

Reduced-Order-Modelling (ROM) for Hydroelastic Floating Structures by Using a Unified Finite Element Formulation

MSc Thesis Report

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Reduced-Order-Modelling (ROM) for Hydroelastic Floating Structures by Using a Unified Finite Element Formulation

by

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Preface

After two years of study in the Offshore and Dredging Engineering (ODE) master's program at TU Delft, this thesis project closes the journey. It has not been an easy one, but that never made it impossible. The whole learning process in this master's program has shaped me into a more resilient person.

Reviewing my motivation for pursuing my master's studies in this program makes me realise that I have been through many circumstances. The progression is not as ideal as what I wrote in my essay back then. Initially, I planned to work on tidal energy conversion because I am interested in ocean energy, and it is one of the potential renewable energy sources in my home country. However, I found that this topic was no longer available at TU Delft. Fortunately, since the focus of ocean energy development in my home country has shifted from tidal to floating PV, and offshore floating PV (OPV) has become a growing research topic at TU Delft, I changed my interest to OPV. I am really thankful to the Computational Multiphysics in Offshore Engineering (CMOE) TU Delft research group, particularly to my supervisor Oriol Colomés, for providing me the opportunity to shape my knowledge in numerical analysis for flexible floating structures that is applicable to OPV applications.

This thesis project focused on a monolithic finite element approach to hydroelastic problems. It completes my learning curve as a structural engineer in understanding the deeper fundamentals of finite element analysis. Moreover, the application of this topic is very extensive, especially in offshore renewable energy such as offshore PV and wave energy. This research builds on the monolithic FE framework developed by Oriol Colomés and later extends it to include the moored case, Mindlin theory formulation, and modal superposition analysis. The development of this method is aimed at providing a legacy solver for other researchers interested in this topic. As a side quest in my future engineering work, I am keen on conducting further research on this topic.

Lastly, it has been a very long journey for me working on this finite-element topic. I am deeply grateful to Oriol, my supervisor, for all the guidance and patience he has given. As I conclude my master's study, he and I will have two published papers (soon). Something that I wished for at the beginning of the study, and he led the way to make it happen. I am also grateful to Andrei Faragau for assessing this thesis project as the thesis committee. Thank you to everyone who has crossed paths with me during my time here. I wish kindness upon all of us, and until we cross paths again somewhere in the future. My last gratitude to the Indonesia Endowment Fund for Education (LPDP) for the financial support that helps me complete my study at TU Delft. I hope this modest contribution to the hydroelastic research can give something back to Indonesia's offshore energy development.

*Dewa Gede Pradnyanata
Delft, July 2026*

NB:

I intentionally wrote this preface, for the last time, here at the TU Delft bibliotheek, the place where I spend more time than in my own room. The place that saw my whole development, from not knowing how to make a function in Python to coding heavily in Julia. I can see myself visiting this place again in the future. Thank you for the free massage!

Abstract

The demand for hydroelastic studies in floating structures applications has increased in recent years. Their applicability extends the potential for floating structure development in offshore environments. For instance, it is applicable to offshore photovoltaic (OPV), wave energy converter (WEC), floating airport, and future floating city development. However, this branch of fluid-structure interaction (FSI) problem requires a solid foundation in understanding the two-way interaction between the floating structure and the fluid.

Computational methods are the most suitable option for studying hydroelasticity due to the ease of problem setup and are comparatively cheaper than experiments. Despite this advantage, this method also faces problems in the coupling strategy that lead to high computational cost. According to a previous study, monolithic coupling shows better computational performance than partitioned coupling. But since it solves a high-order matrix system, it may still require a long computation time. This research aims to investigate the applicability of a reduced-order system in solving the hydroelastic problem by using the monolithic finite element framework. Modal superposition methods will be employed within the framework to compare the computational time between the full-order and reduced-order models.

A reduced-order-modelling (ROM) framework through modal superposition is developed in this research, including a new structural formulation using Mindlin theory to cover a wide range of floating structure problems in a unified formulation. The analysis is conducted using the Gridap FEM library in the Julia programming language and comprises: 1) dry and wet modal analysis, 2) frequency-dependent added mass, added damping, and force calculation, 3) modal superposition analysis, and 4) full-order monolithic FEM analysis.

This research has developed two modal superposition frameworks, namely the matrix and integration methods. These methods have been proven to accurately estimate the full-order solutions and are in good agreement with the existing literature. The matrix method shows better computational performance compared to the full-order solution, up to hundreds of times faster, and up to thousands of times faster than the integration method. The integration method requires longer running time than the full-order solution due to the expanded weak formulation. Furthermore, this research also compared the dry and wet modal solutions and found that the dry modal solution performs better than the wet solution. The wet modal analysis is solved by using a nonlinear eigenvalue problem (NLEP). However, the method used in this research still needs to be validated with a more robust NLEP method for subsequent research.

Overall, this research shows good applicability of the reduced-order model using the monolithic FEM framework. Both methods give a good estimation of the full-order solutions. Despite the heavy computational time, the integration method provides a good implementation of modal superposition directly to the weak formulation within the Gridap ecosystem. The newly developed unified structural formulation also shows good applicability in covering a wide range of VLFS hydroelastic cases.

Keywords: Reduced-order modelling (ROM), modal superposition, dry and wet modes, Mindlin theory, hydroelasticity, very large floating structure (VLFS), fluid-structure interaction, monolithic finite element formulation

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Nomenclature

Abbreviations

Abbreviation	Definition
BEM	Boundary Element Method
DFSBC	Dynamic Free Surface Boundary Condition
EB Beam	Euler-Bernoulli Beam
FEM	Finite Element Method
FSI	Fluid-Structure Interaction
KBBC	Kinematic Bottom Boundary Condition
KFSBC	Kinematic Free Surface Boundary Condition
LHS	Left-Hand Side
NLEP	Non-linear Eigenvalue Problem
OPV	Offshore Photovoltaic
PDE	Partial Differential Equation
QEP	Quadratic Eigenvalue Problem
RAO	Response Amplitude Operator
RB	Reduced-Basis
RHS	Right-Hand Side
ROM	Reduced Order Modelling
TPOD	Truncated Proper Orthogonal Decomposition
VLFS	Very Large Floating Structure
WEC	Wave Energy Converter

Introduction

1.1. Background

In recent years, the demand for renewable energy has increased significantly. Following the success of the offshore wind industry, the path is now moving toward offshore floating photovoltaic (OPV). Solar panel technology for land use has been progressively developed. However, because it requires a large installation area, land availability poses a challenge for long-term development. Therefore, many projects have been developed on water bodies, for instance, lakes, dams, and seas. Since the available area on lakes and dams is also limited, this technology is being advanced for offshore applications (Fan et al., 2025). Several types of structures that have been utilised in the offshore environment include pontoon, semi-submersible, fixed platform, and membrane structures.

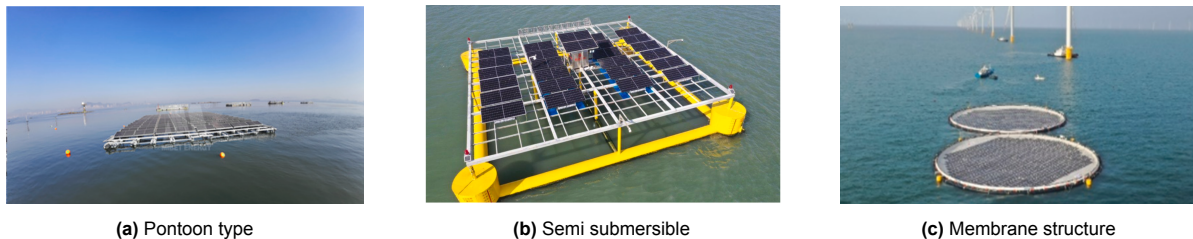


Figure 1.1: Type of Offshore Floating PV (Fan et al., 2025)

Solar PV technology requires a large area to increase the power plant capacity. This means that it also requires a wide base structure in offshore applications. In this case, a fixed platform is rarely the solution due to the financial and manufacturing constraints of constructing a massive steel structure. Therefore, the floating foundation becomes a solution. One potential type of structure that can be utilised for the OPV is the very large floating structure (VLFS) that is typically tens to hundreds of meters long. This type of structure is selected due to its advantages in lower construction cost and ease of offshore installation (Jiang et al., 2023). The pontoon, semi-submersible, and membrane structures, as in Fig. (1.1), are examples of VLFS structures.

Geometrically, a very large floating structure (VLFS) has a thin thickness relative to its width and length, with a very large length and width-to-thickness ratio. This large ratio can result in very low bending stiffness and flexible behaviour when subjected to oscillating wave loads. Due to its flexibility, the structure can deform in response to wave motion, following the wave pattern. If the excitation wave frequency matches the structure's natural frequency, a resonance response can be expected.

Moreover, the interaction between the structure and the fluid is a complex physical phenomenon. When the structure is deflected by the wave motion, its wide area can displace a large mass of water beneath the structure, producing a large added-mass effect on the structure. This displacement pushes the water, reflecting energy back from the wave and affecting the surrounding fluid field. This reflection

of wave energy by the structure's displacement radiates waves that propagate into the surrounding fluid. Therefore, the dynamic interaction between the fluid and the structure is a two-way process that affects each other's behaviour. This type of interaction phenomenon requires a hydroelastic study to understand the underlying physics.

The hydroelastic study focuses on the reciprocal influence between hydrodynamic loading and a deformable structure in a two-way interaction. Unlike one-way analysis, which uses hydrodynamic loading solely as a load input, this hydroelastic approach captures how hydrodynamic loads affect the structure's deformation, while this deformation also affects the behaviour of the surrounding fluid, introducing added mass, added damping, and hydrodynamic stiffness effects to the structure, as discussed previously. Analysing the structural response by including these influences can give a more accurate result than using the one-way interaction method. As a result, this hydroelastic analysis requires coupled models using a tight-coupling strategy that accounts for the behaviour of both the fluid and the structure (Jiang et al., 2023). Additionally, when the hydroelastic problem deals with oscillating forces (e.g., wave loads) at a specific excitation frequency, the likelihood of resonance must be investigated by estimating the structure's natural frequency through modal analysis. Unlike the traditional method that uses structural terms only (i.e., dry natural frequency), including fluid terms in the modal analysis (i.e., wet natural frequency) can provide a more accurate estimate of the actual natural frequency. Thus, the hydroelastic approach becomes more favourable in studying the behaviour of a flexible structure under wave loading, especially for very large floating structures (VLFS).

The demand for the hydroelastic study of VLFS is not limited to offshore photovoltaic (OPV). It was initially developed to study the behaviour of a ship under hydrodynamic loading (Bishop and Price, 1976) and later developed for wave energy (Babarit, Gendron, et al., 2013; Chao Wang et al., 2024; Simone Michele et al., 2020), floating breakwaters (Loukogeorgaki, Michailides, and Angelides, 2012), and floating tunnel (Ge et al., 2010; Ji et al., 2026) applications. In the 1990s and early 2000s, hydroelastic research, especially for VLFS applications, saw significant growth, coinciding with the success of a full-scale floating airport prototype installation in Japan. It was successfully tested for takeoff and landing, though it was later disassembled. This groundbreaking project enabled new prospective opportunities to explore the use of VLFS. For instance, floating container terminals, floating cities, floating tourism facilities, and waste processing facilities (Suzuki, 2005). Therefore, the advancement of this VLFS hydroelastic study will be an interesting research area in the upcoming years.

1.2. Research Problem

The study of hydroelasticity has covered experimental, analytical, and numerical analyses. Experimental studies can provide better visualisation of fluid-structure interaction behaviour. However, the cost of the experimental setup is relatively high. It is also difficult to scale a multiphysics problem when the fluid and structural parameters have different orders of dimensional units. An improper scaling method can lead to inaccurate structural behaviour and fail to fully reflect the actual condition. These issues are limiting the availability of experimental studies. Alternatively, researchers have developed computational methods for hydroelastic problems using analytical and numerical approaches.

The application of numerical and analytical methods has been proven effective in studying the hydroelastic behaviour of flexible floating structures. It helps promote research on hydroelasticity because it is flexible in its problem setup, relatively inexpensive compared to experiments, and can provide accurate estimates. Several analytical and numerical methods that have been developed by the researchers for hydroelasticity analysis are as follows:

For the analytical method, there are modal expansion method (S. Michele et al., 2024; T. I. Khabakhpasheva and A. A. Korobkin, 2002), Wiener-Hopf method (Williams and Meylan, 2012; Zhao, C. Hu, et al., 2008; Tkacheva, 2003), boundary integral equation method (BIEM) (S. C. Mohapatra and Soares, 2021), and eigenfunction expansion/matching method (A. Korobkin, T. Khabakhpasheva, and Shishmarev, 2023; Amouzadrad, S. Mohapatra, and Guedes Soares, 2026; Xu and Wellens, 2022; Praveen, Karmakar, and Guedes Soares, 2020; Meylan, Luke G. Bennetts, and Peter, 2017; L. G. Bennetts, Biggs, and Porter, 2007; Wu, Watanabe, and Utsunomiya, 1995).

For the numerical method, there are finite element method coupled with boundary element method (FEM - BEM) (Amouzadrad, S. Mohapatra, and Guedes Soares, 2026; Nguyen et al., 2018; Meylan,

Luke G. Bennetts, and Peter, 2017; Gao et al., 2011; Ge et al., 2010; Riyansyah, C.M. Wang, and Choo, 2010; Utsunomiya et al., 1995), FEM modal superposition coupled with BIEM (Michailides, Loukogeorgaki, and Angelides, 2013; Loukogeorgaki, Michailides, and Angelides, 2012; Ertekin et al., 1993), FEM modal superposition coupled with diffraction-radiation problem for fluid (A. E. Karperaki and K. A. Belibassakis, 2021; Babarit, Singh, et al., 2017), monolithic FEM for direct coupling between structural FEM and fluid dynamic pressure (Agarwal, Colomés, and Metrikine, 2024; Colomés, Verdugo, and Akkerman, 2023; Servan-Camas et al., 2021), finite different method (FDM) coupled with BEM (Hemant and Karmakar, 2025), discrete multi body FEM coupled with boundary element method (DMB - BEM) (Jin, Lee, et al., 2025; Jin, Bakti, and M. Kim, 2020), and with Morison force formulation (Ji et al., 2026), FEM coupled with computational fluid dynamic (CFD) (Chao Wang et al., 2024; Wei et al., 2024; Huang, Z. Hu, and Chen, 2023; Gu et al., 2023), and fully smoothed particle hydrodynamic (SPH) (Ioannou et al., 2025; Ma, Iijima, and Oka, 2020).

In subsequent research, the focus shifts to developing an efficient computational framework that accurately estimates hydroelastic behaviour without excessive computational time. Although it is less expensive than an experimental study, achieving high solution accuracy in the computational analysis requires tight fluid-structure coupling, which can be computationally expensive. The coupling strategy becomes the primary challenge in the computational hydroelastic analysis. A comparative study from Servan-Camas et al. (2021) has compared the computational performance between partitioned and monolithic coupling. It is shown that the monolithic coupling method outperforms the partitioned coupling method in terms of computational time. But the development of this monolithic coupling lags behind that of the partitioned method due to the fewer "legacy" solvers available in this method (Hou, J. Wang, and Layton, 2012). Studies by Colomés, Verdugo, and Akkerman (2023) and Agarwal, Colomés, and Metrikine (2024) have proposed a novel framework for the monolithic finite element method (FEM) and have validated it against existing literature. This research will focus on developing the monolithic FEM framework. To address the challenge of high computational time, this research will develop a reduced-order modelling (ROM) analysis for the monolithic FEM framework using the modal superposition method. This method can reduce the size of the monolithic system from the order of thousands of elements to just the matrix system with the order of tens. In addition, this research will develop a unified structural formulation that can cover a wide range of VLFS problems.

1.3. Research Objectives

This research aims to investigate the effectiveness of reduced-order modelling (ROM) through modal superposition in optimising the computational cost of monolithic FEM for solving complex coupled fluid-structure interaction problems. A unified structural formulation will be developed to cover a wide range of VLFS problems by using Mindlin theory.

1.4. Research Scopes

This research will utilise the monolithic finite element method to solve the hydroelastic problem, based on the framework developed by Colomés, Verdugo, and Akkerman (2023) and Agarwal, Colomés, and Metrikine (2024). The development of this method will focus on computational efficiency through reduced-order modelling (ROM) by using modal superposition.

Moreover, a new mathematical formulation for a unified model using Mindlin theory will be developed. This theory is used because of its capability to recover the Euler-Bernoulli and membrane formulations by adjusting the physical properties in the formulation. The formulation derivation of this mathematical model will also impose the Neumann boundary conditions on the structural boundary terms using linear spring terms in translational and rotational directions. Additionally, internal joint terms will be added to the formulation to complete the mathematical framework for solving a wide range of hydroelastic problems.

Three structural setups will be used to evaluate the effectiveness of the ROM method. Those are free-floating structures, structures with mooring attachments, and multi-module structures with flexible joint connections. These structural setups will be combined with Mindlin beam theory, Euler-Bernoulli beam theory, and membrane structure.

For the numerical setup, this research will focus on the two-dimensional problem with a one-dimensional

structural domain. This research is performed in the frequency domain, focusing on the steady-state solutions. This frequency domain analysis is selected to directly fit the governing physics, where hydroelasticity problems are frequency-dependent. It helps in understanding the structural behaviour under specific wave loading and to make a direct connection to the modal analysis.

1.5. Research Questions

In addressing the research objective, the main research question is formulated as:

“How does the reduced-order modelling (ROM) computational performance outweigh the full-order monolithic FEM in solving complex coupled fluid-structure interaction problems?”

This main research question will be addressed through the following sub-questions:

SQ1: What strategy is used for the reduced-order modelling?

SQ2: How does the wet natural frequency differ from the dry natural frequency for each mode?

SQ3: How does the performance of dry and wet modal superposition differ in estimating the full solution?

SQ4: How does the complexity of the structure affect the number of required modes?

SQ5: How effective is the ROM in reducing the computational time compared to the full monolithic FEM?

1.6. Thesis Outlines

This thesis report is structured into six (6) chapters. The outline of this report is as follows:

Chapter 1: Introduction

This chapter presents the background reasoning for the selected research problem and defines the research objectives to address the problem.

Chapter 2: Theory

This chapter provides the underlying mathematical theories used in this research. Consisting of the general setup of the hydroelastic problem of a very large floating structure, fluid and structure governing equations, frequency domain analysis, fundamentals of modal analysis and the strategy used in reduced-order modelling through modal superposition.

Chapter 3: Numerical Formulation

This chapter derives the weak formulation of the governing equation, discretises the formulation, and validates it by performing a convergence study.

Chapter 4: Problem Setup

The problems used to evaluate the ROM performance are explained in this chapter, including the schematic overview, the respective discrete weak formulation, and the parameters used in the analysis.

Chapter 5: Results and Discussion

The results of the analysis of all the problems in the previous chapter will be provided and discussed in this chapter.

Chapter 6: Conclusions and Recommendations

This chapter will summarise and conclude the research findings. Provide reflections on the research approach and suggest directions for future research.

2

Theories

This chapter will discuss the underlying mathematical theories used in the research. Initially, the general problem setup, including the assumptions, will be explained, followed by the governing equations, modal analysis, and reduced-order modelling via modal superposition.

2.1. Research Setup

The problem is set on a two-dimensional Cartesian grid, with the wave propagating in the positive x -direction and the depth along the vertical axis. This research uses a two-dimensional fluid domain with a one-dimensional structural domain. The fluid domain Ω is bounded with inlet boundary Γ_{in} on the left side, outlet boundary Γ_{out} on the right side, seabed boundary Γ_{sb} at the bottom, free surface Γ_{fs} and structural Γ_b boundary on the top. Two wave damping areas are defined near the inlet and outlet to stabilise the wave generation in the simulation. The structure has boundaries on the left side where the wave first contacts the structure, and on the right side, denoted as $\Gamma_l, \Gamma_r \subset \Gamma_{BC}$. In the case of multi-body structures with joint connections, the structure will have interface joint connections denoted as Λ_j . Moreover, this research will use the notation of ϕ for the potential flow, κ for the fluid free surface, and η for the structural deflection solutions. The schematic view of the general setup is presented in Fig. (2.1)

In the formulation derived in this research, the following assumptions will be applied:

Assumption 1. The fluid is assumed to be ideal, which is irrotational, inviscid, and incompressible.

The wave is assumed linear with a very small amplitude compared to the wavelength and water depth.

Assumption 2. The structure is assumed to have a tight interface with the bounded fluid. No air gap is formed at the interface.

Assumption 3. The mooring terms at the structural boundary and joint are assumed to be modelled with a linear spring model in vertical and rotational directions.

Assumption 4. The problem is set in the two-dimensional domain and uses frequency-domain analysis, focusing on the steady-state solution.

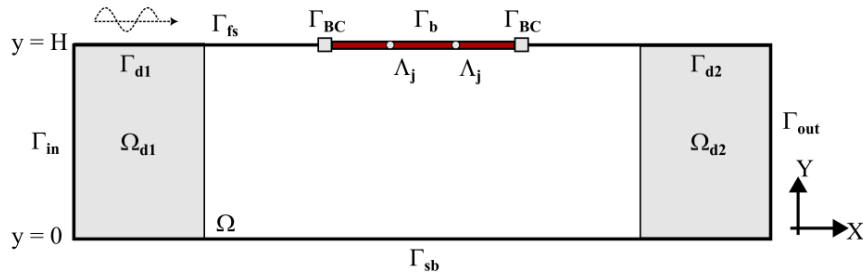


Figure 2.1: Schematic view of the general setup

2.2. Fluid Governing Equation

The fluid is governed by a linear potential flow theory. Following the Assumption 1, the fluid flow satisfies the irrotational and inviscid conditions. This flow is defined as a velocity vector $\mathbf{u}$, which is the first derivative of the potential field ϕ .

$$\mathbf{u} = \nabla \phi \quad (2.1)$$

The incompressibility of the assumed fluid satisfies the continuity equation. This means that the flow of the fluid through a control volume equals the fluid outflow of the volume, ensuring there is no fluid being compressed.

$$\nabla \cdot \mathbf{u} = 0 \quad (2.2)$$

Combining these ideal fluid conditions by substituting Eq. (2.1) into Eq. (2.2), the fluid governing equation is obtained in the following partial differential equation (PDE):

$$\Delta \phi = 0 \quad \text{in } \Omega \quad (2.3)$$

2.2.1. Boundary Conditions

The governing fluid equation in Eq. (2.3) is solved using appropriate kinematic and dynamic boundary conditions. The boundary terms will appear naturally after reducing the PDE order in the weak formulation. The detailed steps will be derived in Section 3.2. The kinematic boundary conditions for the fluid are provided in Eq. (2.4)

$$\mathbf{n} \cdot \nabla \phi = 0 \quad \text{on } \Gamma_{sb} \quad (2.4a)$$

$$\mathbf{n} \cdot \nabla \phi = u_{in} \quad \text{on } \Gamma_{in} \quad (2.4b)$$

$$\mathbf{n} \cdot \nabla \phi = u_{out} \quad \text{on } \Gamma_{out} \quad (2.4c)$$

$$\mathbf{n} \cdot \nabla \phi = \frac{\partial \kappa}{\partial t} \quad \text{on } \Gamma_{fs} \quad (2.4d)$$

$$\mathbf{n} \cdot \nabla \phi = \frac{\partial \eta}{\partial t} \quad \text{on } \Gamma_b \quad (2.4e)$$

The kinematic bottom boundary condition (KBBC) is expressed in Eq. (2.4a). On the bottom boundary, no normal velocity passes through the seabed. It is assumed to be impermeable, and the normal velocity is set to zero. The boundary conditions in Eqs. (2.4b) and (2.4c) define and control the flow condition at the inlet and outlet boundaries. On the surface, the kinematic free surface boundary condition (KFSBC) is governed by the normal velocity of the fluid free surface and the vertical structural deflection velocity as expressed in Eq. (2.4d) and (2.4e). The normal velocity on the surface is equal to the first time derivative of the free surface κ and the structure deflection η .

The dynamic boundary condition is applied to the surface boundary and is governed by Bernoulli's pressure equation. Assuming linear potential flow, the higher-order term in Bernoulli's pressure equation can be neglected. Therefore, the pressure equation can be simplified as in Eq. (2.5).

$$p = -\rho_w \frac{\partial \phi}{\partial t} - \rho_w g \kappa - \frac{1}{2} \rho_w (\nabla \phi)^2 \approx -\rho_w \frac{\partial \phi}{\partial t} - \rho_w g \kappa \quad (2.5)$$

Similar to KFSBC, the dynamic free surface boundary condition (DFSBC) is also applied to both the free surface domain Γ_{fs} and the structural domain Γ_b . For the dynamic free-surface boundary, the pressure equals atmospheric pressure, where $p = p_{atm} = 0$. On the beam interface, this pressure equals the pressure exerted by the fluid on the structure. Therefore, this term will be coupled with the structure and substituted into the structural governing equation. The dynamic boundary condition is expressed as follows:

$$\frac{\partial \phi}{\partial t} + g \kappa = 0 \quad \text{on } \Gamma_{fs} \quad (2.6a)$$

$$-\rho_w \frac{\partial \phi}{\partial t} - \rho_w g \eta = p \quad \text{on } \Gamma_b \quad (2.6b)$$

2.2.2. Wave Generation and Damping Zone

The wave is generated in the inlet boundary. It is modelled by using the linear wave theory (Eq. 2.7a) and the wave potential (Eq. 2.7b), in accordance with Assumption 1. The expression of wave potential in Eq. (2.7b) incorporates the dispersion relation. Since this research employs a two-dimensional case, the gradient expresses the derivation in the x-direction, $\nabla = [\partial/\partial x]$. Therefore, the inlet velocity is calculated from the potential derivative in the x-direction as in Eq. (2.7c). On the outlet boundary condition, Eq. (2.4c), this quantity is set to zero, $u_{\text{out}} = 0$.

$$\kappa_{\text{in}}(x, t) = \kappa_0 \exp(ik_\lambda x) \exp(-i\omega t) \quad (2.7a)$$

$$\phi_{\text{in}}(x, y, t) = -i \frac{\kappa_0 \omega}{k_\lambda} \frac{\cosh(k_\lambda y)}{\sinh(k_\lambda H)} \exp(ik_\lambda x) \exp(-i\omega t) \quad (2.7b)$$

$$u_{\text{in}} = \frac{\partial \phi_{\text{in}}(x, y, t)}{\partial x} = (\kappa_0 \omega) \frac{\cosh(k_\lambda y)}{\sinh(k_\lambda H)} \exp(ik_\lambda x) \exp(-i\omega t) \quad (2.7c)$$

Numerical wave generation simulation in a wave-tank-like domain faces challenges in simulating open-sea conditions. Wave reflection at the boundary can cause inaccurate and unstable results. To address this problem, the formulation must ensure that the wave generated at the inlet is stable before it reaches the structure and that the wave at the outlet is not reflected back. The study by M. W. Kim, Koo, and Hong (2014) demonstrated the effectiveness of wave damping implementation in the upstream and downstream domains for the free-surface solution. This method stabilises wave generation in the upstream zone and effectively absorbs the outgoing wave at the outlet boundary, preventing reflection. There are five methods available, proposed by M. W. Kim, Koo, and Hong (2014). This research will use the $\phi_n - \kappa$ damping scheme (Method 5) and the associated ramp function of Shape 1. This combination of method and ramp shape has proven to be the optimal solution with the minimal reflection disturbance (M. W. Kim, Koo, and Hong, 2014). The updated damping-zone kinematic and dynamic boundary conditions for Method 5 are given by Eq. (2.8).

$$\begin{aligned} \frac{\partial \kappa}{\partial t} &= \mathbf{n} \cdot \nabla \phi - \mu_2 (\kappa - \kappa_{\text{in}}) \\ \mathbf{n} \cdot \nabla \phi &= \frac{\partial \kappa}{\partial t} + \mu_2 (\kappa - \kappa_{\text{in}}) \quad \text{on } \Gamma_d \quad (\text{KFSBC}) \end{aligned} \quad (2.8a)$$

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= -g\kappa - \mu_1 (\nabla \phi \cdot \mathbf{n} - \nabla \phi_{\text{in}} \cdot \mathbf{n}) \\ \frac{\partial \phi}{\partial t} + g\kappa + \mu_1 (\nabla \phi \cdot \mathbf{n} - \nabla \phi_{\text{in}} \cdot \mathbf{n}) &= 0 \quad \text{on } \Gamma_d \quad (\text{DFSBC}) \end{aligned} \quad (2.8b)$$

Where μ_1 and μ_2 are the damping coefficients. μ_1 is expressed as the ramp function, which in this research will implement the Shape 1 function, and μ_2 is related to μ_1 with the wavenumber k_λ . The expression of μ_1 and μ_2 are as follows:

$$\mu_1(x) = \begin{cases} \mu_0 \left[1 - \sin \left(\frac{\pi}{2} \frac{x}{L_d} \right) \right], & \text{if } x < x_{d,\text{in}}, \\ \mu_0 \left[1 - \cos \left(\frac{\pi}{2} \frac{(x-x_d)}{L_d} \right) \right], & \text{if } x > x_{d,\text{out}}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_2(x) = k_\lambda \mu_1(x)$$

Another strategy is to incorporate upstream wave damping and enforce a radiation boundary at the outlet, allowing the wave to bypass the boundary without reflection (Agarwal, Colomés, and Metrikine, 2024). The radiation boundary is expressed as follows:

$$\mathbf{n} \cdot \nabla \phi = ik_\lambda \phi \quad \text{on } \Gamma_{\text{out}} \quad (2.9)$$

2.3. Structural Governing Equation

In this research, a novel unified monolithic finite element method (FEM) formulation will be developed based on the framework proposed by Colomés, Verdugo, and Akkerman (2023) and Agarwal, Colomés, and Metrikine (2024), using the Gridap FEM library (Badia and Verdugo, 2020). This FEM formulation will be developed from the Mindlin plate theory, which is equivalent to the Timoshenko beam theory for the one-dimensional problem. Although this research will use a one-dimensional structural problem, the structural formulation will be maintained within the Mindlin (1951) framework to enable scalable future research on the plate case. Compared to the Timoshenko model, the Mindlin model simplifies the unknown variables only for the beam deflection η . This helps recover the formulation of another structural governing equation, such as the Euler-Bernoulli beam or membrane structure, by setting the appropriate mechanical properties. The equation is provided in Eq. (2.10).

$$\left(\nabla^2 - \frac{\rho}{G'} \frac{\partial^2}{\partial t^2}\right) \left(D \nabla^2 - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2}\right) \eta + \rho h \frac{\partial^2 \eta}{\partial t^2} = \left(1 - \frac{D \nabla^2}{G' h} + \frac{\rho h^2}{12 G'} \frac{\partial^2}{\partial t^2}\right) p. \quad (2.10)$$

Where ∇ is gradient operator along the structure domain Γ_b , for this 2D cases with 1D structural domain, $\nabla = [\partial/\partial x]$, ρ is the density of the plate, $G' = \varepsilon G$ is the factorized shear modulus, $\varepsilon = \pi^2/12$ is the shear reduction factor used by Mindlin (1951), $D = Eh^3/12(1-\nu^2)$ is the structural rigidity, $\nu = 0.3$ is the Poisson ratio, h is the plate thickness, η is the plate deflection, and p is the external pressure for structural coupling term with fluid.

2.3.1. Coupled Governing Equation

The dynamic boundary condition in Eq. (2.6b) will be substituted into the right-hand side (RHS) of the governing equation above. Before the derivation, new terms for the rotational inertia and shear modulus are introduced for simplification as $J = h^3/12$ (the rotational inertia) and $C = G'h = \varepsilon Gh$, the factorised shear modulus including the plate thickness h .

First, substitute the pressure term from the dynamic boundary condition and decompose the right-hand side (RHS) of the governing equation.

$$\begin{aligned} RHS &:= \left(1 - \frac{D \nabla^2}{G' h} + \frac{\rho h^2}{12 G'} \frac{\partial^2}{\partial t^2}\right) \left(-\rho_w \frac{\partial \phi}{\partial t} - \rho_w g \eta\right) \\ &= \left(-\rho_w \frac{\partial \phi}{\partial t} - \rho_w g \eta\right) - \frac{D \nabla^2}{C} \left(-\rho_w \frac{\partial \phi}{\partial t} - \rho_w g \eta\right) + \frac{\rho J}{C} \frac{\partial^2}{\partial t^2} \left(-\rho_w \frac{\partial \phi}{\partial t} - \rho_w g \eta\right) \\ &= \left(-\rho_w \frac{\partial \phi}{\partial t} - \rho_w g \eta\right) + \left(\rho_w \frac{D}{C} \nabla^2 \frac{\partial \phi}{\partial t} + \rho_w g \frac{D}{C} \nabla^2 \eta\right) + \left(-\rho_w \frac{\rho J}{C} \frac{\partial^3 \phi}{\partial t^3} - \rho_w g \frac{\rho J}{C} \frac{\partial^2 \eta}{\partial t^2}\right) \\ &= \rho_w g \frac{D}{C} \nabla^2 \eta - \rho_w g \frac{\rho J}{C} \frac{\partial^2 \eta}{\partial t^2} - \rho_w g \eta + \rho_w \frac{D}{C} \nabla^2 \frac{\partial \phi}{\partial t} - \rho_w \frac{\rho J}{C} \frac{\partial^3 \phi}{\partial t^3} - \rho_w \frac{\partial \phi}{\partial t} \\ &= \left(\rho_w g \frac{D}{C}\right) \nabla^2 \eta - \left(\rho_w g \frac{\rho J}{C} \frac{\partial^2}{\partial t^2} + \rho_w g\right) \eta + \left(\rho_w \frac{D}{C} \frac{\partial}{\partial t}\right) \nabla^2 \phi - \left(\rho_w \frac{\rho J}{C} \frac{\partial^3}{\partial t^3} + \rho_w \frac{\partial}{\partial t}\right) \phi \end{aligned}$$

Next, evaluate the left-hand side (LHS)

$$\begin{aligned} LHS &:= \left(\nabla^2 - \frac{\rho}{G'} \frac{\partial^2}{\partial t^2}\right) \left(D \nabla^2 - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2}\right) \eta + \rho h \frac{\partial^2 \eta}{\partial t^2} \\ &= D \nabla^4 \eta - \rho J \nabla^2 \frac{\partial^2 \eta}{\partial t^2} - \frac{\rho D h}{C} \nabla^2 \frac{\partial^2 \eta}{\partial t^2} + \frac{\rho^2 J h}{C} \frac{\partial^4 \eta}{\partial t^4} + \rho h \frac{\partial^2 \eta}{\partial t^2} \\ &= D \nabla^4 \eta + \left(-\rho J - \frac{\rho D h}{C}\right) \nabla^2 \frac{\partial^2 \eta}{\partial t^2} + \rho h \frac{\partial^2 \eta}{\partial t^2} + \frac{\rho^2 J h}{C} \frac{\partial^4 \eta}{\partial t^4} \\ &= D \nabla^4 \eta + \left(-\rho J \frac{\partial^2}{\partial t^2} - \frac{\rho D h}{C} \frac{\partial^2}{\partial t^2}\right) \nabla^2 \eta + \left(\rho h \frac{\partial^2}{\partial t^2} + \frac{\rho^2 J h}{C} \frac{\partial^4}{\partial t^4}\right) \eta \end{aligned}$$

Now combine both sides, collect terms by spatial variable, and divide both sides by the fluid density ρ_w . The coupled structure-governing equation, also known as the strong form, is given by Eq. (2.11).

$$\begin{aligned} \nabla^2 (D_\rho \nabla^2 \eta) + \left(-\frac{\rho}{\rho_w} J \frac{\partial^2}{\partial t^2} - \frac{D_\rho}{C} \rho h \frac{\partial^2}{\partial t^2} - \frac{D_\rho}{C} \rho_w g \right) \nabla^2 \eta + \left(\frac{\rho^2 J h}{\rho_w C} \frac{\partial^4}{\partial t^4} + \frac{\rho h}{\rho_w} \frac{\partial^2}{\partial t^2} + \frac{\rho g J}{C} \frac{\partial^2}{\partial t^2} + g \right) \eta \\ - \left(\frac{D_\rho}{C} \rho_w \frac{\partial}{\partial t} \right) \nabla^2 \phi + \left(\frac{\rho J}{C} \frac{\partial^3}{\partial t^3} + \frac{\partial}{\partial t} \right) \phi = 0 \end{aligned} \quad (2.11)$$

Where $D_\rho = D/\rho_w$.

2.3.2. Frequency Domain Analysis

The strong form in Eq. (2.11) is still in the time domain. Following Assumption 4, the frequency-domain formulation will be derived by evaluating the temporal derivative terms. The frequency-domain problem considers steady-state solutions, in which the separation-of-variables solution is applied for the fluid ϕ , free-surface κ and structure η . The separation-of-variables solutions of all unknowns are written as follows:

$$\begin{aligned} \phi(x, y, t) &= \bar{\phi}(x, y) \exp(-i\omega t) \\ \kappa(x, y, t) &= \bar{\kappa}(x, y) \exp(-i\omega t) \\ \eta(x, y, t) &= \bar{\eta}(x, y) \exp(-i\omega t) \end{aligned} \quad (2.12)$$

The bar notation expresses the amplitude of the response. From these expressions, the time derivative can then be evaluated. The frequency term will appear in the time derivative, as shown in the example of Eq. (2.13). Because all variables are functions of the temporal term $\exp(-i\omega t)$, the solution can be expressed solely in terms of the amplitude.

$$\begin{aligned} \ddot{\phi}(x, y, t) &= -\omega^2 \bar{\phi}(x, y) \exp(-i\omega t) \approx -\omega^2 \bar{\phi}(x, y) \\ \ddot{\kappa}(x, y, t) &= -\omega^2 \bar{\kappa}(x, y) \exp(-i\omega t) \approx -\omega^2 \bar{\kappa}(x, y) \\ \ddot{\eta}(x, y, t) &= -\omega^2 \bar{\eta}(x, y) \exp(-i\omega t) \approx -\omega^2 \bar{\eta}(x, y) \end{aligned} \quad (2.13)$$

For clarity in the formulation, the bar notation will be dropped in the following derivation. Therefore, by implementing this separation of variables solution and expressing the solution using the response amplitude, the time domain governing equation in Eq. (2.11) can be written in the frequency domain as:

$$\begin{aligned} \nabla^2 (D_\rho \nabla^2 \eta) + \left(\frac{\rho}{\rho_w} J \omega^2 + \frac{D_\rho}{C} \rho h \omega^2 - \frac{D_\rho}{C} \rho_w g \right) \nabla^2 \eta + \left(\frac{\rho^2 J h}{\rho_w C} \omega^4 - \frac{\rho h}{\rho_w} \omega^2 - \frac{\rho g J}{C} \omega^2 + g \right) \eta \\ + \left(i \frac{D_\rho}{C} \rho_w \omega \right) \nabla^2 \phi + \left(i \frac{\rho J}{C} \omega^3 - i\omega \right) \phi = 0 \quad \text{on } \Gamma_b \end{aligned} \quad (2.14)$$

The same approach is also applied to the boundary terms with time-derivative components.

