DELFT UNIVERSITY OF TECHNOLOGY

DEPARTMENT OF AEROSPACE ENGINEERING

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STRESSES AROUND PIN-LOADED HOLES IN ELASTICALLY ORTHOTROPIC PLATES WITH ARBITRARY LOAD DIRECTION

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SUMMARY

Stresses have been calculated for infinite orthotropic plates with a circular hole, loaded by a frictionless rigid pin of the same diameter under various angles with the symmetry axes of the material. The calculations are based on the analytical method of complex stress functions.

A numerical approach was used for satisfying the displacement boundary conditions of the contact area between pin and hole. Stress concentration factors, based on the nominal bearing stress are presented graphically for six laminates of carbon fibre reinforced plastic. A quadratic failure criterion was used to predict the bearing stress at which first significant damage occurs. Peak stresses, places where they occur and bearing strength show directional sensitivity.

CONTENTS

1. INTRODUCTION

All published work on the theoretical investigation of pin-loaded holes in anisotropic plates is related to problems in which the load direction coincides with one of the principal axes of the laminate. In real structures, however, the requirement that load direction and one of the material axes coincide will not always be met. Since composite materials are anisotropic the bearing strength will vary with the load direction. In Reference [1] this variation of bearing strength was investigated experimentally for 0°/±45° lay-ups of three different composites, all showing sensitivity for the load direction. It is the purpose of this report to determine theoretically the stresses around pin-loaded holes in orthotropic plates in the case of an arbitrary load direction with respect to the material axes, using the analytical method of Reference [2].

Reference [2] deals with an unreinforced hole in an infinite orthotropic plate. The hole is loaded on a part of its edge by an infinitely rigid pin, while the other part of the edge is free. The contact between pin and plate material is assumed to be frictionless. The loading force and one of the material axes are supposed to coincide with the Y-axis.
The load distribution is represented by a sine series with unknown coefficients a_n. This distribution is symmetric with respect to the Y-axis, hence the series consists of odd terms only.

In Reference [2] the relevant comples stress functions are evaluated from a boundary condition problem of the first kind. The unknown coefficients a_n, present in the stress functions, are solved from the prescribed displacements of a discrete number of points of the contact area.

The same method will be used for the problem treated in this report;

a schematic representation is given in Figure 1. In this figure the α–β–axes are the principal material axes, the angle between these axes and the coordinate axes is φ. In the general case the edge stress distribution is not symmetric with respect to the Y–axis, so the sine series representing the distribution will consist of odd and even terms.

In the general case the direction of the load resultant will not coincide with the displacement vector. This suggests two different approaches:

- the direction of the loading force is known. Then the displacements of the pin are unknown.
- the pin is given a known displacement; then the value and the direction of the loading force are unknown.

In this report the second approach has been chosen. The reason is that for the formulation of the displacement boundary conditions for the discrete number of points of the contact area the displacements of the pin must be known. It is then plausible to give the pin a known displacement in Y–direction. The direction of the load resultant will then follow from the calculation.

The choice of the coordinate axes X–Y with respect to laminate orientation, pin displacement and load direction is free. It has been made so as to facilitate the mathematical formulation of the problem. By taking the displacement direction of the pin along the Y–axis the entire loaded area, i.e. the contact area, is between $\theta = 0°$ and π.

Numerical values of the stresses are obtained for five carbon fibre reinforced laminates that are relevant as structural materials. The unidirectional material has been added because of its extremely anisotropic character. The elastic properties of the laminates are given in Table 1, as well as two complex material parameters s_1 and s_2, needed in the theory of complex stress functions. Most of the properties of Table 1 are measured, some are calculated from measured values.

From the calculated stress pattern a strength prediction has been made by applying the Tsai-Hill failure criterion, neglecting important practical aspects such as three-dimensional stresses, non-linear elasticity and joint geometry. Therefore the calculated bearing strength is only an indication for the first significant (or initial) damage around a single pin in a large plate. Nevertheless it may be useful for the designer in the choice of some of the parameters mentioned in Reference [3], e.g. lamina orientation and lay-up.

2. BASIC EQUATIONS

The general formulae for the boundary conditions of an orthotropic plate in a plane stress situation are according to Reference [4]

$$2 \text{ Re} \sum_{k=1,2} \phi_k(z_k) = \int_0^s Y \, ds + K_1$$

$$2 \text{ Re} \sum_{k=1,2} S_{k_\varphi} \phi_k(z_k) = - \int_0^s X \, ds + K_2 \qquad (2.1)$$

in which X and Y are external forces and $\phi_k(z_k)$ are complex functions of

$$z_k = x + s_{k_\varphi} y \qquad k = 1, 2 \qquad (2.2)$$

After solving $\phi_k(z_k)$ the stresses can be calculated with

$$\sigma_x = 2 \text{ Re} \sum_{k=1,2} s_{k_\varphi}^2 \phi_k'(z_k)$$

$$\sigma_y = 2 \text{ Re} \sum_{k=1,2} \phi_k'(z_k) \qquad (2.3)$$

$$\tau_{xy} = -2 \text{ Re} \sum_{k=1,2} S_{k_\varphi} \phi_k'(z_k)$$

and the displacements with

$$u = 2 \text{ Re} \sum_{k=1,2} u_{k_\varphi} \phi_k(z_k) + K_3 y + K_4$$

$$v = 2 \text{ Re} \sum_{k=1,2} v_{k_\varphi} \phi_k(z_k) + K_5 x + K_6 \qquad (2.4)$$

The index φ denotes the angle φ between material axes and coordinate axes. According to Reference [4] φ has a positive value when the material axes are rotated in clockwise direction relative to the coordinate axes. So, in Figure 1 φ has a negative value.
K_1 ... K_6 are integration constants; K_3 and K_5 are zero when no rigid rotation of the plate is allowed. K_4 and K_6 represent a translation of the plate as a rigid body.

The complex constants S_{k_φ} are defined by

$$S_{k_\varphi} = \frac{S_k \cos\varphi - \sin\varphi}{S_k \sin\varphi + \cos\varphi} \qquad k = 1,\ 2 \qquad (2.5)$$

in which S_k can be solved from

$$S_1{}^2 S_2{}^2 = \frac{S_{22}}{S_{11}}$$

$$(2.6)$$

$$S_1{}^2 + S_2{}^2 = -\frac{2S_{12} + S_{66}}{S_{11}}$$

In (2.4) is

$$u_{k_\varphi} = S_{11_\varphi} S_{k_\varphi}^{\ 2} + S_{12_\varphi} - S_{16_\varphi} S_{k_\varphi}$$

$$v_{k_\varphi} = \frac{S_{12} \ S_{k_\varphi}^{\ 2} + S_{22_\varphi} - S_{26_\varphi} S_{k_\varphi}}{S_{k_\varphi}} \qquad k = 1,2 \qquad (2.7)$$

S_{ij} are the material compliances in the principal material directions, S_{ij_φ} are the compliances in the coordinate directions. S_k is either complex or imaginary.

The general expressions for the functions $\phi_k(z_k)$ in the case of a pin loaded hole in an infinite plate with zero stresses at infinity are

$$\phi_k(z_k) = A_k \ln \zeta_k + \phi_k^{\ o}(z_k) + B_k \qquad k = 1,\ 2 \qquad (2.8)$$

in which

$$\zeta_k = \frac{z_k + \sqrt{z_k^{\ 2} - S_{k_\varphi}^{\ 2} - 1}}{1 - is_{k_\varphi}} \qquad (2.9)$$

$\phi_k^{\ o}(z_k)$ in (2.8) is a function, holomorphic outside the hole. The function $A_k \ln \zeta_k$ is multivalued. Its presence in (2.8) is necessary since the resultant forces on the edge of the hole are not zero. A_k can be solved from a set of equations, given in Reference [4]. These equations are

$$\sum_{k=1,2} (u_{k_\varphi} A_k - \overline{u_{k_\varphi}}\ \overline{A_k}) = 0$$

$$\sum_{k=1,2} (v_{k_\varphi} A_k - \overline{v_{k_\varphi}}\,\overline{A_k}) = 0$$

$$\sum_{k=1,2} (\quad A_k - \quad \overline{A_k}) = \frac{R_y}{2\pi i} \tag{2.10}$$

$$\sum_{k=1,2} (S_{k_\varphi} A_k - \overline{S_{k_\varphi}}\,\overline{A_k}) = - \frac{R_x}{2\pi i}$$

yielding

$$A_k = [R_y\{S_{k_\varphi}(S_{\ell_\varphi}\overline{S_{\ell_\varphi}} + S_{\ell_\varphi}\overline{S_{k_\varphi}} + \overline{S_{\ell_\varphi}}\,\overline{S_{k_\varphi}} - \frac{S_{12_\varphi}}{S_{11_\varphi}}) - \frac{S_{26_\varphi}}{S_{11_\varphi}}\} +$$

$$R_x\ \{S_{k_\varphi}(S_{\ell_\varphi} + \overline{S_{\ell_\varphi}} + \overline{S_{k_\varphi}} - \frac{S_{16_\varphi}}{S_{11_\varphi}}) - \frac{S_{12_\varphi}}{S_{11_\varphi}}\}] \ /$$

$$\{2\pi i(S_{\ell_\varphi} - S_{k_\varphi})\ (\overline{S_{\ell_\varphi}} - S_{k_\varphi})\ (S_{k_\varphi} - \overline{S_{k_\varphi}})\} \tag{2.11}$$

$$k = 1,2 \qquad \ell = 3-k$$

in which R_x and R_y are the resultant forces in X- and Y-direction respectively. In Appendix 4 a further evaluation of (2.11) is given.

3. THE MULTIVALUED FUNCTIONS $A_k \ln \zeta_k$

The radius of the hole is taken unity. Then ζ_1 and ζ_2 in (2.9) have the same value at the opening edge:

$$\zeta_1 = \zeta_2 = \sigma = \cos\theta + i \sin\theta = e^{i\theta} \qquad (3.1)$$

Substitution of (2.8) in (2.1) yields for the edge

$$2 \operatorname{Re} \sum_{k=1,2} \{ B_k + A_k \ln \sigma + \phi_k{}^o(z_k)\} = \int_0^s Y\, ds + K_1$$

$$(3.2)$$

$$2 \operatorname{Re} \sum_{k=1,2} \{S_{k_\varphi} B_k + S_{k_\varphi} A_k \ln \sigma + S_{k_\varphi} \phi_k{}^o(z_k)\} = -\int_0^s X\, ds + K_2$$

in which, following from the third and fourth equation of (2.10)

$$2 \operatorname{Re} \sum_{k=1,2} A_k \ln \sigma = \frac{R_y \theta}{2\pi}$$

$$2 \operatorname{Re} \sum_{k=1,2} S_{k_\varphi} A_k \ln \sigma = -\frac{R_x \theta}{2\pi}$$

So, equations (3.2) become

$$2 \operatorname{Re} \sum_{k=1,2} \{ B_k + \phi_k{}^o(z_k)\} = \int_0^s Y\, ds + K_1 - \frac{R_y \theta}{2\pi} \qquad (3.3)$$

$$2 \operatorname{Re} \sum_{k=1,2} \{S_{k_\varphi} B_k + S_{k_\varphi} \phi_k{}^o(z_k)\} = -\int_0^s X\, ds + K_2 + \frac{R_x \theta}{2\pi} \qquad (3.3)$$

In (3.3) the left hand sides are single-valued, hence in spite of the multi-valued parts $\frac{R_y \theta}{2\pi}$ and $\frac{R_x \theta}{2\pi}$ the right hand sides must be single-valued too. So $\int_0^s Y\, ds$ and $-\int_0^s X\, ds$ must contain multi-valued parts from which the increase for a complete circuit of integration cancels the increase of $-\frac{R_y \theta}{2\pi}$ and $\frac{R_x \theta}{2\pi}$ respectively.

The expression for the radial compression load on the edge of the hole has to obey the next requirements:

$$P_r = p_o \sum_{n=1,2,3}^{\infty} a_n \sin n\theta \qquad \text{for } \theta < \theta < \pi$$

$$(3.4)$$

$$P_r = 0 \qquad \text{for } \pi < \theta < 2\pi$$

This expression can be found by multiplying a sine series

$$P_r' = p_o \sum_{n=1,2,3}^{\infty} a_n \sin n\theta$$

continuous on the whole contour, by a step function

$$f(\alpha) = \frac{1}{2} + \frac{2}{\pi} \sum_{m=1,3}^{\infty} \frac{\sin m\theta}{m} = \begin{matrix} 1 \text{ for } 0 < \theta < \pi \\ 0 \text{ for } \pi < \theta < 2\pi \end{matrix}$$

resulting in

$$P_r = p_o \left(\frac{1}{2} + \frac{2}{\pi} \sum_{m=1,3}^{\infty} \frac{\sin m\theta}{m}\right) \sum_{n=1,2,3}^{\infty} a_n \sin n\theta \qquad (3.5)$$

(3.5) can be converted to

$$P_r = p_o \left[\frac{1}{2} \sum_{n=1,2,3}^{\infty} a_n \sin n\theta + \frac{1}{\pi} \left\{ \sum_{n=1,3}^{\infty} \frac{a_n}{n} + \right.\right.$$

$$\left.\left. + \sum_{m,n}^{*} a_n \left(\frac{1}{n-m} + \frac{1}{n+m}\right) \cos m\theta \right\}\right] \qquad (3.6)$$

in which

$$\sum_{m,n}^{*} = \sum_{n=1,3}^{\infty} \sum_{m=2,4}^{\infty} + \sum_{n=2,4}^{\infty} \sum_{m=1,3}^{\infty} \qquad (3.7)$$

Expression (3.6) is continuous on the entire contour of the hole and obeys (3.4). The terms with odd values of n represent the symmetric part of the load on the edge, the term with even n the asymmetric part.

With

$$X = P_r \cos \theta$$

and

$$Y = P_r \sin \theta$$

it can be derived from (3.6)

$$\int_0^s X\,ds = -f_1 + \frac{R_x\theta}{2\pi} + \frac{R_x}{4} + \frac{p_o}{4}\sum_{n=1,3}^{\infty}\frac{a_n + a_{n+2}}{n+1} \qquad (3.8)$$

$$\int_0^s Y\,ds = f_2 + \frac{R_y\theta}{2\pi} + K \qquad (3.9)$$

where, with a_1 chosen unity

$$R_x = \int_0^{2\pi} X\,ds = p_o\,\{a_2 + \sum_{n=2,4}^{\infty}\frac{a_n + a_{n+2}}{n+1}\} \qquad (3.10)$$

$$R_y = \int_0^{2\pi} Y\,ds = \frac{p_o}{2}\,\pi \qquad (3.11)$$

$$f_1 = p_o\,[\frac{1}{4}\,\{a_2\cos\theta + \sum_{n=1,2,3,4}^{\infty}\frac{a_n + a_{n+2}}{n+1}\cos(n+1)\,\theta\} -$$

$$\frac{\sin 2\theta}{4\pi}\,\frac{R_x}{p_o} - \frac{\sin\theta}{\pi}\sum_{n=1,3,5}^{\infty}\frac{a_n}{n} -$$

$$\frac{1}{2\pi}\,(\sum_{n=1,3}^{\infty}\sum_{m=2,4}^{\infty} + \sum_{n=2,4}^{\infty}\sum_{m=3,5}^{\infty})\,a_n\,(\frac{1}{n+m} + \frac{1}{n-m})\,\cdot$$

$$\{\frac{\sin(m-1)\,\theta}{m-1} + \frac{\sin(m+1)\,\theta}{m+1}\}] \qquad (3.12)$$

$$f_2 = p_o\,[\frac{1}{4}\,\{a_2\sin\theta - \sum_{n=1,2,3,4}^{\infty}\frac{a_n - a_{n+2}}{n+1}\sin(n+1)\,\theta\} -$$

$$\frac{\cos 2\theta}{4\pi}\frac{R_x}{p_o} - \frac{\cos\theta}{4\pi}\sum_{n=1,3,5}^{\infty}\frac{a_n}{n} +$$

$$\frac{1}{2\pi}\left(\sum_{n=1,3}^{\infty}\sum_{m=2,4}^{\infty} + \sum_{n=2,4}^{\infty}\sum_{m=3,5}^{\infty}\right) a_n \left(\frac{1}{n+m} + \frac{1}{n-m}\right) \cdot$$

$$\left\{\frac{\cos(m-1)\,\theta}{m-1} - \frac{\cos(m+1)\,\theta}{m+1}\right\}\Big] \tag{3.13}$$

$$K = p_o \Big[\frac{1}{\pi}\sum_{n=1,3}^{\infty}\frac{a_n}{n} - \frac{1}{2\pi}\left(\sum_{n=1,3}^{\infty}\sum_{m=2,4}^{\infty} + \sum_{n=2,4}^{\infty}\sum_{m=3,5}^{\infty}\right)$$

$$a_n \left(\frac{1}{n+m} + \frac{1}{n-m}\right)\left(\frac{1}{m-1} - \frac{1}{m+1}\right) + \frac{R_x}{4\pi p_o}\Big] \tag{3.14}$$

With (3.8) and (3.9) boundary conditions (3.3) become

$$2\,\mathrm{Re}\sum_{k=1,2}^{\infty}\{\ B_k + \ \phi_k{}^{o}(z_k)\} = f_2 + K + K_1$$

$$(3.15)$$

$$2\,\mathrm{Re}\sum_{k=1,2}^{\infty}\{S_{k\varphi} B_k + S_{k\varphi}\,\phi_k{}^{o}(z_k)\} = f_1 - \frac{R_x}{4} - \frac{p_o}{4}\sum_{n=1,3}^{\infty}\frac{a_n + a_{n+2}}{n+1} + K_2$$

in which the multi-valued parts of (3.8) and (3.9) have eliminated the multi-valued parts in the right hand sides of (3.3). So, both right hand sides of (3.15) are single-valued. With f_1 and f_2 continuous on the entire contour the functions $\phi_k{}^{o}(z_k)$ can be solved from (3.15).

