n=1}^{N} \|S_n\|$. This example can be simplified by taking $M_n = 1$ for all n. In this case we have (bounded) linear maps $S_n : A_n \to B$ (with $A_n = A_{1,n}$), which we combine to obtain a (bounded) multilinear map $\bigtimes_{n=1}^{N} A_n \to B$ given by $(x_1,\ldots,x_N) \mapsto \prod_{n=1}^{N} S_n(x_n)$. Variations on these examples can be obtained by replacing the multiplication map $(x_1,\ldots,x_N) \mapsto \prod_{n=1}^{N} x_n$ by $(x_1,\ldots,x_N) \mapsto \prod_{n=1}^{N} x_{\tau(n)}$, where τ is a permutation of $\{1,\ldots,N\}$.

Remark 2.5. Note that if $A_1,\ldots,A_n,B$ are vector spaces and $T \in \mathcal{L}^0(A_1,\ldots,A_n;B)$ is a multilinear map, then we can obtain new multilinear maps from T by fixing one or more of the variables. Because we will make use of this later on, we introduce special notation for this construction. Let $J \subset \{1,\ldots,n\}$ be a strict subset and $x_j \in A_j$ for $j \in J$. Then we write $T\left(\cdot|J;(x_j)_{j\in J}\right)$ for the map $\bigtimes_{\{1,\ldots,n\}\setminus J} A_j \to B$ given by

$$
(x_j)_{j\in\{1,\ldots,n\}\setminus J} \mapsto T\left((x_j)_{j\in\{1,\ldots,n\}\setminus J}|J;(x_j)_{j\in J}\right) := T(x_1,\ldots,x_n). \tag{27}
$$

Here we note that the Cartesian product $\bigtimes_{\{1,\ldots,n\}\setminus J} A_j$ is considered to be taken in order of increasing j, or in other words the order of the variables $x_j \in A_j$ for $j \in \{1,\ldots,n\}\setminus J$ is preserved when considering the map $T\left(\cdot|J;(x_j)_{j\in J}\right)$. From Definition 2.1 it is clear that $T\left(\cdot|J;(x_j)_{j\in J}\right)$ is again a multilinear map, i.e. $T\left(\cdot|J;(x_j)_{j\in J}\right) \in \mathcal{L}^0\left((A_j)_{j\in\{1,\ldots,n\}\setminus J};B\right)$, where the tuple $(A_j)_{j\in\{1,\ldots,n\}\setminus J}$ is considered to be in order of increasing subscript j. Note that $T\left(\cdot|J;(x_j)_{j\in J}\right) = T$ if J is empty. If $A_1,\ldots,A_n,B$ are normed spaces, then it can be seen from Definition 2.1 (or Remark 2.1) that $T\left(\cdot|J;(x_j)_{j\in J}\right)$ is bounded, hence $T\left(\cdot|J;(x_j)_{j\in J}\right) \in \mathcal{L}\left((A_j)_{j\in\{1,\ldots,n\}\setminus J};B\right)$. In particular we have that

$$\left\|T\left(\cdot|J;(x_j)_{j\in J}\right)\right\| \leq \|T\| \prod_{j\in J} \|x_j\|. \tag{28}$$

Finally we note that this fixing of variables can be done by either fixing at once all variables that are to be fixed, or in several steps. In other words, if $I, J \subset \{1,\ldots,n\}$ are disjoint with $I \cup J$ a strict subset of $\{1,\ldots,n\}$, $x_j \in A_j$ for $j \in I \cup J$ and $S = T\left(\cdot|J;(x_j)_{j\in J}\right)$. Then we have that

$$S\left(\cdot|I;(x_j)_{j\in I}\right) = T\left(\cdot|I\cup J;(x_j)_{j\in I\cup J}\right). \tag{29}$$

This simple observation is useful, because it ensures that fixing several variables can be viewed as fixing one variable at a time.

The following result generalizes a well-known result about linear maps.

Theorem 2.1. *Let $n \in \mathbb{N}$ and $A_1,\ldots,A_n$ normed vector spaces. Let B be a Banach space. Then $\mathcal{L}(A_1,\ldots,A_n;B)$ is a Banach space.*

Proof. We have already seen that $\mathcal{L}(A_1,\ldots,A_n;B)$ is a normed vector space in Remark 2.2, so it remains to show completeness. Let $(T_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}(A_1,\ldots,A_n;B)$. Then by Remark 2.3, $(T_m(x_1,\ldots,x_n))_{m\in\mathbb{N}}$ is a Cauchy sequence in B for all $x_1 \in A_1,\ldots,x_n \in A_n$. It follows from the completeness of B that this sequence converges to some limit (in B). So we can define a function $T : \bigtimes_{j=1}^n A_j \to B$ by setting

$$T(x_1,\ldots,x_n) = \lim_{m\to\infty} T_m(x_1,\ldots,x_n). \tag{30}$$

We will show that $T \in \mathcal{L}(A_1,\ldots,A_n;B)$ and $T_m \to T$ in norm. Multilinearity of T follows immediately from multilinearity of each T_m and linearity of limits. To prove boundedness of T, note that by the inverse triangle inquality we have that

$$|\|T_m\| - \|T_k\|| \leq \|T_m - T_k\| \tag{31}$$

holds for all $m,k \in \mathbb{N}$. So $(\|T_m\|)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, hence it is bounded (and convergent). So $C := \sup_{k\in\mathbb{N}} \|T_k\| < \infty$. Now for all $x_1 \in A_1,\ldots,x_n \in A_n$ we have that

$$\|T_m(x_1,\ldots,x_n)\| \leq \|T_m\| \prod_{j=1}^n \|x_j\| \leq C \prod_{j=1}^n \|x_j\|. \tag{32}$$

It follows that

$$\|T(x_1,\ldots,x_n)\| = \lim_{m\to\infty} \|T_m(x_1,\ldots,x_n)\| \leq C \prod_{j=1}^n \|x_j\|. \tag{33}$$

This shows that T is bounded ($\|T\| \leq C$). To show convergence of T_m to T, let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $\|T_m - T_k\| < \frac{1}{2}\epsilon$ for all $m,k \geq N$. Then for all $x_1 \in A_1,\ldots,x_n \in A_n$ and $m,k \geq N$ we have that

$$\|T_m(x_1,\ldots,x_n) - T_k(x_1,\ldots,x_n)\| \leq \|T_m - T_k\| \prod_{j=1}^n \|x_j\| \leq \frac{1}{2}\epsilon \prod_{j=1}^n \|x_j\|. \tag{34}$$

It follows that for all $x_1 \in A_1,\ldots,x_n \in A_n$ and $m \geq N$ we have that

$$\|T_m(x_1,\ldots,x_n) - T(x_1,\ldots,x_n)\| = \lim_{k\to\infty} \|T_m(x_1,\ldots,x_n) - T_k(x_1,\ldots,x_n)\| \leq \frac{1}{2}\epsilon \prod_{j=1}^n \|x_j\| \tag{35}$$

It follows that $\|T_m - T\| \leq \frac{1}{2}\epsilon < \epsilon$ for all $m \geq N$. This shows that

$$\lim_{m\to\infty} \|T_m - T\| = 0. \tag{36}$$

So, T_m converges to T in norm, which shows that $\mathcal{L}(A_1,\ldots,A_n;B)$ is complete, hence a Banach space. $\square$

2.2 Boundedness of amplifications

In this subsection we introduce amplifications of multilinear maps. These amplifications are also multilinear maps, but they are defined on spaces of matrices. By considering the potential boundedness of these amplifications we obtain subspaces of multilinear maps that become progressively smaller as we increase the dimensions of the matrices we consider. The way we define amplifications generalizes the usual definition for linear maps. We will use some of the theory of C^*-algebras and for this reason we start by stating the relevant definitions.

Definition 2.2. Algebra

An (associative) algebra is a vector space A equipped with a bilinear, associative multiplication map $A \times A \to A$,

$$(x, y) \mapsto xy. \tag{37}$$

A linear subspace B of A is called a subalgebra if $xy \in B$ for all $x, y \in B$. If A is also equipped with a norm $\|\cdot\|$ that is submultiplicative, i.e.

$$\|xy\| \leq \|x\|\|y\| \tag{38}$$

for all $x, y \in A$, then A is called a normed algebra. In case A is also complete, it is called a Banach algebra.

Definition 2.3. Involutive algebra

An involutive algebra is an algebra A, equipped with a conjugate-linear map $A \to A$, $x \mapsto x^*$ that satisfies $x^{**} = x$ and $(xy)^* = y^*x^*$ for all $x, y \in A$. A subalgebra B of A is called a $*$-subalgebra if $x^* \in B$ for all $x \in B$. A Banach-$*$ algebra is an involutive Banach algebra A such that $\|x^*\| = \|x\|$ for all $x \in A$. A C^*-algebra is an involutive Banach algebra A such that $\|x^*x\| = \|x\|^2$ for all $x \in A$.

Remark 2.6. It can be shown that every C^*-algebra is a Banach-$*$ algebra (see Lemma 2.1.3 in [23]). If H is a Hilbert space, then $\mathcal{B}(H)$ (the space of bounded linear operators on H) is a C^*-algebra (see Example 2.1.3 in [23]). For more theory on C^*-algebras, we refer to [23].

Definition 2.4. Homomorphism

Let A, B be algebras. A linear map $\phi : A \to B$ is called an algebra homomorphism if it satisfies

$$\phi(xy) = \phi(x)\phi(y) \tag{39}$$

for all $x, y \in A$. If A, B are involutive algebras, an algebra homomorphism $\phi : A \to B$ is called a $*$-homomorphism if it also satisfies

$$\phi(x^*) = \phi(x)^* \tag{40}$$

for all $x \in A$. If a $(*-)$homomorphism is bijective, it is called a $(*-)$isomorphism.

Let H be a Hilbert space and E a linear subspace of $\mathcal{B}(H)$. Define for $n \in \mathbb{N}$

$$M_n(E) = \{(x_{i,j})_{i,j=1}^n : x_{i,j} \in E\}. \tag{41}$$

Equipped with element-wise addition and scalar multiplication, $M_n(E)$ becomes a vector space and a linear subspace of $M_n(B(H))$. If E is a subalgebra of $B(H)$, then $M_n(E)$ is also a subalgebra of $M_n(B(H))$ when both are equipped with the product

$$(x_{i,j})_{i,j=1}^n (y_{j,k})_{j,k=1}^n = \left(\sum_{j=1}^n x_{i,j} y_{j,k}\right)_{i,k=1}^n. \tag{42}$$

In particular if E is a $*$-subalgebra of $B(H)$, then $M_n(E)$ is also a $*$-subalgebra of $M_n(B(H))$ when both are equipped with the involution

$$\left((x_{i,j})_{i,j=1}^n\right)^* = (x_{j,i}^*)_{i,j=1}^n. \tag{43}$$

Let $l_2^n(H)$ be the Hilbert space direct sum of n copies of H. So,

$$l_n^2(H) = \{(h_j)_{j=1}^n : h_j \in H\} \tag{44}$$

equipped with entry-wise addition and scalar multiplication and the inner product

$$\left\langle (g_j)_{j=1}^n, (h_j)_{j=1}^n \right\rangle = \sum_{j=1}^n \langle g_j, h_j \rangle . \tag{45}$$

This is again a Hilbert space. We define the map $\phi : M_n(\mathcal{B}(H)) \to \mathcal{B}(l_2^n(H))$, $(T_{i,j})_{i,j=1}^n \mapsto T$, where T is given by

$$T((h_j)_{j=1}^n) = \left(\sum_{j=1}^n T_{i,j} h_j \right)_{i=1}^n . \tag{46}$$

Note that

$$\begin{aligned}
\left\| T((h_j)_{j=1}^n) \right\| &= \left\| \left(\sum_{j=1}^n T_{i,j} h_j \right)_{i=1}^n \right\| \le \sum_{j=1}^n \left\| (T_{i,j} h_j)_{i=1}^n \right\| = \sum_{j=1}^n \sqrt{\sum_{i=1}^n \|T_{i,j} h_j\|^2} \\
&\le \sum_{j=1}^n \sqrt{\sum_{i=1}^n \|T_{i,j}\|^2 \|h_j\|^2} = \sum_{j=1}^n \|h_j\| \sqrt{\sum_{i=1}^n \|T_{i,j}\|^2} \le \sum_{k=1}^n \|h_k\| \sqrt{\sum_{l=1}^n \left(\max_{i,j \in \{1,\dots,n\}} \|T_{i,j}\| \right)^2} \\
&= n^{\frac{1}{2}} \max_{i,j \in \{1,\dots,n\}} \|T_{i,j}\| \sum_{k=1}^n \|h_k\| = n^{\frac{1}{2}} \max_{i,j \in \{1,\dots,n\}} \|T_{i,j}\| \|(\|h_k\|)_{k=1}^n\|_1 \\
&\le n^{\frac{1}{2}} \max_{i,j \in \{1,\dots,n\}} \|T_{i,j}\| \|(\|h_k\|)_{k=1}^n\|_2 \|(1)_{k=1}^n\|_2 = n \max_{i,j \in \{1,\dots,n\}} \|T_{i,j}\| \sqrt{\sum_{k=1}^n \|h_k\|^2} \\
&= n \max_{i,j \in \{1,\dots,n\}} \|T_{i,j}\| \|(h_k)_{k=1}^n\|.
\end{aligned} \tag{47}$$

This shows that T is bounded with $\|T\| \le n \max_{i,j \in \{1,\dots,n\}} \|T_{i,j}\|$. So, our map is well-defined. The map we defined above is clearly linear and it also preserves products as can be seen from

$$S(T((h_j)_{j=1}^n)) = S\left(\left(\sum_{j=1}^n T_{i,j} h_j \right)_{i=1}^n \right) = \left(\sum_{i=1}^n S_{k,i} \sum_{j=1}^n T_{i,j} h_j \right)_{k=1}^n = \left(\sum_{j=1}^n \left(\sum_{i=1}^n S_{k,i} T_{i,j} \right) h_j \right)_{k=1}^n , \tag{48}$$

where we note that $(\sum_{i=1}^n S_{k,i} T_{i,j})_{k,j=1}^n$ is the matrix product of $(S_{k,i})_{k,i=1}^n$ and $(T_{i,j})_{i,j=1}^n$.

Our map also preserves the involution. To see this, let $(T_{i,j})_{i,j=1}^n \in M_n(B(H))$, $T = \phi\left((T_{i,j})_{i,j=1}^n \right)$ and $(x_j)_{j=1}^n, (y_i)_{i=1}^n \in l_2^n(H)$. We have that

$$\begin{aligned}
\left\langle T((x_j)_{j=1}^n), (y_i)_{i=1}^n \right\rangle &= \left\langle \left(\sum_{j=1}^n T_{i,j} x_j \right)_{i=1}^n, (y_i)_{i=1}^n \right\rangle = \sum_{i=1}^n \sum_{j=1}^n \langle T_{i,j} x_j, y_i \rangle = \sum_{i=1}^n \sum_{j=1}^n \left\langle x_j, T_{i,j}^* y_i \right\rangle \\
&= \left\langle (x_j)_{j=1}^n, \left(\sum_{i=1}^n T_{i,j}^* y_i \right)_{j=1}^n \right\rangle = \left\langle (x_j)_{j=1}^n, \phi\left(\left((T_{i,j})_{i,j=1}^n \right)^* \right) (y_i)_{i=1}^n \right\rangle .
\end{aligned} \tag{49}$$

This shows that $\phi\left(\left((T_{i,j})_{i,j=1}^n \right)^* \right) = T^*$. So, our map preserves the involution.

Note that if $\left(\sum_{j=1}^n T_{i,j}h_j\right)_{i=1}^n = 0$ for all $h_1, \ldots, h_n \in H$, then we can in particular for each $k \in \{1, \ldots, n\}$ and $h \in H$ choose $h_j = \delta_{j,k}h$. For this choice of h_j we see that

$$0 = \left(\sum_{j=1}^n T_{i,j}h_j\right)_{i=1}^n = (T_{i,k}h)_{i=1}^n . \tag{50}$$

It follows that $T_{i,k}h = 0$ for all $i, k \in \{1, \ldots, n\}$ and for all $h \in H$. But then $T_{i,j} = 0$ for all $i, j \in \{1, \ldots, n\}$, hence $(T_{i,j})_{i,j=1}^n = 0$. This shows that the map is injective (because it is linear). To show surjectivity of this map first note that for all $k \in \{1, \ldots, n\}$, the map $\pi_k : l_2^n(H) \to l_2^n(H)$ given by $\pi_k((h_j)_{j=1}^n) = ((\delta_{j,k}h_j)_{j=1}^n)$ is an orthogonal projection with image $\pi_k(l_2^n(H)) = \left\{(h_j)_{j=1}^n : h_j = 0 \quad \forall j \neq k\right\}$. Also note that $\sum_{k=1}^n \pi_k = \mathrm{id}_{l_2^n(H)}$. It follows that $T = \sum_{i=1}^n \sum_{j=1}^n \pi_i T \pi_j$ for all $T \in \mathcal{B}(l_2^n(H))$. Also note that for $k \in \{1, \ldots, n\}$, $\iota_k : H \to \pi_k(l_2^n(H))$ given by $\iota_k(h) = (\delta_{j,k}h)_{j=1}^n$ defines an isometric isomorphism between H and $\pi_k(l_2^n(H))$. For $i, j \in \{1, \ldots, n\}$ let $T_{i,j} = \iota_i^{-1} \circ \pi_i \circ T \circ \iota_j$. Then $T_{i,j} \in \mathcal{B}(H)$ with $\|T_{i,j}\| \leq \|T\|$. So $(T_{i,j})_{i,j=1}^n \in M_n(\mathcal{B}(H))$. Now $T = \sum_{i=1}^n \sum_{j=1}^n \iota_i \circ T_{i,j} \circ \iota_j^{-1} \circ \pi_j$. This means that

$$\begin{aligned}
T((h_k)_{k=1}^n) &= \sum_{i=1}^n \sum_{j=1}^n \iota_i(T_{i,j}(\iota_j^{-1}(\pi_j((h_k)_{k=1}^n)))) = \sum_{i=1}^n \sum_{j=1}^n \iota_i(T_{i,j}(\iota_j^{-1}((\delta_{j,k}h_j)_{k=1}^n))) \\
&= \sum_{i=1}^n \sum_{j=1}^n \iota_i(T_{i,j}(h_j)) = \sum_{i=1}^n \sum_{j=1}^n (\delta_{i,k}T_{i,j}(h_j))_{k=1}^n = \left(\sum_{i=1}^n \sum_{j=1}^n \delta_{i,k}T_{i,j}(h_j)\right)_{k=1}^n \\
&= \left(\sum_{j=1}^n T_{k,j}(h_j)\right)_{k=1}^n = \phi\left((T_{i,j})_{i,j=1}^n\right)((h_k)_{k=1}^n) .
\end{aligned} \tag{51}$$

This shows that our map ϕ is surjective, hence bijective. We have also identified its inverse, namely $T \mapsto \left(\iota_i^{-1} \circ \pi_i \circ T \circ \iota_j\right)_{i,j=1}^n$.

We have shown that our map is a $*$-isomorphism; hence it allows $M_n(\mathcal{B}(H))$ to be identified with $\mathcal{B}(l_2^n(H))$ as $*$-algebras. Under this identification $M_n(\mathcal{B}(H))$ and its subspace $M_n(E)$ inherit the norm from $\mathcal{B}(l_2^n(H))$, denoted as $\|\cdot\|_n$. Note that $M_1(E)$ can just be identified with E and $\|\cdot\|_1$ is just the original norm $\|\cdot\|$ on $\mathcal{B}(H)$. We restate some of our results

Theorem 2.2. *Let H be a Hilbert space and $n \in \mathbb{N}$. The map $M_n(\mathcal{B}(H)) \to \mathcal{B}(l_2^n(H))$ given by*

$$(T_{i,j})_{i,j=1}^n \mapsto \left((h_j)_{j=1}^n \mapsto \left(\sum_{j=1}^n T_{i,j}h_j\right)_{i=1}^n\right) \tag{52}$$

is a $*$-isomorphism. Using this isomorphism, $M_n(\mathcal{B}(H))$ is equipped with the norm of $l_2^n(H)$, which turns $M_n(\mathcal{B}(H))$ into a C^*-algebra.

Theorem 2.3. *Let H be a Hilbert space and $n \in \mathbb{N}$. For $T = (T_{i,j})_{i,j=1}^n \in M_n(\mathcal{B}(H))$ we have that*

$$\|T_{i,j}\| \leq \|T\|_n \tag{53}$$

for all $i, j \in \{1, \ldots, n\}$. We also have that

$$\|T\|_n \leq n \max_{i,j \in \{1,\ldots,n\}} \|T_{i,j}\|. \tag{54}$$

In particular we have that $\|\cdot\|_n$ and $T \mapsto \max_{i,j\in\{1,\ldots,n\}} \|T_{i,j}\|$ are equivalent norms on $M_n(\mathcal{B}(H))$.

Remark 2.7. Note that if $S \in \mathcal{B}(H)$, then $(\delta_{i,j}S)_{i,j=1}^n \in M_n(\mathcal{B}(H))$ with

$$\left\|(\delta_{i,j}S)_{i,j=1}^n\right\|_n = \|S\|. \tag{55}$$

Indeed, if $h_1, \dots, h_n \in H$, then

$$(\delta_{i,j} S)_{i,j=1}^n (h_j)_{j=1}^n = \left(\sum_{j=1}^n \delta_{i,j} S h_j \right)_{i=1}^n = (S h_i)_{i=1}^n . \tag{56}$$

It follows that

$$\left\| (\delta_{i,j} S)_{i,j=1}^n (h_j)_{j=1}^n \right\|^2 = \|(S h_i)_{i=1}^n\|^2 = \sum_{i=1}^n \|S h_i\|^2 \leq \|S\|^2 \left\| (h_j)_{j=1}^n \right\|^2 . \tag{57}$$

This shows that $\left\| (\delta_{i,j} S)_{i,j=1}^n \right\|_n \leq \|S\|$. For $\epsilon > 0$ we can choose $h_j = 0$ for all $j \geq 2$ and h_1 such that $\|S h_1\| > (\|S\| - \epsilon)\|h_1\|$. For this choice of $h_1, \dots, h_n$ we have that

$$\left\| (\delta_{i,j} S)_{i,j=1}^n (h_j)_{j=1}^n \right\| = \|S h_1\| > (\|S\| - \epsilon)\|h_1\| = (\|S\| - \epsilon) \left\| (h_j)_{j=1}^n \right\| . \tag{58}$$

This shows that $\left\| (\delta_{i,j} S)_{i,j=1}^n \right\|_n > \|S\| - \epsilon$ for any $\epsilon > 0$. It follows that $\left\| (\delta_{i,j} S)_{i,j=1}^n \right\|_n \geq \|S\|$ and our claim follows.

Theorem 2.3 has a few useful consequences. We refer to subsection 3.1 for the topological definitions in Remark 2.8 and Remark 2.9.

Remark 2.8. If $\left(T^{(m)}\right)_{m \in \mathbb{N}} = \left(\left(T_{i,j}^{(m)} \right)_{i,j=1}^n \right)_{m \in \mathbb{N}}$ is a sequence in $M_n(\mathcal{B}(H))$, then $\left(T_{i,j}^{(m)} \right)_{m \in \mathbb{N}}$ is a sequence in $\mathcal{B}(H)$ for all $i, j \in \{1, \dots, n\}$. Because $\|\cdot\|_n$ and $T \mapsto \max_{i,j \in \{1,\dots,n\}} \|T_{i,j}\|$ are equivalent norms on $M_n(\mathcal{B}(H))$, it follows that $\left(T^{(m)}\right)_{m \in \mathbb{N}}$ is Cauchy if and only if $\left(T_{i,j}^{(m)} \right)_{m \in \mathbb{N}}$ is Cauchy for all $i, j \in \{1, \dots, n\}$. Similarly if $T = (T_{i,j})_{i,j=1}^n \in M_n(\mathcal{B}(H))$, then $\left(T^{(m)}\right)_{m \in \mathbb{N}}$ converges to T if and only if $\left(T_{i,j}^{(m)} \right)_{m \in \mathbb{N}}$ converges to $T_{i,j}$ for all $i, j \in \{1, \dots, n\}$. An analogous result holds for convergence of nets.

Remark 2.9. The norm $T \mapsto \max_{i,j \in \{1,\dots,n\}} \|T_{i,j}\|$ generates the product topology on $\bigtimes_{(i,j) \in \{1,\dots,n\}^2} \mathcal{B}(H)$. In light of Theorem 2.3 this means that $M_n(\mathcal{B}(H))$ (equipped with $\|\cdot\|_n$) and $\bigtimes_{(i,j) \in \{1,\dots,n\}^2} \mathcal{B}(H)$ (equipped with the product topology) are one and the same as topological spaces. It follows that for any topological space $\mathcal{T}$ and function $f : \mathcal{T} \to M_n(\mathcal{B}(H))$, f is continuous if and only if $x \mapsto f(x)_{i,j}$ is continuous for all $i, j \in \{1, \dots, n\}$. In particular the projections $M_n(\mathcal{B}(H)) \to \mathcal{B}(H)$, $T \mapsto T_{i,j}$ are continuous. From the definition of the product topology it is clear that if $i, j \in \{1, \dots, n\}$ and $A_{i,j} \subseteq \mathcal{B}(H)$, then $\{T \in \mathcal{B}(H) : T_{i,j} \in A_{i,j}\}$ is open if $A_{i,j}$ is open and is closed if $A_{i,j}$ is closed. By taking finite intersections it follows that if $A_{i,j} \subseteq \mathcal{B}(H)$ for all $i, j \in \{1, \dots, n\}$, then $\bigtimes_{(i,j) \in \{1,\dots,n\}^2} A_{i,j}$ is open in $M_n(\mathcal{B}(H))$ if each $A_{i,j}$ is open and is closed in $M_n(\mathcal{B}(H))$ if each $A_{i,j}$ is closed.

Corollary 2.4. *Let H be a Hilbert space, E a closed linear subspace of $\mathcal{B}(H)$ and $n \in \mathbb{N}$. Then $M_n(E)$ is closed.*

Proof. We give 2 short proofs, based on Remark 2.8 and Remark 2.9 respectively. For the first proof, let $\left(T^{(m)}\right)_{m \in \mathbb{N}} = \left(\left(T_{i,j}^{(m)} \right)_{i,j=1}^n \right)_{m \in \mathbb{N}}$ be a sequence in $M_n(E)$ that converges to $T = (T_{i,j})_{i,j=1}^n \in M_n(\mathcal{B}(H))$. By Remark 2.8, for each $i, j \in \{1, \dots, n\}$, $T_{i,j}^{(m)}$ is a sequence in E that converges to $T_{i,j}$. Hence $T_{i,j} \in E$ because E is closed. So we have $T \in M_n(E)$, which shows that $M_n(E)$ is closed. For the second proof note that $M_n(E) = \bigtimes_{(i,j) \in \{1,\dots,n\}^2} E$ as a subset of $M_n(\mathcal{B}(H))$. Because E is closed, Remark 2.9 implies that $M_n(E)$ is closed. ☐

Note that if $m \leq n$, we can isometrically embed $M_m(E)$ into $M_n(E)$. We will show one such embedding.

Theorem 2.5. *Let H be a Hilbert space and E a linear subspace of $\mathcal{B}(H)$. For $m, n \in \mathbb{N}$ with $m \leq n$ define $\sigma_{n,m} : M_m(\mathcal{B}(H)) \to M_n(\mathcal{B}(H))$ by $\sigma_{n,m}\left((T_{k,l})_{k,l=1}^m \right)_{i,j} = T_{i,j}$ if $i, j \leq m$ and $\sigma_{n,m}\left((T_{k,l})_{k,l=1}^m \right)_{i,j} = 0$ otherwise. Then $\sigma_{n,m}$ is a an isometric $*$-homomorphism and $\sigma_{n,m}(M_m(E)) \subseteq M_n(E)$.*

Proof. From the definition it is immediately clear that $\sigma_{n,m}(M_m(E)) \subseteq M_n(E)$ and that $\sigma_{n,m}$ is injective. Note that $l^2_n(H) \simeq l^2_m(H) \oplus l^2_{n-m}(H)$ as Hilbert spaces with a unitary isomorphism given by $\phi((h_j)_{j=1}^n) = ((h_j)_{j=1}^m, (h_{j+m})_{j=1}^{n-m})$. This induces a $*$-isomorphism $\Phi : \mathcal{B}(l^2_m(H) \oplus l^2_{n-m}(H)) \to \mathcal{B}(l^2_n(H))$ given by $\Phi(S) = \phi^{-1} \circ S \circ \phi$. For $T = (T_{i,j})_{i,j=1}^m$ note that $\Phi^{-1}(\sigma_{n,m}(T)) = T \oplus 0$. This clearly shows that $\Phi^{-1} \circ \sigma_{n,m}$ is a $*$-homomorphism, hence so is $\sigma_{n,m}$. $\sigma_{n,m}$ is an injective $*$-homomorphism between C^*-algebras. It follows from Theorem 3.1.5. in [23] that $\sigma_{n,m}$ must be isometric. So, $\sigma_{n,m}$ is a an isometric $*$-homomorphism. The fact that $\sigma_{n,m}$ is isometric can also be directly verified in a straightforward manner. $\square$

Remark 2.10. Note that if $m \leq k \leq n$, then $\sigma_{n,k} \circ \sigma_{k,m} = \sigma_{n,m}$. This follows directly from the definition of the embedding in Theorem 2.5. We will sometimes write $\sigma^E_{n,m}$ for the embedding $\sigma_{n,m} : M_m(E) \to M_n(E)$ to prevent ambiguity.

Definition 2.5. Amplification

Let $N \in \mathbb{N}$ and for $n \in \{1, \ldots, N\}$ let H_n be a Hilbert space and $E_n \subseteq \mathcal{B}(H_n)$ a linear subspace of $\mathcal{B}(H_n)$. Also let K be a Hilbert space and $F \subseteq \mathcal{B}(K)$ a linear subspace of $\mathcal{B}(K)$. We consider an N-linear map $\phi : \bigtimes_{n=1}^N E_n \to F$. For $m \in \mathbb{N}$ we define the map $\phi_m : \bigtimes_{n=1}^N M_m(E_n) \to M_m(F)$ by

$$\phi_m\left(T^{(1)}, \ldots, T^{(N)}\right) = \left(\sum_{i_1=1}^m \cdots \sum_{i_{N-1}=1}^m \phi\left(T^{(1)}_{i_0,i_1}, \ldots, T^{(N)}_{i_{N-1},i_N}\right)\right)_{i_0,i_N=1}^m, \tag{59}$$

where if $N = 1$ we interpret $\sum_{i_1=1}^m \cdots \sum_{i_{N-1}=1}^m \phi\left(T^{(1)}_{i_0,i_1}, \ldots, T^{(N)}_{i_{N-1},i_N}\right)$ as $\phi\left(T^{(1)}_{i_0,i_1}\right)$. So, no summation takes place if $N = 1$. For those readers familiar with amplifications in the context of linear maps, note that the case $N = 1$ is simply the linear case. Note that N-linearity of ϕ implies that ϕ_m is also N-linear. Also note that ϕ_1 can just be identified with ϕ itself by identifying $M_1(E_n)$ with E_n and $M_1(F)$ with F.

Note that the above amplification defines a map

$$\mathcal{L}^0(E_1, \ldots, E_N; F) \to \mathcal{L}^0(M_m(E_1), \ldots, M_m(E_N); M_m(F)). \tag{60}$$

From the above definition of amplification it is clear that the amplification map is linear. Recall that $\mathcal{L}^0(M_m(E_1), \ldots, M_m(E_N); M_m(F))$ has a linear subspace, $\mathcal{L}(M_m(E_1), \ldots, M_m(E_N); M_m(F))$, which is a normed space. Let $\mathcal{L}^m(E_1, \ldots, E_N; F)$ be the preimage of $\mathcal{L}(M_m(E_1), \ldots, M_m(E_N); M_m(F))$ under the amplification map. So,

$$\mathcal{L}^m(E_1, \ldots, E_N; F) = \{\phi \in \mathcal{L}^0(E_1, \ldots, E_N; F) : \phi_m \in \mathcal{L}(M_m(E_1), \ldots, M_m(E_N); M_m(F))\}. \tag{61}$$

Note that from the identification of ϕ_1 with ϕ it is clear that $\mathcal{L}^1(E_1, \ldots, E_N; F) = \mathcal{L}(E_1, \ldots, E_N; F)$. For general $m \in \mathbb{N}$ it follows from the linearity of the amplification map that $\mathcal{L}^m(E_1, \ldots, E_N; F)$ is a linear subspace of $\mathcal{L}^0(E_1, \ldots, E_N; F)$ and that $\|\phi\|_m = \|\phi_m\|$ defines a seminorm on $\mathcal{L}^m(E_1, \ldots, E_N; F)$. We will show that $\|\cdot\|_m$ is in fact a norm. This will be a consequence of the following stronger result.

Lemma 2.6. *Let $N \in \mathbb{N}$ and for $j \in \{1, \ldots, N\}$ let H_j be a Hilbert space and E_j a linear subspace of $\mathcal{B}(H_j)$. Also let K be a Hilbert space and F a linear subspace of $\mathcal{B}(K)$. Let $m, n \in \mathbb{N}$ with $m \leq n$. Then we have that*

$$\mathcal{L}^n(E_1, \ldots, E_N; F) \subseteq \mathcal{L}^m(E_1, \ldots, E_N; F) \tag{62}$$

and $\|\phi\|_m \leq \|\phi\|_n$ for all $\phi \in \mathcal{L}^m(E_1, \ldots, E_N; F)$.

Proof. Let $E_1, \ldots, E_N, F$ and m, n be as in the above statement. For $A \in \{E_1, \ldots, E_N, F\}$ let $\sigma^A_{n,m}$ be the isometric linear embedding of $M_m(A)$ into $M_n(A)$ we introduced in Theorem 2.5. So $\sigma^A_{n,m}\left((T_{k,l})_{k,l=1}^m\right)_{i,j} = T_{i,j}$ if $i, j \leq m$ and $\sigma^A_{n,m}\left((T_{k,l})_{k,l=1}^m\right)_{i,j} = 0$ otherwise. We will first show that

$$\phi_n\left(\sigma^{E_1}_{n,m}\left(T^{(1)}\right), \ldots, \sigma^{E_N}_{n,m}\left(T^{(N)}\right)\right) = \sigma^F_{n,m}\left(\phi_m\left(T^{(1)}, \ldots, T^{(N)}\right)\right) \tag{63}$$

for all $T^{(1)} \in M_m(E_1), \dots, T^{(N)} \in M_m(E_N)$. Let $T^{(1)} \in M_m(E_1), \dots, T^{(N)} \in M_m(E_N)$. Then we have that

$$\phi_n\left(\sigma_{n,m}^{E_1}\left((T^{(1)}\right), \dots, \sigma_{n,m}^{E_N}\left(T^{(N)}\right)\right) = \\ \left(\sum_{i_1=1}^{n} \cdots \sum_{i_{N-1}=1}^{n} \phi\left(\left(\sigma_{n,m}^{E_1}\left(T^{(1)}\right)\right)_{i_0,i_1}, \dots, \left(\sigma_{n,m}^{E_N}\left(T^{(N)}\right)\right)_{i_{N-1},i_N}\right)\right)_{i_0,i_N=1}^{n}. \tag{64}$$

Now note that $\phi((\sigma_{n,m}^{E_1}(T^{(1)}))_{i_0,i_1}, \dots, (\sigma_{n,m}^{E_N}(T^{(N)}))_{i_{N-1},i_N})$ will be non-zero only if all of its entries are non-zero, which can only occur if $i_j \leq m$ for all $j \in \{0, \dots, N\}$. It follows that

$$\phi_n\left(\sigma_{n,m}^{E_1}\left(T^{(1)}\right), \dots, \sigma_{n,m}^{E_N}\left(T^{(N)}\right)\right)_{i_0,i_N} = 0 \tag{65}$$

if $i_0 > m$ or $i_N > m$. On the other hand if $i_0, i_N \leq m$, then we have that

$$\begin{aligned} &\phi_n\left(\sigma_{n,m}^{E_1}\left(T^{(1)}\right), \dots, \sigma_{n,m}^{E_N}\left(T^{(N)}\right)\right)_{i_0,i_N} = \\ &\sum_{i_1=1}^{n} \cdots \sum_{i_{N-1}=1}^{n} \phi\left(\left(\sigma_{n,m}^{E_1}\left(T^{(1)}\right)\right)_{i_0,i_1}, \dots, \left(\sigma_{n,m}^{E_N}\left(T^{(N)}\right)\right)_{i_{N-1},i_N}\right) = \\ &\sum_{i_1=1}^{m} \cdots \sum_{i_{N-1}=1}^{m} \phi\left(\left(\sigma_{n,m}^{E_1}\left(T^{(1)}\right)\right)_{i_0,i_1}, \dots, \left(\sigma_{n,m}^{E_N}\left(T^{(N)}\right)\right)_{i_{N-1},i_N}\right) = \\ &\sum_{i_1=1}^{m} \cdots \sum_{i_{N-1}=1}^{m} \phi\left(T_{i_0,i_1}^{(1)}, \dots, T_{i_{N-1},i_N}^{(N)}\right) = \phi_m\left(T^{(1)}, \dots, T^{(N)}\right)_{i_0,i_N}. \end{aligned} \tag{66}$$

This shows that indeed

$$\phi_n\left(\sigma_{n,m}^{E_1}\left(T^{(1)}\right), \dots, \sigma_{n,m}^{E_N}\left(T^{(N)}\right)\right) = \sigma_{n,m}^{F}\left(\phi_m\left(T^{(1)}, \dots, T^{(N)}\right)\right). \tag{67}$$

In other words we have that

$$\phi_n \circ \times_{k=1}^{N} \sigma_{n,m}^{E_k} = \sigma_{n,m}^{F} \circ \phi_m \tag{68}$$

Using that $\sigma_{n,m}^{E_1}, \dots, \sigma_{n,m}^{E_N}, \sigma_{n,m}^{F}$ are isometric it follows that

$$\|\phi_m\| = \left\|\sigma_{n,m}^{F} \circ \phi_m\right\| = \left\|\phi_n \circ \times_{k=1}^{N} \sigma_{n,m}^{E_k}\right\| \leq \|\phi_n\| \prod_{k=1}^{N} \left\|\sigma_{n,m}^{E_k}\right\| = \|\phi_n\|. \tag{69}$$

This shows that

$$\mathcal{L}^n(E_1, \dots, E_N; F) \subseteq \mathcal{L}^m(E_1, \dots, E_N; F) \tag{70}$$

and

$$\|\phi\|_m = \|\phi_m\| \leq \|\phi_n\| = \|\phi\|_n. \tag{71}$$

□

Corollary 2.7. *Let $N \in \mathbb{N}$ and for $j \in \{1, \dots, N\}$ let H_j be a Hilbert space and E_j a linear subspace of $\mathcal{B}(H_j)$. Also let K be a Hilbert space and F a linear subspace of $\mathcal{B}(K)$. For all $m \in \mathbb{N}$ we have that $\|\cdot\|_m$ is a norm on $\mathcal{L}^m(E_1, \dots, E_N; F)$.*

Proof. We already know that the statement is true for $m = 1$. We also know that for all $m \in \mathbb{N}$, $\|\cdot\|_m$ is a seminorm on $\mathcal{L}^m(E_1, \dots, E_N; F)$. It remain to show that $\|\phi\|_m = 0$ only if $\phi = 0$. But $\|\phi\|_1 \leq \|\phi\|_m$ follows from Lemma 2.6, so $\|\phi\|_m = 0$ implies $\|\phi\|_1 = 0$, which in turn implies $\phi = 0$, finishing the proof. □

We summarize some of the preceding results in the following definition.

Definition 2.6. m-boundedness for multilinear maps

Let $N \in \mathbb{N}$ and for $j \in \{1, \ldots, N\}$ let H_j be a Hilbert space and E_j a linear subspace of $\mathcal{B}(H_j)$. Also let K be a Hilbert space and F a linear subspace of $\mathcal{B}(K)$. For $m \in \mathbb{N}$ we call an N-linear map $\phi \in \mathcal{L}^0(E_1, \ldots, E_N; F)$ m-bounded if ϕ_m is bounded as a multilinear map. $\|\phi\|_m = \|\phi_m\|$ defines a norm on the vector space

$$\mathcal{L}^m(E_1, \ldots, E_N; F) = \{\phi \in \mathcal{L}^0(E_1, \ldots, E_N; F) : \phi_m \in \mathcal{L}(M_m(E_1), \ldots, M_m(E_N); M_m(F))\} \tag{72}$$

of m-bounded N-linear maps. We have

$$\mathcal{L}^n(E_1, \ldots, E_N; F) \subseteq \mathcal{L}^m(E_1, \ldots, E_N; F) \tag{73}$$

when $m \leq n$ and the inclusion map is contractive.

Recall from Remark 2.4 that if $S_1, \ldots, S_N$ are bounded multilinear maps and T is an N-linear map defined on the Cartesian product of the codomains of the maps $S_1, \ldots, S_N$, we can construct a new bounded multilinear map $T \circ \times_{n=1}^N S_n$ with norm no greater than $\|T\| \prod_{n=1}^N \|S_n\|$. A natural question is if this construction can be applied to m-bounded multilinear maps to obtain a new m-bounded multilinear map. To prove that this is indeed the case we need the following lemma, which asserts that this construction interacts with amplifications in a convenient way.