2.3.3. Pretensioned Term and Mooring Boundary Conditions

In this unified formulation, additional pretensioned and mooring terms will be added to the equation. This improves the formulation's versatility across a wide range of hydroelastic cases. The pretensioned term is added according to the membrane formulation from Agarwal, Colomés, and Metrikine (2024), and is defined as follows:

$$\text{Pretensioned term} := -T_\rho \nabla^2 \eta$$

This term corresponds to the second term of the governing equation in Eq. (2.14) with the same spatial

term order of $\nabla^2 \eta$. Therefore, it updates the structural governing equation in Eq. (2.15).

$$\begin{aligned} \nabla^2 (D_\rho \nabla^2 \eta) + \left(\frac{\rho}{\rho_w} J \omega^2 + \frac{D_\rho}{C} \rho h \omega^2 - \frac{D_\rho}{C} \rho_w g - T_\rho \right) \nabla^2 \eta + \left(\frac{\rho^2 J h}{\rho_w C} \omega^4 - \frac{\rho h}{\rho_w} \omega^2 - \frac{\rho g J}{C} \omega^2 + g \right) \eta \\ + \left(i \frac{D_\rho}{C} \rho_w \omega \right) \nabla^2 \phi + \left(i \frac{\rho J}{C} \omega^3 - i \omega \right) \phi = 0 \quad \text{on } \Gamma_b \end{aligned} \quad (2.15)$$

The terms inside the brackets are constant values. Therefore, for clarity, the constants can be denoted using A_1, A_2, A_3, A_4 , and A_5 .

$$\nabla^2 (A_1 \nabla^2 \eta) + A_2 \nabla^2 \eta + A_3 \eta + A_4 \nabla^2 \phi + A_5 \phi = 0 \quad \text{on } \Gamma_b \quad (2.16)$$

with

$$A_1 = D_\rho, \quad (2.17a)$$

$$A_2 = \left(\frac{\rho}{\rho_w} J \omega^2 + \frac{D_\rho}{C} \rho h \omega^2 - \frac{D_\rho}{C} \rho_w g - T_\rho \right), \quad (2.17b)$$

$$A_3 = \left(\frac{\rho^2 J h}{\rho_w C} \omega^4 - \frac{\rho h}{\rho_w} \omega^2 - \frac{\rho g J}{C} \omega^2 + g \right), \quad (2.17c)$$

$$A_4 = \left(i \frac{D_\rho}{C} \rho_w \omega \right) \text{ and} \quad (2.17d)$$

$$A_5 = \left(i \frac{\rho J}{C} \omega^3 - i \omega \right). \quad (2.17e)$$

Furthermore, this formulation will incorporate mooring applications, modelled as linear springs at the structural boundary Γ_{BC} and at joint connections Λ_j , where applicable. This boundary term will naturally be imposed in the weak-form formulation after reducing the high-order terms of the structural governing equation (strong form). For the 1-D case, the boundary can be differentiated as the left boundary Γ_l and the right boundary Γ_r , where $\Gamma_l, \Gamma_r \subset \Gamma_{BC}$. Thus, the mooring boundaries can be expressed as in Eq. (2.18), where the detailed derivation will be explained in Section 3.3.

$$M := -D_\rho \nabla^2 \eta = k_{r\rho} \nabla \eta \cdot \mathbf{n} \quad \text{on } \Gamma_l, \Gamma_r \subset \Gamma_{BC} \quad (\text{Rotational spring term}) \quad (2.18a)$$

$$V := \nabla (D_\rho \nabla^2 \eta) + A_2 \nabla \eta + A_4 \nabla \phi = k_{s\rho} \eta \cdot \mathbf{n} \quad \text{on } \Gamma_l, \Gamma_r \subset \Gamma_{BC} \quad (\text{Vertical spring term}) \quad (2.18b)$$

Where $k_{r\rho} = k_r / \rho_w$ is the rotational spring stiffness and $k_{s\rho} = k_s / \rho_w$ is the vertical spring stiffness.

2.3.4. Flexible Internal Joint Interface Condition

For the case of a structure with internal joint connections Λ_j as depicted in Fig. (2.1), discontinuity is introduced in the interface. The deflection of the structure η must be continuous, while the rotation (the derivative) may not be continuous. The continuity in the joint connection can be imposed weakly by introducing the jump operator, $\llbracket \cdot \rrbracket$, and mean operator, $\langle \cdot \rangle$, into the joint interface. These operators are defined as follows:

$$\llbracket v \rrbracket := v^+ - v^-, \quad \text{and} \quad \langle v \rangle := \frac{1}{2}(v^+ + v^-). \quad (2.19)$$

Where $(\cdot)^+$ and $(\cdot)^-$ denote the evaluation of the quantities at each side of the joint. Thus, the interface conditions at the joints can be defined in Eq. (2.20).

$$\llbracket -D_\rho \nabla^2 \eta \cdot \mathbf{n} \rrbracket = 0 \quad \text{on } \Lambda_j \quad (\text{Bending moment continuity}) \quad (2.20a)$$

$$\llbracket \nabla \eta \cdot \mathbf{n} \rrbracket = 0 \quad \text{on } \Lambda_j \quad (\text{Derivative continuity}) \quad (2.20b)$$

$$\llbracket \nabla \phi \cdot \mathbf{n} \rrbracket = 0 \quad \text{on } \Lambda_j \quad (\text{Derivative continuity}) \quad (2.20c)$$

$$\langle -D_\rho \nabla^2 \eta_h \rangle = k_{r\rho} \llbracket \nabla \eta \cdot \mathbf{n} \rrbracket \quad \text{on } \Lambda_j \quad (\text{Rotational spring term}) \quad (2.20d)$$

$$\llbracket \nabla (D_\rho \nabla^2 \eta) \cdot \mathbf{n} + (A_2 \nabla \eta) \cdot \mathbf{n} + (A_4 \nabla \phi) \cdot \mathbf{n} \rrbracket = k_{s\rho} \langle \eta \rangle \quad \text{on } \Lambda_j \quad (\text{Vertical spring term}) \quad (2.20e)$$

In the interface conditions Eq. (2.20), the derivatives have been weakly imposed. In the case that only vertical mooring is applicable, the rotational spring term in Eq. (2.20d) is not imposed, and the mean bending moment term $\langle -D_\rho \nabla^2 \eta_h \rangle$ is preserved in the equation. Detailed derivation of this discontinuous term will be explained in Section 3.4.

2.4. Modal Analysis

In the modal superposition method, the modal shape for each natural frequency shall be determined by modal analysis. The modal analysis for the hydroelastic case can be calculated with and without the influence of the surrounding fluid. In a general structural dynamic problem, modal analysis is performed by neglecting external influences, such as fluid and loading effects, and is based solely on the structural mechanical properties. This type of analysis is known as “Dry” modal analysis. In the hydroelastic case, the influence of the surrounding fluid determines the structure’s natural dynamic behaviour through hydrostatic restoring forces, which act as distributed stiffness forces on the structure (Bishop and Price, 1976). According to this, the influence of the fluid shall be accounted for in the modal analysis through “Wet” modal analysis.

In this research, the modal analysis will be limited to the undamped problem. The general equation of motion of a dynamic problem in the frequency domain can be written as in Eq. (2.21).

$$\begin{aligned} \mathbf{M} \ddot{\eta} + \mathbf{K} \eta &= \mathbf{F} \\ (-i\omega)^2 \mathbf{M} \eta + \mathbf{K} \eta &= \mathbf{F} \\ (-\omega^2 \mathbf{M} + \mathbf{K}) \eta &= \mathbf{F} \end{aligned} \quad (2.21)$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{K}$ is the stiffness matrix, and $\mathbf{F}$ is the force vector. The modal shapes and natural frequencies are obtained by solving the eigenvalue problem for the free-vibration system as in Eq. (2.22).

$$(-\omega^2 \mathbf{M} + \mathbf{K}) \eta = 0 \quad (2.22)$$

However, considering the final structural strong formulation in Eq. (2.15), the equation has an ω^4 term in the A_3 constant. This component is a secondary mass term for the structure and, consequently, the standard method for solving the eigenvalue problem in Eq. (2.22) becomes inapplicable. If the terms are collected based on the wave frequency component and the coupling terms are moved to the force term, the general equation of motion in Eq. (2.21) can be rewritten as:

$$(\omega^4 \mathbf{S} - \omega^2 \mathbf{M} + \mathbf{K}) \eta = \mathbf{F}(\omega) \quad (2.23)$$

The equation of motion is now in ω^2 quadrature formulation. To solve the modal analysis for this formulation, the quadratic eigenvalue problem (QEP) approach will be applied (Tisseur and Meerbergen, 2001). While this QEP is commonly used for a damped system, its mathematical expression can be adopted to perform the modal analysis for the Mindlin beam model.

Following the modal analysis method of Agarwal, Colomés, and Metrikine (2024), the monolithic formulation can be expressed in matrix form. Including the secondary mass term $\mathbf{S}$ into the matrix, the matrix system is written as follows:

$$\begin{bmatrix} \omega^4 \mathbf{S}_{11} - \omega^2 \mathbf{M}_{11} + \mathbf{K}_{11}^d + \mathbf{K}_{11}^w & -i\omega \mathbf{C}_{12} & 0 \\ i\omega \mathbf{C}_{21} & \mathbf{K}_{22} & \mathbf{C}_{23} \\ 0 & \mathbf{C}_{32} & \mathbf{K}_{33} \end{bmatrix} \begin{bmatrix} \eta_p \\ \phi_p \\ \kappa_p \end{bmatrix} = \begin{bmatrix} \mathbf{F}_\eta \\ \mathbf{F}_\phi \\ \mathbf{F}_\kappa \end{bmatrix} \quad (2.24)$$

To solve the eigenvalue problem of this monolithic formulation, the matrix in Eq. (2.24) must first be rewritten in η_p by eliminating the variables as follows:

First, derive the free surface component, κ_p , in the third row:

$$\begin{aligned} \mathbf{C}_{32} \phi_p + \mathbf{K}_{33} \kappa_p &= \mathbf{F}_\kappa \\ \kappa_p &= \mathbf{K}_{33}^{-1} \mathbf{F}_\kappa - \mathbf{K}_{33}^{-1} \mathbf{C}_{32} \phi_p \end{aligned} \quad (2.25)$$

Secondly, break down the potential component, ϕ_p , in the second row and substitute the free surface term, κ_p , from Eq. (2.25) into the equation:

$$\begin{aligned} i\omega \mathbf{C}_{21} \eta_p + \mathbf{K}_{22} \phi_p + \mathbf{C}_{23} \kappa_p &= \mathbf{F}_\phi \\ i\omega \mathbf{C}_{21} \eta_p + \mathbf{K}_{22} \phi_p + \mathbf{C}_{23} (\mathbf{K}_{33}^{-1} \mathbf{F}_\kappa - \mathbf{K}_{33}^{-1} \mathbf{C}_{32} \phi_p) &= \mathbf{F}_\phi \\ \mathbf{K}_{22} \phi_p - \mathbf{C}_{23} \mathbf{K}_{33}^{-1} \mathbf{C}_{32} \phi_p &= \mathbf{F}_\phi - \mathbf{C}_{23} \mathbf{K}_{33}^{-1} \mathbf{F}_\kappa - i\omega \mathbf{C}_{21} \eta_p \\ (\mathbf{K}_{22} - \mathbf{C}_{23} \mathbf{K}_{33}^{-1} \mathbf{C}_{32}) \phi_p &= \mathbf{F}_\phi - \mathbf{C}_{23} \mathbf{K}_{33}^{-1} \mathbf{F}_\kappa - i\omega \mathbf{C}_{21} \eta_p \end{aligned} \quad (2.26)$$

On the left-hand side, the additional term appears and will be denoted $\mathbf{M}_\phi$. Thus, the potential term can be written as

$$\phi_p = \mathbf{M}_\phi^{-1} \mathbf{F}_\phi - \mathbf{M}_\phi^{-1} \mathbf{C}_{23} \mathbf{K}_{33}^{-1} \mathbf{F}_\kappa - i\omega \mathbf{M}_\phi^{-1} \mathbf{C}_{21} \eta_p \quad (2.27)$$

where

$$\mathbf{M}_\phi = \mathbf{K}_{22} - \mathbf{C}_{23} \mathbf{K}_{33}^{-1} \mathbf{C}_{32}$$

Lastly, break down the structural deflection component, η_p , and substitute Eq. (2.27) into the derivation.

$$\begin{aligned} (\omega^4 \mathbf{S}_{11} - \omega^2 \mathbf{M}_{11} + \mathbf{K}_{11}^d + \mathbf{K}_{11}^w) \eta_p - i\omega \mathbf{C}_{12} \phi_p &= \mathbf{F}_\eta \\ (\omega^4 \mathbf{S}_{11} - \omega^2 \mathbf{M}_{11} + \mathbf{K}_{11}^d + \mathbf{K}_{11}^w) \eta_p - i\omega \mathbf{C}_{12} (\mathbf{M}_\phi^{-1} \mathbf{F}_\phi - \mathbf{M}_\phi^{-1} \mathbf{C}_{23} \mathbf{K}_{33}^{-1} \mathbf{F}_\kappa - i\omega \mathbf{M}_\phi^{-1} \mathbf{C}_{21} \eta_p) &= \mathbf{F}_\eta \\ (\omega^4 \mathbf{S}_{11} - \omega^2 \mathbf{M}_{11} + \mathbf{K}_{11}^d + \mathbf{K}_{11}^w) \eta_p - i\omega \mathbf{C}_{12} \mathbf{M}_\phi^{-1} \mathbf{F}_\phi + i\omega \mathbf{C}_{12} \mathbf{M}_\phi^{-1} \mathbf{C}_{23} \mathbf{K}_{33}^{-1} \mathbf{F}_\kappa + (i\omega)^2 \mathbf{C}_{12} \mathbf{M}_\phi^{-1} \mathbf{C}_{21} \eta_p &= \mathbf{F}_\eta \\ (\omega^4 \mathbf{S}_{11} - \omega^2 \mathbf{M}_{11} + \mathbf{K}_{11}^d + \mathbf{K}_{11}^w) \eta_p + (-\omega^2) \mathbf{C}_{12} \mathbf{M}_\phi^{-1} \mathbf{C}_{21} \eta_p &= \mathbf{F}_\eta + i\omega \mathbf{C}_{12} \mathbf{M}_\phi^{-1} \mathbf{F}_\phi - i\omega \mathbf{C}_{12} \mathbf{M}_\phi^{-1} \mathbf{C}_{23} \mathbf{K}_{33}^{-1} \mathbf{F}_\kappa \end{aligned}$$

From the derivation, an additional mass term appears on the left-hand side, $-\omega^2 (\mathbf{C}_{12} \mathbf{M}_\phi^{-1} \mathbf{C}_{21}) \eta_p$, which is equivalent to the added mass. This term will be denoted as $\mathbf{A}$, and the final equation becomes:

$$(\omega^4 \mathbf{S}_{11} - \omega^2 (\mathbf{M}_{11} + \mathbf{A}) + (\mathbf{K}_{11}^d + \mathbf{K}_{11}^w)) \eta_p = \mathbf{F}_\eta + i\omega \mathbf{C}_{12} \mathbf{M}_\phi^{-1} \mathbf{F}_\phi - i\omega \mathbf{C}_{12} \mathbf{M}_\phi^{-1} \mathbf{C}_{23} \mathbf{K}_{33}^{-1} \mathbf{F}_\kappa \quad (2.28)$$

where,

$$\mathbf{A} = \mathbf{C}_{12} \mathbf{M}_\phi^{-1} \mathbf{C}_{21}$$

The dry modal analysis can then be calculated using the free-vibration case where the external forcing terms of $\mathbf{F}_\eta$, $\mathbf{F}_\phi$, and $\mathbf{F}_\kappa$ are 0, and neglecting the added mass, $\mathbf{A}$ and the restoring stiffness term, $\mathbf{K}_{11}^w$. Solving the eigenvalue problem of Eq.(2.29)

$$(\omega^4 \mathbf{S}_{11} - \omega^2 \mathbf{M}_{11} + \mathbf{K}_{11}^d) \{\eta_p\} = 0 \quad (2.29)$$

The wet modal frequency is solved by including the influence of the added mass $\mathbf{A}$ and restoring stiffness, $\mathbf{K}_{11}^w$, into the eigenvalue problem in Eq. (2.30).

$$(\omega^4 \mathbf{S}_{11} - \omega^2 (\mathbf{M}_{11} + \mathbf{A}) + (\mathbf{K}_{11}^d + \mathbf{K}_{11}^w)) \{\eta_p\} = 0 \quad (2.30)$$

Where ω is the natural frequency of the system and $\{\eta_p\}$ is the modal shape of the system, which later in this research will be denoted as $[\Phi]$, a modal shape matrix with $m \times n$ size, where m is the total degree of freedom of the structure and n is the number of modes.

Implementation of QEP:

Now look for the original expression of the eigenvalue problem for a damped system using QEP:

$$(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K}) \eta_p = 0 \quad (2.31)$$

Where $\lambda = -i\omega$. However, substituting the value of $\lambda = \omega^2$ into the dry eigenvalue system Eq. (2.29), the same quadratic eigenvalue expression to solve the dry modal superposition is obtained.

$$(\lambda^2 \mathbf{S}_{11} - \lambda \mathbf{M}_{11} + \mathbf{K}_{11}^d) \{\eta_p\} = 0 \quad (2.32)$$

Following the study from Tisseur and Meerbergen (2001), this eigenvalue problem can be solved through linearization using the generalised eigenvalue problem (GEP) method, employing an auxiliary variable $u = \lambda \eta_p$.

$$\begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{K}_{11}^d & \mathbf{M}_{11} \end{bmatrix} \begin{bmatrix} \eta_p \\ u \end{bmatrix} - \lambda \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{S}_{11} \end{bmatrix} \begin{bmatrix} \eta_p \\ u \end{bmatrix} = 0 \quad (2.33)$$

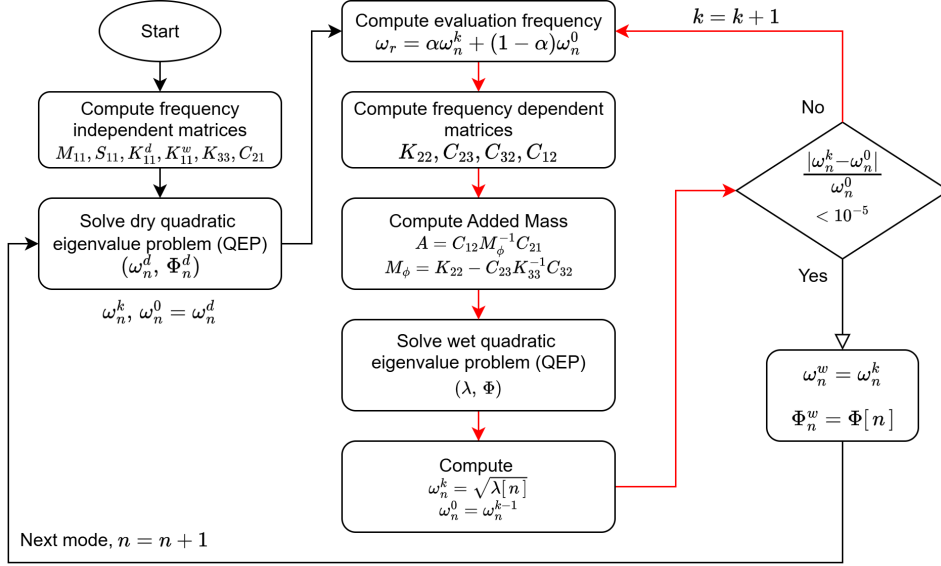


Figure 2.2: Iterative procedure of wet modal analysis for the unified monolithic FEM

This system can be rewritten as a standard eigenvalue problem below:

$$\mathbf{A}\Phi - \lambda\mathbf{B}\Phi = 0$$

Where the eigenvalue obtained from solving the EVP above is $\lambda = \omega^2$ and the eigenvector is $\Phi = [\eta_p; u]$. Since u is an auxiliary variable that is a scaled term of η_p , only the first half of Φ , that is, the mode shape components η_p , will be considered.

The wet modal superposition is calculated using the same framework as in Agarwal, Colomés, and Metrikine (2024), solving a nonlinear eigenvalue problem. This method uses the dry natural frequency as the initial value for the iteration at each n -th natural frequency and applies the QEP to solve the eigenvalue problem, now considering the restoring stiffness $\mathbf{K}_{11}^w$ and added mass $\mathbf{A}$ in the calculation of Eq. (2.33). In modal analysis, wave generation is not required. Therefore, the damping zone term is replaced by the radiation boundary $\mathbf{n} \cdot \nabla \phi = ik\phi$, applied on the inlet Γ_{in} and outlet Γ_{out} boundary integrals, at the matrix $\mathbf{K}_{22}$. This damping term will be required for the modal superposition to capture all terms in the monolithic FE formulation. The procedure for the iterative wet modal analysis is depicted in Fig. (2.2). The detailed formulation of defining the matrices for modal analysis will be explained in Section (3.7) after the discrete weak formulation has been derived.

2.5. Modal Superposition

The modal analysis in the previous section solves the natural frequencies $\{\omega_n\}$ and the modal shapes $[\Phi_n]$ of the total N modes. The modal shapes obtained from the analysis will later be used in the reduced-order modelling (ROM) of modal superposition to transform the mass matrix, stiffness matrix, and force vector from the physical domain to the modal domain. The initial study of the hydroelasticity problem involving modal superposition was performed by Hamamoto and Fujita (2002). Following the framework from that study, the structure deflection can be defined as the superposition of the N modal solutions, which is expressed as follows:

$$\{\eta\} = [\Phi]\{p\} = \sum_{m=1}^N p_m \cdot \Phi_m \quad (2.34)$$

Where p is the modal amplitude obtained from solving the modal system and N is the total number of modes used in the modal superposition. The modal system is obtained by substituting Eq. (2.34) into

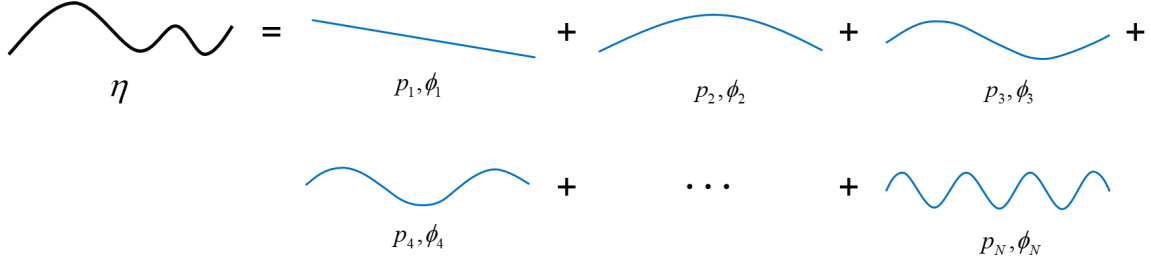


Figure 2.3: Modal superposition illustration for the hydroelastic case

the undamped system in Eq. (2.23) and premultiplying with $[\Phi]^T$, resulting to:

$$[\Phi]^T (\omega^4 \mathbf{S} - \omega^2 \mathbf{M} + \mathbf{K}) [\Phi] \{p\} = [\Phi]^T \mathbf{F} \quad (2.35)$$

$$(\omega^4 [\Phi]^T \mathbf{S} [\Phi] - \omega^2 [\Phi]^T \mathbf{M} [\Phi] + [\Phi]^T \mathbf{K} [\Phi]) \{p\} = [\Phi]^T \mathbf{F}$$

Where $\mathbf{M}$ is the total mass matrix of the system, $\mathbf{S}$ is the secondary mass, $\mathbf{K}$ is the total stiffness of the system, and $\mathbf{F}$ is the total force of the system. Therefore, the modal matrices are constructed as follows:

$$[S_{mod}] = [\Phi]^T [\mathbf{S}_{11}] [\Phi] \quad (2.36a)$$

$$[M_{mod}] = [\Phi]^T [\mathbf{M}_{11} + \mathbf{A}(\omega)] [\Phi] \quad (2.36b)$$

$$[K_{mod}] = [\Phi]^T [\mathbf{K}_{11}^d + \mathbf{K}_{11}^w] [\Phi] \quad (2.36c)$$

$$\{Q_{mod}\} = [\Phi]^T \{\mathbf{F}_{tot}\} \quad (2.36d)$$

The matrix $\mathbf{A}$ depends on the wave frequency and is a function of matrices $\mathbf{C}_{12}$, $\mathbf{C}_{21}$, $\mathbf{C}_{23}$, $\mathbf{C}_{32}$, $\mathbf{K}_{22}$, and $\mathbf{K}_{33}$. To fully capture all physical terms in the monolithic system, both dry and wet modal superposition cases include the restoring stiffness $\mathbf{K}_{11}^w$ and added mass $\mathbf{A}$. The modal equation of motion in the frequency domain can be rewritten as:

$$\omega^4 [S_{mod}] \{p\} - \omega^2 [M_{mod}] \{p\} + [K_{mod}] \{p\} = \{Q_{mod}\} \quad (2.37)$$

$$[\omega^4 S_{mod} - \omega^2 M_{mod} + K_{mod}] \{p\} = \{Q_{mod}\}$$

These modal matrices are dense matrices with an $N \times N$ size, where N is the number of modal shapes used in the analysis. The size of this matrix system is extremely smaller than the full matrices that have an $N_{dof} \times N_{dof}$ size, where the value of N is typically taken as $10 \sim 15$ while N_{dof} can be in the order of hundreds to thousands. Finally, the modal amplitude solution can be obtained by solving Eq. (2.38). Solving this system is extremely cheap and can be done almost instantaneously for $N = 10 \sim 15$.

$$\{p\} = [\omega^4 S_{mod} - \omega^2 M_{mod} + K_{mod}]^{-1} \{Q_{mod}\} \quad (2.38)$$

The structure deflection solution can be calculated by substituting this modal amplitude solution and the mode shapes obtained from the modal analysis into Eq. (2.34). Fig. (2.3) depicts the illustration of the modal superposition up to mode N . Detailed formulation for determining each matrix and implementation in the monolithic formulation are provided in Section 3.8.

Numerical Formulation

This chapter will derive the weak formulation of the governing equation from the previous chapter. The implementation of the unified monolithic FEM formulation into the modal analysis and modal superposition are discussed later in this chapter.

3.1. Weak Formulation

To solve the fluid and structural governing equations in Eq. (2.3) and (2.15) through monolithic FEM, a variational formulation is required by deriving the weak form of the governing equation (strong form). The weak form can be derived by using the Galerkin method, that is, multiplying the strong form by a test function and integrating over the domain. Before applying the Galerkin method, the function space using the Hilbert space is first introduced. Define a functional space of $L^r(\Omega)$, $1 \leq r < \infty$ that is, its r^{th} power order is integrable in the domain Ω . This research will use the Hilbert space with an inner product, where $r = 2$. This function of space $L^2(\Omega)$ is a square-integrable function and corresponds to an L^2 norm of $\|u\|_{L^2(\Omega)} \equiv \|u\|_{\Omega} = (u, u)_{\Omega}^{1/2}$. Additionally, another function space at the boundary Γ is also defined, where $\Gamma \subset \Omega$. This function also corresponds to its respective L^2 norm. These functions are expressed in the following notations:

$$(u, v)_{\Omega} = \int_{\Omega} u(\mathbf{x}) v(\mathbf{x}) d\Omega \quad (u, v)_{\Gamma} = \int_{\Gamma} u(\mathbf{x}) v(\mathbf{x}) d\Gamma \quad (3.1)$$

Where $u(\mathbf{x})$ represents the trial function where the unknown solutions are being solved, and $v(\mathbf{x})$ is the test function. Moreover, one more function of space needs to be defined. The function where the distributions of its derivative in the domain Ω up to order m are in $L^2(\Omega)$, denoted as $H^m(\Omega)$. This research will focus on $m = 1$, for which the first derivative is square-integrable.

The next step is to define the variational space $\mathcal{V}$ in the fluid domain, that is, $\mathcal{V} = H^1(\Omega)$, and the trace spaces in the free surface $\mathcal{V}_{\Gamma_{fs}}$ and structure domains $\mathcal{V}_{\Gamma_b}$. These trace functions are also part of the variational space $\mathcal{V}$, where $\mathcal{V}_{\Gamma_{fs}} = \{v|_{\Gamma_{fs}} : v \in \mathcal{V}\}$ and $\mathcal{V}_{\Gamma_b} = \{v|_{\Gamma_b} : v \in \mathcal{V}\}$. In this formulation, the unknown functions of velocity potential, free surface, and structural deflection are defined on the following:

- Velocity potential, denoted as ϕ with an associated test function of w with $\phi, w \in \mathcal{V}$
- Free surface, denoted as κ with an associated test function of u with $\kappa, u \in \mathcal{V}_{\Gamma_{fs}}$
- Structural deflection, denoted as η with an associated test function of v with $\eta, v \in \mathcal{V}_{\Gamma_b}$

The second-order fluid governing equation in Eq. (2.3) for solving ϕ and κ requires $H^1(\Omega)$ space, while the fourth-order structural governing equation in Eq. (2.15) requires $H^2(\Gamma_b)$ space. It indicates that this hydroelastic case solves a mixed-dimensional problem. To solve this problem, this research will adopt the approach used in Colomés, Verdugo, and Akkerman (2023) by using the $H^1(\Gamma_b)$ space for η and implementing the Continuous/Discontinuous Galerkin (C/DG) method to weakly enforce the

higher-order derivatives. After defining the function spaces, the variational problem of this monolithic formulation reads:

Find $[\phi, \kappa, \eta] \in \mathcal{V} \times \mathcal{V}_{\Gamma_{fs}} \times \mathcal{V}_{\Gamma_b}$ such that:

$$B_\omega([\phi, \kappa, \eta], [w, u, v]) = L_\omega([w, u, v]) \quad \forall [w, u, v] \in \mathcal{V} \times \mathcal{V}_{\Gamma_{fs}} \times \mathcal{V}_{\Gamma_b}$$

Where B_ω is the bilinear form and L_ω is the linear form in the frequency domain. These forms are obtained after deriving the weak form of the governing equation. The following parts present the detailed weak formulation.

3.2. Fluid Weak Formulation

The fluid weak form is derived by taking the strong form in Eq. (2.3), multiplying by a test function w and integrating over the domain Ω .

$$\int_{\Omega} \Delta \phi w \, d\Omega = 0$$

Apply integration by parts to reduce the order of the partial differential equation (PDE), then combine with the divergence method, yielding the following.

$$\begin{aligned} \int_{\Omega} \nabla(w \nabla \phi) \, d\Omega &= 0 \\ \int_{\Gamma} w(\mathbf{n} \cdot \nabla \phi) \, d\Gamma - \int_{\Omega} \nabla w \nabla \phi \, d\Omega &= 0 \end{aligned}$$

Following the notation in Eq. (3.1), the equation above can be rewritten as:

$$(\nabla \phi, \nabla w)_{\Omega} - (\mathbf{n} \cdot \nabla \phi, w)_{\Gamma} = 0 \quad (3.2)$$

Here, Γ is the boundary term of the fluid domain Ω , where $\Gamma_{in}, \Gamma_{out}, \Gamma_{sb}, \Gamma_{fs}, \Gamma_b, \Gamma_d \in \Gamma$. Therefore, the kinematic and dynamic boundary terms in Eq. (2.4), (2.6a), and (2.8) are imposed into the weak formulation. However, since the terms in the previous equations are still in the time domain, they need to be converted to the frequency domain by evaluating the temporal derivatives and using only the amplitude of the solution, neglecting the temporal terms in the separation-of-variables solution as used in Subsection 2.3.2. The summary of the boundary conditions in the frequency domain for the fluid is expressed as follows:

$$\mathbf{n} \cdot \nabla \phi = 0 \quad \text{on } \Gamma_{sb} \quad (3.3a)$$

$$\mathbf{n} \cdot \nabla \phi = u_{in} \quad \text{on } \Gamma_{in} \quad (3.3b)$$

$$\mathbf{n} \cdot \nabla \phi = u_{out} \quad \text{on } \Gamma_{out} \quad (3.3c)$$

$$\mathbf{n} \cdot \nabla \phi = -i\omega\kappa \quad \text{on } \Gamma_{fs} \quad (\text{KFSBC}) \quad (3.3d)$$

$$\mathbf{n} \cdot \nabla \phi = -i\omega\eta \quad \text{on } \Gamma_b \quad (\text{KFSBC}) \quad (3.3e)$$

$$-i\omega\phi + g\kappa = 0 \quad \text{on } \Gamma_{fs} \quad (\text{DFSBC}) \quad (3.3f)$$

$$\mathbf{n} \cdot \nabla \phi = -i\omega\kappa + \mu_2(\kappa - \kappa_{in}) \quad \text{on } \Gamma_{d1}, \Gamma_{d2} \quad (\text{KFSBC}) \quad (3.3g)$$

$$-i\omega\phi + g\kappa + \mu_1(\nabla \phi \cdot \mathbf{n} - \nabla \phi_{in} \cdot \mathbf{n}) = 0 \quad \text{on } \Gamma_{d1}, \Gamma_{d2} \quad (\text{DFSBC}) \quad (3.3h)$$

Where u_{in}, κ_{in} are in Eq. (2.7c) and (2.7a) without the temporal term, and u_{out} is set to zero since the damping zone at the outlet is implemented instead of the radiation boundary. The kinematic boundary conditions above can naturally be substituted into Eq. (3.2).

$$\begin{aligned} (\nabla \phi, \nabla w)_{\Omega} - (0, w)_{\Gamma_{sb}} - (u_{in}, w)_{\Gamma_{in}} - (0, w)_{\Gamma_{out}} - (-i\omega\kappa, w)_{\Gamma_{fs}} - (-i\omega\eta, w)_{\Gamma_b} \\ - (-i\omega\kappa + \mu_2(\kappa - \kappa_{in}), w)_{\Gamma_{d1}, \Gamma_{d2}} = 0 \end{aligned}$$

Evaluate the bilinear and linear terms. The value of κ_{in} at the outlet damping Γ_{d2} is set to zero to fully absorb all the waves.

$$\begin{aligned} (\nabla\phi, \nabla w)_{\Omega} + (i\omega\eta, w)_{\Gamma_b} + (i\omega\kappa, w)_{\Gamma_{\text{fs}}} + (i\omega\kappa, w)_{\Gamma_{\text{d1}}} + (i\omega\kappa, w)_{\Gamma_{\text{d2}}} - (\mu_2\kappa, w)_{\Gamma_{\text{d1}}} - (\mu_2\kappa, w)_{\Gamma_{\text{d2}}} \\ = (u_{\text{in}}, w)_{\Gamma_{\text{in}}} - (\mu_2\kappa_{\text{in}}, w)_{\Gamma_{\text{d1}}} \end{aligned} \quad (3.4)$$

At the dynamic free surface boundary for the damping zone Γ_{d1} and Γ_{d2} , there is a prescribed term of $\nabla\phi_{\text{in}} \cdot \mathbf{n}$ that is the velocity in the normal direction of the surface at the damping zone. This value is equal to the first time derivative of the surface elevation, that is:

$$\nabla\phi_{\text{in}} \cdot \mathbf{n} = \frac{\partial\kappa_{\text{in}}}{\partial t} = -i\omega\kappa_{\text{in}} \quad (3.5)$$

For the dynamic boundary conditions, the weak form is derived separately. Here, a modified test function of $\beta_h(\alpha_f w + u)$ is used to ensure the coercivity and the stabilisation of the formulation, applied on the free surface (Colomés, Verdugo, and Akkerman, 2023). The value of α_f in frequency domain analysis is defined by using the following equation:

$$\alpha_f = \frac{(-i\omega)}{g} \left(\frac{1 - \beta_h}{\beta_h} \right) \quad (3.6)$$

Where $\beta_h = 0.5$ is set as a fixed parameter. Now apply the Galerkin method for the dynamic boundary in Eq. (3.3f) and (3.3h) using the modified test function.

$$\begin{aligned} \int_{\Gamma_{\text{fs}}} \beta_h(-i\omega\phi + g\kappa)(\alpha_f w + u) d\Gamma_{\text{fs}} + \int_{\Gamma_{\text{d1}}} \beta_h(-i\omega\phi + g\kappa)(\alpha_f w + u) d\Gamma_{\text{d1}} \\ + \int_{\Gamma_{\text{d2}}} \beta_h(-i\omega\phi + g\kappa)(\alpha_f w + u) d\Gamma_{\text{d2}} \\ + \int_{\Gamma_{\text{d1}}} (\mu_1(\nabla\phi \cdot \mathbf{n} - \nabla\phi_{\text{in}} \cdot \mathbf{n}))(\alpha_f w + u) d\Gamma_{\text{d1}} \\ + \int_{\Gamma_{\text{d2}}} (\mu_1(\nabla\phi \cdot \mathbf{n} - \nabla\phi_{\text{in}} \cdot \mathbf{n}))(\alpha_f w + u) d\Gamma_{\text{d2}} = 0 \end{aligned}$$

Rewrite with the notation as in Eq. (3.1).

$$\begin{aligned} \beta_h(-i\omega\phi + g\kappa, \alpha_f w + u)_{\Gamma_{\text{fs}}} + \beta_h(-i\omega\phi + g\kappa, \alpha_f w + u)_{\Gamma_{\text{d1}}} + (\mu_1 \nabla\phi \cdot \mathbf{n} - \mu_1 \nabla\phi_{\text{in}} \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{\text{d1}}} \\ + \beta_h(-i\omega\phi + g\kappa, \alpha_f w + u)_{\Gamma_{\text{d2}}} + (\mu_1 \nabla\phi \cdot \mathbf{n} - \mu_1 \nabla\phi_{\text{in}} \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{\text{d2}}} = 0 \end{aligned}$$

Separate the bilinear and linear terms.