4. SOLUTION OF THE HOLOMORPHIC FUNCTIONS $\Phi_k{}^o(z_k)$

The functions $\phi_k{}^o(z_k)$ in (2.8) are holomorphic outside the hole, so they can be presented as series of negative powers of z_k with unknown coefficients. These series are replaced by power series of ζ_k

$$\phi_k{}^o(z_k) = \sum_{n=1}^{\infty} g_n{}^{(k)} \, z_k{}^{-n} = \sum_{n=1}^{\infty} h_n{}^{(k)} \, \zeta_k{}^{-n} \tag{4.1}$$

On the edge of the hole, according to (3.1)

$$\zeta_1 = \zeta_2 = \sigma$$

Hence, on the edge both functions $\phi_1{}^o(z_1)$ and $\phi_2{}^o(z_2)$ are expressed in the same coordinate σ, so boundary conditions (3.15) now become

$$\begin{aligned} &2 \,\mathrm{Re} \sum_{k=1,2} \{ \quad B_k + \quad \phi_k{}^o(\sigma)\} = f_2 + K + K_1 \\ &2 \,\mathrm{Re} \sum_{k=1,2} \{S_{k_\varphi} B_k + S_{k_\varphi} \phi_k{}^o(\sigma)\} = \\ &\qquad = f_1 - \frac{R_x}{4} - \frac{p_o}{4} \sum_{n=1,3}^{\infty} \frac{a_n + a_{n+2}}{n+1} + K_2 \end{aligned} \tag{4.2}$$

In (4.2) the constants in the right hand sides determine the constants B_k and, according to (2.4), the translation of the plate as a rigid body. However, in (2.4) K_4 and K_6 may be chosen so that these translations are zero. This will not affect the computation of the unknown coefficients a_n in (3.6) since they will be evaluated from a boundary condition for the displacements that result from elastic deformations only. Equations (4.2) now simplify to:

$$2 \text{ Re} \sum_{k=1,2} \phi_k^{\,o}(\sigma) = f_2$$

$$2 \text{ Re} \sum_{k=1,2} S_{k_\varphi} \phi_k^{\,o}(\sigma) = f_1$$

or, written in another form

$$(S_{\ell_\varphi} - S_{k_\varphi})\,\phi_k^{\,o}(\sigma) + (S_{\ell_\varphi} - \overline{S_{k_\varphi}})\,\overline{\phi_k^{\,o}(\sigma)} + (S_{\ell_\varphi} - \overline{S_{\ell_\varphi}})\,\overline{\phi_\ell^{\,o}(\sigma)}$$
$$= S_{\ell_\varphi} f_2 - f_1 \qquad k = 1,2 \qquad \ell = 3-k \qquad (4.3)$$

The coefficients $h_n^{(k)}$ in (4.1) can be solved directly from (4.3) by the use of a Cauchy integral. This is possible since $\phi_k^{\,o}(\sigma)$, continuous on the edge of the hole, are the boundary values of functions $\phi_k^{\,o}(\zeta_k)$, holomorphic outside the hole, while $\phi_k^{\,o}(\infty) = 0$.

Cauchy's integral for the infinite region then gives

$$\frac{1}{2\pi i} \oint \frac{\phi_k^{\,o}(\sigma)}{\sigma - \zeta_k}\, d\sigma = \phi_k^{\,o}(\zeta_k)$$

$$\frac{1}{2\pi i} \oint \frac{\overline{\phi_k^{\,o}(\sigma)}}{\sigma - \zeta_k}\, d\sigma = 0 \qquad (4.4)$$

Applying (4.4) to (4.3) results in

$$-(S_{\ell_\varphi} - S_{k_\varphi})\ \phi_k{}^{o}(\zeta_k) = \frac{S_\ell}{2\pi i} \oint \frac{f_2}{\sigma - \zeta_k}\, d\sigma - \frac{1}{2\pi i} \oint \frac{f_1}{\sigma - \zeta_k}\, d\sigma \quad (4.5)$$

in which the - sign is necessary since the direction of integration is counter clockwise.

In Appendix 1 the integrals in the right hand side of (4.5) have been worked out, resulting in

$$\phi_k{}^{o}(\zeta_k) = \frac{p_o}{2\pi i(S_{\ell_\varphi} - S_{k_\varphi})} \{ \sum_{m,n} \frac{2na_n(m^2 - n^2 + 1 + 2i\, S_{\ell_\varphi} m)}{N_{m,n}} \zeta_k{}^{-m} - \frac{\pi i}{4} \sum_{n=0,1,2}^{\infty} \frac{a_n(1 + iS_{\ell_\varphi}) + a_{n+2}(1 - iS_{\ell_\varphi})}{n+1} \zeta_k{}^{-n-1} \} \quad (4.6)$$

where

$$\sum_{m,n} = \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} + \sum_{m=2,4}^{\infty} \sum_{n=2,4}^{\infty} \quad (4.7)$$

$$a_0 = 0 \quad (4.8)$$

$$N_{m,n} = m\{(m+1)^2 - n^2\}\ \{(m-1)^2 - n^2\} \quad (4.9)$$

$$k = 1,2 \qquad \ell = 3-k$$

Except for the coefficients a_n the complex stress functions $\phi_k(z_k)$

are now known. The unknown coefficients a_n will be solved from a displacement boundary condition for a number of points on the loaded part of the edge.

5. DISPLACEMENTS ON THE EDGE OF THE HOLE

Application of formulae (2.4) to the multi-valued parts $A_k \ln \sigma$ yields for the edge of the hole the first two equations of (2.10), so these parts do not contribute to the displacements.

With (4.6) it can easily be derived

$$\sum_{k=1,2} u_{k_\varphi} \phi_k{}^o(\sigma) = \frac{-p_o}{2\pi i} \Big[\sum_{m,n} \frac{2na_n\{(m^2-n^2+1)\, C_1 + 2mC_2\}\, \sigma^{-m}}{N_{m,n}}$$

$$- \frac{\pi i}{4} \sum_{n=0,1,2}^{\infty} \frac{\{(a_n+a_{n+2})\, C_1 + (a_n-a_{n+2})\, C_2\}\, \sigma^{-n-1}}{n+1} \Big] \qquad (5.1)$$

in which,with (2.7)

$$C_1 = \frac{u_{1_\varphi} - u_{2_\varphi}}{S_{1_\varphi} - S_{2_\varphi}} = S_{11_\varphi}(S_{1_\varphi} + S_{2_\varphi}) - S_{16_\varphi} \qquad (5.2)$$

$$C_2 = i\, \frac{u_{1_\varphi} S_{2_\varphi} - u_{2_\varphi} S_{1_\varphi}}{S_{1_\varphi} - S_{2_\varphi}} = i(S_{11_\varphi} S_{1_\varphi} S_{2_\varphi} - S_{12_\varphi}) \qquad (5.3)$$

So the displacement u for the edge of the hole will be

$$u = \sum_{k=1,2} \{u_{k_\varphi} \phi_k{}^o(\sigma) + \overline{u_{k_\varphi} \phi_k{}^o(\sigma)}\}$$

$$= \frac{p_o}{4} \{ \sum_{n=0,1,2}^{\infty} F_3{}^{(n)} \sin(n+1)\,\theta - \frac{8}{\pi} \sum_{m,n} F_1{}^{(m,n)} \cos m\theta$$

$$+ \sum_{n=0,1,2}^{\infty} F_3^{*(n)} \cos(n+1)\,\theta + \frac{8}{\pi} \sum_{m,n} F_1^{*(m,n)} \sin m\theta \}$$

(5.4)

where

$$F_1^{(m,n)} = \frac{na_n\{(m^2 - n^2 + 1)\, C_5 - 2m\, C_6\}}{N_{m,n}} \quad (5.5)$$

$$F_1^{*(m,n)} = \frac{2na_n C_4}{\{(m-1)^2 - n^2\}\,\{(m+1)^2 - n^2\}} \quad (5.6)$$

$$F_3^{(n)} = \frac{a_n(C_5 - C_6) + a_{n+2}(C_5 + C_6)}{n+1} \quad (5.7)$$

$$F_3^{*(n)} = \frac{C_4(a_n - a_{n+2})}{n+1} \quad (5.8)$$

and, according to Appendix 2

$$\overline{C}_1 + C_1 = 0$$

$$\overline{C}_2 + C_2 = 2S_{11}(S_1S_2 + 1) \sin\varphi \cos\varphi \; \mathrm{Im}(S_1 + S_2) = 2C_4 \quad (5.9)$$

$$\overline{C}_1 - C_1 = 2iS_{11}(S_1S_2 \sin^2\varphi - \cos^2\varphi)\; \mathrm{Im}(S_1 + S_2) = -2iC_5$$

(5.10)

$$\overline{C}_2 - C_2 = 2i(S_{12} - S_{11}S_1S_2) = 2iC_6 \quad (5.11)$$

The expression for the displacement V can be found in the same way. With (4.6)

$$\sum_{k=1,2} v_{k_\varphi} \phi_k{}^o(\sigma) = \frac{-p_o}{2\pi i}\Big[\sum_{m,n} \frac{2na_n\{(m^2-n^2+1)(-i\bar{C}_2) + 2imC_3\}\ \sigma^{-m}}{N_{m,n}}$$

$$- \frac{\pi i}{4} \sum_{n=0,1,2}^{\infty} \frac{\{(a_n+a_{n+2})(-i\bar{C}_2) + i(a_n-a_{n+2})\ C_3\}\ \sigma^{-n-1}}{n+1}\Big] \tag{5.12}$$

in which, with (2.7)

$$C_3 = \frac{v_{1_\varphi} S_{2_\varphi} - V_{2_\varphi} S_{1_\varphi}}{S_{1_\varphi} - S_{2_\varphi}} = S_{26_\varphi} - \frac{S_{22_\varphi}}{S_{1_\varphi} S_{2_\varphi}} (S_{1_\varphi} + S_{2_\varphi}) \tag{5.13}$$

The displacement V on the edge of the hole is:

$$v = \sum_{k=1,2} \{v_{k_\varphi} \phi_k{}^o(\sigma) + \overline{v_{k_\varphi} \phi_k{}^o(\sigma)}\}$$

$$= \frac{p_o}{4} \{ \sum_{n=0,1,2}^{\infty} F_4{}^{(n)} \cos(n+1)\ \theta + \frac{8}{\pi} \sum_{m,n} F_2{}^{(m,n)} \sin m\theta$$

$$- \sum_{n=0,1,2}^{\infty} F_4{}^{*(n)} \sin(n+1)\ \theta + \frac{8}{\pi} \sum_{m,n} F_2{}^{*(m,n)} \cos m\theta\} \tag{5.14}$$

where

$$F_2^{(m,n)} = \frac{na_n\{(m^2 - n^2 + 1)\ C_6 - 2mC_7\}}{N_{m,n}} \qquad (5.15)$$

$$F_2^{*(m,n)} = \frac{na_n(m^2 - n^2 + 1)\ C_4}{N_{m,n}} \qquad (5.16)$$

$$F_4^{(n)} = \frac{a_n(C_6 - C_7) + a_{n+2}(C_6 + C_7)}{n + 1} \qquad (5.17)$$

$$F_4^{*(n)} = \frac{C_4(a_n + a_{n+2})}{n + 1} \qquad (5.18)$$

and according to Appendix 2:

$$\overline{C}_3 + C_3 = 0$$

$$\overline{C}_3 - C_3 = 2iS_{11}(S_1S_2\cos^2\varphi - \sin^2\varphi)\ \mathrm{Im}(S_1 + S_2) = -2iC_7 \qquad (5.19)$$

In the boundary condition formula for the displacements the pin will be given the displacement in Y-direction of the point belonging to $\theta = 90°$. This displacement will be denoted by V_i.

6. BOUNDARY CONDITION FOR THE DISPLACEMENTS

The displacement of a point on the loaded part of the edge of the plate consists of two parts:

- a part equal to the displacement of the pin as a rigid body. The components of this displacement are made V_i in Y-direction and zero in X-direction.
- a displacement relative to the pin with components u_r and v_r. For the loaded part the radial displacement of the plate material with respect to the pin must be zero. For small deformations this requirement results in

$$\frac{u_r}{v_r} = -\text{tg}\ \theta \qquad (6.1)$$

With

$$u = u_r$$

$$v = v_i + v_r$$

expression (6.1) becomes

$$u \cos\theta + (v - v_i) \sin\theta = 0 \qquad (6.2)$$

(6.2) is the boundary condition for the displacements which will be imposed on the loaded part of the edge.

N.B. In Appendix 2 it is discussed that constants $C_4 - C_7$ are inde-

pendent of φ for materials with equal S_{11} and S_{22}.
For (quasi) isotropic materi als C_4 - C_7 only depend on the values of S_{11} and S_{12}. As a result Poisson's ratio μ is the only material constant in boundary condition (6.2). This implies that for isotropic materials the constants a_n - hence the radial stress distribution - is dependent on the material constant μ only.

7. DETERMINATION OF THE CONSTANTS a_n

Some of the series in the displacement formulae (5.4) and (5.14) can be replaced by analytic expressions that are only valid for $0 < \theta < \pi$. This has been done in Appendix 3 resulting in the displacement formulae (A3.8) and (A3.15) for the upper half of the edge. These formulae, substituted in condition (6.2) yield

$$a_{1\theta} + \sum_{n=2,3,4}^{\infty} a_{n\theta}\, a_n = 0 \tag{7.1}$$

in which $a_{1\theta}$ and $a_{n\theta}$ are known coefficients.

With

$$\begin{aligned} C_8 &= C_5 \cos\theta - C_4 \sin\theta \\ C_9 &= C_4 \cos\theta - C_7 \sin\theta \\ C_{10} &= C_8 \cos\theta - C_9 \sin\theta \\ C_{11} &= -C_8 \sin\theta - C_9 \cos\theta \end{aligned} \tag{7.2}$$

they become:

$$a_{1\theta} = \frac{C_8}{2} \sin 2\theta + \frac{C_9}{2} \cos 2\theta + C_6 \left(\theta - \frac{\pi}{2}\right) \cos\theta - \frac{C_7}{2} \sin\theta$$
$$+ \frac{8}{\pi} \sum_{m=1,3}^{\infty} \frac{2C_9 \sin m\theta - C_8\, m \cos m\theta - 2C_7 (-1)^{\frac{m+1}{2}} \sin\theta}{m^2 (m-2)(m+2)} \tag{7.3}$$

For n is even:

$$a_{n\theta} = \frac{2}{n^2-1}\Big[C_{10}\, n \sin n\theta + C_{11} \cos n\theta$$

$$+ 2C_6 \sin n\theta - \frac{C_6 n}{\pi}\{\frac{4}{n^2-1} \cos\theta - (2\theta-\pi)\sin\theta\} - C_4(-1)^{\frac{n}{2}} \sin\theta\Big]$$

$$+ \frac{8}{\pi}\sum_{m=2,4}^{\infty} \frac{2C_9 mn \sin m\theta - C_8 n(m^2-n^2+1)\cos m\theta - C_4 n(m^2-n^2+1)(-1)^{\frac{m}{2}}\sin\theta}{N_{m,n}} \qquad (7.4)$$

For n is odd:

$$a_{n\theta} = \frac{2}{n^2-1}\Big[C_{10}\, n \sin n\theta + C_{11} \cos n\theta$$

$$+ 2C_6 \sin n\theta - \{2C_6(-1)^{\frac{n-1}{2}} + C_7 n(-1)^{\frac{n-1}{2}}\} \sin\theta\Big]$$

$$+ \frac{8}{\pi}\sum_{m=1,3}^{\infty} \frac{2C_9 mn \sin m\theta - C_8 n(m^2-m^2+1)\cos m\theta - 2mnC_7(-1)^{\frac{m+1}{2}}\sin\theta}{N_{m,n}} \qquad (7.5)$$

The terms with θ do not make (7.3) and (7.4) multivalued since these expressions are only valid for $0 < \theta < \pi$. For the highest value of m in the series any large number giving sufficient accuracy may be taken.