Lemma 2.8. *Let $N \in \mathbb{N}$ and for $n \in \{1, \ldots, N\}$ let $M_n \in \mathbb{N}$. For $n \in \{1, \ldots, N\}$ and $m_n \in \{1, \ldots, M_n\}$ let $H_{m_n,n}$ be a Hilbert space and $E_{m_n,n} \subseteq \mathcal{B}(H_{m_n,n})$ a linear subspace of $\mathcal{B}(H_{m_n,n})$. For $n \in \{1, \ldots, N\}$, let K_n be a Hilbert space and $F_n \subseteq \mathcal{B}(K_n)$ a linear subspace of $\mathcal{B}(K_n)$. Also let L be a Hilbert space and $G \subseteq \mathcal{B}(L)$ a linear subspace of $\mathcal{B}(L)$. If $\psi^{(n)} \in \mathcal{L}^0(E_{1,n}, \ldots, E_{M_n,n}; F_n)$ for $n \in \{1, \ldots, N\}$ and if $\phi \in \mathcal{L}^0(F_1, \ldots, F_N; G)$, then for all $k \in \mathbb{N}$ we have that*

$$\left(\phi \circ \times_{n=1}^N \psi^{(n)}\right)_k = \phi_k \circ \times_{n=1}^N \psi_k^{(n)}. \tag{74}$$

Proof. To improve readability we introduce the following notation. Suppose $A = \{j_1, \ldots, j_l\} \subseteq \mathbb{N}$ is a finite subset of $\mathbb{N}$ with $j_1, \ldots, j_l$ distinct and $m \in \mathbb{N}$. Then we write $\sum_{i_A=1}^m f(i_{j_1}, \ldots, i_{j_l})$ as a shorthand for $\sum_{i_{j_1}=1}^m \cdots \sum_{i_{j_l}=1}^m f(i_{j_1}, \ldots, i_{j_l})$ for any function $f : \{1, \ldots, m\}^l \to V$, where V is a vector space. It is clear that if $A_1, \ldots, A_s \subseteq \mathbb{N}$ (with $s \in \mathbb{N}$) are finite and disjoint and $A = \bigcup_{j=1}^s A_j$, then

$$\sum_{i_A=1}^m f((i_j)_{j\in A}) = \sum_{i_{A_1}=1}^m \cdots \sum_{i_{A_s}=1}^m f((i_j)_{j\in A}). \tag{75}$$

Let

$$T^{(m_n,n)} = \left(T_{i,j}^{(m_n,n)}\right)_{i,j=1}^k \in M_k(E_{m_n,n}) \tag{76}$$

for all $n \in \{1, \ldots, N\}$ and $m_n \in \{1, \ldots, M_n\}$. Then we have that

$$\psi_k^{(n)}\left(\left(T^{(m_n,n)}\right)_{m_n=1}^{M_n}\right) = \left(\sum_{i_{A_n}=1}^k \psi^{(n)}\left(\left(T^{(m_n,n)}_{i_{m_n-1+\sum_{j=1}^{n-1} M_j}, i_{m_n+\sum_{j=1}^{n-1} M_j}}\right)_{m_n=1}^{M_n}\right)\right)_{i_{\sum_{j=1}^{n-1} M_j}, i_{\sum_{j=1}^{n} M_j}=1}^k, \tag{77}$$

where $A_n = \{\sum_{j=1}^{n-1} M_j + 1, \sum_{j=1}^{n-1} M_j + 2, \ldots, \sum_{j=1}^{n} M_j - 1\}$. Note that $A_1, \ldots, A_n$ are all disjoint and together with $B = \{\sum_{j=1}^{1} M_j, \sum_{j=1}^{2} M_j, \ldots, \sum_{j=1}^{N-1} M_j\}$ they form a partition of $A = \{1, 2, \ldots, \sum_{j=1}^{N} M_j - 1\}$. It follows that

$$\left(\phi_k \circ \times_{n=1}^N \psi_k^{(n)}\right)\left(\left(\left(T^{(m_n,n)}\right)_{m_n=1}^{M_n}\right)_{n=1}^{N}\right) = \phi_k\left(\left(\psi_k^{(n)}\left(\left(T^{(m_n,n)}\right)_{m_n=1}^{M_n}\right)\right)_{n=1}^{N}\right) =$$

$$\left(\sum_{i_B=1}^{k} \phi\left(\left(\left(\psi_k^{(n)}\left(\left(T^{(m_n,n)}\right)_{m_n=1}^{M_n}\right)\right)_{i_{\sum_{j=1}^{n-1} M_j}, i_{\sum_{j=1}^{n} M_j}}\right)_{n=1}^{N}\right)\right)_{i_0, i_{\sum_{j=1}^{N} M_j}=1}^{k} =$$

$$\left(\sum_{i_B=1}^{k} \phi\left(\left(\sum_{i_{A_n}=1}^{k} \psi^{(n)}\left(\left(T^{(m_n,n)}_{i_{m_n-1+\sum_{j=1}^{n-1} M_j}, i_{m_n+\sum_{j=1}^{n-1} M_j}}\right)_{m_n=1}^{M_n}\right)\right)_{n=1}^{N}\right)\right)_{i_0, i_{\sum_{j=1}^{N} M_j}=1}^{k} =$$

$$\left(\sum_{i_B=1}^{k} \sum_{i_{A_1}=1}^{k} \cdots \sum_{i_{A_N}=1}^{k} \phi\left(\left(\psi^{(n)}\left(\left(T^{(m_n,n)}_{i_{m_n-1+\sum_{j=1}^{n-1} M_j}, i_{m_n+\sum_{j=1}^{n-1} M_j}}\right)_{m_n=1}^{M_n}\right)\right)_{n=1}^{N}\right)\right)_{i_0, i_{\sum_{j=1}^{N} M_j}=1}^{k} = \tag{78}$$

$$\left(\sum_{i_A=1}^{k} \phi\left(\left(\psi^{(n)}\left(\left(T^{(m_n,n)}_{i_{m_n-1+\sum_{j=1}^{n-1} M_j}, i_{m_n+\sum_{j=1}^{n-1} M_j}}\right)_{m_n=1}^{M_n}\right)\right)_{n=1}^{N}\right)\right)_{i_0, i_{\sum_{j=1}^{N} M_j}=1}^{k} =$$

$$\left(\sum_{i_A=1}^{k} \left(\phi \circ \times_{n=1}^N \psi^{(n)}\right)\left(\left(\left(T^{(m_n,n)}_{i_{m_n-1+\sum_{j=1}^{n-1} M_j}, i_{m_n+\sum_{j=1}^{n-1} M_j}}\right)_{m_n=1}^{M_n}\right)_{n=1}^{N}\right)\right)_{i_0, i_{\sum_{j=1}^{N} M_j}=1}^{k} =$$

$$\left(\phi \circ \times_{n=1}^N \psi^{(n)}\right)_k\left(\left(\left(T^{(m_n,n)}\right)_{m_n=1}^{M_n}\right)_{n=1}^{N}\right).$$

This shows that $\left(\phi_k \circ \times_{n=1}^N \psi_k^{(n)}\right) = \left(\phi \circ \times_{n=1}^N \psi^{(n)}\right)_k$, completing the proof. □

Using Lemma 2.8 we can now prove the following theorem.

Theorem 2.9. *Let $N \in \mathbb{N}$ and for $n \in \{1, \dots, N\}$ let $M_n \in \mathbb{N}$. For $n \in \{1, \dots, N\}$ and $m_n \in \{1, \dots, M_n\}$ let $H_{m_n,n}$ be a Hilbert space and $E_{m_n,n} \subseteq \mathcal{B}(H_{m_n,n})$ a linear subspace of $\mathcal{B}(H_{m_n,n})$. For $n \in \{1, \dots, N\}$, let K_n be a Hilbert space and $F_n \subseteq \mathcal{B}(K_n)$ a linear subspace of $\mathcal{B}(K_n)$. Also let L be a Hilbert space and $G \subseteq \mathcal{B}(L)$ a linear subspace of $\mathcal{B}(L)$. Let $k \in \mathbb{N}$. If $\psi^{(n)} \in \mathcal{L}^k(E_{1,n}, \dots, E_{M_n,n}; F_n)$ for $n \in \{1, \dots, N\}$ and if $\phi \in \mathcal{L}^k(F_1, \dots, F_N; G)$, then $\phi \circ \times_{n=1}^N \psi^{(n)} \in \mathcal{L}^k(E_{1,1}, \dots, E_{M_1,1}, \dots, E_{1,N}, \dots, E_{M_N,N}; G)$ with*

$$\left\|\phi \circ \times_{n=1}^N \psi^{(n)}\right\|_k \leq \|\phi\|_k \prod_{n=1}^{N} \left\|\psi^{(n)}\right\|_k. \tag{79}$$

Proof. We already know, from Remark 2.4, that the theorem holds for $k = 1$. Using this and Lemma 2.8 we find that

$$\left(\phi \circ \times_{n=1}^N \psi^{(n)}\right)_k = \phi_k \circ \times_{n=1}^N \psi_k^{(n)} \tag{80}$$

is bounded, so $\phi \circ \times_{n=1}^N \psi^{(n)} \in \mathcal{L}^k(E_{1,1}, \dots, E_{M_1,1}, \dots, E_{1,N}, \dots, E_{M_N,N}; G)$, and

$$\left\|\phi \circ \times_{n=1}^N \psi^{(n)}\right\|_k = \left\|\left(\phi \circ \times_{n=1}^N \psi^{(n)}\right)_k\right\| = \left\|\phi_k \circ \times_{n=1}^N \psi_k^{(n)}\right\| \leq \|\phi_k\| \prod_{n=1}^{N} \left\|\psi_k^{(n)}\right\| = \|\phi\|_k \prod_{n=1}^{N} \left\|\psi^{(n)}\right\|_k. \tag{81}$$

This completes the proof. □

In Remark 2.5 we have seen that from a (bounded) multilinear map, new bounded multilinear maps can be constructed by fixing some of the variables. Our next result shows that this construction preserves m-boundedness.

Theorem 2.10. *Let $N \in \mathbb{N}$ and for $j \in \{1, \ldots, N\}$ let H_j be a Hilbert space and E_j a linear subspace of $\mathcal{B}(H_j)$. Also let K be a Hilbert space and F a linear subspace of $\mathcal{B}(K)$. Let $m \in \mathbb{N}$ and $\phi \in \mathcal{L}^m(E_1, \ldots, E_N; F)$. Let $J \subset \{1, \ldots, N\}$ be a strict subset and $S_j \in E_j$ for all $j \in J$. Then we have that $\phi\left(\cdot|J; (S_j)_{j\in J}\right) \in \mathcal{L}^m\left((E_j)_{j\in\{1,\ldots,N\}\setminus J}; F\right)$ with*

$$\left\|\phi\left(\cdot|J; (S_j)_{j=1}^N\right)\right\|_m \leq \|\phi\|_m \prod_{j\in J} \|S_j\| \tag{82}$$

Proof. Let N, E_j, F, m, ϕ be as in the statement of the theorem. Note that the theorem is clearly true when J is empty. Also note that, because we can fix the variables one at a time (see Remark 2.5), it is sufficient to prove the theorem in case J is a singleton set. The general statement then follows by repeatedly applying the case where only one variable is fixed. So, we assume that $N \geq 2$ and $J = \{k\}$ for some $k \in \{1, \ldots, N\}$. Let $S \in E_k$. Let $T^{(k)} \in M_m(E_k)$ be given by $T^{(k)} = (\delta_{i,j}S)_{i,j=1}^n$. For $n \in \{1, \ldots, N\}$ with $n \neq k$ choose $T^{(n)} \in M_m(E_n)$ arbitrarily. For $i_{k-1}, i_k \in \{1, \ldots, m\}$ note that

$$T^{(k)}_{i_{k-1},i_k} = \delta_{i_{k-1},i_k} S. \tag{83}$$

It follows that

$$\begin{aligned}
\phi_m\left(T^{(1)}, \ldots, T^{(N)}\right) &= \left(\sum_{i_1=1}^m \cdots \sum_{i_{N-1}=1}^m \phi\left(T^{(1)}_{i_0,i_1}, \ldots, T^{(N)}_{i_{N-1},i_N}\right)\right)_{i_0,i_N=1}^m \\
&= \left(\sum_{i_1=1}^m \cdots \sum_{i_{N-1}=1}^m \phi\left(T^{(1)}_{i_0,i_1}, \ldots, T^{(k-1)}_{i_{k-2},i_{k-1}}, \delta_{i_{k-1},i_k} S, T^{(k+1)}_{i_k,i_{k+1}} \ldots, T^{(N)}_{i_{N-1},i_N}\right)\right)_{i_0,i_N=1}^m \\
&= \left(\sum_{i_1=1}^m \cdots \sum_{i_{k-1}=1}^m \sum_{i_{k+1}=1}^m \cdots \sum_{i_{N-2}=1}^m \phi\left(T^{(1)}_{i_0,i_1}, \ldots, T^{(k-1)}_{i_{k-2},i_{k-1}}, S, T^{(k+1)}_{i_{k-1},i_{k+1}} \ldots, T^{(N)}_{i_{N-1},i_N}\right)\right)_{i_0,i_N=1}^m \\
&= \psi_m\left(\left(T^{(n)}\right)_{n\in\{1,\ldots,N\}\setminus\{k\}}\right),
\end{aligned} \tag{84}$$

where $\psi = \phi(\cdot|\{k\}; S)$. Using Remark 2.7 it follows that

$$\begin{aligned}
\left\|\psi_m\left(\left(T^{(n)}\right)_{n\in\{1,\ldots,N\setminus\{k\}}\right)\right\|_m = \left\|\phi_m\left(T^{(1)}, \ldots, T^{(N)}\right)\right\|_m &\leq \|\phi\|_m \prod_{n=1}^N \left\|T^{(n)}\right\|_m \\
&= \|\phi\|_m \|S\| \prod_{n\in\{1,\ldots,N\}\setminus\{k\}} \left\|T^{(n)}\right\|_m.
\end{aligned} \tag{85}$$

This shows that $\phi(\cdot|\{k\}; S) \in \mathcal{L}^m\left((E_j)_{j\in\{1,\ldots,N\}\setminus\{k\}}; F\right)$ with

$$\|\phi(\cdot|\{k\}; S)\|_m \leq \|\phi\|_m \|S\|. \tag{86}$$

This proves the theorem in case J is a singleton set and as we have already argued, by repeated application of this result (fixing one variable at a time), the general statement of the theorem follows. □

We have seen in Theorem 2.1 that $\mathcal{L}(A_1, \ldots, A_N; B)$ is a Banach space if $A_1, \ldots, A_N$ are normed vector spaces and B is a Banach space. We will show that $\mathcal{L}^k(E_1, \ldots, E_N; F)$ is a Banach space if $E_1, \ldots, E_N$ are linear subspaces of spaces of bounded operators on a Hilbert space and F is a closed linear subspace of the space of bounded operators on a Hilbert space. The following lemma will be used to this end.

Lemma 2.11. *Let $N \in \mathbb{N}$ and for $j \in \{1, \ldots, N\}$ let H_j be a Hilbert space and E_j a linear subspace of $\mathcal{B}(H_j)$. Also let K be a Hilbert space and F a linear subspace of $\mathcal{B}(K)$. Let $(\phi^{(m)})_{m\in\mathbb{N}}$ be a sequence in $\mathcal{L}^0(E_1, \ldots, E_N; F)$ that converges pointwise to $\phi \in \mathcal{L}^0(E_1, \ldots, E_N; F)$. Then for all $k \in \mathbb{N}$, the sequence $(\phi_k^{(m)})_{m\in\mathbb{N}}$ in $\mathcal{L}^0(M_k(E_1), \ldots, M_k(E_N); M_k(F))$ converges pointwise to ϕ_k.*

Proof. For $n \in \{1, \ldots, N\}$, let $T^{(n)} \in M_k(E_n)$. From the pointwise convergence of $(\phi^{(m)})_{m\in\mathbb{N}}$ to ϕ it is clear that

$$\sum_{i_1=1}^{k} \cdots \sum_{i_{N-1}=1}^{k} \phi^{(m)}\left(T^{(1)}_{i_0,i_1}, \ldots, T^{(N)}_{i_{N-1},i_N}\right) \to \sum_{i_1=1}^{k} \cdots \sum_{i_{N-1}=1}^{k} \phi\left(T^{(1)}_{i_0,i_1}, \ldots, T^{(N)}_{i_{N-1},i_N}\right) \tag{87}$$

for all $i_0, i_N \in \{1, \ldots, k\}$. As can be seen from Definition 2.5, this means that the matrix entries of $\phi_k^{(m)}(T^{(1)}, \ldots, T^{(N)})$ converge to the matrix entries of $\phi_k(T^{(1)}, \ldots, T^{(N)})$. It follows from Remark 2.8, that $\phi_k^{(m)}(T^{(1)}, \ldots, T^{(N)})$ converges to $\phi_k(T^{(1)}, \ldots, T^{(N)})$. So, $\phi_k^{(m)}$ converges pointwise to ϕ_k. □

Theorem 2.12. *Let $N \in \mathbb{N}$ and for $j \in \{1, \ldots, N\}$ let H_j be a Hilbert space and E_j a linear subspace of $\mathcal{B}(H_j)$. Also let K be a Hilbert space and F a closed linear subspace of $\mathcal{B}(K)$. Then $\mathcal{L}^k(E_1, \ldots, E_N; F)$ is a Banach space for all $k \in \mathbb{N}$.*

Proof. We already know from Corollary 2.7 that $\mathcal{L}^k(E_1, \ldots, E_N; F)$ is a normed vector space. So, it remains to prove completeness. We also know from Theorem 2.1 that the theorem holds for $k = 1$, because F is a Banach space. Let $k \in \mathbb{N}$ and $(\phi^{(m)})_{m\in\mathbb{N}}$ a Cauchy sequence in $\mathcal{L}^k(E_1, \ldots, E_N; F)$ (with respect to $\|\cdot\|_k$). The inequality

$$\left\|\phi^{(m)} - \phi^{(l)}\right\| = \left\|\phi^{(m)} - \phi^{(l)}\right\|_1 \leq \left\|\phi^{(m)} - \phi^{(l)}\right\|_k, \tag{88}$$

which holds by Lemma 2.6, shows that $(\phi^{(m)})_{m\in\mathbb{N}}$ is also Cauchy in $\mathcal{L}(E_1, \ldots, E_N; F)$.

Because $\mathcal{L}(E_1, \ldots, E_N; F)$ is complete, we have that $\phi^{(m)}$ converges in $\|\cdot\|$ to some $\phi \in \mathcal{L}(E_1, \ldots, E_N; F)$. It follows that $\phi^{(m)}$ also converges to ϕ pointwise. By Lemma 2.11 we have that $\phi_k^{(m)}$ converges pointwise to ϕ_k. The equality

$$\left\|\phi_k^{(m)} - \phi_k^{(l)}\right\| = \left\|\phi^{(m)} - \phi^{(l)}\right\|_k \tag{89}$$

shows that $(\phi_k^{(m)})_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}(M_k(E_1), \ldots, M_k(E_N); M_k(F))$. Note that $M_k(F)$ is a closed linear subspace of $M_k(\mathcal{B}(K))$ by Corollary 2.4, because F is closed. Hence $M_k(F)$ is a Banach space, because $M_k(\mathcal{B}(K))$ is a Banach space. This implies that $\mathcal{L}(M_k(E_1), \ldots, M_k(E_N); M_k(F))$ is a Banach space. So, the Cauchy sequence $(\phi_k^{(m)})_{m\in\mathbb{N}}$ must converge (in norm) to some $\psi \in \mathcal{L}(M_k(E_1), \ldots, M_k(E_N); M_k(F))$. It follows that $\phi_k^{(m)}$ also converges to ψ pointwise. But then we must have that $\psi = \phi_k$. So $\phi_k \in \mathcal{L}(M_k(E_1), \ldots, M_k(E_N); M_k(F))$ and $\phi_k^{(m)}$ converges to ϕ_k in norm. In other words we have that $\phi \in \mathcal{L}^k(E_1, \ldots, E_N; F)$ and $\phi^{(m)}$ converges to ϕ in $\|\cdot\|_k$. This shows that $\mathcal{L}^k(E_1, \ldots, E_N; F)$ is a Banach space. □

2.3 Completely bounded multilinear maps

In this subsection we will define completely bounded multilinear maps in a way that generalizes completely bounded linear maps. These are the multilinear maps ϕ that are k-bounded for all $k \in \mathbb{N}$ with $\sup_{k\in\mathbb{N}} \|\phi\|_k < \infty$. As we shall see, the results we proved about k-bounded multilinear maps about completeness and construction of multilinear maps will carry over to completely bounded maps. In fact, we have already gone through most of the technical details in subsection 2.2. We note that our definition of complete boundedness for multilinear maps is not the only way to generalize the complete boundedness of linear maps. Complete boundedness for multilinear maps is sometimes defined differently in the literature. For example, in [14], what we call completely bounded multilinear maps are instead called multiplicatively bounded and the term completely bounded has another meaning. We have chosen to call our definition completely bounded, to avoid having different names for a similar property of linear and multilinear maps. We stress that when combining or comparing our results with those from the literature, it should be checked that the same (or equivalent) definitions are used.

Definition 2.7. Completely bounded multilinear maps

Let $N \in \mathbb{N}$ and for $j \in \{1, \ldots, N\}$ let H_j be a Hilbert space and E_j a linear subspace of $\mathcal{B}(H_j)$. Also let K be a Hilbert space and F a linear subspace of $\mathcal{B}(K)$. We call a multilinear map $\phi \in \mathcal{L}^0(E_1, \ldots, E_N; F)$ completely bounded if $\phi \in \mathcal{L}^k(E_1, \ldots, E_N; F)$ for all $k \in \mathbb{N}$ and

$$\|\phi\|_{CB} := \sup_{k \in \mathbb{N}} \|\phi\|_k < \infty. \tag{90}$$

Note that $\|\phi\|_k \leq \|\phi\|_{CB}$ for all $k \in \mathbb{N}$. Let $CB(E_1, \ldots, E_N; F)$ denote that set of all completely bounded multilinear maps in $\mathcal{L}^0(E_1, \ldots, E_N; F)$.

Theorem 2.13. *Let $N \in \mathbb{N}$ and for $j \in \{1, \ldots, N\}$ let H_j be a Hilbert space and E_j a linear subspace of $\mathcal{B}(H_j)$. Also let K be a Hilbert space and F a linear subspace of $\mathcal{B}(K)$. Then $CB(E_1, \ldots, E_N; F)$ is a vector space and $\|\cdot\|_{CB}$ is a norm on $CB(E_1, \ldots, E_N; F)$.*

Proof. Let $\phi, \psi \in CB(E_1, \ldots, E_N; F)$ and $c \in \mathbb{C}$. Then we have that

$$\|\phi + \psi\|_{CB} = \sup_{k \in \mathbb{N}} \|\phi + \psi\|_k \leq \sup_{k \in \mathbb{N}} (\|\phi\|_k + \|\psi\|_k) \leq \sup_{k \in \mathbb{N}} \|\phi\|_k + \sup_{k \in \mathbb{N}} \|\psi\|_k = \|\phi\|_{CB} + \|\psi\|_{CB} < \infty \tag{91}$$

and

$$\|c\phi\|_{CB} = \sup_{k \in \mathbb{N}} \|c\phi\|_k = \sup_{k \in \mathbb{N}} |c| \|\phi\|_k = |c| \sup_{k \in \mathbb{N}} \|\phi\|_k = |c| \|\phi\|_{CB} < \infty. \tag{92}$$

This shows that $CB(E_1, \ldots, E_N; F)$ is a vector space and $\|\cdot\|_{CB}$ is a seminorm on $CB(E_1, \ldots, E_N; F)$. If $\|\phi\|_{CB} = 0$, then $\|\phi\| = 0$ as well, hence $\phi = 0$. This shows that $\|\cdot\|_{CB}$ is a norm. □

Some of the results we have shown for k-bounded multilinear maps have analogous results that hold for completely bounded multilinear maps.

Theorem 2.14. *Let $N \in \mathbb{N}$ and for $n \in \{1, \ldots, N\}$ let $M_n \in \mathbb{N}$. For $n \in \{1, \ldots, N\}$ and $m_n \in \{1, \ldots, M_n\}$ let $H_{m_n,n}$ be a Hilbert space and $E_{m_n,n} \subseteq \mathcal{B}(H_{m_n,n})$ a linear subspace of $\mathcal{B}(H_{m_n,n})$. For $n \in \{1, \ldots, N\}$, let K_n be a Hilbert space and $F_n \subseteq \mathcal{B}(K_n)$ a linear subspace of $\mathcal{B}(K_n)$. Also let L be a Hilbert space and $G \subseteq \mathcal{B}(L)$ a linear subspace of $\mathcal{B}(L)$. If $\psi^{(n)} \in CB(E_{1,n}, \ldots, E_{M_n,n}; F_n)$ for $n \in \{1, \ldots, N\}$ and if $\phi \in CB(F_1, \ldots, F_N; G)$, then $\phi \circ \times_{n=1}^N \psi^{(n)} \in CB(E_{1,1}, \ldots, E_{M_1,1}, \ldots, E_{1,N}, \ldots, E_{M_N,N}; G)$ with*

$$\left\| \phi \circ \times_{n=1}^N \psi^{(n)} \right\|_{CB} \leq \|\phi\|_{CB} \prod_{n=1}^N \left\| \psi^{(n)} \right\|_{CB}. \tag{93}$$

Proof. Using Theorem 2.9 we have that $\phi \circ \times_{n=1}^N \psi^{(n)} \in \mathcal{L}^k(E_{1,1}, \ldots, E_{M_1,1}, \ldots, E_{1,N}, \ldots, E_{M_N,N}; G)$ with

$$\left\| \phi \circ \times_{n=1}^N \psi^{(n)} \right\|_k \leq \|\phi\|_k \prod_{n=1}^N \left\| \psi^{(n)} \right\|_k \leq \|\phi\|_{CB} \prod_{n=1}^N \left\| \psi^{(n)} \right\|_{CB} < \infty \tag{94}$$

for all $k \in \mathbb{N}$. But then

$$\left\| \phi \circ \times_{n=1}^N \psi^{(n)} \right\|_{CB} = \sup_{k \in \mathbb{N}} \left\| \phi \circ \times_{n=1}^N \psi^{(n)} \right\|_k \leq \|\phi\|_{CB} \prod_{n=1}^N \left\| \psi^{(n)} \right\|_{CB} < \infty, \tag{95}$$

which also shows that $\phi \circ \times_{n=1}^N \psi^{(n)} \in CB(E_{1,1}, \ldots, E_{M_1,1}, \ldots, E_{1,N}, \ldots, E_{M_N,N}; G)$. □

Theorem 2.15. *Let $N \in \mathbb{N}$ and for $j \in \{1, \ldots, N\}$ let H_j be a Hilbert space and E_j a linear subspace of $\mathcal{B}(H_j)$. Also let K be a Hilbert space and F a linear subspace of $\mathcal{B}(K)$. Let $\phi \in CB(E_1, \ldots, E_N; F)$. Let $J \subset \{1, \ldots, N\}$ be a strict subset and $S_j \in E_j$ for all $j \in J$. Then we have that $\phi\left(\cdot | J; (S_j)_{j \in J}\right) \in CB\left((E_j)_{j \in \{1, \ldots, N\} \setminus J}; F\right)$ with*

$$\left\| \phi\left(\cdot | J; (S_j)_{j \in J}\right) \right\|_{CB} \leq \|\phi\|_{CB} \prod_{j \in J} \|S_j\|. \tag{96}$$

Proof. Let N, E_j, F, ϕ, J, S_j be as in the statement of the theorem. Then it follows from Theorem 2.10 that for all $m \in \mathbb{N}$: $\phi\left(\cdot|J;(S_j)_{j\in J}\right) \in \mathcal{L}^m\left((E_j)_{j\in\{1,\ldots,N\}\setminus J};F\right)$ with

$$\left\|\phi\left(\cdot|J;(S_j)_{j\in J}\right)\right\|_m \leq \|\phi\|_m \prod_{j\in J}\|S_j\| \leq \|\phi\|_{CB} \prod_{j\in J}\|S_j\|. \tag{97}$$

This implies that $\phi\left(\cdot|J;(S_j)_{j\in J}\right) \in CB\left((E_j)_{j\in\{1,\ldots,N\}\setminus J};F\right)$ with

$$\left\|\phi\left(\cdot|J;(S_j)_{j\in J}\right)\right\|_{CB} \leq \|\phi\|_{CB} \prod_{j\in J}\|S_j\|. \tag{98}$$

□

Theorem 2.16. *Let $N \in \mathbb{N}$ and for $j \in \{1,\ldots,N\}$ let H_j be a Hilbert space and E_j a linear subspace of $\mathcal{B}(H_j)$. Also let K be a Hilbert space and F a closed linear subspace of $\mathcal{B}(K)$. Then $CB(E_1,\ldots,E_N;F)$ is a Banach space.*

Proof. We already know that $CB(E_1,\ldots,E_N;F)$ is a normed vector space by Theorem 2.13. So, it remains to show completeness. We also recall from Theorem 2.12 that $\mathcal{L}^k(E_1,\ldots,E_N;F)$ is a Banach space for all $k \in \mathbb{N}$. Let $(\phi^{(m)})_{m\in\mathbb{N}}$ be a Cauchy sequence in $CB(E_1,\ldots,E_N;F)$ (with respect to $\|\cdot\|_{CB}$). It follows from the reverse triangle inequality that $(\|\phi^{(m)}\|_{CB})_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, hence it is bounded. So $C := \sup_{m\in\mathbb{N}}\|\phi^{(m)}\|_{CB} < \infty$. Because $\|\psi\|_k \leq \|\psi\|_{CB}$ for all $k \in \mathbb{N}$ and all $\psi \in CB(E_1,\ldots,E_N;F)$, it follows that for all $k \in \mathbb{N}$ the sequence $(\phi^{(m)})_{m\in\mathbb{N}}$ is Cauchy with respect to $\|\cdot\|_k$ and bounded in $\|\cdot\|_k$ by C. In particular we have that $(\phi^{(m)})_{m\in\mathbb{N}}$ is Cauchy with respect to $\|\cdot\|$. By completeness of $\mathcal{L}(E_1,\ldots,E_N;F)$, $\phi^{(m)}$ converges to some $\phi \in \mathcal{L}(E_1,\ldots,E_N,F)$ in $\|\cdot\|$. Let $k \in \mathbb{N}$. Because $(\phi^{(m)})_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}^k(E_1,\ldots,E_N;F)$ with respect to $\|\cdot\|_k$, it converges to some $\psi \in \mathcal{L}^k(E_1,\ldots,E_N;F)$ with respect to $\|\cdot\|_k$. It follows from

$$\left\|\phi^{(m)} - \psi\right\| \leq \left\|\phi^{(m)} - \psi\right\|_k \tag{99}$$

that $\phi^{(m)}$ also converges to ψ in $\|\cdot\|$. But then $\psi = \phi$ by uniqueness of limits. So, we have that $\phi^{(m)}$ converges to $\phi \in \mathcal{L}^k(E_1,\ldots,E_N;F)$ in $\|\cdot\|_k$. This holds for all $k \in \mathbb{N}$. For all $k, m \in \mathbb{N}$ we have that

$$\left\|\phi^{(m)}\right\|_k \leq C, \tag{100}$$

hence also

$$\|\phi\|_k = \lim_{m\to\infty}\left\|\phi^{(m)}\right\|_k \leq C \tag{101}$$

for all $k \in \mathbb{N}$. This shows that $\phi \in CB(E_1,\ldots,E_N;F)$ (with $\|\phi\|_{CB} \leq C$). To shows convergence of $\phi^{(m)}$ to ϕ in $\|\cdot\|_{CB}$, let $\epsilon > 0$ and choose $M \in \mathbb{N}$ such that for all $m, l \geq M$ we have that

$$\left\|\phi^{(m)} - \phi^{(l)}\right\|_{CB} < \frac{1}{2}\epsilon. \tag{102}$$

Then for all $k \in \mathbb{N}$ and $m, l \geq M$ we have that

$$\left\|\phi^{(m)} - \phi^{(l)}\right\|_k \leq \left\|\phi^{(m)} - \phi^{(l)}\right\|_{CB} < \frac{1}{2}\epsilon. \tag{103}$$

It follows that

$$\left\|\phi^{(m)} - \phi\right\|_k = \lim_{l\to\infty}\left\|\phi^{(m)} - \phi^{(l)}\right\|_k \leq \frac{1}{2}\epsilon \tag{104}$$

for all $k \in \mathbb{N}$ and $m \geq M$. But then we have that

$$\left\|\phi^{(m)} - \phi\right\|_{CB} = \sup_{k\in\mathbb{N}}\left\|\phi^{(m)} - \phi\right\|_k \leq \frac{1}{2}\epsilon < \epsilon \tag{105}$$

for all $m \geq M$. This shows that $\phi^{(m)}$ converges to ϕ in $\|\cdot\|_{CB}$. We conclude that $CB(E_1, \ldots, E_N; F)$ is a Banach space. $\square$

We present the following example of a completely bounded multilinear map.

Lemma 2.17. *Let H be a Hilbert space and $E \subseteq \mathcal{B}(H)$ a subalgebra of $\mathcal{B}(H)$. Let $N \in \mathbb{N}$ and consider the map $\phi : E^{\times N} \to E$ given by*

$$\phi(x_1, \ldots, x_N) = \prod_{n=1}^{N} x_n. \tag{106}$$

Then ϕ is a completely bounded multilinear map with $\|\phi\|_{CB} \leq 1$.

Proof. Note that ϕ is a bounded multilinear map with $\|\phi\| \leq 1$ because the norm on $\mathcal{B}(H)$ is submultiplicative. Let $m \in \mathbb{N}$. Then $\phi_m : M_m(E)^{\times N} \to M_m(E)$ is given by

$$\begin{aligned} \phi_m\left(T^{(1)}, \ldots, T^{(N)}\right) &= \left(\sum_{i_1=1}^{m} \cdots \sum_{i_{N-1}=1}^{m} \phi\left(T^{(1)}_{i_0,i_1}, \ldots, T^{(N)}_{i_{N-1},i_N}\right)\right)_{i_0,i_N=1}^{m} \\ &= \left(\sum_{i_1=1}^{m} \cdots \sum_{i_{N-1}=1}^{m} \prod_{n=1}^{N} T^{(n)}_{i_{n-1},i_n}\right)_{i_0,i_N=1}^{m} \\ &= \prod_{n=1}^{N} T^{(n)}. \end{aligned} \tag{107}$$

Because the norm on $M_m(E)$ is submultiplicative, it follows that

$$\|\phi\|_m = \|\phi_m\| \leq 1. \tag{108}$$

Since this holds for all $m \in \mathbb{N}$, we conclude that ϕ is completely bounded with $\|\phi\|_{CB} \leq 1$. $\square$

With this we conclude our treatment of completely bounded multilinear maps. We will apply the results from this section in subsection 4.2, to prove results about multilinear Fourier multipliers. There are other applications of completely bounded multilinear maps, besides Fourier multipliers. In [14] it is shown that what we call completely bounded multilinear maps (and what they call multiplicatively bounded multilinear maps) are closely related to the Haagerup tensor product of operator spaces. In fact this tensor product is such that what we have defined as completely bounded multilinear maps can be viewed as completely bounded linear maps on the Haagerup tensor product. This Haagerup tensor product and other related tensor products play an important role in the study of Schur multipliers, as can be see in [21]. While the Haagerup tensor product is beyond the scope of this thesis, we will encounter Schur multipliers in subsection 4.3, where we establish their connection to Fourier multipliers.

3 Locally compact groups

In this section we introduce locally compact groups and some results from the general theory on locally compact groups. These results will be used in section 4 and section 5 to define and work with Fourier multipliers. Our treatment of locally compact groups is based on [17] (mostly chapter 2), to which we will refer for most proofs. After going though the general theory of locally compact groups, we will state and prove some of our own results in subsection 3.5 that do not rely on Fourier multipliers. Before defining locally compact groups we will first go over some notation for groups to avoid ambiguity. We will also briefly state several definitions and results from general topology, because the theory of locally compact groups has a significant topological component.

3.1 Topological groups

Remark 3.1. Let G be a group. Unless specified otherwise we will write the group operation using juxtaposition:

$$(x, y) \mapsto xy. \tag{109}$$

The identity element of G will be written as e_G or simply e if the group G is clear from the context. For $x \in G$ the inverse of x will be written as x^{-1}. If $x \in G$ and $A, B \subseteq G$ we write

$$xA := \{xy : y \in A\}, \tag{110}$$

$$Ax := \{yx : y \in A\}, \tag{111}$$

$$AB := \{xy : x \in A, y \in B\} \tag{112}$$

and

$$A^{-1} := \{x^{-1} : x \in A\}. \tag{113}$$

If $A_1, \ldots, A_n \subseteq G$, we write

$$\prod_{j=1}^{n} A_j := A_1 A_2 \ldots A_n = \{x_1 x_2 \ldots x_n : x_j \in A_j \, \forall j \in \{1, \ldots, n\}\}, \tag{114}$$

where the associativity of the group operation ensures that this notation is unambiguous. For $A \subseteq G$ and $n \in \mathbb{N}$ we write

$$A^n := \prod_{j=1}^{n} A = \{x_1 x_2 \ldots x_n : x_j \in A \, \forall j \in \{1, \ldots, n\}\}. \tag{115}$$

This last notation is perhaps less standard and we stress that A^n as we have defined it should not be confused with $\{x^n : x \in A\}$. We also write

$$A^{\times n} := \bigtimes_{j=1}^{n} A \tag{116}$$

for the Cartesian product of n copies of A. This is to distinguish it from A^n as defined above. We note that if $G_1, \ldots, G_n$ are groups, then the Cartesian product $\bigtimes_{j=1}^{n} G_j$ is again a group when the group operations of multiplication and inversion are defined component-wise. This results in the direct product $\bigtimes_{j=1}^{n} G_j$.

Definition 3.1. Let G be a group, $y \in G$ and $f : G \to \mathbb{C}$ a function on G. We define functions $L_y f : G \to \mathbb{C}$ and $R_y f : G \to \mathbb{C}$ by

$$L_y f(x) = f(y^{-1}x) \tag{117}$$

and

$$R_y f(x) = f(xy). \tag{118}$$

Remark 3.2. Note that L_y and R_y are linear operators on the vector space of complex valued functions on G. In fact $y \mapsto L_y$ and $y \mapsto R_y$ are group homomorphisms.

To define and work with locally compact groups, several concepts from topology are needed. So, before defining locally compact groups, we will first state some definitions and results from topology that we will be using going forward. Most of these results can either directly be found in [15] or are relatively straightforward to prove from the given definitions and previous results.

Definition 3.2. Topological space

Let X be a set. A topology on X is a collection $\mathcal{T}$ of subsets of X that satisfies the following properties:

1. $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.
2. If $\mathcal{U} \subseteq \mathcal{T}$, then $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$, i.e. any union of elements of $\mathcal{T}$ is again an element of $\mathcal{T}$.
3. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$, i.e. the intersection of any two elements of $\mathcal{T}$ is again an element of $\mathcal{T}$ (and by induction $\mathcal{T}$ contains the intersection of any finite number of elements of $\mathcal{T}$).

The pair $(X, \mathcal{T})$ is called a topological space. If the chosen topology is clear, X itself is often also called a topological space. The elements of $\mathcal{T}$ are called open sets (with respect to the topology $\mathcal{T}$).

An important subcategory of topological spaces are the metric spaces (with normed vector spaces as a further subcategory), where the open sets of a metric space (as defined by the metric) form a topology. Many definitions for metric spaces can be stated in terms of open sets and these can usually be stated analogously for topological spaces.

Definition 3.3. Neighbourhood

If $x \in X$ and $A \subseteq X$, then A is called a neighbourhood of x if there is an open set U such that $x \in U \subseteq A$. In case A itself is open we can always choose $U = A$. So, an open set A is an open neighbourhood of x if and only if $x \in A$.

Definition 3.4. Closed set and closure

Let X be a topological space. A set $A \subseteq X$ is called closed if its complement $X \backslash A$ is open. Finite unions of closed sets are again closed and arbitrary intersections of closed sets are again closed. Every set $A \subseteq X$ has a smallest closed set $\overline{A}$ that contains A. This set is called the closure of A and is given by

$$\overline{A} = \bigcap_{\{F \subseteq X : F \text{ is closed}, A \subseteq F\}} F. \tag{119}$$

One can equivalently characterize $\overline{A}$ as the set of all $x \in X$ such that $A \cap U \neq \emptyset$ for any open neighbourhood U of x. Note that A is closed if and only if $A = \overline{A}$. If $\overline{A} = X$, then A is called dense (in X). If X has a countable, dense subset, then X is called separable.

Definition 3.5. Continuity

Let X and Y be topological spaces and $f : X \to Y$ a function. The function f is called continuous if the pre-image $f^{-1}(U) \subseteq X$ is open for all open $U \subseteq Y$. This definition agrees with the definition of continuity for functions between metric spaces. If $x \in X$, then f is called continuous at x if for every open neighbourhood V of $f(x)$ there exists an open neighbourhood U of x such that $f(U) \subseteq V$. It can be shown that f is continuous if and only if it is continuous at every $x \in X$. If f is bijective and both f and its inverse f^{-1} are continuous, then f is called a homeomorphism. Note that a composition of continuous functions is continuous.

Definition 3.6. Directed set

Let I be a non-empty set and $\preceq$ a relation on I. Suppose $\preceq$ satisfies the following properties:

1. Reflexivity: $\iota \preceq \iota$ for all $\iota \in I$.
2. Transitivity: For all $\iota_1, \iota_2, \iota_3 \in I$, if $\iota_1 \preceq \iota_2$ and $\iota_2 \preceq \iota_3$, then $\iota_1 \preceq \iota_3$.
3. Every pair of elements of I has a shared upper bound: If $\iota_1, \iota_2 \in I$, then there exists a $\iota \in I$ such that $\iota_1 \preceq \iota$ and $\iota_2 \preceq \iota$ (by induction any finite number of elements of I has a shared upper bound).

Then $(I, \preceq)$ (and often I itself if the relation $\preceq$ is clear) is called a directed set. We often write $\iota_1 \succeq \iota_2$ as an equivalent way of stating $\iota_2 \preceq \iota_1$. Note that $(\mathbb{N}, \leq)$ is a directed set.

Definition 3.7. Net

Let X be a topological space. A net in X is a function $I \to X$, where I is a directed set. In particular any sequence in X is a net. We will usually write the net in sequence notation, i.e. we will write x_ι for the image of $\iota \in I$ and write the net as $(x_\iota)_{\iota \in I}$ or simply (x_ι) without explicitly mentioning the directed set I.

A net $(x_\iota)_{\iota \in I}$ in X is said to converge to an element $x \in X$ if for every (open) neighbourhood U of x, there exists a $\iota_0 \in I$ such that $x_\iota \in U$ for all $\iota \succeq \iota_0$. In this case x is called a limit of the net x_ι. This definition of convergence applies to sequences in particular and in case X is a metric space this definition of convergence of a sequence is equivalent to convergence of the sequence in the metric. Note that while the limit of a convergent sequence in a metric space is always unique, a net or sequence in a topological space can have more than one limit. Uniqueness of limits can be ensured by imposing additional restrictions on the topology, see Remark 3.12.

Remark 3.3. A number of notions in metric spaces can be characterized in terms of sequences. In topological spaces, sequences are generally not enough to characterize the analogues of these notions, but nets frequently are. The following are some topological notions that can be characterized in terms of nets:

- If X is a topological space and $A \subseteq X$, then $\overline{A}$ consists of all points in X that are limits of nets in A. In particular, A is closed if and only if whenever a net in A has a limit in X, that limit lies in A.
- If X and Y are topological spaces and $f : X \to Y$ a function, then f is continuous if and only if whenever x_ι is a net in X and $x \in X$ is a limit of that net, then $f(x_\iota)$ converges to $f(x)$.

Nets are therefore frequently used in proofs in cases where sequences would be used if the topological spaces involved would be metric spaces. In particular nets are often used to prove continuity of functions.

Definition 3.8. Subspace topology

Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$ a subset. Then $\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}\}$ is a topology on Y called the subspace topology and $(Y, \mathcal{T}_Y)$ is called a subspace of $(X, \mathcal{T})$. Note that a subset of $A \subseteq Y$ that is open in the subspace topology $\mathcal{T}_Y$ is not necessarily open in the topology $\mathcal{T}$ of X, unless Y is open in X. The same statement holds if we replace every instance of “open” with “closed”. Unless specified otherwise, whenever $Y \subseteq X$ is considered as a topological space, the chosen topology is assumed to be the subspace topology. If X is a metric space and $Y \subseteq X$, the subspace topology on Y inherited from the metric topology on X will be the same topology as the metric topology on Y (where the metric on Y is obtained from the metric on X by restriction).

Remark 3.4. The subspace topology behaves well with respect to continuity. Let X and Y be topological spaces and $f : X \to Y$ a function. If $A \subseteq X$ and f is continuous, then $f|_A : A \to Y$ will also be continuous. If $f(X) \subseteq B \subseteq Y$, then $f : X \to Y$ is continuous if and only if it is continuous when viewed as a function $X \to B$.

The subspace topology also behaves well with respect to nets. Let X be a topological space and $Y \subseteq X$ equipped with the subspace topology. If x_ι is a net in Y and $x \in Y$, then x_ι converges to x in the topology of X if and only if it converges to x in the subspace topology.

Definition 3.9. Topological base

Let $(X, \mathcal{T})$ be a topological space. A base for the topology $\mathcal{T}$ is a subset $\mathcal{B} \subseteq \mathcal{T}$ such that either of the following two equivalent statements hold:

1. Every $U \in \mathcal{T}$ can be written as the union of elements of $\mathcal{B}$, i.e. $U = \bigcup_{B \in \mathcal{B}'} B$ for some $\mathcal{B}' \subseteq \mathcal{B}$.
2. For every $U \in \mathcal{T}$ and $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Note that $\mathcal{T}$ itself is a base for $\mathcal{T}$. Every base $\mathcal{B} \subseteq \mathcal{T}$ satisfies the following requirements (which follow from the definition):

1. $\bigcup_{B \in \mathcal{B}} B = X$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

Conversely, if X is any set and $\mathcal{B}$ is a collection of subsets of X that satisfies the above two conditions, then there is a unique topology $\mathcal{T}$ on X such that $\mathcal{B}$ is a base for $\mathcal{T}$. This topology is said to be generated by the base $\mathcal{B}$ and consists of all unions of elements of $\mathcal{B}$. Note that while a topology is uniquely determined by its base, a topology can have multiple bases.

Remark 3.5. Bases are often used to define a topology on some set. In fact the topology of a metric space is defined in this way. If (X, d) is a metric space, the metric topology on X is generated by the base consisting of all sets $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$, where $x \in X$ and $\epsilon > 0$. Another application of bases is that in some cases it is sufficient to check a certain property only for sets in the base instead of all open sets. This occurs for example when checking if a function is continuous, where it is sufficient to check that $f^{-1}(B)$ is open when B is a set in the base. A special case of this is the ϵ-δ definition of continuity for functions between metric spaces.

Definition 3.10. Second-countability
A topological space $(X, \mathcal{T})$ is said to be second-countable if there is a countable base for $\mathcal{T}$.

Definition 3.11. Product space
Let $n \in \mathbb{N}$ and let $(X_1, \mathcal{T}_1), \ldots, (X_n, \mathcal{T}_n)$ be topological spaces. Let $\mathcal{B}$ consist of all subsets of $\bigtimes_{j=1}^n X_j$ of the form $\bigtimes_{j=1}^n U_j$, where $U_j \in \mathcal{T}_j$. $\mathcal{B}$ satisfies the requirements stated in Definition 3.9 to generate a topology on $\bigtimes_{j=1}^n X_j$. This topology is called the product topology of $\mathcal{T}_1, \ldots, \mathcal{T}_n$ and $\bigtimes_{j=1}^n X_j$ equipped with this product topology is called a product space. Unless specified otherwise, we will always assume that $\bigtimes_{j=1}^n X_j$ is equipped with the product topology. This definition of product topology can be generalized to Cartesian products of infinitely many topological spaces, but we will not need this.