$$\begin{aligned} \beta_h(-i\omega\phi + g\kappa, \alpha_f w + u)_{\Gamma_{\text{fs}}} + \beta_h(-i\omega\phi + g\kappa, \alpha_f w + u)_{\Gamma_{\text{d1}}} + (\mu_1 \nabla\phi \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{\text{d1}}} \\ + \beta_h(-i\omega\phi + g\kappa, \alpha_f w + u)_{\Gamma_{\text{d2}}} + (\mu_1 \nabla\phi \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{\text{d2}}} \\ = (\mu_1 \nabla\phi_{\text{in}} \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{\text{d1}}} + (\mu_1 \nabla\phi_{\text{in}} \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{\text{d2}}} \end{aligned}$$

Similar to the kinematic boundary, the ϕ_{in} at the outlet damping Γ_{d2} is also set to zero.

$$\begin{aligned} \beta_h(-i\omega\phi + g\kappa, \alpha_f w + u)_{\Gamma_{\text{fs}}} + \beta_h(-i\omega\phi + g\kappa, \alpha_f w + u)_{\Gamma_{\text{d1}}} + (\mu_1 \nabla\phi \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{\text{d1}}} \\ + \beta_h(-i\omega\phi + g\kappa, \alpha_f w + u)_{\Gamma_{\text{d2}}} + (\mu_1 \nabla\phi \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{\text{d2}}} \\ = (\mu_1 \nabla\phi_{\text{in}} \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{\text{d1}}} \end{aligned} \quad (3.7)$$

Finally, combining the Eq. (3.4) and (3.7), the bilinear B_ω and linear L_ω terms of the fluid weak form

are as below:

$$\begin{aligned}
 B_\omega([\phi, \kappa, \eta], [w, u, v]) &= (\nabla \phi, \nabla w)_\Omega + (i\omega \kappa, w)_{\Gamma_{fs}} + \beta_h(-i\omega \phi + g\kappa, \alpha_f w + u)_{\Gamma_{fs}} \\
 &+ (i\omega \kappa, w)_{\Gamma_{d1}} + \beta_h(-i\omega \phi + g\kappa, \alpha_f w + u)_{\Gamma_{d1}} - (\mu_2 \kappa, w)_{\Gamma_{d1}} + (\mu_1 \nabla \phi \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{d1}} \\
 &+ (i\omega \kappa, w)_{\Gamma_{d2}} + \beta_h(-i\omega \phi + g\kappa, \alpha_f w + u)_{\Gamma_{d2}} - (\mu_2 \kappa, w)_{\Gamma_{d2}} + (\mu_1 \nabla \phi \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{d2}} \\
 &+ (i\omega \eta, w)_{\Gamma_b}
 \end{aligned} \tag{3.8a}$$

$$L_\omega([w, u, v]) = (u_{in}, w)_{\Gamma_{in}} - (\mu_2 \kappa_{in}, w)_{\Gamma_{d1}} + (\mu_1 \nabla \phi_{in} \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{d1}} \tag{3.8b}$$

3.3. Structural Weak Formulation

The terms inside the bracket of the structural strong form in Eq. (2.15) are constant values. Therefore, for clarity and simplification, in this derivation the constants will be denoted A_1, A_2, A_3, A_4 , and A_5 as in Eq. (2.16). Apply the Galerkin method, multiplying the strong form by the test function v and integrating over the domain Γ_b .

$$\begin{aligned}
 \int_{\Gamma_b} (A_1 \nabla^4 \eta + A_2 \nabla^2 \eta + A_3 \eta + A_4 \nabla^2 \phi + A_5 \phi) v \, d\Gamma_b &= 0 \\
 (A_1 \nabla^4 \eta, v)_{\Gamma_b} + (A_2 \nabla^2 \eta, v)_{\Gamma_b} + (A_3 \eta, v)_{\Gamma_b} + (A_4 \nabla^2 \phi, v)_{\Gamma_b} + (A_5 \phi, v)_{\Gamma_b} &= 0
 \end{aligned}$$

Evaluate the higher-order terms by applying the integration by parts twice for the fourth-order component and once for the second-order components:

$$\begin{aligned}
 &(\nabla^2(A_1 \nabla^2 \eta), v)_{\Gamma_b} + (A_2 \nabla^2 \eta, v)_{\Gamma_b} + (A_4 \nabla^2 \phi, v)_{\Gamma_b} \\
 &= (A_1 \nabla^2 \eta, \nabla^2 v)_{\Gamma_b} + (\nabla(A_1 \nabla^2 \eta) \cdot \mathbf{n}, v)_{\Gamma_{BC}, \Lambda_j} - (A_1 \nabla^2 \eta \cdot \mathbf{n}, \nabla v)_{\Gamma_{BC}, \Lambda_j} \\
 &\quad - (A_2 \nabla \eta, \nabla v)_{\Gamma_b} + ((A_2 \nabla \eta) \cdot \mathbf{n}, v)_{\Gamma_{BC}, \Lambda_j} - (A_4 \nabla \phi, \nabla v)_{\Gamma_b} + ((A_4 \nabla \phi) \cdot \mathbf{n}, v)_{\Gamma_{BC}, \Lambda_j} \\
 &= (A_1 \nabla^2 \eta, \nabla^2 v)_{\Gamma_b} - (A_2 \nabla \eta, \nabla v)_{\Gamma_b} - (A_4 \nabla \phi, \nabla v)_{\Gamma_b} \\
 &\quad + (\nabla(A_1 \nabla^2 \eta) \cdot \mathbf{n} + (A_2 \nabla \eta) \cdot \mathbf{n} + (A_4 \nabla \phi) \cdot \mathbf{n}, v)_{\Gamma_{BC}, \Lambda_j} - (A_1 \nabla^2 \eta \cdot \mathbf{n}, \nabla v)_{\Gamma_{BC}, \Lambda_j}
 \end{aligned} \tag{3.9}$$

The boundary terms arise naturally upon integrating by parts the higher-order PDE. The boundary terms here are applicable to the edge boundaries (Γ_l and $\Gamma_r \subset \Gamma_{BC}$) and to the internal interface if joint connections exist (Λ_j). Substituting these higher-order terms into the weak forms and enforcing the mooring boundary in Eq. (2.18) will result in the structural bilinear terms.

$$\begin{aligned}
 B_\omega([\phi, \kappa, \eta], [w, u, v]) &= (A_1 \nabla^2 \eta, \nabla^2 v)_{\Gamma_b} - (A_2 \nabla \eta, \nabla v)_{\Gamma_b} + (A_3 \eta, v)_{\Gamma_b} - (A_4 \nabla \phi, \nabla v)_{\Gamma_b} \\
 &\quad + (A_5 \phi, v)_{\Gamma_b} + (k_{s\rho} \eta \cdot \mathbf{n}, v)_{\Gamma_{BC}} + (k_{r\rho} \nabla \eta \cdot \mathbf{n}, \nabla v)_{\Gamma_{BC}}
 \end{aligned} \tag{3.10}$$

For the free-floating cases, where the moment and shear at both ends are zero, the value of the mooring stiffnesses $k_{s\rho}$ and $k_{r\rho}$ can be set to zero.

If internal joint connections are present, the rotation at the interface can be discontinuous due to multi-body hydroelastic interactions. However, the deflection at the interface must be continuous. Therefore, a jump formulation (see Eq. 2.19) will be implemented in the joint terms to capture discontinuities at the left (-) and right (+) sides of the joint and weakly enforce deflection continuity. Additionally, introducing the jump property as follows:

$$[[a \cdot b]] = [[a]] \langle b \rangle + \langle a \rangle [[b]] \tag{3.11}$$

Now, apply the jump operator and use the jump property to the joint terms in Eq. (3.9).

$$\begin{aligned}
 &([[\nabla(A_1 \nabla^2 \eta) \cdot \mathbf{n} + (A_2 \nabla \eta) \cdot \mathbf{n} + (A_4 \nabla \phi) \cdot \mathbf{n}], v])_{\Lambda_j} - ([[A_1 \nabla^2 \eta \cdot \mathbf{n}, \nabla v]])_{\Lambda_j} \\
 &= ([[\nabla(A_1 \nabla^2 \eta) \cdot \mathbf{n} + (A_2 \nabla \eta) \cdot \mathbf{n} + (A_4 \nabla \phi) \cdot \mathbf{n}], \langle v \rangle])_{\Lambda_j} \\
 &\quad + (\langle \nabla(A_1 \nabla^2 \eta) + (A_2 \nabla \eta) + (A_4 \nabla \phi) \rangle, [[v \cdot \mathbf{n}]]_{\Lambda_j})_{\Lambda_j} \\
 &\quad - ([[A_1 \nabla^2 \eta \cdot \mathbf{n}], \langle \nabla v \rangle])_{\Lambda_j} - (\langle A_1 \nabla^2 \eta \rangle, [[\nabla v \cdot \mathbf{n}]]_{\Lambda_j})_{\Lambda_j}
 \end{aligned} \tag{3.12}$$

The interface conditions in Eq. (2.20) can now be imposed into the weak formulation in Eq. (3.12). Additionally, the continuity of the test function can be imposed by setting $\llbracket v \cdot \mathbf{n} \rrbracket = 0$. Therefore, the bilinear term of the structure with joint connection is updated as follows:

$$\begin{aligned} B_\omega([\phi, \kappa, \eta], [w, u, v]) &= (A_1 \nabla^2 \eta, \nabla^2 v)_{\Gamma_b} - (A_2 \nabla \eta, \nabla v)_{\Gamma_b} + (A_3 \eta, v)_{\Gamma_b} - (A_4 \nabla \phi, \nabla v)_{\Gamma_b} \\ &\quad + (A_5 \phi, v)_{\Gamma_b} + (k_{s\rho} \eta \cdot \mathbf{n}, v)_{\Gamma_{BC}} + (k_{r\rho} \nabla \eta \cdot \mathbf{n}, \nabla v)_{\Gamma_{BC}} \\ &\quad + (k_{s\rho} \langle \eta \rangle, \langle v \rangle)_{\Lambda_j} + (k_{r\rho} \llbracket \nabla \eta \cdot \mathbf{n} \rrbracket, \llbracket \nabla v \cdot \mathbf{n} \rrbracket)_{\Lambda_j} \end{aligned} \quad (3.13)$$

For the case that there is no vertical spring term at the joint, the continuity of the jump term in Eq. (2.20e) can be enforced by setting the value to zero. Therefore, the vertical spring term $(k_{s\rho} \langle \eta \rangle, \langle v \rangle)_{\Lambda_j}$ is neglected. However, if there is no rotational spring term, the interface condition cannot be set to zero. The term $\langle -A_1 \nabla^2 \eta \rangle$ must be preserved, and the penalisation method is applied to stabilise the solution. The penalisation method is detailed in Section 3.4.

The complete weak formulation is obtained by combining the bilinear and linear terms of the fluid (Eq. 3.8) and structure (Eq. 3.13). Thus, the variational formulation of the hydroelastic problem for a structure with joints, flexible connections, and mooring lines reads:

Find $[\phi, \kappa, \eta] \in \mathcal{V} \times \mathcal{V}_{\Gamma_{fs}} \times \mathcal{V}_{\Gamma_b}$ such that:

$$B_\omega([\phi, \kappa, \eta], [w, u, v]) = L_\omega([w, u, v]) \quad \forall [w, u, v] \in \mathcal{V} \times \mathcal{V}_{\Gamma_{fs}} \times \mathcal{V}_{\Gamma_b} \quad (3.14)$$

With,

$$\begin{aligned} B_\omega([\phi, \kappa, \eta], [w, u, v]) &= (\nabla \phi, \nabla w)_\Omega + (i\omega \kappa, w)_{\Gamma_{fs}} + \beta_h(-i\omega \phi + g\kappa, \alpha_f w + u)_{\Gamma_{fs}} \\ &\quad + (i\omega \kappa, w)_{\Gamma_{d1}} + \beta_h(-i\omega \phi + g\kappa, \alpha_f w + u)_{\Gamma_{d1}} - (\mu_2 \kappa, w)_{\Gamma_{d1}} + (\mu_1 \nabla \phi \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{d1}} \\ &\quad + (i\omega \kappa, w)_{\Gamma_{d2}} + \beta_h(-i\omega \phi + g\kappa, \alpha_f w + u)_{\Gamma_{d2}} - (\mu_2 \kappa, w)_{\Gamma_{d2}} + (\mu_1 \nabla \phi \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{d2}} \\ &\quad + (A_1 \nabla^2 \eta, \nabla^2 v)_{\Gamma_b} - (A_2 \nabla \eta, \nabla v)_{\Gamma_b} + (A_3 \eta, v)_{\Gamma_b} - (A_4 \nabla \phi, \nabla v)_{\Gamma_b} + (A_5 \phi, v)_{\Gamma_b} + (i\omega \eta, w)_{\Gamma_b} \\ &\quad + (k_{s\rho} \eta \cdot \mathbf{n}, v)_{\Gamma_{BC}} + (k_{r\rho} \nabla \eta \cdot \mathbf{n}, \nabla v)_{\Gamma_{BC}} \\ &\quad + (k_{s\rho} \langle \eta \rangle, \langle v \rangle)_{\Lambda_j} + (k_{r\rho} \llbracket \nabla \eta \cdot \mathbf{n} \rrbracket, \llbracket \nabla v \cdot \mathbf{n} \rrbracket)_{\Lambda_j} \end{aligned} \quad (3.15a)$$

$$L_\omega([w, u, v]) = (u_{in}, w)_{\Gamma_{in}} - (\mu_2 \kappa_{in}, w)_{\Gamma_{d1}} + (\mu_1 \nabla \phi_{in} \cdot \mathbf{n}, \alpha_f w + u)_{\Gamma_{d1}} \quad (3.15b)$$

3.4. Spatial Discretisation and Discontinuous Formulation

The formulation in Eq. (3.15) is a continuous formulation. To solve the finite solution, the formulation has to be discretised over the domain. A discretised space of the domain Ω is defined as a triangulation Ω_h where the finite variational space $\mathcal{V}_h \subset \mathcal{V}$ and the trace functions for the structural deflection $\mathcal{V}_{b,h} \subset \mathcal{V}_b$ and free surface $\mathcal{V}_{fs,h} \subset \mathcal{V}_{fs}$ are constructed to solve the unknowns of the finite quantities of velocity potential ϕ_h , free surface κ_h , and structure deflection η_h . Moreover, the discretised facet term needs to be defined at the boundaries, as the entity with one lower-order dimension than the domain. For the fluid domain Ω_h , the facets are $\Gamma_{fs,h}$ for the free surface, $\Gamma_{b,h}$ for the structural beam, $\Gamma_{in,h}$ for the inlet face, $\Gamma_{out,h}$ for the outlet face, and $\Gamma_{sb,h}$ for the sea bed. Additionally, the facet term of the structure is defined and denoted by $\Lambda_{b,h}$. In this 2D problem with a 1D structural domain, the facet of the structure will be a 0-dimensional quantity of nodes, see Fig. (3.1).

Using this notation, the finite variational formulation equivalent to Eq. (3.15), reads:

Find $[\phi_h, \kappa_h, \eta_h] \in \mathcal{V}_h \times \mathcal{V}_{\Gamma_{fs,h}} \times \mathcal{V}_{\Gamma_{b,h}}$ such that:

$$B_h([\phi_h, \kappa_h, \eta_h], [w_h, u_h, v_h]) = L_h([w_h, u_h, v_h]) \quad \forall [w_h, u_h, v_h] \in \mathcal{V}_h \times \mathcal{V}_{\Gamma_{fs,h}} \times \mathcal{V}_{\Gamma_{b,h}} \quad (3.16)$$

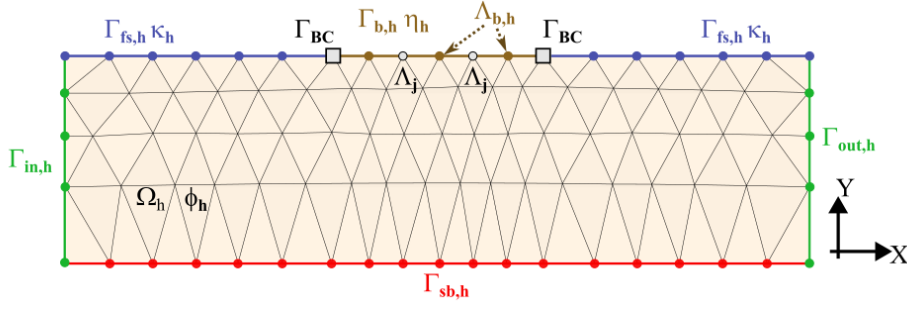


Figure 3.1: Discretised Scheme

With,

$$B_h([\phi_h, \kappa_h, \eta_h], [w_h, u_h, v_h]) = (\nabla \phi_h, \nabla w_h)_{\Omega_h} \quad (3.17a)$$

$$\begin{aligned} & + (i\omega \kappa_h, w_h)_{\Gamma_{fs,h}} + \beta_h(-i\omega \phi_h + g \kappa_h, \alpha_f w_h + u_h)_{\Gamma_{fs,h}} \\ & + (i\omega \kappa_h, w_h)_{\Gamma_{d1,h}} + \beta_h(-i\omega \phi_h + g \kappa_h, \alpha_f w_h + u_h)_{\Gamma_{d1,h}} \\ & + (i\omega \kappa_h, w_h)_{\Gamma_{d2,h}} + \beta_h(-i\omega \phi_h + g \kappa_h, \alpha_f w_h + u_h)_{\Gamma_{d2,h}} \\ & - (\mu_2 \kappa_h, w_h)_{\Gamma_{d1,h}} + (\mu_1 \nabla \phi_h \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d1,h}} \\ & - (\mu_2 \kappa_h, w_h)_{\Gamma_{d2,h}} + (\mu_1 \nabla \phi_h \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d2,h}} \\ & + (A_1 \nabla^2 \eta_h, \nabla^2 v_h)_{\Gamma_{b,h}} - (A_2 \nabla \eta_h, \nabla v_h)_{\Gamma_{b,h}} + (A_3 \eta_h, v_h)_{\Gamma_{b,h}} \\ & - (A_4 \nabla \phi_h, \nabla v_h)_{\Gamma_{b,h}} + (A_5 \phi_h, v_h)_{\Gamma_{b,h}} + (i\omega \eta_h, w_h)_{\Gamma_{b,h}} \\ & + (k_{s\rho} \eta_h \cdot \mathbf{n}, v_h)_{\Gamma_{BC,h}} + (k_{r\rho} \nabla \eta_h \cdot \mathbf{n}, \nabla v_h)_{\Gamma_{BC,h}} \\ & + (k_{s\rho} \langle \eta_h \rangle, \langle v_h \rangle)_{\Lambda_{j,h}} + (k_{r\rho} \llbracket \nabla \eta_h \cdot \mathbf{n} \rrbracket, \llbracket \nabla v_h \cdot \mathbf{n} \rrbracket)_{\Lambda_{j,h}} \end{aligned}$$

$$L_h([w_h, u_h, v_h]) = (u_{in}, w_h)_{\Gamma_{in,h}} - (\mu_2 \kappa_{in}, w_h)_{\Gamma_{d1,h}} + (\mu_1 \nabla \phi_{in} \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d1,h}} \quad (3.17b)$$

The variational space of $\mathcal{V}_{\Gamma_{b,h}}$ for the structural element in the weak formulation above is defined in $H^2(\Gamma_b)$ with C^1 continuity. However, the variational space for the fluid domain $\mathcal{V}_h$ and free surface $\mathcal{V}_{fs,h}$ can be defined in lower order space of $H^1(\Omega/\Gamma_{fs})$. In this research, an alternative approach from Colomés, Verdugo, and Akkerman (2023) is used, employing a continuous/discontinuous Galerkin (C/DG) approach for a fourth-order operator. This approach generalises the FE space for the structure using the same order as the fluid, which is defined in $H^1(\Gamma_b)$ space by using Lagrange polynomials.

The continuity of the derivative is enforced by using the interior penalty approach. This penalty method is applied to the higher-order terms at the structural facet $\Lambda_{b,h}$. This facet term is obtained after reducing the order by integration by parts, as in Eq. (3.9). This condition is applicable because $\Lambda_{b,h} \subset \Gamma_{b,h}$. To weakly enforce the derivative continuity, the following conditions are applied:

$$\llbracket v_h \cdot \mathbf{n} \rrbracket = 0 \quad \text{on } \Lambda_{b,h} \quad (\text{Test function continuity}) \quad (3.18a)$$

$$\llbracket -D_\rho \nabla^2 \eta_h \cdot \mathbf{n} \rrbracket = 0 \quad \text{on } \Lambda_{b,h} \quad (\text{Bending moment continuity}) \quad (3.18b)$$

$$\llbracket \nabla \eta_h \cdot \mathbf{n} \rrbracket = 0 \quad \text{on } \Lambda_{b,h} \quad (\text{Derivative continuity}) \quad (3.18c)$$

$$\llbracket \nabla \phi_h \cdot \mathbf{n} \rrbracket = 0 \quad \text{on } \Lambda_{b,h} \quad (\text{Derivative continuity}) \quad (3.18d)$$

$$\llbracket \nabla(D_\rho \nabla^2 \eta_h) \cdot \mathbf{n} + (A_2 \nabla \eta_h) \cdot \mathbf{n} + (A_4 \nabla \phi_h) \cdot \mathbf{n} \rrbracket = 0 \quad \text{on } \Lambda_{b,h} \quad (\text{Shear continuity}) \quad (3.18e)$$

Use the same approach as when deriving the joint term (Eq. 3.12), employing the jump property to

weakly enforce the continuity on the facet and enforce the facet conditions in Eq. (3.18).

$$\begin{aligned}
& (\llbracket \nabla(A_1 \nabla^2 \eta_h) \cdot \mathbf{n} + (A_2 \nabla \eta_h) \cdot \mathbf{n} + (A_4 \nabla \phi_h) \cdot \mathbf{n}, v_h \rrbracket)_{\Lambda_{b,h}} - (\llbracket A_1 \nabla^2 \eta_h \cdot \mathbf{n}, \nabla v_h \rrbracket)_{\Lambda_{b,h}} \\
& = (\llbracket \nabla(A_1 \nabla^2 \eta_h) \cdot \mathbf{n} + (A_2 \nabla \eta_h) \cdot \mathbf{n} + (A_4 \nabla \phi_h) \cdot \mathbf{n} \rrbracket, \langle v_h \rangle)_{\Lambda_{b,h}} \\
& + (\langle \nabla(A_1 \nabla^2 \eta_h) + (A_2 \nabla \eta_h) + (A_4 \nabla \phi_h) \rangle, \llbracket v_h \cdot \mathbf{n} \rrbracket)_{\Lambda_{b,h}} \\
& - (\llbracket A_1 \nabla^2 \eta_h \cdot \mathbf{n} \rrbracket, \langle \nabla v_h \rangle)_{\Lambda_{b,h}} - (\langle A_1 \nabla^2 \eta_h \rangle, \llbracket \nabla v_h \cdot \mathbf{n} \rrbracket)_{\Lambda_{b,h}} \\
& = - (\langle A_1 \nabla^2 \eta_h \rangle, \llbracket \nabla v_h \cdot \mathbf{n} \rrbracket)_{\Lambda_{b,h}}
\end{aligned} \tag{3.19}$$

Then apply the penalty method to the facet term by adding symmetric and penalisation terms, as follows:

$$\begin{aligned}
(-\langle A_1 \nabla^2 \eta_h \rangle, \llbracket \nabla v_h \cdot \mathbf{n} \rrbracket)_{\Lambda_{b,h}} & := (-\langle A_1 \nabla^2 \eta_h \rangle, \llbracket \nabla v_h \cdot \mathbf{n} \rrbracket)_{\Lambda_{b,h}} + (-\langle A_1 \nabla^2 v_h \rangle, \llbracket \nabla \eta_h \cdot \mathbf{n} \rrbracket)_{\Lambda_{b,h}} \\
& - \frac{\gamma(-A_1)}{h} (\llbracket \nabla v_h \cdot \mathbf{n} \rrbracket, \llbracket \nabla \eta_h \cdot \mathbf{n} \rrbracket)_{\Lambda_{b,h}}
\end{aligned} \tag{3.20}$$

The first term in Eq. (3.20) above is the consistency term, from the derivation after weakly enforcing the continuity. The second term is the symmetric term, and the third term is the penalised term, where γ is a constant to ensure stability, and h is the characteristic element size. The stability of this method has been proven in the study by Colomés, Verdugo, and Akkerman (2023).

3.5. Unified Monolithic FEM Weak Formulation

After deriving the discontinuous term for the structure, the final weak formulation of the Unified Monolithic FEM can be obtained by incorporating Eq. (3.20) into Eq. (3.17a). Therefore, the weak formulation reads:

Find $[\phi_h, \kappa_h, \eta_h] \in \mathcal{V}_h \times \mathcal{V}_{\Gamma_{fs},h} \times \mathcal{V}_{\Gamma_b,h}$ such that:

$$B_h([\phi_h, \kappa_h, \eta_h], [w_h, u_h, v_h]) = L_h([w_h, u_h, v_h]) \quad \forall [w_h, u_h, v_h] \in \mathcal{V}_h \times \mathcal{V}_{\Gamma_{fs},h} \times \mathcal{V}_{\Gamma_b,h} \quad (3.21)$$

With,

$$B_h([\phi_h, \kappa_h, \eta_h], [w_h, u_h, v_h]) = (\nabla \phi_h, \nabla w_h)_{\Omega_h} \quad (3.22a)$$

$$\begin{aligned} &+ (i\omega \kappa_h, w_h)_{\Gamma_{fs,h}} + \beta_h(-i\omega \phi_h + g \kappa_h, \alpha_f w_h + u_h)_{\Gamma_{fs,h}} \\ &+ (i\omega \kappa_h, w_h)_{\Gamma_{d1,h}} + \beta_h(-i\omega \phi_h + g \kappa_h, \alpha_f w_h + u_h)_{\Gamma_{d1,h}} \\ &+ (i\omega \kappa_h, w_h)_{\Gamma_{d2,h}} + \beta_h(-i\omega \phi_h + g \kappa_h, \alpha_f w_h + u_h)_{\Gamma_{d2,h}} \\ &- (\mu_2 \kappa_h, w_h)_{\Gamma_{d1,h}} + (\mu_1 \nabla \phi_h \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d1,h}} \\ &- (\mu_2 \kappa_h, w_h)_{\Gamma_{d2,h}} + (\mu_1 \nabla \phi_h \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d2,h}} \\ &+ (A_1 \nabla^2 \eta_h, \nabla^2 v_h)_{\Gamma_{b,h}} + (A_2 \nabla \eta_h, \nabla v_h)_{\Gamma_{b,h}} + (A_3 \eta_h, v_h)_{\Gamma_{b,h}} \\ &+ (A_4 \nabla \phi_h, \nabla v_h)_{\Gamma_{b,h}} + (A_5 \phi_h, v_h)_{\Gamma_{b,h}} + (i\omega \eta_h, w_h)_{\Gamma_{b,h}} \\ &+ (-\langle A_1 \nabla^2 \eta_h \rangle, [\nabla v_h \cdot \mathbf{n}])_{\Lambda_{b,h}} + (-\langle A_1 \nabla^2 v_h \rangle, [\nabla \eta_h \cdot \mathbf{n}])_{\Lambda_{b,h}} \\ &+ \frac{\gamma A_1}{h} ([\nabla v_h \cdot \mathbf{n}], [\nabla \eta_h \cdot \mathbf{n}])_{\Lambda_{b,h}} \\ &+ (k_{s\rho} \eta_h \cdot \mathbf{n}, v_h)_{\Gamma_{BC,h}} + (k_{r\rho} \nabla \eta_h \cdot \mathbf{n}, \nabla v_h)_{\Gamma_{BC,h}} \\ &+ (k_{s\rho} \langle \eta_h \rangle, \langle v_h \rangle)_{\Lambda_{j,h}} + (k_{r\rho} [\nabla \eta_h \cdot \mathbf{n}], [\nabla v_h \cdot \mathbf{n}])_{\Lambda_{j,h}} \end{aligned}$$

$$L_h([w_h, u_h, v_h]) = (u_{in}, w_h)_{\Gamma_{in,h}} - (\mu_2 \kappa_{in}, w_h)_{\Gamma_{d1,h}} + (\mu_1 \nabla \phi_{in} \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d1,h}} \quad (3.22b)$$

Where,

$$\begin{aligned} A_1 &= D_\rho, \\ A_2 &= -\left(\frac{\rho}{\rho_w} J \omega^2 + \frac{D_\rho}{C} \rho h \omega^2 - \frac{D_\rho}{C} \rho_w g - T_\rho\right), \\ A_3 &= \left(\frac{\rho^2 J h}{\rho_w C} \omega^4 - \frac{\rho h}{\rho_w} \omega^2 - \frac{\rho g J}{C} \omega^2 + g\right), \\ A_4 &= -\left(i \frac{D_\rho}{C} \rho_w \omega\right) \text{ and} \\ A_5 &= \left(i \frac{\rho J}{C} \omega^3 - i\omega\right). \end{aligned}$$

Since this formulation is aimed at being applicable to various hydroelastic problems, specific conditions and procedures must be defined as follows:

1. Present formulation:

This formulation uses the condition of the Mindlin model with internal joint connections in vertical and rotational directions. At the edge boundary, linear springs in vertical and rotational directions are available. The pretensioned term is provided at the A_2 coefficient.

2. Euler-Bernoulli (EB) recovery:

To recover the EB beam, neglect the rotational inertia J by setting the value to zero and increase the shear modulus to infinity $C \rightarrow \infty$ to prevent shear deformation.

3. Membrane recovery:

To recover membrane structure, use the same procedure as the EB beam and set the bending rigidity D_ρ to zero. The stiffness of the membrane will come from the pretensioned term T_ρ .

4. **For free-floating structure:**

The moment and shear at the edges are zero. This is equivalent to setting the mooring stiffness value in translational and rotational directions to zero.

5. **For joint without vertical spring:**

This means that the continuity of the vertical shear force must be enforced. The jump property of the shear force in the interface condition Eq. (2.20e) shall be set to zero. In this formulation, it can be done equivalently either by removing the vertical spring term at the joint or by setting the vertical spring stiffness at the joint to zero.

6. **For joint without rotational spring:**

If the rotational spring connection is not present, and the connection is not a hinged connection where $k_{rp} = 0$, this means that the bending moment must be continuous at the joint. In this case, the bending term in Eq. (2.20d) must be preserved. Looking back at Eq. (3.12), the last term $-(\langle A_1 \nabla^2 \eta \rangle, [\nabla v \cdot \mathbf{n}])_{\Lambda_j}$ must be remained in the formulation and now becomes the facet term of the structure since $\Lambda_j \subset \Gamma_b$. Therefore, this term must also be penalised for the discontinuous formulation, using the exact same method as in Eq. (3.20). Thus, if this case applies, replace the following joint rotational spring term with the joint penalised term.

$$(k_{rp} [\nabla \eta_h \cdot \mathbf{n}], [\nabla v_h \cdot \mathbf{n}])_{\Lambda_{j,h}} \rightarrow (-\langle A_1 \nabla^2 \eta_h \rangle, [\nabla v_h \cdot \mathbf{n}])_{\Lambda_{j,h}} + (-\langle A_1 \nabla^2 v_h \rangle, [\nabla \eta_h \cdot \mathbf{n}])_{\Lambda_{j,h}} + \frac{\gamma A_1}{h} ([\nabla v_h \cdot \mathbf{n}], [\nabla \eta_h \cdot \mathbf{n}])_{\Lambda_{j,h}}$$

7. **For structure without joint:**

This can simply be enforced by removing all joint terms. The continuity of the bending moment will be enforced through the structure facet terms $\Lambda_{b,h}$.

3.6. Convergence Analysis

The convergence analysis is performed by assessing the L^2 norm of the structural deflection error. The error rate on mesh refinement is compared with the optimal error rate $\mathcal{O}(r+1)$ to evaluate the efficiency of the numerical problem in Eq. (3.22). The analysis uses a free-floating structure with a uniform rectangular mesh in the x and y directions, keeping a fixed aspect ratio of 1.250. The error is calculated using Eq. (3.23), where η_{ref} is the deflection solution for the referenced case and η_i is the solution for the evaluated case.

$$\text{Error} = \|\eta_i - \eta_{ref}\|_{L^2(\Gamma_b)} := \sqrt{\int_{\Gamma_b} |\eta_i - \eta_{ref}|^2 d\Gamma_b} \quad (3.23)$$

The reference uses the finest discretisation, using $r_{ref} = 4$ with the number of elements in the x and y directions of $[n_x, n_y] = [256, 64]$ that correspond to the mesh size $h_{ref} = 0.023$ m. The evaluated cases use $r_i = [2, 3, 4]$ with $[n_x] = [15, 24, 32, 48, 64, 128]$ and $[n_y] = [4, 6, 8, 12, 16, 32]$, that correspond with mesh size $h_i = [0.625, 0.417, 0.313, 0.208, 0.156, 0.078]$ m. The problem is set in $H = 2.0$ m water depth, with a wave frequency of $\omega = 8.976$ rad/sec and associated wavelength $\lambda_w = 0.765$ m. The damping length is applied near the inlet-outlet boundary, where the length is equal to $2L_b$. Between the damping zone and the structure, free surface zones are set on both sides of the structure for $L_{fs} = L_b$. The structure length is $L_b = 6$ m, which is 3 times the water depth, and the thickness is $h_p = 0.038$ m. The structural density is $\rho = 220$ kg/m³, and Young's modulus is 103 MPa. These parameters are selected to effectively capture the structure's deflection without introducing a large added-mass effect. The result in Fig. (3.2) shows that the L^2 -norm error rate converges at the optimal rate of $\mathcal{O}(r+1)$.

3.7. Modal Analysis Formulation

This section defines the matrices used in the modal analysis, as shown in Eq. (2.29) for dry modal analysis and in Eq. (2.30) for wet modal analysis. Recalling the monolithic matrix form in Eq. (2.24):

$$\begin{bmatrix} \omega^4 \mathbf{S}_{11} - \omega^2 \mathbf{M}_{11} + \mathbf{K}_{11}^d + \mathbf{K}_{11}^w & -i\omega \mathbf{C}_{12} & 0 \\ i\omega \mathbf{C}_{21} & \mathbf{K}_{22} & \mathbf{C}_{23} \\ 0 & \mathbf{C}_{32} & \mathbf{K}_{33} \end{bmatrix} \begin{bmatrix} \eta_p \\ \phi_p \\ \kappa_p \end{bmatrix} = \begin{bmatrix} \mathbf{F}_\eta \\ \mathbf{F}_\phi \\ \mathbf{F}_\kappa \end{bmatrix}$$

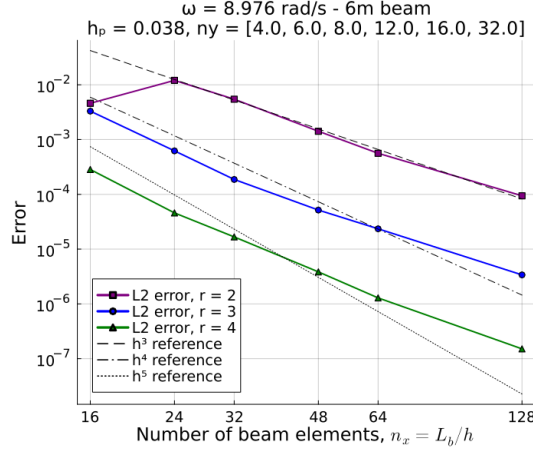


Figure 3.2: The evolution of the L2-norm of the beam deflection error for a 6 m beam with varying number of elements and element order

Where $\eta_p = \{\eta_{P,i}\}$, $\phi_p = \{\phi_{P,i}\}$, and $\kappa_p = \{\kappa_{P,i}\}$ are the degrees of freedom of the discretized system, Eq. (3.22). The estimation of all the unknowns is given as:

$$\eta_h = \sum_i v_{h,i} \eta_{p,i}, \quad \phi_h = \sum_j w_{h,j} \phi_{p,j}, \quad \text{and}, \quad \kappa_h = \sum_k u_{h,k} \kappa_{p,k}, \quad (3.24)$$

From the matrix system above, each matrix term depends on the specific degree of freedom and the associated test function. The terms can be expressed as:

$$\begin{aligned} \mathbf{S}_{11}(\eta_h, v_h) & \quad \mathbf{M}_{11}(\eta_h, v_h) & \quad \mathbf{K}_{11}^d(\eta_h, v_h) & \quad \mathbf{K}_{11}^w(\eta_h, v_h) & \quad \mathbf{C}_{12}^d(\phi_h, v_h) \\ \mathbf{C}_{21}(\eta_h, w_h) & \quad \mathbf{K}_{22}(\phi_h, w_h) & \quad \mathbf{C}_{23}(\kappa_h, w_h) & \quad \mathbf{C}_{32}(\phi_h, u_h) & \quad \mathbf{K}_{33}(\kappa_h, u_h) \end{aligned} \quad (3.25)$$

Collecting the terms in the weak formulation Eq. (3.22) according to the pair of unknown and test function, and breaking down the constants A_1 to A_5 according to the ω order, the matrix for the modal analysis can be defined by using the notation in Eq. (3.24) as follow:

$$\begin{aligned} \mathbf{K}_{11}^w &= (g v_h, v_h)_{\Gamma_b} & \mathbf{K}_{22} &= (\nabla w_h, \nabla w_h)_{\Omega} - (i k w_h, w_h)_{\Gamma_{in}, \Gamma_{out}} & \mathbf{K}_{33} &= (g u_h, u_h)_{\Gamma_{fs}} \\ \mathbf{C}_{21} &= (v_h, w_h)_{\Gamma_b} & \mathbf{C}_{23} &= (i \omega u_h, w_h)_{\Gamma_{fs}} & \mathbf{C}_{32} &= (-i \omega w_h, u_h)_{\Gamma_{fs}} \end{aligned} \quad (3.26)$$

$$\begin{aligned} \mathbf{K}_{11}^d &= (D_\rho \nabla^2 v_h, \nabla^2 v_h)_{\Gamma_b} + \left(\left[\frac{D_\rho}{C} \rho_w g + T_\rho \right]_{A_2} \nabla v_h, \nabla v_h \right)_{\Gamma_b} \\ &\quad - (\langle \mathbf{D}_\rho \nabla^2 v_h \rangle, [\nabla v_h \cdot \mathbf{n}_\Lambda])_{\Lambda_b} - ([\nabla v_h \cdot \mathbf{n}_\Lambda], \langle \mathbf{D}_\rho \nabla^2 v_h \rangle)_{\Lambda_b} + \frac{\gamma D_\rho}{h} ([\nabla v_h \cdot \mathbf{n}_\Lambda], [\nabla v_h \cdot \mathbf{n}_\Lambda])_{\Lambda_b} \\ &\quad + (k_{s\rho} v_h, v_h)_{\Gamma_{BC}} + (k_{r\rho} \nabla v_h, \nabla v_h)_{\Gamma_{BC}} \\ &\quad + (k_{s\rho} \langle v_h \rangle, \langle v_h \rangle)_{\Lambda_j} + (k_{r\rho} [\nabla v_h \cdot \mathbf{n}], [\nabla v_h \cdot \mathbf{n}])_{\Lambda_j} \\ \mathbf{M}_{11} &= \left(\left[\frac{\rho}{\rho_w} J + \frac{D_\rho}{C} \rho h \right]_{A_2} \nabla(v_h), \nabla(v_h) \right)_{\Gamma_b} + \left(\left[\frac{\rho h}{\rho_w} + \frac{\rho g J}{C} \right]_{A_3} v_h, v_h \right)_{\Gamma_b} \\ \mathbf{S}_{11} &= \left(\left[\frac{\rho^2 J h}{\rho_w C} \right]_{A_3} v_h, v_h \right)_{\Gamma_b} \\ \mathbf{C}_{12} &= (w_h, v_h)_{\Gamma_b} + \left(\left[\frac{D_\rho}{C} \rho_w \right]_{A_4} \nabla w_h, \nabla v_h \right)_{\Gamma_b} + \left(\left[-\frac{\rho J}{C} \omega^2 \right]_{A_5} w_h, v_h \right)_{\Gamma_b} \end{aligned} \quad (3.27)$$

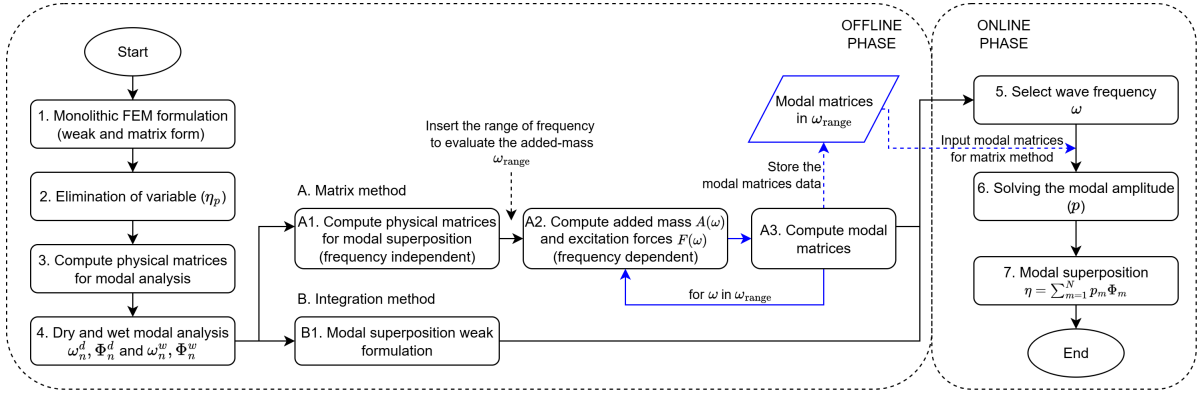


Figure 3.3: Modal superposition procedure

As explained in Section 2.4, modal analysis does not require wave generation. Therefore, the wave damping term is not included in the matrix definition. One thing worth mentioning in this definition, using the Mindlin theory, is that the coupling term between the structure and the fluid, C_{12} , includes an ω^2 term, which arises from the $i\omega^3$ term in the A_5 constant. This is used in the added-mass calculation, as derived in Eq. (2.28). Using this matrix definition, the modal analysis for dry and wet modes can be performed by following the iterative method in Fig. (2.2)

3.8. Modal Superposition Formulation

From the previous step, the natural frequencies (ω_n^d and ω_n^w) and mode shapes (Φ_n^d and Φ_n^w) for both the dry and wet modes are obtained. Moreover, to obtain the structural deflection solution of the reduced problem, the modal superposition method is performed. There are two approaches that can be used for the monolithic FEM framework. The first approach is to assemble the matrices and convert them into the modal domain, as presented in Section 2.5. This approach reduces the system size to only $N \times N$, where N is the number of modes used in the superposition. Generally, it can be taken as around 10–15. For subsequent use, this method will be referred to as the Matrix method. The second approach is to implement the modal superposition solution, as in Eq. (2.34), into the monolithic FEM weak formulation, as in Eq. (3.22). The fluid terms in this approach will remain in the physical domain. Consequently, by substituting Eq. (2.34) and its respective test function into the weak form, it will expand the bilinear form to solve the modal amplitude p_m for N modes, instead of directly solving the deflection η_h . This is equivalent to the Matrix method in the step of Eq. (2.38). After the modal amplitudes of p_m are solved, the deflection solution can be solved by using Eq. (2.34). For subsequent use, this method will be referred to as the Integration method, since the modal superposition solution is applied to the structure integration boundary $d\Gamma_b$.