Condition (7.1) is imposed on N points of the loaded part of the edge. The N values of θ, defining those points, give in (7.1) N equations from which N unknown coefficients a_n can be solved by inversion of the matrix $[a_{n\theta}]$ in

$$[a_{n\theta}]\ \{a_n\} = \{-a_{1\theta}\} \tag{7.6}$$

With the coefficients a_n solved the holomorphic parts of the complex stress functions are completely known, as well as the radial edge stress given by:

$$\sigma_r = -p_o \sum_{n=1,2}^{\infty} a_n \sin n\theta \tag{7.7}$$

8. THE STRESSES

The complex stress functions:

$$\phi_k(z_k) = A_k \ln \zeta_k + \phi_k^{\,o}(\zeta_k)$$

are solved as functions of ζ_k. Before substitution in the stress formulae (2.3) the functions must be differentiated with respect to z_k

$$\phi_k'(z_k) = \frac{d}{dz_k}\{\phi_k(\zeta_k)\} = \frac{d}{d\zeta_k}\{\phi_k(\zeta_k)\}\frac{d\zeta_k}{dz_k}$$

in which, following from (2.9)

$$\frac{d\zeta_k}{dz_k} = \frac{\zeta_k}{\sqrt{z_k^{\,2} - S_{k_\varphi}^{\,2} - 1}} \qquad (8.1)$$

So, with (4.6)

$$\phi_k'(z_k) = [A_k + \frac{p_o}{2\pi i(S_{k_\varphi} - S_{\ell_\varphi})}\{\sum_{m,n}\frac{2na_n(m^2-n^2+1+2iS_{\ell_\varphi}m)}{\{(m+1)^2-n^2\}\{(m-1)^2-n^2\}}\zeta_k^{-m}$$

$$-\frac{\pi i}{4}\sum_{n=0,1,2}^{N}(a_n(1+iS_{\ell_\varphi}) + a_{n+2}(1-iS_{\ell_\varphi}))\,\zeta_k^{-n-1}\}]\,/\sqrt{z_k^{\,2}-S_{k_\varphi}^{\,2}-1} \qquad (8.2)$$

With (8.2) substituted in (2.3) the stresses can be calculated in every point of the plate.

For the edge of the hole, where

$$z_k = \cos\theta + S_{k_\varphi} \sin\theta$$

$$\zeta_k = \cos\theta + i \sin\theta = \sigma$$

expression (8.1) becomes

$$\frac{d\sigma}{dz_k} = \frac{i\sigma}{S_{k_\varphi} \cos\theta - \sin\theta}$$

and

$$\phi_k{}'(z_k) = \frac{1}{S_{k_\varphi} \cos\theta - \sin\theta} \Big[iA_k + \frac{p_o}{2\pi(S_{k_\varphi} - S_{\ell_\varphi})} \cdot \Big\{ \sum_{m,n} \frac{2na_n(m^2 - n^2 + 1 + 2iS_{\ell_\varphi} m)\,\sigma^{-m}}{\{(m+1)^2 - n^2\}\{(m-1)^2 - n^2\}} - \frac{\pi i}{4} \sum_{n=0,1,2}^{N} \left(a_n(1 + iS_{\ell_\varphi}) + a_{n+2}(1 - iS_{\ell_\varphi})\right) \sigma^{-n-1} \Big\} \Big] \tag{8.4}$$

With the known radial stress and one of the stresses σ_x, σ_y or τ_{xy} the stress-situation on the edge of the hole is completely known. When for instance σ_y is calculated, the other stresses can easily be derived with

$$\sigma_t \cos^2 \theta = \sigma_y - \sigma_r \sin^2 \theta$$

$$\sigma_x = \sigma_r \cos^2 \theta + \sigma_t \sin^2 \theta \qquad (8.5)$$

$$\tau_{xy} = (\sigma_r - \sigma_t) \sin \theta \cos \theta$$

where σ_t is the tangential edge stress.

The expression (8.4) cannot be used if $S_{k_\varphi} - S_{\ell_\varphi} = 0$ as is the case for isotropic materials. Therefore in Appendix 5 the following expression for σ_y on the edge of the hole has been derived:

$$\frac{\sigma_y}{p_o} = \mathrm{Re}\ \frac{1}{\pi(S_{1_\varphi} \cos \theta - \sin \theta)(S_{2_\varphi} \cos \theta - \sin \theta)} \cdot$$

$$\Big[\frac{R_y}{p_o} \{\tfrac{1}{2}(S_{1_\varphi}+S_{2_\varphi}) \cos\theta - \tfrac{1}{2}\sin\theta + 2iA(S_{1_\varphi} S_{2_\varphi} C_5 \cos\theta + C_4 \sin\theta)\}$$

$$+ \frac{R_x}{p_o} \{\tfrac{1}{2}\cos\theta + 2iA(C_4 \cos\theta + (S_{1_\varphi}+S_{2_\varphi}) C_5 \cos\theta - C_5 \sin\theta)\}$$

$$- \sum_{m,n} \left\{\frac{(m^2-n^2+1) \cos\theta + 2im(S_{1_\varphi}+S_{2_\varphi}) \cos\theta - 2im \sin\theta}{\{(m+1)^2-n^2\}\{(m-1)^2-n^2\}}\right\} 2na_n\ \sigma^{-m}$$

$$- \frac{\pi}{4} \sum_{n=0,1,2} \{((S_{1_\varphi} + S_{2_\varphi} - i) \cos\theta - \sin\theta)\ a_n -$$

$$((S_{1_\varphi} + S_{2_\varphi} + i) \cos\theta - \sin\theta\ )\ a_{n+2}\}\ \sigma^{-n-1}\Big] \qquad (8.6)$$

For (quasi) isotropic materials the whole stress pattern only depends on the material constant μ.

9. NUMERICAL CALCULATIONS

The edge of the hole consists of a region of separation between pin and plate and a region of contact on which the boundary condition for the displacements must be imposed. The latter region does not generally coincide with the entire upper half of the hole.
For all investigated materials and angles between material axes and coordinate axes it appears to be smaller than 180°, leaving small areas of clearance in the neighbourhood of $\theta = 0°$ and 180°. In fact the contact angle is unknown and must be determined by the requirement that tractions between pin and plate are (physically) impossible. Since an iteration procedure magnifies the computing work considerably a standard contact angle $11° < \theta < 169°$ has been adopted, yielding sufficient accuracy.

The displacement boundary condition is imposed to 22 points of the contact area, resulting in 22 coefficients a_n. The calculations have been made for the following material orientations:

90° unidirectional	$\varphi = 0°$ and 30°
$90_n°/\pm45°$, n = 0, 1, 2, 4	$\varphi = 0°$, 30° and 45°
90°/0°/±45° (quasi-isotr.)	$\varphi = 0°$

The coefficients a_n are listed in Tables 2-6. According to (3.10) R_x can be calculated from the coefficients a_n with even index n. With $R_y = \frac{\pi}{2} p_o$ the angle δ between the resultant R and the Y-axis is easily found. δ is given in Figures 2-7.

The tangential and radial stresses on the upper half of the edge of the hole are shown in Figures 2-7. The stresses are made dimensionless with the bearing stress p, which is, according to the classical definition

$$p = \frac{R}{Dt}$$

For the present calculations, with D = 2 and t = 1, this results in

$$p = \tfrac{1}{2}\sqrt{R_x^{\,2} + R_y^{\,2}} = \tfrac{1}{2}p_o \sqrt{\left\{\sum_{n=2,4}^{22} \frac{2na_n}{n^2 - 1}\right\}^2 + \frac{\pi^2}{4}}$$

The strength values of the laminates, used in the failure criterion, are given in Table 7.
The predicted bearing strength $\bar{p}$ is listed in Table 8, together with the angle θ where first significant damage occurs.

All calculations have been made with a computer program in Fortran IV which has the capability of handling complex expressions.

10. DISCUSSION OF RESULTS

As may be concluded from Tables 2-6 the convergence of the series of 22 coefficients a_n, together with the known $a_1 = 1$, is sufficient in all the investigated cases. For the symmetric loading, i.e. for $\varphi = 0$, all coefficients with even n should be zero; the small values of these coefficients resulting from the limited accuracy of the solution procedure do not significantly affect the results.

As discussed in Appendix 2 the displacements of the edge of the hole in a laminate with equal S_{11} and S_{22} are independent on a rotation of the material axes. For the ±45° angle-ply this results in equal sets of coefficients a_n, as shown in Table 6. So, the radial stress

$$\sigma_r = -p_o \sum_{n=1,2}^{23} a_n \sin n\theta$$

is independent of the loading angle. Although this phenomenon is mathematically clear it is by no means obvious physically. It is noted that this applies to the radial stress only and neither to the tangential stress nor to the bearing strength.

It is obvious that for (quasi) isotropic materials the entire stress field is independent of φ. As discussed in Chapter 8 the stresses in these materials depend on the value of Poisson's ratio only. For the quasi-isotropic material chosen in this report as well as for the ±45° angle-ply a sine distribution for the radial stress is a good approximation, except for the small areas near $\theta = 0°$ and 180°. This is in agreement with one of the conclusions of Reference [6].

Peak radial stresses are generally developed near the points on the edge of the hole where the direction of the highest Young's modulus of the plate material is perpendicular to the hole boundary, both for $\varphi = 0$ and for the other values of φ. The maximum tangential stresses do not necessarily occur in the plate net area perpendicular to the load direction. For $\varphi = 0$ they are generated at the ends of the contact area, except for the ±45° angle-ply where they occur at $\theta = 45°$ and 135°. For other values of φ no conclusion can be drawn about the various places of the maximum tangential stresses, except for the ±45° angle-ply after rotation over $\varphi = 45°$, showing the maximum stresses at the ends of the contact area. The compressive tangential stresses occurring in the ±45° angle-ply are remarkable. They may result from the high value of Poisson's ratio μ.

The radial and tangential stresses are strongly influenced by the anisotropy of the material. As a result the bearing strength $\overline{p}$ shows a great variety of values for the different laminates and loading angles. Generally $\overline{p}$ decreases with increasing φ, except for the ±45° angle-ply. There seems to be a tendency that the materials with the highest values of $\overline{p}$ exhibit the greatest variation of bearing strength with direction of load.

The calculated values of the bearing strength are low as compared with most experimental results, found in literature. This cannot be explained by the fundamental difference between the infinite plate model used and the finite dimensions of the test specimen, certainly not for those with high width to pin-diameter ratios. One reason for this discrepancy may be the influence of lateral clamping pressure in the pin area, used in most experiments. Reference [7] shows a strong relation between joint strength and degree of clamping. Another reason may be the presence of friction between pin and plate. This will reduce the stresses near the top of the hole where most of the failures occur, resulting in a higher load level at first significant damage.

11. CONCLUDING REMARKS

Stresses were calculated in infinite orthotropic plates loaded without friction by an infinite rigid pin in a circular hole of the same diameter. The pin was given a displacement under different angles with the material axes.

In all investigated cases the contact angle between pin and hole boundary is smaller than 180°. The anisotropy and the angle between load direction and material axes have a strong influence on the contact pressure and the induced tangential stress. The maximum tangential stress does not necessarily occur in the plate net area perpendicular to the load direction. In general the maximum radial stresses are found near points with maximum stiffness perpendicular to the hole boundary. For isotropic materials a sine distribution for the contact pressure seems to be a good approximation, except for the small areas near $\theta = 0°$ and 180°.

The theoretical bearing strength varies with load direction. A consistent conclusion for this relation cannot be given, except that the directional sensitivity is higher for materials with high bearing strength.

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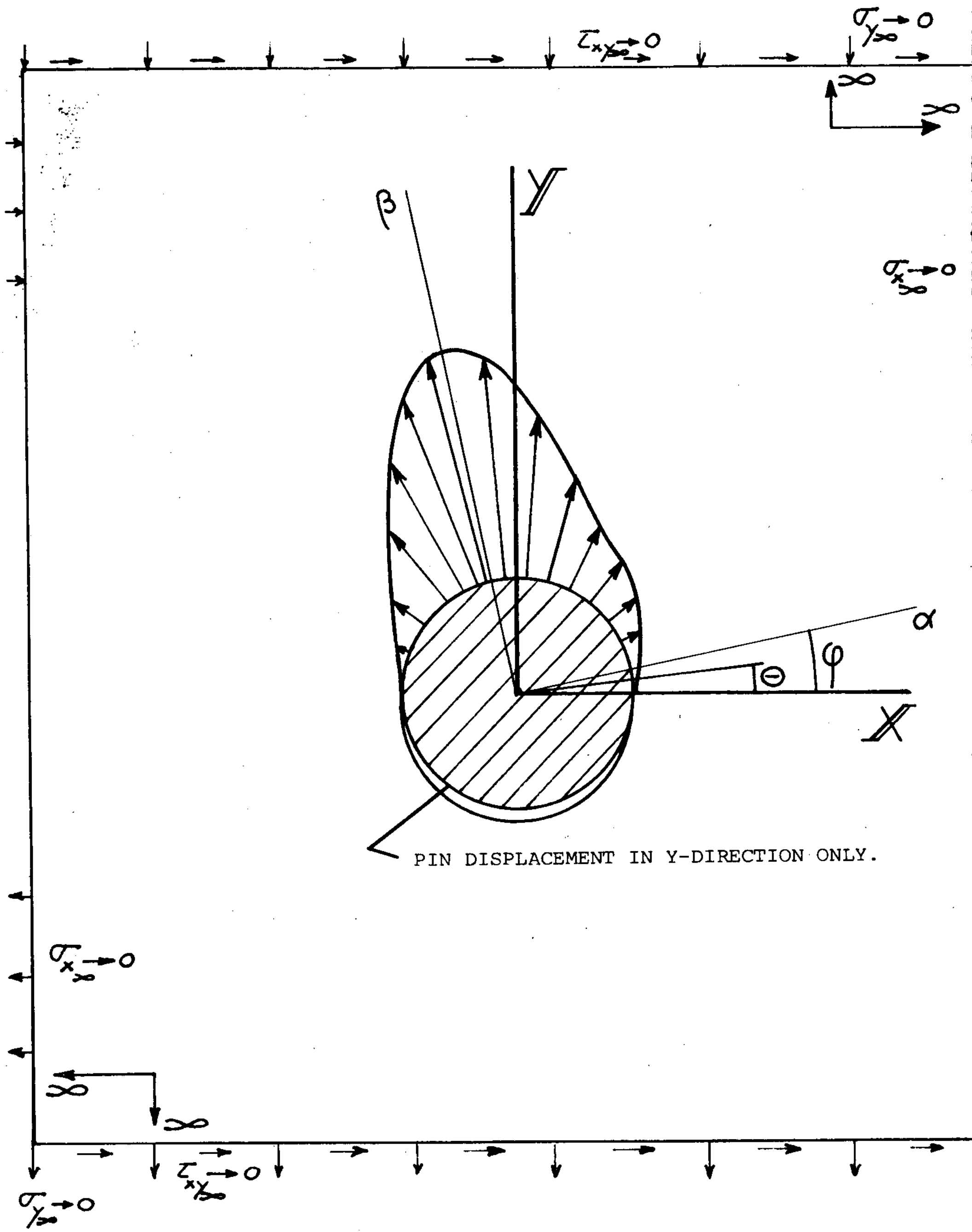

FIGURE 1: SCHEMATIC REPRESENTATION OF THE PROBLEM OF A PIN-LOADED HOLE IN AN ORTHOTROPIC MATERIAL. THE $\alpha - \beta$ AXES ARE THE MATERIAL SYMMETRY-AXES.