Remark 3.6. The product topology is constructed in such a way that many properties of the product space can be derived from properties of its factors and vice versa. Let $X_1, \ldots, X_n$ be topological spaces. We mention several important properties of the product space $X := \bigtimes_{j=1}^n X_j$.

- For any $k \in \{1, \ldots, n\}$, the projection $\pi_k : X \to X_k$, $\pi_k(x_1, \ldots, x_n) = x_k$ is continuous.
- If Y is a topological space and $f : Y \to X$ is a function, then f is continuous if and only if $\pi_k \circ f$ is continuous for all $k \in \{1, \ldots, n\}$.
- Let $A_j \subseteq X_j$ for all $j \in \{1, \ldots, n\}$. We equip each A_j with the subspace topology it inherits from X_j. Then we can equip $\bigtimes_{j=1}^n A_j$ with the corresponding product topology. We also have that $\bigtimes_{j=1}^n A_j \subseteq \bigtimes_{j=1}^n X_j$, so $\bigtimes_{j=1}^n A_j$ can also be equipped with the subspace topology it inherits from $\bigtimes_{j=1}^n X_j$. These two topologies on $\bigtimes_{j=1}^n A_j$ are the same topology.
- If $A_j \subseteq X_j$ is open for all $j \in \{1, \ldots, n\}$, then $\bigtimes_{j=1}^n A_j$ is open in the topology of $\bigtimes_{j=1}^n X_j$.
- If $A_j \subseteq X_j$ for all $j \in \{1, \ldots, n\}$, then $\overline{\bigtimes_{j=1}^n A_j} = \bigtimes_{j=1}^n \overline{A_j}$, where the first closure is taken in the topology of $\bigtimes_{j=1}^n X_j$. In particular, if each A_j is a closed subset of X_j, then $\bigtimes_{j=1}^n A_j$ is closed in the topology of $\bigtimes_{j=1}^n X_j$.
- If each X_j is separable, then so is $\bigtimes_{j=1}^n X_j$.

- If each X_j is second-countable, then so is $\bigtimes_{j=1}^n X_j$.

Definition 3.12. Compactness

Let X be a topological space. A subset $A \subseteq X$ is called compact if any open cover of A has a finite subcover. i.e. if $\mathcal{U}$ is a collection of open subsets of X such that $A \subseteq \bigcup_{U\in\mathcal{U}} U$, then there exists a finite subcollection $\{U_1, \ldots, U_n\} \subseteq \mathcal{U}$ such that $A \subseteq \bigcup_{j=1}^n U_j$.

Remark 3.7. We list some important properties of compact sets, some of which may be familiar from the context of metric spaces.

- A finite set is always compact.
- A union of finitely many compact sets is compact.
- Let X be a topological space and $Y \subseteq X$ equipped with the subspace topology. Let $A \subseteq Y$. Then A is compact as a subset of Y (with respect to the subspace topology) if and only if it is compact as a subset of X. In particular $A \subseteq X$ is compact as a subset of X if and only if it is compact as a subset of itself when A is equipped with the subspace topology.
- A closed subset of a compact set is compact.
- Let X and Y be topological spaces and $f : X \to Y$ continuous. If $A \subseteq X$ is compact, then $f(A) \subseteq Y$ is compact.
- Let $X_1, \ldots, X_n$ be topological spaces. If $A_j \subseteq X_j$ is compact for all $j \in \{1, \ldots, n\}$, then $\bigtimes_{j=1}^n A_j$ is compact (in the product topology).

Remark 3.8. Let X be topological space. We write $C(X)$ for the set of all continuous functions $X \to \mathbb{C}$. $C(X)$ is an algebra under pointwise addition, scalar multiplication and product. $f \mapsto \overline{f}$ defines an involution on $C(X)$, turning it into an involutive algebra. Let $C_b(X) \subseteq C(X)$ be the set of all bounded continuous functions $X \to \mathbb{C}$. $C_b(X)$ is a $*$-subalgebra of $C(X)$ and can be equipped with the supremum norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$. With this norm $C_b(X)$ is a C^*-algebra. We write $C_c(X)$ for those functions $f \in C(X)$ such that for some compact $K \subseteq X$, we have that $f(x) = 0$ for all $x \notin K$. If $f \in C(X)$ and $K \subseteq X$ is compact, then $f(K) \subseteq \mathbb{C}$ is compact, hence bounded. It follows that $C_c(X)$ is a $*$-subalgebra of $C_b(X)$.

Definition 3.13. Topological group

A topological group is a group G equipped with a topology such that the group operations of multiplication:

$$(x, y) \mapsto xy \tag{120}$$

and inversion:

$$x \mapsto x^{-1} \tag{121}$$

are continuous maps $G\times G \to G$ and $G \to G$ respectively. Here $G\times G$ is equipped with the product topology.

Remark 3.9. If G is a topological group and $A, B \subseteq G$ are compact, then $A \times B \subseteq G^{\times 2}$ is compact in the product topology, hence AB is compact by continuity of multiplication. It follows that if $A_1, \ldots, A_n \subseteq G$ are compact, then so is $\prod_{j=1}^n A_j$. Continuity of inversion implies that A^{-1} is compact whenever $A \subseteq G$ is compact.

Remark 3.10. Note that if G is a topological group and $y \in G$, then L_y and R_y are linear isometries on $C_b(G)$.

Definition 3.14. Uniform continuity

Let G be a topological group and $f : G \to \mathbb{C}$ a bounded function. f is called left uniformly continuous if $y \mapsto L_y f$ is continuous on G. Here we equip the vector space of bounded complex-valued functions on G with the supremum norm. Similarly, f is called right uniformly continuous if $y \mapsto R_y f$ is continuous on G.

Remark 3.11. It is straightforward to check that $y \mapsto L_y f$ and $y \mapsto R_y f$ are continuous on G if and only if they are continuous in the identity $e \in G$. So, our definition of uniform continuity is equivalent to the one used in [17]. It can also be checked that any bounded function $f : G \to \mathbb{C}$ that is either left or right uniformly continuous must be a continuous function.

Lemma 3.1. *Let G be a topological group and $f \in C_c(G)$. Then f is both left and right uniformly continuous.*

For the proof of the above lemma we refer to Proposition 2.6 in [17].

3.2 Locally compact groups and the Haar measure

To arrive at the definition of a locally compact group from a topological group, we need to impose some additional conditions on the topology.

Definition 3.15. Hausdorff space

Let $(X, \mathcal{T})$ be a topological space. The topology $\mathcal{T}$ is called Hausdorff and $(X, \mathcal{T})$ is called a Hausdorff space if for every distinct $x, y \in X$, there exist disjoint open neighbourhoods U_x of x and U_y of y.

Remark 3.12. Note that every metric space is a Hausdorff space. Hausdorff spaces have several (often desirable) properties that are not true for general topological spaces. In many contexts, topological spaces are required to be Hausdorff spaces or even stronger requirements are imposed. We list several important properties of Hausdorff spaces.

- Let X be a Hausdorff space and $A \subseteq X$ a compact set. Then A is closed in the topology of X. In particular any finite subset of X is closed, which includes singleton sets.
- Let X be a Hausdorff space. If x_ι is a net in X, then it has at most one limit.

When working with Hausdorff spaces, it useful to know whether topological spaces constructed from Hausdorff spaces are also Hausdorff spaces. For the following constructions this is indeed the case.

- Let X be a Hausdorff space and $Y \subseteq X$. Then Y, equipped with the subspace topology, is again a Hausdorff space.
- Let $X_1, \ldots, X_n$ be Hausdorff spaces and $X = \bigtimes_{j=1}^n X_j$ the product space. Then X is again a Hausdorff space.

Definition 3.16. Local compactness

Let X be a Hausdorff space. X is called a locally compact Hausdorff space if either one of the following two equivalent conditions holds.

1. For all $x \in X$ and all open neighbourhoods U of x, there exists a compact neighbourhood K of x with $x \in K \subseteq U$.
2. Every $x \in X$ has a compact neighbourhood.

The first condition clearly implies the second, but the converse is not trivial and relies on the fact that a compact Hausdorff space is normal (defined in Definition 3.25). A compact Hausdorff space is always locally compact. Note that the above two equivalent conditions that define a locally compact Hausdorff space also make sense for general topological spaces, but in this case they are not necessarily equivalent. For a general topological space, multiple reasonable definitions of local compactness can be formulated, but these are generally not equivalent. For this reason it is more convenient to work with locally compact Hausdorff spaces.

Remark 3.13. Locally compact Hausdorff spaces exhibit the following behaviour under taking subspaces and products.

- Let X be a locally compact Hausdorff space. If $U, F \subseteq X$ with U open and F closed, then $U \cap F$ is a locally compact Hausdorff space when equipped with the subspace topology. In particular any open or closed subset of X is a locally compact Hausdorff space.

- Let $X_1, \dots, X_n$ be locally compact Hausdorff spaces. Then the product space $X = \bigtimes_{j=1}^{n} X_j$ is a locally compact Hausdorff space.

Definition 3.17. $C_0(X)$

Let X be a locally compact Hausdorff space. Let $C_0(X)$ be the set of all $f \in C(X)$ such that for any $\epsilon > 0$ there exists a compact set $K \subseteq X$ such that $|f(x)| < \epsilon$ for all $x \notin K$. Then $C_0(X)$ is a closed $*$-subalgebra of $C_b(X)$ and therefore a C^*-algebra. $C_c(X)$ is a dense $*$-subalgebra of $C_0(X)$.

Definition 3.18. Locally compact group

A topological group G is called a locally compact group if it is a locally compact Hausdorff space.

Remark 3.14. As mentioned before, the Cartesian product $\bigtimes_{j=1}^{n} G_j$ of groups $G_1, \dots, G_n$ is again a group, the direct product of $G_1, \dots, G_n$. If $G_1, \dots, G_n$ are topological groups, then $\bigtimes_{j=1}^{n} G_j$ is also a topological space when equipped with the product topology. Because the group operations on the direct product are defined component-wise, it is straightforward to check that $\bigtimes_{j=1}^{n} G_j$ is indeed a topological group. If $G_1, \dots, G_n$ are locally compact groups, then their direct product is again a locally compact group (Remark 3.13).

In order to study locally compact groups from an analytic point of view, they can be equipped with a Borel measure that possesses certain properties. Such a measure is called a Haar measure.

Definition 3.19. Radon measure

Let X be a locally compact Hausdorff space. A (positive) Radon measure on X is a positive Borel measure μ on X such that $\mu(K) < \infty$ for any compact $K \subseteq X$ and such that the following properties are satisfied:

1. Outer regularity: $\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open}\}$ for all Borel sets $E \subseteq X$.

2. Inner regularity: $\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ compact}\}$ for all open sets $U \subseteq X$.

Definition 3.20. Haar measure

Let G be a locally compact group. A left Haar measure on G is a nonzero Radon measure μ on G that is left-invariant, i.e. $\mu(xE) = \mu(E)$ for any $x \in G$ and any Borel set $E \subseteq G$. Similarly, a right Haar measure on G is a nonzero Radon measure μ on G that is right-invariant, i.e. $\mu(Ex) = \mu(E)$ for any $x \in G$ and any Borel set $E \subseteq G$.

Remark 3.15. If μ is a Radon measure on G, then $\tilde{\mu}(E) = \mu(E^{-1})$ defines another Radon measure on G. If μ is a left Haar measure, then $\tilde{\mu}$ is a right Haar measure and vice versa. Because of this $1:1$ correspondence between left- and right Haar measure one can choose to study one or the other [17]. Left Haar measure is more commonly studied and we shall refer to left Haar measure simply as Haar measure.

From the definition of a Haar measure, it is not immediately clear if every locally compact group has a Haar measure. In some cases it is relatively straightforward to point out a Haar measure. For example $\mathbb{R}$ and $\mathbb{Z}$ are both locally compact groups when equipped with addition as the group operation with Lebesgue measure and counting measure respectively as a Haar measure. Even in cases like these where at least one Haar measure is known, it is not immediately clear how many other Haar measures exist. What can be seen from the definition is that any positive scalar multiple of a Haar measure is again a Haar measure. So, Haar measure is not completely unique. While non-trivial to prove, it turns out that any locally compact group has a Haar measure and all other Haar measures on this locally compact group can be obtained from a single Haar measure by multiplication with a positive scalar. We restate this in the following theorem, for the proof of which we refer to Theorems 2.10 and 2.20 in [17].

Theorem 3.2. *Let G be a locally compact group. There exists a Haar measure μ on G. Moreover, if μ and ν are Haar measures on G, then $\nu = c\mu$ for some constant $c > 0$.*

The following is a useful property of the Haar measure and is proved in Proposition 2.19 in [17].

Lemma 3.3. *Let G be a locally compact group and μ a (left) Haar measure on G. If $U \subseteq G$ is a non-empty open set, then $\mu(U) > 0$.*

We know that a product of locally compact groups is again a locally compact group (Remark 3.14). A natural question to ask is how the Haar measure on the product is related to the Haar measures of the separate factors. The answer to this question turns out to be somewhat subtle and a proper discussion requires a few additional definitions.

Definition 3.21. σ-compact space

Let X be a topological space. X is called a σ-compact space, if there exist compact sets $K_n \subseteq X$ for $n \in \mathbb{N}$ such that $X = \bigcup_{n=1}^{\infty} K_n$, i.e. X is the union of countably many compact sets.

Lemma 3.4. *Let X be a second-countable, locally compact Hausdorff space. Then X is σ-compact. Moreover, X has a countable base such that every element of this countable base has compact closure.*

Proof. Let $\mathcal{B}$ be a countable base for the topology on X. Let $x \in X$ and $U \subseteq X$ an open neighbourhood of x. X is locally compact, so there exists a compact neighbourhood K of x such that $x \in K \subseteq U$. Hence there exists an open set V with $x \in V \subseteq \overline{V} \subseteq K \subseteq U$. Because $\mathcal{B}$ is a base, there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq V \subseteq \overline{V} \subseteq K \subseteq U$. Note that $\overline{B} \subseteq K$ is closed, hence $\overline{B}$ is compact by Remark 3.7. This shows that $\mathcal{B}' = \{B \in \mathcal{B} : \overline{B} \text{ is compact}\}$ is a base for the topology of X. $\mathcal{B}'$ is countable and each of its elements has compact closure. It follows from Definition 3.9 that

$$\bigcup_{B \in \mathcal{B}'} B = X. \tag{122}$$

But then also

$$\bigcup_{B \in \mathcal{B}'} \overline{B} = X. \tag{123}$$

This shows that X is σ-compact. □

Definition 3.22. σ-finite measure space

Let X be a measure space with positive measure μ. μ (or the space X) is called σ-finite if there exist measurable sets $A_n \subseteq X$ for $n \in \mathbb{N}$ with $\mu(A_n) < \infty$ and such that $X = \bigcup_{n=1}^{\infty} A_n$.

Lemma 3.5. *Let X be a σ-compact locally compact Hausdorff space equipped with a Radon measure μ. Then μ is a σ-finite measure.*

Proof. X is σ-compact. So, there exist compact sets $K_n \subseteq X$ for $n \in \mathbb{N}$ such that $X = \bigcup_{n=1}^{\infty} K_n$. μ is a Radon measure, so each K_n is measurable and $\mu(K_n) < \infty$. This shows that μ is a σ-finite measure. □

Remark 3.16. We know that the direct product of locally compact groups $G_1, \ldots, G_n$ is again a locally compact group. We have established that every locally compact group has a Haar measure and that this Haar measure is unique up to multiplication with a positive scalar. Is the product measure of Haar measures $\mu_1, \ldots, \mu_n$ on $G_1, \ldots, G_n$ again a Haar measure on the direct product? This question is most easily answered in case each of the Haar measures $\mu_1, \ldots, \mu_n$ is σ-finite. In this case a unique product measure exists on $\bigtimes_{j=1}^{n} G_j$. A second-countable locally compact group equipped with a Haar measure is σ-finite. This follows from Lemma 3.4 and Lemma 3.5. If $G_1, \ldots, G_n$ are all second-countable, then it turns out that the product measure of Haar measures $\mu_1, \ldots, \mu_n$ is again a Haar measure on the direct product $\bigtimes_{j=1}^{n} G_j$ [17]. We note that the σ-finiteness of $\mu_1, \ldots, \mu_n$ also ensures that Fubini's theorem can be applied to non-negative or integrable functions on $\bigtimes_{j=1}^{n} G_j$. For some functions a version of Fubini's theorem can still be applied even without the assumption of σ-finiteness on the measures. We refer to section 2.3 in [17] for more details on these and other technicalities that occur without the assumption of σ-finiteness. We will mainly be working with groups that are second-countable, so we will be able to use Fubini's theorem and other results that rely on σ-finiteness without issue.

While not true in general, in some cases a left Haar measure is also a right Haar measure. Because the left Haar measures on a locally compact group are all positive scalar multiples of each other, if one left Haar measure is also a right Haar measure, then the same holds for all left Haar measures on the same locally compact group. For this reason the fact that left Haar measures are also right Haar measures can be viewed as a property of the group.

Definition 3.23. Unimodularity
A locally compact group G is called unimodular if one (or equivalently all) of its left Haar measures is also a right Haar measure.

More generally, it can be quantified to what degree a left Haar measure fails to be a right Haar measure. This is done through the modular function.

Definition 3.24. Modular function
Let G be a locally compact group. There exists a (unique) function $\Delta : G \to (0, \infty)$ such that

$$\mu(Ex) = \Delta(x)\mu(E) \tag{124}$$

for all $x \in G$, Borel sets $E \subseteq G$ and (left) Haar measures μ on G.

From the definition and existence of the modular function, it is clear that a locally compact group G is unimodular if and only if its modular function is the constant 1 function. As we will see, some results that hold for locally compact groups take on a simpler form for unimodular groups and some results that hold for unimodular groups do not hold in general. We refer to section 2.4 in [17] for more details on the modular function.

Remark 3.17. The following classes of locally compact groups consist entirely of unimodular groups:

- Abelian locally compact groups are unimodular because $xE = Ex$ for any Borel set E and group element x.
- Discrete groups are unimodular. A group is discrete if it is equipped with the discrete topology, i.e. every set is open. In such a topology, a set is compact if and only if it is finite. Using this property, it is straightforward to check that a discrete group is always locally compact and that the counting measure is both a left and right Haar measure.
- Compact Hausdorff groups are always unimodular locally compact groups. The proof of this fact relies on the fact that the modular function $\Delta : G \to (0, \infty)$ is a continuous group homomorphism, where $(0, \infty)$ is equipped with multiplication as the group operation. Hence if G is compact, then $\Delta(G)$ should be a compact subgroup of $(0, \infty)$ and $\{1\}$ is the only compact subgroup of $(0, \infty)$. See also Proposition 2.24, Proposition 2.27 and Corollary 2.28 in [17].

When integrating against a Haar measure, certain changes of variables can be performed as stated in the next lemma. Some of the expressions simplify when G is unimodular

Lemma 3.6. *Let G be a locally compact group and μ a Haar measure on G. For all measurable $f : G \to \mathbb{C}$ and $y \in G$ we have that*

$$\int_G f(yx)d\mu(x) = \int_G f(x)d\mu(x), \tag{125}$$

$$\int_G f(xy)d\mu(x) = \Delta(y^{-1}) \int_G f(x)d\mu(x) \tag{126}$$

and

$$\int_G f(x^{-1})d\mu(x) = \int_G \Delta(x^{-1})f(x)d\mu(x). \tag{127}$$

In each case if one of the integrals exists, then so does the other.

We will make frequent use of these change of variables formulas, especially when G is unimodular, in which case the second and third formulas simplify. We refer to Proposition 2.9, Proposition 2.24 and Proposition 2.31 in [17] for proofs of each of the change of variables formulas in Lemma 3.6.

Remark 3.18. Once we have chosen and fixed a Haar measure μ on a locally compact group G, we can as usual define the L^p-spaces $L^p(G) = L^p(G, \mathcal{B}(G), \mu)$ for $1 \leq p \leq \infty$, which are Banach spaces. Because all Haar measures on G are positive scalar multiples of each other, the spaces $L^p(G, \mathcal{B}(G), \mu)$ as vector spaces do not depend on μ, but their norms do and are positive scalar multiples of each other. If a fixed Haar measure is chosen, we will write $\|\cdot\|_p$ for the norm on $L^p(G, \mathcal{B}(G), \mu)$.

Remark 3.19. Let G be a locally compact group, $y \in G$ and $1 \leq p \leq \infty$. Note that L_y and R_y are bounded linear maps on $L^p(G)$ with

$$\|L_y f\|_p = \|f\|_p \tag{128}$$

and

$$\|R_y f\|_p = \Delta(y^{-1})^{\frac{1}{p}} \|f\|_p \tag{129}$$

for all $f \in L_p(G)$. This follows from Definition 3.24 and Lemma 3.6.

Lemma 3.7. *Let G be a locally compact group, $1 \leq p < \infty$ and $f \in L^p(G)$. Then $y \mapsto L_y f$ and $y \mapsto R_y f$ are continuous maps $G \to L^p(G)$.*

Using Remark 3.19, it can be shown that it is sufficient to prove continuity of $y \mapsto L_y f$ and $y \mapsto R_y f$ in the identity element e. For the remainder of the proof we refer to Proposition 2.41 in [17].

3.3 Density of $C_c(X)$ and separability of $L^p(X)$

In this subsection we prove two important results about locally compact Hausdorff spaces equipped with a Radon measure. The first is that for $1 \leq p < \infty$, the space $C_c(X)$ is a dense subspace of $L^p(X)$. The second is that $L^p(X)$ is separable if X is also second-countable. The density of $C_c(G)$ in $L^p(G)$ is also used in [17] and [22], but neither book gives a proof. More generally, the density of $C_c(X)$ in $L^p(X)$ seems to be a widely known result, but many texts that use it either do not prove it or only prove a special case, such as when $X = \mathbb{R}^n$. The separability of $L^p(X)$ will be used when applying Corollary 4.16 to identify certain functions as Fourier multipliers. Before proving these results we need to introduce an additional topological definition.

Definition 3.25. Normal space

Let $(X, \mathcal{T})$ be a topological space. X is called normal if it satisfies the following conditions:

1. For all distinct $x, y \in X$, there exists an open $U \subseteq X$ such that $x \notin U$ and $y \in U$.
2. For all disjoint and closed $A, B \subseteq X$, there exist disjoint open $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

Remark 3.20. The second condition in Definition 3.25 seemingly implies the first if we take A and B to be singleton sets. However, singleton sets are not necessarily closed in a general topological space. In fact the first condition (which also holds in Hausdorff spaces) is equivalent to the statement that every singleton subset of X is closed in X (see section 1.5 in [15]). In light of this equivalence it follows that every normal space is a Hausdorff space. The converse is not true, but every compact Hausdorff space is normal (see Theorem 3.1.9 in [15]).

An important result for normal spaces is Urysohn's lemma (see Theorem 1.5.11 in [15] for a proof).

Theorem 3.8. *Let X be a normal topological space and $A, B \subseteq X$ disjoint closed sets. There exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

Remark 3.21. The existence of the continuous functions in Urysohn's lemma is equivalent to the second statement in the definition of a normal topological space.

Urysohn's lemma is very useful to show the existence of certain continuous functions. Unfortunately the topological spaces/groups we are interested in are not necessarily normal. So, Urysohn's lemma cannot be applied directly. We will show that Urysohn's lemma implies a similar result that holds for locally compact Hausdorff spaces. To prove this result, we first need the following lemma.

Lemma 3.9. *Let X be a locally compact Hausdorff space, $U \subseteq X$ open and $K \subseteq U$ compact. Then there exists an open set $V \subseteq X$ such that $\overline{V}$ is compact and $K \subseteq V \subseteq \overline{V} \subseteq U$.*

Proof. Note that for every $x \in K$, there exists an open set V_x with $\overline{V_x}$ compact and $x \in V_x \subseteq \overline{V_x} \subseteq U$. Then $\{V_x : x \in K\}$ forms an open cover of K. K is compact. So, there must exist a finite subcover $\{V_{x_1}, \ldots, V_{x_N}\}$. Let $V = \bigcup_{n=1}^N V_{x_n}$. Then we have that V is open and

$$K \subseteq \bigcup_{n=1}^{N} V_{x_n} = V \subseteq \overline{V} \subseteq \bigcup_{n=1}^{N} \overline{V_{x_n}} \subseteq U. \tag{130}$$

Since each $\overline{V_{x_n}}$ is compact, we have that $\bigcup_{n=1}^N \overline{V_{x_n}}$ is compact. Since $\overline{V}$ is closed, it follows that $\overline{V}$ is compact. □

Our next result, "Urysohn's lemma for locally compact Hausdorff spaces", has the following two equivalent formulations.

Theorem 3.10. *Let X be a locally compact Hausdorff space, $F \subseteq X$ closed and $K \subseteq X$ compact with F and K disjoint. Then there exists an $f \in C_c(X)$ with $f(X) \subseteq [0,1]$, $f(x) = 1$ for all $x \in K$ and $f(x) = 0$ for all $x \in F$.*

Theorem 3.11. *Let X be a locally compact Hausdorff space, $U \subseteq X$ open and $K \subseteq U$ compact. Then there exists an $f \in C_c(X)$ with $f(X) \subseteq [0,1]$, $f(x) = 1$ for all $x \in K$ and $f(x) = 0$ for all $x \notin U$.*

Proof. The equivalence of Theorem 3.10 and Theorem 3.11 follows by considering the complements of F and U. So, it is enough to prove either one of the equivalent formulations. We will prove Theorem 3.11.

Let X be locally compact Hausdorff space with $K \subseteq U \subseteq X$, where K is compact and U is open. Let $V \subseteq X$ be open such that $\overline{V}$ is compact and $K \subseteq V \subseteq \overline{V} \subseteq U$. Also let $W \subseteq X$ be open such that $\overline{W}$ is compact and $K \subseteq W \subseteq \overline{W} \subseteq V$. Note that V and W exist by Lemma 3.9. Note that $\overline{V}$ is a compact Hausdorff space, and therefore a normal space, when equipped with the subspace topology it inherits from X. Note that K and $\overline{V} \backslash W$ are disjoint subsets of $\overline{V}$ that are closed in the topology of X, hence also in the topology of $\overline{V}$. It follows from Urysohn's Lemma that there exists a continuous function $f : \overline{V} \to [0,1]$ such that $f(x) = 1$ for all $x \in K$ and $f(x) = 0$ for all $f \in \overline{V} \backslash W$. In other words, the support of f is contained in W. We define the function $g : X \to [0,1]$ by

$$g(x) = \begin{cases} f(x), & x \in \overline{V} \\ 0, & x \notin \overline{V} \end{cases}. \tag{131}$$

Then we have that $g|_{\overline{V}} = f$. Because f is continuous, we have that $g|_V = f|_V$ is a continuous function. So, for every open $A \subseteq [0,1]$ we have that $(g|_V)^{-1}(A) \subseteq V$ is open in the subspace topology of V. Because V is open in X, it follows that $(g|_V)^{-1}(A)$ is also an open subset of X. Now $X \backslash \overline{W}$ is an open subset of X and $g(x) = 0$ for all $x \in X \backslash \overline{W}$. Since $\overline{W} \subseteq V$ we have that $V \cup (X \backslash \overline{W}) = X$. It follows that for any open $A \subseteq [0,1]$:

$$g^{-1}(A) = \begin{cases} (g|_V)^{-1}(A), & 0 \notin A \\ (g|_V)^{-1}(A) \cup (X \backslash \overline{W}), & 0 \in A \end{cases} \tag{132}$$

is an open subset of X. This shows that $g : X \to [0,1]$ is a continuous function. Since $g(x) = 1$ for $x \in K$ and the support of X is contained in $W \subseteq \overline{W} \subseteq U$ with $\overline{W}$ compact, we have that $f \in C_c(X)$ satisfies all requirements to prove Urysohn's lemma for locally compact Hausdorff spaces. □

We can now prove the density of $C_c(X)$ in $L^p(X)$.

Theorem 3.12. *Let X be a locally compact Hausdorff space equipped with a Radon measure μ. Let $1 \leq p < \infty$. Then $C_c(X) \subseteq L^p(X)$ and this inclusion is dense in $\|\cdot\|_p$.*

Proof. If $f \in C_c(X)$, then there is a compact set $K \subseteq X$ such that the support of f is contained in K. Since $f(K) \subseteq \mathbb{C}$ is compact, we have that f is a bounded function. Let $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in X$. Because μ is a Radon measure we have that $\mu(K) < \infty$. It follows that

$$\|f\|_p^p = \int_X |f(x)|^p d\mu(x) = \int_K |f(x)|^p d\mu(x) \leq \int_K M^p d\mu(x) = M^p \mu(K) < \infty. \tag{133}$$

This shows that $C_c(X) \subseteq L^p(X)$. As is the case for more general measures, we have that the simple functions are dense in $L^p(X)$ (see for example Lemma 4.2.1 in [1]). Since $C_c(X)$ is a vector space, it remains to show that $\mathbb{1}_A$ can be approximated by functions in $C_c(X)$ for any Borel set $A \subseteq X$ with finite measure. Let A be such a Borel set. Because μ is a Radon measure, given $\epsilon > 0$, there exists an open set $A \subseteq U \subseteq X$ such that $\mu(U) < \mu(A) + \epsilon$. It follows that $\mu(U \backslash A) < \epsilon$. Hence we have that

$$\|\mathbb{1}_U - \mathbb{1}_A\|_p^p = \left\|\mathbb{1}_{U\backslash A}\right\|_p^p = \int_X \left|\mathbb{1}_{U\backslash A}(x)\right|^p d\mu(x) = \mu(U\backslash A) < \epsilon. \tag{134}$$

This shows that for any Borel set $A \subseteq X$ with finite measure, $\mathbb{1}_A$ can be approximated by indicator functions $\mathbb{1}_U$, where U is open and has finite measure. So it remains to prove that $\mathbb{1}_U$, where U is open and has finite measure, can be approximated in $\|\cdot\|_p$ by functions in $C_c(X)$. Let U be such a set and $\epsilon > 0$. μ is a Radon measure, so there exists a compact set $K \subseteq U$ such that $\mu(K) > \mu(U) - \epsilon$. Then we have that $\mu(U \backslash K) < \epsilon$. Urysohn's lemma for locally compact Hausdorff spaces implies that there exists a function $f \in C_c(X)$ such that $f(X) \subseteq [0, 1]$, $f(x) = 1$ for all $x \in K$ and the support of f is contained in U. It follows that the support of $\mathbb{1}_U - f$ is contained in $U \backslash K$ and we have that

$$\|\mathbb{1}_U - f\|_p^p = \int_X |\mathbb{1}_U(x) - f(x)|^p d\mu(x) = \int_{U\backslash K} |\mathbb{1}_U(x) - f(x)|^p d\mu(x) \leq \int_{U\backslash K} d\mu(x) = \mu(U\backslash K) < \epsilon. \tag{135}$$

This shows that $\mathbb{1}_U$ can be approximated in $\|\cdot\|_p$ be functions in $C_c(X)$ and we can therefore conclude that $C_c(X)$ is dense in $L^p(X)$. □

Now we work towards proving the separability of $L^p(X)$. As a first step we show that $C_0(X)$ is separable. Our proof depends on the Stone-Weierstrass theorem (see section V.8 and in particular Corollary V.8.3 in [5]):

Theorem 3.13. *Let X be a locally compact Hausdorff space and $A \subseteq C_0(X)$ a closed subalgebra of $C_0(X)$ such that the following conditions are satisfied:*

1. *$\overline{f} \in A$ for all $f \in A$.*
2. *For each $x \in X$ there exists an $f \in A$ such that $f(x) \neq 0$*
3. *A separates the points of X: for all distinct $x, y \in X$ there is an $f \in A$ such that $f(x) \neq f(y)$.*

Then $A = C_0(X)$.

Theorem 3.14. *Let X be a second-countable, locally compact Hausdorff space. Then $C_0(X)$ is separable. Moreover, $C_0(X)$ has a countable dense subset consisting of functions in $C_c(X)$.*

Proof. If X has a finite number of elements, then any subset of X is compact, hence closed (Remark 3.12). So, any subset of X is also open, i.e. the topology on X is discrete. It follows that any function $f : X \to \mathbb{C}$ is automatically continuous and compactly supported. So, $C_0(X) = C_c(X)$ coincides with the vector space of all functions $X \to \mathbb{C}$. Because X is finite, $C_0(X) = C_c(X)$ is finite-dimensional, hence separable. Indeed, a finite-dimensional vector space has a finite basis and the $\mathbb{Q}$-linear span of this basis is countable and dense. This proves the theorem in case X is finite.

Now assume instead that X is not finite. In particular X has more than one element. Let $\mathcal{B}$ be a countable base for the topology of X such that $\overline{B}$ is compact for all $B \in \mathcal{B}$. Such a base exists by Lemma 3.4. Note that $\mathcal{B}$ must contain at least two sets. Indeed, if $\mathcal{B}$ is empty, then X must be empty. If $\mathcal{B} = \{B\}$, then we must have $X = B$ and X and $\emptyset$ are the only open sets. This contradicts the assumption that X is Hausdorff, because X has more than one element. Let $\mathcal{A} \subseteq \mathcal{B}^{\times 2}$ denote the set of ordered pairs $(B_1, B_2) \in \mathcal{B}^{\times 2}$ such that $\overline{B_1}$ and $\overline{B_2}$ are disjoint. Since $\mathcal{B}$ is countable, we have that $\mathcal{A}$ is countable. By Theorem 3.10, for each $(B_1, B_2) \in \mathcal{A}$, there exists a function $f_{B_1,B_2} \in C_c(X)$, taking values in $[0,1]$ and such that $f_{B_1,B_2}(x) = 1$ for all $x \in \overline{B_1}$ and $f_{B_1,B_2}(x) = 0$ for all $x \in \overline{B_2}$. We fix one such function f_{B_1,B_2} for all $(B_1, B_2) \in \mathcal{A}$. Then $\{f_{B_1,B_2} : (B_1, B_2) \in \mathcal{A}\} \subseteq C_c(X)$ is countable. The set of all finite products of functions of the form f_{B_1,B_2} is then also a countable subset of $C_c(X)$. Let $A \subseteq C_c(X)$ be the finite linear span of all finite products of functions of the form f_{B_1,B_2}. Also let $A_0 \subseteq A$ consist of all finite linear combinations of finite products of functions of the form f_{B_1,B_2}, where all the coefficients in the linear combination are elements of $\mathbb{Q} + i\mathbb{Q} \subseteq \mathbb{C}$. Because $\mathbb{Q} + i\mathbb{Q}$ is a countable, dense subset of $\mathbb{C}$, it follows that A_0 is countable and that

$$A_0 \subseteq A \subseteq \overline{A_0} \subseteq C_0(X). \tag{136}$$

From this it follows that $\overline{A_0} = \overline{A}$. If we can show that $\overline{A} = C_0(X)$, then we are done. We note that A is a subalgebra of $C_0(X)$. Because each f_{B_1,B_2} is non-negative, it follows that $\overline{f} \in A$ for all $f \in A$. Given distinct $x, y \in X$, there exist disjoint open neighbourhoods U_x of x and U_y of y. Because X is locally compact, there exist compact neighbourhoods K_x of x and K_y of y with $K_x \subseteq U_x$ and $K_y \subseteq U_y$. So, K_x and K_y are disjoint. Because $\mathcal{B}$ is a base, there exist $B_x, B_y \in \mathcal{B}$ with $x \in B_x \subseteq K_x$ and $y \in B_y \subseteq K_y$. Then $\overline{B_x} \subseteq K_x$ and $\overline{B_y} \subseteq K_y$, hence $\overline{B_x}$ and $\overline{B_y}$ are disjoint. Now we have that $f_{B_x,B_y} \in A$ with $f_{B_x,B_y}(x) = 1$ and $f_{B_x,B_y}(y) = 0$. This shows that A separates the points of X. Because X has more than one element, this also shows that A satisfies condition (2) in Theorem 3.13. It follows that $\overline{A}$ is a closed subalgebra of $C_0(X)$ that satisfies all conditions in Theorem 3.13. So, we conclude that $\overline{A} = C_0(X)$ and $A_0 \subseteq C_c(X)$ is a countable, dense subset of $C_0(X)$. $\square$

Theorem 3.15. *Let X be a second-countable, locally compact Hausdorff space equipped with a Radon measure μ. Let $1 \leq p < \infty$. Then $L^p(X)$ is separable.*

Proof. Let $A \subseteq C_c(X)$ be countable and dense (in $\|\cdot\|_\infty$) in $C_0(X)$ (A exists by Theorem 3.14). X is σ-compact by Lemma 3.4. So, there exist compact sets $K_n \subseteq X$ for $n \in \mathbb{N}$ such that $\bigcup_{n=1}^{\infty} K_n = X$. Without loss of generality we can assume that $K_n \subseteq K_{n+1}$ for all $n \in \mathbb{N}$. Indeed, if $K_n \subseteq K_{n+1}$ does not hold we can instead work with the compact sets $K'_n := \bigcup_{j=1}^{n} K_j$. These sets cover X and satisfy $K'_n \subseteq K'_{n+1}$. So, we assume that $K_n \subseteq K_{n+1}$ for all $n \in \mathbb{N}$. Note that $\mu(K_n) < \infty$ for all $n \in \mathbb{N}$.

Let $f \in L^p(X)$ and $\epsilon > 0$. Then $|f - \mathbb{1}_{K_n} f|^p$ converges pointwise to 0 as $n \to \infty$ and is bounded in absolute value by $|f|^p$, which is an integrable function because $f \in L^p(X)$. It follows by the dominated convergence theorem that

$$\|f - \mathbb{1}_{K_n} f\|_p^p = \int_X |f(x) - \mathbb{1}_{K_n}(x) f(x)|^p d\mu(x) \tag{137}$$

converges to 0. So $\mathbb{1}_{K_n} f$ converges to f in $\|\cdot\|_p$. Choose $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $\|f - \mathbb{1}_{K_n} f\|_p < \frac{\epsilon}{3}$. Then in particular we have that

$$\|f - \mathbb{1}_{K_N} f\|_p < \frac{\epsilon}{3}. \tag{138}$$

By density of $C_c(X)$ in $L^p(X)$, there exists a $g \in C_c(X)$ such that

$$\|\mathbb{1}_{K_N} f - \mathbb{1}_{K_N} g\| \leq \|\mathbb{1}_{K_N} f - g\|_p < \frac{\epsilon}{3}. \tag{139}$$

A is dense in $C_0(X)$ (in $\|\cdot\|_\infty$). So, there exists an $h \in C_c(X)$, such that

$$\|g - h\|_\infty < \frac{\epsilon}{3\mu(K_N)^{\frac{1}{p}} + 1}. \tag{140}$$

It follows that

$$\|\mathbb{1}_{K_N} g - \mathbb{1}_{K_N} h\|_p \leq \|g - h\|_\infty \|\mathbb{1}_{K_N}\|_p = \|g - h\|_\infty \mu(K_N)^{\frac{1}{p}} < \frac{\epsilon}{3}. \tag{141}$$

It follows by the triangle inequality that

$$
\begin{aligned}
\|f - \mathbb{1}_{K_N} h\|_p &= \|f - \mathbb{1}_{K_N} f + \mathbb{1}_{K_N} f - \mathbb{1}_{K_N} g + \mathbb{1}_{K_N} g - \mathbb{1}_{K_N} h\|_p \\
&\leq \|f - \mathbb{1}_{K_N} f\|_p + \|\mathbb{1}_{K_N} f - \mathbb{1}_{K_N} g\|_p + \|\mathbb{1}_{K_N} g - \mathbb{1}_{K_N} h\|_p < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned} \tag{142}
$$

This shows that $\{\mathbb{1}_{K_n} h : h \in A, n \in \mathbb{N}\}$ is dense in $L^p(X)$. Since $\{\mathbb{1}_{K_n} h : h \in A, n \in \mathbb{N}\}$ is countable, it follows that $L^p(X)$ is separable. $\square$

Theorem 3.15 implies the following result for spaces of L^p-functions that take values in a separable Hilbert space.

Theorem 3.16. *Let X be a second-countable, locally compact Hausdorff space equipped with a Radon measure μ and let H be a separable Hilbert space. Then $L^p(X; H)$ is separable for $1 \leq p < \infty$.*

Proof. Let $1 \leq p < \infty$. We first prove the theorem for finite-dimensional H. We know that $L^p(X)$ is separable by Theorem 3.15. So, let $A \subseteq L^p(X)$ be a countable, dense subset. H is finite-dimensional. So, it has an orthonormal basis $(h_1, \dots, h_N)$, where N is the dimension of H. Each $h \in H$ can be uniquely written as

$$
h = \sum_{n=1}^{N} c_n h_n, \tag{143}
$$

where $c_n \in \mathbb{C}$ is given by

$$
c_n = \langle h, h_n \rangle. \tag{144}
$$

Moreover, we have that

$$
\|h\|^2 = \sum_{n=1}^{N} |c_n|^2 = \sum_{n=1}^{N} |\langle h, h_n \rangle|^2. \tag{145}
$$

For $f : X \to H$ and $n \in \{1, \dots, N\}$, write f_n for the function

$$
f_n(x) = \langle f(x), h_n \rangle. \tag{146}
$$

Then

$$
f(x) = \sum_{n=1}^{N} f_n(x) h_n \tag{147}
$$

and

$$
\|f\|_p = \left(\int_X \|f(x)\|^p dx \right)^{\frac{1}{p}} = \left(\int_X \left(\sum_{n=1}^{N} |f_n(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \leq \left(\int_X \left(\sum_{n=1}^{N} |f_n(x)| \right)^{p} dx \right)^{\frac{1}{p}} \leq \sum_{n=1}^{N} \|f_n\|_p. \tag{148}
$$

This shows that $f \in L^p(X; H)$ if and only if $f_n \in L^p(X)$ for all $n \in \mathbb{N}$. Given $f \in L^p(X; H)$ and $\epsilon > 0$, choose $g_n \in A \subseteq L^p(X)$ for all $n \in \mathbb{N}$, such that $\|f_n - g_n\|_p < \frac{\epsilon}{N}$. Then $g = \sum_{n=1}^{N} g_n h_n \in L^p(X; H)$ and it follows that

$$
\|f - g\|_p \leq \sum_{n=1}^{N} \|f_n - g_n\|_p < \sum_{n=1}^{N} \frac{\epsilon}{N} = \epsilon. \tag{149}
$$

This shows that $\{\sum_{n=1}^{N} g_n h_n : g_n \in A\}$ is dense in $L^p(X; H)$. Since $\{\sum_{n=1}^{N} g_n h_n : g_n \in A\}$ is countable, this shows that $L^p(X; H)$ is separable. This proves the theorem when H is finite-dimensional.

Now assume instead that H is infinite-dimensional. Because H is separable, it has a countable orthonormal basis $(h_n)_{n\in\mathbb{N}}$. Each $h \in H$ can be uniquely written as

$$h = \sum_{n=1}^{\infty} c_n h_n, \tag{150}$$

where the above series converges in norm and $c_n \in \mathbb{C}$ is given by

$$c_n = \langle h, h_n \rangle . \tag{151}$$

Moreover, we have that

$$\|h\|^2 = \sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} |\langle h, h_n\rangle|^2. \tag{152}$$

For $N \in \mathbb{N}$, let H_N be the linear span of $h_1, \dots, h_N$. Then $H_N \subseteq H$ is a finite-dimensional Hilbert space with orthonormal basis $(h_1, \dots, h_N)$. Let $P_N : H \to H_N \subseteq H$ be the orthogonal projection given by

$$P_N(h) = \sum_{n=1}^{N} \langle h, h_n \rangle h_n. \tag{153}$$

Then $P_N(h)$ converges to h in norm as $N \to \infty$ for all $h \in H$ and we have

$$\|P_N(h)\| \leq \|h\| \tag{154}$$

and

$$\|h - P_N(h)\| \leq \|h\|. \tag{155}$$

Let $\tilde{P}_N : L^p(X; H) \to L^p(X; H_N) \subseteq L^p(X; H)$ be given by

$$(\tilde{P}_N f)(x) = P_N(f(x)). \tag{156}$$

Since $\|P_N(f(x))\| \leq \|f(x)\|$, we indeed have that $\tilde{P}_N$ maps $L^p(X;H)$ into $L^p(X;H_N)$. For $f \in L^p(X;H)$ and $x \in X$, we have that $\left\|(\tilde{P}_N f)(x) - f(x)\right\|^p$ converges to 0 and is bounded by $\|f(x)\|^p$. The dominated convergence theorem now implies that $\tilde{P}_N f$ converges to f in $\|\cdot\|_p$. It follows that $\bigcup_{N=1}^{\infty} L^p(X;H_N)$ is dense in $L^p(X;H)$. We know that $L^p(X;H_N)$ is separable, so it has a countable dense subset A_N. Then $\bigcup_{N=1}^{\infty} A_N$ is a countable, dense subset of $\bigcup_{N=1}^{\infty} L^p(X;H_N)$. It follows that $\bigcup_{N=1}^{\infty} A_N$ is also dense in $L^p(X;H)$. We conclude that $L^p(X;H)$ is also separable if H is infinite-dimensional. □

If X is a second-countable, locally compact Hausdorff space equipped with Radon measure μ and H is a separable Hilbert space, then Theorem 3.16 shows that $L^p(X;H)$ is separable for $1 \leq p < \infty$. If $\Omega \subseteq X$ is a measurable set, then we can restrict the measure μ to Ω to obtain another measure space. The following corollary shows that $L^p(\Omega;H)$ is also separable.