The procedure of this modal superposition using both methods is provided in Fig. (3.3). It is generally divided into two phases, the offline and online phases. For the matrix method, the modal matrix assembly, including the calculation of frequency-dependent added mass, added damping, and forces, is performed in the offline phase. The online phase starts with selecting the wave frequency, calculating the modal amplitude, and calculating the modal superposition. For the integration method, the offline phase is for formulating the weak formulation and is then followed by the online phase after selecting the evaluated wave frequency. The added mass and added damping effects are directly applied within the weak formulation.

Matrix method:

Assembling the physical matrices for the modal superposition (step A1) in the offline phase uses the same mechanism as in the modal analysis. However, in this step, wave generation is crucial for accurately capturing all terms in the problem. Therefore, the wave damping terms must be included. The differences are only in the fluid and the free surface matrix of K_{22} , C_{23} , C_{32} , and K_{33} , while the matrices for the structure terms remain. In addition, the force vector also needs to be defined, and the total force is obtained from the right-hand side (RHS) of Eq. (2.28). The definitions of the new matrices with

the wave damping inclusion and the force vector are as follows:

$$\begin{aligned}
\mathbf{K}_{22} &= (\nabla w_h, \nabla w_h)_\Omega + (\mu_1 \nabla_n w_h, \alpha_f(\omega) w_h)_{\Gamma_{d1, d2}} + (-i\omega \beta_h w_h, \alpha_f(\omega) w_h)_{\Gamma_{fs, \Gamma_{d1, d2}}} \\
\mathbf{C}_{23} &= (i\omega u_h, w_h)_{\Gamma_{fs, \Gamma_{d1, d2}}} + (\beta_h g \kappa_h, \alpha_f(\omega) w_h)_{\Gamma_{fs, \Gamma_{d1, d2}}} + (-\mu_2(k) u_h, w_h)_{\Gamma_{d1, d2}} \\
\mathbf{C}_{32} &= (\mu_1 \nabla_n w_h, u_h)_{\Gamma_{d1, d2}} + (-i\omega \beta_h w_h, u_h)_{\Gamma_{fs, \Gamma_{d1, d2}}} \\
\mathbf{K}_{33} &= (\beta_h g u_h, u_h)_{\Gamma_{fs, \Gamma_{d1, d2}}} \\
\mathbf{F}_\eta &= (0, v_h)_{\Gamma_b} \\
\mathbf{F}_\phi &= (u_{in}, w_h)_{\Gamma_{in}} - (\mu_2 \kappa_{in}, w_h)_{\Gamma_{d1}} + (\mu_1 \nabla_n \phi_{in}, \alpha_f(\omega) w_h)_{\Gamma_{d1}} \\
&= (u_{in}, w_h)_{\Gamma_{in}} - (\mu_2 \kappa_{in}, w_h)_{\Gamma_{d1}} + (-i\omega \mu_1 \kappa_{in}, \alpha_f(\omega) w_h)_{\Gamma_{d1}} \\
\mathbf{F}_\kappa &= (\mu_1 \nabla_n \phi_{in}, u_h)_{\Gamma_{d1}} = (-i\omega \mu_1 \kappa_{in}, u_h)_{\Gamma_{d1}} \\
\mathbf{F}_{tot} &= \mathbf{F}_\eta + i\omega \mathbf{C}_{12} \mathbf{M}_\phi^{-1} \mathbf{F}_\phi - i\omega \mathbf{C}_{12} \mathbf{M}_\phi^{-1} \mathbf{C}_{23} \mathbf{K}_{33}^{-1} \mathbf{F}_\kappa
\end{aligned} \tag{3.28}$$

Here, in the damping terms, the coefficient α_f depends on the frequency as in Eq. (3.6), and the coefficient μ_2 depends on the wave number k as in Eq. (2.8). For the detailed derivation of these matrices, see Appendix A. After constructing the physical matrices, the next step A2 is to convert them to the modal domain. The size of these modal matrices and the modal force vector is reduced significantly to only $N \times N$, with $N = 10 \sim 15$.

$$\begin{aligned}
[M_{mod}] &= [\Phi]^T [\mathbf{M}_{11}] [\Phi] & [A_{mod}] &= [\Phi]^T [\mathbf{A}(\omega)] [\Phi] \\
[K_{mod}] &= [\Phi]^T [\mathbf{K}_{11}^d + \mathbf{K}_{11}^w] [\Phi] & [S_{mod}] &= [\Phi]^T [\mathbf{S}_{11}] [\Phi] \\
\{F_{mod}\} &= [\Phi]^T \{F_{tot}\}
\end{aligned} \tag{3.29}$$

Adopting the solution in Eq. (2.38), the modal amplitude of this method can be obtained as follows:

$$\{p\} = [\omega^4 S_{mod} - \omega^2 M_{mod} - \omega^2 A_{mod} + K_{mod}]^{-1} \{F_{mod}\} \tag{3.30}$$

Finally, solve the deflection solution by using Eq. (2.34).

Integration method:

In this integration method, all fluid terms are preserved in the formulation, including the fluid-structural coupling terms. Therefore, the added-mass effect is directly accounted within the formulation. The modal superposition term, as the truncated solution of η_h , and its respective test function are defined in Eq. (3.31).

$$\eta_{mod}(p) := \sum_{m=1}^N p_m \cdot \Phi_m \approx \eta_h \tag{3.31a}$$

$$v_{mod}(q) := \sum_{n=1}^N q_n \cdot \Phi_n \approx v_h \tag{3.31b}$$

In this formulation, the spatial derivative of these quantities can be applied directly by deriving the mode shapes within the superposition. This is applicable due to the linearity of the superposition method. The derivatives are as follows:

$$\begin{aligned}
\nabla \eta_{mod}(p) &:= \nabla \eta_h \approx \sum_{m=1}^N p_m \cdot \nabla \Phi_m, & \Delta \eta_{mod}(p) &:= \nabla^2 \eta_h \approx \sum_{m=1}^N p_m \cdot \nabla^2 \Phi_m \\
\nabla v_{mod}(q) &:= \nabla v_h \approx \sum_{n=1}^N q_n \cdot \nabla \Phi_n, & \Delta v_{mod}(q) &:= \nabla^2 v_h \approx \sum_{n=1}^N q_n \cdot \nabla^2 \Phi_n
\end{aligned}$$

The modal amplitude p_m is a constant quantity. Before applying the term to the weak form, it must be defined in the FE space by using 'ConstantFESpace' in the Gridap library. Therefore, after defining the

constant quantity p_m in the FE Space and substituting the terms into the weak form in Eq. (3.22), the updated weak form for the modal superposition of the integration method reads:

Find $[\phi_h, \kappa_h, p...] \in \mathcal{V}_h \times \mathcal{V}_{\Gamma_{fs},h} \times \mathcal{V}_{\Gamma_b,h}$ such that:

$$B_h([\phi_h, \kappa_h, p...], [w_h, u_h, q...]) = L_h([w_h, u_h, q...]) \quad \forall [w_h, u_h, q...] \in \mathcal{V}_h \times \mathcal{V}_{\Gamma_{fs},h} \times \mathcal{V}_{\Gamma_b,h} \quad (3.32)$$

With,

$$B_h([\phi_h, \kappa_h, p...], [w_h, u_h, q...]) = (\nabla \phi_h, \nabla w_h)_{\Omega_h} \quad (3.33a)$$

$$\begin{aligned} & + (i\omega \kappa_h, w_h)_{\Gamma_{fs},h} + \beta_h(-i\omega \phi_h + g\kappa_h, \alpha_f w_h + u_h)_{\Gamma_{fs},h} \\ & + (i\omega \kappa_h, w_h)_{\Gamma_{d1},h} + \beta_h(-i\omega \phi_h + g\kappa_h, \alpha_f w_h + u_h)_{\Gamma_{d1},h} \\ & + (i\omega \kappa_h, w_h)_{\Gamma_{d2},h} + \beta_h(-i\omega \phi_h + g\kappa_h, \alpha_f w_h + u_h)_{\Gamma_{d2},h} \\ & - (\mu_2 \kappa_h, w_h)_{\Gamma_{d1},h} + (\mu_1 \nabla \phi_h \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d1},h} \\ & - (\mu_2 \kappa_h, w_h)_{\Gamma_{d2},h} + (\mu_1 \nabla \phi_h \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d2},h} \\ & + (A_1 \Delta \eta_{\text{mod}}(p), \Delta v_{\text{mod}}(q))_{\Gamma_{b,h}} + (A_2 \nabla \eta_{\text{mod}}(p), \nabla v_{\text{mod}}(q))_{\Gamma_{b,h}} + (A_3 \eta_{\text{mod}}(p), v_{\text{mod}}(q))_{\Gamma_{b,h}} \\ & + (A_4 \nabla \phi_h, \nabla v_{\text{mod}}(q))_{\Gamma_{b,h}} + (A_5 \phi_h, v_{\text{mod}}(q))_{\Gamma_{b,h}} + (i\omega \eta_{\text{mod}}(p), w_h)_{\Gamma_{b,h}} \\ & + (-\langle A_1 \Delta \eta_{\text{mod}}(p) \rangle, \llbracket \nabla v_{\text{mod}}(q) \cdot \mathbf{n} \rrbracket)_{\Lambda_{b,h}} + (-\langle A_1 \Delta v_{\text{mod}}(q) \rangle, \llbracket \nabla \eta_{\text{mod}}(p) \cdot \mathbf{n} \rrbracket)_{\Lambda_{b,h}} \\ & + \frac{\gamma A_1}{h} (\llbracket \nabla v_{\text{mod}}(q) \cdot \mathbf{n} \rrbracket, \llbracket \nabla \eta_{\text{mod}}(p) \cdot \mathbf{n} \rrbracket)_{\Lambda_{b,h}} \\ & + (k_{s\rho} \eta_{\text{mod}}(p) \cdot \mathbf{n}, v_{\text{mod}}(q))_{\Gamma_{BC},h} + (k_{r\rho} \nabla \eta_{\text{mod}}(p) \cdot \mathbf{n}, \nabla v_{\text{mod}}(q))_{\Gamma_{BC},h} \\ & + (k_{s\rho} \langle \eta_{\text{mod}}(p) \rangle, \langle v_{\text{mod}}(q) \rangle)_{\Lambda_{j,h}} + (k_{r\rho} \llbracket \nabla \eta_{\text{mod}}(p) \cdot \mathbf{n} \rrbracket, \llbracket \nabla v_{\text{mod}}(q) \cdot \mathbf{n} \rrbracket)_{\Lambda_{j,h}} \end{aligned}$$

$$L_h([w_h, u_h, q...]) = (u_{\text{in}}, w_h)_{\Gamma_{\text{in},h}} - (\mu_2 \kappa_{\text{in}}, w_h)_{\Gamma_{d1},h} + (\mu_1 \nabla \phi_{\text{in}} \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d1},h} \quad (3.33b)$$

This FE formulation yields solutions for the fluid ϕ_h , free surface κ_h , and modal amplitude p . For the final step, to solve the structural deflection, substitute the modal amplitude p solution and the modal shapes Φ into Eq. (2.34).

4

Problem Setup

This chapter describes the detailed setup for each case used to evaluate the effectiveness of the ROM method using the Unified Monolithic Formulation. There are five cases used in this research. Those are free-end floating structures at high frequency, free-end floating structures at low frequency, floating structures with mooring attachment, multi-body floating structures with flexible joint connections, and floating membranes.

4.1. Case 1 - Free-End Floating Beam in High Frequency

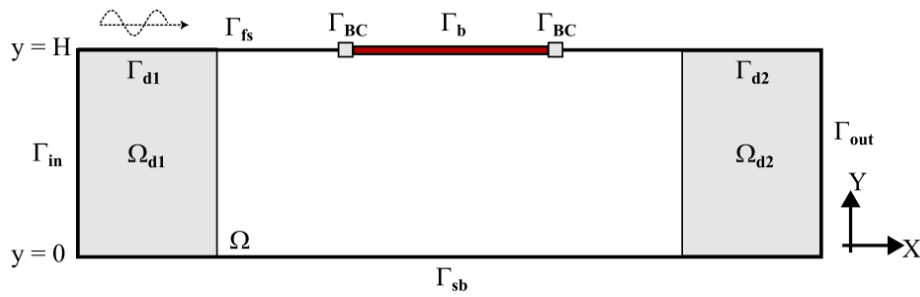


Figure 4.1: Schematic view of free-end floating structure

Case 1 uses the Mindlin beam model with the free-end boundary without mooring attachments. It uses a continuous beam without an internal joint. The schematic view of this problem is presented in Fig. (4.1).

Table 4.1: Case 1 parameters

Parameter	Symbol	Value	Units
Water depth	H	1.1	m
Wave frequency	ω	8.976	rad/sec
Damping zone length	L_d	$2\lambda_w$	-
Structure length	L_b	10	m
Structure thickness	h_p	0.038	m
Structure density	ρ	220	kg/m ³
Young's modulus	E	103	MPa
Mesh size	h	0.2	m
Element order	r	4	-
Translational mooring stiffness	k_s	0.0	N/m
Rotational mooring stiffness	k_r	0.0	N·m

The case will be compared with the existing literature, including Zhao, C. Hu, et al. (2008), which uses the Wiener-Hopf method, and A. E. Karperaki and K. A. Belibassakis (2021), which uses FEM coupling with the diffraction-radiation problem. The parameters used in this analysis are given in Table 4.1. This structure does not use a joint connection. Therefore, in the weak formulation, this term is taken out. The mooring at the end boundaries is all set to zero. The weak form of this problem is as follows:

$$\begin{aligned}
B_h([\phi_h, \kappa_h, \eta_h], [w_h, u_h, v_h]) = & (\nabla \phi_h, \nabla w_h)_{\Omega_h} \\
& + (i\omega \kappa_h, w_h)_{\Gamma_{fs,h}} + \beta_h(-i\omega \phi_h + g\kappa_h, \alpha_f w_h + u_h)_{\Gamma_{fs,h}} \\
& + (i\omega \kappa_h, w_h)_{\Gamma_{d1,h}} + \beta_h(-i\omega \phi_h + g\kappa_h, \alpha_f w_h + u_h)_{\Gamma_{d1,h}} \\
& + (i\omega \kappa_h, w_h)_{\Gamma_{d2,h}} + \beta_h(-i\omega \phi_h + g\kappa_h, \alpha_f w_h + u_h)_{\Gamma_{d2,h}} \\
& - (\mu_2 \kappa_h, w_h)_{\Gamma_{d1,h}} + (\mu_1 \nabla \phi_h \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d1,h}} \\
& - (\mu_2 \kappa_h, w_h)_{\Gamma_{d2,h}} + (\mu_1 \nabla \phi_h \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d2,h}} \\
& + (A_1 \nabla^2 \eta_h, \nabla^2 v_h)_{\Gamma_{b,h}} + (A_2 \nabla \eta_h, \nabla v_h)_{\Gamma_{b,h}} + (A_3 \eta_h, v_h)_{\Gamma_{b,h}} \\
& + (A_4 \nabla \phi_h, \nabla v_h)_{\Gamma_{b,h}} + (A_5 \phi_h, v_h)_{\Gamma_{b,h}} + (i\omega \eta_h, w_h)_{\Gamma_{b,h}} \\
& + (-\langle A_1 \nabla^2 \eta_h \rangle, [\nabla v_h \cdot \mathbf{n}])_{\Lambda_{b,h}} + (-\langle A_1 \nabla^2 v_h \rangle, [\nabla \eta_h \cdot \mathbf{n}])_{\Lambda_{b,h}} \\
& + \frac{\gamma A_1}{h} ([\nabla v_h \cdot \mathbf{n}], [\nabla \eta_h \cdot \mathbf{n}])_{\Lambda_{b,h}} \\
& + (k_{s\rho} \eta_h \cdot \mathbf{n}, v_h)_{\Gamma_{BC,h}} + (k_{r\rho} \nabla \eta_h \cdot \mathbf{n}, \nabla v_h)_{\Gamma_{BC,h}} \\
\\
L_h([w_h, u_h, v_h]) = & (u_{in}, w_h)_{\Gamma_{in,h}} - (\mu_2 \kappa_{in}, w_h)_{\Gamma_{d1,h}} + (\mu_1 \nabla \phi_{in} \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d1,h}}
\end{aligned}$$

From this formulation, the matrices and the weak formulation for the modal superposition method can be directly derived by using the method in Section 3.8.

4.2. Case 2 - Free-End Floating Beam in Low Frequency

Case 2 uses the same setting as in Case 1 with the same weak formulation, but at a lower frequency of $\omega = 2.185$ rad/sec. The lower frequency requires a longer damping zone length. In this case, it is set to $L_d = 4\lambda_w$. This case will also be compared with the results in Zhao, C. Hu, et al. (2008) and A. E. Karperaki and K. A. Belibassakis (2021).

Table 4.2: Case 2 parameters

Parameter	Symbol	Value	Units
Water depth	H	1.1	m
Wave frequency	ω	2.185	rad/sec
Damping zone length	L_d	$4\lambda_w$	-
Structure length	L_b	10	m
Structure thickness	h_p	0.038	m
Structure density	ρ	220	kg/m ³
Young's modulus	E	103	MPa
Mesh size	h	0.2	m
Element order	r	4	-
Translational mooring stiffness	k_s	0.0	N/m
Rotational mooring stiffness	k_r	0.0	N·m

4.3. Case 3 - Structural with Mooring Attachment

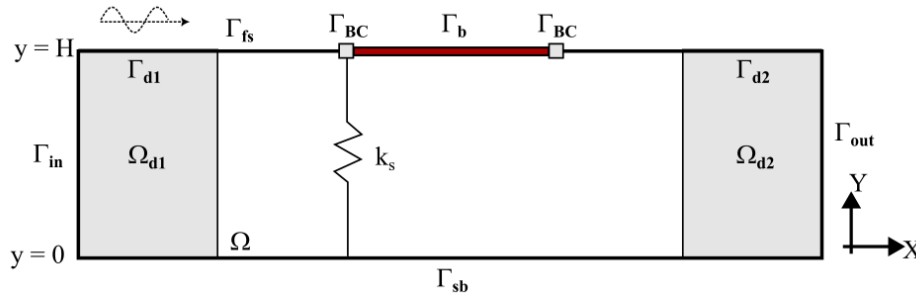


Figure 4.2: Schematic view of moored floating structure

Case 3 uses the Euler-Bernoulli beam model that is recovered from the Unified formulation. The structure is connected with mooring attachments at the left boundary, where the wave first contacts the structure. It uses a continuous beam without an internal joint. The schematic view of this problem is presented in Fig. (4.2). The case will be compared with the existing literature, including Nguyen et al. (2018), which uses the FEM-BEM coupling, and A. Karperaki, K. Belibassakis, and Papathanasiou (2016), which uses FEM coupling with the shallow water approximation. The parameters used in this analysis are given in Table 4.3.

The joint term in the weak formulation is neglected, and only mooring at the left boundary is considered in the weak formulation. The boundary integral at the right boundary will be omitted to prevent double counting. To recover the EB beam, the rotational inertia J is set to zero and the shear modulus is increased to infinity, $C \rightarrow \infty \sim 10^{32}$, to prevent shear deformation.

The weak form of this problem is as follows:

$$\begin{aligned}
B_h([\phi_h, \kappa_h, \eta_h], [w_h, u_h, v_h]) = & (\nabla \phi_h, \nabla w_h)_{\Omega_h} \\
& + (i\omega \kappa_h, w_h)_{\Gamma_{fs,h}} + \beta_h(-i\omega \phi_h + g\kappa_h, \alpha_f w_h + u_h)_{\Gamma_{fs,h}} \\
& + (i\omega \kappa_h, w_h)_{\Gamma_{d1,h}} + \beta_h(-i\omega \phi_h + g\kappa_h, \alpha_f w_h + u_h)_{\Gamma_{d1,h}} \\
& + (i\omega \kappa_h, w_h)_{\Gamma_{d2,h}} + \beta_h(-i\omega \phi_h + g\kappa_h, \alpha_f w_h + u_h)_{\Gamma_{d2,h}} \\
& - (\mu_2 \kappa_h, w_h)_{\Gamma_{d1,h}} + (\mu_1 \nabla \phi_h \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d1,h}} \\
& - (\mu_2 \kappa_h, w_h)_{\Gamma_{d2,h}} + (\mu_1 \nabla \phi_h \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d2,h}} \\
& + (A_1 \nabla^2 \eta_h, \nabla^2 v_h)_{\Gamma_{b,h}} + (A_2 \nabla \eta_h, \nabla v_h)_{\Gamma_{b,h}} + (A_3 \eta_h, v_h)_{\Gamma_{b,h}} \\
& + (A_4 \nabla \phi_h, \nabla v_h)_{\Gamma_{b,h}} + (A_5 \phi_h, v_h)_{\Gamma_{b,h}} + (i\omega \eta_h, w_h)_{\Gamma_{b,h}} \\
& + (-\langle A_1 \nabla^2 \eta_h \rangle, [\nabla v_h \cdot \mathbf{n}])_{\Lambda_{b,h}} + (-\langle A_1 \nabla^2 v_h \rangle, [\nabla \eta_h \cdot \mathbf{n}])_{\Lambda_{b,h}} \\
& + \frac{\gamma A_1}{h} ([\nabla v_h \cdot \mathbf{n}], [\nabla \eta_h \cdot \mathbf{n}])_{\Lambda_{b,h}} \\
& + (k_{sp} \eta_h \cdot \mathbf{n}, v_h)_{\Gamma_{1,h}} + (k_{rp} \nabla \eta_h \cdot \mathbf{n}, \nabla v_h)_{\Gamma_{1,h}}
\end{aligned}$$

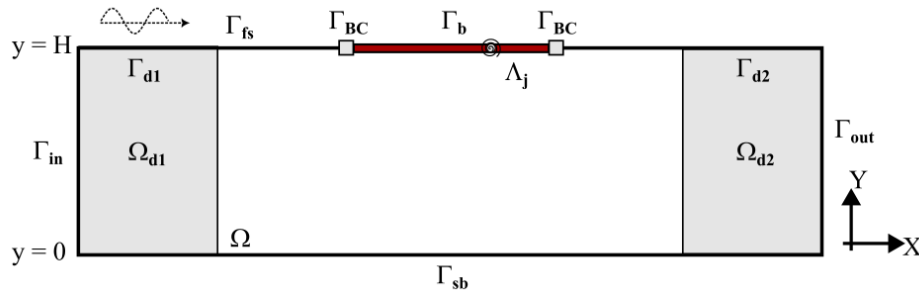
$$L_h([w_h, u_h, v_h]) = (u_{\text{in}}, w_h)_{\Gamma_{\text{in},h}} - (\mu_2 \kappa_{\text{in}}, w_h)_{\Gamma_{\text{d1},h}} + (\mu_1 \nabla \phi_{\text{in}} \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{\text{d1},h}}$$

From this formulation, the matrices and the weak formulation for the modal superposition can be directly derived by using the method in Section 3.8.

Table 4.3: Case 3 parameters

Parameter	Symbol	Value	Units
Water depth	H	0.25	m
Wave frequency	ω	2.144	rad/sec
Damping zone length	L_d	$4\lambda_w$	-
Structure length	L_b	10	m
Structure thickness	h_p	0.038	m
Structure density	ρ	220	kg/m ³
Young's modulus	E	103	MPa
Mesh size	h	0.2	m
Element order	r	4	-
Translational mooring stiffness	k_s	24500	N/m
Rotational mooring stiffness	k_r	0.0	N·m

4.4. Case 4 - Multi-Body Structure with Flexible Joint Connection

**Figure 4.3:** Schematic view of multi-body floating structure with flexible joint connection

Case 4 uses the multi-body Mindlin beam formulation, connected by a flexible joint in the rotational direction. The schematic view of this problem is presented in Fig. (4.3). The case will be compared with the existing literature, including Zhao, Hao, et al. (2015), which uses the Wiener-Hopf method, and T. I. Khabakhpasheva and A. A. Korobkin (2002), which uses the modal expansion method. The parameters used in this analysis are given in Table 4.4.

Table 4.4: Case 4 parameters

Parameter	Symbol	Value	Units
Water depth	H	1.1	m
Wave frequency	ω	4.397	rad/sec
Damping zone length	L_d	$2\lambda_w$	-
Structure length	L_b	10	m
Structure thickness	h_p	0.038	m
Structure density	ρ	220	kg/m ³
Young's modulus	E	103	MPa
Mesh size	h	0.1	m
Element order	r	4	-
Translational mooring stiffness	k_s	0.0	N/m
Rotational mooring stiffness	k_r	0.0	N·m

The joint connection is located at $0.65L_b$ from the left boundary. This case uses a hinge connection with $k_r = 0$ N·m. The mesh size is reduced to $h = 0.1$ m to accurately capture the joint location. Otherwise, the joint location can deviate.

The weak form of this problem is as follows:

$$\begin{aligned}
B_h([\phi_h, \kappa_h, \eta_h], [w_h, u_h, v_h]) = & (\nabla \phi_h, \nabla w_h)_{\Omega_h} \\
& + (i\omega \kappa_h, w_h)_{\Gamma_{fs,h}} + \beta_h(-i\omega \phi_h + g\kappa_h, \alpha_f w_h + u_h)_{\Gamma_{fs,h}} \\
& + (i\omega \kappa_h, w_h)_{\Gamma_{d1,h}} + \beta_h(-i\omega \phi_h + g\kappa_h, \alpha_f w_h + u_h)_{\Gamma_{d1,h}} \\
& + (i\omega \kappa_h, w_h)_{\Gamma_{d2,h}} + \beta_h(-i\omega \phi_h + g\kappa_h, \alpha_f w_h + u_h)_{\Gamma_{d2,h}} \\
& - (\mu_2 \kappa_h, w_h)_{\Gamma_{d1,h}} + (\mu_1 \nabla \phi_h \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d1,h}} \\
& - (\mu_2 \kappa_h, w_h)_{\Gamma_{d2,h}} + (\mu_1 \nabla \phi_h \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d2,h}} \\
& + (A_1 \nabla^2 \eta_h, \nabla^2 v_h)_{\Gamma_{b,h}} + (A_2 \nabla \eta_h, \nabla v_h)_{\Gamma_{b,h}} + (A_3 \eta_h, v_h)_{\Gamma_{b,h}} \\
& + (A_4 \nabla \phi_h, \nabla v_h)_{\Gamma_{b,h}} + (A_5 \phi_h, v_h)_{\Gamma_{b,h}} + (i\omega \eta_h, w_h)_{\Gamma_{b,h}} \\
& + (-\langle A_1 \nabla^2 \eta_h \rangle, [\nabla v_h \cdot \mathbf{n}])_{\Lambda_{b,h}} + (-\langle A_1 \nabla^2 v_h \rangle, [\nabla \eta_h \cdot \mathbf{n}])_{\Lambda_{b,h}} \\
& + \frac{\gamma A_1}{h} ([\nabla v_h \cdot \mathbf{n}], [\nabla \eta_h \cdot \mathbf{n}])_{\Lambda_{b,h}} \\
& + (k_{s\rho} \eta_h \cdot \mathbf{n}, v_h)_{\Gamma_{BC,h}} + (k_{r\rho} \nabla \eta_h \cdot \mathbf{n}, \nabla v_h)_{\Gamma_{BC,h}} \\
& + (k_{s\rho} \langle \eta_h \rangle, \langle v_h \rangle)_{\Lambda_{j,h}} + (k_{r\rho} [\nabla \eta_h \cdot \mathbf{n}], [\nabla v_h \cdot \mathbf{n}])_{\Lambda_{j,h}} \\
L_h([w_h, u_h, v_h]) = & (u_{in}, w_h)_{\Gamma_{in,h}} - (\mu_2 \kappa_{in}, w_h)_{\Gamma_{d1,h}} + (\mu_1 \nabla \phi_{in} \cdot \mathbf{n}, \alpha_f w_h + u_h)_{\Gamma_{d1,h}}
\end{aligned}$$

4.5. Case 5 - Floating Membrane

Case 5 uses a similar beam model as in Case 1. However, since it uses a membrane structure, the parameters in Case 1 are adjusted to recover the membrane model from the Unified formulation. The result will be compared with the Agarwal, Colomés, and Metrikine (2024) result, which uses the same monolithic formulation framework but solves a second-order membrane PDE. The parameter used in this case is taken from the Trivedi and Koley (2023) setup, which are given in Table 4.5.

Table 4.5: Case 5 parameters

Parameter	Symbol	Value	Units
Water depth	H	10	m
Wave frequency	ω	2.4	rad/sec
Damping zone length	L_d	$2\lambda_w$	-
Structure length	L_b	20	m
Structure thickness	h_p	0.001	m
Structure density	ρ	922500	kg/m ³
Young's modulus	E	0	MPa
Mesh size	h	0.4	m
Element order	r	4	-
Translational mooring stiffness	k_s	0.0	N/m
Rotational mooring stiffness	k_r	0.0	N·m

The membrane recovery uses the same method as the Euler-Bernoulli beam, with $J = 0$, $C \rightarrow \infty \sim 10^{32}$, and sets the rigidity to zero by using $E = 0$. To match the formulation in this research, the plate thickness is assumed to be very small; $h_p = 0.001$ m is selected. From this assumption, the structural density and pre-tensioned term can be calculated as follows:

$$m_\rho = 0.9 = \frac{\rho h_p}{\rho_w}, \quad \rho = \frac{0.9 \rho_w}{h_p} = 922500 \text{ kg/m}^3$$

$$T = 0.1 \rho_w g H^2 = 1.0055e5 \text{ N/m}$$

5

Result and Discussion

This chapter presents the analysis of all cases and discusses the results. The discussion will cover the natural frequencies and modal shapes, the modal-amplitude solution, the discrepancy between the full solution and the truncated ROM solution, and the computational time comparison. The cases here are performed for the first 15 modes to evaluate the accuracy of the truncated solution up to the higher order. The analysis is executed on a system with an Intel(R) Core(TM) i7-13620H processor and 64 GB of RAM. The code used in this research is available via the following GitHub repository **"Monolithic_FEM_Modal_Superposition"**

5.1. Case 1 - Free-End Floating Beam in High Frequency

5.1.1. Modal Analysis Result

Case 1 is the free-end floating beam with no boundary constraints, evaluated for a high-frequency wave with $\omega = 8.976$ rad/sec. See the detailed setup in Section 4.1. The first 6 dry and wet mode shapes of this structure are presented in Fig. (5.1) and (5.2). In the dry mode shapes, the rigid modes are observed in modes 1 and 2. In the wet mode, all mode shapes are observed to be flexible.

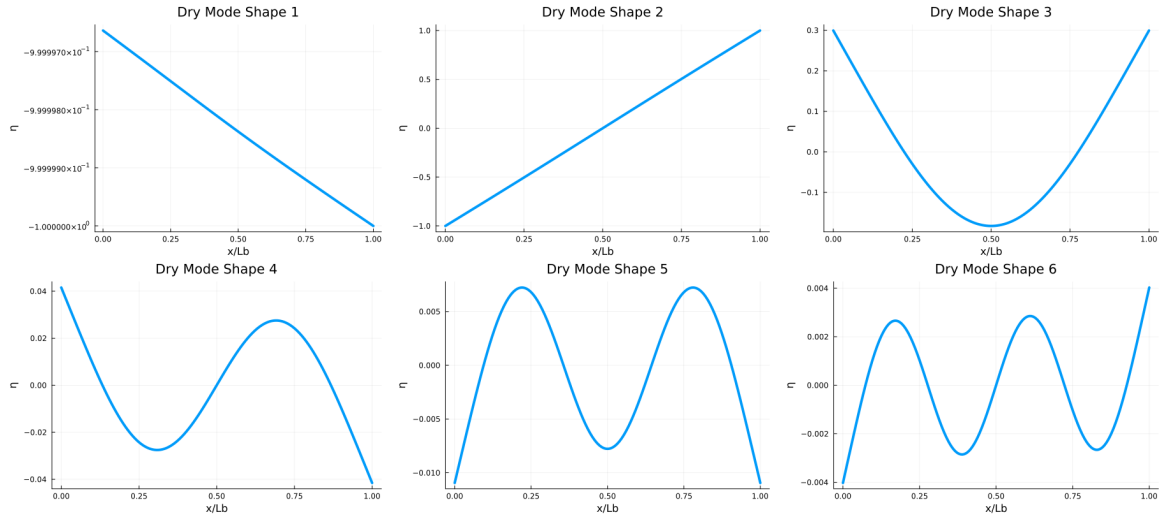
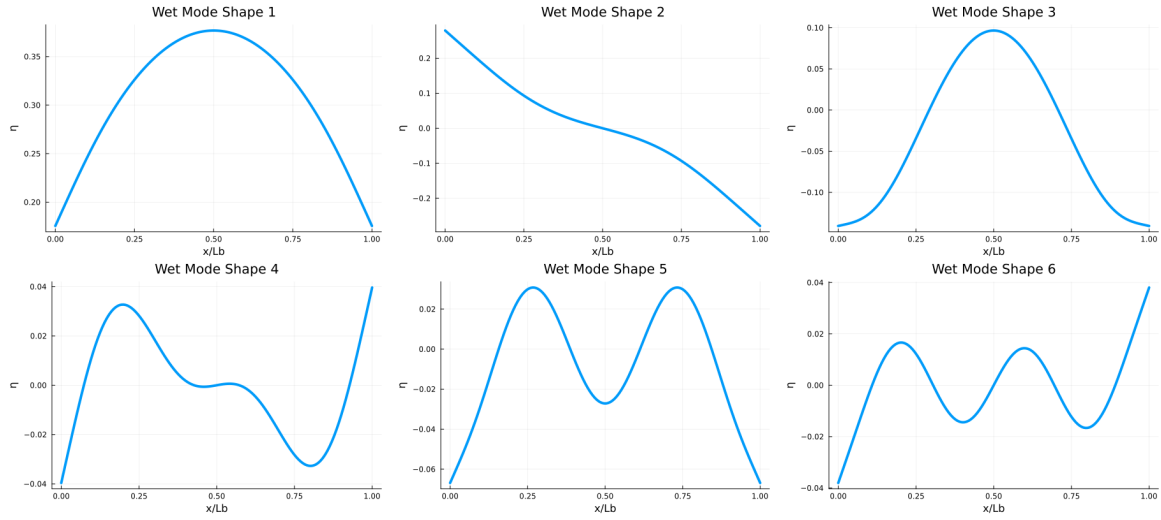


Figure 5.1: First 6th dry modal shapes

Figure 5.2: First 6th wet modal shapes

The dry and wet natural frequencies for case 1 are shown in Table (5.1). From the data, the wet natural frequency does not change significantly, unlike the dry natural frequency. This is due to the significant influence of added mass on the structure. The ratio between the real part of the dry and wet natural frequency tends to be around ~ 5 at the higher frequency.