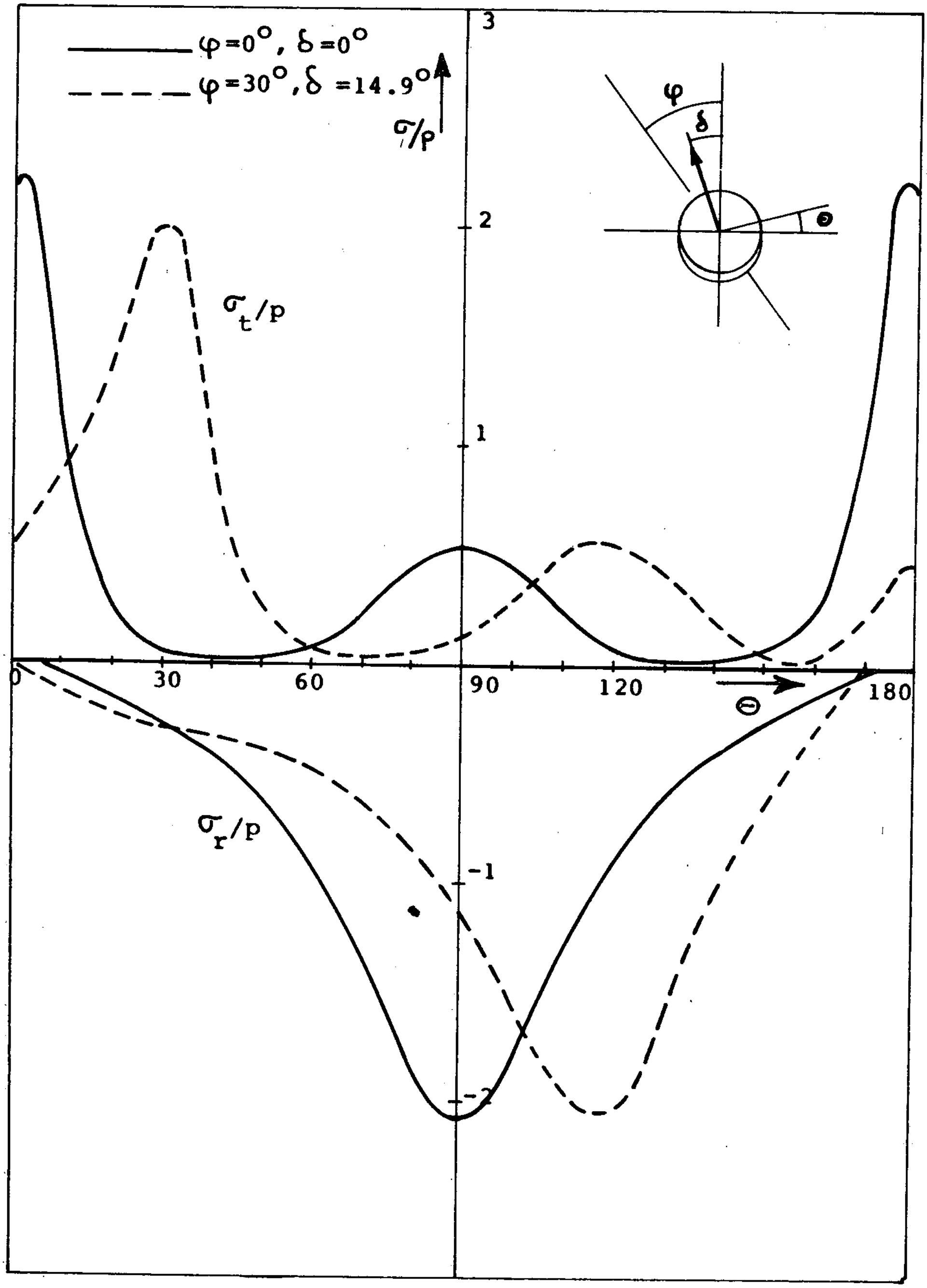

FIGURE 2: THE STRESS DISTRIBUTION AROUND A PIN-LOADED HOLE IN A UNIDIRECTIONAL C.F.R.P. LAMINATE.

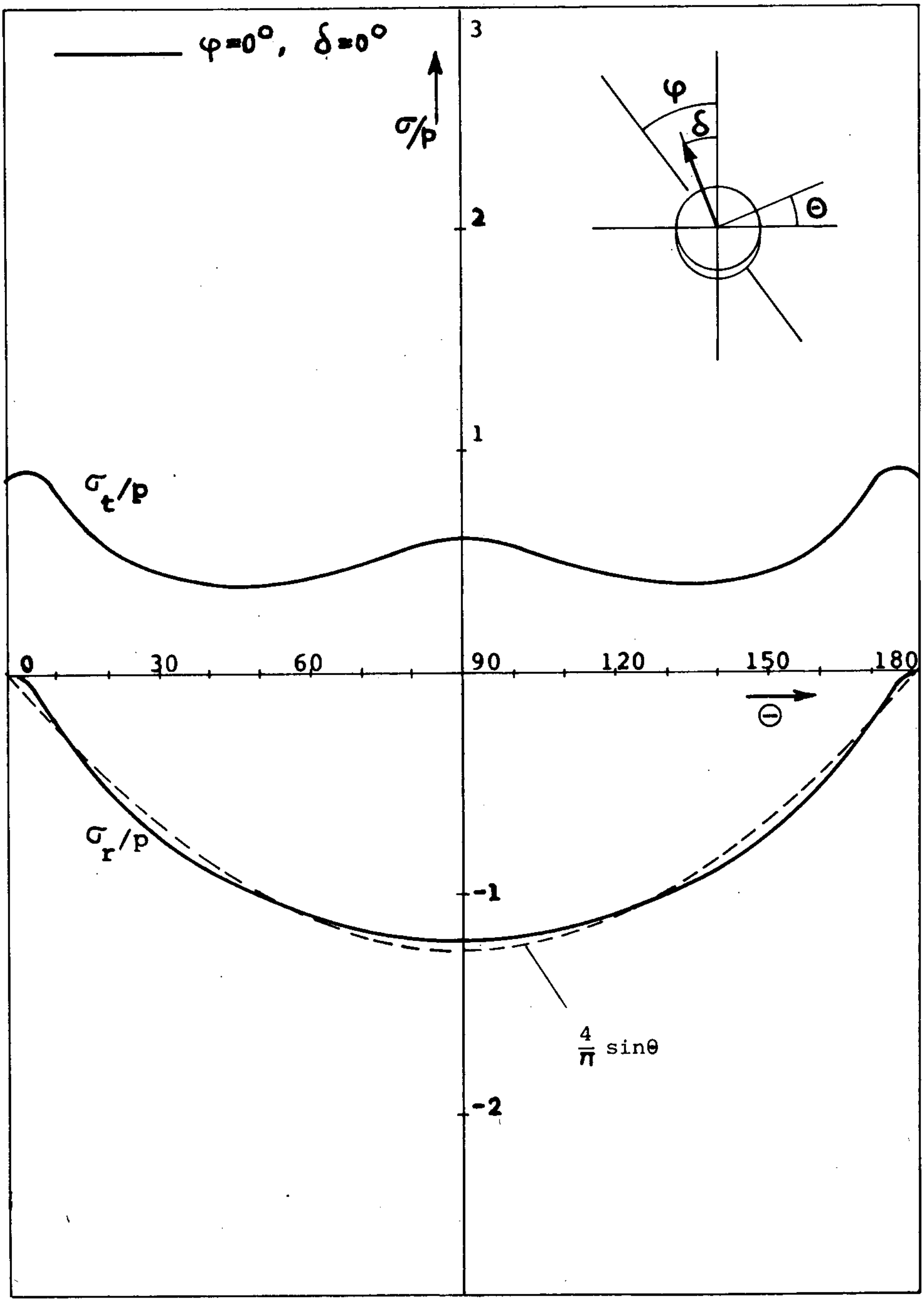

FIGURE 3: THE STRESS DISTRIBUTION AROUND A PIN-LOADED HOLE IN A QUASI-ISOTROPIC C.F.R.P. LAMINATE.

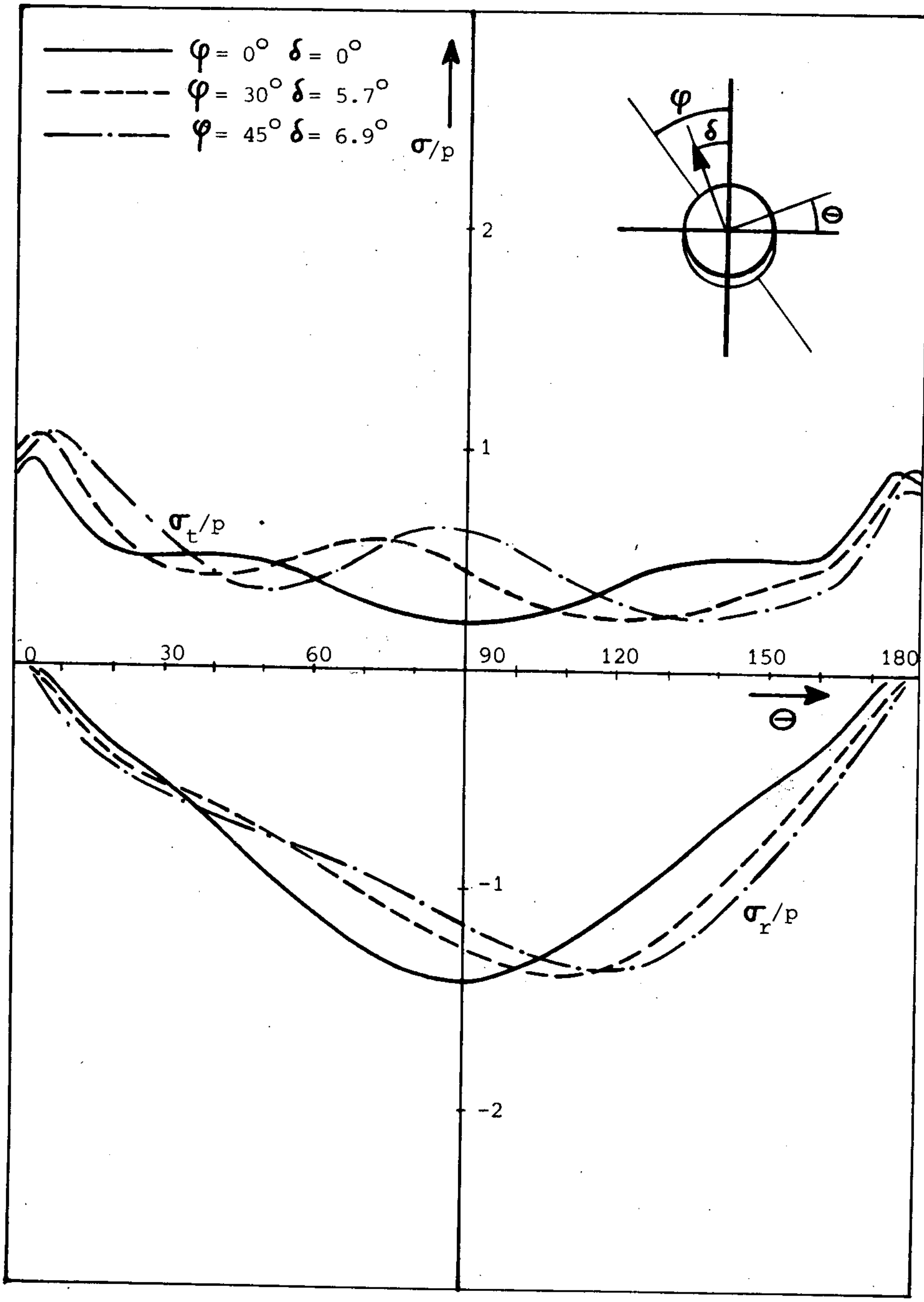

FIGURE 4: THE STRESS DISTRIBUTION AROUND A PIN-LOADED HOLE IN A $(90^\circ/\pm45^\circ)_s$ C.F.R.P. LAMINATE.

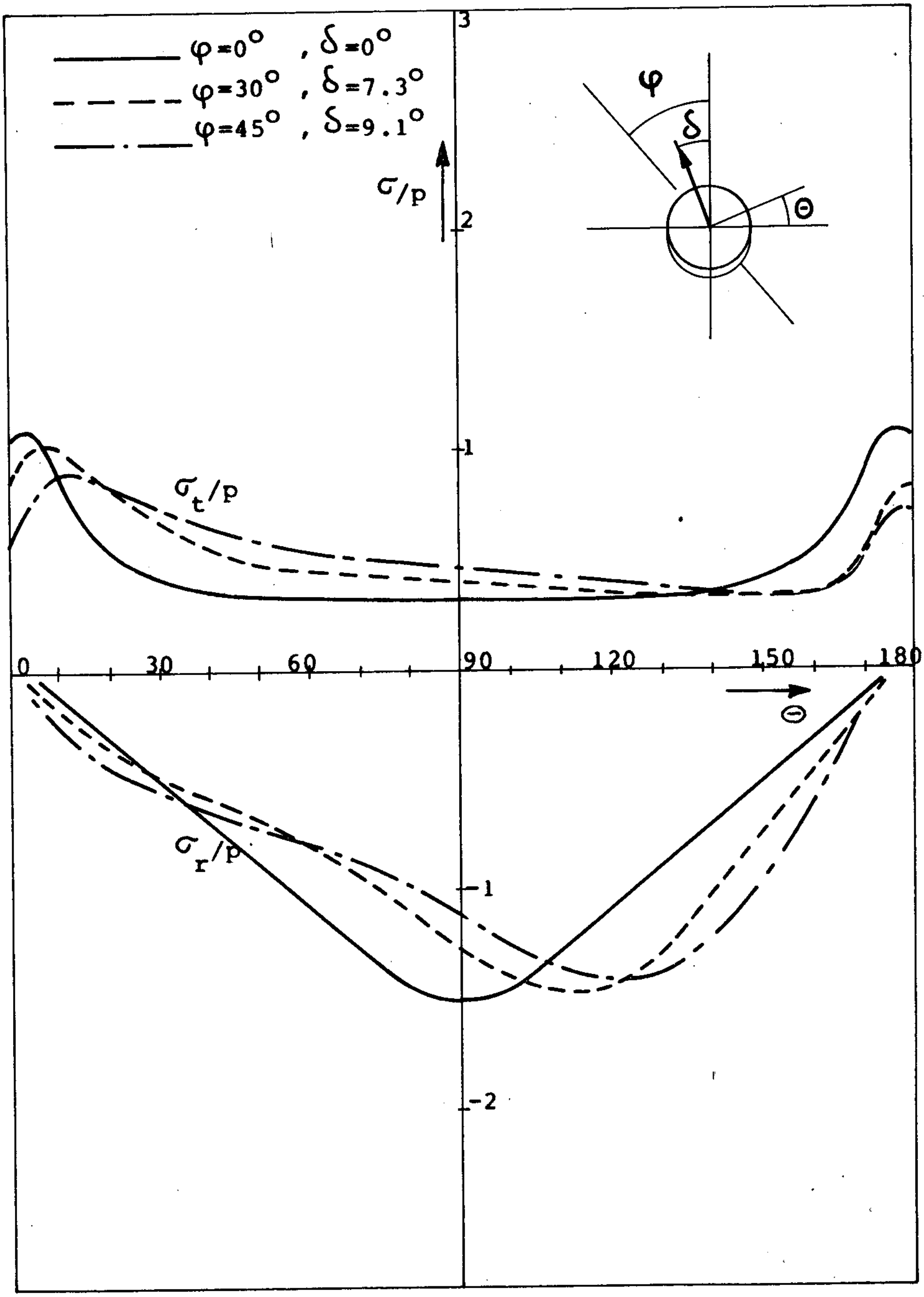

FIGURE 5: THE STRESS DISTRIBUTION AROUND A PIN-LOADED HOLE IN A $(90^{\circ}_{2}/\pm 45^{\circ})_{s}$ C.F.R.P. LAMINATE.