Corollary 3.17. *Let X be a second-countable, locally compact Hausdorff space equipped with a Radon measure μ and let H be a separable Hilbert space. Let $\Omega \subseteq X$ be measurable and equip Ω with the restriction of μ. Then $L^p(\Omega;H)$ is separable for $1 \leq p < \infty$.*

Proof. We know from Theorem 3.16, that $L^p(X;H)$ is separable. So, let $A \subseteq L^p(X;H)$ be a countable dense subset. Any function $f \in L^p(\Omega;H)$ can be trivially extended to $L^p(X;H)$ by setting

$$f(x) = 0 \tag{157}$$

for $x \notin \Omega$. Note that this extension preserves pointwise operations (such as addition and scalar multiplication) and the norm $\|\cdot\|_p$. So this extension is an isometry $L^p(\Omega;H) \to L^p(X;H)$, which we use to identify $L^p(\Omega;H)$ with a subspace of $L^p(X;H)$. Let $g \in A$. For each $n \in \mathbb{N}$ we choose and fix a function $g_n \in L^p(\Omega;H)$ such that

$$\|g - g_n\|_p < \frac{1}{n}, \tag{158}$$

assuming such a function exists. Note that for $n \leq m$, g_n exists if g_m exists. We set N_g equal to the largest $n \in \mathbb{N}$ such that g_n exists. In case g_n does not exist for any $n \in \mathbb{N}$, we set $N_g = 0$. In case g_n exists for all $n \in \mathbb{N}$, we set $N_g = \infty$. Then we have that g_n exists for all $n \in \mathbb{N}$ with $n \leq N_g$. Let A' be the set consisting of all g_n for $g \in A$ and $n \in \mathbb{N}$ with $n \leq N_g$. Then $A' \subseteq L^p(\Omega; H)$ is countable because A and $\mathbb{N}$ are countable. To show that A' is dense in $L^p(\Omega; H)$, let $f \in L^p(\Omega; H)$ and $\epsilon > 0$. Choose $n \in \mathbb{N}$ with $n \geq \frac{2}{\epsilon}$, which ensures that $\frac{1}{n} \leq \frac{\epsilon}{2}$. A is dense in $L^p(X; H)$. So, we can choose a $g \in A$ such that

$$\|f - g\|_p < \frac{1}{n}. \tag{159}$$

Because $f \in L^p(\Omega; H)$, we must have that $n \leq N_g$ and $g_n \in A'$ is such that

$$\|g - g_n\|_p < \frac{1}{n}. \tag{160}$$

It follows that

$$\|f - g_n\|_p = \|f - g + g - g_n\|_p \leq \|f - g\|_p + \|g - g_n\|_p < \frac{1}{n} + \frac{1}{n} \leq \epsilon. \tag{161}$$

This shows that A' is dense in $L^p(\Omega; H)$. □

3.4 Convolution

Similar to $\mathbb{R}^n$, we can define convolution on a locally compact group G. As we will see, many results that hold for convolution on $\mathbb{R}^n$ will still be true in this more general context, but some results will not fully generalize.

Definition 3.26. Convolution

Let G be a locally compact group with a fixed Haar measure. Let $f, g : G \to \mathbb{C}$ be measurable functions. We define the convolution $f * g : G \to \mathbb{C}$ by

$$\begin{aligned}
(f * g)(x) &= \int_G f(y) g(y^{-1}x) dy \\
&= \int_G f(xy) g(y^{-1}) dy \\
&= \int_G f(y^{-1}) g(yx) \Delta(y^{-1}) dy \\
&= \int_G f(xy^{-1}) g(y) \Delta(y^{-1}) dy,
\end{aligned} \tag{162}$$

if the integrals on the right-hand-side exist. Here integration is against a fixed Haar measure and the four integrals on the right-hand-side are all equal because of the change of variables formulas in Lemma 3.6. Note that choosing a different Haar measure will change the value of $f * g$.

Remark 3.22. We note that, in contrast to convolution on $\mathbb{R}^n$, convolution on G is in general not commutative. Convolution is commutative when the group G is abelian, e.g. $G = \mathbb{R}^n$.

The following results give some examples where $f * g$ is well-defined. For proofs we refer to Propositions 2.39 and 2.40 in [17] respectively.

Theorem 3.18. *Let G be a locally compact group with fixed Haar measure. Let $1 \leq p \leq \infty$, $f \in L^1(G)$ and $g \in L^p(G)$. Then $f * g \in L^p(G)$ is well-defined with*

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p. \tag{163}$$

If $p = \infty$, then $f * g$ is continuous. In case the group G is unimodular, $g * f \in L^p(G)$ is well-defined with

$$\|g * f\|_p \leq \|f\|_1 \|g\|_p. \tag{164}$$

If $p = \infty$, then $g * f$ is continuous.

Theorem 3.19. *Let G be a unimodular locally compact group with fixed Haar measure. Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(G)$ and $g \in L^q(G)$, then $f * g \in C_0(G)$ with*

$$\|f * g\|_\infty \leq \|f\|_p \|g\|_q. \tag{165}$$

Remark 3.23. The convolution map $L^1(G) \times L^1(G) \to L^1(G)$ is associative and turns $L^1(G)$ into a Banach algebra. One can define an involution $f \mapsto f^*$ on $L^1(G)$ by setting

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}. \tag{166}$$

This turns $L^1(G)$ into a Banach-$*$ algebra. See section 2.5 in [17] for more details.

Remark 3.24. Convolutions can also be defined between complex Radon measures. Let $M(G)$ be the space of complex Radon measures on G equipped with the total variation norm. This is a Banach space. For $\mu, \nu \in M(G)$, there exists a unique measure $\mu * \nu \in M(G)$ that satisfies

$$\int_G \phi(z) d(\mu * \nu)(z) = \int_G \int_G \phi(xy) d\mu(x) d\nu(y) = \int_G \int_G \phi(xy) d\nu(y) d\mu(x) \tag{167}$$

for all $\phi \in C_0(G)$. This new measure satisfies $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$. Note that to every $f \in L^1(G)$, one can associate a complex Radon measure μ_f such that

$$\int_G \phi(x) d\mu_f(x) = \int_G \phi(x) f(x) dx, \tag{168}$$

where the integral on the right-hand-side is against the Haar measure. The convolution between Radon measures extends the definition of convolution between functions in $L^1(G)$ in the sense that $\mu_f * \mu_g = \mu_{f*g}$ for all $f, g \in L^1(G)$. Moreover, the measure resulting from the convolution between any complex Radon measure and a Radon measure derived from a function in $L^1(G)$ (in either order) has an associated function in $L^1(G)$. We refer to section 2.5 in [17] for more details on convolutions between measures.

3.5 Pseudo-convolution

In this subsection, we will cover some more specialized material that will be used in section 5 when constructing Fourier multipliers. Nevertheless, nothing we introduce in this subsection is reliant on the definition of Fourier multipliers, and might have other applications as well. Some of the material we introduce here takes inspiration from section 2 in [18].

For this subsection we fix an arbitrary locally compact group G that is assumed to be second-countable. This allows us to apply Fubini's theorem without issues. The condition that G should be second-countable will also be imposed on G in section 5 and is also present in section 2 of [18]. We also fix a (left) Haar measure μ on G. By Remark 3.16 we have that the product measure μ^n is a Haar measure on $G^{\times n}$ for all $n \in \mathbb{N}$. We fix this Haar measure on $G^{\times n}$. We will suppress μ in the integration notation, i.e. we will write $\int_G f(x)dx$ instead of $\int_G f(x)d\mu(x)$. Similarly we will suppress μ^n in the integration notation.

Most of this subsection is devoted to the definition of a "pseudo-convolution" $L^1(G^{\times(n+1)}) \times L^\infty(G^{\times n}) \to C_b(G^{\times n})$, $(F, \psi) \mapsto F \tilde{*} \psi$ (see Definition 3.29) and to proving several properties of this map. We start by introducing some notation.

Definition 3.27. For $n \in \mathbb{N}$ and $f : G^{\times n} \to \mathbb{C}$ we define the function $\tilde{f} : G^{\times(n+1)} \to \mathbb{C}$ by

$$\tilde{f}(x_0, \ldots, x_n) = f\left(\left(x_{j-1}x_j^{-1}\right)_{j=1}^n\right). \tag{169}$$

Remark 3.25. Note that the function $G^{\times n} \to G^{\times(n+1)}$, $(s_1, \ldots, s_n) \mapsto \left(\prod_{m=j}^{n} s_m\right)_{j=1}^{n+1}$ is continuous and

$$\tilde{f}\left(\left(\prod_{m=j}^{n} s_m\right)_{j=1}^{n+1}\right) = f(s_1, \ldots, s_n) \tag{170}$$

for any function $f : G^{\times n} \to \mathbb{C}$. $G^{\times(n+1)} \to G^{\times n}$, $(x_0, \ldots, x_n) \mapsto \left(x_{j-1}x_j^{-1}\right)_{j=1}^{n}$ is also continuous. It follows that $\tilde{f}$ is continuous if and only if f is continuous, $\tilde{f}$ is bounded if and only if f is bounded and $\tilde{f}$ is measurable if and only if f is measurable.

Definition 3.28. For $n \in \mathbb{N}$ and $f_1, \ldots, f_n : G \to \mathbb{C}$ we define the function $\otimes_{j=1}^{n} f_j : G^{\times n} \to \mathbb{C}$ given by

$$\left(\otimes_{j=1}^{n} f_j\right)(x_1, \ldots, x_n) = \prod_{j=1}^{n} f_j(x_j). \tag{171}$$

Remark 3.26. If $f_1, \ldots, f_n : G \to \mathbb{C}$ are all continuous, then so is $\otimes_{j=1}^{n} f_j$. If $f_1, \ldots, f_n$ are all measurable, then so is $\otimes_{j=1}^{n} f_j$. If $1 \leq p \leq \infty$ and $f_1, \ldots, f_n \in L^p(G)$, then $\otimes_{j=1}^{n} f_j \in L^p(G^{\times n})$ with

$$\left\|\otimes_{j=1}^{n} f_j\right\|_p = \prod_{j=1}^{n} \|f_j\|_p. \tag{172}$$

We will now introduce a bilinear map reminiscent of a convolution that will be very useful for creating continuous functions and improving certain convergence properties.

Definition 3.29. Let $n \in \mathbb{N}$, $F \in L^1(G^{\times(n+1)})$ and $\psi : G^{\times n} \to \mathbb{C}$ a bounded, measurable function. We define a function $F \tilde{*} \psi : G^{\times n} \to \mathbb{C}$ given by

$$(F \tilde{*} \psi)(s_1, \ldots, s_n) = \int_{G^{\times(n+1)}} \psi\left(\left(t_{j-1}^{-1} s_j t_j\right)_{j=1}^{n}\right) F(t_0, \ldots, t_n) d(t_0, \ldots, t_n). \tag{173}$$

Lemma 3.20. *For any $n \in \mathbb{N}$, $(F, \psi) \mapsto F \tilde{*} \psi$ defines a bounded bilinear map $L^1(G^{\times(n+1)}) \times L^\infty(G^{\times n}) \to C_b(G^{\times n})$ with*

$$\|F \tilde{*} \psi\|_\infty \leq \|F\|_1 \|\psi\|_\infty. \tag{174}$$

Moreover, we have that

$$(F \tilde{*} \psi)\tilde{} = F * \tilde{\psi}. \tag{175}$$

Proof. Choose and fix a bounded representative of ψ such that $|\psi| \leq \|\psi\|_\infty$ holds pointwise. We will later argue that $F \tilde{*} \psi$ does not depend on the chosen representative (see Remark 3.27 and Remark 3.30). For this choice of representative, the integrand in Definition 3.29 is integrable. The integrand is also measurable as a function of the variables t_j and s_j. Hence $F \tilde{*} \psi$ is measurable and bounded with

$$\|F \tilde{*} \psi\|_\infty \leq \|F\|_1 \|\psi\|_\infty. \tag{176}$$

Bilinearity of $(F, \psi) \mapsto F \tilde{*} \psi$ is clear from the definition. To prove the continuity of $F \tilde{*} \psi$ we consider the functions $\tilde{\psi}, (F \tilde{*} \psi)\tilde{} \in L^\infty(G^{\times(n+1)})$ and recall from Remark 3.25 that $F \tilde{*} \psi$ is continuous if and only if $(F \tilde{*} \psi)\tilde{}$ is continuous. We have that

$$\begin{aligned}
(F \tilde{*} \psi)\tilde{}(x_0, \ldots, x_n) &= (F \tilde{*} \psi)\left(\left(x_{j-1}x_j^{-1}\right)_{j=1}^{n}\right) = \int_{G^{n+1}} \psi\left(\left(t_{j-1}^{-1} x_{j-1} x_j^{-1} t_j\right)_{j=1}^{n}\right) F(t_0, \ldots, t_n) d(t_0, \ldots, t_n) \\
&= \int_{G^{n+1}} \psi\left(\left((t_{j-1}^{-1} x_{j-1})(t_j^{-1} x_j)^{-1}\right)_{j=1}^{n}\right) F(t_0, \ldots, t_n) d(t_0, \ldots, t_n) \\
&= \int_{G^{n+1}} \tilde{\psi}\left(\left(t_j^{-1} x_j\right)_{j=0}^{n}\right) F(t_0, \ldots, t_n) d(t_0, \ldots, t_n) \\
&= (F * \tilde{\psi})(x_0, \ldots, x_n).
\end{aligned} \tag{177}$$

Because $F \in L^1(G^{\times(n+1)})$ and $\tilde{\psi} \in L^\infty(G^{\times(n+1)})$ it follows from Theorem 3.18 that $(F\tilde{*}\psi)\breve{}$ is continuous. Hence $F\tilde{*}\psi$ is also continuous. $\square$

Remark 3.27. Note that in Definition 3.29, the function ψ is specified to be a bounded, measurable function $G^{\times n} \to \mathbb{C}$ and not a $L^\infty(G^{\times n})$ function. The subtle distinction here is that $L^\infty(G^{\times n})$-functions are identified with each other when they differ on a set with (Haar) measure 0 and are allowed to be essentially bounded, instead of bounded on all of $G^{\times n}$. Allowing $\psi \in L^\infty(G^{\times n})$ leads to the problem that the integral defining $F\tilde{*}\psi$ might depend on the chosen representative of ψ and might not be well-defined when the chosen representative is not bounded on all of $G^{\times n}$. The given proof of Lemma 3.20 works for a fixed bounded representative of ψ. So, for now we should, strictly speaking, assume that ψ is a bounded function and not an equivalence class of essentially bounded functions whenever working with Definition 3.29 and Lemma 3.20. Nevertheless $F\tilde{*}\psi$ is well-defined for $\psi \in L^\infty(G^{\times n})$ and Lemma 3.20 holds as stated and we will prove this in Remark 3.30.

Remark 3.28. Note that for any $F \in L^1(G^{\times(n+1)})$ we have that

$$(F\tilde{*}1)(s_1, \ldots, s_n) = \int_{G^{\times(n+1)}} F(t_0, \ldots, t_n) d(t_0, \ldots, t_n) \tag{178}$$

and

$$(|F|\tilde{*}1)(s_1, \ldots, s_n) = \int_{G^{\times(n+1)}} |F(t_0, \ldots, t_n)| d(t_0, \ldots, t_n) = \|F\|_1. \tag{179}$$

To state how the pseudo-convolution introduced in Definition 3.29 affects convergence properties, we first need to introduce some topologies.

Definition 3.30. Topological vector space

A topological vector space is a vector space V equipped with a topology such that the operations of addition:

$$(x, y) \mapsto x + y \tag{180}$$

and scalar multiplication:

$$(\alpha, x) \mapsto \alpha x \tag{181}$$

are continuous maps $V \times V \to V$ and $\mathbb{C} \times V \to V$.

Definition 3.31. Locally convex space

Let V be a vector space and $\mathcal{P}$ a family of seminorms on V. We equip V with the topology with all finite intersections of sets of the form $\{x \in V : p(x - x_0) < \epsilon\}$ as a base, where $p \in \mathcal{P}$, $x_0 \in V$ and $\epsilon > 0$. If for all $x \in V$ such that $p(x) = 0$ for all $p \in \mathcal{P}$ we have that $x = 0$, then we call V (equipped with this topology) a locally convex space.

Remark 3.29. A locally convex space is always a Hausdorff topological vector space (see for example section IV.1 in [5]). If V is a locally convex space with topology generated by the family $\mathcal{P}$ of seminorms, then a net (x_ι) in V converges to $x \in V$ if and only if $p(x_\iota - x)$ converges to 0 for all $p \in \mathcal{P}$.

Many commonly used topologies on vector spaces fit within the framework of locally convex spaces. For example if V is a normed vector space with norm $\|\cdot\|$, then $\{\|\cdot\|\}$ generates the norm-topology on V. The topologies we will introduce in this section will also fall within this framework. For more details on the topic of locally convex spaces we refer to chapter IV in [5].

The first topology we introduce (aside from norm-topologies) is the $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology on $L^\infty(G^{\times n})$. Every $f \in L^1(G^{\times n})$ defines a bounded linear functional T_f on $L^\infty(G^{\times n})$ given by

$$T_f(\psi) = \int_{G^{\times n}} f(x_1, \ldots, x_n)\psi(x_1, \ldots, x_n) d(x_1, \ldots, x_n) \tag{182}$$

with $\|T_f\| = \|f\|_1$. Note that $f \mapsto T_f$ is a linear isometry. The $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology on $L^\infty(G^{\times n})$ is the topology generated by the family of seminorms $\{\psi \mapsto |T_f(\psi)| : f \in L^1(G^{\times n})\}$. Since

$$\|\psi\|_\infty = \sup_{\|f\|_1 \leq 1} |T_f(\psi)|, \tag{183}$$

we have that $L^\infty(G^{\times n})$ equipped with this topology is a locally convex space. Note that a net (ψ_ι) in $L^\infty(G^{\times n})$ converges to $\psi \in L^\infty(G^{\times n})$ in the $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology if and only if $T_f(\psi_\iota)$ converges to $T_f(\psi)$ for all $f \in L^1(G^{\times n})$.

The second topology we introduce, this one on $C_b(G^{\times n})$, is the topology of uniform convergence on compact sets. This is the topology generated by the family consisting of the seminorms $\psi \mapsto \sup_{x\in K} |\psi(x)|$ for all compact $K \subseteq G^{\times n}$. Because singleton sets are compact, it is clear that $\psi = 0$ if $\sup_{x\in K} |\psi(x)| = 0$ for all compact sets K. So, $C_b(G^{\times n})$ equipped with this topology is a locally convex space and a net (ψ_ι) in $C_b(G^{\times n})$ converges to $\psi \in C_b(G^{\times n})$ if and only if $\sup_{x\in K} |\psi_\iota(x) - \psi(x)|$ converges to 0 for all compact sets $K \subseteq G^{\times n}$.

Lemma 3.21. *Let (ψ_ι) be a net in $C_b(G^{\times n})$ that is bounded in $\|\cdot\|_\infty$ and converges to $\psi \in C_b(G^{\times n})$ uniformly on compact sets. Then (ψ_ι) also converges to ψ in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology.*

Proof. Let $k > 0$ be such that $\|\psi_\iota\|_\infty \leq k$ for all ι. Note that ψ_ι converges to ψ pointwise, hence it follows that also $\|\psi\|_\infty \leq k$. Let $f \in L^1(G^{\times n})$ and $\eta > 0$. Let $\epsilon = \frac{\eta}{4k+1} > 0$. By density of $C_c(G^{\times n})$ in $L^1(G^{\times n})$ (Theorem 3.12), there exists a $g \in C_c(G^{\times n})$ such that $\|f - g\|_1 < \epsilon$. Note that

$$\|g\|_1 \leq \|g - f\|_1 + \|f\|_1 = \|f\|_1 + \epsilon. \tag{184}$$

g is compactly supported. So, there exists a compact set $K \subseteq G^{\times n}$ such that $g(x) = 0$ for $x \notin K$. Let $\delta = \frac{\eta}{2\|f\|_1 + 2\epsilon}$. Because ψ_ι converges to ψ uniformly on compact sets, there exists a ι_0 such that

$$|\psi_\iota(x) - \psi(x)| < \delta \tag{185}$$

for all $x \in K$ and all $\iota \succeq \iota_0$. It follows that for all $\iota \succeq \iota_0$:

$$\begin{aligned}
|T_f(\psi_\iota) - T_f(\psi)| &= |T_f(\psi_\iota - \psi)| \leq \|f(\psi_\iota - \psi)\|_1 \leq \|(f-g)(\psi_\iota - \psi)\|_1 + \|g(\psi_\iota - \psi)\|_1 \\
&\leq \|f - g\|_1 \|\psi_\iota - \psi\|_\infty + \|g(\psi_\iota - \psi)\|_1 \leq 2k\epsilon + \|g(\psi_\iota - \psi)\|_1 \\
&= 2k\epsilon + \int_K |g(x_1, \ldots, x_n)||\psi_\iota(x_1, \ldots, x_n) - \psi(x_1, \ldots, x_n)| d(x_1, \ldots, x_n) \\
&\leq 2k\epsilon + \int_K |g(x_1, \ldots, x_n)|\delta d(x_1, \ldots, x_n) = 2k\epsilon + \|g\|_1 \delta \leq 2k\epsilon + (\|f\|_1 + \epsilon)\delta \\
&< \frac{1}{2}\eta + \frac{1}{2}\eta = \eta.
\end{aligned} \tag{186}$$

This shows that $T_f(\psi_\iota)$ converges to $T_f(\psi)$ for all $f \in L^1(G^{\times n})$, which completes the proof. □

We will now work towards proving that if a bounded net ψ_ι in $L^\infty(G^{\times n})$ converges in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology to ψ and $F \in L^1(G^{\times (n+1)})$, then $F \tilde{*} \psi_\iota$ converges to $F \tilde{*} \psi$ uniformly on compact sets. Our argument is inspired by parts of the proof of Lemma 2.2 in [18].

Lemma 3.22. *Let (ψ_ι) be a net in $L^\infty(G^{\times n})$ that is bounded in $\|\cdot\|_\infty$ and converges to $\psi \in L^\infty(G^{\times n})$ in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology. Then this convergence is uniform on compact subsets of $L^1(G^{\times n})$, i.e. if $K \subseteq L^1(G^{\times n})$ is compact (in norm topology) and $\epsilon > 0$, then there exists a ι_0 such that for all $\iota \succeq \iota_0$ and all $f \in K$ we have that*

$$|T_f(\psi_\iota) - T_f(\psi)| < \epsilon. \tag{187}$$

Proof. Let (ψ_ι) and ψ be as in the statement of the lemma. Let $k > 0$ be such that $\|\psi\|_\infty \leq k$ and $\|\psi_\iota\|_\infty \leq k$ for all ι. Fix $\eta > 0$ and choose $\epsilon = \frac{\eta}{2k+1} > 0$. For every $f \in L^1(G^{\times n})$ there exists a ι_f such that for all $\iota \succeq \iota_f$:

$$|T_f(\psi_\iota) - T_f(\psi)| < \epsilon. \tag{188}$$

If $f \in L^1(G^{\times n})$ and $g \in B_\epsilon(f) := \{g \in L^1(G^{\times n}) : \|f-g\|_1 < \epsilon\}$, then we have that

$$|T_f(\phi) - T_g(\phi)| = |T_{f-g}(\phi)| \leq \|f-g\|_1 \|\phi\|_\infty \leq \epsilon \|\phi\|_\infty \tag{189}$$

for any $\phi \in L^\infty(G^{\times n})$. It follows that

$$|T_f(\psi) - T_g(\psi)| < k\epsilon \tag{190}$$

and

$$|T_f(\psi_\iota) - T_g(\psi_\iota)| < k\epsilon \tag{191}$$

for all ι. It follows that for all $g \in B_\epsilon(f)$ and all $\iota \succeq \iota_f$:

$$\begin{aligned} |T_g(\psi_\iota) - T_g(\psi)| &= |T_g(\psi_\iota) - T_f(\psi_\iota) + T_f(\psi_\iota) - T_f(\psi) + T_f(\psi) - T_g(\psi)| \\ &\leq |T_g(\psi_\iota) - T_f(\psi_\iota)| + |T_f(\psi_\iota) - T_f(\psi)| + |T_f(\psi) - T_g(\psi)| \\ &< (2k+1)\epsilon = \eta. \end{aligned} \tag{192}$$

Now let $K \subseteq L^1(G^{\times n})$ be a compact set. Then $\{B_\epsilon(f) : f \in K\}$ is an open cover of K. By compactness, this cover has a finite subcover. So, there exist $f_1, \ldots, f_N \in K$ such that $K \subseteq \bigcup_{j=1}^N B_\epsilon(f_j)$. Let ι_0 be such that $\iota_0 \succeq \iota_{f_j}$ for all $j \in \{1, \ldots, N\}$. Then, for all $g \in K$ we have that $g \in B_\epsilon(f_j)$ for some $j \in \{1, \ldots, N\}$, hence for all $\iota \succeq \iota_0 \succeq \iota_{f_j}$ we have that

$$|T_g(\psi_\iota) - T_g(\psi)| < \eta. \tag{193}$$

This shows that the convergence of ψ_ι to ψ in the $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology is uniform on compact subsets of $L^1(G^{\times n})$. ∎

Lemma 3.23. *Let $\psi \in L^\infty(G^{\times n})$ and $F \in L^1(G^{\times (n+1)})$. For all $s_1, \ldots, s_n \in G$ we have that*

$$(F \tilde{*} \psi)(s_1, \ldots, s_n) = T_{F_{(s_1,\ldots,s_n)}}(\psi). \tag{194}$$

Here $F_{(s_1,\ldots,s_n)} \in L^1(G^{\times n})$ is given by

$$F_{(s_1,\ldots,s_n)}(t_1, \ldots, t_n) = \int_G F\left(\left(\left(\prod_{m=j+1}^{n} s_m\right)\left(\prod_{m=0}^{j} t_m\right)\right)_{j=0}^{n}\right) dt_0 \tag{195}$$

with $\left\|F_{(s_1,\ldots,s_n)}\right\|_1 \leq \|F\|_1$. The map $G^{\times n} \to L^1(G^{\times n})$, $(s_1, \ldots, s_n) \mapsto F_{(s_1,\ldots,s_n)}$ is continuous (with respect to the norm topology on $L^1(G^{\times n})$).

Proof. Let $\psi \in L^\infty(G^{\times n})$ and $F \in L^1(G^{\times (n+1)})$. We consider a fixed bounded representative of ψ. We will argue in Remark 3.30, that $F \tilde{*} \psi$ does not depend on the chosen representative for ψ. Note that the integrand in the definition of $F \tilde{*} \psi$ (see Definition 3.29) is an integrable function on $G^{\times (n+1)}$. This allows us to integrate separately over each variable in any order:

$$\begin{aligned} (F \tilde{*} \psi)(s_1, \ldots, s_n) &= \int_{G^{\times (n+1)}} \psi\left(\left(t_{j-1}^{-1} s_j t_j\right)_{j=1}^{n}\right) F\left((t_j)_{j=0}^{n}\right) d(t_0, \ldots, t_n) \\ &= \int_G \cdots \int_G \psi\left(\left(t_{j-1}^{-1} s_j t_j\right)_{j=1}^{n}\right) F\left((t_j)_{j=0}^{n}\right) dt_n \ldots dt_0. \end{aligned} \tag{196}$$

To rewrite this expression we will perform several changes of variables, justified by Lemma 3.6. First we substitute t_j by $\left(\prod_{m=j+1}^n s_m\right) t_j$ for all $j \in \{0, \ldots, n\}$ (note that nothing happens when $j = n$). We note that these substitutions lead to the substitution of $t_{j-1}^{-1} s_j t_j$ by

$$\left(\left(\prod_{m=j}^{n} s_m\right) t_{j-1}\right)^{-1} s_j \left(\prod_{m=j+1}^{n} s_m\right) t_j = t_{j-1}^{-1} \left(\prod_{m=j}^{n} s_m\right)^{-1} \left(\prod_{m=j}^{n} s_m\right) t_j = t_{j-1}^{-1} t_j \tag{197}$$

for all $j \in \{1, \dots, n\}$. It follows that

$$(F \tilde{*} \psi)(s_1, \dots, s_n) = \int_G \cdots \int_G \psi\left(\left(t_{j-1}^{-1} t_j\right)_{j=1}^{n}\right) F\left(\left(\left(\prod_{m=j+1}^{n} s_m\right) t_j\right)_{j=0}^{n}\right) dt_n \dots dt_0. \tag{198}$$

Next we perform more changes of variables. For these substitutions we note that for $j \in \{1, \dots, n\}$, the variable t_{j-1} can be treated as a constant when evaluating the integral over t_j. So we can substitute t_j by $t_{j-1} t_j$ for all $j \in \{1, \dots, n\}$. We note that the order in which we perform these substitutions matters. In particular we will perform these substitutions in order of decreasing j. This leads to the overall substitution of t_j by $\prod_{m=0}^{j} t_m$ and hence $t_{j-1}^{-1} t_j$ by t_j. From this it follows that

$$\begin{aligned}(F \tilde{*} \psi)(s_1, \dots, s_n) &= \int_G \cdots \int_G \psi\left((t_j)_{j=1}^{n}\right) F\left(\left(\left(\prod_{m=j+1}^{n} s_m\right)\left(\prod_{m=0}^{j} t_m\right)\right)_{j=0}^{n}\right) dt_n \dots dt_0 \\ &= \int_{G^{\times n}} \psi\left((t_j)_{j=1}^{n}\right) \int_G F\left(\left(\left(\prod_{m=j+1}^{n} s_m\right)\left(\prod_{m=0}^{j} t_m\right)\right)_{j=0}^{n}\right) dt_0 d(t_1, \dots, t_n) \\ &= T_{F_{(s_1, \dots, s_n)}}(\psi).\end{aligned} \tag{199}$$

For the second equality we again applied Fubini's theorem. When ψ, F are non-negative the use of Fubini's theorem is justified and by replacing ψ and F by $|\psi|$ and $|F|$ in the above calculations, the use of Fubini's theorem in the general case can be justified by verifying that the integrand is an integrable function. The above calculations with ψ replaced by 1 and F by $|F|$ show that $F_{(s_1, \dots, s_n)}$ is an integrable function. Indeed, we have that

$$\begin{aligned}\left\|F_{(s_1, \dots, s_n)}\right\|_1 &= \int_{G^{\times n}} \left|\int_G F\left(\left(\left(\prod_{m=j+1}^{n} s_m\right)\left(\prod_{m=0}^{j} t_m\right)\right)_{j=0}^{n}\right) dt_0\right| d(t_1, \dots, t_n) \\ &\leq \int_{G^{\times n}} \int_G \left|F\left(\left(\left(\prod_{m=j+1}^{n} s_m\right)\left(\prod_{m=0}^{j} t_m\right)\right)_{j=0}^{n}\right)\right| dt_0 d(t_1, \dots, t_n) \\ &= (|F| \tilde{*} 1)(s_1, \dots, s_n) = \|F\|_1.\end{aligned} \tag{200}$$

In particular this also shows that the definition of $F_{(s_1, \dots, s_n)}$ does not depend on the chosen version of F. To prove the continuity of the map $G^{\times n} \to L^1(G^{\times n})$, $(s_1, \dots, s_n) \mapsto F_{(s_1, \dots, s_n)}$, we will show that it is a composition of continuous maps. Note that $G^{\times n} \to G^{\times (n+1)}$,

$$(s_1, \dots, s_n) \mapsto \left(\prod_{m=j+1}^{n} s_m\right)_{j=0}^{n} \tag{201}$$

is a continuous map. The map $G^{\times (n+1)} \to L^1(G^{\times (n+1)})$

$$(x_0, \dots, x_n) \mapsto L_{(x_0^{-1}, \dots, x_n^{-1})} F \tag{202}$$

is continuous by Lemma 3.7. The map $L^1(G^{\times (n+1)}) \to L^1(G^{\times n})$

$$f \mapsto \left((t_1, \ldots, t_n) \mapsto \int_G f \left(\left(\prod_{m=0}^{j} t_m \right)_{j=0}^{n} \right) dt_0 \right) \tag{203}$$

is a linear contraction, hence continuous. This follows from our earlier calculations, because the above map is just $f_{(e,\ldots,e)}$. We note that $(s_1, \ldots, s_n) \mapsto F_{(s_1,\ldots,s_n)}$ is the composition of these three continuous maps, hence is itself a continuous map. □

Remark 3.30. Note that, just like with Lemma 3.20, the proof of Lemma 3.23 given above works if we assume that ψ is a bounded function instead of an equivalence class of essentially bounded functions. Most of the steps in this proof can also be performed without this assumption and doing this will show that this assumption is in fact not necessary. To justify the use of Fubini's theorem we initially assume that ψ and F are non-negative. We also need to choose a representative of ψ, because at this point it is not clear that $F\tilde{*}\psi$ does not depend on the chosen version of ψ. Under these assumptions we can follow the same steps as in the proof above to show that

$$(F\tilde{*}\psi)(s_1, \ldots, s_n) = \int_{G^{\times n}} \psi\left((t_j)_{j=1}^{n} \right) \int_G F \left(\left(\left(\prod_{m=j+1}^{n} s_m \right) \left(\prod_{m=0}^{j} t_m \right) \right)_{j=0}^{n} \right) dt_0 d(t_1, \ldots, t_n). \tag{204}$$

In particular, by choosing $\psi = 1$ it follows that

$$\begin{aligned} &\int_{G^{\times n}} \int_G F \left(\left(\left(\prod_{m=j+1}^{n} s_m \right) \left(\prod_{m=0}^{j} t_m \right) \right)_{j=0}^{n} \right) dt_0 d(t_1, \ldots, t_n) = (F\tilde{*}1)(s_1, \ldots, s_n) \\ &= \int_{G^{\times(n+1)}} F(t_0, \ldots, t_n) d(t_0, \ldots, t_n) < \infty. \end{aligned} \tag{205}$$

Using this and the essential boundedness of ψ it follows that $(F\tilde{*}\psi)(s_1, \ldots, s_n)$ is finite for any non-negative F and any non-negative representative of ψ. It follows that $(|F|\tilde{*}|\psi|)(s_1, \ldots, s_n)$ is finite for general F and any representative of ψ. This can be used to justify the use of Fubini's theorem without the non-negativity assumption because the integrand is integrable. It follows that

$$(F\tilde{*}\psi)(s_1, \ldots, s_n) = \int_{G^{\times n}} \psi\left((t_j)_{j=1}^{n} \right) \int_G F \left(\left(\left(\prod_{m=j+1}^{n} s_m \right) \left(\prod_{m=0}^{j} t_m \right) \right)_{j=0}^{n} \right) dt_0 d(t_1, \ldots, t_n) \tag{206}$$

for any representative of ψ. In this form it is clear that $F\tilde{*}\psi$ does not depend on the chosen representative of ψ. Therefore one can always assume that a representative of ψ is chosen such that $|\psi| \leq \|\psi\|_\infty$ holds pointwise on all of $G^{\times n}$. From this it follows that Lemma 3.20 and Lemma 3.23 hold as stated.

Theorem 3.24. *Let (ψ_ι) be a net in $L^\infty(G^{\times n})$ that is bounded in $\|\cdot\|_\infty$ and converges to $\psi \in L^\infty(G^{\times n})$ in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology. Let $F \in L^1(G^{\times(n+1)})$. Then $F\tilde{*}\psi_\iota$ converges to $F\tilde{*}\psi$ uniformly on compact sets.*

Proof. Let (ψ_ι), ψ and F be as in the statement of the theorem. From Lemma 3.23 we know that

$$(F\tilde{*}\psi_\iota)(s_1, \ldots, s_n) = T_{F_{(s_1,\ldots,s_n)}}(\psi_\iota) \tag{207}$$

and

$$(F\tilde{*}\psi)(s_1, \ldots, s_n) = T_{F_{(s_1,\ldots,s_n)}}(\psi), \tag{208}$$

where $F_{(s_1,\ldots,s_n)} \in L^1(G^{\times n})$. This already implies that $F\tilde{*}\psi_\iota$ converges to $F\tilde{*}\psi$ pointwise. To prove that this convergence is uniform on compact sets, let $K \subseteq G^{\times n}$ be compact and $\epsilon > 0$. The map $(s_1, \ldots, s_n) \mapsto$

$F_{(s_1,\dots,s_n)}$ is continuous by Lemma 3.23, hence $\{F_{(s_1,\dots,s_n)} : (s_1,\dots,s_n) \in K\}$ is a compact subset of $L^1(G^{\times n})$. It now follows from Lemma 3.22 that there exists a ι_0 such that for all $\iota \succeq \iota_0$ and all $(s_1,\dots,s_n) \in K$ we have that:

$$|(F \tilde{*} \psi_\iota)(s_1,\dots,s_n) - (F \tilde{*} \psi)(s_1,\dots,s_n)| = \left| T_{F_{(s_1,\dots,s_n)}}(\psi_\iota) - T_{F_{(s_1,\dots,s_n)}}(\psi) \right| < \epsilon. \tag{209}$$

This shows that $F \tilde{*} \psi_\iota$ converges to $F \tilde{*} \psi$ uniformly on compact sets. ☐

This theorem has the following two corollaries:

Corollary 3.25. *Let (ψ_ι) be a net in $L^\infty(G^{\times n})$ that is bounded in $\|\cdot\|_\infty$ and converges to $\psi \in L^\infty(G^{\times n})$ in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology. Let $F \in L^1(G^{\times(n+1)})$. Then $F \tilde{*} \psi_\iota$ converges in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology to $F \tilde{*} \psi$.*

Proof. This follows by combining Theorem 3.24 and Lemma 3.21, where we note that $F \tilde{*} \psi_\iota$ is bounded in $\|\cdot\|_\infty$ by Lemma 3.20. ☐

Corollary 3.26. *Let (ψ_ι) be a net in $L^\infty(G^{\times n})$ that is bounded in $\|\cdot\|_\infty$ and converges to $1 \in L^\infty(G^{\times n})$ in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology. Let $F \in L^1(G^{\times(n+1)})$ be such that $\int_{G^{\times(n+1)}} F(t_0,\dots,t_n) d(t_0,\dots,t_n) = 1$. Then $F \tilde{*} \psi_\iota$ converges to 1 uniformly on compact sets.*

Proof. This is a special case of Theorem 3.24, where we note that $F \tilde{*} 1 = 1$ by Remark 3.28. ☐

Lemma 3.27. *Let $\psi \in L^\infty(G^{\times n})$ and $F \in L^1(G^{\times(n+1)})$ be compactly supported, i.e. there exist compact sets $K_\psi \subseteq G^{\times n}$, $K_F \subseteq G^{\times(n+1)}$ such that $\psi = \mathbb{1}_{K_\psi}\psi$ and $F = \mathbb{1}_{K_F} F$ almost everywhere. Then $F \tilde{*} \psi \in C_c(G^{\times n})$.*

Proof. From Definition 3.29 it is clear that $F \tilde{*} \psi$ does not depend on the chosen representative of F. Similarly Remark 3.30 shows that $F \tilde{*} \psi$ also does not depend on the chosen representative of ψ. So without loss of generality we can choose representatives of ψ and F that always take the value 0 outside of K_ψ and K_F respectively. Because the coordinate projections $G^{\times n} \to G$ and $G^{\times(n+1)} \to G$ are continuous functions, we can find a compact set $K_0 \subseteq G$ such that $K_\psi \subseteq K_0^{\times n}$ and $K_F \subseteq K_0^{\times(n+1)}$. We recall that

$$(F \tilde{*} \psi)(s_1,\dots,s_n) = \int_{G^{\times(n+1)}} \psi\left((t_j)_{j=1}^n\right) F\left(\left(\left(\prod_{m=j+1}^n s_m\right)\left(\prod_{m=0}^j t_m\right)\right)_{j=0}^n\right) d(t_0,\dots,t_n) \tag{210}$$

and note that the integrand equals 0 unless $t_1,\dots,t_n \in K_0$ because the support of ψ is contained in $K_0^{\times n}$. The integrand also equals 0 unless $\prod_{m=0}^n t_m \in K_0$ (observe the last variable of F in the above expression). It follows that the integrand equals 0 unless $\prod_{m=0}^j t_m \in K_0(K_0^{n-j})^{-1}$ for all $j \in \{0,\dots,n\}$. The integrand also equals 0 unless $\prod_{m=j+1}^n s_m \in K_0\left(\prod_{m=0}^j t_m\right)^{-1}$ for all $j \in \{0,\dots,n-1\}$ (observe the first n variables of F). It follows that the integrand equals 0 for all $t_0,\dots,t_n \in G$, and therefore $(F \tilde{*} \psi)(s_1,\dots,s_n) = 0$, unless

$$\prod_{m=j+1}^n s_m \in K_0^{n+1-j} K_0^{-1} \tag{211}$$

for all $j \in \{0,\dots,n\}$, where we note that this condition is automatically true for $j = n$. This implies that $(F \tilde{*} \psi)(s_1,\dots,s_n) = 0$ unless

$$s_j = \left(\prod_{m=j}^n s_m\right)\left(\prod_{m=j+1}^n s_m\right)^{-1} \in K_0^{n+2-j} K_0^{-1} K_0 (K_0^{n+1-j})^{-1} \tag{212}$$

for all $j \in \{1,\dots,n\}$. So the support of $F \tilde{*} \psi$ is contained in $K^{\times n}$, where $K \subseteq G$ is the compact set

$$K := \bigcup_{j=1}^n K_0^{n+2-j} K_0^{-1} K_0 (K_0^{n+1-j})^{-1}. \tag{213}$$

This shows that $(F\tilde{*}\psi)$ is compactly supported if ψ and F are compactly supported. Combined with Lemma 3.20, it follows that $(F\tilde{*}\psi) \in C_c(G^{\times n})$. $\square$

4 Fourier multipliers

In this section we introduce (multilinear) Fourier multipliers (subsection 4.2) and related concepts such as the group von Neumann algebra (subsection 4.1), Schur multipliers (subsection 4.3), the Fourier algebra (subsection 4.4) and weak amenability (subsection 4.5). In contrast to the approach we have taken in the introduction (section 1), in this section we will first define mutlilinear Fourier multipliers in terms of the group von Neumann algebra and make the connection with the Fourier algebra afterwards. We refer to the introduction (section 1) for a different approach that might be more intuitive for readers familiar with classical Fourier multipliers.

4.1 Group von Neumann algebra

In this subsection we define the group von Neumann algebra $VN(G)$ of a locally compact group G. The Fourier multipliers, which we will define in subsection 4.2, will be multilinear maps $VN(G)^{\times n} \to VN(G)$. We start by introducing several topologies that will be relevant.

Definition 4.1. Strong operator topology (SOT)

Let H be a Hilbert space. The strong operator topology (SOT) is the topology on $\mathcal{B}(H)$ generated by the family of seminorms $\{A \mapsto \|Ax\| : x \in H\}$.

Definition 4.2. Weak operator topology (WOT)

Let H be a Hilbert space. The weak operator topology (WOT) is the topology on $\mathcal{B}(H)$ generated by the family of seminorms $\{A \mapsto |\langle Ax, y\rangle| : x, y \in H\}$.

Remark 4.1. Note that $\mathcal{B}(H)$ equipped with either one of the SOT and the WOT is a locally convex space. If (A_ι) is a net in $\mathcal{B}(H)$ and $A \in \mathcal{B}(H)$, then A_ι converges to A in the SOT if and only if $A_\iota x$ converges to Ax in norm for all $x \in H$. Similarly A_ι converges to A in the WOT if and only if $\langle A_\iota x, y\rangle$ converges to $\langle Ax, y\rangle$ for all $x, y \in H$. It is straightforward to check that any set that is open in the WOT is also open in the SOT and any set that is open in the SOT is open in the norm topology. Therefore, we have that if A_ι converges to A in norm, then it also converges in the SOT and if it converges in the SOT, then it also converges in the WOT.