Table 5.1: Case 1 dry and wet natural frequencies

Mode	ω_n^d	ω_n^w	Ratio $\Re(\omega_n^d/\omega_n^w)$
1	0.00033 - 0.0im	0.39190 - 0.31760im	0.00084
2	0.24216 - 0.0im	1.45993 - 0.50013im	0.16587
3	1.82767 - 0.0im	2.32988 - 0.51421im	0.78444
4	4.90586 - 0.0im	3.04429 - 0.51192im	1.61149
5	9.55625 + 0.0im	3.68968 - 0.51751im	2.58999
6	15.7577 + 0.0im	4.39951 - 0.50267im	3.58170
7	23.5077 - 0.0im	5.33325 - 0.46487im	4.40775
8	32.8027 - 0.0im	6.61360 - 0.42645im	4.95987
9	43.6393 - 0.0im	8.30004 - 0.38755im	5.25772
10	56.0137 - 0.0im	10.4194 - 0.34363im	5.37592
11	69.9217 - 0.0im	12.9894 - 0.29737im	5.38296
12	85.3587 - 0.0im	16.0194 - 0.25201im	5.32845
13	102.320 - 0.0im	19.5525 - 0.18209im	5.23307
14	120.799 - 0.0im	23.5215 - 4.2e-24im	5.13567
15	140.790 - 0.0im	28.1445 - 0.30390im	5.00240

5.1.2. Modal Superposition Result

Modal amplitude for each mode for the Matrix and Integration methods is presented in Fig. (5.3). The plots from both methods show similar results. The maximum amplitude solution occurs in mode 9 and is close to mode 10 for both dry and wet solutions. This mode is related to the natural frequency of $\omega_n^d = 43.6393$ and 56.0137 rad/sec and $\omega_n^w = 8.30004$ and 10.4194 rad/sec. Considering that the excitation wave in this case has a frequency of $\omega = 8.976$ rad/sec, the wet natural frequency shows a good estimation by showing that the maximum amplitude of the solution occurs on the mode that is closest to the excitation wave frequency. From this information, the likelihood of resonance can be predicted by using the wet natural frequency.

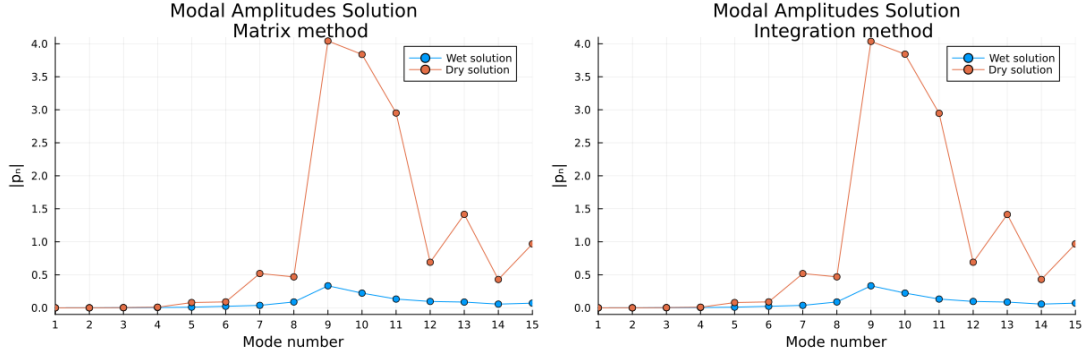


Figure 5.3: Modal Amplitude Solution

Using this modal amplitude solution and the mode shape, the modal superposition solution can be calculated. The truncated solution from this ROM method will be compared to the full solution by using root mean squared error (RMSE). The numerical results of the structural deflection for both truncated and full solutions exist on the same space function order. Thus, the error can be calculated directly by calculating the deflection deviation for each node. The RMSE is calculated as

$$RMSE = \sqrt{\left(\frac{\sum_{i=1}^N (\eta_{f,i} - \eta_{r,i})^2}{N} \right)} \quad (5.1)$$

Where $\eta_{f,i}$ is the full solution at node i and $\eta_{r,i}$ is the truncated ROM solution at node i . N is the total number of degrees of freedom (nodes) in the structure. The RMSE for this case is presented in Fig. (5.4). Both the Matrix and Integration methods have similar magnitudes. The figure shows that the dry solution yields a better result with a smaller error. It can be seen that the error has the order of $\mathcal{O}(10^{-3})$ at $N = 11$ for the dry solution and $N = 12$ for the wet mode solution.

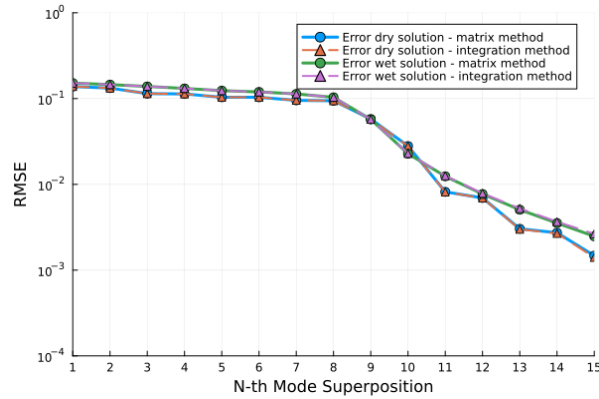
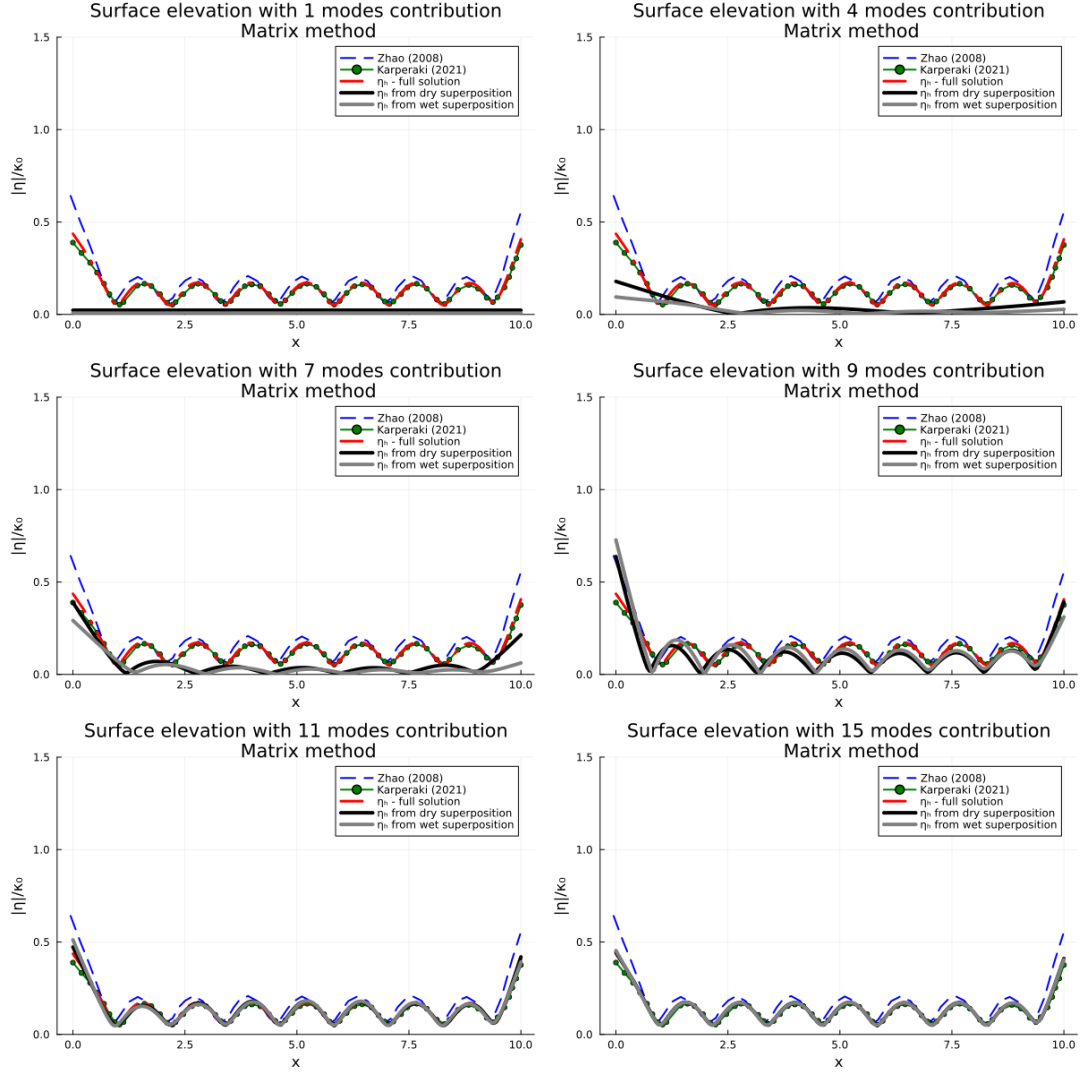


Figure 5.4: RMS Error for Case 1

Because the RMSEs are the same for the matrix and integration methods, the superposition solution of the two methods will be the same or differ only by an immeasurable amount. Therefore, only the modal superposition from the matrix solution will be plotted in Fig. (5.5). The complete plots are available in Appendix B. From the plots, both mode solutions provide good estimates of the full solution at higher-mode contributions, $N > 11$. Even though the solution between dry and wet modes is closed, the dry mode still gives better results than the wet mode. The RMSE for the dry mode at the higher mode is lower than that for the wet mode.

Figure 5.5: First 6th dry and wet modal superpositions

5.1.3. Computational Time Performance

The computational cost of this ROM method is compared with that of the full solution in this subsection. The computational time for the modal superposition only accounts for the online phase used to solve for the modal amplitude p and the structural deflection η . The summary of the computational time is presented in Table 5.2. The results show that modal superposition using the matrix approach performs better than the full solution and the integration method. It is around 67 – 100 times faster than the full solution and 1100 – 1600 times faster than the integration method.

Table 5.2: Case 1: Computation Time

Analysis Type	Running Time (sec)			
	Warm up	Dry	Wet	Full
Full FEM solution	352.66	~	~	1.99
Modal Analysis	155.08	909.88		~
Modal matrices construction	199.08	183.04	182.46	~
Modal superposition - Matrix	2.27	0.02	0.03	~
Modal superposition - Integration	3209.00	32.01	33.69	~

5.2. Case 2 - Free-End Floating Beam in Low Frequency

5.2.1. Modal Analysis Result

Case 2 uses the same geometrical problem as in Case 1, only differs in the wave frequency and, consequently, the damping zone length. The wave frequency $\omega = 2.185$ rad/sec is used in this analysis. See the detailed setup explanation in Section 4.2. The first 6 dry and wet mode shapes of this structure are presented in Fig. (5.6) and (5.7). In the dry mode shapes, the rigid modes are observed in modes 1 and 2. In the wet mode, the first mode shows a strange plot. However, since the value is very small, it can be interpreted as the rigid mode. The dry mode shapes are similar to those in case 1. However, there are discrepancies in the wet modes due to the different domain size of the damping zone when calculating the wet modal analysis.

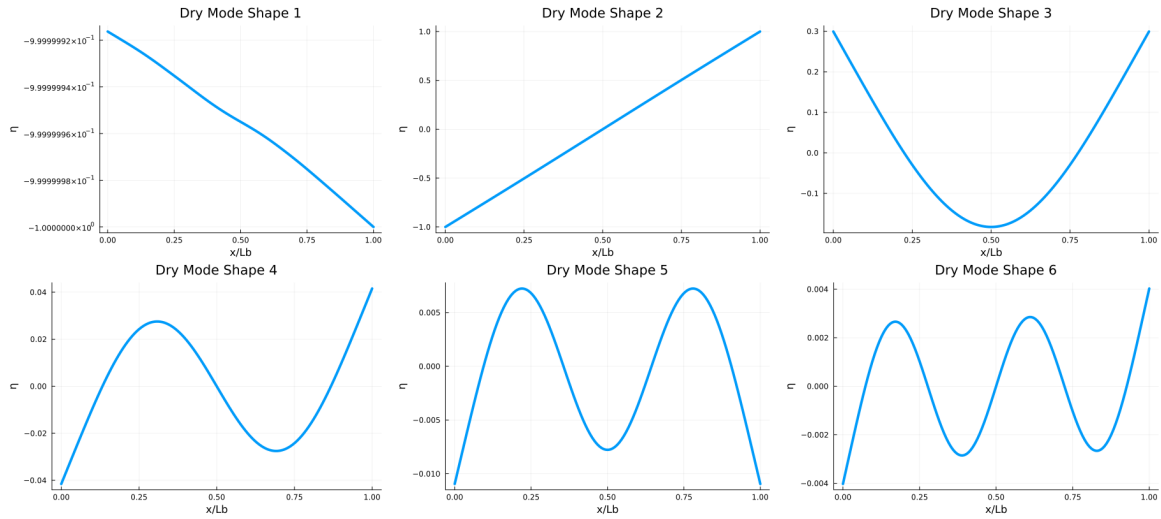


Figure 5.6: First 6th dry modal shapes

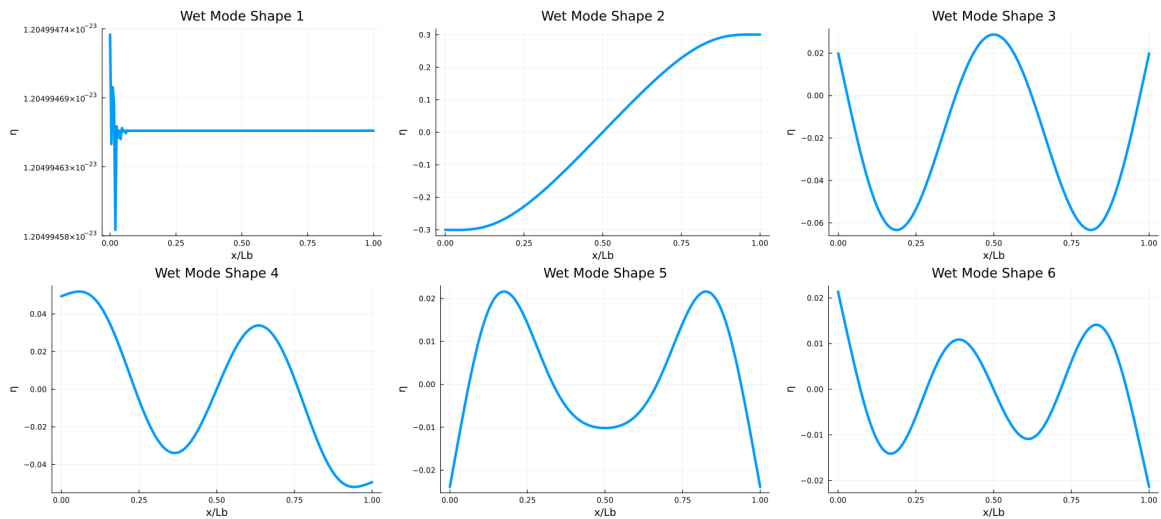


Figure 5.7: First 6th wet modal shapes

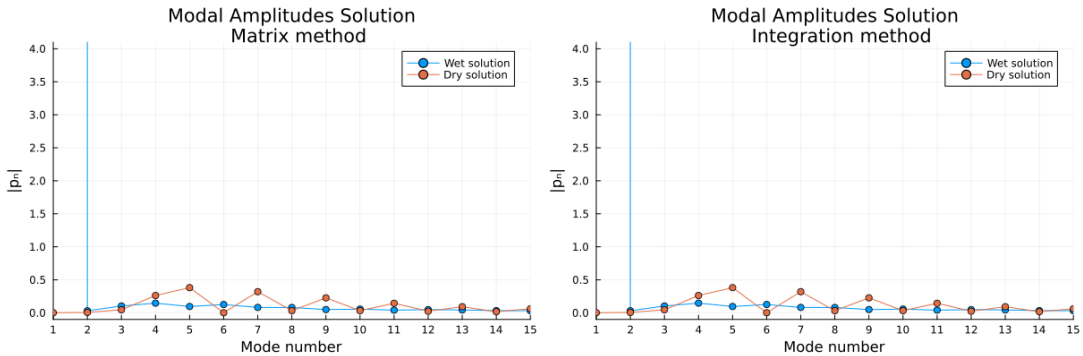
The dry and wet natural frequencies for case 2 are shown in Table (5.3). The results are similar to those in case 1 because it uses the same structural model and excludes external forces in the modal analysis. Small discrepancies are observed in the wet natural frequency for the same reason as in the modal shape differences, but are not significant for the physical interpretation. The ratio between the real part of the dry and wet natural frequency also tends to be around ~ 5 at the higher frequency.

Table 5.3: Case 2 dry and wet natural frequencies

Mode	ω_n^d	ω_n^w	Ratio $\Re(\omega_n^d/\omega_n^w)$
1	0.00000 + 0.0001im	0.00000 + 2.88e11im	NaN
2	0.24216 - 0.0im	1.45992 - 0.50013im	0.16587
3	1.82766 - 0.0im	2.32987 - 0.51421im	0.78444
4	4.90586 - 0.0im	3.04429 - 0.51192im	1.61149
5	9.55625 + 0.0im	3.68968 - 0.51751im	2.58999
6	15.7577 + 0.0im	4.39951 - 0.50267im	3.58170
7	23.5076 - 0.0im	5.33325 - 0.46487im	4.40775
8	32.8026 - 0.0im	6.61360 - 0.42645im	4.95987
9	43.6392 - 0.0im	8.30003 - 0.38755im	5.25772
10	56.0137 - 0.0im	10.4193 - 0.34363im	5.37592
11	69.9217 - 0.0im	12.9894 - 0.29879im	5.38294
12	85.3586 - 0.0im	16.0168 - 0.26036im	5.32931
13	102.319 - 0.0im	19.5502 - 0.19199im	5.23367
14	120.798 - 0.0im	23.5215 - 5.5e-55im	5.13567
15	140.790 - 0.0im	27.7615 - 0.28849im	5.07141

5.2.2. Modal Superposition Result

Modal amplitudes for each mode of the Matrix and Integration methods are presented in Fig. (5.8). The plots from both methods show similar results. A very large amplitude is observed at the contribution of rigid mode 1 for the wet modal superposition. This happens because the mode shapes for mode 1 are numerically very small, with the value tending to zero. This is a spurious result for the physical interpretation. The wave frequency in this case is $\omega = 2.185$ rad/sec. The peak of the wet modal amplitude does not match the wave frequency, indicating that no resonant response is observed in modes close to the wave frequency. This can be validated by observing the deflection plots in Fig. (5.10), where the ratio between the structure deflection and the wave amplitude is close to one with no amplified deflection. Therefore, the estimation result for this wet natural frequency cannot be the sole basis for interpretation of the resonance phenomenon.

**Figure 5.8:** Modal Amplitude Solution

Moreover, the RMSE and the modal superposition still need to be checked. The RMSE is presented in Fig. (5.9). The figure shows a decent result where the order of error $\mathcal{O}(10^{-3})$ is achieved at the lower modes. For the dry solution, it is reached at $N = 7$, and for the wet solution, it is at $N = 10$. This is consistent with the fact that at lower frequencies, convergence can be reached at lower modes because the deflection solution does not exhibit a fluctuating response, as in the higher-frequency case. Although the modal amplitude solution does not provide a good physical interpretation, the RMSE error provides better confirmation for this low-frequency case.

The modal superposition result of this case is given in Fig. (5.10). Consistent with the RMSE calculation, the dry superposition result has converged at $N = 7$, and beyond $N = 11$, the wet solutions have also

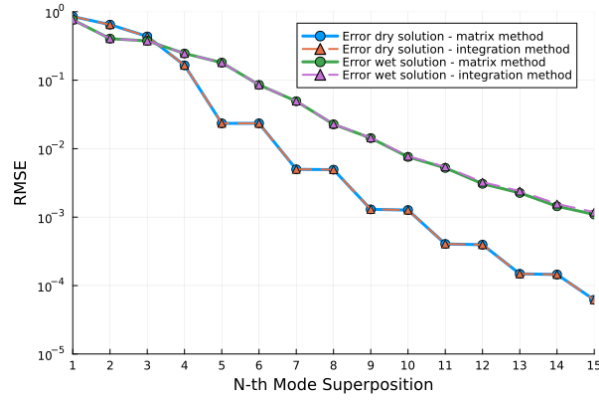
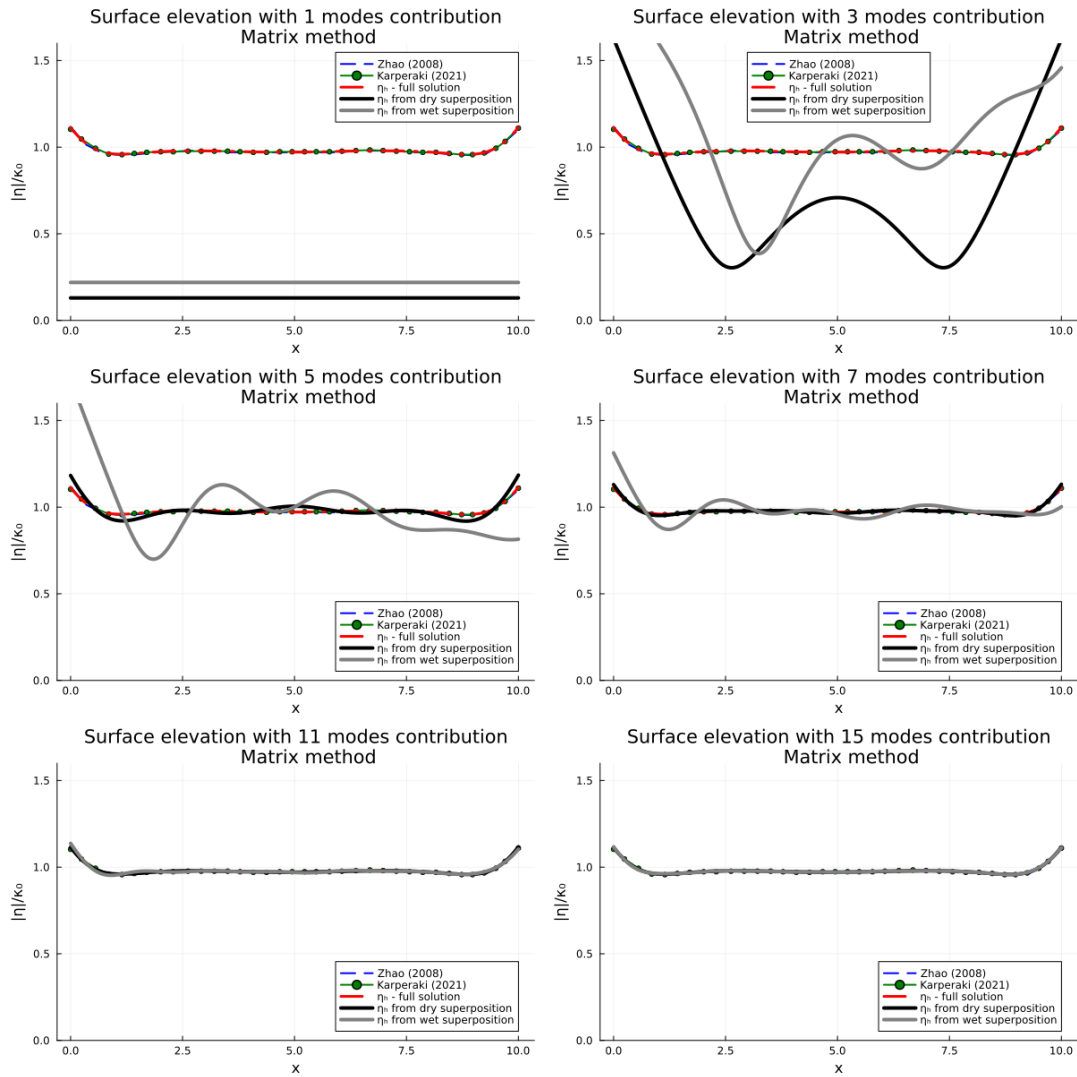


Figure 5.9: RMS Error for Case 2

converged. Despite the spurious result found in the modal analysis, the modal superposition results provide a good estimate and, physically, offer a good interpretation for the low-frequency case. The accuracy of the dry solution is better than that of the wet solution.

Figure 5.10: First 6th dry and wet modal superpositions

5.2.3. Computational Time Performance

The computational cost of this case is summarised in Table 5.4. The same pattern is shown in case 1, where the modal superposition using the matrix approach performs better than the full solution and the integration method. It is around 60 – 80 times faster than the full solution and 900 – 1200 times faster than the integration method.

Table 5.4: Case 2: Computation Time

Analysis Type	Running Time (sec)			
	Warm up	Dry	Wet	Full
Full FEM solution	348.55	~	~	2.49
Modal Analysis	200.55	5140.02	~	~
Modal matrices construction	125.96	1541.73	1560.56	~
Modal superposition - Matrix	1.10	0.04	0.03	~
Modal superposition - Integration	3165.78	36.11	34.51	~

5.3. Case 3 - Structural with Mooring Attachment

5.3.1. Modal Analysis Result

In case 3, a different model is used. The structure is now attached using mooring and uses the Euler-Bernoulli beam model. A low-frequency wave is also used in this case with $\omega = 2.144$ rad/sec. See the detailed setup explanation in Section 4.3. The first 6 dry and wet mode shapes of this structure are presented in Fig. (5.11) and (5.12). In the dry mode shapes, the rigid modes are observed in mode 1. In the wet mode, all mode shapes are flexible.

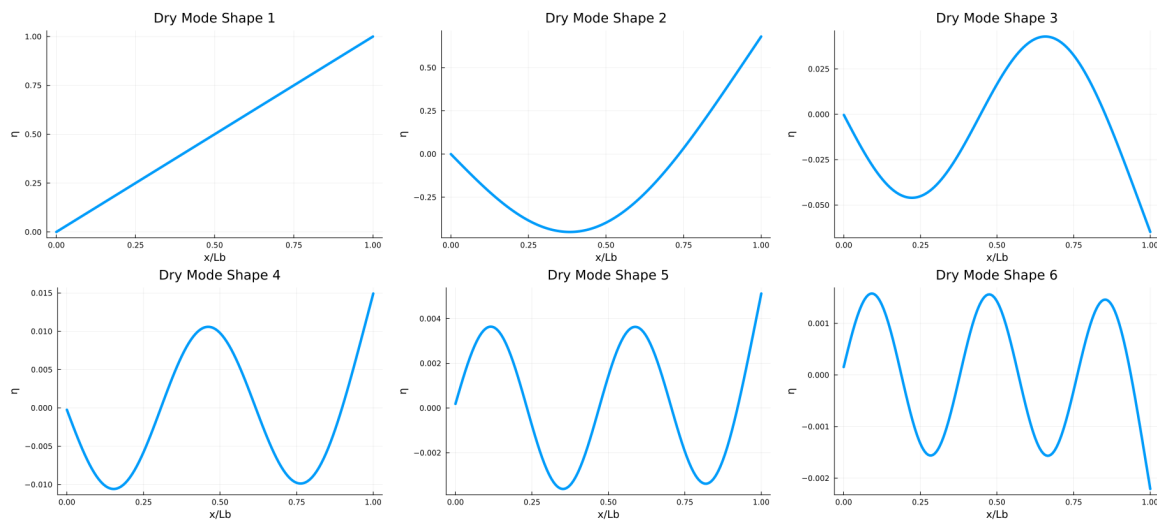


Figure 5.11: First 6th dry modal shapes

The dry and wet natural frequencies in this case are completely different across all modes compared to the previous cases, shown in Table (5.5). Even though it has the same structural length and thickness, adding a mooring line at one edge gives additional stiffness to the system, resulting in a change to the system's natural frequency. Interestingly, the ratio between the real part of the dry and wet natural frequency at the higher frequency has the same value as the previous model, around ~ 5 .

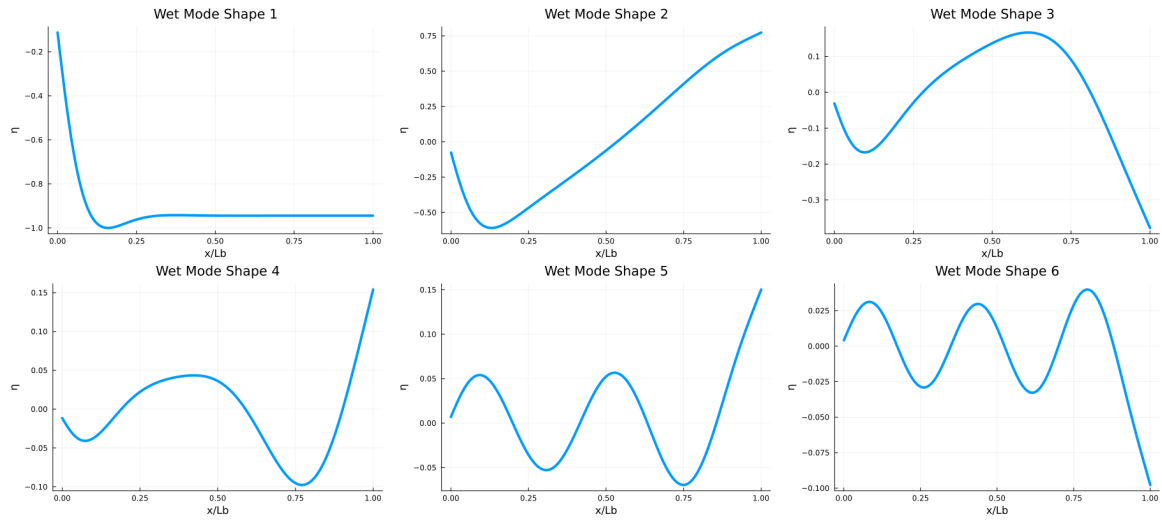
Figure 5.12: First 6th wet modal shapes

Table 5.5: Case 3 dry and wet natural frequencies

Mode	ω_n^d	ω_n^w	Ratio $\Re(\omega_n^d/\omega_n^w)$
1	0.00000 + 0.0002im	0.00000 + 5.03e-6im	NaN
2	1.21272 + 0.0im	0.73363 - 0.25390im	1.65303
3	3.92722 + 0.0im	1.24831 - 0.29164im	3.14600
4	8.18429 + 0.0im	1.77998 - 0.33099im	4.59794
5	13.9729 - 0.0im	2.37154 - 0.37954im	5.89192
6	21.2765 + 0.0im	3.11399 - 0.41212im	6.83254
7	30.0711 - 0.0im	4.13096 - 0.42516im	7.27945
8	40.3225 - 0.0im	5.50122 - 0.42540im	7.32973
9	51.9811 + 0.0im	7.26597 - 0.40556im	7.15404
10	64.9770 - 0.0im	9.44183 - 0.36221im	6.88182
11	79.2186 - 0.0im	12.0385 - 0.31171im	6.58039
12	94.6108 + 0.0im	15.0756 - 0.26967im	6.27573
13	111.112 - 0.0im	18.5801 - 0.20899im	5.98016
14	128.809 + 0.0im	22.5970 - 5.0e-27im	5.70030
15	147.911 - 0.0im	27.1094 - 0.24325im	5.45606

5.3.2. Modal Superposition Result

Modal amplitudes for each mode of the Matrix and Integration methods are presented in Fig. (5.13). The plots from both methods show similar results. The peak amplitudes of both methods are in the same mode, that is, $N = 5$. This mode is associated with natural frequencies of $\omega_n^d = 13.9729$ rad/sec and $\omega_n^w = 2.37154$ rad/sec. The wet natural frequency at $N = 5$ is close to the wave frequency. In this case, the wet natural frequency gives a good estimation of the structural natural frequency.

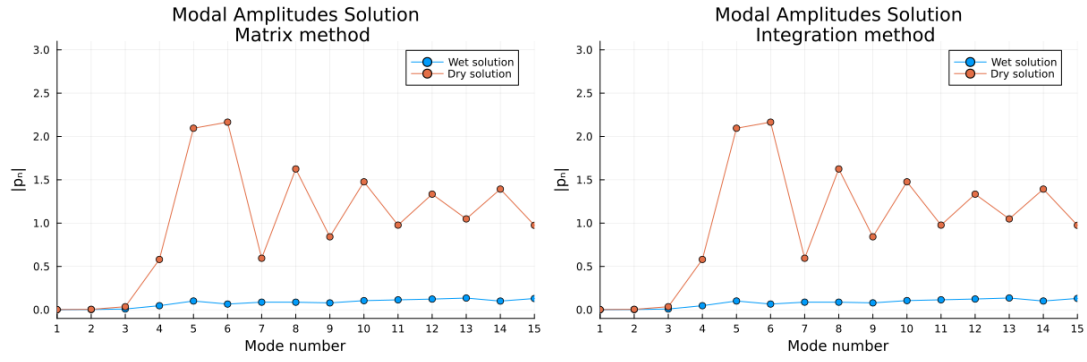


Figure 5.13: Modal Amplitude Solution

The RMSE in this case is presented in Fig. (5.14). The figure shows a decent trend, but the result differs from case 2, even though it operates at the same low frequency. In this case, the order of error $\mathcal{O}(10^{-3})$ is achieved at $N = 10$ for the dry solution and at $N = 11$ for the wet solution, whereas in the free-ends model it is reached at $N = 7$ for the dry solution. The gap error between the dry and wet is observed to be tight, indicating that both dry and wet solutions perform at a very close order of accuracy.

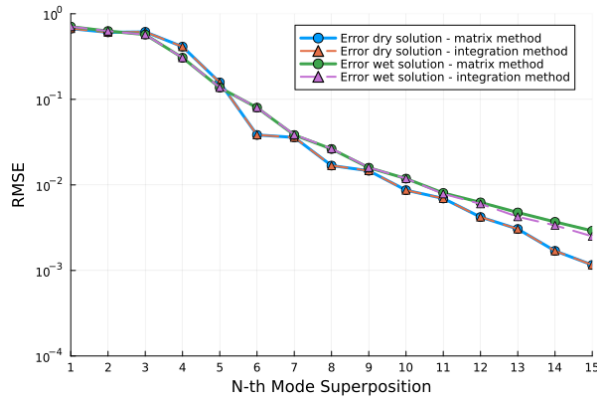
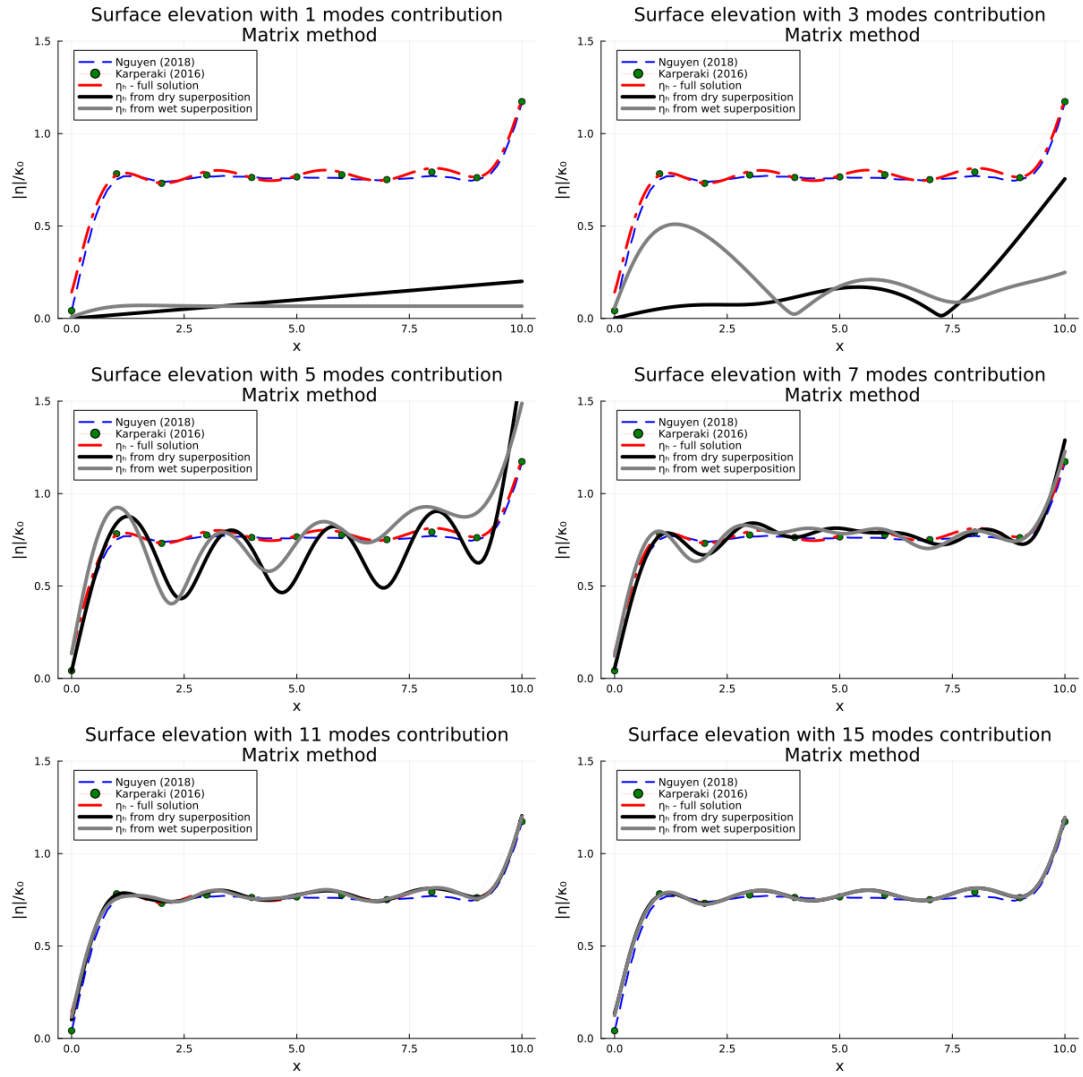


Figure 5.14: RMS Error for Case 3

The modal superposition result of this case is given in Fig. (5.15). Consistent with the RMSE calculation, the dry-and-wet superposition result yields a close solution for $N > 7$, although the dry modal superposition still gives better accuracy.