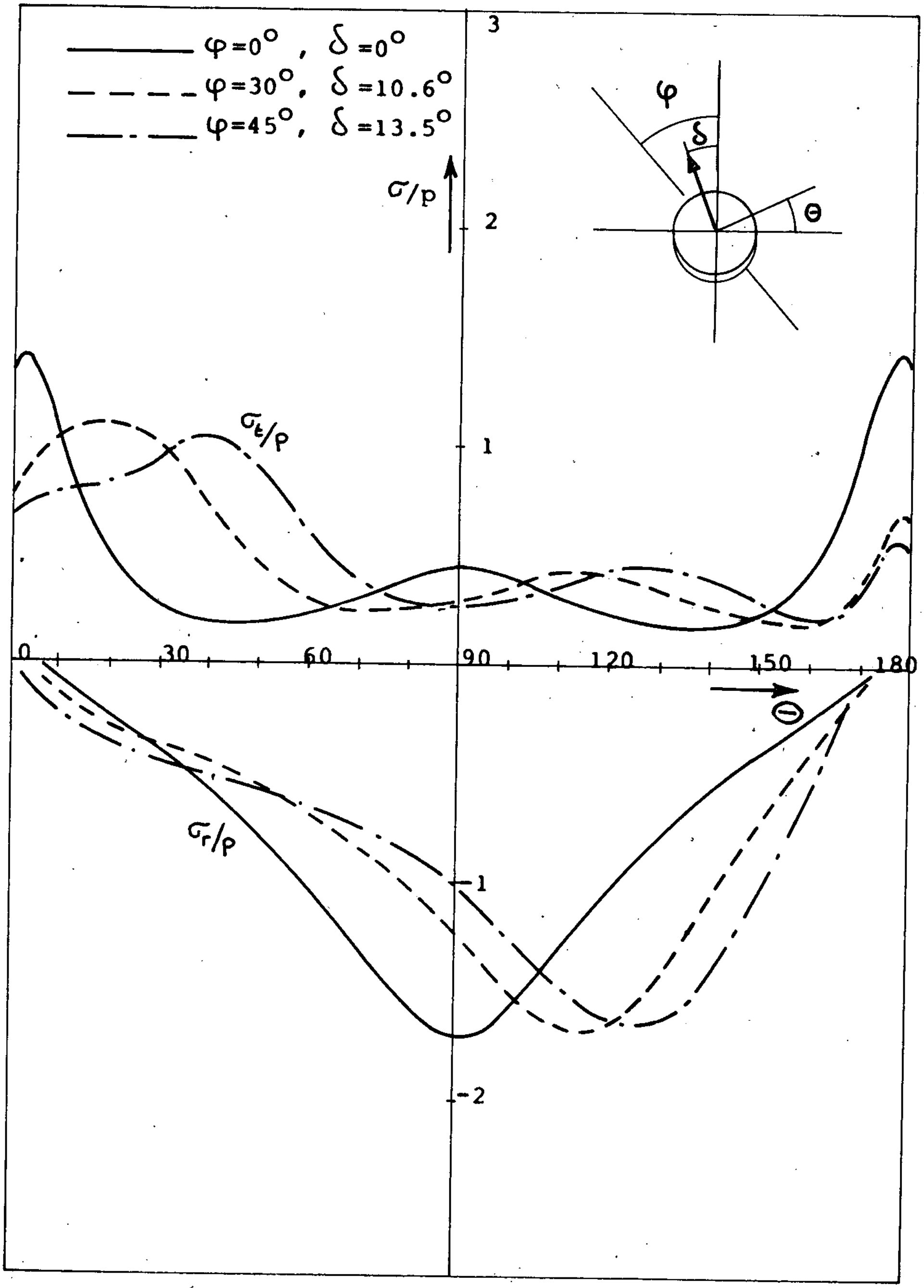

FIGURE 6: THE STRESS DISTRIBUTION AROUND A PIN-LOADED HOLE IN A $(90^{o}_{4}/\pm 45^{o})_{s}$ C.F.R.P. LAMINATE.

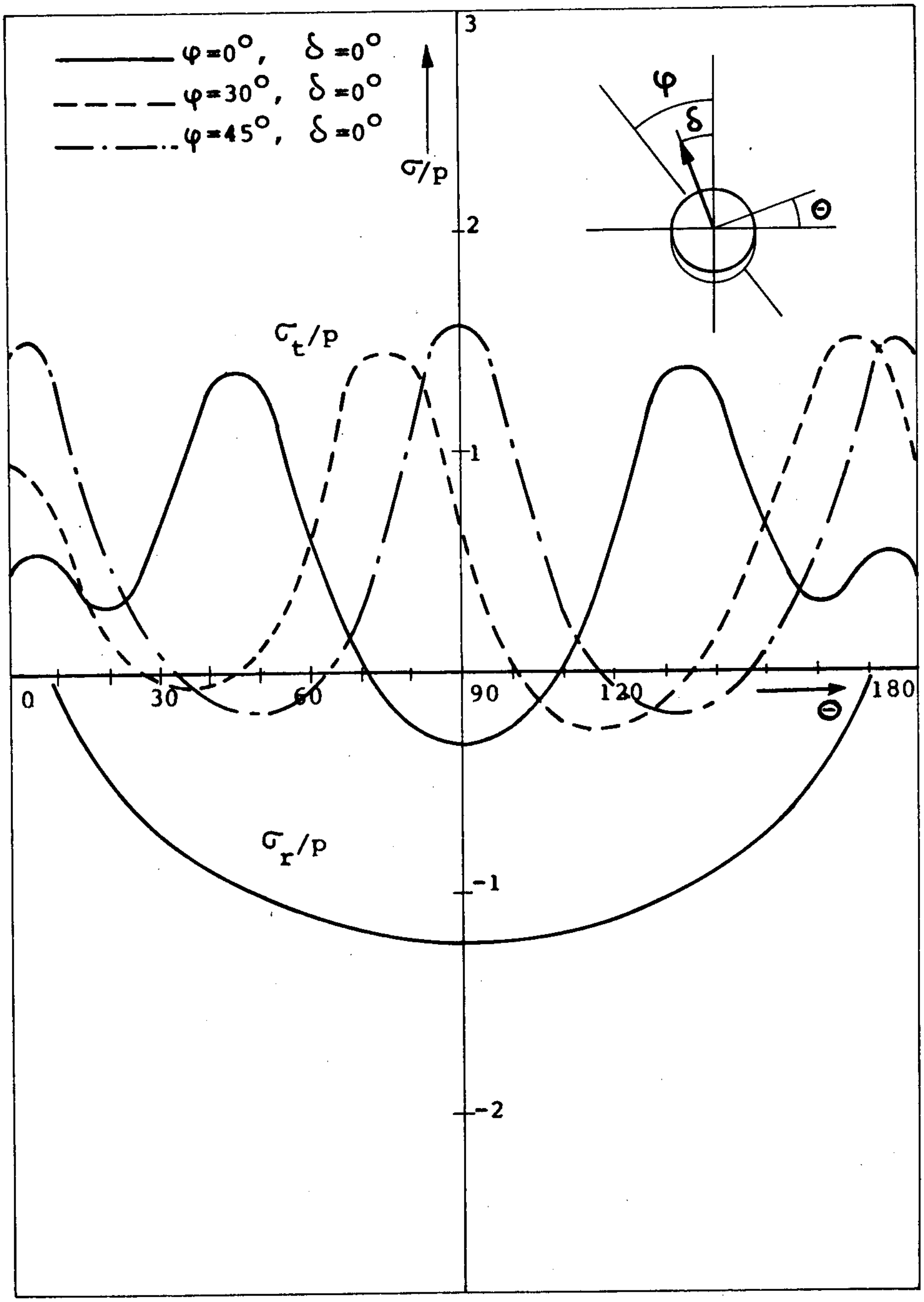

FIGURE 7: THE STRESS DISTRIBUTION AROUND A PIN-LOADED HOLE IN A $(\pm 45^{\circ})_{s}$ C.F.R.P. LAMINATE.

Laminate type	E_x GPa	E_y GPa	G_{xy} GPa	μ_{xy}	s_1	s_2
unidirectional	9.11	156.39	5.35	0.0198	0.189i	1.280i
$(90^\circ/\pm45^\circ/0^\circ)_s$	quasi-isotropic			0.2237	i	i
$(90^\circ/\pm45^\circ)_s$	25.78	66.12	20.27*	0.2701	0.360 + 0.704i	-0.360 + 0.704i
$(90^\circ_2/\pm45^\circ)_s$	23.62	80.03	16.55*	0.1650	0.687i	0.790i
$(90^\circ_4/\pm45^\circ)_s$	18.75	121.31	12.81*	0.0958	0.369i	1.070i
$(\pm45^\circ)_s$	20.43	20.43	27.74	0.7287	0.825 + 0.565i	-0.825 + 0.565i

Table 1: The laminate elastic constants and the complex material parameters s.

* values are calculated from the unidirectional and $(\pm45^\circ)_s$ values.

	U.D. $\varphi = 0^{\circ}$	U.D. $\varphi = 30^{\circ}$	Q.I. $\varphi = 0^{\circ}$
a(1)	0.100d+01	0.100d+01	0.100d+01
a(2)	-0.111d-05	-0.427d+00	0.205d-08
a(3)	-0.382d+00	-0.725d-01	0.677d-01
a(4)	0.467d-07	0.270d+00	0.543d-09
a(5)	0.142d+00	-0.124d+00	0.104d-01
a(6)	-0.159d-05	0.230d-02	0.261d-08
a(7)	-0.637d-01	0.438d-01	-0.708d-02
a(8)	-0.268d-05	-0.107d-01	0.592d-08
a(9)	0.109d-01	-0.161d-01	-0.130d-01
a(10)	-0.393d-05	0.244d-01	0.960d-08
a(11)	-0.158d-01	-0.171d-01	-0.142d-01
a(12)	-0.460d-05	0.875d-02	0.128d-07
a(13)	-0.466d-02	-0.704d-02	-0.129d-01
a(14)	-0.468d-05	0.598d-02	0.149d-07
a(15)	-0.657d-02	-0.769d-02	-0.104d-01
a(16)	-0.407d-05	0.559d-02	0.152d-07
a(17)	-0.359d-02	-0.549d-02	-0.742d-02
a(18)	-0.295d-05	0.307d-02	0.132d-07
a(19)	-0.243d-02	-0.296d-02	-0.441d-02
a(20)	-0.161d-05	0.148d-02	0.895d-08
a(21)	-0.974d-03	-0.135d-02	-0.191d-02
a(22)	-0.478d-06	0.452d-03	0.318d-08
a(23)	-0.257d-03	-0.318d-03	-0.427d-03

Table 2: The coefficients a_n for the unidirectional and the quasi-isotropic laminate.

	φ = 0°	φ = 30°	φ = 45°
a(1)	0.100d+01	0.100d+01	0.100d+01
a(2)	0.209d-08	-0.153d+00	-0.181d+00
a(3)	-0.972d-01	-0.781d-02	0.752d-01
a(4)	0.281d-10	0.682d-01	0.589d-01
a(5)	0.113d-01	-0.148d-01	-0.176d-01
a(6)	0.236d-08	0.126d-01	0.265d-01
a(7)	-0.158d-01	-0.127d-01	-0.163d-01
a(8)	0.548d-08	0.967d-02	0.143d-01
a(9)	-0.159d-01	-0.179d-01	-0.195d-01
a(10)	0.896d-08	0.781d-02	0.109d-01
a(11)	-0.159d-01	-0.174d-01	-0.195d-01
a(12)	0.120d-07	0.589d-02	0.834d-02
a(13)	-0.134d-01	-0.151d-01	-0.171d-01
a(14)	0.140d-07	0.436d-02	0.615d-01
a(15)	-0.105d-01	-0.118d-01	-0.135d-01
a(16)	0.144d-07	0.302d-02	0.425d-02
a(17)	-0.735d-02	-0.828d-02	-0.946d-02
a(18)	0.125d-07	0.186d-02	0.261d-02
a(19)	-0.433d-02	-0.488d-02	-0.558d-02
a(20)	0.849d-08	0.911d-03	0.128d-02
a(21)	-0.187d-02	-0.211d-02	-0.241d-02
a(22)	0.304d-08	0.252d-03	0.352d-03
a(23)	-0.416d-03	-0.469d-03	-0.536d-03

Table 3: The coefficients a_n for the $(90^\circ/\pm 45^\circ)_s$-laminate.

	$\varphi = 0^{\circ}$	$\varphi = 30^{\circ}$	$\varphi = 45^{\circ}$
a(1)	0.100d+01	0.100d+01	0.100d+01
a(2)	0.168d-08	-0.195d+00	-0.233d+00
a(3)	-0.133d+00	-0.148d-01	0.939d-01
a(4)	-0.595d-10	0.909d-01	0.719d-01
a(5)	0.264d-01	-0.179d-01	-0.266d-01
a(6)	0.182d-08	0.126d-01	0.348d-01
a(7)	-0.129d-01	-0.489d-02	-0.111d-01
a(8)	0.415d-08	0.100d-01	0.164d-01
a(9)	-0.104d-01	-0.127d-01	-0.135d-01
a(10)	0.678d-08	0.891d-02	0.126d-01
a(11)	-0.112d-01	-0.128d-01	-0.147d-01
a(12)	0.911d-08	0.660d-02	0.986d-02
a(13)	-0.977d-02	-0.112d-01	-0.132d-01
a(14)	0.106d-07	0.489d-02	0.728d-02
a(15)	-0.775d-02	-0.896d-02	-0.105d-01
a(16)	0.108d-07	0.340d-02	0.503d-02
a(17)	-0.544d-02	-0.630d-02	-0.747d-02
a(18)	0.944d-08	0.209d-02	0.309d-02
a(19)	-0.322d-02	-0.373d-02	-0.443d-02
a(20)	0.640d-08	0.102d-02	0.151d-02
a(21)	-0.139d-02	-0.161d-02	-0.192d-02
a(22)	0.229d-08	0.283d-03	0.418d-03
a(23)	-0.311d-03	-0.360d-03	-0.428d-03

Table 4: The coefficients a_n for the $(90^{\circ}_2/\pm 45^{\circ})_s$- laminate.

	$\varphi = 0^\circ$	$\varphi = 30^\circ$	$\varphi = 45^\circ$
a(1)	0.100d+01	0.100d+01	0.100d+01
a(2)	0.159d-08	-0.292d+00	-0.354d+00
a(3)	-0.237d+00	-0.454d-01	0.126d+00
a(4)	-0.282d-09	0.157d+00	0.106d+00
a(5)	0.609d-01	-0.517d-01	-0.644d-01
a(6)	0.164d-08	0.112d-01	0.654d-01
a(7)	-0.254d-01	0.507d-02	-0.195d-01
a(8)	0.362d-08	0.831d-02	0.208d-01
a(9)	-0.684d-02	-0.140d-01	-0.121d-01
a(10)	0.597d-08	0.129d-01	0.164d-01
a(11)	-0.113d-01	-0.133d-01	-0.155d-01
a(12)	0.802d-08	0.840d-02	0.142d-01
a(13)	-0.872d-02	-0.107d-01	-0.142d-01
a(14)	0.935d-08	0.626d-02	0.105d-01
a(15)	-0.706d-02	-0.865d-02	-0.111d-01
a(16)	0.955d-08	0.443d-02	0.723d-02
a(17)	-0.488d-02	-0.605d-02	-0.777d-02
a(18)	0.833d-08	0.271d-02	0.445d-02
a(19)	-0.288d-02	-0.356d-02	-0.460d-02
a(20)	0.566d-08	0.133d-02	0.218d-02
a(21)	-0.124d-02	-0.154d-02	-0.199d-02
a(22)	0.203d-08	0.370d-03	0.604d-03
a(23)	-0.278d-03	-0.343d-03	-0.443d-03

Table 5: The coefficients a_n for the $(90^\circ_4/\pm 45^\circ)_s$ - laminate.

	$\varphi = 0^{\circ}$	$\varphi = 30^{\circ}$	$\varphi = 45^{\circ}$
a(1)	0.100d+01	0.100d+01	0.100d+01
a(2)	0.328d-08	0.328d-08	0.328d-08
a(3)	0.367d-01	0.367d-01	0.367d-01
a(4)	0.732d-09	0.732d-09	0.732d-09
a(5)	-0.184d-01	-0.184d-01	-0.184d-01
a(6)	0.461d-08	0.461d-08	0.461d-08
a(7)	-0.327d-01	-0.327d-01	-0.327d-01
a(8)	0.107d-07	0.107d-07	0.107d-07
a(9)	-0.352d-01	-0.352d-01	-0.352d-01
a(10)	0.176d-07	0.176d-07	0.176d-07
a(11)	-0.327d-01	-0.327d-01	-0.327d-01
a(12)	0.237d-07	0.237d-07	0.237d-07
a(13)	-0.277d-01	-0.277d-01	-0.277d-01
a(14)	0.277d-07	0.277d-07	0.277d-07
a(15)	-0.214d-01	-0.214d-01	-0.214d-01
a(16)	0.283d-07	0.283d-07	0.283d-07
a(17)	-0.148d-01	-0.148d-01	-0.148d-01
a(18)	0.247d-07	0.247d-07	0.247d-07
a(19)	-0.871d-02	-0.871d-02	-0.871d-02
a(20)	0.167d-07	0.167d-07	0.167d-07
a(21)	-0.374d-02	-0.374d-02	-0.374d-02
a(22)	0.600d-08	0.600d-08	0.600d-08
a(23)	-0.831d-03	-0.831d-03	-0.831d-03

Table 6: The coefficients a_n for the $(\pm 45^{\circ})_s$- laminate.