Definition 4.3. σ-weak operator topology

Let H be a Hilbert space. The σ-weak operator topology on $\mathcal{B}(H)$ is the topology generated by the family of all seminorms of the form

$$A \mapsto \left|\sum_{n=1}^{\infty} \langle Ax_n, y_n\rangle\right| \tag{214}$$

where $x_n, y_n \in H$ for $n \in \mathbb{N}$ are such that

$$\sum_{n=1}^{\infty} \|x_n\|^2 < \infty \tag{215}$$

and

$$\sum_{n=1}^{\infty} \|y_n\|^2 < \infty. \tag{216}$$

Remark 4.2. Note that the family of seminorms that generates the σ-weak operator topology contains the family that generates the WOT (this can be seen by choosing $x_n = y_n = 0$ for all $n \geq 2$). So the σ-weak operator topology is a locally convex toplogy on $\mathcal{B}(H)$ and any set that is open in the WOT is also open in the σ-weak operator topology. It follows that any net A_ι in $\mathcal{B}(H)$ that converges to $A \in \mathcal{B}(H)$ in the σ-weak operator topology, also converges to A in the WOT. Note that A_ι converges to A in the σ-weak operator topology if and only if $\sum_{n=1}^{\infty} \langle A_\iota x_n, y_n\rangle$ converges to $\sum_{n=1}^{\infty} \langle Ax_n, y_n\rangle$ for all $x_n, y_n \in H$ such that $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|y_n\|^2 < \infty$. It is straightforward to check that for any fixed $B \in \mathcal{B}(H)$, the maps $A \mapsto AB$ and $A \mapsto BA$ are continuous maps $\mathcal{B}(H) \to \mathcal{B}(H)$ with respect to the SOT, WOT and σ-weak operator topology. We refer to chapter II in [27] for more details on the SOT, WOT and σ-weak operator topology as well as several other important topologies on $\mathcal{B}(H)$.

Definition 4.4. Von Neumann algebra

Let H be a Hilbert space. A von Neumann algebra on H is a $*$-subalgebra of $\mathcal{B}(H)$ that is closed in the SOT.

Definition 4.5. Commutant

Let H be a Hilbert space and $C \subseteq \mathcal{B}(H)$. The commutant of C is defined as

$$C' := \{B \in \mathcal{B}(H) : AB = BA \, \forall A \in C\}. \tag{217}$$

Remark 4.3. For any $C \subseteq \mathcal{B}(H)$, we have that C' is a closed subalgebra of $\mathcal{B}(H)$. If C is self-adjoint (i.e. $C = C^* := \{A^* : A \in C\}$), then C' is also a $*$-subalgebra of $\mathcal{B}(H)$. We always have $C \subseteq C''$ and $C' = C'''$. See also section 4.1 in [23].

Theorem 4.1. *Let H be a Hilbert space and A a $*$-subalgebra of $\mathcal{B}(H)$ containing id_H. Then the following are equivalent:*

1. *A is a von Neumann algebra.*
2. *A is closed in the WOT.*
3. *$A = A''$.*

Remark 4.4. For the proof of Theorem 4.1 we refer to Theorems 4.1.5 and 4.2.5 in [23]. Note that for any $C \subseteq \mathcal{B}(H)$, there is a smallest von Neumann algebra that contains C. This is called the von Neumann algebra generated by C. In case C is self-adjoint and contains id_H, we have that the von Neumann algebra generated by C is equal to C'', see also section 4.1 in [23].

To define Fourier multipliers, we need to introduce a specific von Neumann algebra, known as the group von Neumann algebra. Its definition relies on certain representations, which we will introduce first.

Definition 4.6. Continuous unitary representation

Let G be a locally compact group and H a Hilbert space. A continuous unitary representation π of G on H is a group homomorphism from G to the group $\mathcal{U}(H) \subseteq \mathcal{B}(H)$ of unitary operators on H, with π continuous with respect to the SOT on $\mathcal{B}(H)$.

Definition 4.7. $*$-representation

Let A be an involutive algebra and H a Hilbert space. A $*$-representation π of A on H is a $*$-homomorphism $A \to \mathcal{B}(H)$.

Theorem 4.2. *Let G be a locally compact group, H a Hilbert space and $\pi : G \to \mathcal{U}(H)$ a continuous unitary representation. For each $f \in L^1(G)$, there exists a unique $\pi'(f) \in \mathcal{B}(H)$ such that*

$$\langle \pi'(f)u, v\rangle = \int_G f(x) \langle \pi(x)u, v\rangle \, dx \tag{218}$$

for all $u, v \in H$, where integration is against the Haar measure. This defines a $*$-representation $\pi' : G \to \mathcal{B}(H)$.

For a proof of the above result we refer to Theorem 3.9 in [17] or section 13.3 in [9]. Of particular interest to us are the left-regular representations of G and $L^1(G)$.

Definition 4.8. Left regular representation

Let G be a locally compact group. The left-regular representation of G is the continuous unitary representation $\lambda : G \to \mathcal{U}(L^2(G))$ given by

$$\lambda(x)(f) = L_x(f). \tag{219}$$

This representation induces (through Theorem 4.2) the left-regular representation of $L^1(G)$, which is the $*$-representation $\lambda' : L^1(G) \to \mathcal{B}(L^2(G))$ given by

$$\lambda'(f)(g) = f * g. \tag{220}$$

Remark 4.5. It is straightforward to check that $\lambda(x)^* = \lambda(x^{-1}) = \lambda(x)^{-1}$, hence $\lambda(x)$ is unitary. We have already seen that $x \mapsto L_x$ is a group homomorphism (Remark 3.2) and is continuous in the SOT (Lemma 3.7). So λ is indeed a continuous unitary representation. See section 3.2. in [17], section 13.3 in [9] or section 1.6 in [22] for why λ' is given by the convolution: $\lambda'(f)g = f * g$.

Definition 4.9. Group von Neumann algebra

Let G be a locally compact group. The group von Neumann algebra $VN(G)$ of G is the von Neumann algebra

$$VN(G) = \{\lambda(x) : x \in G\}'' = \{\lambda'(f) : f \in L^1(G)\}''. \tag{221}$$

For the proof of the second equality in Definition 4.9, we refer to section 13.3 in [9].

We note that the group von Neumann algebra $VN(G)$, like any other von Neumann algebra, can be equipped with the relative SOT, WOT or σ-weak operator topology. The following theorem, for the proof of which we refer to Corollary 2.4.4 in [22], asserts that the latter two are the same for the group von Neumann algebra.

Theorem 4.3. *Let G be a locally compact group. The relative WOT and σ-weak operator topology on $VN(G) \subseteq \mathcal{B}(L^2(G))$ are the same.*

Lemma 4.4. *Let G be a locally compact group (with Haar measure μ). Then the finite linear span of $\{\lambda(x) : x \in G\}$ is dense in $VN(G)$ in the relative SOT, WOT and σ-weak operator topology. Also, $\{\lambda(x) : x \in G\}$ is a linearly independent set.*

Proof. Because $\lambda(x)\lambda(y) = \lambda(xy)$ and $\lambda(x)^* = \lambda(x)^{-1} = \lambda(x^{-1})$, we have that the finite linear span of $\{\lambda(x) : x \in G\}$ is a $*$-subalgebra of $\mathcal{B}(L^2(G))$. Hence, by Lemma 4.1.4 and Theorem 4.2.5 in [23], the finite linear span of $\{\lambda(x) : x \in G\}$ is dense in its second commutant in the relative SOT and WOT. Clearly, the second commutant of the finite linear span of $\{\lambda(x) : x \in G\}$ is equal to

$$\{\lambda(x) : x \in G\}'' = VN(G). \tag{222}$$

So the finite linear span of $\{\lambda(x) : x \in G\}$ is dense in $VN(G)$ in the relative SOT and WOT. By Theorem 4.3 it is also dense in the relative σ-weak operator topology. To prove linear independence of $\{\lambda(x) : x \in G\}$, let $x_1, \ldots, x_n \in G$ distinct and $c_1, \ldots, c_n \in \mathbb{C}$ such that $\sum_{j=1}^n c_j \lambda(x_j) = 0$. If $n = 1$, then we must have $c_1 = 0$, because $\lambda(x_1) \neq 0$. So assume $n \geq 2$. Because G is a Hausdorff space, each pair of distinct points in G has disjoint open neighbourhoods. Applying this to the pairs $\{x_i, x_j\}$ with $i \neq j$, it follows that for each $i, j \in \{1, \ldots, n\}$ with $i \neq j$ we can find an open set $U_{i,j} \subseteq G$ such that $x_i \in U_{i,j}$ (and $x_j \in U_{j,i}$) and such that $U_{i,j} \cap U_{j,i} = \emptyset$. In other words, $U_{i,j}$ and $U_{j,i}$ are disjoint open neighbourhoods of x_i and x_j, respectively. Define, for each $i \in \{1, \ldots, n\}$, the open set

$$V_i := \bigcap_{j \in \{1,\ldots,n\} \setminus \{i\}} U_{i,j}. \tag{223}$$

Then we have that $x_i \in V_i$ for all $i \in \{1, \ldots, n\}$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. Let $W_i := x_i^{-1} V_i$. Then W_i is an open neighbourhood of the identity e for all $i \in \{1, \ldots, n\}$. Define

$$W := \bigcap_{i=1}^n W_i. \tag{224}$$

Then W is an open neighbourhood of e. Note that $\mu(W) > 0$ because W is open and non-empty (Lemma 3.3). Also note that for all $i \in \{1, \ldots, n\}$:

$$x_i W \subseteq x_i W_i = V_i. \tag{225}$$

So, the sets $x_i W$ for $i \in \{1, \ldots, n\}$ are pairwise disjoint. Let $K \subseteq W$ be a compact neighbourhood of e, which exists because G is locally compact. Then $0 < \mu(K) < \infty$, because K is compact and contains a non-empty open set. We also have that the sets $x_i K$ for $i \in \{1, \ldots, n\}$ are pairwise disjoint. We consider the function $\mathbb{1}_K \in L^2(\mathcal{B}(G))$ and note that

$$\|\mathbb{1}_K\|_2^2 = \mu(K). \tag{226}$$

Because $\sum_{j=1}^n c_j \lambda(x_j) = 0$, we have that

$$\sum_{j=1}^n c_j \lambda(x_j)(\mathbb{1}_K) = 0. \tag{227}$$

However,

$$\sum_{j=1}^n c_j \lambda(x_j)(\mathbb{1}_K) = \sum_{j=1}^n c_j \mathbb{1}_{x_j K}. \tag{228}$$

Because the sets $x_i K$ for $i \in \{1, \ldots, n\}$ are pairwise disjoint, it follows that

$$0 = \left\| \sum_{j=1}^n c_j \lambda(x_j)(\mathbb{1}_K) \right\|_2^2 = \left\| \sum_{j=1}^n c_j \mathbb{1}_{x_j K} \right\|_2^2 = \sum_{j=1}^n |c_j|^2 \mu(x_j K) = \mu(K) \sum_{j=1}^n |c_j|^2. \tag{229}$$

Since $\mu(K) > 0$, we must have that $c_j = 0$ for all $j \in \{1, \ldots, n\}$. This shows that $\{\lambda(x) : x \in G\}$ is linearly independent. □

Corollary 4.5. *Let G be a locally compact group, $n \in \mathbb{N}$ and V a vector space. Any map $\{\lambda(x) : x \in G\}^{\times n} \to V$ has a unique n-linear extension to the Cartesian product of n copies of the finite linear span of $\{\lambda(x) : x \in G\}$.*

Proof. Let $f : \{\lambda(x) : x \in G\}^{\times n} \to V$ be a map. By linear independence every element of the finite linear span of $\{\lambda(x) : x \in G\}$ can be uniquely written as $\sum_{x \in G} c_x \lambda(x)$, where $c_x \in \mathbb{C}$ and $c_x = 0$ for all but finitely many x. An n-linear extension of f maps $\left(\sum_{x_j \in G} c_{x_j,j} \lambda(x_j)\right)_{j=1}^n$ to $\sum_{(x_1,\ldots,x_n) \in G^{\times n}} \left(\prod_{j=1}^n c_{x_j,j}\right) f(\lambda(x_1), \ldots, \lambda(x_n))$, establishing uniqueness. Since $\left(\sum_{x_j \in G} c_{x_j,j} \lambda(x_j)\right)_{j=1}^n \mapsto \sum_{(x_1,\ldots,x_n) \in G^{\times n}} \left(\prod_{j=1}^n c_{x_j,j}\right) f(\lambda(x_1), \ldots, \lambda(x_n))$ defines an n-linear map, we also have existence. □

Lemma 4.6. *Let X and Y be topological spaces with Y Hausdorff, $n \in \mathbb{N}$ and $X_0 \subseteq X$ dense. Let $f, g : X^{\times n} \to Y$ be functions that are continuous in each variable and that agree on $X_0^{\times n}$. Then $f = g$.*

Proof. We have that

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \tag{230}$$

for all $x_1, \ldots, x_n \in X_0$. We claim that for all $m \in \{0, \ldots, n\}$:

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \tag{231}$$

for all $x_1, \ldots, x_m \in X$ and $x_{m+1}, \ldots, x_n \in X_0$. This is clearly true for $m = 0$. Assuming this holds for some $m \in \{0, \ldots, n-1\}$, we fix arbitrary $x_1, \ldots, x_{m+1} \in X$ and $x_{m+2}, \ldots, x_n \in X_0$. By density of X_0 in X, there exists a net y_ι in X_0 that converges to x_{m+1}. By continuity of f and g in the $(m+1)$-th variable, we have that $f(x_1, \ldots, x_m, y_\iota, x_{m+2}, \ldots, x_n)$ converges to $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_m, y_\iota, x_{m+2}, \ldots, x_n)$ converges to $g(x_1, \ldots, x_n)$. But $f(x_1, \ldots, x_m, y_\iota, x_{m+2}, \ldots, x_n) = g(x_1, \ldots, x_m, y_\iota, x_{m+2}, \ldots, x_n)$ for all ι, hence it follows that the claim holds for $m + 1$ by uniqueness of limits (Y is Hausdorff). The result now follows by induction. □

Corollary 4.7. *Let G be a locally compact group, $n \in \mathbb{N}$ and V a locally convex space. Let $f, g : VN(G)^{\times n} \to V$ be multilinear functions that agree on $\{\lambda(x) : x \in G\}$ and such that for either one of the SOT, WOT or σ-weak operator topology, both f and g are continuous in each variable with respect to this topology. Then $f = g$.*

Proof. By Corollary 4.5, f and g must agree on the Cartesian product of n copies of the finite linear span of $\{\lambda(x) : x \in G\}$. Because this linear span is dense in $VN(G)$, it follows by Lemma 4.6 that $f = g$. □

Note that Corollary 4.7 implies that any map $\phi : \{\lambda(x) : x \in G\}^{\times n} \to V$ has at most one multilinear extension to $VN(G)^{\times n}$ that is continuous with respect to the SOT, WOT or σ-weak operator topology in each variable.

4.2 Fourier multipliers

In this subsection we define Fourier multipliers and cover several results about Fourier multipliers. In particular we show several methods to construct new Fourier multipliers from existing ones, based on the methods introduced in section 2 to construct multilinear maps.

Definition 4.10. Fourier multiplier

Let G be a locally compact group and $n \in \mathbb{N}$. Let $\phi \in C_b(G^{\times n})$. If the map $\{\lambda(x) : x \in G\}^{\times n} \to VN(G)$ given by

$$(\lambda(x_j))_{j=1}^n \mapsto \phi(x_1, \ldots, x_n) \prod_{j=1}^n \lambda(x_j) = \phi(x_1, \ldots, x_n) \lambda\left(\prod_{j=1}^n x_j\right) \tag{232}$$

extends to a bounded multilinear map $M_\phi : VN(G)^{\times n} \to VN(G)$ that is continuous in each variable with respect to the σ-weak operator topology, then M_ϕ is called a Fourier multiplier and ϕ is called its symbol. Sometimes ϕ itself is also called a Fourier multiplier. We write $M^n A(G)$ for the set of all $\phi \in C_b(G^{\times n})$ such that M_ϕ is a Fourier multiplier. If M_ϕ is completely bounded as a multilinear map, then we call M_ϕ a completely bounded Fourier multiplier. We write $M_{cb}^n A(G)$ for the set of all $\phi \in C_b(G^{\times n})$ such that M_ϕ is a completely bounded Fourier multiplier. In case $n = 1$ we usually do not write the superscript n.

Remark 4.6. Note that by Corollary 4.7, the map M_ϕ is unique if it exists.

Theorem 4.8. *Let G be a locally compact group and $n \in \mathbb{N}$. $M^n A(G)$ and $M_{cb}^n A(G)$ are normed spaces when equipped with the norms*

$$\|\phi\|_{M^n A(G)} := \|M_\phi\| \tag{233}$$

and

$$\|\phi\|_{M_{cb}^n A(G)} := \|M_\phi\|_{CB}, \tag{234}$$

respectively. $MA(G)$ and $M_{cb}A(G)$ are normed algebras. The inclusions of $M_{cb}^n A(G)$ into $M^n A(G)$ and of $M^n A(G)$ into $C_b(G^{\times n})$ are contractive, i.e.

$$\|\phi\|_\infty \leq \|\phi\|_{M^n A(G)} \leq \|\phi\|_{M_{cb}^n A(G)}. \tag{235}$$

Proof. Let $\phi, \psi \in M^n A(G)$ and $c \in \mathbb{C}$. It is clear that $M_\phi + M_\psi$ and cM_ϕ are bounded multilinear maps $VN(G)^{\times n} \to VN(G)$ that are continuous in each variable with respect to the σ-weak operator topology. Moreover they are also completely bounded in case $\phi, \psi \in M_{cb}^n A(G)$. From Definition 4.10 and Remark 4.6 it is clear that $M_\phi + M_\psi = M_{\phi+\psi}$ and $cM_\phi = M_{c\phi}$. This shows that $M^n A(G)$ and $M_{cb}^n A(G)$ are vector spaces and $\phi \mapsto M_\phi$ is linear. It follows that $\|\cdot\|_{M^n A(G)}$ and $\|\cdot\|_{M_{cb}^n A(G)}$ are seminorms. Let $\phi \in M^n A(G)$. For $x_1, \ldots, x_n \in G$ we have that

$$|\phi(x_1,\ldots,x_n)| = \left\| \phi(x_1,\ldots,x_n)\lambda\left(\prod_{j=1}^n x_j\right)\right\| = \|M_\phi(\lambda(x_1),\ldots,\lambda(x_n))\| \leq \|M_\phi\| \prod_{j=1}^n \|\lambda(x_j)\| = \|\phi\|_{M^nA(G)}. \tag{236}$$

This shows that $\|\phi\|_\infty \leq \|\phi\|_{M^nA(G)}$. For $\phi \in M^n_{cb}A(G)$ we have that $\|M_\phi\| \leq \|M_\phi\|_{CB}$, hence $\|\phi\|_{M^nA(G)} \leq \|\phi\|_{M^n_{cb}A(G)}$. It follows that $\|\cdot\|_{M^nA(G)}$ and $\|\cdot\|_{M^n_{cb}A(G)}$ are norms and the inclusions of $M^n_{cb}A(G)$ into $M^nA(G)$ and of $M^nA(G)$ into $C_b(G^{\times n})$ are contractive. If $\phi,\psi \in MA(G)$, then $M_\phi, M_\psi : VN(G) \to VN(G)$ are bounded and continuous with respect to the σ-weak operator topology. The composition $M_\phi M_\psi$ is also bounded with

$$\|M_\phi M_\psi\| \leq \|M_\phi\|\|M_\psi\| \tag{237}$$

and continuous with respect to the σ-weak operator topology. Moreover, in case $\phi,\psi \in M_{cb}A(G)$, we have that M_ϕ and M_ψ are completely bounded. It follows by Theorem 2.14 (in particular the linear case) that the composition $M_\phi M_\psi$ is completely bounded, with

$$\|M_\phi M_\psi\|_{CB} \leq \|M_\phi\|_{CB}\|M_\psi\|_{CB}. \tag{238}$$

It remains to show that $M_\phi M_\psi = M_{\phi\psi}$. For any $x \in G$ we have that

$$M_\phi(M_\psi(\lambda(x))) = M_\phi(\psi(x)\lambda(x)) = \psi(x)M_\phi(\lambda(x)) = \phi(x)\psi(x)\lambda(x). \tag{239}$$

This shows that $M_\phi M_\psi = M_{\phi\psi}$ (hence M_ϕ and M_ψ commute) and we conclude that $MA(G)$ and $M_{cb}A(G)$ are normed algebras. □

Theorem 4.9. *Let G be a locally compact group. Let $k \in \mathbb{N}$ and $n_1,\ldots,n_k \in \mathbb{N}$. Let $\phi_j \in C_b(G^{\times n_j})$ for all $j \in \{1,\ldots,k\}$. Let $N = \sum_{j=1}^k n_j$ and define $\Phi \in C_b(G^{\times N})$ by*

$$\Phi(x_1,\ldots,x_N) := \prod_{j=1}^k \phi_j\left(\left(x_{\sum_{l=1}^{j-1} n_l + i}\right)_{i=1}^{n_j}\right). \tag{240}$$

If $\phi_j \in M^{n_j}A(G)$ for all $j \in \{1,\ldots,k\}$, then we have that $\Phi \in M^NA(G)$ with

$$\|\Phi\|_{M^NA(G)} \leq \prod_{j=1}^k \|\phi_j\|_{M^{n_j}A(G)}. \tag{241}$$

If $\phi_j \in M^{n_j}_{cb}A(G)$ for all $j \in \{1,\ldots,k\}$, then we have that $\Phi \in M^N_{cb}A(G)$ with

$$\|\Phi\|_{M^N_{cb}A(G)} \leq \prod_{j=1}^k \|\phi_j\|_{M^{n_j}_{cb}A(G)}. \tag{242}$$

Proof. Note that it is clear that $\Phi \in C_b(G^{\times N})$ by continuity of multiplication in $\mathbb{C}$. Suppose that $\phi_j \in M^{n_j}A(G)$ for all $j \in \{1,\ldots,k\}$. Then $M_{\phi_j} : VN(G)^{\times n_j} \to VN(G)$ is bounded for $j \in \{1,\ldots,k\}$ with

$$\|M_{\phi_j}\| = \|\phi_j\|_{M^{n_j}A(G)}. \tag{243}$$

We define the map $M : VN(G)^{\times N} \to VN(G)$ by

$$M(A_1,\ldots,A_N) := \prod_{j=1}^k M_{\phi_j}\left(\left(A_{\sum_{l=1}^{j-1} n_l + i}\right)_{i=1}^{n_j}\right). \tag{244}$$

Then, by Remark 2.4 and Lemma 2.17, we have that M is a bounded multilinear map with

$$\|M\| \leq \prod_{j=1}^{k} \|\phi_j\|_{M^{n_j}A(G)}. \tag{245}$$

Moreover in case $\phi_j \in M_{cb}^{n_j}A(G)$ for all $j \in \{1,\ldots,k\}$, then it follows by Theorem 2.14 that M is completely bounded with

$$\|M\|_{CB} \leq \prod_{j=1}^{k} \|\phi_j\|_{M_{cb}^{n_j}A(G)}. \tag{246}$$

Because each M_{ϕ_j} is continuous in each of its variables with respect to the σ-weak operator topology and because products in $\mathcal{B}(H)$ are continuous separately in each factor with respect to the σ-weak operator topology (see Remark 4.2), it follows that M is continuous in each variable with respect to the σ-weak operator topology. It remains to prove that $M = M_\Phi$. To that end let $x_1, \ldots, x_N \in G$. We have that

$$\begin{aligned}
M(\lambda(x_1),\ldots,\lambda(x_N)) &= \prod_{j=1}^{k} M_{\phi_j}\left(\left(\lambda\left(x_{\sum_{l=1}^{j-1} n_l+i}\right)\right)_{i=1}^{n_j}\right) \\
&= \prod_{j=1}^{k}\left(\phi_j\left(\left(x_{\sum_{l=1}^{j-1} n_l+i}\right)_{i=1}^{n_j}\right)\prod_{i=1}^{n_j}\lambda\left(x_{\sum_{l=1}^{j-1} n_l+i}\right)\right) \\
&= \left(\prod_{j=1}^{k}\phi_j\left(\left(x_{\sum_{l=1}^{j-1} n_l+i}\right)_{i=1}^{n_j}\right)\right)\left(\prod_{j=1}^{k}\prod_{i=1}^{n_j}\lambda\left(x_{\sum_{l=1}^{j-1} n_l+i}\right)\right) \\
&= \Phi(x_1,\ldots,x_n)\prod_{j=1}^{N}\lambda(x_j).
\end{aligned} \tag{247}$$

This shows that $M = M_\Phi$, completing the proof. □

The following corollary is a special case of Theorem 4.9 where $n_j = 1$ for all j.

Corollary 4.10. *Let G be a locally compact group. Let $n \in \mathbb{N}$ and $\phi_j \in C_b(G)$ for all $j \in \{1,\ldots,n\}$. If $\phi_j \in MA(G)$ for all $j \in \{1,\ldots,n\}$, then we have that $\bigotimes_{j=1}^n \phi_j \in M^nA(G)$ with*

$$\left\|\bigotimes_{j=1}^{n}\phi_j\right\|_{M^nA(G)} \leq \prod_{j=1}^{n}\|\phi_j\|_{MA(G)}. \tag{248}$$

If $\phi_j \in M_{cb}A(G)$ for all $j \in \{1,\ldots,n\}$, then we have that $\bigotimes_{j=1}^n \phi_j \in M_{cb}^nA(G)$ with

$$\left\|\bigotimes_{j=1}^{n}\phi_j\right\|_{M_{cb}^nA(G)} \leq \prod_{j=1}^{n}\|\phi_j\|_{M_{cb}A(G)}. \tag{249}$$

Theorem 4.11. *Let G be a locally compact group and $n \in \mathbb{N}$. Let $J \subset \{1,\ldots,n\}$ be a strict subset. Let $x_j = e$ for $j \in J$. Let $\phi \in C_b(G^{\times n})$ and define $\psi \in C_b(G^{\times(n-|J|)})$ by*

$$\psi\left((x_j)_{j\in\{1,\ldots,n\}\setminus J}\right) = \phi(x_1,\ldots,x_n), \tag{250}$$

where the tuple $(x_j)_{j\in\{1,\ldots,n\}\setminus J}$ is considered to be ordered in order of increasing j. If $\phi \in M^nA(G)$, then $\psi \in M^{n-|J|}A(G)$ with

$$\|\psi\|_{M^{n-|J|}A(G)} \leq \|\phi\|_{M^nA(G)}. \tag{251}$$

If $\phi \in M_{cb}^nA(G)$, then $\psi \in M_{cb}^{n-|J|}A(G)$ with

$$\|\psi\|_{M_{cb}^{n-|J|}A(G)} \leq \|\phi\|_{M_{cb}^nA(G)}. \tag{252}$$

Proof. Let ϕ, ψ be as in the statement of the theorem. Let $x_j = e$ for $j \in J$. Suppose that $\phi \in M^nA(G)$. Then $M_\phi : VN(G)^{\times n} \to VN(G)$ is a bounded multilinear map and continuous in each variable with respect to the σ-weak operator topology. We consider the map $M := M_\phi(\cdot|J; (\lambda(x_j))_{j\in J}) : VN(G)^{\times(n-|J|)} \to VN(G)$ as defined in Remark 2.5. We know from Remark 2.5 that M is a bounded multilinear map with

$$\|M\| \leq \|M_\phi\| \prod_{j\in J} \|\lambda(x_j)\| = \|\phi\|_{M^nA(G)}. \tag{253}$$

Because M is obtained from M_ϕ by fixing some of the variables, we have that M is continuous in each of its variables with respect to the σ-weak operator topology. We also know, from Theorem 2.15, that if $\phi \in M_{cb}^nA(G)$, then M is completely bounded with

$$\|M\|_{CB} \leq \|M_\phi\|_{CB} \prod_{j\in J} \|\lambda(x_j)\| = \|\phi\|_{M_{cb}^nA(G)}. \tag{254}$$

It remains to show that $M = M_\psi$. To this end let $x_j \in G$ for $j \in \{1, \dots, n\}\backslash J$ (and recall that $x_j = e$ for $j \in J$). We have that

$$\begin{aligned} M\left((\lambda(x_j))_{j\in\{1,\dots,n\}\backslash J}\right) &= M_\phi(\lambda(x_1), \dots, \lambda(x_n)) = \phi(x_1, \dots, x_n) \prod_{j=1}^n \lambda(x_j) \\ &= \psi\left((x_j)_{j\in\{1,\dots,n\}\backslash J}\right) \prod_{j\in\{1,\dots,n\}\backslash J} \lambda(x_j), \end{aligned} \tag{255}$$

where all tuples and products are considered to be in order of increasing j. This shows that $M = M_\psi$, completing the proof. ☐

The following corollary is a special case of Theorem 4.11 where we fix all but one variable.

Corollary 4.12. *Let G be a locally compact group and $n \in \mathbb{N}$. Let $\phi \in C_b(G^{\times n})$ and define for $j \in \{1, \dots, n\}$ the function $\phi_j \in C_b(G)$ by*

$$\phi_j(x_j) = \phi(x_1, \dots, x_n), \tag{256}$$

where $x_i = e$ for $i \neq j$. If $\phi \in M^nA(G)$, then $\phi_j \in MA(G)$ with

$$\|\phi_j\|_{MA(G)} \leq \|\phi\|_{M^nA(G)}. \tag{257}$$

If $\phi \in M_{cb}^nA(G)$, then $\phi_j \in M_{cb}A(G)$ with

$$\|\phi_j\|_{M_{cb}A(G)} \leq \|\phi\|_{M_{cb}^nA(G)}. \tag{258}$$

4.3 Schur multipliers

In this subsection we introduce a different class of multilinear maps known as Schur multipliers. We will see that every completely bounded Fourier multiplier has an associated completely bounded Schur multiplier [3]. This opens up the theory of Schur multipliers for the study of Fourier multipliers. In particular we will use a result about Schur multipliers that will help us to identify Fourier multipliers [21]. Our definition of Schur multipliers is taken from [3], but we will need to briefly define several prerequisite notions before presenting this definition.

Definition 4.11. Spectrum

Let A be a C^*-algebra with multiplicative unit $I \in A$. For $a \in A$, the spectrum $\sigma(a) \subseteq \mathbb{C}$ of a is defined as the set of all $\lambda \in \mathbb{C}$ such that $a - \lambda I$ has no muliplicative inverse in A.

Definition 4.12. Let A be a C^*-algebra with multiplicative unit. Then $a \in A$ is called positive if it is hermitian ($a^* = a$) and $\sigma(a) \subseteq [0, \infty)$.

Lemma 4.13. *Let A be a C^*-algebra with multiplicative unit. Given $a \in A$, there exists a unique positive element $|a| \in A$ such that $|a|^2 = a^*a$.*

We refer to section 2.2 in [23] for the proof of Lemma 4.13 and more details on positive elements of C^*-algebras. In particular the above definitions and result apply to $\mathcal{B}(H)$ for any Hilbert space H, because $\mathcal{B}(H)$ is a C^*-algebra with id_H as multiplicative identity.

Definition 4.13. Operator trace
Let H be a Hilbert space with orthonormal basis E. Let $a \in \mathcal{B}(H)$ be positive. Define

$$\mathrm{Tr}_E(a) := \sum_{e \in E} \langle ae, e \rangle . \tag{259}$$

The positivity of a ensures that all terms in the above sum are non-negative (see Theorem 2.3.5 in [23]), to ensure that the sum is well-defined (but not necessarily finite). Note that E is not necessarily finite or even countable, so the above sum is to be understood as the limit of the net consisting of $\sum_{e \in F} \langle ae, e \rangle$ for all finite $F \subseteq E$ and ordered by inclusion of the sets F (see also section 2.4 in [23]). $\mathrm{Tr}_E(a)$ can be shown to not depend on the orthonormal basis E, hence we just write $\mathrm{Tr}(a)$.

Definition 4.14. Schatten p-operators
Let H be a Hilbert space. For $a \in \mathcal{B}(H)$ and $1 \leq p < \infty$ we define

$$\|a\|_p = (\mathrm{Tr}(|a|^p))^{\frac{1}{p}} \in [0, \infty]. \tag{260}$$

Let $S_p(H)$ denote the set of $a \in \mathcal{B}(H)$ such that $\|a\|_p$ is finite. $S_p(H)$ is a Banach space when equipped with $\|\cdot\|_p$. Operators in $S_p(H)$ are called Schatten-p operators on H. Also let $S_\infty(H)$ denote the compact operators on H (i.e. operators that map any bounded set onto a set with compact closure) equipped with the operator norm. Then $S_\infty(H)$ is also a Banach space. For $1 \leq p \leq q \leq \infty$ we have that $S_p(H) \subseteq S_q(H)$ densely.

The following definition is taken from [3].

Definition 4.15. Schur multipliers
Let X be a measure space. There is a bijective linear isometry $L^2(X \times X) \to S_2(L^2(X))$ given by

$$A \mapsto \left(\xi \mapsto \left(t \mapsto \int_X A(t, s)\xi(s) ds \right) \right). \tag{261}$$

Using this bijection, we identify $L^2(X \times X)$ and $S_2(L^2(X))$ as Banach spaces. For $n \in \mathbb{N}$ and $\phi \in L^\infty(X^{\times(n+1)})$ we define the bounded multilinear map $S_\phi : S_2(L^2(X))^{\times n} \to S_2(L^2(X))$ by

$$S_\phi(A_1, \ldots, A_n)(t_0, t_n) = \int_{X^{\times(n-1)}} \phi(t_0, \ldots, t_n) \prod_{j=1}^{n} A_j(t_{j-1}, t_j) d(t_1, \ldots, t_{n-1}). \tag{262}$$

For $p_1, \ldots, p_n \in [1, \infty]$ and $p^{-1} = \sum_{j=1}^n p_j^{-1}$ we consider the restriction of S_ϕ where for all $i \in \{1, \ldots, n\}$, the i-th variable of S_ϕ is restricted to $S_2(L^2(X)) \cap S_{p_i}(L^2(X))$. If this restriction of S_ϕ can be extended to a bounded multilinear map $\bigtimes_{j=1}^n S_{p_j}(L^2(X)) \to S_p(L^2(X))$, which is also denoted by S_ϕ, then S_ϕ is called a $(p_1, \ldots, p_n)$-Schur multiplier and ϕ its symbol.

More on the topic of Schatten p-operators and/or Schur multipliers can be found in appendix D of [20], [10], chapter 6 of [24] and [26].

We will mainly be interested in Schur multipliers for which $p_j = \infty$ for all $j \in \{1, \ldots, n\}$ (and therefore also $p = \infty$) and going forward every Schur multiplier is an $(\infty, \ldots, \infty)$-Schur multiplier unless specified otherwise. Our interest in these Schur multipliers is due to the following transference result (Proposition 2.3 in [3]).

Theorem 4.14. *Let G be a locally compact group and $\phi \in C_b(G^{\times n})$. Then $S_{\tilde{\phi}} : S_\infty(L^2(G))^{\times n} \to S_\infty(L^2(G))$ is a completely bounded Schur multiplier if and only if $\phi \in M^n_{cb}A(G)$. In this case, we have that*

$$\left\| S_{\tilde{\phi}} \right\|_{CB} = \|\phi\|_{M^n_{cb}A(G)}. \tag{263}$$

For the definition of $\tilde{\phi}$ we refer to Definition 3.27. Theorem 4.14 allows us to use the theory of Schur multipliers when studying Fourier multipliers. In particular we will make use of the following result, which is a slightly less general version of Theorem 3.4 in [21].

Theorem 4.15. *Let X be a σ-finite measure space, $\psi \in L^\infty(X^{\times(n+1)})$ and $r > 0$. Then $S_\psi : S_\infty(L^2(X))^{\times n} \to S_\infty(L^2(X))$ is a completely bounded Schur multiplier with*

$$\|S_\psi\|_{CB} < r \tag{264}$$

if and only if there exist essentially bounded functions $a_0, a_n : X \to l^2(\mathbb{N})$ and $a_j : X \to \mathcal{B}(l^2(\mathbb{N}))$ for $j \in \{1, \ldots, n-1\}$ such that

$$\mathit{esssup}_{(x_0,\ldots,x_n)\in X^{\times(n+1)}} \prod_{j=0}^{n} \|a_j(x_j)\| < r \tag{265}$$

and for almost all $x_0, \ldots, x_n \in X$ we have that

$$\psi(x_0, \ldots, x_n) = \left\langle a_n(x_n), \left(\prod_{j=1}^{n-1} a_{n-j}(x_{n-j}) \right) (a_0(x_0)) \right\rangle. \tag{266}$$

Theorem 4.14 and Theorem 4.15 together imply the following corollary, which we will use in section 5 to show that certain functions are Fourier multipliers.

Corollary 4.16. *Let G be a locally compact group for which the Haar measure is σ-finite, $\phi \in C_b(G^{\times n})$ and $r > 0$. Then ϕ is a completely bounded Fourier multiplier with*

$$\|\phi\|_{M^n_{cb}A(G)} < r \tag{267}$$

if and only if there exist essentially bounded functions $a_0, a_n : G \to l^2(\mathbb{N})$ and $a_j : G \to \mathcal{B}(l^2(\mathbb{N}))$ for $j \in \{1, \ldots, n-1\}$ such that

$$\mathit{esssup}_{(x_0,\ldots,x_n)\in G^{\times(n+1)}} \prod_{j=0}^{n} \|a_j(x_j)\| < r \tag{268}$$

and for almost all $x_0, \ldots, x_n \in G$ we have that

$$\tilde{\phi}(x_0, \ldots, x_n) = \left\langle a_n(x_n), \left(\prod_{j=1}^{n-1} a_{n-j}(x_{n-j}) \right) (a_0(x_0)) \right\rangle. \tag{269}$$

Remark 4.7. In the statement of Theorem 4.15 and Corollary 4.16, $l^2(\mathbb{N})$ can be replaced by an arbitrary infinite-dimensional separable Hilbert space H. To see why, let H_1, H_2 be arbitrary infinite-dimensional separable Hilbert spaces. Let $(e_j)_{j\in\mathbb{N}}$ and $(f_j)_{j\in\mathbb{N}}$ be (ordered) orthonormal bases for H_1 and H_2 respectively. Then there exists a unique bounded linear map $U : H_1 \to H_2$ such that $U(e_j) = f_j$ for all $j \in \mathbb{N}$. U is a unitary and $U^{-1} = U^*$ sends f_j to e_j. Note that $\mathcal{B}(H_1) \to \mathcal{B}(H_2)$, $B \mapsto UBU^*$ is an isometric isomorphism of C^*-algebras and its inverse is given by $B \mapsto U^*BU$. Therefore, if $a_0, a_n : X \to H_1$ and $a_j : X \to \mathcal{B}(H_1)$ for $j \in \{1, \ldots, n-1\}$ are essentially bounded functions, then so are $b_0 : X \to H_2$ given by

$$b_0(x) = U(a_0(x)), \tag{270}$$

$b_n : X \to H_2$ given by

$$b_n(x) = U(a_n(x)) \tag{271}$$

and $b_j : X \to \mathcal{B}(H_2)$ given by

$$b_j(x) = U a_j(x) U^*. \tag{272}$$

for $j \in \{1, \ldots, n-1\}$. Here, for all $j \in \{0, \ldots, n\}$, b_j and a_j have the same essential supremum. Moreover, we have that

$$\begin{aligned}\left\langle b_n(x_n), \left(\prod_{j=1}^{n-1} b_{n-j}(x_{n-j})\right)(b_0(x_0))\right\rangle &= \left\langle U(a_n(x_n)), \left(\prod_{j=1}^{n-1} U a_{n-j}(x_{n-j}) U^*\right)(U(a_0(x_0)))\right\rangle \\ &= \left\langle U(a_n(x_n)), U\left(\prod_{j=1}^{n-1} a_{n-j}(x_{n-j})\right) U^*(U(a_0(x_0)))\right\rangle \\ &= \left\langle U(a_n(x_n)), U\left(\prod_{j=1}^{n-1} a_{n-j}(x_{n-j})\right)(a_0(x_0))\right\rangle \\ &= \left\langle a_n(x_n), \left(\prod_{j=1}^{n-1} a_{n-j}(x_{n-j})\right)(a_0(x_0))\right\rangle.\end{aligned} \tag{273}$$

If H is an arbitrary infinite-dimensional separable Hilbert space, this result with $H_1 = H$ and $H_2 = l^2(\mathbb{N})$ implies that $l^2(\mathbb{N})$ can be replaced with H in the "if" part of Theorem 4.15 and Corollary 4.16. Similarly, this result with $H_2 = H$ and $H_1 = l^2(\mathbb{N})$ implies that $l^2(\mathbb{N})$ can be replaced with H in the "only if" part of Theorem 4.15 and Corollary 4.16.

The following result due to Bozejko and Fendler [2] is similar to the $n = 1$ case of Corollary 4.16. A proof of this result can also be found in the appendix of [18] (Theorem 3.2) and in [22] (Theorem 5.4.6).

Theorem 4.17. *Let G be a locally compact group, $k \geq 0$ and $\phi \in C(G)$. Then $\phi \in M_{cb}A(G)$ with $\|\phi\|_{M_{cb}A(G)} \leq k$ if and only if there exists a Hilbert space H and bounded maps $\xi, \eta : G \to H$ such that*

$$\phi(y^{-1}x) = \langle \xi(x), \eta(y) \rangle \tag{274}$$

for all $x, y \in G$ and such that

$$\sup_{x \in G} \|\xi(x)\| \sup_{y \in G} \|\eta(y)\| \leq k. \tag{275}$$

The Hilbert space H and the functions ξ, η can be chosen such that ξ and η are continuous.

The following result will be useful when applying Corollary 4.16.

Lemma 4.18. *Let G be a second-countable locally compact group and H a Hilbert space. Let $f \in L^1(G)$. Let $a : G \to H$ and $b : G \to \mathcal{B}(H)$ be bounded, i.e.*

$$\|a\|_\infty := \sup_{x \in G} \|a(x)\| < \infty \tag{276}$$

and

$$\|b\|_\infty := \sup_{x \in G} \|b(x)\| < \infty. \tag{277}$$

Also assume that for all $g, h \in H$, the functions

$$x \mapsto \langle g, a(x) \rangle \tag{278}$$

and

$$x \mapsto \langle g, b(x)(h) \rangle \tag{279}$$

are measurable. Then we have that the following hold:

(1) There exist unique functions $f * a : G \to H$, $f * b : G \to \mathcal{B}(H)$ such that

$$\langle g, (f * a)(x) \rangle = (f * \langle g, a(\cdot) \rangle)(x) := \int_G f(t) \left\langle g, a(t^{-1}x) \right\rangle dt \tag{280}$$

and

$$\langle g, (f * b)(x)(h) \rangle = (f * \langle g, b(\cdot)(h) \rangle)(x) := \int_G f(t) \left\langle g, b(t^{-1}x)(h) \right\rangle dt \tag{281}$$

for all $g, h \in H$.

(2) The functions $f * a$ and $f * b$ are continuous (with respect to the norm topologies on H and $\mathcal{B}(H)$) and bounded with

$$\|f * a\|_\infty := \sup_{x \in G} \|(f * a)(x)\| \leq \|f\|_1 \|a\|_\infty \tag{282}$$

and

$$\|f * b\|_\infty := \sup_{x \in G} \|(f * b)(x)\| \leq \|f\|_1 \|b\|_\infty. \tag{283}$$

(3) The maps $a \mapsto f * a$ and $b \mapsto f * b$ are linear and the maps $f \mapsto f * a$ and $f \mapsto f * b$ are conjugate-linear. The following identities hold for all $g, h \in H$:

$$\langle (f * a)(x), g \rangle = (\overline{f} * \langle a(\cdot), g \rangle)(x) := \int_G \overline{f(t)} \left\langle a(t^{-1}x), g \right\rangle dt \tag{284}$$

and

$$\langle (f * b)(x)(h), g \rangle = (\overline{f} * \langle b(\cdot)(h), g \rangle)(x) := \int_G \overline{f(t)} \left\langle b(t^{-1}x)(h), g \right\rangle dt. \tag{285}$$

(4) If $f_1, \ldots, f_n \in L^1(G)$ and $b_1, \ldots, b_n : G \to \mathcal{B}(H)$ are bounded with $x \mapsto \langle g, b_j(x)(h) \rangle$ measurable for all $j \in \{1, \ldots, n\}$ and $g, h \in H$, then we have that

$$\begin{aligned} \left\langle g, \left(\prod_{j=1}^n (f_j * b_j)(x_j) \right)(h) \right\rangle &= \left((\otimes_{j=1}^n f_j) * \left((y_1, \ldots, y_n) \mapsto \left\langle g, \left(\prod_{j=1}^n b_j(y_j) \right)(h) \right\rangle \right) \right)(x_1, \ldots, x_n) \\ &:= \int_{G^{\times n}} \prod_{j=1}^n f_j(t_j) \left\langle g, \left(\prod_{j=1}^n b_j(t_j^{-1}x_j) \right)(h) \right\rangle d(t_1, \ldots, t_n) \end{aligned} \tag{286}$$

for all $x_1, \ldots, x_n \in G$ and $g, h \in H$.