Figure 5.15: First 6th dry and wet modal superpositions

5.3.3. Computational Time Performance

The computational cost of case 3 is summarised in Table 5.6. The modal superposition using the matrix approach performs better than the full solution and the integration method. It is around 50 – 90 times faster than the full solution and 600 – 1000 times faster than the integration method.

Table 5.6: Case 3: Computation Time

Analysis Type	Running Time (sec)			
	Warm up	Dry	Wet	Full
Full FEM solution	349.84	~	~	2.88
Modal Analysis	188.70	1714.04		~
Modal matrices construction	123.61	466.95	457.21	~
Modal superposition - Matrix	1.15	0.03	0.05	~
Modal superposition - Integration	3103.24	30.67	30.21	~

5.4. Case 4 - Multi-Body Structure with Flexible Joint Connection

5.4.1. Modal Analysis Result

Case 4 uses the most complex geometry compared to the previous cases. This model has two flexible bodies connected by a flexible joint. This case uses an intermediate wave frequency of $\omega = 4.397$ rad/sec. See the detailed setup explanation in Section 4.4. The first 6 dry and wet mode shapes of this structure are presented in Fig. (5.16) and (5.17). In the dry mode shapes, the rigid modes are observed in modes 1 to 3. In wet mode, all mode shapes are flexible modes.

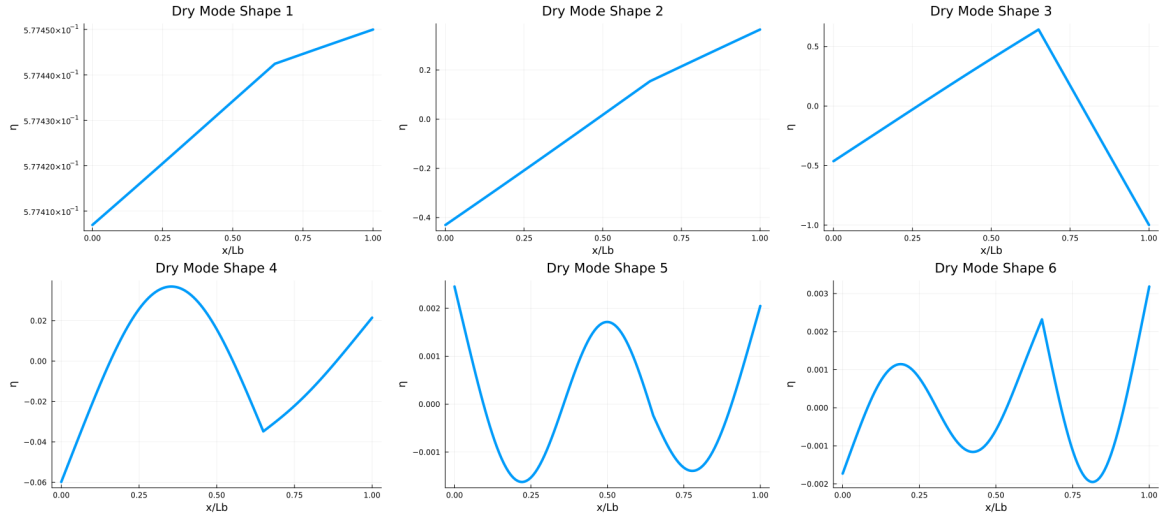


Figure 5.16: First 6th dry modal shapes

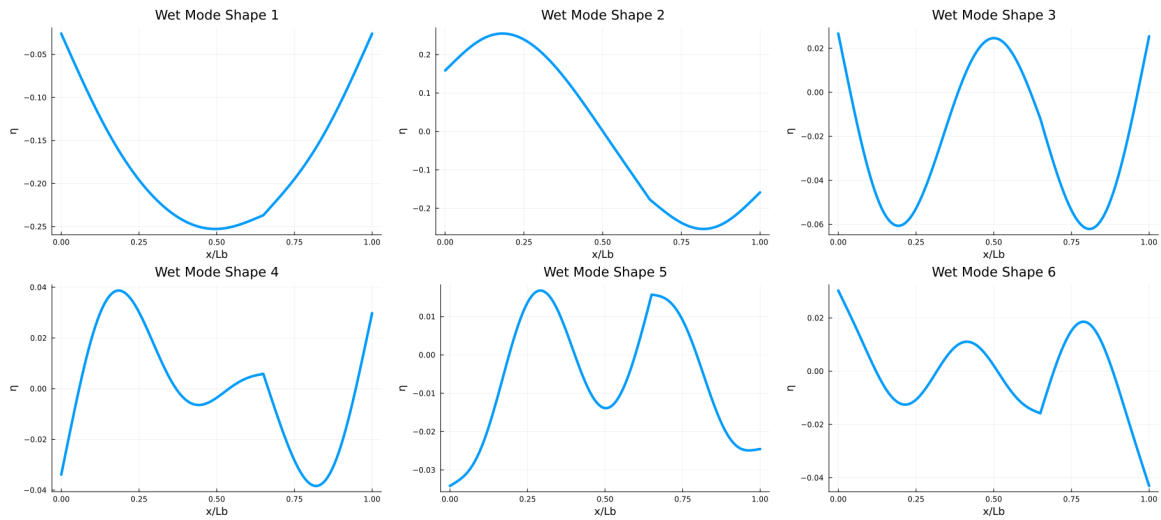


Figure 5.17: First 6th wet modal shapes

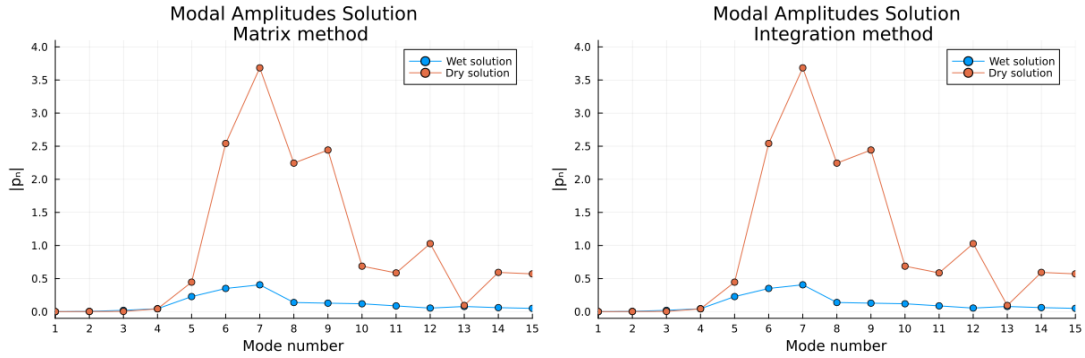
The dry and wet natural frequencies for case 4 are shown in Table (5.7). The ratio between the real parts of the dry and wet natural frequencies at the higher frequency in this case is consistent with the previous cases, that is, around ~ 5 .

Table 5.7: Case 4 dry and wet natural frequencies

Mode	ω_n^d	ω_n^w	Ratio $\Re(\omega_n^d/\omega_n^w)$
1	0.00050 + 6.01e-6im	0.39191 - 0.31761im	0.00128
2	0.23911 + 2.93e-8im	1.46016 - 0.50002im	0.16375
3	0.53874 + 1.41e-8im	2.32805 - 0.51405im	0.23141
4	3.60205 - 8.31e-9im	3.02293 - 0.49316im	1.19157
5	9.52778 - 4.66e-10im	3.68876 - 0.50111im	2.58291
6	12.9526 + 1.12e-9im	4.31424 - 0.52007im	3.00230
7	21.0791 - 4.59e-11im	5.09761 - 0.42119im	4.13510
8	32.7608 + 5.45e-10im	6.61633 - 0.41434im	4.95150
9	38.2004 + 2.34e-10im	7.77100 - 0.43956im	4.91576
10	52.9587 - 1.55e-11im	9.89663 - 0.30533im	5.35118
11	69.3598 + 3.46e-10im	12.9839 - 0.30880im	5.34197
12	77.4164 - 1.77e-11im	14.8458 - 0.29091im	5.21467
13	99.1883 + 2.40e-10im	18.8643 - 0.18991im	5.25798
14	118.818 - 2.21e-10im	23.4321 - 0.20357im	5.07073
15	130.820 - 3.65e-10im	26.3233 - 0.15334im	4.96976

5.4.2. Modal Superposition Result

Modal amplitudes for each mode of the Matrix and Integration methods are presented in Fig. (5.18). The plots from both methods show similar results with the peak amplitudes at $N = 7$. For the wet solution, the peak values for both modes $N = 6$ and 7 are very close. These modes are associated with natural frequencies of $\omega_n^w = 4.31$ and 5.09 rad/sec, which are close to the excitation wave frequency. Even though a significant modal amplitude is not observed, the peak amplitude on that mode can explain the resonance phenomenon. In this result, the wet natural frequency gives a better estimation than the dry modes.

**Figure 5.18:** Modal Amplitude Solution

The RMSE in this case is presented in Fig. (5.19). The figure shows a good decline rate as in the previous cases. In this case, the order of error $\mathcal{O}(10^{-3})$ is achieved at $N = 10$ for the dry solution and at $N = 11$ for the wet solution. The dry solution gives a better estimate than the wet modes, with a lower RMSE value. The modal superposition result of this case is given in Fig. (5.20).

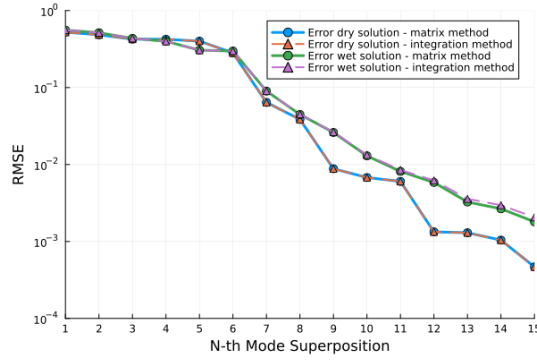
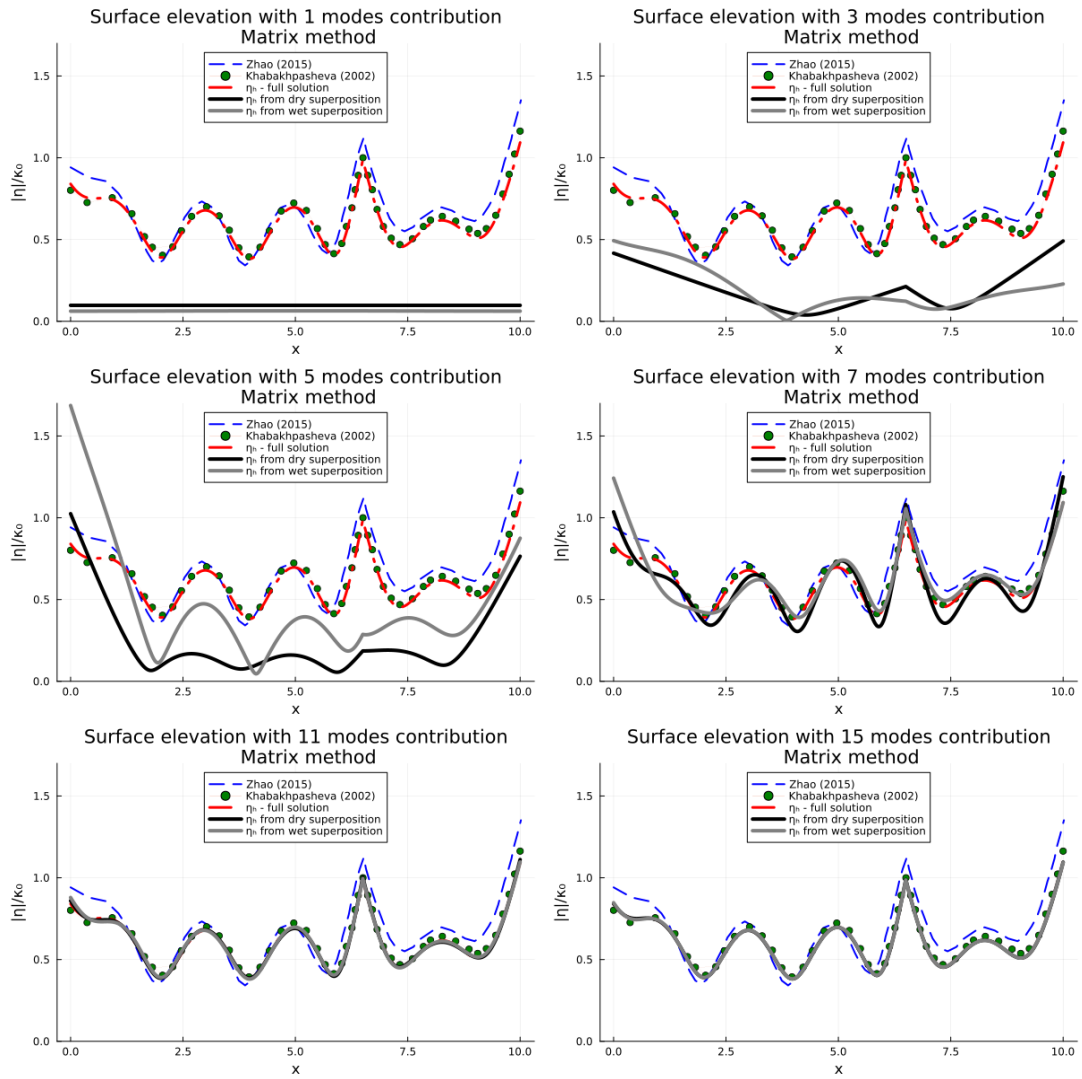


Figure 5.19: RMS Error for Case 4

Figure 5.20: First 6th dry and wet modal superpositions

5.4.3. Computational Time Performance

The computational cost of case 4 is summarised in Table 5.8. The modal superposition using the matrix approach performs better than the full solution and the integration method. It is around 110 – 180 times faster than the full solution and 3000 – 4500 times faster than the integration method.

Table 5.8: Case 4: Computation Time

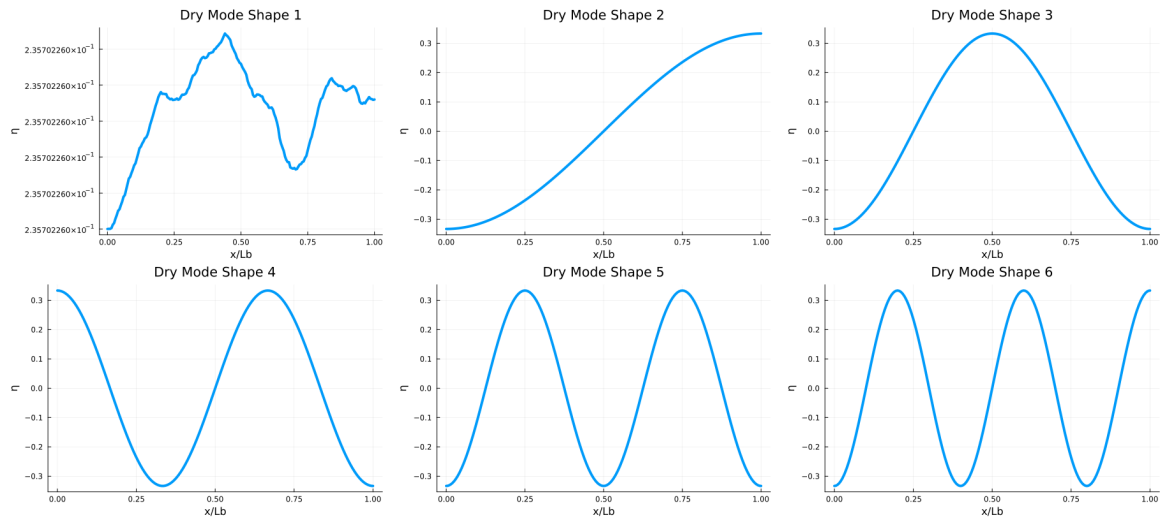
Analysis Type	Running Time (sec)			
	Warm up	Dry	Wet	Full
Full FEM solution	399.49	~	~	3.52
Modal Analysis	238.03	5317.30	~	~
Modal matrices construction	142.79	1042.20	1071.08	~
Modal superposition - Matrix	1.13	0.03	0.02	~
Modal superposition - Integration	4335.92	88.02	90.82	~

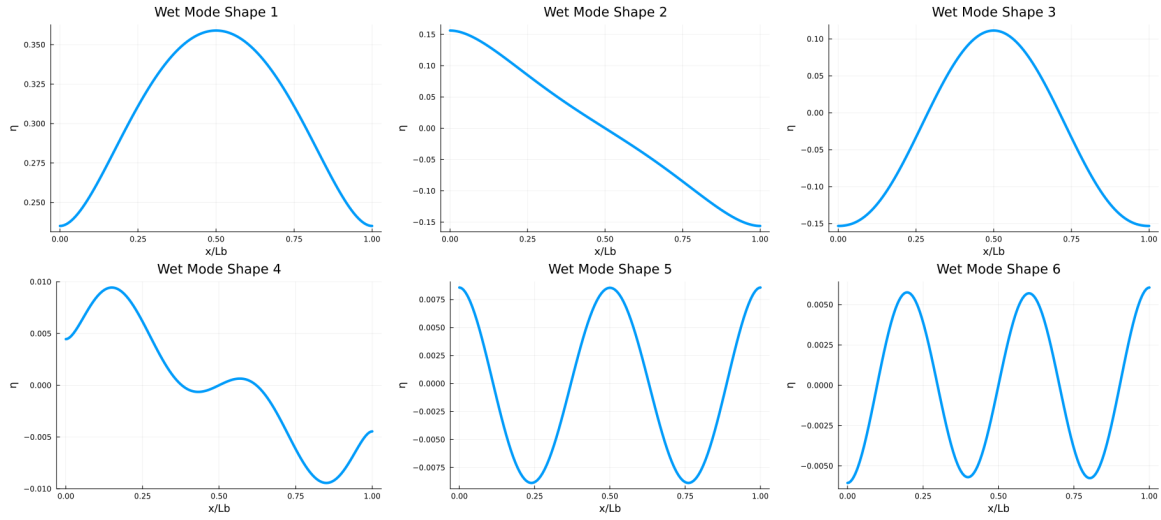
5.5. Case 5 - Floating Membrane

5.5.1. Modal Analysis Result

In case 5, the unified monolithic formulation returns the membrane structure. In this case, the bending rigidity and rotational inertia are set to 0. Consequently, leaving the stiffness only at the pretensioned term. Mathematically, this method may yield unstable results because the system is reduced from the fourth-order structural governing equation to a second-order one. Most of the terms in the system go to zero. To stabilise the calculation of the eigenvalue problem, the modal analysis is updated from using the quadratic eigenvalue problem (QEP) to the standard eigenvalue problem if the norm of the secondary mass S_{11} matrix is very small, close to zero. For the detailed setup explanation, see Section 4.5.

The first 6 dry and wet mode shapes of this structure are presented in Fig. (5.21) and (5.22). In the dry mode shapes, the first mode is spurious, but its magnitude is relatively small. Therefore, it can be assumed to be rigid. In wet mode, all mode shapes are flexible modes.

**Figure 5.21:** First 6th dry modal shapes

Figure 5.22: First 6th wet modal shapes

The dry and wet natural frequencies for case 5 are shown in Table (5.9). The ratio between the real parts of the dry and wet natural frequencies at the higher frequency in this case is different from that in the previous cases, which is around ~ 1 . However, the value of this natural frequency matches the result from Agarwal, Colomés, and Metrikine (2024).

Table 5.9: Case 5 dry and wet natural frequencies

Mode	ω_n^d	ω_n^w	Ratio $\Re(\omega_n^d/\omega_n^w)$
1	2.79937 + 0.0im	0.67471 - 0.40901im	4.15E-06
2	1.63995 + 0.0im	1.55390 - 0.27343im	1.05537
3	3.27991 + 0.0im	2.41402 - 0.20497im	1.35869
4	4.91987 + 0.0im	3.46778 - 0.15374im	1.41873
5	6.55983 + 0.0im	4.64292 - 0.11277im	1.41286
6	8.19979 + 0.0im	5.90799 - 0.08476im	1.38791
7	9.83975 + 0.0im	7.23751 - 0.06467im	1.35954
8	11.4797 + 0.0im	8.61813 - 0.05080im	1.33204
9	13.1196 + 0.0im	10.0365 - 0.04069im	1.30719
10	14.7596 + 0.0im	11.4859 - 0.03260im	1.28501
11	16.3995 + 0.0im	12.9605 - 0.02547im	1.26534
12	18.0395 + 0.0im	14.4608 - 0.01386im	1.24747
13	19.6795 + 0.0im	15.9691 - 2.7e-10im	1.23234
14	21.3194 + 0.0im	17.4778 - 1.1e-10im	1.21980
15	22.9594 + 0.0im	19.0185 - 0.00572im	1.20721

5.5.2. Modal Superposition Result

Modal amplitudes for each mode of the Matrix and Integration methods are presented in Fig. (5.23). The plots from both methods show similar results. The peak amplitudes of both methods can be seen in the mode $N = 3$, where the excitation wave frequency exactly matches the wet natural frequency, that is $\omega = 2.40$ rad/sec. The wet natural frequency shows a better estimation.

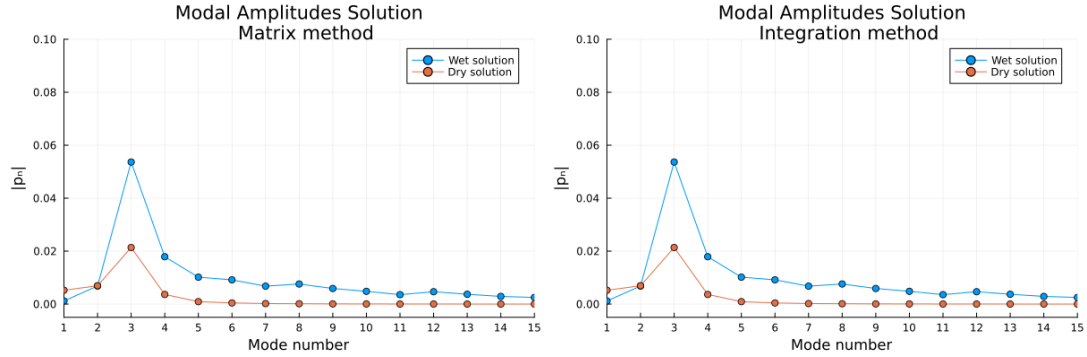


Figure 5.23: Modal Amplitude Solution

The RMSE in this case is presented in Fig. (5.24). The figure shows a good rate and a very close gap between the wet and dry solutions. This is validated by the modal superposition result in Fig. (5.25), where the solutions of dry and wet modes are very typical. In this case, the order of error $\mathcal{O}(10^{-3})$ is achieved with fewer modes contributing compared to the previous cases, at $N = 5$ for both solutions. This shows that for a very flexible structure, the wet and dry solutions are typically similar and can converge with fewer contributing modes.

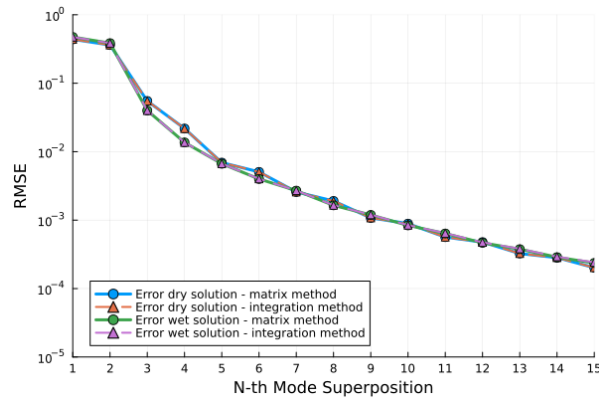
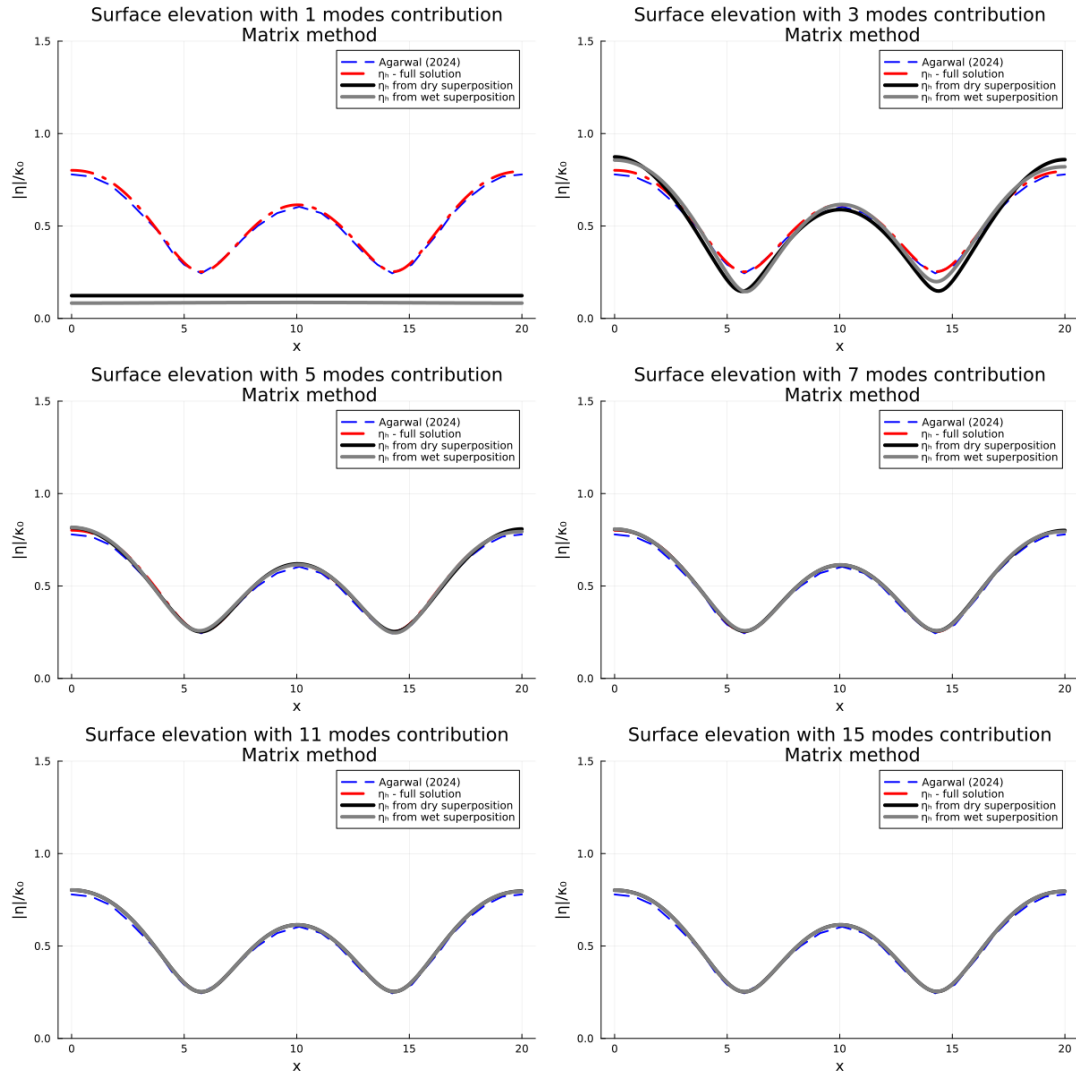


Figure 5.24: RMS Error for Case 5

Figure 5.25: First 6th dry and wet modal superpositions

5.5.3. Computational Time Performance

The computational cost of case 5 is summarised in Table 5.10. The modal superposition using the matrix approach performs better than the full solution and the integration method, similar to the previous cases. The runtime is 45 – 120 times faster than the full solution and 700 – 1800 times faster than the integration method.

Table 5.10: Case 5: Computation Time

Analysis Type	Running Time (sec)			
	Warm up	Dry	Wet	Full
Full FEM solution	429.42	~	~	2.34
Modal Analysis	239.40	599.07		~
Modal matrices construction	130.97	279.58	272.36	~
Modal superposition - Matrix	1.18	0.02	0.05	~
Modal superposition - Integration	3268.25	35.81	36.23	~

5.6. Added Mass and Added Damping Contribution Study

This study aims to observe the added mass and added damping contributions to the hydroelasticity of VLFS. The added mass calculated in Eq. (2.28) has complex values.

$$(-\omega^2) A(\omega) = (-\omega^2) \mathbf{C}_{12}(\omega) \mathbf{M}_\phi^{-1}(\omega) \mathbf{C}_{21}(\omega)$$

The contribution is evaluated by using the modal projection for each mode that can be obtained by the following:

$$\begin{aligned} (-\omega^2) A_i &= (-\omega^2) (\Phi_i^T \cdot \mathbf{A} \cdot \Phi_i) \\ &= (-\omega^2) (\tilde{A} + i\tilde{B}) \\ &= -\omega^2 \tilde{A} - i\omega \tilde{B} \end{aligned}$$

Where A_i is the scalar amplitude of the added mass matrix at mode i and is calculated over a range of frequency ω , $\tilde{A}$ is the real amplitude associated with the modal added mass, and $\tilde{B} = \omega B$ is the amplitude associated with the modal added damping. The mode shapes Φ include the dry and wet solutions. The contribution of the added mass and added damping will be evaluated by using dry and wet modes separately. Using the problem in Case 1 (Sec. 4.1), the contribution of the added mass and added damping is provided in Fig. (5.26) and (5.27). Complete plots are available in Appendix C.

For case 1, the added mass contribution is more significant than the structural mass. The magnitude of the added mass and added damping is reduced significantly at the higher modes. It is observed that the value of added mass and added damping at mode 1 are in the order of $\mathcal{O}(10^2)$, and at mode 6, the value are in the order of $\mathcal{O}(10^{-5})$ for the dry solution and $\mathcal{O}(10^{-3})$ for the wet solution. This indicates that the rigid mode has the dominant contribution for the added mass and added damping. The dominance of the added mass over the structural mass is expected since Case 1 has 1.1 m water depth and 10 m beam length. The ratio between the beam and the depth is almost 10. In this ratio, the structure is relatively very large compared to the water depth. More water is displaced by the structure, and the higher added mass influences the structure.

Moreover, fluctuating modal added mass amplitudes are observed at lower frequencies in modes 1 and 2 for the dry solution and in all modes for the wet solution. Interestingly, negative added mass can be observed at mode 1. This negative added mass effect can result from the low-frequency pumping mode (piston or Helmholtz mode) on the floating structure that partially covers the free surface area. This phenomenon can also trigger a low-frequency rigid-motion resonance due to rapid variations in added mass (M. McIver and P. McIver, 2016). However, since this frequency region and topic are not in the scope of this research, further research is needed to study the effect of added mass contribution in the very low frequency.

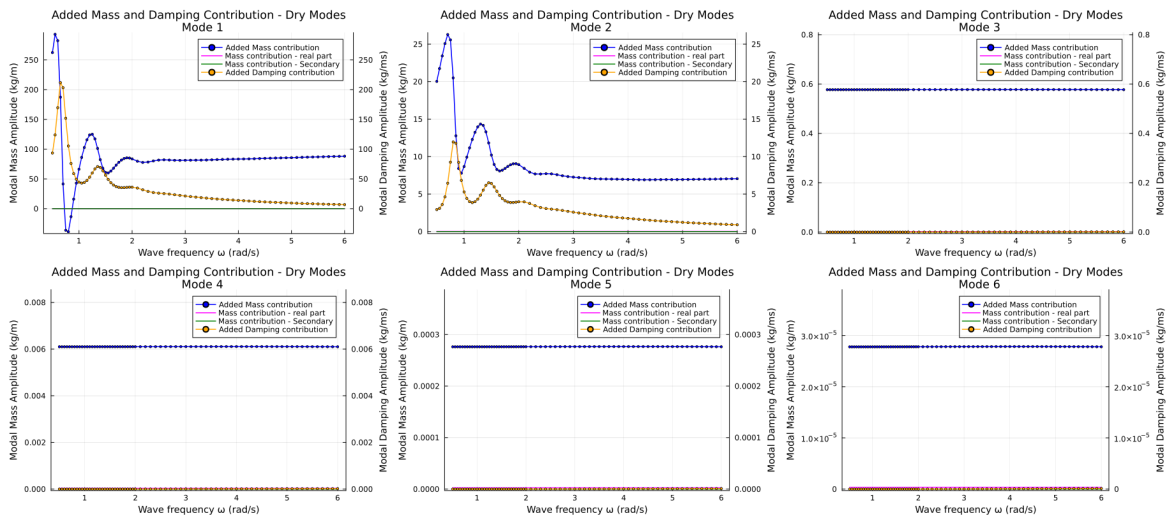


Figure 5.26: First 6th modes contribution - Dry solution - Case 1

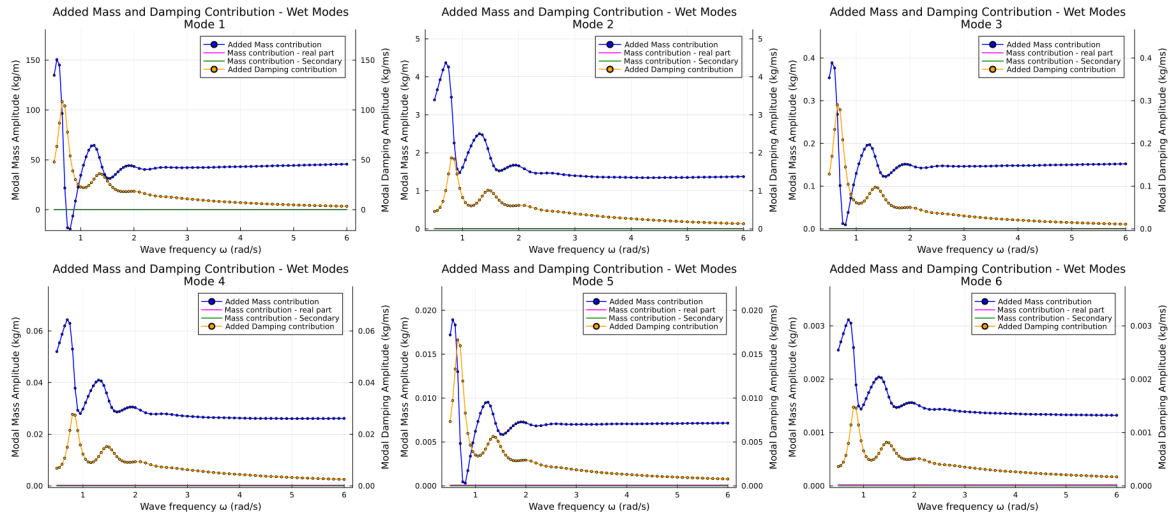


Figure 5.27: First 6th modes contribution - Wet solution - Case 1

The added damping also follows the same pattern as the added mass. It is associated with radiation damping, the energy dissipated by the oscillating floating structure to the fluid. This radiation damping helps to reduce the deflection of the structure due to the excitation wave, especially in the resonance response. The added damping values for all modes are non-negative and decrease at higher frequencies. At higher frequencies and higher mode contributions, the value tends to zero. For the added mass amplitude, it tends to be a constant value.

In addition, the added mass and damping for case 5 will also be extracted. This case uses a very flexible membrane with a 20 meter length on a 10 meter water depth. This case is expected to have a lower added mass effect. The modal projection plots of the added mass and damping amplitude for case 5 are depicted in Fig. (5.28) and (5.29).