Laminate type	$\bar{\sigma}_x$ MPa Tension	Compr.	$\bar{\sigma}_y$ MPa Tension	Compr.	$\bar{\tau}_{xy}$ MPa
unidirectional	64	212	1600	1042	70
$(90^\circ/\pm45^\circ/0^\circ)_s$	620	485	620	485	217
$(90^\circ/\pm45^\circ)_s$	255	354	790	690	310
$(90^\circ_2/\pm45^\circ)_s$	212	286	1072	805	217
$(90^\circ_4/\pm45^\circ)_s$	72	236	1418	823	168
$(\pm45^\circ)_s$	177	207	177	207	429

Table 7: Laminate strength values. $\bar{\tau}_{xy}$ is calculated from the individual layer values.

Laminate type	Rotation angle of the laminate $\varphi = 0^\circ$ $\bar{p}$ MPa	Θ°	$\varphi = 30^\circ$ $\bar{p}$ MPa	Θ°	$\varphi = 45^\circ$ $\bar{p}$ MPa	Θ°
unidirectional	123	90	119	100, 110		
$(90^\circ/\pm45^\circ/0^\circ)_s$	299	60, 120				
$(90^\circ/\pm45^\circ)_s$	437	60, 120	365	80	338	90
$(90^\circ_2/\pm45^\circ)_s$	339	60, 120	286	80	264	90
$(90^\circ_4/\pm45^\circ)_s$	191	90	190	120	186	130
$(\pm45^\circ)_s$	189	90	199	110	213	60, 120

Table 8: The theoretical bearing strength $\bar{p}$ for three angles of rotation of the laminate with respect to the coordinate axes. Θ denotes the point on the edge of the hole where the predicted first significant damage occurs.

APPENDIX 1. THE HOLOMORPHIC PARTS $\phi_k^{\,o}(z_k)$ OF THE COMPLEX STRESS FUNCTIONS

The boundary condition (4.3) for the holomorphic parts $\phi_k^{\,o}(z_k)$ of the complex stress functions results in

$$-(S_{\ell_\varphi} - S_{k_\varphi})\ \phi_k^{\,o}(\zeta_k) = \frac{S_{\ell_\varphi}}{2\pi i} \oint \frac{f_2}{\sigma - \zeta_k}\,d\sigma - \frac{1}{2\pi i} \oint \frac{f_1}{\sigma - \zeta_k}\,d\sigma \qquad (4.5)$$

in which f_1 and f_2 are given in (3.12) and (3.13) respectively. In the expressions for f_1 and f_2 all sines and cosines are replaced by powers of σ according to

$$\cos N\theta = \frac{\sigma^N + \sigma^{-N}}{2} \quad \text{and} \quad \sin N\theta = \frac{\sigma^N - \sigma^{-N}}{2i}$$

In the general term $\oint \frac{\sigma^N}{\sigma - \zeta_k}\,d\sigma$ in (4.5) $\frac{\sigma^N}{\sigma - \zeta_k}$ is the boundary value of a function $\frac{\zeta_k^{*N}}{\zeta_k^{*} - \zeta_k}$, analytic <u>inside</u> the unit circle. For ζ_k outside this circle one finds

$$\oint \frac{\sigma^N}{\sigma - \zeta_k}\,d\sigma = 0 \qquad (A1.1)$$

In the other general term $\oint \frac{\sigma^{-N}}{\sigma - \zeta_k}\,d\sigma$ in (4.5) σ^{-N} is the boundary value of a function ζ_k^{-N}, analytic outside the unit circle and zero at infinity. So Cauchy's integral for the infinite region gives

$$\frac{1}{2\pi i}\oint\frac{\sigma^{-N}}{\sigma-\zeta_k}\,d\sigma = -\frac{1}{\zeta_k{}^N} \tag{A1.2}$$

As a result of (A1.1) and (A1.2) the integrals in (4.5) become

$$\frac{1}{2\pi i}\oint\frac{f_1 d\sigma}{\sigma-\zeta_k} = p_o\ [\tfrac{1}{8}\ \{\frac{-a_2}{\zeta_k} - \sum_{n=1,2}^{\infty}\frac{a_n+a_{n+2}}{n+1}\frac{1}{\zeta_k^{n+1}}\} -$$

$$\frac{R_x}{p_o}\frac{1}{8\pi i\zeta_k{}^2} - \frac{1}{2\pi i\zeta_k}\sum_{n=1,3}^{\infty}\frac{a_n}{n} -$$

$$\frac{1}{2\pi}\left(\sum_{n=1,3}^{\infty}\ \sum_{m=2,4}^{\infty} + \sum_{n=2,4}^{\infty}\ \sum_{m=3,5}^{\infty}\right) a_n\ \left(\frac{1}{n+m}+\frac{1}{n-m}\right)\cdot$$

$$\{\frac{1}{2i(m-1)\ \zeta_k{}^{m-1}} + \frac{1}{2i(m+1)\ \zeta_k{}^{m+1}}\}]$$

or, after using exp ression (3.10) for R_x and rearranging the double summations

$$\frac{1}{2\pi i}\oint\frac{f_1 d\sigma}{\sigma-\zeta_k} = p_o\ [\tfrac{1}{8}\ \{\frac{-a_2}{\zeta_k} - \sum_{n=1,2}^{\infty}\frac{a_n+a_{n+2}}{n+1}\frac{1}{\zeta_k{}^{n+1}}\} \tag{A1.3}$$

$$+\frac{1}{\pi i}\left(\sum_{n=1,3}^{\infty}\ \sum_{m=1,3}^{\infty} + \sum_{n=2,4}^{\infty}\ \sum_{m=2,4}^{\infty}\right)\frac{na_n(m^2-n^2+1)\ \zeta_k{}^{-m}}{m\{(m+1)^2-n^2\}\{(m-1)^2-n^2\}}]$$

$$\frac{1}{2\pi i}\oint\frac{f_2 d\sigma}{\sigma-\zeta_k} = p_o\ [\tfrac{1}{8i}\ \{\frac{a_2}{\zeta_k} - \sum_{n=1,2}^{\infty}\frac{a_n-a_{n+2}}{n+1}\frac{1}{\zeta_k{}^{n+1}}\} \tag{A1.4}$$

$$-\frac{2}{\pi}\left(\sum_{n=1,3}^{\infty}\ \sum_{m=1,3}^{\infty} + \sum_{n=2,4}^{\infty}\ \sum_{m=2,4}^{\infty}\right)\frac{na_n m\ \zeta_k{}^{-m}}{m\{(m+1)^2-n^2\}\{(m-1)^2-n^2\}}]$$

Introduction of

$$N_{m,n} = m\{(m + 1)^2 - n^2\}\{(m - 1)^2 - n^2\}$$

$$\sum_{m,n} = \sum_{n=1,3}^{\infty} \sum_{m=1,3}^{\infty} + \sum_{n=2,4}^{\infty} \sum_{m=2,4}^{\infty}$$

$$a_0 = 0$$

and substitution of (A1.3) and (A1.4) in (4.5) results in

$$\phi_k^{o}(\zeta_k) = \frac{p_o}{2\pi i(S_{\ell_\varphi} - S_{k_\varphi})} \Big[\sum_{m,n} \frac{2na_n(m^2-n^2+1+2iS_{\ell_\varphi} m)\ \zeta_k^{-m}}{N_{m,n}} - \frac{\pi i}{4} \sum_{n=0,1,2}^{\infty} \frac{a_n(1+iS_{\ell_\varphi}) + a_{n+2}(1-iS_{\ell_\varphi})}{n + 1}\ \zeta_k^{-n-1} \Big] \quad \text{(A1.5)}$$

For the calculation of the stresses the complex stress functions must be differentiated with respect to z_k. For the second series of (A1.5) that is allowed only if after differentiation the general term

$$\{a_n(1+iS_{\ell_\varphi}) + a_{n+2}(1-iS_{\ell_\varphi})\}\ \zeta_k^{-n-1} \ / \ (z_k^{\ 2} - S_k^{\ 2} - 1)^{\frac{1}{2}}$$

has a limit for $n \to \infty$. For points outside the hole, where $|\zeta_k| > 1$, this limit exists. For points on the edge of the hole, where $|\zeta_k| = 1$ this limit only exists if $\lim\limits_{n\to\infty} a_n = 0$.

APPENDIX 2. EXPRESIONS FOR THE CONSTANTS $C_4 - C_7$

The transformation formulae for the material compliances S_{ij_φ}, relevant for this report, will be expressed first in the complex material parameters S_k as presented in (2.6).

$$S_{11_\varphi} = S_{11} \cos^4\varphi + S_{22} \sin^4\varphi + (2S_{12}+S_{66}) \sin^2\varphi \cos^2\varphi$$

$$= S_{11} \{\cos^4\varphi + S_1{}^2 S_2{}^2 \sin^4\varphi - (S_1{}^2+S_2{}^2) \sin^2\varphi \cos^2\varphi\}$$

$$= S_{11} (S_1{}^2 \sin^2\varphi - \cos^2\varphi)(S_2{}^2 \sin^2\varphi - \cos^2\varphi) \qquad \text{(A2.1)}$$

$$S_{22_\varphi} = S_{11} (S_1{}^2 \cos^2\varphi - \sin^2\varphi)(S_2{}^2 \cos^2\varphi - \sin^2\varphi) \qquad \text{(A2.2)}$$

$$S_{12_\varphi} = S_{12} + S_{11}(1+S_1{}^2)(1+S_2{}^2) \sin^2\varphi \cos^2\varphi \qquad \text{(A2.3)}$$

$$S_{16_\varphi} = S_{11}\{(2S_1{}^2S_2{}^2+S_1{}^2+S_2{}^2) \sin^3\varphi \cos\varphi -$$

$$(2+S_1{}^2+S_2{}^2) \sin\varphi \cos^3\varphi\} \qquad \text{(A2.4)}$$

$$S_{26_\varphi} = S_{11}\{(2S_1{}^2S_2{}^2+S_1{}^2+S_2{}^2) \sin\varphi \cos^3\varphi -$$

$$(2+S_1{}^2+S_2{}^2) \sin^3\varphi \cos\varphi\} \qquad \text{(A2.5)}$$

Since $S_1{}^2S_2{}^2 = \bar{S}_1{}^2\bar{S}_2{}^2$ and $S_1{}^2 + S_2{}^2 = \bar{S}_1{}^2 + \bar{S}_2{}^2$ the complex parameters S_k in (A2.1) - (A2.5) can be replaced by their conjugated complex values.

From (2.5) and (A2.1) - (A2.5) it is easily derived that

$$S_{1_\varphi} S_{2_\varphi} = \frac{(S_1 \cos\varphi - \sin\varphi)(S_2 \cos\varphi - \sin\varphi)(S_1 \sin\varphi - \cos\varphi)(S_2 \sin\varphi - \cos\varphi)}{(S_1{}^2 \sin^2\varphi - \cos^2\varphi)(S_2{}^2 \sin^2\varphi - \cos^2\varphi)}$$

$$= \frac{\{(S_1{}^2+1) \sin\varphi \cos\varphi - S_1\}\{(S_2{}^2+1) \sin\varphi \cos\varphi - S_2\}}{S_{11_\varphi}/S_{11}}$$

$$= \frac{S_{12_\varphi} - S_{12} - S_{11}(S_1S_2+1)(S_1+S_2) \sin\varphi \cos\varphi + S_{11}S_1S_2}{S_{11_\varphi}} \qquad \text{(A2.6)}$$

In (A2.6) S_1S_2 is real and $S_1 + S_2$ is imaginary, hence

$$\text{Re } S_{1_\varphi} S_{2_\varphi} = \frac{S_{12_\varphi} - S_{12} + S_{11}S_1S_2}{S_{11_\varphi}}$$

and (A2.7)

$$\text{Im } S_{1_\varphi} S_{2_\varphi} = \frac{-S_{11}(S_1S_2+1) \sin\varphi \cos\varphi \ \text{Im } (S_1+S_2)}{S_{11_\varphi}}$$

In the same way is found

$$S_{1_\varphi} + S_{2_\varphi} = \frac{S_{16} - S_{11}S_1S_2(S_1+S_2) \sin^2\varphi + S_{11}(S_1+S_2) \cos^2\varphi}{S_{11_\varphi}} \qquad \text{(A2.8)}$$

in which

$$\text{Re } (S_{1_\varphi} + S_{2_\varphi}) = \frac{S_{16_\varphi}}{S_{11_\varphi}}$$

and (A2.9)

$$\mathrm{Im}\,(S_{1_\varphi} + S_{2_\varphi}) = \frac{S_{11}(\cos^2\varphi - S_1S_2 \sin^2\varphi)\ \mathrm{Im}(S_1+S_2)}{S_{11_\varphi}}$$

$$\frac{1}{S_{1_\varphi}} + \frac{1}{S_{2_\varphi}} = \frac{S_{26_\varphi} + S_{11}S_1S_2(S_1+S_2)\cos^2\varphi - S_{11}(S_1+S_2)\sin^2\varphi}{S_{22_\varphi}} = \frac{S_{1_\varphi}+S_{2_\varphi}}{S_{1_\varphi}S_{2_\varphi}} \quad \text{(A2.10)}$$

in which

$$\mathrm{Re}\,\left(\frac{S_{1_\varphi} + S_{2_\varphi}}{S_{1_\varphi}S_{2_\varphi}}\right) = \frac{S_{26_\varphi}}{S_{22_\varphi}}$$

and (A2.11)

$$\mathrm{Im}\,\left(\frac{S_{1_\varphi}+S_{2_\varphi}}{S_{1_\varphi}S_{2_\varphi}}\right) = \frac{S_{11}(S_1S_2\cos^2\varphi - \sin^2\varphi)\ \mathrm{Im}(S_1+S_2)}{S_{22_\varphi}}$$

The expressions for $C_4 - C_7$ are found by substituting (A2.9), (A2.7) and (A2.11) in (5.2), (5.3) and (5.13) respectively.

$$2C_4 = \overline{C}_2+C_2 = 2\ \mathrm{Re}\ C_2 = 2\ \mathrm{Re}\ \{i(S_{11_\varphi}S_{1_\varphi}S_{2_\varphi} - S_{12_\varphi})\}$$

$$= \underline{2S_{11}(S_1S_2+1)\sin\varphi\cos\varphi\ \mathrm{Im}(S_1+S_2)} \quad \text{(A2.12)}$$

$$-2iC_5 = \overline{C}_1-C_1 = -2i\ \mathrm{Im}\ C_1 = -2i\ \mathrm{Im}\ \{S_{11_\varphi}(S_{1_\varphi}+S_{2_\varphi}) - S_{16_\varphi}\}$$

$$= \underline{2iS_{11}(S_1S_2\sin^2\varphi - \cos^2\varphi)\ \mathrm{Im}(S_1+S_2)}$$

(A2.13)

$$2iC_6 = \overline{C}_2 - C_2 = -2i\ \mathrm{Im}\ C_2 = -2i\ \mathrm{Im}\ \{i(S_{11_\varphi}S_{1_\varphi}S_{2_\varphi} - S_{12_\varphi})\}$$

$$= \underline{2i(S_{12} - S_{11}S_1S_2)} \qquad \text{(A2.14)}$$

$$-2iC_7 = \overline{C}_3 - C_3 = -2i\ \mathrm{Im}\ C_3 = -2i\ \mathrm{Im}\ (S_{26_\varphi} - S_{22_\varphi}\frac{S_{1_\varphi}+S_{2_\varphi}}{S_{1_\varphi}S_{2_\varphi}})$$

$$= \underline{2iS_{11}(S_1S_2\cos^2\varphi - \sin^2\varphi)\ \mathrm{Im}(S_1+S_2)}$$

(A2.15)

$$\overline{C}_1 + C_1 = 2\ \mathrm{Re}\ \{S_{11_\varphi}(S_{1_\varphi}+S_{2_\varphi}) - S_{16_\varphi}\} = 2\ \mathrm{Re}(S_{16_\varphi} - S_{16_\varphi}) = 0$$

$$\overline{C}_3 + C_3 = 2\ \mathrm{Re}\ \{S_{26_\varphi} - S_{22_\varphi}\frac{S_{1_\varphi}+S_{2_\varphi}}{S_{1_\varphi}S_{2_\varphi}}\} = 2\ \mathrm{Re}(S_{26_\varphi} - S_{26_\varphi}) = 0$$

Note: For materials with equal S_{11} and S_{22}, resulting in $S_1S_2 = -1$, it is easily seen that expressions (A2.12), (A2.13) and (A2.15) are invariant, just as (A2.14); so for these materials C_4, C_5 and C_7 do not depend on the angle φ between material axes and coordinate axes. This implies that the displacements u and v on the edge of the hole are independent of a rotation of these materials!