(5) If $f_0, \ldots, f_n \in L^1(G)$, $a_0, a_n : G \to H$ and $a_1, \ldots, a_{n-1} : G \to \mathcal{B}(H)$ are bounded with $x \mapsto \langle g, a_0(x) \rangle$, $x \mapsto \langle g, a_n(x) \rangle$ and $x \mapsto \langle g, b_j(x)(h) \rangle$ measurable for all $j \in \{1, \ldots, n-1\}$, then we have that

$$\left\langle (\overline{f_n} * a_n)(x_n), \left(\prod_{j=1}^{n-1} (f_{n-j} * a_{n-j})(x_{n-j})\right) ((f_0 * a_0)(x_0)) \right\rangle$$

$$= \left((\otimes_{j=0}^{n} f_j) * \left((y_0, \dots, y_n) \mapsto \left\langle a_n(y_n), \left(\prod_{j=1}^{n-1} a_{n-j}(y_{n-j}) \right) (a_0(y_0)) \right\rangle \right) \right) (x_0, \dots, x_n) \tag{287}$$

$$:= \int_{G^{\times(n+1)}} \prod_{j=0}^{n} f_j(t_j) \left\langle a_n(t_n^{-1}x_n), \left(\prod_{j=1}^{n-1} a_{n-j}(t_{n-j}^{-1}x_{n-j}) \right) (a_0(t_0^{-1}x_0)) \right\rangle d(t_0, \dots, t_n)$$

for all $x_0, \dots, x_n \in G$.

Proof. For all $g, h \in H$ and $x \in G$ we have that

$$|\langle g, a(x)\rangle| \le \|g\|\|a(x)\| \le \|g\|\|a\|_\infty \tag{288}$$

and

$$|\langle g, b(x)(h)\rangle| \le \|g\|\|b(x)(h)\| \le \|g\|\|h\|\|b(x)\| \le \|g\|\|h\|\|b\|_\infty. \tag{289}$$

This shows that for fixed $g, h \in H$, $x \mapsto \langle g, a(x)\rangle$ and $x \mapsto \langle g, b(x)(h)\rangle$ are functions in $L^\infty(G)$, because they are measurable and bounded. Because $f \in L^1(G)$, it follows from Proposition 2.39 in [17] that the functions $f * \langle g, a(\cdot)\rangle$ and $f * \langle g, b(\cdot)(h)\rangle$ are continuous and bounded with

$$\|f * \langle g, a(\cdot)\rangle\|_\infty \le \|f\|_1 \|\langle g, a(\cdot)\rangle\|_\infty \le \|f\|_1 \|g\|\|a\|_\infty \tag{290}$$

and

$$\|f * \langle g, b(\cdot)(h)\rangle\|_\infty \le \|f\|_1 \|\langle g, b(\cdot)(h)\rangle\|_\infty \le \|f\|_1 \|g\|\|h\|\|b\|_\infty. \tag{291}$$

For fixed $f \in L^1(G)$ and $x \in G$ define $\alpha_{f,x} : H \to \mathbb{C}$ and $\beta_{f,x} : H \times H \to \mathbb{C}$ by

$$\alpha_{f,x}(g) = (f * \langle g, a(\cdot)\rangle)(x) = \int_G f(t) \left\langle g, a(t^{-1}x) \right\rangle dt \tag{292}$$

and

$$\beta_{f,x}(g, h) = (f * \langle g, b(\cdot)(h)\rangle)(x) = \int_G f(t) \left\langle g, b(t^{-1}x)(h) \right\rangle dt. \tag{293}$$

From these definitions and the boundedness of $f * \langle g, a(\cdot)\rangle$ and $f * \langle g, b(\cdot)(h)\rangle$ we see that $\alpha_{f,x}$ is a bounded linear functional with

$$\|\alpha_{f,x}\| \le \|f\|_1 \|a\|_\infty \tag{294}$$

and $\beta_{f,x}$ is a bounded sesquilinear form (i.e. it is linear in the first component and conjugate-linear in the second) with

$$\|\beta_{f,x}\| \le \|f\|_1 \|b\|_\infty. \tag{295}$$

From the definitions of $\alpha_{f,x}$ and $\beta_{f,x}$, we can see that $\alpha_{f,x}$ depends linearly on f and conjugate-linearly on a. Similarly we have that $\beta_{f,x}$ depends linearly on f and conjugate-linearly on b. It follows from the Riesz representation theorem that there exists a unique element $(f * a)(x) \in H$ such that

$$\langle g, (f * a)(x)\rangle = \alpha_{f,x}(g) = (f * \langle g, a(\cdot)\rangle)(x) \tag{296}$$

with

$$\|(f * a)(x)\| = \|\alpha_{f,x}\| \leq \|f\|_1 \|a\|_\infty. \tag{297}$$

It follows from Theorem 2.3.6. in [23] or Theorem II.2.2. in [5] (as a consequence of the Riesz representation theorem) that there exists a unique element $(f * b)(x) \in \mathcal{B}(H)$ such that

$$\langle g, (f * b)(x)(h)\rangle = \beta_{f,x}(g, h) = (f * \langle g, b(\cdot)(h)\rangle)(x) \tag{298}$$

with

$$\|(f * b)(x)\| = \|\beta_{f,x}\| \leq \|f\|_1 \|b\|_\infty. \tag{299}$$

This shows the existence and uniqueness of $f * a$ and $f * b$ as stated in part (1). We note that the correspondence between a bounded linear functional on a Hilbert space and the associated element in the Hilbert space in the Riesz representation theorem is conjugate-linear. This follows directly from the sesquilinearity of the inner product. Similarly, the correspondence between an element $A \in \mathcal{B}(H)$ and the corresponding sesquilinear form $(g, h) \mapsto \langle g, Ah\rangle$ is also conjugate-linear (it would be linear if we would instead consider the sesquilinear form $(g, h) \mapsto \langle Ag, h\rangle$). This means that the correspondence between $\alpha_{f,x}$ and $(f * a)(x)$ is conjugate-linear and the correspondence between $\beta_{f,x}$ and $(f * b)(x)$ is also conjugate-linear. It follows that $f * a$ depends linearly on a and conjugate-linearly on f and $f * b$ depends linearly on b and conjugate-linearly on f. We note that we have already proved the boundedness of $f * a$ and $f * b$ as stated in part (2) and the statements about (conjugate-)linearity in part (3).

We first prove continuity of $f * a$ and $f * b$ under the assumption that $f \in C_c(G)$ and then use an approximation argument. This approach is similar to the approach used in Proposition 2.39 in [17] to prove that $f * g$ is continuous for $f \in L^1(G)$ and $g \in L^\infty(G)$. So, we assume $f \in C_c(G)$. We know from Proposition 2.6 in [17] that such an f is left uniformly continuous. In other words for any $\epsilon > 0$, there exists an open neighbourhood $V_\epsilon \subseteq G$ of e such that

$$\sup_{x \in G} \left|f(y^{-1}x) - f(x)\right| < \epsilon \tag{300}$$

for all $y \in V_\epsilon$. Given $x \in G$ and $\epsilon > 0$, we have that $V_\epsilon^{-1}x$ is an open neighbourhood of x. For any $y \in V_\epsilon^{-1}x$ we have that $xy^{-1} \in V_\epsilon$, hence

$$\sup_{t \in G} |f(yt) - f(xt)| = \sup_{t \in G} \left|f(yx^{-1}t) - f(t)\right| < \epsilon. \tag{301}$$

Let $K \subseteq G$ be compact and such that $f(t) = 0$ for all $t \in G \backslash K$. It follows that for all $g \in H$ and $y \in V_\epsilon^{-1}x$, we have

$$\begin{aligned}
|\langle g, (f * a)(y) - (f * a)(x)\rangle| &= |\langle g, (f * a)(y)\rangle - \langle g, (f * a)(x)\rangle| \\
&= \left|\int_G f(t) \left\langle g, a(t^{-1}y)\right\rangle dt - \int_G f(t) \left\langle g, a(t^{-1}x)\right\rangle dt\right| \\
&= \left|\int_G f(yt) \left\langle g, a(t^{-1})\right\rangle dt - \int_G f(xt) \left\langle g, a(t^{-1})\right\rangle dt\right| \\
&= \left|\int_G (f(yt) - f(xt)) \left\langle g, a(t^{-1})\right\rangle dt\right| \leq \int_G |f(yt) - f(xt)| \left|\left\langle g, a(t^{-1})\right\rangle\right| dt \\
&\leq \int_{x^{-1}K \cup y^{-1}K} |f(yt) - f(xt)| \left|\left\langle g, a(t^{-1})\right\rangle\right| dt \leq \int_{x^{-1}K \cup y^{-1}K} \epsilon \|g\| \left\|a(t^{-1})\right\| dt \\
&\leq \int_{x^{-1}K \cup y^{-1}K} \epsilon \|g\| \|a\|_\infty dt = \epsilon \|g\| \|a\|_\infty \mu(x^{-1}K \cup y^{-1}K) \\
&\leq \epsilon \|g\| \|a\|_\infty (\mu(x^{-1}K) + \mu(y^{-1}K)) = 2\epsilon \|g\| \|a\|_\infty \mu(K).
\end{aligned} \tag{302}$$

Choosing $g = (f * a)(y) - (f * a)(x)$, we see that

$$\|(f*a)(y)-(f*a)(x)\|^2 \le 2\epsilon\|(f*a)(y)-(f*a)(x)\|\|a\|_\infty\mu(K), \tag{303}$$

hence

$$\|(f*a)(y)-(f*a)(x)\| \le 2\epsilon\|a\|_\infty\mu(K). \tag{304}$$

This shows that $f*a$ is continuous in x for all $x\in G$, hence $f*a$ is continuous when $f\in C_c(G)$. Similarly, for all $g,h\in H$ and $y\in V_\epsilon^{-1}x$ we have that

$$\begin{aligned}
|\langle g,((f*b)(y)-(f*b)(x))(h)\rangle| &= |\langle g,(f*b)(y)(h)\rangle-\langle g,(f*b)(x)(h)\rangle| \\
&= \left|\int_G f(t)\left\langle g,b(t^{-1}y)(h)\right\rangle dt-\int_G f(t)\left\langle g,b(t^{-1}x)(h)\right\rangle dt\right| \\
&= \left|\int_G f(yt)\left\langle g,b(t^{-1})(h)\right\rangle dt-\int_G f(xt)\left\langle g,b(t^{-1})(h)\right\rangle dt\right| \\
&= \left|\int_G (f(yt)-f(xt))\left\langle g,b(t^{-1})(h)\right\rangle dt\right| \\
&\le \int_G |f(yt)-f(xt)|\left|\left\langle g,b(t^{-1})(h)\right\rangle\right| dt \\
&\le \int_{x^{-1}K\cup y^{-1}K} |f(yt)-f(xt)|\left|\left\langle g,b(t^{-1})(h)\right\rangle\right| dt \\
&\le \int_{x^{-1}K\cup y^{-1}K} \epsilon\|g\|\|h\|\left\|b(t^{-1})\right\| dt \\
&\le \int_{x^{-1}K\cup y^{-1}K} \epsilon\|g\|\|h\|\|b\|_\infty dt = \epsilon\|g\|\|h\|\|b\|_\infty\mu(x^{-1}K\cup y^{-1}K) \\
&\le \epsilon\|g\|\|h\|\|b\|_\infty(\mu(x^{-1}K)+\mu(y^{-1}K)) = 2\epsilon\|g\|\|h\|\|b\|_\infty\mu(K).
\end{aligned} \tag{305}$$

Choosing $g=((f*b)(y)-(f*b)(x))(h)$, we see that

$$\|((f*b)(y)-(f*b)(x))(h)\|^2 \le 2\epsilon\|((f*b)(y)-(f*b)(x))(h)\|\|h\|\|b\|\mu(K), \tag{306}$$

hence

$$\|((f*b)(y)-(f*b)(x))(h)\| \le 2\epsilon\|h\|\|b\|\mu(K). \tag{307}$$

This holds for all $h\in H$ and therefore

$$\|(f*b)(y)-(f*b)(x)\| \le 2\epsilon\|b\|\mu(K). \tag{308}$$

This shows that $f*b$ is continuous in x for all $x\in G$, hence $f*b$ is continuous when $f\in C_c(G)$. To prove continuity of $f*a$ and $f*b$ for general $f\in L^1(G)$ we recall that we have already proved that $f\mapsto f*a$ and $f\mapsto f*b$ are conjugate-linear and that

$$\|f*a\|_\infty \le \|f\|_1\|a\|_\infty \tag{309}$$

and

$$\|f*b\|_\infty \le \|f\|_1\|b\|_\infty. \tag{310}$$

Let $\epsilon>0$ and $f\in L^1(G)$. By density of $C_c(G)$ in $L^1(G)$ (Theorem 3.12), there is an $f_0\in C_c(G)$ such that $\|f_0-f\|_1<\epsilon$. Then we have that

$$\|f_0*a-f*a\|_\infty = \|(f_0-f)*a\|_\infty \le \|f_0-f\|_1\|a\|_\infty \le \|a\|_\infty\epsilon \tag{311}$$

and

$$\|f_0 * b - f * b\|_\infty = \|(f_0 - f) * b\|_\infty \leq \|f_0 - f\|_1 \|b\|_\infty \leq \|b\|_\infty \epsilon. \tag{312}$$

Because $f_0 \in C_c(G)$, given $x \in G$, we can choose an open neighbourhood $U_{x,\epsilon}$ of x such that

$$\|(f_0 * a)(y) - (f_0 * a)(x)\| < \epsilon \tag{313}$$

and

$$\|(f_0 * b)(y) - (f_0 * b)(x)\| < \epsilon \tag{314}$$

for all $y \in U_{x,\epsilon}$. It follows that for all $y \in U_{x,\epsilon}$:

$$\begin{aligned} &\|(f * a)(y) - (f * a)(x)\| \\ &\leq \|(f * a)(y) - (f_0 * a)(y)\| + \|(f_0 * a)(y) - (f_0 * a)(x)\| + \|(f_0 * a)(x) - (f * a)(x)\| < (1 + 2\|a\|_\infty)\epsilon \end{aligned} \tag{315}$$

and

$$\begin{aligned} &\|(f * b)(y) - (f * b)(x)\| \\ &\leq \|(f * b)(y) - (f_0 * b)(y)\| + \|(f_0 * b)(y) - (f_0 * b)(x)\| + \|(f_0 * b)(x) - (f * b)(x)\| < (1 + 2\|b\|_\infty)\epsilon. \end{aligned} \tag{316}$$

This shows that $f * a$ and $f * b$ are continuous for all $f \in L^1(G)$, which finishes the proof of part (2). For part (3) note that

$$\langle (f * a)(x), g \rangle = \overline{\langle g, (f * a)(x) \rangle} = \int_G \overline{f(t) \langle g, a(t^{-1}x) \rangle} dt = \int_G \overline{f(t)} \left\langle a(t^{-1}x), g \right\rangle dt \tag{317}$$

and

$$\langle (f * b)(x)(h), g \rangle = \overline{\langle g, (f * b)(x)(h) \rangle} = \int_G \overline{f(t) \langle g, b(t^{-1}x)(h) \rangle} dt = \int_G \overline{f(t)} \left\langle b(t^{-1}x)(h), g \right\rangle dt. \tag{318}$$

Note that in the statement of part (4), the convolution is well-defined because it is the convolution of an $L^1(G)$ function with an $L^\infty(G)$ function. We prove the identity in part (4) using induction on n, where we note that the case $n = 1$ holds by part (1). Suppose the identity holds for some $n \in \mathbb{N}$. Let $f_1, \ldots, f_{n+1}$ and $b_1, \ldots, b_{n+1}$ as in the statement of part (4). Then it follows that

$$\begin{aligned} &\left\langle g, \left(\prod_{j=1}^{n+1} (f_j * b_j)(x_j) \right) (h) \right\rangle = \left\langle (f_1 * b_1)(x_1)^*(g), \left(\prod_{j=2}^{n+1} (f_j * b_j)(x_j) \right) (h) \right\rangle \\ &= \int_{G^{\times n}} \prod_{j=2}^{n+1} f_j(t_j) \left\langle (f_1 * b_1)(x_1)^*(g), \left(\prod_{j=2}^{n+1} b_j(t_j^{-1} x_j) \right) (h) \right\rangle d(t_2, \ldots, t_{n+1}) \\ &= \int_{G^{\times n}} \prod_{j=2}^{n+1} f_j(t_j) \left\langle g, (f_1 * b_1)(x_1) \left(\left(\prod_{j=2}^{n+1} b_j(t_j^{-1} x_j) \right) (h) \right) \right\rangle d(t_2, \ldots, t_{n+1}) \\ &= \int_{G^{\times n}} \prod_{j=2}^{n+1} f_j(t_j) \int_G f_1(t_1) \left\langle g, b_1(t_1^{-1} x_1) \left(\left(\prod_{j=2}^{n+1} b_j(t_j^{-1} x_j) \right) (h) \right) \right\rangle dt_1 d(t_2, \ldots, t_{n+1}) \\ &= \int_{G^{\times (n+1)}} \prod_{j=1}^{n+1} f_j(t_j) \left\langle g, \left(\prod_{j=1}^{n+1} b_j(t_j^{-1} x_j) \right) (h) \right\rangle d(t_1, \ldots, t_n). \end{aligned} \tag{319}$$

Here we used Fubini's theorem in the last step because the integrand is integrable as a function on $G^{\times(n+1)}$ (it is bounded in absolute value by $(\prod_{j=1}^{n+1} |f_j(t_j)|)\|g\|\|h\| \prod_{j=1}^{n+1} \|b_j\|_\infty)$. This proves that the identity in part (4) holds for $n+1$, hence it holds for all $n \in \mathbb{N}$.

Now let $f_0, \dots, f_n$ and $a_0, \dots, a_n$ be as in the statement of part (5). Using part (4), we have that

$$\begin{aligned}
&\left\langle (\overline{f_n} * a_n)(x_n), \left(\prod_{j=1}^{n-1} (f_{n-j} * a_{n-j})(x_{n-j})\right) ((f_0 * a_0)(x_0)) \right\rangle \\
&= \int_G f_n(t_n) \left\langle a_n(t_n^{-1}x_n), \left(\prod_{j=1}^{n-1} (f_{n-j} * a_{n-j})(x_{n-j})\right) ((f_0 * a_0)(x_0)) \right\rangle dt_n \\
&= \int_{G^{\times n}} \prod_{j=1}^{n} f_j(t_j) \left\langle a_n(t_n^{-1}x_n), \left(\prod_{j=1}^{n-1} a_{n-j}(t_{n-j}^{-1}x_{n-j})\right) ((f_0 * a_0)(x_0)) \right\rangle d(t_1, \dots, t_n) \\
&= \int_{G^{\times n}} \prod_{j=1}^{n} f_j(t_j) \left\langle \left(\prod_{j=1}^{n-1} a_{n-j}(t_{n-j}^{-1}x_{n-j})\right)^* (a_n(t_n^{-1}x_n)), (f_0 * a_0)(x_0) \right\rangle d(t_1, \dots, t_n) \\
&= \int_{G^{\times(n+1)}} \prod_{j=0}^{n} f_j(t_j) \left\langle \left(\prod_{j=1}^{n-1} a_{n-j}(t_{n-j}^{-1}x_{n-j})\right)^* (a_n(t_n^{-1}x_n)), a_0(t_0^{-1}x_0) \right\rangle d(t_0, \dots, t_n) \\
&= \int_{G^{\times(n+1)}} \prod_{j=0}^{n} f_j(t_j) \left\langle a_n(t_n^{-1}x_n), \left(\prod_{j=1}^{n-1} a_{n-j}(t_{n-j}^{-1}x_{n-j})\right) (a_0(t_0^{-1}x_0)) \right\rangle d(t_0, \dots, t_n).
\end{aligned} \tag{320}$$

We used Fubini's theorem several times throughout this calculation. This is justified because the integrand in the final expression is an integrable function on $G^{\times(n+1)}$ (it is bounded in absolute value by $(\prod_{j=0}^{n} |f_j(t_j)|) \prod_{j=0}^{n} \|a_j\|_\infty)$. This proves part (5). $\square$

Remark 4.8. In the setting of Lemma 4.18, assume that $x \mapsto \|a(x)\|$ and $x \mapsto \|b(x)\|$ are measurable. Note that from the definition of $f * a$ and $f * b$ it can be seen that all the results are still valid (with the same proof) if we assume that a and b are essentially bounded instead of bounded, i.e. we replace the supremum in the definitions of $\|\cdot\|_\infty$ by the essential supremum. Moreover we have (by part (2) of Lemma 4.18) that $f * a$ and $f * b$ do not change if we replace a and b by functions that are equal to a and b almost everywhere. Also note that the assumption that G is second-countable is only used (by applying Fubini's theorem) when proving parts (4) and (5) of Lemma 4.18. So parts (1)-(3) are valid without the assumption of second-countability. In fact, a careful observation of the proof reveals that if we replace the integrals in parts (4) and (5) by repeated integrals over the variables t_j in order of increasing j, then there is no need to apply Fubini's theorem and the assumption of second-countability can be dropped entirely.

4.4 Fourier algebra

In this subsection we introduce the Fourier algebra $A(G)$ of a locally compact group G. We will see that $VN(G)$ can be identified with the dual space $A(G)^*$ of $A(G)$. From this duality it follows that the symbols of linear Fourier multipliers are exactly the multipliers of $A(G)$. The Fourier algebra will also play an important role in the definition of weak amenability in subsection 4.5. A detailed treatment of the Fourier algebra can be found in [22], while [29] gives a brief and accessible overview.

Definition 4.16. Fourier algebra

Let G be a locally compact group. The Fourier algebra $A(G)$ of G is the set of functions of the form

$$x \mapsto \langle \lambda(x) f, g \rangle = (\overline{g} * f((\cdot)^{-1}))(x), \tag{321}$$

where $f, g \in L^2(G)$. Equipped with pointwise operations and the norm

$$\|a\|_{A(G)} := \inf\{\|f\|_2\|g\|_2 : a(x) = \langle \lambda(x)f, g\rangle\}, \tag{322}$$

$A(G)$ is a Banach algebra.

Remark 4.9. The above characterization of the Fourier algebra can be found in section 3.3 in [29]. It is not at all obvious that the Fourier algebra as defined in Definition 4.16 is indeed a Banach algebra or even a vector space. In fact [29] and [22] define the Fourier algebra in a different way, where it is easier to see that $A(G)$ is a Banach algebra. This definition, however, requires more setup so we have decided to instead define $A(G)$ as in Definition 4.16. We refer the interested reader to either sections 3.2 and 3.3 in [29] for a summarized introduction to the Fourier algebra or sections 2.1 through 2.4 in [22] for a more detailed treatment. In particular Theorem 2.1.11, Proposition 2.3.3 and Theorem 2.4.3 in [22] establish that $A(G)$ is a Banach algebra and can be characterized as in Definition 4.16. Another question that may come to mind is if the norm $\|\cdot\|_{A(G)}$ depends on the chosen Haar measure. Indeed, the above definition of $\|\cdot\|_{A(G)}$ involves the $\|\cdot\|_2$ on $L^2(G)$, which certainly does depend on the chosen Haar measure. Nevertheless, $\|\cdot\|_{A(G)}$ is independent of the chosen Haar measure and this can be seen as follows. Let μ, ν be (left) Haar measures on G. Then $\nu = c\mu$ for some $c > 0$ by Theorem 3.2. Let $a \in A(G)$ and $f, g \in L^2(G)$ be such that

$$a(x) = \langle \lambda(x)f, g\rangle_{L^2(G,\mu)}. \tag{323}$$

Then we have that

$$a(x) = \langle \lambda(x)f, g\rangle_{L^2(G,\mu)} = \int_G f(x^{-1}y)\overline{g(y)}d\mu(y) = \int_G f(x^{-1}y)\overline{g(y)}\frac{1}{c}d\nu(y) = \left\langle \lambda(x)f, \frac{1}{c}g\right\rangle_{L^2(G,\nu)}. \tag{324}$$

It follows that

$$\begin{aligned}\|a\|_{A(G;\nu)} \leq \|f\|_{L^2(G,\nu)}\left\|\frac{1}{c}g\right\|_{L^2(G,\nu)} &= \frac{1}{c}\left(\int_G |f(y)|^2 d\nu(y)\right)^{\frac{1}{2}}\left(\int_G |g(y)|^2 d\nu(y)\right)^{\frac{1}{2}} \\ &= \frac{1}{c}\left(\int_G |f(y)|^2 c d\mu(y)\right)^{\frac{1}{2}}\left(\int_G |g(y)|^2 c d\mu(y)\right)^{\frac{1}{2}} = \|f\|_{L^2(G,\mu)}\|g\|_{L^2(G,\mu)}.\end{aligned} \tag{325}$$

By taking the infimum over all $f, g \in L^2(G)$ such that $a(x) = \langle \lambda(x)f, g\rangle_{L^2(G,\mu)}$, it follows that $\|a\|_{A(G;\nu)} \leq \|a\|_{A(G;\mu)}$. Since μ, ν were arbitrary Haar measures on G, it follows that $\|\cdot\|_{A(G)}$ does not depend on the chosen Haar measure.

Lemma 4.19. *Let G be a locally compact group. Then $A(G) \subseteq C_0(G)$ and this inclusion is contractive and dense. We have $\{x \mapsto \langle \lambda(x)f, g\rangle : f, g \in C_c(G)\} \subseteq A(G) \cap C_c(G)$ and both of these subsets are dense in $A(G)$ (in $\|\cdot\|_{A(G)}$).*

Proof. Let $a \in A(G)$ with $f, g \in L^2(G)$ such that $a(x) = \langle \lambda(x)f, g\rangle$. It follows from Lemma 3.7 and continuity of the inner product that a is a continuous function.

$$|a(x)| = |\langle \lambda(x)f, g\rangle| \leq \|\lambda(x)f\|_2\|g\|_2 = \|f\|_2\|g\|_2 \tag{326}$$

implies that $\|a\|_\infty \leq \|a\|_{A(G)}$. So $A(G) \subseteq C_b(G)$ and this inclusion is contractive. Let $a \in A(G)$ and $f, g \in L^2(G)$ such that $a(x) = \langle \lambda(x)f, g\rangle$. Let $\eta > 0$ and $\epsilon > 0$ such that $\epsilon < 1$ and $\epsilon < \frac{\eta}{\|f\|_2 + \|g\|_2 + 1}$. $C_c(G)$ is dense in $L^2(G)$ (Theorem 3.12), so let $f_0, g_0 \in C_c(G)$ such that $\|f - f_0\|_2 < \epsilon$ and $\|g - g_0\|_2 < \epsilon$. Let $a_0 \in A(G)$ be given by $a_0(x) = \langle \lambda(x)f_0, g_0\rangle$. Then we have that

$$\begin{aligned}
\|a-a_0\|_{A(G)} &= \|x \mapsto \langle \lambda(x)f,g\rangle - \langle \lambda(x)f_0,g_0\rangle\|_{A(G)} \\
&= \|x \mapsto \langle \lambda(x)f,g\rangle - \langle \lambda(x)f_0,g\rangle + \langle \lambda(x)f_0,g\rangle - \langle \lambda(x)f_0,g_0\rangle\|_{A(G)} \\
&= \|x \mapsto \langle \lambda(x)(f-f_0),g\rangle + \langle \lambda(x)f_0,g-g_0\rangle\|_{A(G)} \\
&\leq \|x \mapsto \langle \lambda(x)(f-f_0),g\rangle\|_{A(G)} + \|x \mapsto \langle \lambda(x)f_0,g-g_0\rangle\|_{A(G)} \\
&\leq \|f-f_0\|_2\|g\|_2 + \|f_0\|_2\|g-g_0\|_2 \\
&\leq \epsilon\|g\|_2 + (\|f\|_2+\epsilon)\epsilon = \epsilon(\|f\|_2+\|g\|_2+\epsilon) < \epsilon(\|f\|_2+\|g\|_2+1) \leq \eta.
\end{aligned} \tag{327}$$

This shows that $\{x \mapsto \langle \lambda(x)f,g\rangle : f,g \in C_c(G)\}$ is dense in $A(G)$. Functions in $\{x \mapsto \langle \lambda(x)f,g\rangle : f,g \in C_c(G)\}$ are compactly supported. Indeed, let $f,g \in C_c(G)$ with compact sets K_f, K_g such that $f(x) = 0$ for $x \notin K_f$ and $g(x) = 0$ for $x \notin K_g$. We have

$$\langle \lambda(x)f,g\rangle = \int_G f(x^{-1}y)\overline{g(y)}dy. \tag{328}$$

Note that the integrand in the above integral equals 0 unless $y \in K_g$ and $x^{-1}y \in K_f$. If $y \in K_g$ and $x^{-1}y \in K_f$, then we have that

$$x \in yK_f^{-1} \subseteq K_gK_f^{-1}. \tag{329}$$

It follows that unless $x \in K_gK_f^{-1}$, the integrand equals 0 for all $y \in G$. Hence $\langle \lambda(x)f,g\rangle = 0$ unless $x \in K_gK_f^{-1}$. $K_gK_f^{-1}$ is compact. So, we have that $x \mapsto \langle \lambda(x)f,g\rangle$ is in $C_c(G)$. This means that $\{x \mapsto \langle \lambda(x)f,g\rangle : f,g \in C_c(G)\} \subseteq A(G) \cap C_c(G)$ and both of these subsets are dense in $A(G)$. So, we have that $A(G) \subseteq C_b(G)$ is a contractive inclusion and every element of $A(G)$ can be approximated in $\|\cdot\|_{A(G)}$ by functions in $A(G) \cap C_c(G)$. It follows that every element of $A(G)$ can be uniformly approximated by $C_c(G)$ functions. Since $C_0(G)$ is the uniform closure of $C_c(G)$, it follows that $A(G) \subseteq C_0(G)$. It remains to show that $A(G)$ is uniformly dense in $C_0(G)$. We note that $A(G) \cap C_c(G)$ is a self-adjoint subalgebra of $C_0(G)$ and by Proposition 2.3.2 in [22] it separates the points of G. The uniform density of $A(G)$ in $C_0(G)$ follows from an application of the Stone-Weierstrass theorem (see also Corollary 2.3.5 in [22]). ☐

For the proof of the following theorem we refer to Theorem 2.3.9 in [22].

Theorem 4.20. *Let G be a locally compact group. For every bounded linear functional $\psi \in A(G)^*$ there exists a unique $T_\psi \in VN(G)$ such that*

$$\langle T_\psi(f), g\rangle = \psi(x \mapsto \langle \lambda(x)f,g\rangle) \tag{330}$$

for all $f,g \in L^2(G)$. $\psi \mapsto T_\psi$ is a bijective linear isometry $A(G)^* \to VN(G)$. $\psi \mapsto T_\psi$ is a homeomorphism with respect to the weak-$*$ topology on $A(G)^*$ (i.e. the locally convex topology defined by the family of seminorms $\psi \mapsto |\psi(a)|$ for $a \in A(G)$) and the σ-weak operator topology on $VN(G)$. The inverse of $\psi \mapsto T_\psi$ sends $\lambda(x)$ ($x \in G$) to the linear functional

$$a \mapsto a(x) \tag{331}$$

and sends $\lambda'(f)$ ($f \in L^1(G)$) to the linear functional

$$a \mapsto \int_G f(x)a(x)dx. \tag{332}$$

Definition 4.17. Fourier algebra multiplier

Let G be a locally compact group. We say that a function $\phi \in C_b(G)$ is a multiplier of the Fourier algebra $A(G)$ if $m_\phi : A(G) \to A(G)$ given by

$$m_\phi(f) = \phi f \tag{333}$$

is a well-defined bounded linear map.

Remark 4.10. If $\phi : G \to \mathbb{C}$ is any function such that $\phi f \in A(G)$ for all $f \in A(G)$, then ϕ is necessarily continuous and bounded. Moreover the map m_ϕ will automatically be bounded (this follows from the closed graph theorem because convergence in $\|\cdot\|_{A(G)}$ implies pointwise convergence). The multipliers of the Fourier algebra form a Banach algebra under pointwise operations and the norm $\phi \mapsto \|m_\phi\|$. $A(G)$ is contained in this Banach algebra and this inclusion is contractive. For these results and more on Fourier algebra multipliers, we refer to section 3.4 in [29] and section 5.1 in [22].

The multipliers of the Fourier algebra $A(G)$ turn out to coincide with the symbols of the linear Fourier multipliers, as stated in the next theorem, for the proof of which we refer to Proposition 5.1.2 in [22].

Theorem 4.21. *Let G be locally compact group and $\phi \in C_b(G)$. Then $\phi \in MA(G)$ if and only if ϕ is a multiplier of the Fourier algebra $A(G)$. In this case we have that M_ϕ is the adjoint of m_ϕ if we identify $VN(G)$ and $A(G)^*$ through the linear isometry given in Theorem 4.20. Moreover, we have that*

$$\|\phi\|_{MA(G)} = \|M_\phi\| = \|m_\phi\|. \tag{334}$$

This theorem enables us to prove the following result.

Corollary 4.22. *Let G be a locally compact group. $MA(G)$ and $M_{cb}A(G)$ are Banach algebras.*

Proof. By Theorem 4.21, we know that $MA(G)$ coincides with the Banach algebra of multipliers of $A(G)$. So $MA(G)$ is a Banach algebra. We already know (Theorem 4.8) that $M_{cb}A(G)$ is a normed algebra. It remains to prove that $M_{cb}A(G)$ is complete. To prove completeness let $(\phi_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in $M_{cb}A(G)$. Because $M_{cb}A(G)$ is contractively included in $MA(G)$ (Theorem 4.8), $(\phi_m)_{m\in\mathbb{N}}$ is also a Cauchy sequence in $MA(G)$. By completeness of $MA(G)$, it follows that ϕ_m converges to some $\phi \in MA(G)$ in $\|\cdot\|_{MA(G)}$. It remains to prove that $\phi \in M_{cb}A(G)$ and that this convergence is also in $\|\cdot\|_{M_{cb}A(G)}$. Note that the convergence of ϕ_m to ϕ in $MA(G)$ implies that M_{ϕ_m} converges to M_ϕ in $\mathcal{L}(VN(G); VN(G))$. $(M_{\phi_m})_{m\in\mathbb{N}}$ is also a Cauchy sequence in $CB(VN(G); VN(G))$. It follows from completeness of $CB(VN(G); VN(G))$ (Theorem 2.16) that M_{ϕ_m} converges to some $M \in CB(VN(G); VN(G))$ in $\|\cdot\|_{CB}$. Then M_{ϕ_m} must also converge to M in $\|\cdot\|$. By uniqueness of limits we must have that $M_\phi = M \in CB(VN(G); VN(G))$ and M_{ϕ_m} converges to M_ϕ in $\|\cdot\|_{CB}$. It follows that $\phi \in M_{cb}A(G)$ and ϕ_m converges to ϕ in $\|\cdot\|_{M_{cb}A(G)}$. This shows that $M_{cb}A(G)$ is complete. □

Remark 4.11. We know that $A(G) \subseteq MA(G)$ and $M_{cb}A(G) \subseteq MA(G)$ and that both of these inclusions are contractive. It can be shown that $A(G) \subseteq M_{cb}A(G)$ and that this inclusion is also contractive (see section 3.4 in [29] or section 5.4 (and more specifically Corollary 5.4.11) in [22]). The proof of the contractive inclusion $A(G) \subseteq M_{cb}A(G)$ involves the Fourier-Stieltjes algebra $B(G) \subseteq C_b(G)$ which consists of all functions f of the form

$$f(x) = \langle \pi(x)\xi, \eta \rangle, \tag{335}$$

where π is a continuous unitary representation of G on a Hilbert space H and $\xi, \eta \in H$. $B(G)$ is a Banach algebra when equipped with pointwise operations and the norm $\|f\|_{B(G)} = \inf \|\xi\|\|\eta\|$, where the infimum is taken over all continuous unitary representations π of G on H and all $\xi, \eta \in H$ such that the above expression for f holds. See section 3.2 in [29] or section 2.1 in [22] for these results and more about $B(G)$. $A(G)$ can be equivalently defined as a subalgebra of $B(G)$ (with the same norm) and $A(G)$ is an ideal in $B(G)$ (see section 3.3 in [29] or section 2.3 in [22]). It follows that $A(G) \subseteq B(G)$ isometrically and $B(G) \subseteq MA(G)$ contractively. Corollary 5.4.11 in [22] shows that also $B(G) \subseteq M_{cb}A(G)$ contractively. In summary we have that

$$A(G) \subseteq B(G) \subseteq M_{cb}A(G) \subseteq MA(G) \subseteq C_b(G) \tag{336}$$

with all inclusions being contractive (and the first even isometric).

Remark 4.12. There is another approach to defining multilinear Fourier multipliers that is similar to the characterization of linear Fourier multipliers as multipliers of the Fourier algebra. We briefly explain this approach, which is due to Todorov and Turowska [28]. For $n \in \mathbb{N}$ and G a σ-compact locally compact group, they define the multidimensional Fourier algebra $A^n(G)$ as the space of functions $f \in L^\infty(G^{\times n})$ such that there exists a (unique) completely bounded multilinear map $\Phi : VN(G)^{\times n} \to \mathbb{C}$ that is σ-weakly continuous in each variable and satisfies

$$\Phi(\lambda(x_1), \ldots, \lambda(x_n)) = f(x_1, \ldots, x_n) \tag{337}$$

for all $x_1, \ldots, x_n \in G$. $\|f\|_{A^n(G)} = \|\Phi\|_{CB}$ defines a norm on $A^n(G)$. A function $\phi \in L^\infty(G^{\times n})$ is called a multiplier if for all $f \in A(G)$, the function

$$(x_1, \ldots, x_n) \mapsto \phi(x_1, \ldots, x_n) f\left(\prod_{j=1}^{n} x_j\right) \tag{338}$$

is in $A^n(G)$. The map that sends f to the above function is linear and automatically bounded (by the closed graph theorem). If this map is completely bounded, ϕ is called a completely bounded multiplier. According to Proposition 5.4 in [28], ϕ is a completely bounded multiplier if and only if the map $M_\phi : VN(G)^{\times n} \to VN(G)$ (as we have defined in Definition 4.10) is a completely bounded map that is continuous in each variable with respect to the σ-weak operator topology. A similar result (with a similar proof) might hold if complete boundedness is replaced by boundedness, but it is not explicitly mentioned in [28] if this is the case. This suggests that the definition of (completely bounded) multipliers in [28] might be equivalent to our definition of (completely bounded) Fourier multiplier symbols. In our definition, Fourier multiplier symbols are required to be continuous functions, but the multipliers in [28] are not. In the case of $A(G)$ the continuity of its multipliers is automatic, as mentioned in Remark 4.10. It is not mentioned in [28] if the multipliers they define are automatically continuous or not. If they are, then their definition of multipliers is most likely equivalent to our definition of multilinear Fourier multiplier symbols. If not, then our definition is more restrictive.

4.5 Weak amenability

In this subsection we define weak amenability of locally compact groups, which is an important concept involving both (linear) Fourier multipliers and the Fourier algebra. An overview of the literature on weak amenability can be found in [29], including applications of weak amenability in the study of several kinds of locally compact groups. We will briefly mention several important applications in this subsection. One of the original goals for this thesis was the introduction of a generalization of weak amenability, featuring multilinear Fourier multipliers, and taking first steps to generalize results involving weak amenability. Our proposed generalization of weak amenability turned out to be equivalent to weak amenability itself (see Theorem 4.26 and Remark 4.15).

Definition 4.18. Weak amenability

Let G be a locally compact group. G is called weakly amenable if there exists a constant $k \geq 1$ and a net ϕ_ι in $A(G)$ such that

$$\|\phi_\iota\|_{M_{cb}A(G)} \leq k \tag{339}$$

for all ι and such that ϕ_ι converges to 1 uniformly on compact subsets of G. The infimum of all constants k that can be chosen in the definition of weak amenability is called the Cowling-Haagerup constant $\Lambda(G)$ of G. For locally compact groups that are not weakly amenable, we define $\Lambda(G) = \infty$.

Remark 4.13. The requirement that $k \geq 1$ in the definition of weak amenability can be omitted without changing the value of $\Lambda(G)$. This is because no net ϕ_ι in $A(G)$ can exist that converges to 1 uniformly on compact subsets of G and satisfies

$$\|\phi_\iota\|_{M_{cb}A(G)} \leq k \tag{340}$$

for all ι when $k < 1$. Indeed, because $M_{cb}A(G) \subseteq C_b(G)$ contractively, such a net would be bounded by k in $\|\cdot\|_\infty$. But then this net cannot even converge to 1 pointwise, let alone uniformly on compact sets.

Remark 4.14. An easy example of a weakly amenable group is any compact group. Indeed if G is a compact group, then $0 < \mu(G) < \infty$ for any Haar measure μ. By rescaling the Haar measure if necessary, we can find a Haar measure μ on G such that $\mu(G) = 1$. With respect to this Haar measure we have that the constant 1 function is in $L^2(G)$ and $\|1\|_2 = 1$. Note that

$$\langle \lambda(x)1, 1\rangle = 1, \tag{341}$$

hence $1 \in A(G)$ with $\|1\|_{A(G)} \leq 1$. We also have $\|1\|_{A(G)} \geq \|1\|_\infty = 1$, so $\|1\|_{A(G)} = 1$. Then we must also have $\|1\|_{M_{cb}A(G)} = 1$. So any net that is constantly equal to 1 is valid for the definition of weak amenability with $k = 1$. We conclude that G is weakly amenable with $\Lambda(G) = 1$.

There exist many results in the literature that involve weak amenability. [29] gives a good overview of this topic. We briefly mention several important results from the literature below and refer to [29] for more details and further references.

Theorem 4.23. *Let G be a second-countable, unimodular, locally compact group and Γ a lattice in G. Then Γ is weakly amenable if and only if G is weakly amenable. In this case we have $\Lambda(G) = \Lambda(\Gamma)$.*

A proof of Theorem 4.23 can be found in [18]. For the definition of a lattice we refer to section 5 or [18]. The following result shows an important application of weak amenability in the context of discrete groups.

Theorem 4.24. *Let Γ be a discrete group and $k \geq 1$. The following are equivalent:*

1. *Γ is weakly amenable with $\Lambda(\Gamma) \leq k$.*

2. *There exists a net T_ι of finite rank operators on $C_r^*(\Gamma)$ such that $\|T_\iota\|_{CB} \leq k$ for all ι and $T_\iota x$ converges to x in norm for all $x \in C_r^*(\Gamma)$.*

3. *There exists a net T_ι of finite rank maps on $VN(G)$ that are continuous with respect to the σ-weak operator topology, such that $\|T_\iota\|_{CB} \leq k$ for all ι and $T_\iota x$ converges to x in σ-weak operator topology for all $x \in VN(\Gamma)$.*

Theorem 4.24 relates several different approximation properties. We note that $C_r^*(\Gamma)$ is the C^*-subalgebra of $\mathcal{B}(l^2(\Gamma))$ generated by $\{\lambda(x) : x \in \Gamma\}$. For a proof of Theorem 4.24, we refer to Theorem 2.6 in [18]. The notion of weak amenability also has applications in the context of Lie groups. One result about Lie groups involving weak amenability is the following.

Theorem 4.25. *Let G be a connected simple Lie group. Then G is weakly amenable if and only if it has real rank 0 or 1.*

For the definition of the rank of a Lie group, we refer to section 5 in [29]. The fact that no Lie group with real rank ≥ 2 can be weakly amenable was proved by Haagerup [18] with an additional assumption that was later removed by Dorofaeff [11, 12]. The weak amenability of Lie groups with rank 0 or 1 was proved seperately for different Lie groups by Cowling ([6]), De Cannière and Haagerup ([8]), and Cowling and Haagerup ([7]) (with an additional assumption that was later removed by Hansen [19]). The Cowling-Haagerup constants of the Lie groups of rank 0 and 1 are also known and we refer to Theorem 5.1 in [29] for an overview. We also refer to [29] for more applications of weak amenability.

The following result gives us equivalent characterizations of weak amenability.