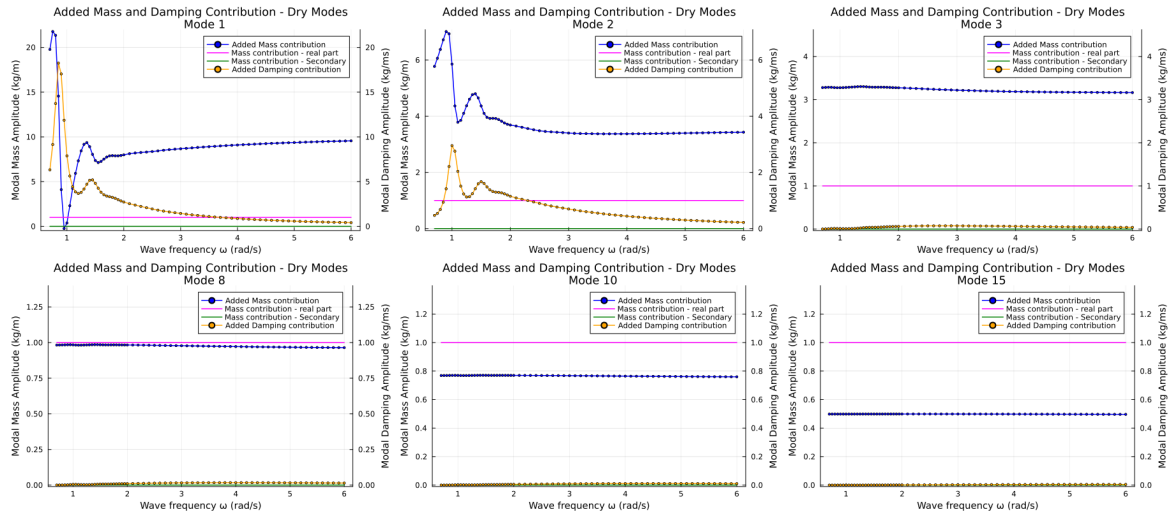


Figure 5.28: First 6th modes contribution - Dry solution - Case 5

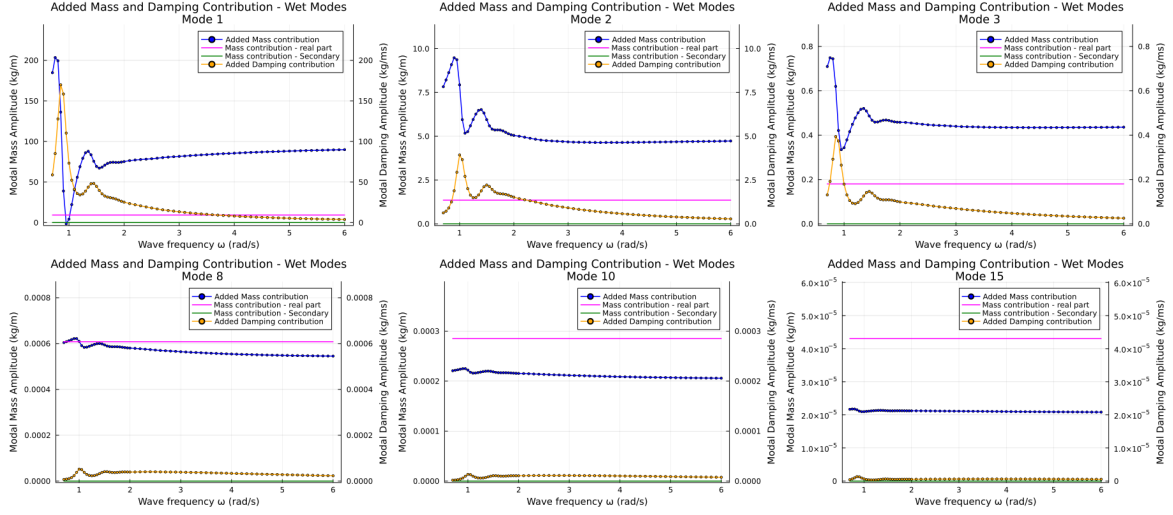
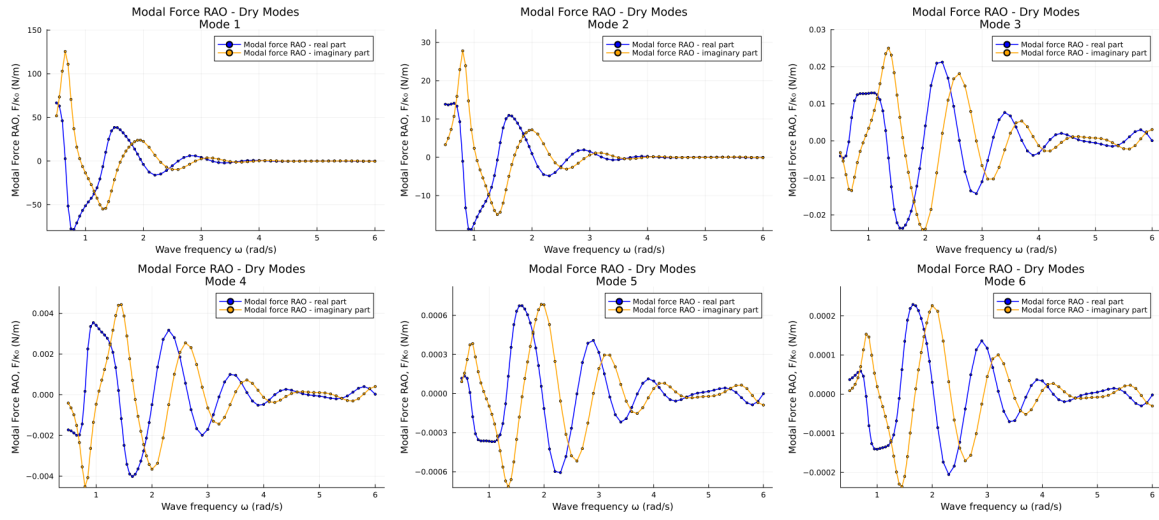
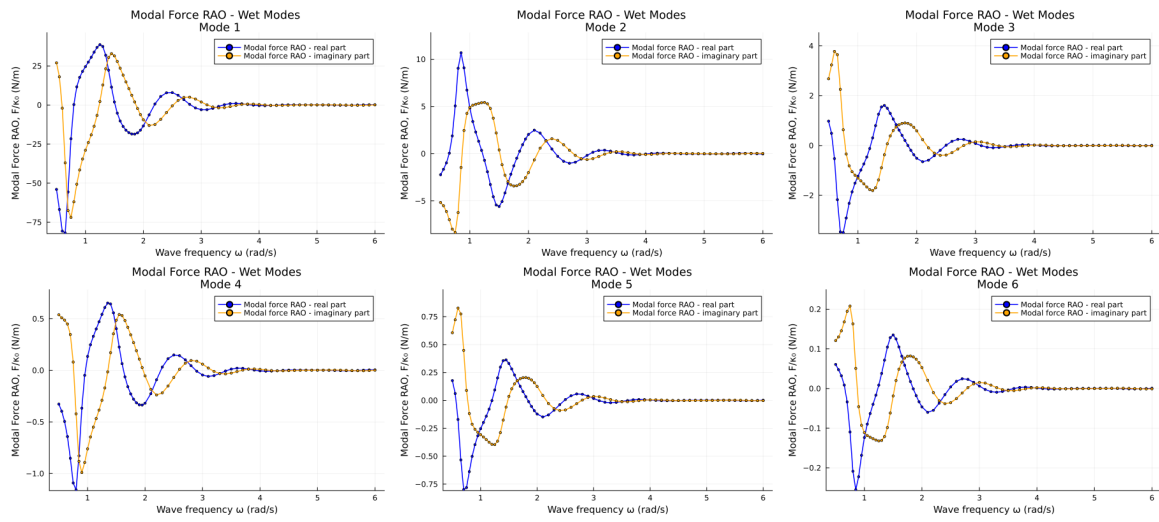


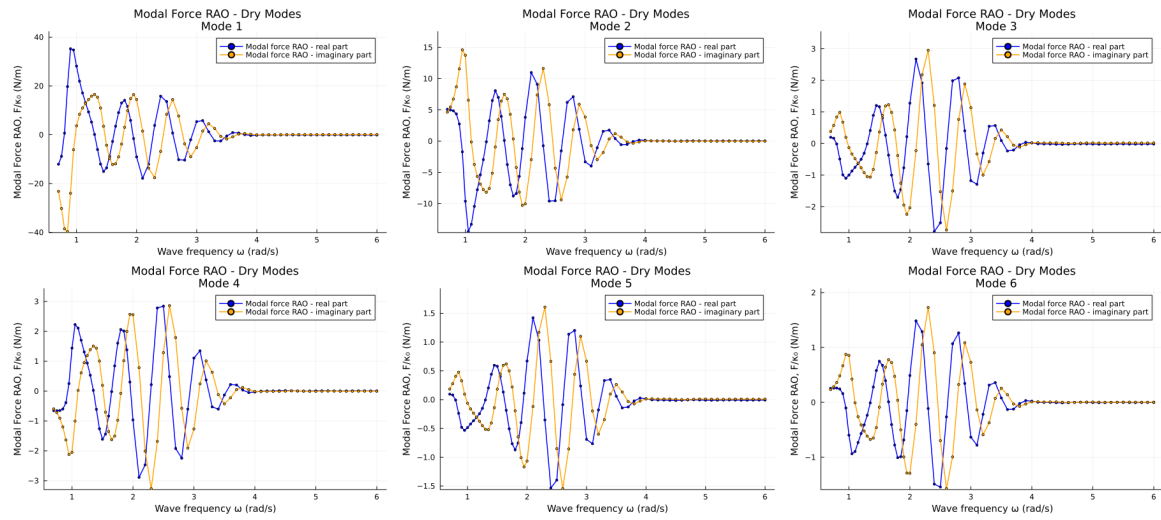
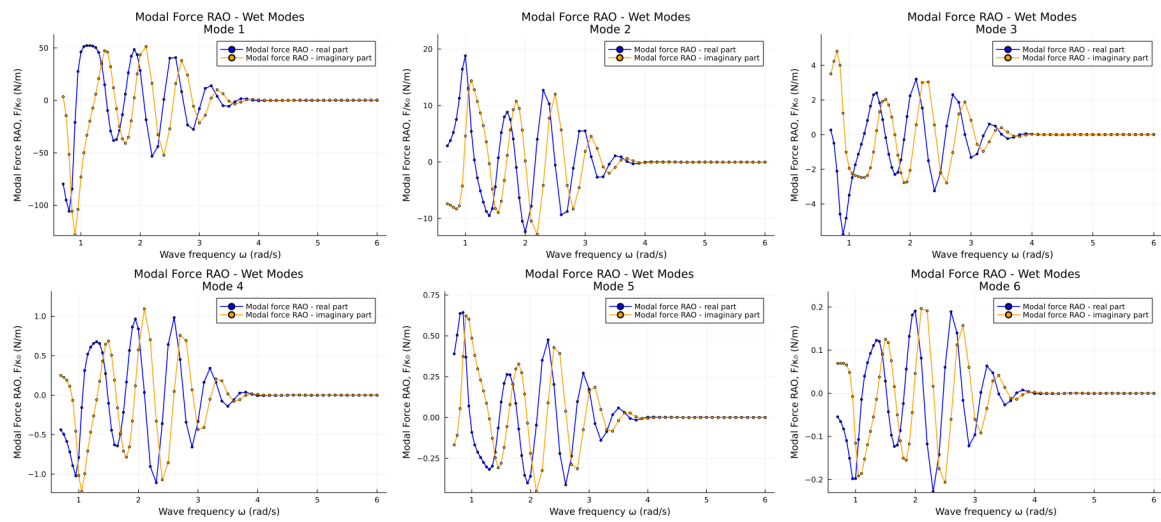
Figure 5.29: First 6th modes contribution - Wet solution - Case 5

The patterns for both added mass and added damping are similar to those in case 1, but with different magnitudes. In this case, the added mass is still dominant for lower modes, but at higher modes (mode 8 and above), the structural mass amplitude is higher than the added mass. This result is expected because the ratio between the structural length and water depth is only 2, compared to case 1, which has a ratio of nearly 10. In case 1, the shallow-water condition strongly confines the fluid between the structure and seabed. This causes a large volume of fluid to move along with the structure's deflection, resulting in a dominant added mass effect. In Case 5, the deeper water gives extra room for the fluid to move after being displaced by the structure's deflection. Consequently, while the added mass remains significant at lower modes, it does not significantly outweigh the structural mass at higher modes, where the modal wavelength becomes comparable to or shorter than the water depth.

When the structural mass becomes significantly larger than the added mass for all modes, the system can be directly approached by using a dry modal solution. However, the fluid coupling terms shall not be directly neglected. The radiation damping effect still needs to be evaluated because it may have a dominant effect on the FSI problem, especially at the near-resonance behaviour. If both of these values are insignificant and the problem does not require the wave transmission and reflection solutions, the problem can then be simplified by using a one-way coupling problem, treating the fluid solely as the external force acting on the structure. In this research, the added mass contribution remains significant at low modes for all cases. Therefore, a fully coupled (two-way) analysis is still required.

In addition to the frequency-dependent added mass and added damping, the modal projection of the force response amplitude operator (RAO) is also plotted in this section. The force terms act on the inlet and upstream wave damping boundaries as derived in the linear terms of the weak form in Eq. (3.22) and the matrix definition in Eq. (3.28). The plots are aimed at observing the influence of the forces for each mode within the frequency range. The plots are depicted in Fig. (5.30) and (5.31) for case 1 and in Fig. (5.32) and (5.33) for case 5. From the figures of all cases, it is observed that the forces decrease with increasing frequency. At the very high frequency, the value is close to zero. The complete plot is available in Appendix C.

Figure 5.30: First 6th modal force RAO - Dry solution - Case 1Figure 5.31: First 6th modal force RAO - Wet solution - Case 1

Figure 5.32: First 6th modal force RAO - Dry solution - Case 5Figure 5.33: First 6th modal force RAO - Wet solution - Case 5

Conclusions and Recommendations

6.1. Conclusions

This research has developed a reduced-order modelling (ROM) formulation for the monolithic FEM by using modal superposition. The formulation used in this research is based on the monolithic framework that was developed in the previous study. The development of this ROM framework aims to investigate an effective computational method for solving the hydroelastic problem.

In addition to developing the ROM framework, this research initially developed a new formulation for the structural term by using Mindlin theory. The initial development is conducted in a one-dimensional structural domain. The aim of developing this theory is to formulate a Unified Formulation that can cover a wide range of hydroelastic problems. The analysis results show that this model performs well in recovering Euler-Bernoulli and membrane structures. Moreover, the implementation of spring connections at the structural boundary and at the structural joint in this formulation provides good applicability for analysing moored and multi-body floating structures.

The ROM analysis framework begins with modal analysis, including dry and wet analyses. The development of this new Unified Structural formulation requires updating the previous modal analysis framework from solving a standard eigenvalue problem to implementing the quadrature eigenvalue problem. This modal analysis helps understand the structural behaviour in its interaction with the surrounding fluid. The results show that the ratio of the dry and wet natural frequencies for the higher modes tends to a constant value. It also shows that the wet natural frequency provides a better estimate of the structure's natural frequency, giving a more accurate prediction of the resonance phenomenon.

Following up on the results of modal analysis, a modal superposition framework is developed to investigate the effectiveness of reduced-order modelling (ROM) within the monolithic FEM framework. Two methods of modal superposition have been developed. The first is the matrix method, which builds the weak-form matrices and solves the modal superposition using the matrix system. The second is the integration method that directly incorporates the modal superposition solution term into the weak formulation. This method aligns with the mathematical formulation used in the monolithic FEM framework and in the Gridap FEM ecosystem. However, since the weak formulation is expanded to solve for the modal amplitudes in N modes, the computation becomes more expensive. As shown in the results, this method can be thousands of times slower than the matrix method. Even the full solution computationally performs better than this approach. Nevertheless, in terms of accuracy, both methods perform well at estimating the full finite-element solution.

The root mean squared (RMS) error calculation shows that the solutions from modal superposition of both methods have reached an order of error $\mathcal{O}(10^{-3})$ and lower at $N = 5 \sim 10$. Comparing the accuracy of dry and wet solutions, dry modal superposition solutions yield higher accuracy than the wet modal solutions at the same number of modes. This means that the dry modal solutions converge faster than the wet modal solutions. This condition is valid as long as the forcing terms accurately capture all physical terms, which will later be incorporated into the dry modal superposition. The wet modal

solution performs worse than the dry solution because of the ill-conditioned nature of the wet modal analysis results. The non-linear eigenvalue problem (NLEP) in this research depends on the initial guess estimate frequency and the relaxation parameter in each iteration. For each iteration, it calculates all mode shape solutions but only considers the mode shape on the respective modes, neglecting the other modes. This method can cause a loss of accuracy in the calculation and slower convergence for the modal superposition. For subsequent research, a more robust method for calculating the NLEP needs to be considered, for instance, a fully rational Krylov method and infinite Arnoldi method (Güttel et al., 2014). However, this Unified Formulation and the modal superposition solution yield results that are consistent with the full-order solution and comparable to those in the existing literature.

The complexity of the structure affects the required number of modes for the truncated solution. But it is not the sole parameter. As depicted in the result, the case with high wave frequency required higher mode contributions than the structure with mooring in lower wave frequency. The required number of modes is more determined by the structure's expected deflection. If the structure has a more complex response, such as fluctuations, it will require more contributing modes. However, in terms of computational time, the complex structure in case 4 requires a longer running time than the other structure.

In conclusion, the two ROM modal superposition approaches that have been developed in this research show two different results in terms of computational performance. The matrix method shows a faster performance than the full solution. But the integration method performs more slowly than the full solution due to the expanded weak formulation in the Gridap framework. Regardless of computational performance, the development of this integration method demonstrates the applicability of the modal superposition method within the variational formulation using the Gridap environment with accurate results.

6.2. Recommendations

The development of reduced-order modelling (ROM) for the monolithic formulation is still in development. This research has built up the fundamental framework for the ROM using the modal superposition method. Several strategies can be explored to develop this method within the monolithic FEM framework, including solving the nonlinear eigenvalue problem (NLEP) using a more robust solution. The NEP-PACK package in Julia can solve the NLEP problem and can be further explored for implementation in this hydroelastic problem since both work in the same Julia programming language (Jarlebring et al., 2018).

The development of modal superposition using the integration method shows that it is ineffective to implement the modal superposition solution directly in the variational problem. Since the writer's knowledge of the Gridap environment is still limited, it is recommended to explore additional features to help formulate the problem more effectively. For the developed matrix method, the formulation can be extended into a three-dimensional problem and into the time domain.

Another ROM method is available in the Gridap environment, using the Reduced-Basis method. The GridapROMs package in the Julia programming language is actively developed and has been proven to solve varying PDE problems (Mueller and Badia, 2026). This method uses a parameterised PDE to produce snapshots for estimating the solution of the reduced model. This method can serve as an alternative to the common modal superposition method.

Additionally, the added mass and added damping phenomenon at very low frequencies can be another interesting topic to be explored. The hydroelastic problems in this frequency region can be further studied in future research. The monolithic FEM framework can mathematically capture this phenomenon.

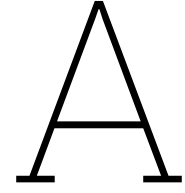
Lastly, the development of the Unified Weak Formulation for the structural terms has shown good results in accommodating a wide range of analyses. For future study, developing this formulation in the three-dimensional case can further validate its applicability.

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Weak Form Matrices Formulation

This appendix presents the derivation of the weak form matrices used in the modal superposition. The matrices are derived from the Unified Formulation weak form as provided in Eq. (3.22) and used in Case 4.4. For the implementation in modal analysis, follow the procedure in Section 3.7, by replacing the damping term and using the radiation boundary for the K_{22} matrix.

For the application of other structural types, the weak formulation in this appendix can be adjusted following the procedure explained in Section 3.5.

```
1 # =====
2 # [A] Full weak formulation
3 # =====
4
5 a((phi,k,eta),(w,u,v)) = (int( grad(w).grad(phi) )dOmega +
6   int( beta_h*(u + alpha_h*w)*(g*k - im*omega*phi) + im*omega*w*k )dGammafs +
7   int( beta_h*(u + alpha_h*w)*(g*k - im*omega*phi) + im*omega*w*k - mu_2in*k*w + mu_1in*grad_n(phi)*(u + alpha_h*w) )dGamma1 +
8   int( beta_h*(u + alpha_h*w)*(g*k - im*omega*phi) + im*omega*w*k - mu_2out*k*w + mu_1out*grad_n(phi)*(u + alpha_h*w) )dGamma2 +
9   int( ( Dp*Delta(v)*Delta(eta) + A2*(grad(v).grad(eta)) + A3*v*eta + A4*(grad(v).grad(phi)) + A5*v*phi ) + im*omega*w*eta )dGammab1 +
10  int( ( Dp*Delta(v)*Delta(eta) + A2*(grad(v).grad(eta)) + A3*v*eta + A4*(grad(v).grad(phi)) + A5*v*phi ) + im*omega*w*eta )dGammab2 +
11  int( (Dp) * ( - jump(grad(v).n*Lambda1) * mean(Delta(eta)) - mean(Delta(v)) * jump(grad(eta).n*Lambda1)
12    + gamma*( jump(grad(v).n*Lambda1) * jump(grad(eta).n*Lambda1) ) ) )dLambda1 +
13  int( (Dp) * ( - jump(grad(v).n*Lambda2) * mean(Delta(eta)) - mean(Delta(v)) * jump(grad(eta).n*Lambda2)
14    + gamma*( jump(grad(v).n*Lambda2) * jump(grad(eta).n*Lambda2) ) ) )dLambda2 +
15  int( (mean(v) * ks2p * mean(eta)) + ( jump(grad(v).n*Lambda1) * kr2p * jump(grad(eta).n*Lambda1) ) )dLambda1 +
16  int( (ksp*(v*eta) + krp*(grad(v).grad(eta)) ) )dGamma_l +
17  int( (ksp*(v*eta) + krp*(grad(v).grad(eta)) ) )dGamma_r )
18 l((w,u,v)) = int( w*v_in )dGamma_in - int( eta*dw - grad_n(phi)*d*(u + alpha_h*w) )dGamma1
19
20 #=
21 (M,K)11 -> (eta, v)    C12 -> (phi, v)
22 K22 -> (phi, w)       C21 -> (eta, w)       C23 -> (kappa, w)
23 K33 -> (kappa, u)      C32 -> (phi, u)
24 -----
25 alpha_h = -im*omega/g * (1-beta_h)/beta_h = -im*omega*alpha_f
26 alpha_f = 1/g * (1-beta_h)/beta_h
27
28 mu_2in(x) = mu_1in(x)*k ; mu_2out(x) = mu_1out(x)*k
29 k*mu_2in(x) = mu_1in(x) ; k*mu_2out(x) = mu_1out(x)
30
31 Nomenclature:
32 xXXf -> function of wave frequency omega ( divided by -im*omega )
33 xXXf2 -> function of wave frequency omega ( divided by (-im*omega)^2 )
34 xXXk -> function of wave number k
35 =#
```



```

36
37 # Overwriting the coefficient to eliminate wave frequency  $\omega$  and number  $k$ 
38  $\alpha_f = (1/g * (1-\beta_h)/\beta_h)$ ;  $\alpha_h = (-im*\omega) * \alpha_f$ 
39  $k\mu_{zin}(x) = \mu_{lin}(x)$ ;  $k\mu_{zout}(x) = \mu_{lout}(x)$ 
40
41 # =====
42 # [B] Breakdown
43 # =====
44 # [1] Fluid domain ~  $d\Omega$ 
45 # -----
46  $\int (\nabla(w) \cdot \nabla(\phi)) d\Omega$ 
47 # a. weak form matrix
48  $k22(\phi, w) = \int (\nabla(w) \cdot \nabla(\phi)) d\Omega$  # ←
49
50 # -----
51 # [2] Free-surface ~  $d\Gamma_f$ 
52 # -----
53  $\int (\beta_h*(u + \alpha_h*w)*(g*\kappa - im*\omega*\phi) + im*\omega*w*\kappa) d\Gamma_f$ 
54  $\int ((\beta_h*u + \beta_h*\alpha_h*w)*(g*\kappa - im*\omega*\phi) + im*\omega*w*\kappa) d\Gamma_f$ 
55  $\int ((\beta_h*u + \beta_h*\alpha_h*w)*g*\kappa - (\beta_h*u + \beta_h*\alpha_h*w)*im*\omega*\phi + im*\omega*w*\kappa) d\Gamma_f$ 
56  $\int (\beta_h*u*g*\kappa + \beta_h*\alpha_h*w*g*\kappa - \beta_h*u*im*\omega*\phi - \beta_h*\alpha_h*w*im*\omega*\phi + im*\omega*w*\kappa) d\Gamma_f$ 
57  $\int (\beta_h*g*(\kappa*u) + \beta_h*\alpha_h*g*(\kappa*w) - \beta_h*im*\omega*(\phi*u) - \beta_h*\alpha_h*im*\omega*(\phi*w) + im*\omega*(\kappa*w)) d\Gamma_f$ 
58
59 # a. Separated by the trial and test functions
60  $k22(\phi, w) = \int (-\beta_h*\alpha_h*im*\omega*(\phi*w)) d\Gamma_f$ 
61  $c23(\kappa, w) = \int (\beta_h*\alpha_h*g*(\kappa*w) + im*\omega*(\kappa*w)) d\Gamma_f$ 
62  $c32(\phi, u) = \int (-\beta_h*im*\omega*(\phi*u)) d\Gamma_f$ 
63  $k33(\kappa, u) = \int (\beta_h*g*(\kappa*u)) d\Gamma_f$  # ←
64
65 # b. Extract the frequency and/or wave number function
66 # The frequency term will be included later in the non-linear modal analysis
67  $k22(\phi, w) = (-im*\omega) * \int (\beta_h*\alpha_h*(\phi*w)) d\Gamma_f$ 
68  $= (-im*\omega)^2 * \int (\beta_h*\alpha_f*(\phi*w)) d\Gamma_f$ 
69  $k22f2(\phi, w) = \int (-\beta_h*\alpha_f*(\phi*w)) d\Gamma_f$  # ←
70
71  $c23(\kappa, w) = \int ((-im*\omega)*\beta_h*\alpha_f*g*(\kappa*w) - (-im*\omega)*( \kappa*w)) d\Gamma_f$ 
72  $= (-im*\omega) * \int (\beta_h*\alpha_f*g*(\kappa*w) - (\kappa*w)) d\Gamma_f$ 
73  $c23f(\kappa, w) = \int (\beta_h*\alpha_f*g*(\kappa*w) - (\kappa*w)) d\Gamma_f$  # ←
74
75  $c32(\phi, u) = (-im*\omega) * \int (\beta_h*(\phi*u)) d\Gamma_f$ 
76  $c32f(\phi, u) = \int (\beta_h*(\phi*u)) d\Gamma_f$  # ←
77
78 # c. Final weak form matrices
79  $k22f2(\phi, w) = \int (\beta_h*\alpha_f*(\phi*w)) d\Gamma_f$ 
80  $c23f(\kappa, w) = \int (\beta_h*\alpha_f*g*(\kappa*w) + (-1)*( \kappa*w)) d\Gamma_f$ 
81  $c32f(\phi, u) = \int (\beta_h*(\phi*u)) d\Gamma_f$ 
82  $k33(\kappa, u) = \int (\beta_h*g*(\kappa*u)) d\Gamma_f$ 
83
84 # -----
85 # [3] Damping terms ~  $d\Gamma_{d1}$ ,  $d\Gamma_{d2}$ 
86 # -----
87  $\int (\beta_h*(u + \alpha_h*w)*(g*\kappa - im*\omega*\phi) + im*\omega*w*\kappa - \mu_{zin}* \kappa*w + \mu_{lin}*\nabla_n(\phi)*(u + \alpha_h*w)) d\Gamma_{d1} +$ 
88  $\int (\beta_h*(u + \alpha_h*w)*(g*\kappa - im*\omega*\phi) + im*\omega*w*\kappa - \mu_{zout}* \kappa*w + \mu_{lout}*\nabla_n(\phi)*(u + \alpha_h*w)) d\Gamma_{d2} +$ 
89
90 # due to similarity, only  $d\Gamma_{d1}$  will be broken down,  $d\Gamma_{d2}$  appears later
91  $\int ((\beta_h*u + \beta_h*\alpha_h*w)*(g*\kappa - im*\omega*\phi) + im*\omega*w*\kappa - \mu_{lin}(x)*(k)*\kappa*w + \mu_{lin}*\nabla_n(\phi)*(u + \alpha_h*w)) d\Gamma_{d1}$ 
92  $\int ((\beta_h*u + \beta_h*\alpha_h*w)*g*\kappa - (\beta_h*u + \beta_h*\alpha_h*w)*im*\omega*\phi + im*\omega*w*\kappa - \mu_{lin}(x)*(k)*\kappa*w$ 
93  $+ \mu_{lin}*\nabla_n(\phi)*u + \mu_{lin}*\nabla_n(\phi)*\alpha_h*w) d\Gamma_{d1}$ 
94  $\int (\beta_h*g*(\kappa*u) + \beta_h*\alpha_h*g*(\kappa*w) - \beta_h*im*\omega*(\phi*u) - \beta_h*\alpha_h*im*\omega*(\phi*w) + im*\omega*(\kappa*w) - \mu_{lin}(x)*(k)*( \kappa*w)$ 
95  $+ \mu_{lin}*(\nabla_n(\phi)*u) + \mu_{lin}*\alpha_h*(\nabla_n(\phi)*w)) d\Gamma_{d1}$ 
96

```

```

97 # a. Seperated by the trial and test functions, includes dΓd2
98 k33(κ,u) = ∫(βn*g*(κ*u))dΓd1 + ∫(βn*g*(κ*u))dΓd2 # ←
99
100 c23(κ,w) = ∫(βn*αn*g*(κ*w) + im*ω*(κ*w) - μlin(x)*(κ)*(κ*w))dΓd1 +
101 ∫(βn*αn*g*(κ*w) + im*ω*(κ*w) - μout(x)*(κ)*(κ*w))dΓd2
102
103 k22(φ,w) = ∫(-βn*αn*im*ω*(φ*w) + μlin*αn*(∇n(φ)*w))dΓd1 +
104 ∫(-βn*αn*im*ω*(φ*w) + μout*αn*(∇n(φ)*w))dΓd2
105
106 c32(φ,u) = ∫(-βn*im*ω*(φ*u) + μlin*(∇n(φ)*u))dΓd1 +
107 ∫(-βn*im*ω*(φ*u) + μout*(∇n(φ)*u))dΓd2
108
109 # b. Extract the frequency and/or wave number function
110 c23(κ,w) = ∫(βn*(-im*ω)*αn*g*(κ*w) + im*ω*(κ*w) - μlin(x)*(κ)*(κ*w))dΓd1 +
111 ∫(βn*(-im*ω)*αn*g*(κ*w) + im*ω*(κ*w) - μout(x)*(κ)*(κ*w))dΓd2
112 = (-im*ω) * ∫(βn*αn*g*(κ*w) - (κ*w))dΓd1 + (κ) * ∫(μlin(x)*(κ*w)*(-1))dΓd1
113 (-im*ω) * ∫(βn*αn*g*(κ*w) - (κ*w))dΓd2 + (κ) * ∫(μout(x)*(κ*w)*(-1))dΓd2
114 c23f(κ,w) = ∫(βn*αn*g*(κ*w) - (κ*w))dΓd1 + ∫(βn*αn*g*(κ*w) - (κ*w))dΓd2 # ←
115 c23k(κ,w) = ∫(μlin(x)*(κ*w)*(-1))dΓd1 + ∫(μout(x)*(κ*w)*(-1))dΓd2 # ←
116
117 k22(φ,w) = ∫(βn*(-im*ω)*αn*(-im*ω)*(φ*w) + μlin*(-im*ω)*αn*(∇n(φ)*w))dΓd1 +
118 ∫(βn*(-im*ω)*αn*(-im*ω)*(φ*w) + μout*(-im*ω)*αn*(∇n(φ)*w))dΓd2
119 = (-im*ω)2 * ∫(βn*αn*g*(φ*w))dΓd1 + (-im*ω) * ∫(μlin*αn*(∇n(φ)*w))dΓd1 +
120 (-im*ω)2 * ∫(βn*αn*g*(φ*w))dΓd2 + (-im*ω) * ∫(μout*αn*(∇n(φ)*w))dΓd2
121 k22f2(φ,w) = ∫(βn*αn*g*(φ*w))dΓd1 + ∫(βn*αn*g*(φ*w))dΓd2 # ←
122 k22f1(φ,w) = ∫(μlin*αn*(∇n(φ)*w))dΓd1 + ∫(μout*αn*(∇n(φ)*w))dΓd2 # ←
123
124 c32(φ,u) = ∫(βn*(-im*ω)*(φ*u) + μlin*(∇n(φ)*u))dΓd1 +
125 ∫(βn*(-im*ω)*(φ*u) + μout*(∇n(φ)*u))dΓd2
126 = (-im*ω) * ∫(βn*g*(φ*u))dΓd1 + ∫(μlin*(∇n(φ)*u))dΓd1 +
127 (-im*ω) * ∫(βn*g*(φ*u))dΓd2 + ∫(μout*(∇n(φ)*u))dΓd2
128 c32f(φ,u) = ∫(βn*g*(φ*u))dΓd1 + ∫(βn*g*(φ*u))dΓd2 # ←
129 c32(φ,u) = ∫(μlin*(∇n(φ)*u))dΓd1 + ∫(μout*(∇n(φ)*u))dΓd2 # ←
130
131 # c. Final weak form matrices
132 k22f1(φ,w) = ∫(μlin*αn*(∇n(φ)*w))dΓd1 + ∫(μout*αn*(∇n(φ)*w))dΓd2
133 k22f2(φ,w) = ∫(βn*αn*g*(φ*w))dΓd1 + ∫(βn*αn*g*(φ*w))dΓd2
134 c23f(κ,w) = ∫(βn*αn*g*(κ*w) + (-1)*(κ*w))dΓd1 + ∫(βn*αn*g*(κ*w) + (-1)*(κ*w))dΓd2
135 c23k(κ,w) = ∫(μlin(x)*(κ*w)*(-1))dΓd1 + ∫(μout(x)*(κ*w)*(-1))dΓd2
136 c32f(φ,u) = ∫(βn*g*(φ*u))dΓd1 + ∫(βn*g*(φ*u))dΓd2
137 c32(φ,u) = ∫(μlin*(∇n(φ)*u))dΓd1 + ∫(μout*(∇n(φ)*u))dΓd2
138 k33(κ,u) = ∫(βn*g*(κ*u))dΓd1 + ∫(βn*g*(κ*u))dΓd2
139
140
141 # -----
142 # [4] Structural terms 1 ~ dΛb1, dΛb2, dΛj1, dΓl, dΓr
143 # -----
144 ∫((Dp) * (- jump(∇(v)·nΛb1) * mean(Δ(η)) - mean(Δ(v)) * jump(∇(η)·nΛb1)
145 + γ*( jump(∇(v)·nΛb1) * jump(∇(η)·nΛb1) ))dΛb1 +
146 ∫((Dp) * (- jump(∇(v)·nΛb2) * mean(Δ(η)) - mean(Δ(v)) * jump(∇(η)·nΛb2)
147 + γ*( jump(∇(v)·nΛb2) * jump(∇(η)·nΛb2) ))dΛb2 +
148 ∫((mean(v) * ks2p * mean(η)) + ( jump(∇(v)·nΛj1) * kr2p * jump(∇(η)·nΛj1) ))dΛj1 +
149 ∫((ksp*(v*η) + krp*(∇(v)·∇(η))))dΓl +
150 ∫((ksp*(v*η) + krp*(∇(v)·∇(η))))dΓr
151
152 # Structural stiffness for dry and wet
153 k11(η,v) = ∫((Dp) * (- jump(∇(v)·nΛb1) * mean(Δ(η)) - mean(Δ(v)) * jump(∇(η)·nΛb1)
154 + γ*( jump(∇(v)·nΛb1) * jump(∇(η)·nΛb1) ))dΛb1 +
155 ∫((Dp) * (- jump(∇(v)·nΛb2) * mean(Δ(η)) - mean(Δ(v)) * jump(∇(η)·nΛb2)
156 + γ*( jump(∇(v)·nΛb2) * jump(∇(η)·nΛb2) ))dΛb2 +
157 ∫((mean(v) * ks2p * mean(η)) + ( jump(∇(v)·nΛj1) * kr2p * jump(∇(η)·nΛj1)))dΛj1 +

```

```

158      ∫((ksp*(v*η) + krp*(∇(v)·∇(η))))dΓl +
159      ∫((ksp*(v*η) + krp*(∇(v)·∇(η))))dΓr
160
161
162
163 # -----
164 # [5] Structural terms 2 ~ dΓb1, dΓb2
165 # -----
166 ∫((A1*Δ(v)*Δ(η) + A2*(∇(v)·∇(η)) + A3*v*η + A4*(∇(v)·∇(φ)) + A5*v*φ) + im*ω*w*η)dΓb1 +
167 ∫((A1*Δ(v)*Δ(η) + A2*(∇(v)·∇(η)) + A3*v*η + A4*(∇(v)·∇(φ)) + A5*v*φ) + im*ω*w*η)dΓb2 +
168
169 # Breakdown the coefficient A1, A2, A3, A4, A5. Separated by the frequency component
170 A1 = Dp
171 A2 = (-ρ0*J*ω2/ρw - Dp*ρ0*hp*ω2/C + Dp*ρw*g/C + Tp)
172      = (-ω2)*(ρ0*J/ρw + Dp*ρ0*hp/C) + (Dp*ρw*g/C + Tp)
173 A3 = ρ02*J*hp*ω4/(ρw*C) - ρ0*hp*ω2/ρw - ρ0*g*J*ω2/C + g
174      = (ω4)*(ρ02*J*hp/(ρw*C)) + (-ω2)*(ρ0*hp/ρw + ρ0*g*J/C) + (g)
175 A4 = (im*Dp*ρw*ω/C) * (-1)
176      = (-im*ω)*(Dp*ρw/C)
177 A5 = im*ρ0*J*ω3/C - im*ω
178      = (-im*ω) * (1 - ρ0*J*ω2/C)
179
180 ∫((A1*Δ(v)*Δ(η) + A2*(∇(v)·∇(η)) + A3*v*η + A4*(∇(v)·∇(φ)) + A5*v*φ) + im*ω*w*η)dΓb1
181 ∫(A1*Δ(v)*Δ(η))dΓb1 + ∫(A2*(∇(v)·∇(η)))dΓb1 + ∫(A3*v*η)dΓb1 + ∫(A4*(∇(v)·∇(φ)) + A5*v*φ)dΓb1
182 + ∫(im*ω*w*η)dΓb1
183 ∫(Dp*Δ(v)*Δ(η))dΓb1 + ∫((( -ω2)*(ρ0*J/ρw + Dp*ρ0*hp/C) + (Dp*ρw*g/C + Tp)) * (∇(v)·∇(η)))dΓb1 +
184 ∫(((ω4)*(ρ02*J*hp/(ρw*C)) + (-ω2)*(ρ0*hp/ρw + ρ0*g*J/C) + (g)) * (v*η))dΓb1 +
185 ∫((( -im*ω)*(Dp*ρw/C)) * (∇(v)·∇(φ)) + (( -im*ω) * (1 - ρ0*J*ω2/C)) * (v*φ))dΓb1 + +
186      ⇨ ∫((im*ω)*w*η)dΓb1
187 ∫(Dp*Δ(v)*Δ(η))dΓb1 +
188 ∫((-ω2)*(ρ0*J/ρw + Dp*ρ0*hp/C)*(∇(v)·∇(η)) + (Dp*ρw*g/C + Tp)*(∇(v)·∇(η)))dΓb1 +
189 ∫((ω4)*(ρ02*J*hp/(ρw*C))*(v*η) + (-ω2)*(ρ0*hp/ρw + ρ0*g*J/C)*(v*η) + (g)*(v*η))dΓb1 +
190 ∫((-im*ω)*(Dp*ρw/C)*(∇(v)·∇(φ)) + (-im*ω)*(1 - ρ0*J*ω2/C)*(v*φ))dΓb1 + ∫((im*ω)*w*η)dΓb1
191 ∫(Dp*Δ(v)*Δ(η))dΓb1 +
192 ∫((-ω2)*(ρ0*J/ρw + Dp*ρ0*hp/C)*(∇(v)·∇(η)))dΓb1 +
193 ∫((Dp*ρw*g/C + Tp)*(∇(v)·∇(η)))dΓb1 +
194 ∫((ω4)*(ρ02*J*hp/(ρw*C))*(v*η))dΓb1 +
195 ∫((-ω2)*(ρ0*hp/ρw + ρ0*g*J/C)*(v*η))dΓb1 +
196 ∫((g)*(v*η))dΓb1 +
197 ∫((-im*ω)*(Dp*ρw/C)*(∇(v)·∇(φ)))dΓb1 + ∫((-im*ω)*(1 - ρ0*J*ω2/C)*(v*φ))dΓb1
198 ∫((im*ω)*w*η)dΓb1
199
200 # a. Separated by the trial and test functions and frequency terms
201 k11(η,v) = ∫(Dp*Δ(v)*Δ(η))dΓb1 + ∫((Dp*ρw*g/C + Tp)*(∇(v)·∇(η)))dΓb1 + # ←
202 kw(η,v) = ∫((g)*(v*η))dΓb1 + # ←
203
204 m11(η,v) = ∫((-ω2)*(ρ0*J/ρw + Dp*ρ0*hp/C)*(∇(v)·∇(η)))dΓb1 +
205      ∫((-ω2)*(ρ0*hp/ρw + ρ0*g*J/C)*(v*η))dΓb1
206
207 s11(η,v) = ∫((ω4)*(ρ02*J*hp/(ρw*C))*(v*η))dΓb1
208
209 c12(φ,v) = ∫((-im*ω)*(Dp*ρw/C)*(∇(v)·∇(φ)))dΓb1 +
210      ∫((-im*ω)*(v*φ))dΓb1 +
211      ∫((-im*ω)*(-ρ0*J*ω2/C)*(v*φ))dΓb1
212
213 c21(η,w) = ∫((im*ω)*w*η)dΓb1
214
215 # b. Extract the frequency and/or wave number function
216 m11(η,v) = [-ω2] * ∫((ρ0*J/ρw + Dp*ρ0*hp/C)*(∇(v)·∇(η)))dΓb1 +
217      [-ω2] * ∫((ρ0*hp/ρw + ρ0*g*J/C)*(v*η))dΓb1