An analogous derivation as used for (A2.6) results in

$$\frac{1}{S_{1_\varphi} S_{2_\varphi}} = \frac{S_{12_\varphi} - S_{12} + S_{11}(S_1S_2+1)(S_1+S_2)\ \sin\varphi\cos\varphi + S_{11}S_1S_2}{S_{22_\varphi}}$$

or, with $S_1 + S_2 = -(\overline{S}_1 + \overline{S}_2)$ and $S_1S_2 = \overline{S}_1\overline{S}_2$:

$$\frac{1}{\overline{S}_{1_\varphi} \overline{S}_{2_\varphi}} = S_1S_2 \frac{S_{11_\varphi}}{S_{22_\varphi}}, \quad \text{so } \underline{S_{1_\varphi} S_{2_\varphi} \overline{S}_{1_\varphi} \overline{S}_{2_\varphi} = \frac{S_{22_\varphi}}{S_{11_\varphi}}} \qquad \text{(A2.16)}$$

Expression (A2.16) will be used in Appendix 4.

APPENDIX 3. TRIGONOMETRIC FUNCTIONS FOR A NUMBER OF SERIES IN THE DISPLACEMENT FORMULAE

Some of these functions are taken from Appendix 2 of Reference [2], where an extensive derivation is given.
In formula (5.4) for the displacement u is

$$-\frac{8}{\pi}\sum_{m,n} F_1^{(m,n)} \cos m\theta = -\frac{8}{\pi}\left(\sum_{n=1,3}^{\infty}\sum_{m=1,3}^{\infty} + \sum_{n=2,4}^{\infty}\sum_{m=2,4}^{\infty}\right)$$

$$\frac{na_n\{(m^2-n^2+1)\ C_5 - 2m\ C_6\}\ \cos m\theta}{N_{m,n}} \tag{A3.1}$$

where for the double series with odd indices according to Reference [2]

$$\frac{8}{\pi}\sum_{n=1,3}^{\infty}\sum_{m=1,3}^{\infty}\frac{na_n\ 2mC_6\ \cos m\theta}{N_{m,n}} = \frac{8}{\pi}\,2C_6\sum_{m=1,3}^{\infty}\frac{\cos m\theta}{m^2(m-2)(m+2)} +$$

$$2C_6\sum_{n=3,5}^{\infty} a_n\,\frac{\sin n\theta\ \cos\theta - n\ \cos n\theta\ \sin\theta}{n^2-1} \tag{A3.2}$$

The first series in (A3.2) can be evaluated further after dividing into partial fractions:

$$\frac{8}{\pi}\,2C_6\sum_{m=1,3}^{\infty}\frac{\cos m\theta}{m^2(m-1)\ (m+2)} =$$

$$\frac{8}{\pi}\,2C_6\sum_{m=1,3}^{\infty}\left\{\frac{1}{16}\left(\frac{1}{m-2}-\frac{1}{m+2}\right)-\frac{1}{4}\,\frac{1}{m^2}\right\}\ \cos m\theta$$

$$= \frac{8}{\pi} 2C_6 \left[\frac{1}{16} \left\{ \sum_{m=1,3}^{\infty} \frac{\cos(m-2)\,\theta \cos 2\theta - \sin(m-2)\,\theta \sin 2\theta}{m - 2} \right.\right.$$

$$\left.\left. - \frac{\cos(m+2)\,\theta \cos 2\theta + \sin(m+2)\,\theta \sin 2\theta}{m + 2} \right\} - \frac{1}{4} \sum_{m=1,3}^{\infty} \frac{\cos m\theta}{m^2} \right]$$

$$= \frac{8}{\pi} 2C_6 \left[- \frac{\sin 2\theta}{8} \sum_{m=1,3}^{\infty} \frac{\sin m\theta}{m} - \frac{1}{4} \sum_{m=1,3}^{\infty} \frac{\cos m\theta}{m^2}\right]$$

where, according to 416.01 and 416.10 of Reference [5]

$$\sum_{m=1,3}^{\infty} \frac{\sin m\theta}{m} = \frac{\pi}{4} \text{ and } \sum_{m=1,3}^{\infty} \frac{\cos m\theta}{m^2} = \frac{\pi^2}{8} - \frac{\pi}{4}\,\theta \text{ for } 0 < \theta < \pi$$

so, for the upper half of the edge of the hole

$$\frac{8}{\pi} 2C_6 \sum_{m=1,3}^{\infty} \frac{\cos m\theta}{m^2 (m-2)\,(m+2)} = C_6 \left(\theta - \frac{\pi}{2} - \frac{\sin 2\theta}{2}\right) \qquad \text{(A3.3)}$$

For the summation with even indices is found

$$\frac{8}{\pi} 2C_6 \sum_{n=2,4}^{\infty} \sum_{m=2,4}^{\infty} \frac{n a_n\, m \cos m\theta}{N_{m,n}}$$

$$= \frac{1}{\pi} C_6 \sum_{n=2,4}^{\infty} a_n \sum_{m=2,4}^{\infty} \left\{ \frac{\cos m\theta}{\left(\frac{m}{2}\right)^2 - \left(\frac{n+1}{2}\right)^2} - \frac{\cos m\theta}{\left(\frac{m}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2} \right\}$$

According to 416.17 of Reference [5] for $x = 2\theta - \pi$, with $0 < \theta < \pi$

$$\sin(n+1)\,\theta = \frac{n+1}{\pi}\left\{\frac{1}{2\left(\frac{n+1}{2}\right)^2} - \sum_{m=2,4}^{\infty} \frac{\cos m\theta}{\left(\frac{m}{2}\right)^2 - \left(\frac{n+1}{2}\right)^2}\right\}$$

$$\sin(n-1)\,\theta = \frac{n-1}{\pi}\left\{\frac{1}{2\left(\frac{n-1}{2}\right)^2} - \sum_{m=2,4}^{\infty} \frac{\cos m\theta}{\left(\frac{m}{2}\right) - \left(\frac{n-1}{2}\right)^2}\right\}$$

so

$$\frac{8}{\pi} 2C_6 \sum_{n=2,4}^{\infty} \sum_{m=2,4}^{\infty} \frac{na_n\, m \cos m\theta}{N_{m,n}}$$

$$= \frac{1}{\pi} C_6 \sum_{n=2,4}^{\infty} a_n \left\{\frac{\pi}{n-1} \sin(n-1)\theta - \frac{\pi}{n+1} \sin(n+1)\theta + \frac{2}{(n+1)^2} - \frac{2}{(n-1)^2}\right\}$$

$$= 2C_6 \sum_{n=2,4}^{\infty} a_n \frac{\sin n\theta \cos 2\theta - n \cos n\theta \sin\theta}{n^2 - 1} - \frac{8}{\pi} C_6 \sum_{n=2,4}^{\infty} \frac{na_n}{(n^2-1)^2} \qquad \text{(A3.4)}$$

(A3.2), (A3.3) and (A3.4) now result in

$$\frac{8}{\pi} \sum_{m,n} \frac{na_n\, 2mC_6 \cos m\theta}{N_{m,n}} = 2C_6 \sum_{n=2,3}^{\infty} \frac{a_n}{n^2-1} (\sin n\theta \cos\theta - n \cos n\theta \sin\theta)$$

$$+ C_6 \left(\theta - \frac{\pi}{2} - \frac{\sin 2\theta}{2}\right) - \frac{8}{\pi} C_6 \sum_{n=2,4}^{\infty} \frac{na_n}{(n^2-1)^2} \qquad \text{(A3.5)}$$

Two other series in (5.4) can simply be modified:

$$\sum_{n=0,1,2}^{\infty} F_3^{(n)} \sin(n+1)\,\theta = \frac{C_5-C_6}{2} \sin 2\theta$$

$$+ 2 \sum_{n=2,3}^{\infty} \frac{a_n}{n^2-1} \{C_5(n \sin n\theta \cos\theta - \cos n\theta \sin\theta) +$$

$$+ C_6(\sin n\theta \cos\theta - n\cos n\theta \sin\theta)\} \qquad \text{(A3.6)}$$

$$\sum_{n=0,1,2}^{\infty} F_3^{*(n)} \cos(n+1)\,\theta = \frac{C_4}{2} \cos\,\theta$$

$$- 2C_4 \sum_{n=2,3}^{\infty} \frac{a_n}{n^2-1} (\cos n\theta \cos\theta + n \sin n\theta \sin\theta) \qquad \text{(A3.7)}$$

With (A3.5), (A3.6) and (A.37) the formula for the displacement u of the points on the upper half of the edge now becomes:

$$\frac{u}{p_o/4} = \{C_6(\theta - \frac{\pi}{2}) + \frac{C_5-2C_6}{2} \sin 2\theta + \frac{C_4}{2} \cos 2\theta +$$

$$\frac{8}{\pi} \sum_{m=1,3}^{\infty} \frac{2C_4 \sin m\theta - mC_5 \cos m\theta}{m^2(m-2)(m+2)}\}$$

$$+ 2 \sum_{n=2,3}^{\infty} a_n \{\frac{C_5(n \sin n\theta \cos\theta - \cos n\theta \sin\theta)}{n^2 - 1} +$$

$$\frac{2C_6(\sin n\theta \cos\theta - n \cos n\theta \sin\theta) - C_4(\cos n\theta \cos\theta + n \sin n\theta \sin\theta)}{n^2 - 1}\}$$

$$+\frac{8}{\pi}\left(\sum_{n=2,4}^{\infty}\sum_{m=2,4}^{\infty}+\sum_{n=3,5}^{\infty}\sum_{m=1,3}^{\infty}\right) a_n \frac{2C_4 mn \sin n m\theta - C_5 n(m^2-n^2+1)\cos m\theta}{N_{m,n}}$$

$$-\frac{8}{\pi} C_6 \sum_{n=2,4}^{\infty} \frac{na_n}{(n^2-1)^2} \tag{A3.8}$$

In formula (5.14) for the displacement V is

$$\frac{8}{\pi}\sum_{m,n} F_2^{(m,n)} \sin n m\theta = \frac{8}{\pi}\left(\sum_{n=1,3}^{\infty}\sum_{m=1,3}^{\infty}+\sum_{n=2,4}^{\infty}\sum_{m=2,4}^{\infty}\right)$$

$$\frac{na_n\{(m^2-n^2+1)\, C_6 - 2mC_7\} \sin m\theta}{N_{m,n}} \tag{A3.9}$$

in which for the odd indices, according to Reference [2]

$$\frac{8}{\pi}\sum_{n=1,3}^{\infty}\sum_{m=1,3}^{\infty} \frac{na_n(m^2-n^2+1)\, C_6 \sin m\theta}{N_{m,n}}$$

$$= -C_6 \sin^2\theta + 2C_6 \sum_{n=3,5}^{\infty} \frac{a_n(n \cos n\theta \cos\theta + \sin n\theta \sin\theta - n)}{n^2-1} \tag{A3.10}$$

For the even indices

$$\frac{8}{\pi}\sum_{n=2,4}^{\infty}\sum_{m=2,4}^{\infty} \frac{na_n(m^2-n^2+1)\, C_6 \sin n m\theta}{m\{(m-1)^2-n^2\}\{(m+1)^2-n^2\}} =$$

$$\frac{8C_6}{\pi}\sum_{n=2,4}^{\infty} na_n \sum_{m=2,4}^{\infty} \frac{(m^2-n^2-1)\sin n m\theta + 2 \sin n m\theta}{m\{m^2-(n-1)^2\}\{m^2-(n+1)^2\}}$$

Repeated division into partial fractions leads to

$$\frac{8}{\pi}\sum_{n=2,4}^{\infty}\sum_{m=2,4}^{\infty}\frac{na_n(m^2-n^2+1)\ C_6\sin m\theta}{m\{(m-1)^2-n^2\}\{(m+1)^2-n^2\}} =$$

$$\frac{2}{\pi}C_6\sum_{n=2,4}^{\infty}na_n\sum_{m=2,4}^{\infty}\left[\frac{1}{n(n+1)}\frac{\frac{m}{2}\sin m\theta}{(\frac{m}{2})^2-(\frac{n+1}{2})^2}+\right.$$

$$\left.\frac{1}{n(n-1)}\frac{\frac{m}{2}\sin m\theta}{(\frac{m}{2})^2-(\frac{n-1}{2})^2}-\frac{2}{n^2-1}\frac{\sin m\theta}{(\frac{m}{2})}\right]$$

or, with 416.16 and 416.08 of Reference [5] for $x = 2\theta - \pi$ and $x = 2\theta$ respectively $(0 < \theta < \pi)$:

$$\frac{8}{\pi}\sum_{n=2,4}^{\infty}\sum_{m=2,4}^{\infty}\frac{na_n(m^2-n^2+1)\ C_6\sin m\theta}{m\{(m-1)^2-n^2\}\{(m+1)^2-n^2\}}$$

$$= \frac{2}{\pi}C_6\sum_{n=2,4}^{\infty}na_n\left[\frac{\pi}{2n(n+1)}\cos(n+1)\ \theta+\right.$$

$$\left.\frac{\pi}{2n(n-1)}\cos(n-1)\ \theta+\frac{2}{n^2-1}\left(\theta-\frac{\pi}{2}\right)\right]$$

$$= 2C_6\sum_{n=2,4}^{\infty}\frac{a_n}{n^2-1}\left\{n\cos n\theta\cos\theta+\sin n\theta\sin\theta+\frac{2n\theta}{\pi}-n\right\} \qquad \text{(A3.11)}$$

(A3.10) and (A3.11) result in

$$\frac{8}{\pi}\sum_{m,n}\frac{na_n(m^2-n^2+1)\ C_6\sin m\theta}{N_{m,n}} = -C_6\sin^2\theta$$

$$+ 2C_6 \sum_{n=2,3}^{\infty} \frac{a_n}{n^2-1} (n \cos n\theta \cos\theta + \sin n\theta \sin\theta - n)$$

$$+ 2C_6 \sum_{n=2,4}^{\infty} \frac{a_n}{n^2-1} \frac{2n\theta}{\pi} \qquad \text{(A3.12)}$$

Two other series in (5.14) are modified into

$$\sum_{n=0,1,2}^{\infty} F_4^{(n)} \cos(n+1)\,\theta = \frac{C_6-C_7}{2} \cos 2\theta$$

$$+ 2 \sum_{n=2,3}^{\infty} \frac{a_n}{n^2-1} \{C_6(n \cos n\theta \cos\theta + \sin n\theta \sin\theta) +$$

$$C_7(\cos n\theta \cos\theta + n \sin n\theta \sin\theta)\} \qquad \text{(A3.13)}$$

$$\sum_{n=0,1,2}^{\infty} F_4^{*(n)} \sin(n+1)\,\theta = \frac{C_4}{2} \sin 2\theta$$

$$+ 2C_4 \sum_{n=2,3}^{\infty} \frac{a_n}{n^2-1} (n \sin n\theta \cos\theta - \cos n\theta \sin\theta) \qquad \text{(A3.14)}$$

(A3.12), (A3.13) and (A3.14) now yield for the displacement v on the upper half of the edge

$$\frac{v}{p_o/4} = \{\frac{C_6-C_7}{2} - (2C_6-C_7) \sin^2\theta - \frac{C_4}{2} \sin 2\theta +$$

$$\frac{8}{\pi} \sum_{m=1,3}^{\infty} \frac{mC_4 \cos m\theta - 2C_7 \sin m\theta}{m^2(m-2)(m+2)}\} +$$

$$2 \sum_{n=2,3}^{\infty} a_n \{\frac{2C_6(n \cos n\theta \cos\theta + \sin n\theta \sin\theta)}{n^2-1} +$$

$$\frac{C_7(\cos n\theta \cos\theta + n \sin n\theta \sin\theta) - C_4(n \sin n\theta \cos\theta - \cos n\theta \sin\theta) - C_6 n}{n^2-1}\}$$

$$+ \frac{8}{\pi} \left(\sum_{n=2,4}^{\infty} \sum_{m=2,4}^{\infty} + \sum_{n=3,5}^{\infty} \sum_{m=1,3}^{\infty} \right) a_n \frac{C_4 n(m^2-n^2+1) \cos m\theta - 2mC_7 n \sin m\theta}{N_{m,n}}$$

$$+ 2 \sum_{n=2,4}^{\infty} a_n \frac{C_6 2n\theta}{(n^2-1)\ \pi} \qquad \text{(A3.15)}$$

The displacement v_i in the point belonging to $\theta = \frac{\pi}{2}$ is derived from (A3.15) rather easily:

$$\frac{v_i}{p_o/4} = \frac{C_7-3C_6}{2} + \frac{8}{\pi} \sum_{m=1,3}^{\infty} \frac{2C_7(-1)^{\frac{m+1}{2}}}{m^2(m-2)(m+2)}$$

$$+ 2 \sum_{n=2,4}^{\infty} a_n \{\frac{C_4(-1)^{\frac{n}{2}}}{n^2-1} + \frac{4}{\pi} \sum_{m=2,4}^{\infty} \frac{C_4 n(m^2-n^2+1)(-1)^{\frac{m}{2}}}{N_{m,n}}\}$$

$$+ 2 \sum_{n=3,5}^{\infty} a_n \{\frac{2C_6(-1)^{\frac{n-1}{2}} + C_7 n(-1)^{\frac{n-1}{n}} - C_6 n}{n^2-1} +$$

$$\frac{4}{\pi} \sum_{m=1,3}^{\infty} \frac{2mnC_7(-1)^{\frac{m+1}{2}}}{N_{m,n}}\} \qquad \text{(A3.16)}$$

Expressions (A3.8), (A3.15) and (A3.16) are substituted in the boundary condition for the displacements (6.2), resulting in the expressions (7.3), (7.4) and (7.5) for $a_{1\theta}$ and $a_{n\theta}$, the first being the components of the known vector and the latter being the components of the coefficient matrix in (7.6).