Theorem 4.26. *Let G be a locally compact group and $n \in \mathbb{N}$. The following statements are equivalent.*

1. *G is weakly amenable.*

2. *There exists a net ϕ_ι in $C_c(G) \cap M_{cb}A(G)$ that converges to 1 uniformly on compact subsets of G and such that $\|\phi_\iota\|_{M_{cb}A(G)} \leq k$ for some constant $k \geq 1$ and all ι.*

3. *There exists a net ϕ_ι in $C_c(G^{\times n}) \cap M_{cb}^n A(G)$ that converges to 1 uniformly on compact subsets of $G^{\times n}$ and such that $\|\phi_\iota\|_{M_{cb}^n A(G)} \leq k$ for some constant $k \geq 1$ and all ι.*

Moreover, if G is weakly amenable statement (2) can be satisfied for any $k > \Lambda(G)$. Vice versa, any k for which statement (2) can be satisfied is an upper bound for $\Lambda(G)$. If statement (2) can be satisfied with constant k, then (3) can be satisfied with constant k^n. If statement (3) can be satisfied with constant k, then so can statement (2).

Proof. We will prove in order the implications (1) $\implies$ (2), (2) $\implies$ (1), (2) $\implies$ (3) and (3) $\implies$ (2). The remaining claims made in the statement of the theorem will be apparent from the constants we use to prove the above four implications.

(1) $\implies$ (2): Suppose G is weakly amenable. Let $k > \Lambda(G)$. Then there exists a net $(\phi_\iota)_{\iota \in I}$ in $A(G)$ that converges to 1 uniformly on compact subsets of G and such that $\|\phi_\iota\| \leq k$ for all ι. It follows from Lemma 4.19 that $A(G) \cap C_c(G)$ is dense in $A(G)$ in $\|\cdot\|_{A(G)}$. Hence if we fix some $\delta > 0$, then for every ι and every $m \in \mathbb{N}$ we can find a

$$\psi_{\iota,m} \in A(G) \cap C_c(G) \subseteq C_c(G) \cap M_{cb}A(G) \tag{342}$$

such that

$$\|\psi_{\iota,m} - \phi_\iota\|_{A(G)} < \frac{\delta}{m}. \tag{343}$$

We consider the set $I \times \mathbb{N}$, which is a directed set when equipped with the relation given by $(\iota_1, m_1) \succeq (\iota_2, m_2)$ if and only if $\iota_1 \succeq \iota_2$ and $m_1 \geq m_2$. This turns $(\psi_{\iota,m})_{(\iota,m)\in I\times\mathbb{N}}$ into a net in $C_c(G) \cap A(G) \subseteq C_c(G) \cap M_{cb}A(G)$. Given any compact $K \subseteq G$ and $\epsilon > 0$ we can choose $\iota_0 \in I$ such that for all $\iota \succeq \iota_0$:

$$|\phi_\iota(x) - 1| < \frac{\epsilon}{2} \tag{344}$$

for all $x \in K$. We can also choose $m_0 \in \mathbb{N}$ such that $\frac{\delta}{m} < \frac{\epsilon}{2}$ for all $m \geq m_0$ (any $m_0 > \frac{2\delta}{\epsilon}$ will work). It follows that for all $(\iota, m) \succeq (\iota_0, m_0)$ and all $x \in K$:

$$\begin{aligned} |\psi_{\iota,m}(x) - 1| &= |\psi_{\iota,m}(x) - \phi_\iota(x) + \phi_\iota(x) - 1| \leq |\psi_{\iota,m}(x) - \phi_\iota(x)| + |\phi_\iota(x) - 1| \\ &\leq \|\psi_{\iota,m} - \phi_\iota\|_\infty + |\phi_\iota(x) - 1| \leq \|\psi_{\iota,m} - \phi_\iota\|_{A(G)} + |\phi_\iota(x) - 1| < \frac{\delta}{m} + \frac{\epsilon}{2} < \epsilon. \end{aligned} \tag{345}$$

This shows that the net $\psi_{\iota,m}$ converges to 1 uniformly on compact subsets of G. We also have that for all $(\iota, m) \in I \times \mathbb{N}$:

$$\begin{aligned} \|\psi_{\iota,m}\|_{M_{cb}A(G)} &= \|\psi_{\iota,m} - \phi_\iota + \phi_\iota\|_{M_{cb}A(G)} \leq \|\psi_{\iota,m} - \phi_\iota\|_{M_{cb}A(G)} + \|\phi_\iota\|_{M_{cb}A(G)} \\ &\leq \|\psi_{\iota,m} - \phi_\iota\|_{A(G)} + \|\phi_\iota\|_{M_{cb}A(G)} \leq \frac{\delta}{m} + k \leq k + \delta. \end{aligned} \tag{346}$$

We conclude that statement (2) is satisfied, hence (1) $\implies$ (2). We also note, because $k > \Lambda(G)$ and $\delta > 0$ were arbitrary, that assuming G is weakly amenable, any $k > \Lambda(G)$ can be chosen in statement (2).

(2) $\implies$ (1): Now we assume instead that statement (2) holds and we let k and ϕ_ι be as in statement (2). Let $f \in C_c(G)$ be a non-negative function with $\|f\|_1 = 1$. Then we have that

$$\int_G f(x)dx = \|f\|_1 = 1. \tag{347}$$

For all ι we have that

$$(f * \phi_\iota)(x) = \langle \lambda(x)\check{\phi}_\iota, f \rangle \tag{348}$$

(see Definition 4.16), where for a function g on G, $\check{g}$ is given by $\check{g}(x) = g(x^{-1})$. Because $\check{\phi}_\iota \in C_c(G)$, we have that $f * \phi_\iota \in A(G) \cap C_c(G)$ (see Lemma 4.19). By Theorem 4.17, for each ι, there exists a Hilbert space H_ι and continuous bounded maps $\xi_\iota, \eta_\iota : G \to H_\iota$ such that

$$\phi_\iota(y^{-1}x) = \langle \xi_\iota(x), \eta_\iota(y) \rangle \tag{349}$$

for all $x, y \in G$ and such that

$$\sup_{x\in G}\|\xi_\iota(x)\| \sup_{y\in G}\|\eta_\iota(y)\| \leq k. \tag{350}$$

For $x, y \in G$ we have that

$$(f * \phi_\iota)(y^{-1}x) = \int_G f(t)\phi_\iota(t^{-1}y^{-1}x)dt = \int_G f(t)\phi_\iota((yt)^{-1}x)dt = \int_G f(t)\left\langle \xi_\iota(x), \eta_\iota(yt)\right\rangle dt. \tag{351}$$

We note that $\check{\eta}_\iota$ satisfies the conditions of Lemma 4.18. So there exists a continuous and bounded function $f * \check{\eta}_\iota : G \to H_\iota$, such that

$$\langle g, (f * \check{\eta}_\iota)(y)\rangle = \int_G f(t)\left\langle g, \check{\eta}_\iota(t^{-1}y)\right\rangle dt = \int_G f(t)\left\langle g, \eta_\iota(y^{-1}t)\right\rangle dt \tag{352}$$

for all $y \in G$ and $g \in H_\iota$. Moreover we have that

$$\|f * \check{\eta}_\iota\|_\infty \leq \|f\|_1\|\check{\eta}_\iota\|_\infty = \|\eta_\iota\|_\infty. \tag{353}$$

It follows that

$$(f * \phi_\iota)(y^{-1}x) = \int_G f(t)\left\langle \xi_\iota(x), \eta_\iota(yt)\right\rangle dt = \langle \xi_\iota(x), (f * \check{\eta}_\iota)\check{}(y)\rangle \tag{354}$$

for all $x, y \in G$. But $(f * \check{\eta}_\iota)\check{}$ is continuous and bounded with

$$\|(f * \check{\eta}_\iota)\check{}\|_\infty = \|f * \check{\eta}_\iota\|_\infty \leq \|\eta_\iota\|_\infty. \tag{355}$$

It now follows from Theorem 4.17 that $f * \phi_\iota \in M_{cb}A(G)$ with

$$\|f * \phi_\iota\|_{M_{cb}A(G)} \leq k \tag{356}$$

for all ι. It remains to prove that $f * \phi_\iota$ converges to 1 uniformly on compact subsets of G. To this end let $K_f \subseteq G$ be compact and such that $f(x) = 0$ for $x \notin K_f$. Let $K \subseteq G$ be compact and let $\epsilon > 0$. Then $K_f^{-1}K \subseteq G$ is compact. Choose ι_0 such that

$$|\phi_\iota(x) - 1| < \epsilon \tag{357}$$

for all $\iota \succeq \iota_0$ and all $x \in K_f^{-1}K$. Then, for all $x \in K$ and $t \in K_f$, we have that $t^{-1}x \in K_f^{-1}K$. It follows that for all $\iota \succeq \iota_0$ and all $x \in K$:

$$\begin{aligned}
|(f * \phi_\iota)(x) - 1| &= \left|\int_G f(t)\phi_\iota(t^{-1}x)dt - \int_G f(t)dt\right| = \left|\int_G f(t)(\phi_\iota(t^{-1}x) - 1)dt\right| \\
&\leq \int_G f(t)\big|\phi_\iota(t^{-1}x) - 1\big|dt = \int_{K_f} f(t)\big|\phi_\iota(t^{-1}x) - 1\big|dt \\
&< \int_{K_f} f(t)\epsilon dt = \epsilon.
\end{aligned} \tag{358}$$

This shows that the net $f * \phi_\iota$ converges to 1 uniformly on compact subsets of G. This shows that G is weakly amenable with $\Lambda(G) \leq k$.

(2) $\implies$ (3): Let ϕ_ι and k be as in statement (2). Define $\Phi_\iota = \otimes_{j=1}^n \phi_\iota$, i.e. $\Phi_\iota : G^{\times n} \to \mathbb{C}$ is given by

$$\Phi_\iota(x_1, \ldots, x_n) = \prod_{j=1}^n \phi_\iota(x_j). \tag{359}$$

Then we know that Φ_ι is continuous (Remark 3.26). Φ_ι is also compactly supported. Indeed, if $K_\iota \subseteq G$ is compact and such that $\phi_\iota(x) = 0$ if $x \notin K_\iota$, then $K_\iota^{\times n}$ is compact and $\Phi_\iota(x_1, \ldots, x_n) = 0$ if $(x_1, \ldots, x_n) \notin K_\iota^{\times n}$. So $\Phi_\iota \in C_c(G)$. From Corollary 4.10 it follows that $\Phi_\iota \in M_{cb}^n A(G)$ with

$$\|\Phi_\iota\|_{M^n_{cb}A(G)} \leq \prod_{j=1}^n \|\phi_\iota\|_{M_{cb}A(G)} \leq k^n. \tag{360}$$

So the net $\Phi_\iota \in C_c(G) \cap M^n_{cb}A(G)$ is bounded in $\|\cdot\|_{M^n_{cb}A(G)}$. It remains to show that Φ_ι converges to 1 uniformly on compact subsets of $G^{\times n}$. To this end, let $K \subseteq G^{\times n}$ compact, $\eta > 0$ and choose $\epsilon = \frac{\eta}{nk^{n-1}} > 0$. Let $\pi_j : G^{\times n} \to G$ be the projection onto the j-th component. π_j is a continuous function, so $K_j := \pi_j(K) \subseteq G$ is compact. It follows that $K_0 := \bigcup_{j=1}^n K_j \subseteq G$ is compact and we note that $K \subseteq K_0^{\times n}$. Now choose ι_0 such that for all $\iota \succeq \iota_0$ and all $x \in K_0$ we have that

$$|\phi_\iota(x) - 1| < \epsilon. \tag{361}$$

It follows that for all $\iota \succeq \iota_0$ and all $(x_1, \ldots, x_n) \in K \subseteq K_0^{\times n}$:

$$\begin{aligned}
|\Phi_\iota(x_1, \ldots, x_n) - 1| &= \left| \prod_{j=1}^n \phi_\iota(x_j) - 1 \right| = \left| \sum_{m=1}^n \prod_{j=1}^m \phi_\iota(x_j) - \sum_{m=1}^n \prod_{j=1}^{m-1} \phi_\iota(x_j) \right| \\
&= \left| \sum_{m=1}^n \left(\prod_{j=1}^m \phi_\iota(x_j) - \prod_{j=1}^{m-1} \phi_\iota(x_j) \right) \right| \leq \sum_{m=1}^n \left| \left(\prod_{j=1}^m \phi_\iota(x_j) - \prod_{j=1}^{m-1} \phi_\iota(x_j) \right) \right| \\
&= \sum_{m=1}^n \left| \left(\prod_{j=1}^{m-1} \phi_\iota(x_j) \right) (\phi_\iota(x_m) - 1) \right| = \sum_{m=1}^n \left(\prod_{j=1}^{m-1} |\phi_\iota(x_j)| \right) |\phi_\iota(x_m) - 1| \\
&\leq \sum_{m=1}^n \|\phi_\iota\|_\infty^{m-1} |\phi_\iota(x_m) - 1| \leq \sum_{m=1}^n \|\phi_\iota\|_{M_{cb}A(G)}^{m-1} |\phi_\iota(x_m) - 1| \\
&\leq \sum_{m=1}^n k^{m-1} |\phi_\iota(x_m) - 1| < \sum_{m=1}^n k^{m-1}\epsilon \leq nk^{n-1}\epsilon = \eta.
\end{aligned} \tag{362}$$

This shows that Φ_ι converges to 1 uniformly on compact subsets of $G^{\times n}$. So, (2) $\implies$ (3) and we note that if statement (2) holds with constant k, then statement (3) holds with constant k^n.

(3) $\implies$ (2): Let ϕ_ι and k be as in statement (3). Define $\psi_\iota : G \to \mathbb{C}$ as the function obtained from ϕ_ι by setting all variables of ϕ_ι except for the first equal to the identity $e \in G$. So ψ_ι is given by

$$\psi_\iota(x) = \phi_\iota(x, e, e, \ldots, e). \tag{363}$$

Then ψ_ι is continuous. ψ_ι is also compactly supported. Indeed, let $K \subseteq G^{\times n}$ compact and such that $\phi_\iota(x_1, \ldots, x_n) = 0$ if $(x_1, \ldots, x_n) \notin K$ and $K_1 = \pi_1(K) \subseteq G$. Then K_1 is compact and $\psi_\iota(x) = 0$ if $x \notin K_1$. So $\psi_\iota \in C_c(G)$. By Corollary 4.12 we have that $\psi_\iota \in M_{cb}A(G)$ with

$$\|\psi_\iota\|_{M_{cb}A(G)} \leq \|\phi_\iota\|_{M^n_{cb}A(G)} \leq k. \tag{364}$$

So the net $\psi_\iota \in C_c(G) \cap M_{cb}A(G)$ is bounded in $\|\cdot\|_{M_{cb}A(G)}$. It remains to show that ψ_ι converges to 1 uniformly on compact subsets of G. To this end, let $K \subseteq G$ be compact and $\epsilon > 0$. Since $\{e\} \subseteq G$ is compact, we have that $K \times \{e\}^{\times(n-1)} \subseteq G^{\times n}$ is compact. Let ι_0 be such that for all $\iota \succeq \iota_0$ and all $(x_1, \ldots, x_n) \in K \times \{e\}^{\times(n-1)}$ we have that

$$|\phi_\iota(x_1, \ldots, x_n) - 1| < \epsilon. \tag{365}$$

Then for all $x \in K$, we have that $(x, e, e, \ldots, e) \in K \times \{e\}^{\times(n-1)}$ (where the variable e is repeated $n-1$ times). It follows that

$$|\psi_\iota(x) - 1| = |\phi_\iota(x, e, e, \ldots, e) - 1| < \epsilon \tag{366}$$

for all $\iota \succeq \iota_0$ and all $x \in K$. This shows that ψ_ι converges to 1 uniformly on compact subsets of G. So, statement (2) holds with constant k, which concludes the proof. $\square$

Remark 4.15. The original intention for this thesis was to use statement (3) in Theorem 4.26 as the definition for "n-weak amenability", which would generalize weak amenability in the sense that 1-weak amenability is equivalent to weak amenability (which is the equivalence of (1) and (2) in Theorem 4.26). Eventually, however, it became apparent that n-weak amenability was equivalent to weak amenability as shown in Theorem 4.26.

5 Lattices in locally compact groups

Our aim in this section is to generalize some of the results in section 2 of Haagerup's 2016 paper [18] to multilinear Fourier multipliers. The results we generalize from Haagerup's paper [18] were mainly used to prove that a lattice in a second-countable, unimodular, locally compact group is weakly amenable if and only if the group itself is weakly amenable. Our aim originally was to prove an analogous result for n-weak amenability. Our proposed definition for n-weak amenability turned out to be equivalent to weak amenability (see Remark 4.15 and Theorem 4.26), so proving this analogue was no longer necessary because it follows directly from Haagerup's result. The results we prove in this section might still be interesting, because they show that symbols of multilinear Fourier multipliers on the lattice can be used to construct multilinear Fourier multipliers on the entire group.

We will first describe the setting for this section, which is the same as in [18], and introduce some notation (subsection 5.1). In subsection 5.2 we generalize the main construction $\phi \mapsto \hat{\phi}$ in Lemma 2.1 of [18] to multivariable functions. In subsection 5.3 we combine this construction with the "pseudo-convolution" defined in subsection 3.5, which will allow us to construct multilinear Fourier multipliers on the entire group based on multilinear Fourier multipliers on the lattice.

5.1 Setting and notation

Let G be a second-countable, locally compact group with identity e. Let $\Gamma \subseteq G$ be a lattice in G, i.e. Γ is a closed discrete subgroup of G and the quotient G/Γ has a finite measure that is invariant under left multiplication with elements of G. Note that this measure is unique up to multiplication with a positive constant as proved in Theorem 2.49 of [17].

Under these conditions, Γ is necessarily countable and G is unimodular. For the unimodularity of G, Haagerup refers to Definition 1.8 and Remark 1.9 in [25]. For the countability of Γ, we note that for every $t \in \Gamma$ there exists an open set $U_t \subseteq G$ such that $U_t \cap \Gamma = \{t\}$. Note that $s \notin U_t$ if $s, t \in \Gamma$ with $s \neq t$. G is second-countable and therefore has a countable base $\mathcal{B}$. So, for every $t \in \Gamma$ there exists a $B_t \in \mathcal{B}$ such that $t \in B_t \subseteq U_t$. Then $s \notin B_t$ if $s, t \in \Gamma$ are distinct. It follows that $B_t \neq B_s$ if s and t are distinct. Since $\mathcal{B}$ is countable, it follows that Γ must be countable.

The quotient map $\rho : G \to G/\Gamma$ has a Borel cross section, i.e. a Borel measurable function $q : G/\Gamma \to G$ such that $\rho \circ q = \mathrm{id}_{G/\Gamma}$. We let Ω be the range of one such fixed Borel cross section. Hence $\rho|_\Omega : \Omega \to G/\Gamma$ is a bijection and a Borel isomorphism ($\rho|_\Omega$ and its inverse are Borel measurable). For every $x \in G$ note that

$$\rho(\rho|_\Omega^{-1}(\rho(x))) = \rho(x). \tag{367}$$

Hence $x\Gamma = \rho|_\Omega^{-1}(\rho(x))\Gamma$ and it follows that

$$x = \rho|_\Omega^{-1}(\rho(x))t \tag{368}$$

for some $t \in \Gamma$. This shows that every $x \in G$ can be written as

$$x = st \tag{369}$$

for some $s \in \Omega$ and $t \in \Gamma$. If $s_1, s_2 \in \Omega$ and $t_1, t_2 \in \Gamma$ are such that

$$s_1 t_1 = x = s_2 t_2, \tag{370}$$

then

$$s_1 = \rho|_\Omega^{-1}(\rho(s_1 t_1)) = \rho|_\Omega^{-1}(\rho(s_2 t_2)) = s_2. \tag{371}$$

Hence also $t_1 = t_2$. This shows that for every $x \in G$ there exist unique $s \in \Omega$ and $t \in \Gamma$ such that $x = st$. It follows that

$$G = \bigcup_{t \in \Gamma} \Omega t \tag{372}$$

and this is a disjoint union. Let μ be a Haar measure on G. μ is right-invariant, Γ is countable and the above union is disjoint, so it follows that $\mu(\Omega) > 0$ (otherwise $\mu(G) = 0$). $A \mapsto \mu(\rho|_\Omega^{-1}(A))$ defines a G-invariant measure on G/Γ.

Indeed, for $x \in G$, $A \subseteq G/\Gamma$ a Borel set and $y \in A$, there is a unique $t \in \Gamma$ such that

$$x\rho|_\Omega^{-1}(y)t \in \Omega \tag{373}$$

and for this t we have that

$$\rho(x\rho|_\Omega^{-1}(y)t) = x\rho|_\Omega^{-1}(y)t\Gamma = x\rho|_\Omega^{-1}(y)\Gamma = x\rho(\rho|_\Omega^{-1}(y)) = xy. \tag{374}$$

So we have that

$$x\rho|_\Omega^{-1}(y)t = \rho|_\Omega^{-1}(xy). \tag{375}$$

It follows that

$$\rho|_\Omega^{-1}(xA) = \bigcup_{t\in\Gamma}(x\rho|_\Omega^{-1}(A)t \cap \Omega). \tag{376}$$

This union is disjoint. Indeed, if $t_1, t_2 \in \Gamma$ with $x\rho|_\Omega^{-1}(A)t_1 \cap x\rho|_\Omega^{-1}(A)t_2 \neq \emptyset$, then there exist $y_1, y_2 \in A$ such that $xs_1t_1 = xs_2t_2$, where $s_j = \rho|_\Omega^{-1}(y_j)$ for $j \in \{1, 2\}$. Then $s_1t_1 = s_2t_2$, which implies that $s_1 = s_2$ and $t_1 = t_2$. So, the above union is disjoint. Now we have that

$$\begin{aligned}\mu(\rho|_\Omega^{-1}(xA)) &= \mu\left(\bigcup_{t\in\Gamma}(x\rho|_\Omega^{-1}(A)t\cap\Omega)\right) = \sum_{t\in\Gamma}\mu(x\rho|_\Omega^{-1}(A)t\cap\Omega) = \sum_{t\in\Gamma}\mu(x\rho|_\Omega^{-1}(A)\cap\Omega t^{-1}) \\ &= \mu\left(\bigcup_{t\in\Gamma}(x\rho|_\Omega^{-1}(A)\cap\Omega t^{-1})\right) = \mu(x\rho|_\Omega^{-1}(A)) = \mu(\rho|_\Omega^{-1}(A)).\end{aligned} \tag{377}$$

This shows that this measure is indeed G-invariant.

Since G/Γ has a finite G-invariant measure and any other G-invariant measure on G/Γ must be a positive multiple of this measure, $A \mapsto \mu(\rho|_\Omega^{-1}(A))$ must be a finite measure. In other words $\mu(\Omega) < \infty$. By rescaling μ if necessary, we can obtain a (unique) Haar measure μ such that $\mu(\Omega) = 1$ and we fix this Haar measure. To shorten notation we will suppress μ and the product measure μ^n when integrating against these measures:

$$\int_{G^{\times n}} f(x_1, \ldots, x_n)d(x_1, \ldots, x_n) = \int_{G^{\times n}} f(x_1, \ldots, x_n)d\mu^n(x_1, \ldots, x_n) \tag{378}$$

Because every $x \in G$ has a unique decomposition of the form $x = st$, where $s \in \Omega$ and $t \in \Gamma$, this allows us to uniquely define functions $\omega : G \to \Omega$ and $\gamma : G \to \Gamma$ such that

$$x = \omega(x)\gamma(x) \tag{379}$$

for all $x \in G$. Our earlier calculations show that

$$\rho|_\Omega^{-1}(\rho(x)) = \omega(x), \tag{380}$$

hence ω is Borel measurable. It follows that $\gamma(x) = \omega(x)^{-1}x$ is also Borel measurable. For $x \in G$ we define $\tau_x : \Omega \to \Omega$ by

$$\tau_x(t) = \omega(xt). \tag{381}$$

Let $y \in G/\Gamma$. Then we have that

$$\begin{aligned}&\rho|_\Omega(\tau_x(\rho|_\Omega^{-1}(y))) = \rho|_\Omega(\omega(x\rho|_\Omega^{-1}(y))) = \omega(x\rho|_\Omega^{-1}(y))\Gamma = \omega(x\rho|_\Omega^{-1}(y))\gamma(x\rho|_\Omega^{-1}(y))\Gamma \\ &= x\rho|_\Omega^{-1}(y)\Gamma = x\rho|_\Omega(\rho|_\Omega^{-1}(y)) = xy.\end{aligned} \tag{382}$$

So we see that $\rho|_\Omega \circ \tau_x \circ \rho|_\Omega^{-1} : G/\Gamma \to G/\Gamma$ is left multiplication with x. For $E \subseteq \Omega$ a Borel set and $x \in G$ we now have that

$$\mu(\tau_x(E)) = \mu(\rho|_\Omega^{-1}(\rho|_\Omega(\tau_x(\rho|_\Omega^{-1}(\rho|_\Omega(E)))))) = \mu(\rho|_\Omega^{-1}(x(\rho|_\Omega(E)))) = \mu(\rho|_\Omega^{-1}(\rho|_\Omega(E))) = \mu(E). \tag{383}$$

Here we used that $A \mapsto \mu(\rho|_\Omega^{-1}(A))$ is G-invariant. This shows that $\mu|_\Omega$ is τ_x-invariant for all $x \in G$. Note that if $x, y \in G$ and $z \in \Omega$, then we have that

$$yx^{-1}\tau_x(z) = yzz^{-1}x^{-1}\omega(xz) = (yz)(xz)^{-1}\omega(xz) = \omega(yz)\gamma(yz)\gamma(xz)^{-1}\omega(xz)^{-1}\omega(xz) = \omega(yz)\gamma(yz)\gamma(xz)^{-1}. \tag{384}$$

This shows that

$$\omega(yx^{-1}\tau_x(z)) = \omega(yz) \tag{385}$$

and

$$\gamma(yx^{-1}\tau_x(z)) = \gamma(yz)\gamma(xz)^{-1}. \tag{386}$$

For $n \in \mathbb{N}$ we let $\mu_{\Gamma^{\times n}}$ be the counting measure on $\Gamma^{\times n}$ and note that this is a Haar measure on $\Gamma^{\times n}$. We will also write $\mu_{\Gamma^{\times n}}$ for the trivial extension of $\mu_{\Gamma^{\times n}}$ to $G^{\times n}$, i.e.

$$\mu_{\Gamma^{\times n}}(E) = \left|E \cap \Gamma^{\times n}\right| \tag{387}$$

when $E \subseteq G^{\times n}$ is a Borel set.

5.2 Definition and properties of $\hat{\phi}$

In Lemma 2.1 of his paper [18], given a bounded function $\phi : \Gamma \to \mathbb{C}$, Haagerup defines a bounded continuous function $\hat{\phi} : G \to \mathbb{C}$ given by

$$\hat{\phi}(x) = \int_\Omega \phi(\gamma(xs))ds. \tag{388}$$

This definition is such that $\hat{\phi} \in M_{cb}A(G)$ whenever $\phi \in M_{cb}A(\Gamma)$ and

$$\left\|\hat{\phi}\right\|_{M_{cb}A(G)} \leq \|\phi\|_{M_{cb}A(\Gamma)}. \tag{389}$$

We generalize the construction of $\hat{\phi}$ to an n-variable setting.

Definition 5.1. Let $n \in \mathbb{N}$ and $\phi : \Gamma^{\times n} \to \mathbb{C}$ a bounded function. We define a function $\hat{\phi} : G^{\times n} \to \mathbb{C}$ given by

$$\hat{\phi}(s_1, \dots, s_n) = \int_\Omega \phi\left(\left(\gamma\left(\prod_{m=j}^{n+1} s_m\right)\gamma\left(\prod_{m=j+1}^{n+1} s_m\right)^{-1}\right)_{j=1}^{n}\right) ds_{n+1}. \tag{390}$$

Remark 5.1. Note that if $n = 1$, our definition of $\hat{\phi}$ in Definition 5.1 agrees with the definition in [18]. This can be seen by noting that in the $n = 1$ case, $\gamma(s_2) = e$ for all $s_2 \in \Omega$.

Lemma 5.1. *For any $n \in \mathbb{N}$, $\phi \mapsto \hat{\phi}$ defines a linear contraction $l^\infty(\Gamma^{\times n}) \to L^\infty(G^{\times n})$.*

Proof. Let $\phi \in l^\infty(\Gamma^{\times n})$ and note that the integrand in Definition 5.1 is a Borel measurable function (because γ is Borel measurable) and is bounded in absolute value by $\|\phi\|_\infty$. Because $\mu(\Omega) = 1$, it follows by Fubini's theorem that $\hat{\phi} : G^{\times n} \to \mathbb{C}$ is a well-defined and measurable function. Moreover we have that

$$\begin{aligned} \left|\hat{\phi}(s_1, \dots, s_n)\right| &= \left|\int_\Omega \phi\left(\left(\gamma\left(\prod_{m=j}^{n+1} s_m\right)\gamma\left(\prod_{m=j+1}^{n+1} s_m\right)^{-1}\right)_{j=1}^n\right) ds_{n+1}\right| \\ &\leq \int_\Omega \left|\phi\left(\left(\gamma\left(\prod_{m=j}^{n+1} s_m\right)\gamma\left(\prod_{m=j+1}^{n+1} s_m\right)^{-1}\right)_{j=1}^n\right)\right| ds_{n+1} \\ &\leq \int_\Omega \|\phi\|_\infty ds_{n+1} = \|\phi\|_\infty. \end{aligned} \tag{391}$$

This shows that $\hat{\phi}$ is bounded with

$$\left\|\hat{\phi}\right\|_\infty \leq \|\phi\|_\infty. \tag{392}$$

From the definition of $\hat{\phi}$, it is clear that $\phi \mapsto \hat{\phi}$ is linear, hence $\hat{\phi}$ is a linear contraction. $\square$

As a direct consequence of Lemma 5.1 and Lemma 3.20 we have the following corollary.

Corollary 5.2. *For any $n \in \mathbb{N}$ and $F \in L^1(G^{\times(n+1)})$, $\phi \mapsto F \tilde{*} \hat{\phi}$ defines a bounded linear map $l^\infty(\Gamma^{\times n}) \to C_b(G^{\times n})$ with*

$$\left\|F \tilde{*} \hat{\phi}\right\|_\infty \leq \|F\|_1 \|\phi\|_\infty. \tag{393}$$

For $\phi \in l^1(\Gamma^{\times n})$, the following lemma can be seen as a motivation for the definition of $\hat{\phi}$.

Lemma 5.3. *For $\phi \in l^1(\Gamma^{\times n})$, let $\phi\mu_{\Gamma^{\times n}}$ be the complex Borel measure on $G^{\times n}$ given by*

$$(\phi\mu_{\Gamma^{\times n}})(E) = \sum_{t \in \Gamma^{\times n} \cap E} \phi(t) = \sum_{t \in \Gamma^{\times n}} \mathbb{1}_E(t)\phi(t). \tag{394}$$

For Borel sets $E \subseteq G^{\times n}$, define

$$\nu_\phi(E) = \int_{G^{\times(n+1)}} \mathbb{1}_{\Omega^{\times(n+1)}}\left((s_j)_{j=1}^{n+1}\right)(\phi\mu_{\Gamma^{\times n}})\left(\left(s_j^{-1}\right)_{j=1}^n E \left(s_{j+1}\right)_{j=1}^n\right) d(s_1, \dots, s_{n+1}). \tag{395}$$

Then ν_ϕ defines a complex Borel measure on $G^{\times n}$ and we have that

$$\nu_\phi(E) = \int_E \hat{\phi}(s_1, \dots, s_n) d(s_1, \dots, s_n). \tag{396}$$

Here we have that $\hat{\phi} \in L^1(G^{\times n})$ and

$$\left\|\hat{\phi}\right\|_1 \leq \nu_{|\phi|}(G^{\times n}) = \|\phi\|_1. \tag{397}$$

Proof. Note that

$$\begin{aligned} &(\phi\mu_{\Gamma^{\times n}})\left(\left(s_j^{-1}\right)_{j=1}^n E \left(s_{j+1}\right)_{j=1}^n\right) = \int_{\Gamma^{\times n}} \mathbb{1}_{\left(s_j^{-1}\right)_{j=1}^n E (s_{j+1})_{j=1}^n}(t)\phi(t) d\mu_{\Gamma^{\times n}}(t) \\ &= \int_{\Gamma^{\times n}} \mathbb{1}_E\left((s_j)_{j=1}^n t \left(s_{j+1}^{-1}\right)_{j=1}^n\right)\phi(t) d\mu_{\Gamma^{\times n}}(t), \end{aligned} \tag{398}$$

hence

$$\nu_\phi(E) = \int_{G^{\times(n+1)}} \int_{\Gamma^{\times n}} \mathbb{1}_{\Omega^{\times(n+1)}} \left((s_j)_{j=1}^{n+1} \right) \mathbb{1}_E \left((s_j)_{j=1}^n \, t \left(s_{j+1}^{-1} \right)_{j=1}^n \right) \phi(t) d\mu_{\Gamma^{\times n}}(t) d(s_1, \dots, s_{n+1}). \tag{399}$$

The integrand in the above expression is a measurable function on $\Gamma^{\times n} \times G^{\times(n+1)}$ and is bounded in absolute value by $(t, s_1, \dots, s_{n+1}) \mapsto \mathbb{1}_{\Omega^{\times(n+1)}} \left((s_j)_{j=1}^{n+1} \right) |\phi(t)|$, which is an integrable function that integrates to $\|\phi\|_1$. So, for each Borel set E, the quantity $\nu_\phi(E) \in \mathbb{C}$ is well-defined. Moreover, because the integrand in the above expression for $\nu_\phi(E)$ is an integrable function on $\Gamma^{\times n} \times G^{\times(n+1)}$, we can apply Fubini's theorem to change the order of integration:

$$\begin{aligned}
\nu_\phi(E) &= \int_{\Gamma^{\times n}} \int_{G^{\times(n+1)}} \mathbb{1}_{\Omega^{\times(n+1)}} \left((s_j)_{j=1}^{n+1} \right) \mathbb{1}_E \left((s_j)_{j=1}^n \, t \left(s_{j+1}^{-1} \right)_{j=1}^n \right) \phi(t) d(s_1, \dots, s_{n+1}) d\mu_{\Gamma^{\times n}}(t) \\
&= \int_{\Gamma^{\times n}} \int_{G^{\times(n+1)}} \mathbb{1}_{\Omega^{\times(n+1)}} \left((s_j)_{j=1}^{n+1} \right) \mathbb{1}_E \left(\left(s_j t_j s_{j+1}^{-1} \right)_{j=1}^n \right) \phi \left((t_j)_{j=1}^n \right) d(s_1, \dots, s_{n+1}) d\mu_{\Gamma^{\times n}}(t_1, \dots, t_n) \\
&= \int_{\Gamma^{\times n}} \int_G \cdots \int_G \mathbb{1}_{\Omega^{\times(n+1)}} \left((s_j)_{j=1}^{n+1} \right) \mathbb{1}_E \left(\left(s_j t_j s_{j+1}^{-1} \right)_{j=1}^n \right) \phi \left((t_j)_{j=1}^n \right) ds_1 \dots ds_{n+1} d\mu_{\Gamma^{\times n}}(t_1, \dots, t_n).
\end{aligned} \tag{400}$$

We recall that the Haar measure μ is right-invariant, because G is unimodular. We will use this to further rewrite our expression for $\nu_\phi(E)$. We first substitute the integration variables s_j by $s_j \left(\prod_{k=j}^n t_k \right)^{-1}$ for each $j \in \{1, \dots, n+1\}$, where an empty product is to be interpreted as the identity element $e \in G$. Note that performing all these substitutions results in the substitution of the expression $s_j t_j s_{j+1}^{-1}$ (for $j \in \{1, \dots, n\}$) by

$$s_j \left(\prod_{k=j}^n t_k \right)^{-1} t_j \left(s_{j+1} \left(\prod_{k=j+1}^n t_k \right)^{-1} \right)^{-1} = s_j \left(\prod_{k=j}^n t_k \right)^{-1} t_j \left(\prod_{k=j+1}^n t_k \right) s_{j+1}^{-1} = s_j s_{j+1}^{-1}. \tag{401}$$

So, by performing these substitutions and using right-invariance of μ we find that

$$\begin{aligned}
&\nu_\phi(E) \\
&= \int_{\Gamma^{\times n}} \int_G \cdots \int_G \mathbb{1}_{\Omega^{\times(n+1)}} \left(\left(s_j \left(\prod_{k=j}^n t_k \right)^{-1} \right)_{j=1}^{n+1} \right) \mathbb{1}_E \left(\left(s_j s_{j+1}^{-1} \right)_{j=1}^n \right) \phi \left((t_j)_{j=1}^n \right) ds_1 \dots ds_{n+1} d\mu_{\Gamma^{\times n}}(t_1, \dots, t_n).
\end{aligned} \tag{402}$$

Next we perform another round of substitutions relying on the right-invariance of μ. Note that in our current expression for ν_ϕ, we integrate over s_j in order of increasing j. So, within the integral over s_j, any s_i with $i > j$ can be treated as a fixed element of G with regard to using the right-invariance of μ. Therefore, we can substitute s_j by $s_j s_{j+1}$ in the integral over s_j. We do this for each $j \in \{1, \dots, n\}$ in order of increasing j. Note that performing all these substitutions results in the overall substitution of every instance of s_j by $\prod_{k=j}^{n+1} s_k$ (for $j \in \{1, \dots, n+1\}$) in the integrand of the expression for $\nu_\phi(E)$. Since

$$\left(\prod_{k=j}^{n+1} s_k \right) \left(\prod_{k=j+1}^{n+1} s_k \right)^{-1} = s_j \left(\prod_{k=j+1}^{n+1} s_k \right) \left(\prod_{k=j+1}^{n+1} s_k \right)^{-1} = s_j \tag{403}$$

for all $j \in \{1, \dots, n\}$, it follows that

$\nu_\phi(E)$

$$= \int_{\Gamma^{\times n}} \int_G \cdots \int_G \mathbb{1}_{\Omega^{\times(n+1)}} \left(\left(\left(\prod_{k=j}^{n+1} s_k \right) \left(\prod_{k=j}^{n} t_k \right)^{-1} \right)_{j=1}^{n+1} \right) \mathbb{1}_E \left((s_j)_{j=1}^n \right) \phi \left((t_j)_{j=1}^n \right) ds_1 \dots ds_{n+1} d\mu_{\Gamma^{\times n}}(t_1, \dots, t_n). \tag{404}$$

Note that the integrand of the above expression for $\nu_\phi(E)$ is still a measurable and integrable function on $\Gamma^{\times n} \times G^{\times(n+1)}$. To check integrability simply note that the absolute value of the integrand is obtained by substituting ϕ by $|\phi|$, which changes the value of the integral from $\nu_\phi(E)$ to $\nu_{|\phi|}(E)$, which we know is finite, hence the integrand in the above expression is integrable as a function on $\Gamma^{\times n} \times G^{\times(n+1)}$. This allows us to use Fubini's Theorem to further rewrite the expression for $\nu_\phi(E)$:

$\nu_\phi(E)$

$$\begin{aligned}
&= \int_{G^{\times(n+1)}} \int_{\Gamma^{\times n}} \mathbb{1}_{\Omega^{\times(n+1)}} \left(\left(\left(\prod_{k=j}^{n+1} s_k \right) \left(\prod_{k=j}^{n} t_k \right)^{-1} \right)_{j=1}^{n+1} \right) \mathbb{1}_E \left((s_j)_{j=1}^n \right) \phi \left((t_j)_{j=1}^n \right) d\mu_{\Gamma^{\times n}}(t_1, \dots, t_n) d(s_1, \dots, s_{n+1}) \\
&= \int_{G^{\times(n+1)}} \mathbb{1}_E \left((s_j)_{j=1}^n \right) \sum_{(t_1,\dots,t_n)\in\Gamma^{\times n}} \mathbb{1}_{\Omega^{\times(n+1)}} \left(\left(\left(\prod_{k=j}^{n+1} s_k \right) \left(\prod_{k=j}^{n} t_k \right)^{-1} \right)_{j=1}^{n+1} \right) \phi \left((t_j)_{j=1}^n \right) d(s_1, \dots, s_{n+1}) \\
&= \int_{G^{\times(n+1)}} \mathbb{1}_E \left((s_j)_{j=1}^n \right) \sum_{(t_1,\dots,t_n)\in\Gamma^{\times n}} \prod_{j=1}^{n+1} \mathbb{1}_\Omega \left(\left(\prod_{k=j}^{n+1} s_k \right) \left(\prod_{k=j}^{n} t_k \right)^{-1} \right) \phi \left((t_j)_{j=1}^n \right) d(s_1, \dots, s_{n+1}).
\end{aligned} \tag{405}$$

We now claim that given $(s_1, \dots, s_{n+1}) \in G^{\times(n+1)}$ and $(t_1, \dots, t_n) \in \Gamma^{\times n}$, the condition

$$\left(\prod_{k=j}^{n+1} s_k \right) \left(\prod_{k=j}^{n} t_k \right)^{-1} \in \Omega \tag{406}$$

holds for all $j \in \{1, \dots, n+1\}$ if and only if $s_{n+1} \in \Omega$ and $t_1, \dots, t_n$ are given by $t_j = \gamma_j(s_1, \dots, s_{n+1})$, where

$$\gamma_j(s_1, \dots, s_{n+1}) = \gamma \left(\prod_{m=j}^{n+1} s_m \right) \gamma \left(\prod_{m=j+1}^{n+1} s_m \right)^{-1}. \tag{407}$$

Note that for $j = n+1$ the statement of the condition simply reads $s_{n+1} \in \Omega$, so $s_{n+1} \in \Omega$ is both necessary and sufficient for the condition to hold for $j = n+1$. Assuming $s_{n+1} \in \Omega$, we have that

$$\prod_{k=j}^{n} \gamma_k(s_1, \dots, s_{n+1}) = \gamma \left(\prod_{m=j}^{n+1} s_m \right) \tag{408}$$

for all $j \in \{1, \dots, n\}$. This follows from the following induction argument. The above expression is clearly true for $j = n$, because $s_{n+1} \in \Omega$, hence $\gamma(s_{n+1}) = e$. Assuming the above expression holds for some $j \in \{2, \dots, n\}$, we have that

$$\prod_{k=j-1}^{n} \gamma_k(s_1, \ldots, s_{n+1}) = \gamma_{j-1}(s_1, \ldots, s_{n+1}) \prod_{k=j}^{n} \gamma_k(s_1, \ldots, s_{n+1})$$
$$= \gamma\left(\prod_{m=j-1}^{n+1} s_m\right) \gamma\left(\prod_{m=j}^{n+1} s_m\right)^{-1} \gamma\left(\prod_{m=j}^{n+1} s_m\right) = \gamma\left(\prod_{m=j-1}^{n+1} s_m\right), \tag{409}$$

so the expression holds for $j-1$. It follows that the expression holds for all $j \in \{1, \ldots, n\}$. This implies that for all $j \in \{1, \ldots, n\}$:

$$\left(\prod_{k=j}^{n+1} s_k\right)\left(\prod_{k=j}^{n} \gamma_k(s_1, \ldots, s_{n+1})\right)^{-1} = \omega\left(\prod_{k=j}^{n+1} s_k\right) \gamma\left(\prod_{k=j}^{n+1} s_k\right) \gamma\left(\prod_{k=j}^{n+1} s_k\right)^{-1} = \omega\left(\prod_{k=j}^{n+1} s_k\right) \in \Omega. \tag{410}$$

This shows that the condition holds for all $j \in \{1, \ldots, n+1\}$ if $s_{n+1} \in \Omega$ and $t_j = \gamma_j(s_1, \ldots, s_{n+1})$. Now suppose that the condition holds for all $j \in \{1, \ldots, n+1\}$. Then $s_{n+1} \in \Omega$ and we have for all $j \in \{1, \ldots, n\}$ that

$$\left(\prod_{k=j}^{n+1} s_k\right) = \left(\prod_{k=j}^{n+1} s_k\right)\left(\prod_{k=j}^{n} t_k\right)^{-1}\left(\prod_{k=j}^{n} t_k\right) \tag{411}$$

with $\left(\prod_{k=j}^{n+1} s_k\right)\left(\prod_{k=j}^{n} t_k\right)^{-1} \in \Omega$ and $\left(\prod_{k=j}^{n} t_k\right) \in \Gamma$. But then

$$\prod_{k=j}^{n} t_k = \gamma\left(\prod_{m=j}^{n+1} s_m\right) \tag{412}$$

for all $j \in \{1, \ldots, n\}$. This identity also holds for $j = n+1$ if we interpret an empty product as the identity e and recall that $\gamma(s_{n+1}) = e$ because $s_{n+1} \in \Omega$. It follows that for all $j \in \{1, \ldots, n\}$ we have that

$$t_j = \left(\prod_{k=j}^{n} t_k\right)\left(\prod_{k=j+1}^{n} t_k\right)^{-1} = \gamma\left(\prod_{m=j}^{n+1} s_m\right) \gamma\left(\prod_{m=j+1}^{n+1} s_m\right)^{-1} = \gamma_j(s_1, \ldots, s_{n+1}). \tag{413}$$

So the condition

$$\left(\prod_{k=j}^{n+1} s_k\right)\left(\prod_{k=j}^{n} t_k\right)^{-1} \in \Omega \tag{414}$$

holds for all $j \in \{1, \ldots, n+1\}$ if and only if $s_{n+1} \in \Omega$ and $t_j = \gamma_j(s_1, \ldots, s_{n+1})$ for all $j \in \{1, \ldots, n\}$. Using this it follows that

$$\nu_\phi(E) = \int_{G^{\times(n+1)}} \mathbb{1}_E\left((s_j)_{j=1}^n\right) \mathbb{1}_\Omega(s_{n+1}) \phi\left((\gamma_j(s_1, \ldots, s_{n+1}))_{j=1}^n\right) d(s_1, \ldots, s_{n+1})$$
$$= \int_E \int_\Omega \phi\left(\left(\gamma\left(\prod_{m=j}^{n+1} s_m\right) \gamma\left(\prod_{m=j+1}^{n+1} s_m\right)^{-1}\right)_{j=1}^{n}\right) ds_{n+1} d(s_1, \ldots, s_n) \tag{415}$$
$$= \int_E \hat{\phi}(s_1, \ldots, s_n) d(s_1, \ldots, s_n).$$

From the above expression we see that $\hat{\phi}$ is integrable. Moreover, we have that

$$
\begin{aligned}
\left\|\hat{\phi}\right\|_1 &= \int_{G^{\times n}} \left|\hat{\phi}(s_1,\ldots,s_n)\right| d(s_1,\ldots,s_n) \\
&= \int_{G^{\times n}} \left| \int_\Omega \phi\left(\left(\gamma\left(\prod_{m=j}^{n+1} s_m\right)\gamma\left(\prod_{m=j+1}^{n+1} s_m\right)^{-1}\right)_{j=1}^n\right) ds_{n+1} \right| d(s_1,\ldots,s_n) \\
&\le \int_{G^{\times n}} \int_\Omega \left| \phi\left(\left(\gamma\left(\prod_{m=j}^{n+1} s_m\right)\gamma\left(\prod_{m=j+1}^{n+1} s_m\right)^{-1}\right)_{j=1}^n\right) \right| ds_{n+1} d(s_1,\ldots,s_n) \\
&= \int_{G^{\times n}} (|\phi|\check{)}(s_1,\ldots,s_n) d(s_1,\ldots,s_n) = \nu_{|\phi|}(G^{\times n}) \\
&= \int_{G^{\times(n+1)}} \int_{\Gamma^{\times n}} \mathbb{1}_{\Omega^{\times(n+1)}}\left((s_j)_{j=1}^{n+1}\right) \mathbb{1}_{G^{\times n}}\left((s_j)_{j=1}^n\, t\, \left(s_{j+1}^{-1}\right)_{j=1}^n\right) |\phi(t)| d\mu_{\Gamma^{\times n}}(t) d(s_1,\ldots,s_{n+1}) \\
&= \int_{\Omega^{\times(n+1)}} \sum_{t\in\Gamma^{\times n}} |\phi(t)| d(s_1,\ldots,s_{n+1}) = \int_{\Omega^{\times(n+1)}} \|\phi\|_1 d(s_1,\ldots,s_{n+1}) = \|\phi\|_1 .
\end{aligned}
\tag{416}
$$

Because $\hat{\phi}$ is integrable, we have that ν_ϕ is indeed a complex bounded Borel measure. □

The following result identifies the adjoint map $L^1(G^{\times n}) \to l^1(\Gamma^{\times n})$ of $l^\infty(\Gamma^{\times n}) \to L^\infty(G^{\times n})$, $\phi \mapsto \hat{\phi}$. The $n = 1$ version of this result is used implicitly in Theorem 2.3 of Haagerup's paper [18].