APPENDIX 4. EVALUATION OF THE COEFFICIENTS A_k

With

$$N_k = 2\pi i(S_{\ell_\varphi}-S_{k_\varphi})(\bar{S}_{\ell_\varphi}-S_{k_\varphi})(S_{k_\varphi}-\bar{S}_{k_\varphi}) \qquad \begin{matrix} k = 1,2 \\ \ell = 3-k \end{matrix}$$

expression (2.11) for A_k is written as

$$N_kA_k = R_y\{S_{k_\varphi}S_{\ell_\varphi}(\bar{S}_{k_\varphi}+\bar{S}_{\ell_\varphi}) - \frac{S_{26_\varphi}}{S_{11_\varphi}} + S_{k_\varphi}(\bar{S}_{k_\varphi}\bar{S}_{\ell_\varphi} - \frac{S_{12_\varphi}}{S_{11_\varphi}})\}$$

$$+ R_x\ \{S_{k_\varphi}S_{\ell_\varphi} - \frac{S_{12_\varphi}}{S_{11_\varphi}} + S_{k_\varphi}(\bar{S}_{k_\varphi} + \bar{S}_{\ell_\varphi} - \frac{S_{16_\varphi}}{S_{11_\varphi}})\} \qquad \text{(A4.1)}$$

With (A2.10) and (A2.16) is

$$\frac{S_{26_\varphi}}{S_{11_\varphi}} = \frac{1}{2}\left(\frac{S_{k_\varphi}+S_{\ell_\varphi}}{S_{k_\varphi}S_{\ell_\varphi}} + \frac{\bar{S}_{k_\varphi}+\bar{S}_{\ell_\varphi}}{\bar{S}_{k_\varphi}\bar{S}_{\ell_\varphi}}\right) S_{k_\varphi}S_{\ell_\varphi}\bar{S}_{k_\varphi}\bar{S}_{\ell_\varphi}$$

$$= \frac{1}{2}\ \{\bar{S}_{k_\varphi}\bar{S}_{\ell_\varphi}(S_{k_\varphi}+S_{\ell_\varphi}) + S_{k_\varphi}S_{\ell_\varphi}(\bar{S}_{k_\varphi}+\bar{S}_{\ell_\varphi})\} \qquad \text{(A4.2)}$$

According to (A2.8) is

$$\frac{S_{16_\varphi}}{S_{11_\varphi}} = \frac{1}{2}\ (S_{k_\varphi} + S_{\ell_\varphi} + \bar{S}_{k_\varphi} + \bar{S}_{\ell_\varphi}) \qquad \text{(A4.3)}$$

and from (A2.6) is derived

$$\frac{S_{12_\varphi}}{S_{11_\varphi}} = \frac{1}{2}\left(S_{k_\varphi}S_{\ell_\varphi} + \overline{S}_{k_\varphi}\overline{S}_{\ell_\varphi}\right) + \frac{S_{12} - S_{11}S_kS_\ell}{S_{11_\varphi}} \tag{A4.4}$$

Substitution of (A4.2), (A4.3) and (A4.4) in (A4.1) results in

$$N_kA_k = -\frac{1}{2}(R_x+S_{\ell_\varphi}R_y)(\overline{S}_{k_\varphi}-S_{k_\varphi})(\overline{S}_{\ell_\varphi}-S_{k_\varphi})-(R_x+S_{k_\varphi}R_y)\,\frac{S_{12}-S_{11}S_kS_\ell}{S_{11_\varphi}}$$

or

$$A_k = \frac{1}{2\pi i(S_{\ell_\varphi}-S_{k_\varphi})}\left\{\frac{1}{2}(R_x+S_{\ell_\varphi}R_y)+(R_x+S_{k_\varphi}R_y)\,\frac{S_{12}-S_{11}S_kS_\ell}{S_{11_\varphi}(\overline{S}_{\ell_\varphi}-S_{k_\varphi})(\overline{S}_{k_\varphi}-S_{k_\varphi})}\right\}$$

$$= \frac{1}{2\pi i(S_{\ell_\varphi}-S_{k_\varphi})}\left\{\frac{1}{2}(R_x+S_{\ell_\varphi}R_y)+(R_x+S_{k_\varphi}R_y)\;\cdot\right.$$
$$\left.\frac{(S_{12}-S_{11}S_kS_\ell)(\overline{S}_{\ell_\varphi}-S_{\ell_\varphi})(\overline{S}_{k_\varphi}-S_{\ell_\varphi})}{S_{11_\varphi}(\overline{S}_{\ell_\varphi}-S_{k_\varphi})(\overline{S}_{k_\varphi}-S_{k_\varphi})(\overline{S}_{\ell_\varphi}-S_{\ell_\varphi})(\overline{S}_{k_\varphi}-S_{\ell_\varphi})}\right\} \tag{A4.5}$$

In the denominator in (A4.5) is

$$(\overline{S}_{k_\varphi}-S_{k_\varphi})(\overline{S}_{\ell_\varphi}-S_{\ell_\varphi}) = S^2_{k_\varphi} + \overline{S}_{k_\varphi}\overline{S}_{\ell_\varphi} - S_{k_\varphi}(\overline{S}_{k_\varphi}+\overline{S}_{\ell_\varphi})$$

in which, according to (A2.6), with $S_k+S_\ell = -(\overline{S}_k+\overline{S}_\ell)$ and $S_kS_\ell = \overline{S}_k\overline{S}_\ell$

$$\overline{S}_{k_\varphi}\overline{S}_{\ell_\varphi} = S_{k_\varphi}S_{\ell_\varphi} + \frac{2S_{11}(S_kS_\ell+1)(S_k+S_\ell)\sin\varphi\cos\varphi}{S_{11_\varphi}} \tag{A4.6}$$

and according to (A2.8)

$$\bar{S}_{k_\varphi}+\bar{S}_{\ell_\varphi} = S_{k_\varphi}+S_{\ell_\varphi} + \frac{2S_{11}S_kS_\ell(S_k+S_\ell)\sin^2\varphi - 2S_{11}(S_k+S_\ell)\cos^2\varphi}{S_{11_\varphi}} \tag{A4.7}$$

So

$$(\bar{S}_{k_\varphi}-S_{k_\varphi})(\bar{S}_{\ell_\varphi}-S_{\ell_\varphi}) = \frac{2S_{11}}{S_{11_\varphi}}(S_k+S_\ell)\ \{(S_kS_\ell+1)\sin\varphi\cos\varphi -$$

$$- S_{k_\varphi}S_kS_\ell\sin^2\varphi + S_{k_\varphi}\cos^2\varphi\}$$

or, with $S_{k_\varphi} = \dfrac{S_k\cos\varphi - \sin\varphi}{S_\ell\sin\varphi + \cos\varphi}$

$$(\bar{S}_{k_\varphi}-S_{k_\varphi})(\bar{S}_{\ell_\varphi}-S_{k_\varphi}) = \frac{2S_{11}}{S_{11_\varphi}}S_k(S_k+S_\ell)\frac{S_\ell\sin\varphi + \cos\varphi}{S_k\sin\varphi + \cos\varphi}$$

where interchanging of k and ℓ directly results in an expression for $(\bar{S}_{\ell_\varphi}-S_{\ell_\varphi})(\bar{S}_{k_\varphi}-S_{\ell_\varphi})$. For the denominator in (A4.5) is now easily found:

$$S_{11}\ (\bar{S}_{\ell_\varphi}-S_{\ell_\varphi})(\bar{S}_{k_\varphi}-S_{\ell_\varphi})(\bar{S}_{k_\varphi}-S_{k_\varphi})(\bar{S}_{\ell_\varphi}-S_{k_\varphi}) = \frac{4S_{11}^2}{S_{11_\varphi}}S_kS_\ell(S_k+S_\ell)^2$$

hence, with

$$A = \frac{S_{12}-S_{11}S_k S_\ell}{4S_{11}{}^2 S_k S_\ell (S_k+S_\ell)^2} = \frac{C_6}{4S_{11}{}^2 S_k S_\ell (S_k+S_\ell)^2} \qquad \text{(A4.9)}$$

$$A_k = \frac{1}{2\pi i(S_{\ell_\varphi}-S_{k_\varphi})} \{\tfrac{1}{2}(R_x+S_{\ell_\varphi}R_y)+S_{11_\varphi}A(R_x+S_{k_\varphi}R_y)(\bar{S}_{\ell_\varphi}-S_{\ell_\varphi})(\bar{S}_{k_\varphi}-S_{\ell_\varphi})\} \qquad \text{(A4.10)}$$

For $\varphi = 0$ expression (A4.10) reduces to the well known

$$A_k = \frac{i}{4\pi(S_k{}^2-S_\ell{}^2)} \{R_x\ (S_k - \frac{\mu_{12}}{S_k}) + R_y\ (S_\ell{}^2 - \mu_{12})\} \qquad \text{(A4.11)}$$

APPENDIX 5. THE STRESS σ_y AT THE EDGE OF THE HOLE

The stresses resulting from the logarithmic parts and the single-valued parts of the complex stress functions $\phi_k(z_k)$ are calculated separately.

From (8.4) one finds for the logarithmic parts on the edge of the hole, after differentiation with respect to z_k:

$$\phi_k'(z_k) = \frac{iA_k}{S_{k_\varphi} \cos\theta - \sin\theta} \qquad k = 1,2$$

which gives after substitution in the second formula of (2.3):

$$\sigma_{y_\ell} = 2 \text{ Re} \sum_k \frac{iA_k}{S_{k_\varphi} \cos\theta - \sin\theta}$$

or, with expression (A4.10) for the coefficient A_k:

$$\sigma_{y_\ell} = \text{Re} \frac{1}{\pi(S_{1_\varphi} \cos\theta - \sin\theta)(S_{2_\varphi} \cos\theta - \sin\theta)} \cdot$$

$$[R_y\{\tfrac{1}{2}(S_{1_\varphi}+S_{2_\varphi}) \cos\theta - \tfrac{1}{2} \sin\theta + AS_{11_\varphi} S_{1_\varphi} S_{2_\varphi} (S_{1_\varphi}+S_{2_\varphi}-\bar{S}_{1_\varphi}-\bar{S}_{2_\varphi}) \cos\theta$$

$$- AS_{11_\varphi}(S_{1_\varphi} S_{2_\varphi} - \bar{S}_{1_\varphi} \bar{S}_{2_\varphi}) \sin\theta\}$$

$$+ R_x\{\tfrac{1}{2} \cos\theta + AS_{11_\varphi}(\bar{S}_{1_\varphi} \bar{S}_{2_\varphi} - S_{1_\varphi} S_{2_\varphi}) \cos\theta$$

$$+ AS_{11_\varphi}(S_{1_\varphi}+S_{2_\varphi})(S_{1_\varphi}+S_{2_\varphi}-\bar{S}_{1_\varphi}-\bar{S}_{2_\varphi})\cos\theta$$

$$- AS_{11_\varphi}(S_{1_\varphi}+S_{2_\varphi}-\bar{S}_{1_\varphi}-\bar{S}_{2_\varphi})\sin\theta\}]$$

where, according to (A4.6), (A4.7), (A2.12) and (A2.13)

$$S_{1_\varphi}S_{2_\varphi}-\bar{S}_{1_\varphi}\bar{S}_{2_\varphi} = -\frac{2S_{11}(S_1S_2+1)(S_1+S_2)\sin\varphi\cos\varphi}{S_{11_\varphi}} = -\frac{2iC_4}{S_{11_\varphi}}$$

and

$$S_{1_\varphi}+S_{2_\varphi}-\bar{S}_{1_\varphi}-\bar{S}_{2_\varphi} = -\frac{2S_{11}(S_1S_2\sin^2\varphi - \cos^2\varphi)(S_1+S_2)}{S_{11_\varphi}} = \frac{2iC_5}{S_{11_\varphi}}$$

So, on the edge of the hole the stress σ_y resulting from the logarithmic parts of the complex stress functions is

$$\sigma_{y_\ell} = \text{Re}\,\frac{1}{\pi(S_{1_\varphi}\cos\theta - \sin\theta)(S_{2_\varphi}\cos\theta - \sin\theta)}\cdot$$

$$[R_y\{\tfrac{1}{2}(S_{1_\varphi}+S_{2_\varphi})\cos\theta - \tfrac{1}{2}\sin\theta + 2iA(S_{1_\varphi}S_{2_\varphi}C_5\cos\theta + C_4\sin\theta)\}$$

$$+ R_x\{\tfrac{1}{2}\cos\theta + 2iA(C_4\cos\theta + (S_{1_\varphi}+S_{2_\varphi})C_5\cos\theta - C_5\sin\theta)\}] \quad (A5.1)$$

Note 1: Substitution of the logarithmic parts of the complex stress functions in the displacement formulae (2.4) yields for the edge of the hole the first two equations of (2.10), as already was mentioned in Chapter 5. Since these equations result in zero displacements, the logarithmic parts represent an infinite rigid circular insert joined to the surrounding orthotropic plate along its whole edge.

Note 2: A more elegant method for the derivation of σ_{y_ℓ} seems to be the rotation of R_x and R_y to the material axes $\alpha-\beta$ and then use the simple expression (A4.11) for A_k, where

$$R_\alpha = R_x \cos\varphi - R_y \sin\varphi$$

$$R_\beta = R_y \cos\varphi + R_x \sin\varphi$$

should be used instead of R_x and R_y respectively. Nevertheless this method results in a rather complicated expression for σ_{y_ℓ}.

For the stress σ_{y_s} resulting from the single-valued parts of the complex stress functions it is easily found from (8.4)

$$\sigma_{y_s} = \frac{p_o}{\pi} \mathrm{Re} \sum_k \Big[\frac{1}{S_{k_\varphi} - S_{\ell_\varphi}} \Big\{ \sum_{m,n} \frac{2na_n(m^2-n^2+1+2iS_{\ell_\varphi} m)\ \sigma^{-m}}{\{(m+1)^2-n^2\}\{(m-1)^2-n^2\}}$$

$$- \frac{\pi i}{4} \sum_{n=0,1,2}^{\infty} (a_n(1+iS_{\ell_\varphi})+a_{n+2}(1-iS_{\ell_\varphi}))\ \sigma^{-n-1}\Big\} \Big/ (S_{k_\varphi}\cos\theta-\sin\theta)\Big]$$

which can be evaluated into

$$\frac{\sigma_{y_s}}{p_o} = \mathrm{Re} \frac{-1}{\pi(S_{1_\varphi}\cos\theta - \sin\theta)(S_{2_\varphi}\cos\theta - \sin\theta)} .$$

$$\left[\sum_{m,n} \left\{ \frac{(m^2-n^2+1)\cos\theta + 2im(S_{1_\varphi}+S_{2_\varphi})\cos\theta - 2im\sin\theta}{\{(m+1)^2-n^2\}\{(m-1)^2-n^2\}} \right\} 2na_n\, \sigma^{-m} \right.$$

$$+ \frac{\pi}{4} \sum_{n=0,1,2}^{\infty} \{((S_{1_\varphi}+S_{2_\varphi}-i)\cos\theta - \sin\theta)\, a_n -$$

$$\left. - ((S_{1_\varphi}+S_{2_\varphi}+i)\cos\theta - \sin\theta)\, a_{n+2}\}\, \sigma^{-n-1} \right] \qquad \text{(A5.2)}$$

It is obvious that σ_y is found by simple superposition of σ_{y_ℓ} and σ_{y_s}.