Lemma 5.4. *For $f \in L^1(G^{\times n})$ define $\check{f} : \Gamma^{\times n} \to \mathbb{C}$ by*

$$
\check{f}(t_1,\ldots,t_n) = \int_{G^{\times(n+1)}} \mathbb{1}_{\Omega^{\times(n+1)}}(s_1,\ldots,s_{n+1}) f\left(\left(s_j t_j s_{j+1}^{-1}\right)_{j=1}^n\right) d(s_1,\ldots,s_{n+1}). \tag{417}
$$

$f \mapsto \check{f}$ is a well-defined linear contraction $L^1(G^{\times n}) \to l^1(\Gamma^{\times n})$. For all $\phi \in l^\infty(\Gamma^{\times n})$ and $f \in L^1(G^{\times n})$ we have that

$$
T_f(\hat{\phi}) = T_{\check{f}}(\phi), \tag{418}
$$

where T_f and $T_{\check{f}}$ are as defined in subsection 3.5.

Proof. Parts of this proof will be very similar to the proof of Lemma 5.3 and we will refer to that proof for several steps of this proof. In particular we will see that in the calculations in this proof, f will play the same role as $\mathbb{1}_E$ did in the proof of Lemma 5.3. Let $\phi \in l^\infty(\Gamma^{\times n})$ and $f \in L^1(G^{\times n})$. For now we will consider a fixed version of f and assume that both ϕ and f take values in $[0,\infty)$. This will allow us to use Fubini's theorem. At some point we will show that we can drop the assumption of non-negativity and that $\check{f}$ does not depend on the chosen version of f. We have that

$$
\begin{aligned}
T_{\check{f}}(\phi) &= \int_{\Gamma^{\times n}} \check{f}(t)\phi(t) d\mu_{\Gamma^{\times n}}(t) \\
&= \int_{\Gamma^{\times n}} \int_{G^{\times(n+1)}} \mathbb{1}_{\Omega^{\times(n+1)}}(s_1,\ldots,s_{n+1}) f\left(\left(s_j t_j s_{j+1}^{-1}\right)_{j=1}^n\right) d(s_1,\ldots,s_{n+1}) \phi(t_1,\ldots,t_n) d\mu_{\Gamma^{\times n}}(t_1,\ldots,t_n) \\
&= \int_{\Gamma^{\times n}} \int_G \cdots \int_G \mathbb{1}_{\Omega^{\times(n+1)}}\left((s_j)_{j=1}^{n+1}\right) f\left(\left(s_j t_j s_{j+1}^{-1}\right)_{j=1}^n\right) \phi\left((t_j)_{j=1}^n\right) ds_1 \ldots ds_{n+1} d\mu_{\Gamma^{\times n}}(t_1,\ldots,t_n).
\end{aligned}
\tag{419}
$$

Here we used Fubini's theorem to write the integral over $G^{\times(n+1)}$ as a repeated integral. Next we apply the same substitutions as in the proof of Lemma 5.3. So, we first substitute the integration variables s_j by $s_j \left(\prod_{k=j}^n t_k\right)^{-1}$ for each $j \in \{1,\ldots,n+1\}$. As in the proof of Lemma 5.3, performing all these substitutions results in the substitution of the expression $s_j t_j s_{j+1}^{-1}$ (for $j \in \{1, \dots, n\}$) by $s_j s_{j+1}^{-1}$. Using that μ is right-invariant (G is unimodular), it follows that

$$
\begin{aligned}
&T_{\check{f}}(\phi) \\
&= \int_{\Gamma^{\times n}} \int_G \cdots \int_G \mathbb{1}_{\Omega^{\times (n+1)}} \left(\left(s_j \left(\prod_{k=j}^{n} t_k \right)^{-1} \right)_{j=1}^{n+1} \right) f\left(\left(s_j s_{j+1}^{-1} \right)_{j=1}^{n} \right) \phi\left((t_j)_{j=1}^{n} \right) ds_1 \dots ds_{n+1} d\mu_{\Gamma^{\times n}}(t_1, \dots, t_n).
\end{aligned}
\tag{420}
$$

Next, just like in the proof of Lemma 5.3 we substitute s_j by $s_j s_{j+1}$ in the integral over s_j. We do this for each $j \in \{1, \dots, n\}$ in order of increasing j. Performing all these substitutions results in the overall substitution of every instance of s_j by $\prod_{k=j}^{n+1} s_k$ (for $j \in \{1, \dots, n+1\}$) in the integrand of the expression for $T_{\check{f}}(\phi)$. Since

$$
\left(\prod_{k=j}^{n+1} s_k \right) \left(\prod_{k=j+1}^{n+1} s_k \right)^{-1} = s_j \left(\prod_{k=j+1}^{n+1} s_k \right) \left(\prod_{k=j+1}^{n+1} s_k \right)^{-1} = s_j \tag{421}
$$

for all $j \in \{1, \dots, n\}$, it follows that

$$
\begin{aligned}
&T_{\check{f}}(\phi) \\
&= \int_{\Gamma^{\times n}} \int_G \cdots \int_G \mathbb{1}_{\Omega^{\times (n+1)}} \left(\left(\left(\prod_{k=j}^{n+1} s_k \right) \left(\prod_{k=j}^{n} t_k \right)^{-1} \right)_{j=1}^{n+1} \right) f\left((s_j)_{j=1}^{n} \right) \phi\left((t_j)_{j=1}^{n} \right) ds_1 \dots ds_{n+1} d\mu_{\Gamma^{\times n}}(t_1, \dots, t_n).
\end{aligned}
\tag{422}
$$

Next we use Fubini's theorem to rewrite the above expression for $T_{\check{f}}(\phi)$.

$$
\begin{aligned}
&T_{\check{f}}(\phi) \\
&= \int_{G^{\times (n+1)}} \int_{\Gamma^{\times n}} \mathbb{1}_{\Omega^{\times (n+1)}} \left(\left(\left(\prod_{k=j}^{n+1} s_k \right) \left(\prod_{k=j}^{n} t_k \right)^{-1} \right)_{j=1}^{n+1} \right) f\left((s_j)_{j=1}^{n} \right) \phi\left((t_j)_{j=1}^{n} \right) d\mu_{\Gamma^{\times n}}(t_1, \dots, t_n) d(s_1, \dots, s_{n+1}) \\
&= \int_{G^{\times (n+1)}} f\left((s_j)_{j=1}^{n} \right) \sum_{(t_1, \dots, t_n) \in \Gamma^{\times n}} \mathbb{1}_{\Omega^{\times (n+1)}} \left(\left(\left(\prod_{k=j}^{n+1} s_k \right) \left(\prod_{k=j}^{n} t_k \right)^{-1} \right)_{j=1}^{n+1} \right) \phi\left((t_j)_{j=1}^{n} \right) d(s_1, \dots, s_{n+1}) \\
&= \int_{G^{\times (n+1)}} f\left((s_j)_{j=1}^{n} \right) \sum_{(t_1, \dots, t_n) \in \Gamma^{\times n}} \prod_{j=1}^{n+1} \mathbb{1}_{\Omega} \left(\left(\prod_{k=j}^{n+1} s_k \right) \left(\prod_{k=j}^{n} t_k \right)^{-1} \right) \phi\left((t_j)_{j=1}^{n} \right) d(s_1, \dots, s_{n+1}).
\end{aligned}
\tag{423}
$$

Just like in the proof of Lemma 5.3, we note that $\prod_{j=1}^{n+1} \mathbb{1}_{\Omega} \left(\left(\prod_{k=j}^{n+1} s_k \right) \left(\prod_{k=j}^{n} t_k \right)^{-1} \right) = 1$ (and equals 0 otherwise) if and only if $s_{n+1} \in \Omega$ and

$$
t_j = \gamma \left(\prod_{m=j}^{n+1} s_m \right) \gamma \left(\prod_{m=j+1}^{n+1} s_m \right)^{-1} \tag{424}
$$

for all $j \in \{1, \ldots, n\}$ (see the proof of Lemma 5.3 for the details). It follows that

$$
\begin{aligned}
T_{\check{f}}(\phi) &= \int_{G^{\times(n+1)}} f\left((s_j)_{j=1}^n\right) \mathbb{1}_\Omega(s_{n+1}) \phi\left(\left(\gamma\left(\prod_{m=j}^{n+1} s_m\right) \gamma\left(\prod_{m=j+1}^{n+1} s_m\right)^{-1}\right)_{j=1}^n\right) d(s_1, \ldots, s_{n+1}) \\
&= \int_{G^{\times n}} f\left((s_j)_{j=1}^n\right) \int_\Omega \phi\left(\left(\gamma\left(\prod_{m=j}^{n+1} s_m\right) \gamma\left(\prod_{m=j+1}^{n+1} s_m\right)^{-1}\right)_{j=1}^n\right) ds_{n+1} d(s_1, \ldots, s_n) \\
&= \int_{G^{\times n}} f\left((s_j)_{j=1}^n\right) \hat{\phi}(s_1, \ldots, s_n) d(s_1, \ldots, s_n) = T_f(\hat{\phi}).
\end{aligned}
\tag{425}
$$

Since $\hat{\phi} \in L^\infty(G^{\times n})$ and $f \in L^1(G^{\times n})$, we know that $T_{\check{f}}(\phi) = T_f(\hat{\phi})$ is finite for non-negative f and ϕ. The preceding argument can now be repeated without the assumption of non-negativity. At any time where Fubini's theorem needs to be used, we can check that the integrand is an integrable function by replacing f and ϕ by their absolute values. This shows that each application of Fubini's theorem is justified. So we have that

$$T_{\check{f}}(\phi) = T_f(\hat{\phi}) \tag{426}$$

for all $f \in L^1(G^{\times n})$ and $\phi \in l^\infty(\Gamma^{\times n})$. It follows that

$$\left|T_{\check{f}}(\phi)\right| = \left|T_f(\hat{\phi})\right| \leq \|f\|_1 \left\|\hat{\phi}\right\|_\infty \leq \|f\|_1 \|\phi\|_\infty \tag{427}$$

for all $f \in L^1(G^{\times n})$ and $\phi \in l^\infty(\Gamma^{\times n})$, where we used Lemma 5.1. From the definition of $\check{f}$ we see that

$$\left|\check{f}\right| \leq \check{|f|} \tag{428}$$

holds pointwise. It follows that for all $f \in L^1(G^{\times n})$ and all $t \in \Gamma^{\times n}$

$$\left|\check{f}(t)\right| \leq \left\|\check{f}\right\|_1 \leq \left\|\check{|f|}\right\|_1 = \left|T_{\check{|f|}}(1)\right| \leq \||f|\|_1 \|1\|_\infty = \|f\|_1. \tag{429}$$

This shows that $\check{f}$ does not depend on the chosen version of f and $f \mapsto \check{f}$ is a well-defined map $L^1(G^{\times n}) \to l^1(\Gamma^{\times n})$. From the definition of $\check{f}$ it is clear that the map $f \mapsto \check{f}$ is linear and the above equation shows that it is a contraction. □

The following result, inspired by part of the proof of Theorem 2.3 in [18], shows that $\phi \mapsto \hat{\phi}$ is continuous with respect to the $\sigma(l^\infty(\Gamma^{\times n}), l^1(\Gamma^{\times n}))$ and $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topologies.

Theorem 5.5. *The map $l^\infty(\Gamma^{\times n}) \to L^\infty(G^{\times n})$, $\phi \mapsto \hat{\phi}$ is continuous with respect to the $\sigma(l^\infty(\Gamma^{\times n}), l^1(\Gamma^{\times n}))$ and $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topologies. In other words if ϕ_ι is a net in $l^\infty(\Gamma^{\times n})$ that converges to $\phi \in l^\infty(\Gamma^{\times n})$ in $\sigma(l^\infty(\Gamma^{\times n}), l^1(\Gamma^{\times n}))$ topology, then $\hat{\phi}_\iota$ converges to $\hat{\phi}$ in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology.*

Proof. Let ϕ_ι be a net in $l^\infty(\Gamma^{\times n})$ that converges to $\phi \in l^\infty(\Gamma^{\times n})$ in $\sigma(l^\infty(\Gamma^{\times n}), l^1(\Gamma^{\times n}))$ topology. So $T_g(\phi_\iota)$ converges to $T_g(\phi)$ for every $g \in l^1(\Gamma^{\times n})$. Let $f \in L^1(G^{\times n})$. Using Lemma 5.4, we have that

$$T_f(\hat{\phi}_\iota) = T_{\check{f}}(\phi_\iota) \tag{430}$$

converges to

$$T_{\check{f}}(\phi) = T_f(\hat{\phi}). \tag{431}$$

This shows that $\hat{\phi}_\iota$ converges to $\hat{\phi}$ in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology, which finishes the proof. □

Theorem 5.5 can be combined with Theorem 3.24 to obtain the following result.

Theorem 5.6. *Let ϕ_ι be a net in $l^\infty(\Gamma^{\times n})$ that is bounded in $\|\cdot\|_\infty$ and converges to $\phi \in l^\infty(\Gamma^{\times n})$ either in $\sigma(l^\infty(\Gamma^{\times n}), l^1(\Gamma^{\times n}))$ topology or uniformly on compact sets. Let $F \in L^1(G^{\times(n+1)})$. Then $F\tilde{*}\hat{\phi}_\iota$ converges to $F\tilde{*}\hat{\phi}$ both in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology and uniformly on compact sets.*

Proof. Let ϕ_ι be a net in $l^\infty(\Gamma^{\times n})$ that is bounded in $\|\cdot\|_\infty$ and let $\phi \in l^\infty(\Gamma^{\times n})$. By Lemma 3.21, if ϕ_ι converges to ϕ uniformly on compact sets, then it also converges to ϕ in $\sigma(l^\infty(\Gamma^{\times n}), l^1(\Gamma^{\times n}))$ topology. So, we assume that ϕ_ι converges to ϕ in $\sigma(l^\infty(\Gamma^{\times n}), l^1(\Gamma^{\times n}))$ topology. By Theorem 5.5 we have that $\hat{\phi}_\iota$ converges to $\hat{\phi}$ in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology. The net $\hat{\phi}_\iota$ is bounded in $\|\cdot\|_\infty$ by Lemma 5.1. It now follows from Theorem 3.24 that $F\tilde{*}\hat{\phi}_\iota$ converges to $F\tilde{*}\hat{\phi}$ uniformly on compact sets. The net $F\tilde{*}\hat{\phi}_\iota$ is bounded in $\|\cdot\|_\infty$ by Lemma 3.20. It follows by Lemma 3.21 that the convergence of $F\tilde{*}\hat{\phi}_\iota$ to $F\tilde{*}\hat{\phi}$ also holds in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology. ☐

In the following lemma we prove a useful identity for $\hat{\phi}$. For $n = 1$ this identity also appears in the proof of Lemma 2.1 of [18] and our proof of this identity is similar.

Lemma 5.7. *For $\phi \in l^\infty(\Gamma^{\times n})$ and $x_0, \dots, x_n \in G$ we have that*

$$\tilde{\hat{\phi}}(x_0, \dots, x_n) = \int_\Omega \tilde{\phi}\left((\gamma(x_j s_{n+1}))_{j=0}^n\right) ds_{n+1}. \tag{432}$$

Proof. Recall that $\mu|_\Omega$ is τ_x invariant for all $x \in G$. Recall also the identity

$$\gamma(yx^{-1}\tau_x(z)) = \gamma(yz)\gamma(xz)^{-1} \tag{433}$$

for $x, y \in G$ and $z \in \Omega$. Note that

$$\prod_{m=j}^n x_{m-1}x_m^{-1} = x_{j-1}x_n^{-1} \tag{434}$$

for all $j \in \{1, \dots, n+1\}$, where an empty product is to be interpreted as e. It follows that

$$\begin{aligned}
&\tilde{\hat{\phi}}(x_0, \dots, x_n) = \hat{\phi}\left(\left(x_{j-1}x_j^{-1}\right)_{j=1}^n\right) = \int_\Omega \phi\left(\left(\gamma\left(x_{j-1}x_n^{-1}s_{n+1}\right)\gamma\left(x_j x_n^{-1}s_{n+1}\right)^{-1}\right)_{j=1}^n\right) ds_{n+1} \\
&= \int_\Omega \phi\left(\left(\gamma\left(x_{j-1}x_n^{-1}\tau_{x_n}(s_{n+1})\right)\gamma\left(x_j x_n^{-1}\tau_{x_n}(s_{n+1})\right)^{-1}\right)_{j=1}^n\right) ds_{n+1} \\
&= \int_\Omega \phi\left(\left(\gamma\left(x_{j-1}s_{n+1}\right)\gamma\left(x_n s_{n+1}\right)^{-1}\gamma\left(x_n s_{n+1}\right)\gamma\left(x_j s_{n+1}\right)^{-1}\right)_{j=1}^n\right) ds_{n+1} \\
&= \int_\Omega \phi\left(\left(\gamma\left(x_{j-1}s_{n+1}\right)\gamma\left(x_j s_{n+1}\right)^{-1}\right)_{j=1}^n\right) ds_{n+1} = \int_\Omega \tilde{\phi}\left((\gamma(x_j s_{n+1}))_{j=0}^n\right) ds_{n+1}.
\end{aligned} \tag{435}$$

☐

Lemma 5.8. *Let H be a Hilbert space and let $a : \Gamma \to H$ and $b : \Gamma \to \mathcal{B}(H)$ be bounded, i.e.*

$$\|a\|_\infty := \sup_{t\in\Gamma} \|a(t)\| < \infty \tag{436}$$

and

$$\|b\|_\infty := \sup_{t\in\Gamma} \|b(t)\| < \infty. \tag{437}$$

Let $\hat{a} : G \to L^2(\Omega; H)$ and $\hat{b} : G \to \mathcal{B}(L^2(\Omega; H))$ be given by

$$\hat{a}(x)(s) = a(\gamma(xs)) \tag{438}$$

and

$$\hat{b}(x)(f)(s) = b(\gamma(xs))(f(s)), \tag{439}$$

where $s \in \Omega$, $x \in G$ and $f \in L^2(\Omega; H)$. Then $\hat{a}$ and $\hat{b}$ are well-defined and bounded with

$$\|\hat{a}\|_\infty := \sup_{x\in G} \|\hat{a}(x)\|_2 := \sup_{x\in G} \left(\int_\Omega \|\hat{a}(x)(s)\|^2 ds \right)^{\frac{1}{2}} \le \|a\|_\infty \tag{440}$$

and

$$\left\|\hat{b}\right\|_\infty := \sup_{x\in G} \left\|\hat{b}(x)\right\| \le \|b\|_\infty. \tag{441}$$

If $b_1, \ldots, b_n : \Gamma \to \mathcal{B}(H)$ are bounded, then for all $f \in L^2(\Omega; H)$, $s \in \Omega$ and $x_1, \ldots, x_n \in G$ we have that

$$\left(\prod_{j=1}^n \hat{b}_j(x_j) \right)(f)(s) = \left(\prod_{j=1}^n b_j(\gamma(x_j s)) \right)(f(s)). \tag{442}$$

Proof. For fixed $x \in G$ note that $\Omega \to H$, $s \mapsto a(\gamma(xs))$ is Borel measurable because γ is measurable (and a is automatically measurable because Γ is discrete). So $\hat{a}(x)$ is a well-defined measurable function $\Omega \to H$. We have that

$$\|\hat{a}(x)\|_2 = \left(\int_\Omega \|\hat{a}(x)(s)\|^2 ds \right)^{\frac{1}{2}} = \left(\int_\Omega \|a(\gamma(xs))\|^2 ds \right)^{\frac{1}{2}} \le \left(\int_\Omega \|a\|_\infty^2 ds \right)^{\frac{1}{2}} = \|a\|_\infty < \infty. \tag{443}$$

So $\hat{a}(x) \in L^2(\Omega; H)$ for all $x \in G$ and the above inequality shows that

$$\|\hat{a}\|_\infty \le \|a\|_\infty. \tag{444}$$

For fixed $x \in G$ and $f \in L^2(\Omega; H)$, the function $\Omega \to H$, $s \mapsto b(\gamma(xs))(f(s))$ is Borel measurable. This is because $s \mapsto b(\gamma(xs))$ is measurable (γ is measurable and b is automatically measurable because Γ is discrete), f is measurable, and the evaluation map $\mathcal{B}(H) \times H$, $(B, h) \mapsto B(h)$ is jointly continuous, hence Borel measurable. So $\hat{b}(x)(f)$ is a well-defined measurable function $\Omega \to H$. We have that

$$\begin{aligned}
\left\|\hat{b}(x)(f)\right\|_2 &= \left(\int_\Omega \left\|\hat{b}(x)(f)(s)\right\|^2 ds \right)^{\frac{1}{2}} = \left(\int_\Omega \|b(\gamma(xs))(f(s))\|^2 ds \right)^{\frac{1}{2}} \le \left(\int_\Omega \|b(\gamma(sx))\|^2 \|f(s)\|^2 ds \right)^{\frac{1}{2}} \\
&\le \left(\int_\Omega \|b\|_\infty^2 \|f(s)\|^2 ds \right)^{\frac{1}{2}} \\
&= \|b\|_\infty \|f\|_2.
\end{aligned} \tag{445}$$

This shows that $\hat{b}(x)(f) \in L^2(\Omega; H)$ for all $x \in G$ and $f \in L^2(\Omega; H)$. Linearity of $\hat{b}(x)$ follows from linearity of b and the above inequality shows that $\hat{b}(x) \in \mathcal{B}(L^2(\Omega; H))$ for all $x \in G$ with

$$\left\|\hat{b}\right\|_\infty = \sup_{x\in G} \left\|\hat{b}(x)\right\| \le \|b\|_\infty. \tag{446}$$

We prove the identity

$$\left(\prod_{j=1}^n \hat{b}_j(x_j) \right)(f)(s) = \left(\prod_{j=1}^n b_j(\gamma(x_j s)) \right)(f(s)) \tag{447}$$

using induction on n. For $n = 1$ the identity follows directly from the definition of $\hat{b}_1$. Assume that for some fixed $n \in \mathbb{N}$ the above identity holds for all bounded $b_1, \ldots, b_n : \Gamma \to \mathcal{B}(H)$ and all $f \in L^2(\Omega; H)$, $s \in \Omega$,

$x_1, \ldots, x_n \in G$. Let $b_1, \ldots, b_{n+1} : \Gamma \to \mathcal{B}(H)$ be bounded and let $f \in L^2(\Omega; H)$, $s \in \Omega$, $x_1, \ldots, x_{n+1} \in G$. Then we have that

$$\begin{aligned}\left(\prod_{j=1}^{n+1} \hat{b}_j(x_j)\right)(f)(s) &= \hat{b}_1(x_1)\left(\left(\prod_{j=2}^{n+1} \hat{b}_j(x_j)\right)(f)\right)(s) = b_1(\gamma(x_1 s))\left(\left(\prod_{j=2}^{n+1} \hat{b}_j(x_j)\right)(f)(s)\right) \\ &= b_1(\gamma(x_1 s))\left(\left(\prod_{j=2}^{n+1} b_j(\gamma(x_j s))\right)(f(s))\right) = \left(\prod_{j=1}^{n+1} b_j(\gamma(x_j s))\right)(f(s)).\end{aligned} \tag{448}$$

This shows that the identity holds for $n+1$, hence it follows by induction that it holds for all $n \in \mathbb{N}$. $\square$

5.3 Construction of Fourier multipliers

Using Lemma 5.7, Corollary 4.16, Remark 4.7 and Lemma 5.8 we can come close to proving that $\hat{\phi}$ is the symbol of a completely bounded Fourier multiplier, whenever ϕ is the symbol of a completely bounded Fourier multiplier. Such an argument would require continuity of $\hat{\phi}$ and measurability of the functions $\hat{a}$ and $\hat{b}$ defined in Lemma 5.8. In the $n = 1$ case the function $\hat{\phi}$ is continuous for any $\phi \in l^\infty(\Gamma)$. This is because $\hat{\phi}$ can be written as the convolution

$$\hat{\phi} = (\phi \circ \gamma) * \mathbb{1}_{\Omega^{-1}} \tag{449}$$

with $\phi \circ \gamma \in L^\infty(G)$ and $\mathbb{1}_{\Omega^{-1}} \in L^1(G)$. This argument does not seem to easily generalize to arbitrary n for general $\phi \in l^\infty(\Gamma^{\times n})$. As such it is not clear if $\hat{\phi}$ is always continuous when $n \geq 2$. To remedy this, we will use $F \tilde{*} \hat{\phi}$ instead of $\hat{\phi}$. Here $F \in L^1(G^{\times(n+1)})$ and $F \tilde{*} \hat{\phi}$ is as defined in Definition 3.29. We will show that a result similar to Lemma 5.7 still holds for $F \tilde{*} \hat{\phi}$. We will also need to modify the functions $\hat{a}$ and $\hat{b}$, which will be done by applying Lemma 4.18.

The following result shows that an identity similar to the one in Lemma 5.7 holds for $F \tilde{*} \hat{\phi}$.

Lemma 5.9. *For $\phi \in l^\infty(\Gamma^{\times n})$, $F \in L^1(G^{\times(n+1)})$ and $x_0, \ldots, x_n \in G$ we have that*

$$(F \tilde{*} \hat{\phi})\tilde{}(x_0, \ldots, x_n) = \int_{G^{\times(n+1)}} \int_\Omega F(t_0, \ldots, t_n) \tilde{\phi}\left(\left(\gamma(t_j^{-1} x_j s_{n+1})\right)_{j=0}^n\right) ds_{n+1} d(t_0, \ldots, t_n). \tag{450}$$

Proof. Combining Lemma 5.7 and Lemma 3.20 we have that

$$\begin{aligned}(F \tilde{*} \hat{\phi})\tilde{}(x_0, \ldots, x_n) &= (F * \tilde{\hat{\phi}})(x_0, \ldots, x_n) = \int_{G^{\times(n+1)}} F(t_0, \ldots, t_n) \tilde{\hat{\phi}}\left(\left(t_j^{-1} x_j\right)_{j=0}^n\right) d(t_0, \ldots, t_n) \\ &= \int_{G^{\times(n+1)}} F(t_0, \ldots, t_n) \int_\Omega \tilde{\phi}\left(\left(\gamma(t_j^{-1} x_j s_{n+1})\right)_{j=0}^n\right) ds_{n+1} d(t_0, \ldots, t_n) \\ &= \int_{G^{\times(n+1)}} \int_\Omega F(t_0, \ldots, t_n) \tilde{\phi}\left(\left(\gamma(t_j^{-1} x_j s_{n+1})\right)_{j=0}^n\right) ds_{n+1} d(t_0, \ldots, t_n).\end{aligned} \tag{451}$$

$\square$

In what follows we will mainly need the function $F \tilde{*} \hat{\phi}$, where F is of the form $F = \otimes_{j=0}^n f_j$ with $f_0, \ldots, f_n \in L^1(G)$.

Remark 5.2. Note that $\hat{a}$ and $\hat{b}$ as defined in Lemma 5.8 satisfy the requirements of Lemma 4.18 to define $f * \hat{a}$ and $f * \hat{b}$ for $f \in L^1(G)$ with H replaced by the Hilbert space $L^2(\Omega; H)$. Indeed $\hat{a}$ and $\hat{b}$ were shown to be bounded and from the definition it is clear that $(x, s) \mapsto \hat{a}(x)(s)$ and $(x, s) \mapsto \hat{b}(x)(h)(s)$ are measurable for any $h \in L^2(\Omega; H)$. It follows by Fubini's theorem that $x \mapsto \langle g, \hat{a}(x) \rangle$ and $x \mapsto \left\langle g, \hat{b}(x)(h) \right\rangle$ are measurable for all $g, h \in L^2(\Omega; H)$.

Remark 5.3. Note that both G and Γ are σ-finite measure spaces, because they are countable unions of the measure 1 sets $\{t\}$ and Ωt respectively for $t \in \Gamma$.

Theorem 5.10. *Let $\phi \in M_{cb}^n A(\Gamma)$ and $f_0, \ldots, f_n \in L^1(G)$, then $(\otimes_{j=0}^n f_j) \tilde{*} \hat{\phi} \in M_{cb}^n A(G)$ with*

$$\left\| (\otimes_{j=0}^n f_j) \tilde{*} \hat{\phi} \right\|_{M_{cb}^n A(G)} \leq \|\phi\|_{M_{cb}^n A(\Gamma)} \prod_{j=0}^n \|f_j\|_1. \tag{452}$$

Proof. Let $\epsilon > 0$ and $r = \|\phi\|_{M_{cb}^n A(\Gamma)} + \epsilon > \|\phi\|_{M_{cb}^n A(\Gamma)}$. Because Γ is a locally compact group with σ-finite Haar measure and $\phi \in C_b(\Gamma^{\times n})$, Corollary 4.16 ensures the existence of bounded functions $a_0, a_n : \Gamma \to l^2(\mathbb{N})$ and $a_j : \Gamma \to \mathcal{B}(l^2(\mathbb{N}))$ for $j \in \{1, \ldots, n-1\}$ such that

$$\sup_{(t_0,\ldots,t_n)\in\Gamma^{\times(n+1)}} \prod_{j=0}^n \|a_j(t_j)\| < r \tag{453}$$

and for all $t_0, \ldots, t_n \in \Gamma$ we have that

$$\tilde{\phi}(t_0, \ldots, t_n) = \left\langle a_n(t_n), \left(\prod_{j=1}^{n-1} a_{n-j}(t_{n-j}) \right) (a_0(t_0)) \right\rangle. \tag{454}$$

Note that compared to Corollary 4.16, we have replaced essential boundedness by boundedness, the essential supremum by a regular supremum and almost everywhere equality in the above identity by pointwise equality. This is allowed because the Haar measure on Γ is the counting measure, hence the only measure 0 subset of Γ is the empty set. We now define $\hat{a}_0, \hat{a}_n : G \to L^2(\Omega; l^2(\mathbb{N}))$ and $\hat{a}_j : G \to \mathcal{B}(L^2(\Omega; l^2(\mathbb{N})))$ for $j \in \{1, \ldots, n-1\}$ as in Lemma 5.8. Then we know from Lemma 5.8 that

$$\|\hat{a}_j\|_\infty \leq \|a_j\|_\infty \tag{455}$$

for all $j \in \{0, \ldots, n\}$. We also know that

$$\left(\left(\prod_{j=1}^{n-1} \hat{a}_{n-j}(x_{n-j}) \right) (h) \right) (s) = \left(\prod_{j=1}^{n-1} a_{n-j}(\gamma(x_{n-j}s)) \right) (h(s)) \tag{456}$$

for all $h \in L^2(\Omega; l^2(\mathbb{N}))$, $x_1, \ldots, x_{n-1} \in G$ and $s \in \Omega$. It follows that

$$\begin{aligned}
\left\langle \hat{a}_n(x_n), \left(\prod_{j=1}^{n-1} \hat{a}_{n-j}(x_{n-j}) \right) (\hat{a}_0(x_0)) \right\rangle &= \int_\Omega \left\langle \hat{a}_n(x_n)(s), \left(\left(\prod_{j=1}^{n-1} \hat{a}_{n-j}(x_{n-j}) \right) (\hat{a}_0(x_0)) \right) (s) \right\rangle ds \\
&= \int_\Omega \left\langle \hat{a}_n(x_n)(s), \left(\prod_{j=1}^{n-1} a_{n-j}(\gamma(x_{n-j}s)) \right) (\hat{a}_0(x_0)(s)) \right\rangle ds \\
&= \int_\Omega \left\langle a_n(\gamma(x_n s)), \left(\prod_{j=1}^{n-1} a_{n-j}(\gamma(x_{n-j}s)) \right) (a_0(\gamma(x_0 s))) \right\rangle ds \\
&= \int_\Omega \tilde{\phi} \left((\gamma(x_j s)_{j=0}^n \right) ds.
\end{aligned} \tag{457}$$

We now consider $f_0 * \hat{a}_0 : G \to L^2(\Omega; l^2(\mathbb{N}))$, $\overline{f_n} * \hat{a}_n : G \to L^2(\Omega; l^2(\mathbb{N}))$ and $f_j * \hat{a}_j : G \to \mathcal{B}(L^2(\Omega; l^2(\mathbb{N})))$ for $j \in \{1, \ldots, n-1\}$ defined as in Lemma 4.18 (see Remark 5.2). These functions are all continuous, hence measurable, and bounded with

$$\|f_0 * \hat{a}_0\|_\infty \leq \|f_0\|_1 \|\hat{a}_0\|_\infty \leq \|f_0\|_1 \|a_0\|_\infty, \tag{458}$$

$$\left\|\overline{f_n} * \hat{a}_n\right\|_\infty \leq \left\|\overline{f_n}\right\|_1 \|\hat{a}_n\|_\infty \leq \|f_n\|_1 \|a_n\|_\infty \tag{459}$$

and

$$\|f_j * \hat{a}_j\|_\infty \le \|f_j\|_1 \|\hat{a}_j\|_\infty \le \|f_j\|_1 \|a_j\|_\infty \tag{460}$$

for all $j \in \{1, \ldots, n-1\}$. Using Lemma 5.9 and part (5) of Lemma 4.18 we also have that

$$\begin{aligned}
&\left\langle (\overline{f_n} * \hat{a}_n)(x_n), \left(\prod_{j=1}^{n-1} (f_{n-j} * \hat{a}_{n-j})(x_{n-j})\right) ((f_0 * \hat{a}_0)(x_0)) \right\rangle \\
&= \int_{G^{\times(n+1)}} \prod_{j=0}^{n} f_j(t_j) \left\langle \hat{a}_n(t_n^{-1}x_n), \left(\prod_{j=1}^{n-1} \hat{a}_{n-j}(t_{n-j}^{-1}x_{n-j})\right) (\hat{a}_0(t_0^{-1}x_0)) \right\rangle d(t_0, \ldots, t_n) \\
&= \int_{G^{\times(n+1)}} \prod_{j=0}^{n} f_j(t_j) \int_\Omega \tilde{\phi}\left(\left(\gamma(t_j^{-1}x_j s)\right)_{j=0}^{n}\right) ds d(t_0, \ldots, t_n) \\
&= \int_{G^{\times(n+1)}} \int_\Omega \prod_{j=0}^{n} f_j(t_j) \tilde{\phi}\left(\left(\gamma(t_j^{-1}x_j s)\right)_{j=0}^{n}\right) ds d(t_0, \ldots, t_n) \\
&= ((\otimes_{j=0}^{n} f_j)\tilde{*}\hat{\phi})\check{}(x_0, \ldots, x_n).
\end{aligned} \tag{461}$$

Note that $L^2(\Omega; l^2(\mathbb{N}))$ is separable (by Corollary 3.17). Also note that

$$\|\overline{f_n} * \hat{a}_n\|_\infty \prod_{j=0}^{n-1} \|f_j * \hat{a}_j\|_\infty \le \prod_{j=0}^{n} \|f_j\|_1 \|\hat{a}_j\|_\infty \le \left(\prod_{j=0}^{n} \|f_j\|_1\right)\left(\prod_{j=0}^{n} \|a_j\|_\infty\right) \le \left(\prod_{j=0}^{n} \|f_j\|_1\right) r. \tag{462}$$

Because $(\otimes_{j=0}^{n} f_j)\tilde{*}\hat{\phi} \in C_b(G^{\times n})$, it now follows from Corollary 4.16 that $(\otimes_{j=0}^{n} f_j)\tilde{*}\hat{\phi} \in M_{cb}^n A(G)$ with

$$\left\|(\otimes_{j=0}^{n} f_j)\tilde{*}\hat{\phi}\right\|_{M_{cb}^n A(G)} < \left(\prod_{j=0}^{n} \|f_j\|_1\right) r + \epsilon = \left(\prod_{j=0}^{n} \|f_j\|_1\right) \|\phi\|_{M_{cb}^n A(\Gamma)} + \left(\prod_{j=0}^{n} \|f_j\|_1\right) \epsilon + \epsilon. \tag{463}$$

This holds for all $\epsilon > 0$, so we have that

$$\left\|(\otimes_{j=0}^{n} f_j)\tilde{*}\hat{\phi}\right\|_{M_{cb}^n A(G)} \le \left(\prod_{j=0}^{n} \|f_j\|_1\right) \|\phi\|_{M_{cb}^n A(\Gamma)}. \tag{464}$$

□

Theorem 5.10 can be combined with Theorem 5.6 to obtain the following result.

Theorem 5.11. *Let ϕ_ι be a net in $M_{cb}^n A(\Gamma)$ such that for some constant $k > 0$:*

$$\|\phi_\iota\|_{M_{cb}^n A(\Gamma)} \le k \tag{465}$$

for all ι. Let $\phi \in l^\infty(\Gamma^{\times n})$ and $f_0, \ldots, f_n \in L^1(G)$. Suppose that ϕ_ι converges to ϕ either in $\sigma(l^\infty(\Gamma^{\times n}), l^1(\Gamma^{\times n}))$ topology or uniformly on compact sets. Then $(\otimes_{j=0}^{n} f_j)\tilde{*}\hat{\phi}_\iota$ is a net in $M_{cb}^n A(G)$ bounded in $\|\cdot\|_{M_{cb}^n A(G)}$ by $k\prod_{j=0}^{n} \|f_j\|_1$ that converges to $(\otimes_{j=0}^{n} f_j)\tilde{*}\hat{\phi}$ both in $\sigma(L^\infty(G^{\times n}), L^1(G^{\times n}))$ topology and uniformly on compact sets.

Proof. The boundedness of the net $(\otimes_{j=0}^{n} f_j)\tilde{*}\hat{\phi}_\iota$ in $\|\cdot\|_{M_{cb}^n A(G)}$ follows from Theorem 5.10. Note that the net ϕ_ι is also bounded in $\|\cdot\|_\infty$, because it is bounded in $\|\cdot\|_{M_{cb}^n A(\Gamma)}$, so the rest of the statement follows by invoking Theorem 5.6. □

The following is a special case of Theorem 5.11.

Corollary 5.12. *Let ϕ_ι be a net in $M^n_{cb}A(\Gamma)$ such that for some constant $k \geq 1$:*

$$\|\phi_\iota\|_{M^n_{cb}A(\Gamma)} \leq k \tag{466}$$

for all ι. Let $f_0, \ldots, f_n \in L^1(G)$ be non-negative with $\|f_j\|_1 = 1$ for all $j \in \{0, \ldots, n\}$. Suppose that ϕ_ι converges to 1 uniformly on compact sets. Then $(\otimes^n_{j=0} f_j)\tilde{*}\hat{\phi}_\iota$ is a net in $M^n_{cb}A(G)$ bounded in $\|\cdot\|_{M^n_{cb}A(G)}$ by k that converges to 1 uniformly on compact sets.

Proof. Note that $\otimes^n_{j=0} f_j$ is a non-negative function, hence

$$\int_{G^{\times(n+1)}} (\otimes^n_{j=0} f_j)(t_0, \ldots, t_n) d(t_0, \ldots, t_n) = \left\|\otimes^n_{j=0} f_j\right\|_1 = \prod_{j=0}^{n} \|f_j\|_1 = 1. \tag{467}$$

So $(\otimes^n_{j=0} f_j)\tilde{*}1 = 1$ by Remark 3.28. Since $\hat{1} = 1$ the result now follows from Theorem 5.11. □

Remark 5.4. Corollary 5.12 and the results it depends on are similar in structure to the proof in [18] that G is weakly amenable if the lattice Γ is weakly amenable. The main differences are as follows:

- The map $\phi \mapsto \hat{\phi}$ as defined in [18] (which agrees with our definition for $n = 1$) maps $A(\Gamma)$ contractively into $A(G)$ and maps $M_{cb}A(\Gamma)$ contractively into $M_{cb}A(G)$ (see Lemma 2.1 in [18]). So if ϕ_ι is a net in $A(\Gamma)$ that is bounded in $\|\cdot\|_{M_{cb}A(\Gamma)}$, then $\hat{\phi}_\iota$ is a net in $A(G)$ that is bounded in $\|\cdot\|_{M_{cb}A(G)}$ with the same norm.

- If the bounded net ϕ_ι converges to 1 uniformly on compact sets, then $\hat{\phi}_\iota$ converges in $\sigma(L^\infty(G), L^1(G))$ topology to 1 through an argument similar to Theorem 5.5 (see Theorem 2.3 in [18]).

- To obtain a net in $A(G)$ that converges to 1 uniformly on compact sets, the net $\hat{\phi}_\iota$ is convolved with an appropriate function (see Lemma 2.2. in [18]). This convolution also preserves the boundedness of the net in $\|\cdot\|_{M_{cb}A(G)}$. This convolution essentially replaces our "pseudo-convolution" as defined in Definition 3.29.

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