```

```

218 s11(η,v) = [ω^4] * ∫((ρ₀^2*J*h_p/(ρw*C))*(v*η))dΓb1
219
220
221 c12(φ,v) = [-im*ω] * ∫((Dρ*ρw/C)*(∇(v)·∇(φ)))dΓb1 +
222 [-im*ω] * ∫((v*φ))dΓb1 +
223 [-im*ω] * (-ω^2) * ∫((ρ₀*J/C)*(v*φ))dΓb1
224
225 c21(η,w) = [im*ω] * ∫(w*η)dΓb1
226
227 #=
228 From the matrix definition in Eq. (2.24), the frequency terms in the squared bracket [-]
229 will be taken out from the matrices definition. Only applicable for the following matrices:
230 M11, S11, C12, C21
231 =#
232
233 m11(η,v) = ∫((ρ₀*J*/ρw + Dρ*ρ₀*h_p/C)*(∇(v)·∇(η)))dΓb1 + # ←
234 ∫((ρ₀*h_p*/ρw + ρ₀*g*J/C)*(v*η))dΓb1
235
236 s11(η,v) = ∫((ρ₀^2*J*h_p/(ρw*C))*(v*η))dΓb1 # ←
237
238 c12(φ,v) = ∫((Dρ*ρw/C)*(∇(v)·∇(φ)))dΓb1 + ∫((v*φ))dΓb1 # ←
239
240 c12f2(φ,v) = ∫((ρ₀*J/C)*(v*φ))dΓb1 # ←
241
242 c21(η,w) = ∫(w*η)dΓb1 # ←
243
244 # Now include the second beam term. It has the same term with the beam 1 due to the same properties
245 # c. Final weak form matrices
246 k11(η,v) = ∫(Dρ*Δ(v)*Δ(η))dΓb1 + ∫((Dρ*ρw*g/C + Tρ)*(∇(v)·∇(η)))dΓb1 +
247 ∫(Dρ*Δ(v)*Δ(η))dΓb2 + ∫((Dρ*ρw*g/C + Tρ)*(∇(v)·∇(η)))dΓb2
248 kw(η,v) = ∫((g)*(v*η))dΓb1 + ∫((g)*(v*η))dΓb2
249 m11(η,v) = ∫((ρ₀*J*/ρw + Dρ*ρ₀*h_p/C)*(∇(v)·∇(η)))dΓb1 +
250 ∫((ρ₀*h_p*/ρw + ρ₀*g*J/C)*(v*η))dΓb1 +
251 ∫((ρ₀*J*/ρw + Dρ*ρ₀*h_p/C)*(∇(v)·∇(η)))dΓb2 +
252 ∫((ρ₀*h_p*/ρw + ρ₀*g*J/C)*(v*η))dΓb2
253 s11(η,v) = ∫((ρ₀^2*J*h_p/(ρw*C))*(v*η))dΓb1 + ∫((ρ₀^2*J*h_p/(ρw*C))*(v*η))dΓb2
254 c12(φ,v) = ∫((Dρ*ρw/C)*(∇(v)·∇(φ)))dΓb1 + ∫((v*φ))dΓb1 +
255 ∫((Dρ*ρw/C)*(∇(v)·∇(φ)))dΓb2 + ∫((v*φ))dΓb2
256 c12f2(φ,v) = ∫((ρ₀*J/C)*(v*φ))dΓb1 + ∫((ρ₀*J/C)*(v*φ))dΓb1
257 c21(η,w) = ∫(w*η)dΓb1 + ∫(w*η)dΓb1
258
259 # -----
260 # [5] Forcing term
261 # -----
262 ∫( w*v_i_n )dΓin - ∫( ηd*w - ∇_nφd*(u + α_n*w) )dΓd1
263 ∫( w*v_i_n )dΓin - ∫( ηd*w - ∇_nφd*u - ∇_nφd*α_n*w )dΓd1
264
265 fη(v) = ∫( θ*v )dΓb1 + ∫( θ*v )dΓb2
266 fφ(w) = ∫( w*v_i_n )dΓin - ∫( ηd*w - ∇_nφd*(α_n*w) )dΓd1
267 fκ(u) = ∫( ∇_nφd*(u)*(1) )dΓd1
268
269
270 # =====
271 # [C] Collect all terms
272 # Weak matrices formulation
273 # =====
274 m11(η,v) = ∫((ρ₀*J*/ρw + Dρ*ρ₀*h_p/C)*(∇(v)·∇(η)))dΓb1 +
275 ∫((ρ₀*h_p*/ρw + ρ₀*g*J/C)*(v*η))dΓb1 +
276 ∫((ρ₀*J*/ρw + Dρ*ρ₀*h_p/C)*(∇(v)·∇(η)))dΓb2 +
277 ∫((ρ₀*h_p*/ρw + ρ₀*g*J/C)*(v*η))dΓb2
278 s11(η,v) = ∫((ρ₀^2*J*h_p/(ρw*C))*(v*η))dΓb1 + ∫((ρ₀^2*J*h_p/(ρw*C))*(v*η))dΓb2

```

```

279 k11d(η,v) = ∫(Dρ*Δ(v)*Δ(η))dΓb1 + ∫( (Dρ*ρw*g/C + Tρ)*(∇(v)·∇(η)) )dΓb1 +
280 ∫(Dρ*Δ(v)*Δ(η))dΓb2 + ∫( (Dρ*ρw*g/C + Tρ)*(∇(v)·∇(η)) )dΓb2
281 ∫((Dρ) * (- jump(∇(v)·nΛb1) * mean(Δ(η)) - mean(Δ(v)) * jump(∇(η)·nΛb1) # ←
282 + γ*( jump(∇(v)·nΛb1) * jump(∇(η)·nΛb1) ) ) )dΛb1 +
283 ∫((Dρ) * (- jump(∇(v)·nΛb2) * mean(Δ(η)) - mean(Δ(v)) * jump(∇(η)·nΛb2)
284 + γ*( jump(∇(v)·nΛb2) * jump(∇(η)·nΛb2) ) ) )dΛb2 +
285 ∫((mean(v) * ks2ρ * mean(η)) + ( jump(∇(v)·nΛj1) * kr2ρ * jump(∇(η)·nΛj1)))dΛj1 +
286 ∫((ksp*(v*η) + krρ*(∇(v)·∇(η))))dΓl +
287 ∫((ksp*(v*η) + krρ*(∇(v)·∇(η))))dΓr
288 k11w(η,v) = ∫(Dρ*Δ(v)*Δ(η))dΓb1 + ∫( (Dρ*ρw*g/C + Tρ)*(∇(v)·∇(η)) )dΓb1 +
289 ∫(Dρ*Δ(v)*Δ(η))dΓb2 + ∫( (Dρ*ρw*g/C + Tρ)*(∇(v)·∇(η)) )dΓb2
290 ∫((Dρ) * (- jump(∇(v)·nΛb1) * mean(Δ(η)) - mean(Δ(v)) * jump(∇(η)·nΛb1) # ←
291 + γ*( jump(∇(v)·nΛb1) * jump(∇(η)·nΛb1) ) ) )dΛb1 +
292 ∫((Dρ) * (- jump(∇(v)·nΛb2) * mean(Δ(η)) - mean(Δ(v)) * jump(∇(η)·nΛb2)
293 + γ*( jump(∇(v)·nΛb2) * jump(∇(η)·nΛb2) ) ) )dΛb2 +
294 ∫((mean(v) * ks2ρ * mean(η)) + ( jump(∇(v)·nΛj1) * kr2ρ * jump(∇(η)·nΛj1)))dΛj1 +
295 ∫((ksp*(v*η) + krρ*(∇(v)·∇(η))))dΓl +
296 ∫((ksp*(v*η) + krρ*(∇(v)·∇(η))))dΓr +
297 ∫((v*g*η))dΓb1 + ∫((v*g*η))dΓb2
298
299 c12(φ,v) = ∫((Dρ*ρw/C)*(∇(v)·∇(φ)))dΓb1 + ∫((v*φ))dΓb1 +
300 ∫((Dρ*ρw/C)*(∇(v)·∇(φ)))dΓb2 + ∫((v*φ))dΓb2
301 c12f2(φ,v) = ∫((ρθ*J/C)*(v*φ))dΓb1 + ∫((ρθ*J/C)*(v*φ))dΓb1
302 c21(η,w) = ∫(w*η)dΓb1 + ∫(w*η)dΓb1
303
304 k22(φ,w) = ∫(∇(w)·∇(φ))dΩ
305 k22f1(φ,w) = ∫(μin*αf*(∇n(φ)*w))dΓd1 + ∫(μout*αf*(∇n(φ)*w))dΓd2
306 k22f2(φ,w) = ∫( βh*αf*(φ*w) )dΓfs + ∫(βh*αf*(φ*w))dΓd1 + ∫(βh*αf*(φ*w))dΓd2
307
308 c23f(κ,w) = ∫(βh*αf*g*(κ*w) + (-1)*(κ*w))dΓfs +
309 ∫(βh*αf*g*(κ*w) + (-1)*(κ*w))dΓd1 + ∫(βh*αf*g*(κ*w) + (-1)*(κ*w))dΓd2
310 c23k(κ,w) = ∫(μin(x)*(κ*w)*(-1))dΓd1 + ∫(μout(x)*(κ*w)*(-1))dΓd2
311 c32(φ,u) = ∫(μin*∇n(φ)*u))dΓd1 + ∫(μout*∇n(φ)*u))dΓd2
312 c32f(φ,u) = ∫(βh*∇(φ)*u))dΓfs + ∫(βh*∇(φ)*u))dΓd1 + ∫(βh*∇(φ)*u))dΓd2
313
314 k33(κ,u) = ∫(βh*g*(κ*u))dΓfs + ∫(βh*g*(κ*u))dΓd1 + ∫(βh*g*(κ*u))dΓd2
315
316 fη(v) = ∫( θ*v )dΓb1 + ∫( θ*v )dΓb2
317 fφ(w) = ∫( w*vin )dΓin - ∫( ηd*w - ∇nφd*(αh*w) )dΓd1
318 fκ(u) = ∫( ∇nφd*(u)*(1))dΓd1

```

B

Modal Superposition Output

This appendix provides the plots of modal superposition for all cases, including the matrix and integration methods.

- B.1 Case 1: Free-end floating beam in high frequency
- B.2 Case 2: Free-end floating beam in low frequency
- B.3 Case 3: Structural with mooring attachment
- B.4 Case 4: Multi-body structure with flexible joint connection
- B.5 Case 5: Floating membrane

B.1. Case 1: Free-End Floating Beam in High Frequency

B.1.1. Matrix Method

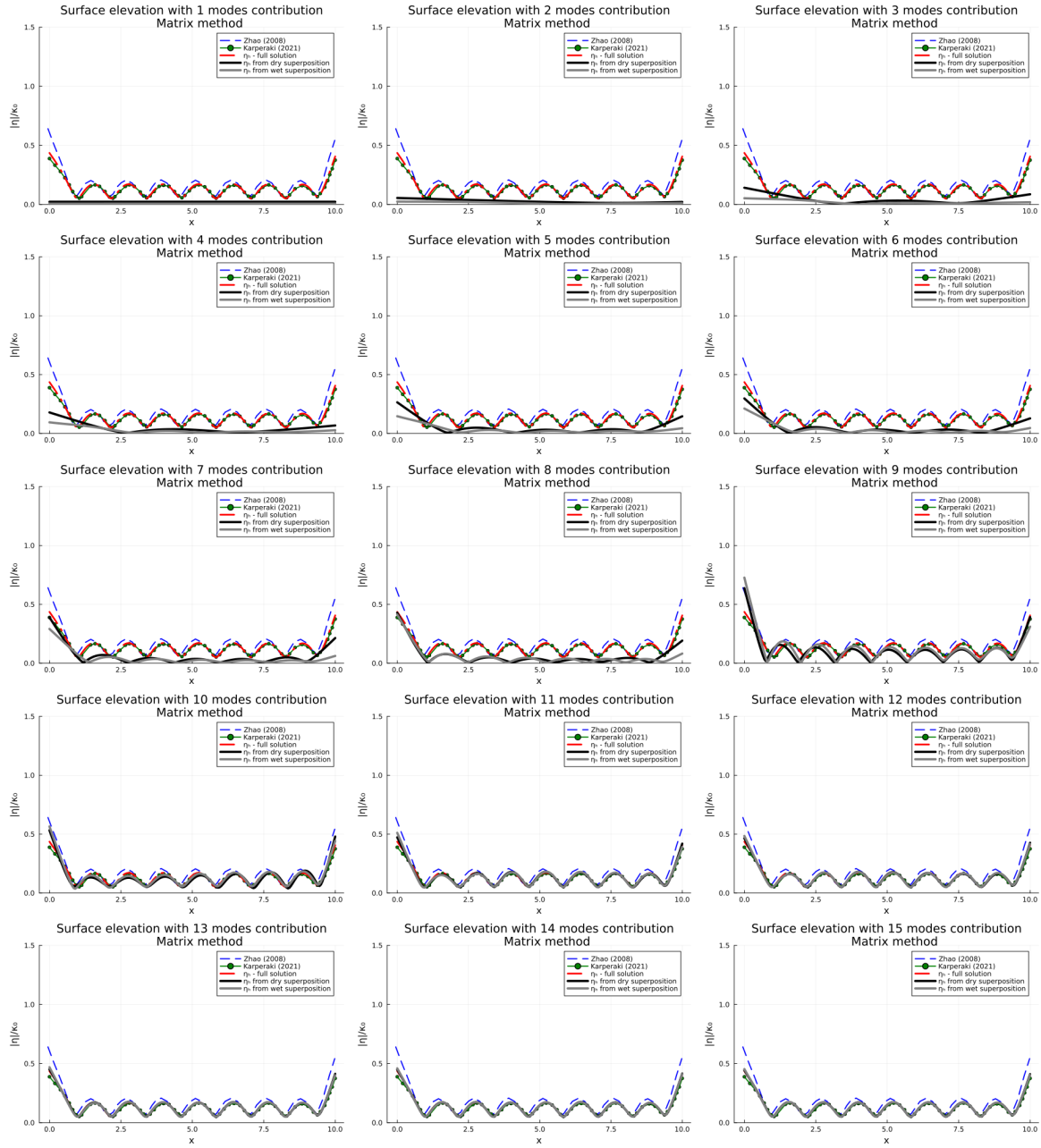


Figure B.1: Matrix Method

B.1.2. Integration Method

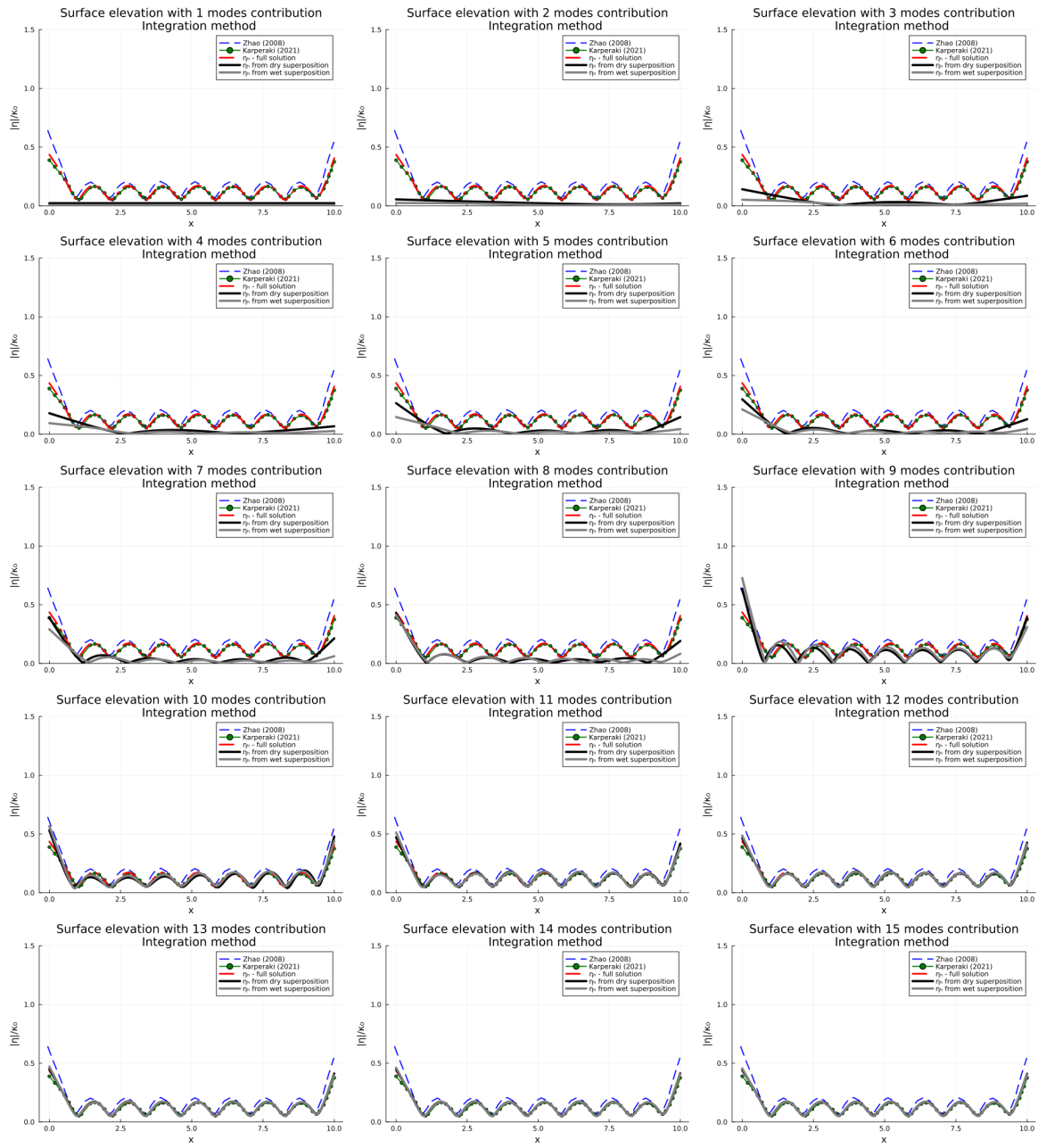


Figure B.2: Integration Method

B.2. Case 2: Free-End Floating Beam in Low Frequency

B.2.1. Matrix Method

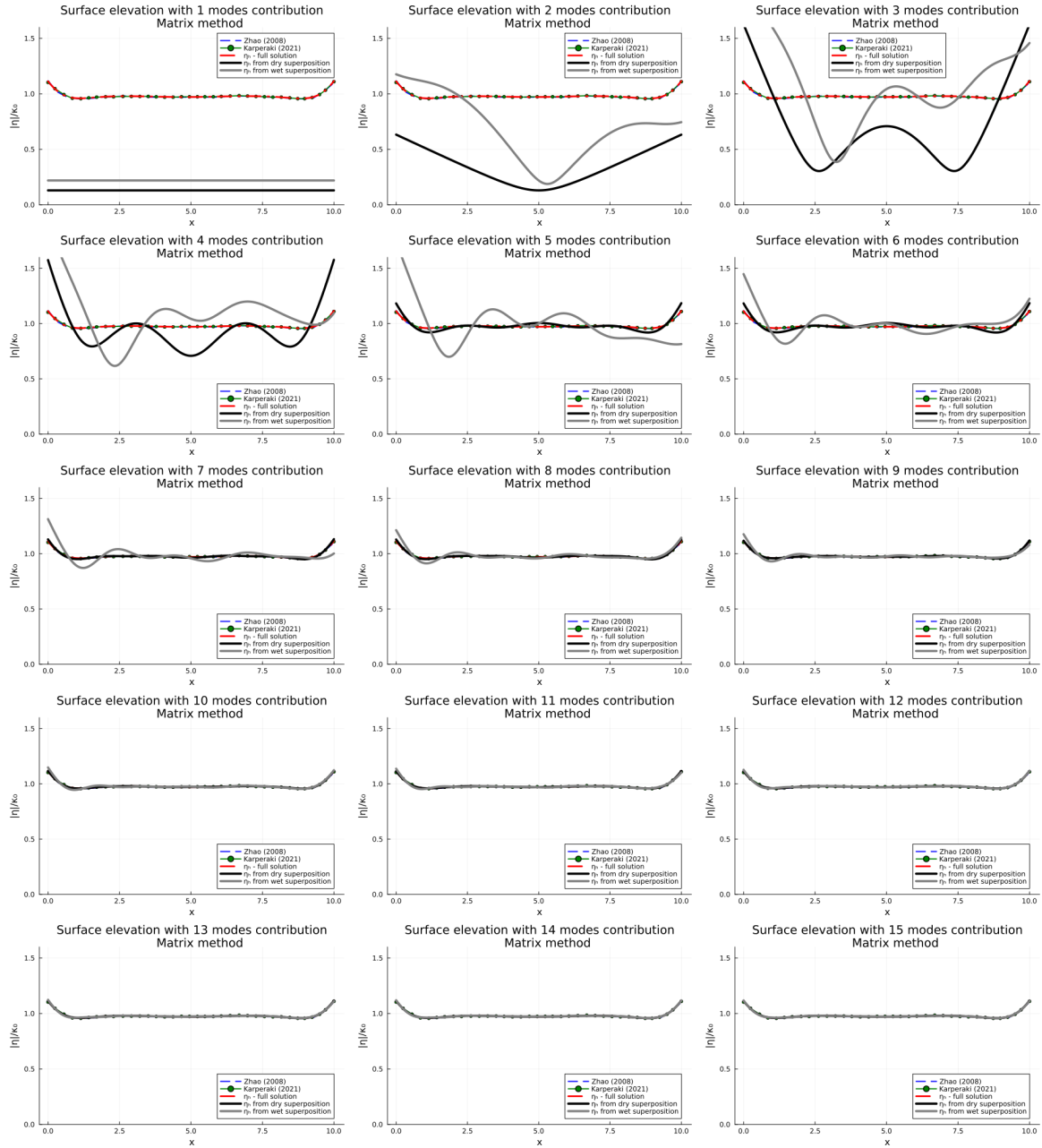


Figure B.3: Matrix Method

B.2.2. Integration Method

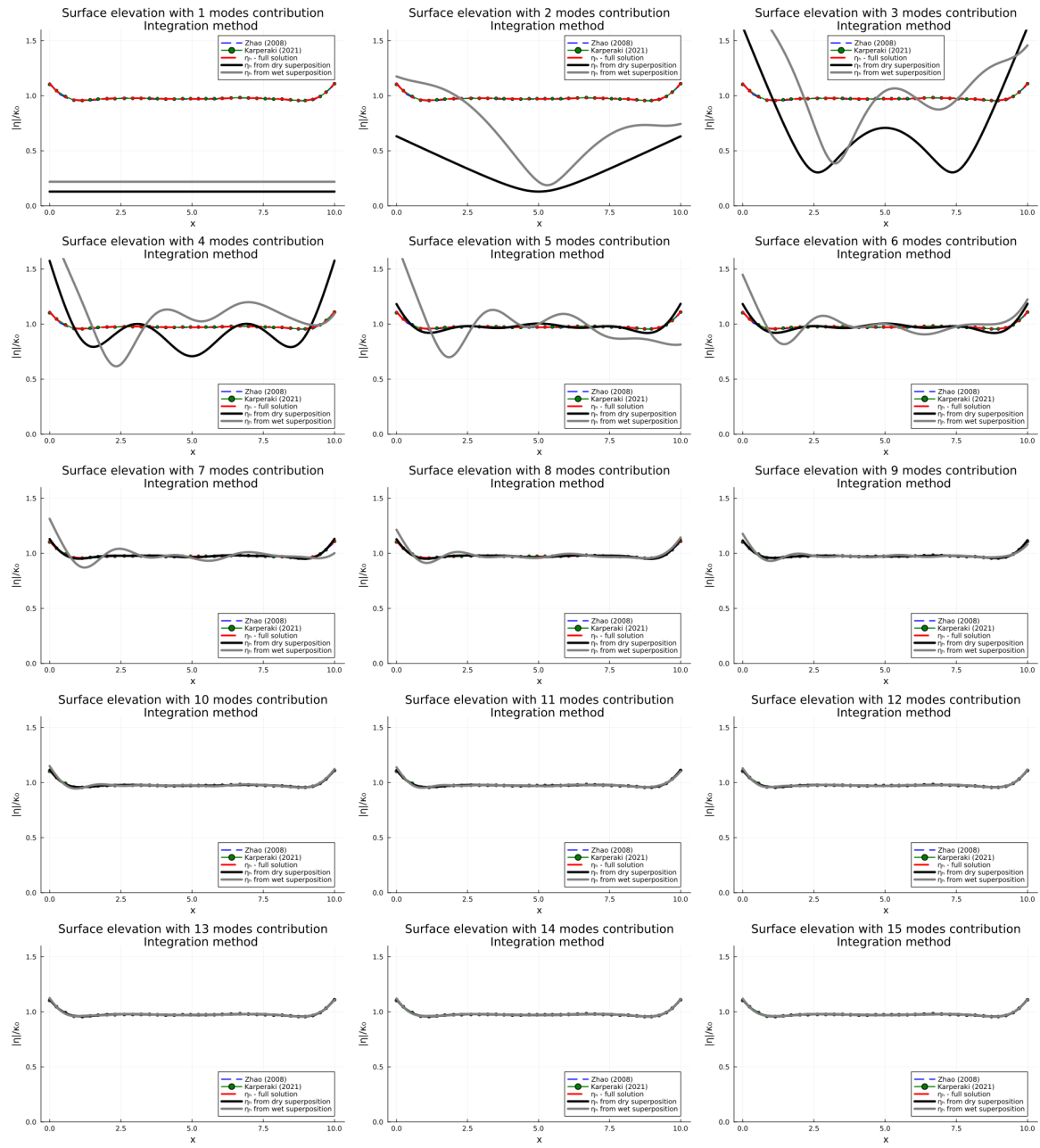


Figure B.4: Integration Method

B.3. Case 3: Structural with Mooring Attachment

B.3.1. Matrix Method

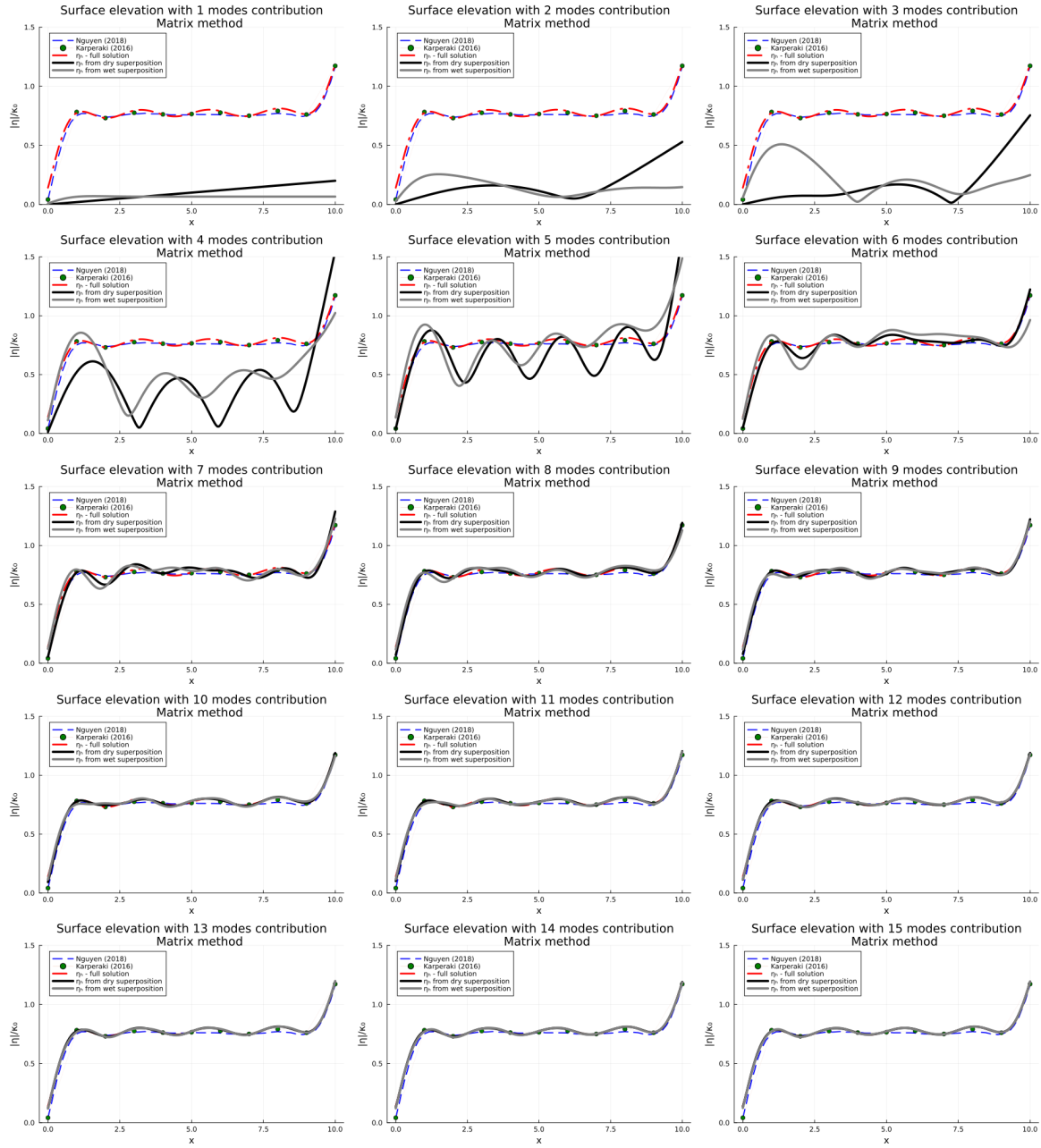


Figure B.5: Matrix Method

B.3.2. Integration Method

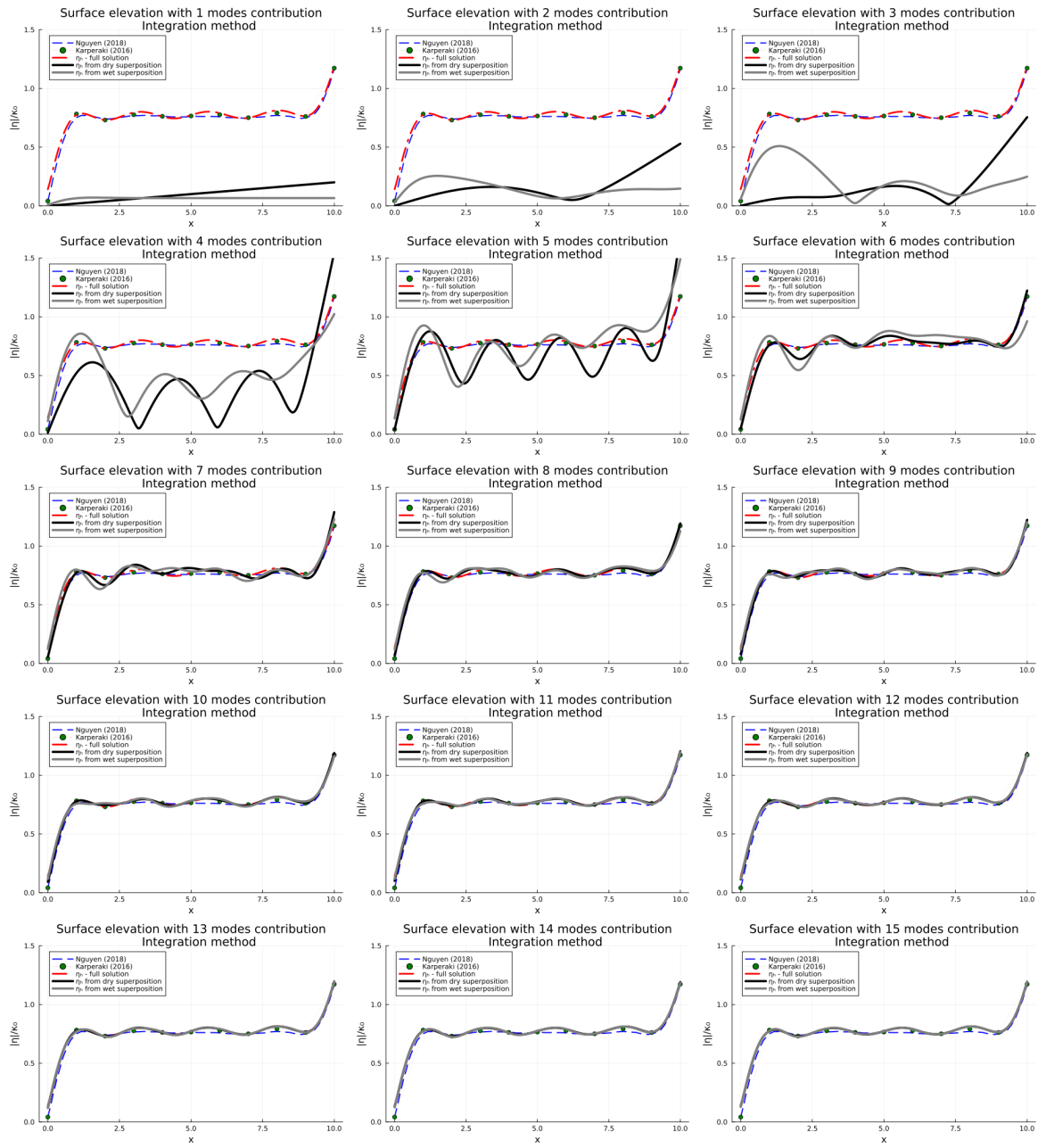


Figure B.6: Integration Method

B.4. Case 4: Multi-Body Structure with Flexible Joint Connection

B.4.1. Matrix Method

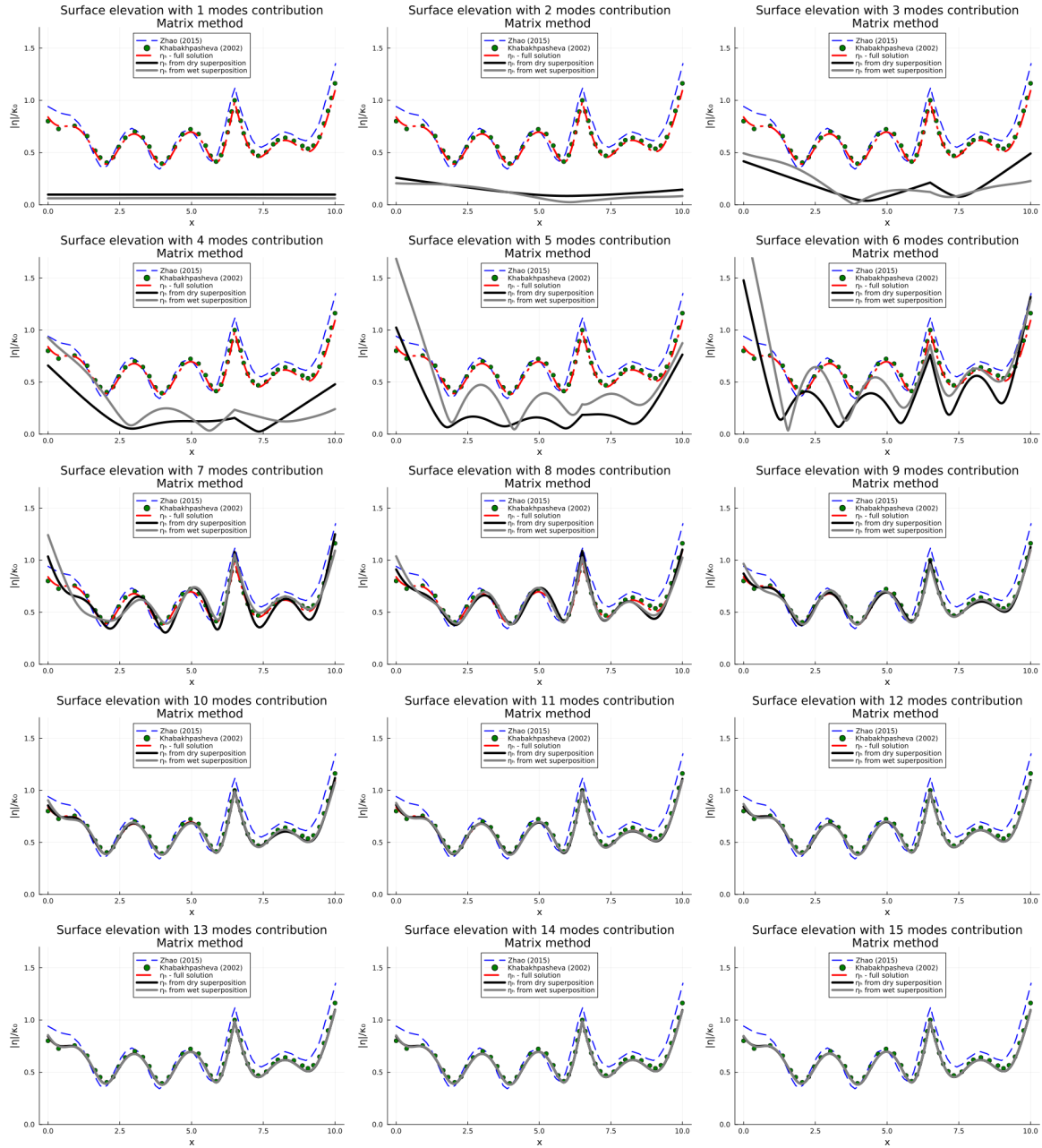


Figure B.7: Matrix Method

B.4.2. Integration Method

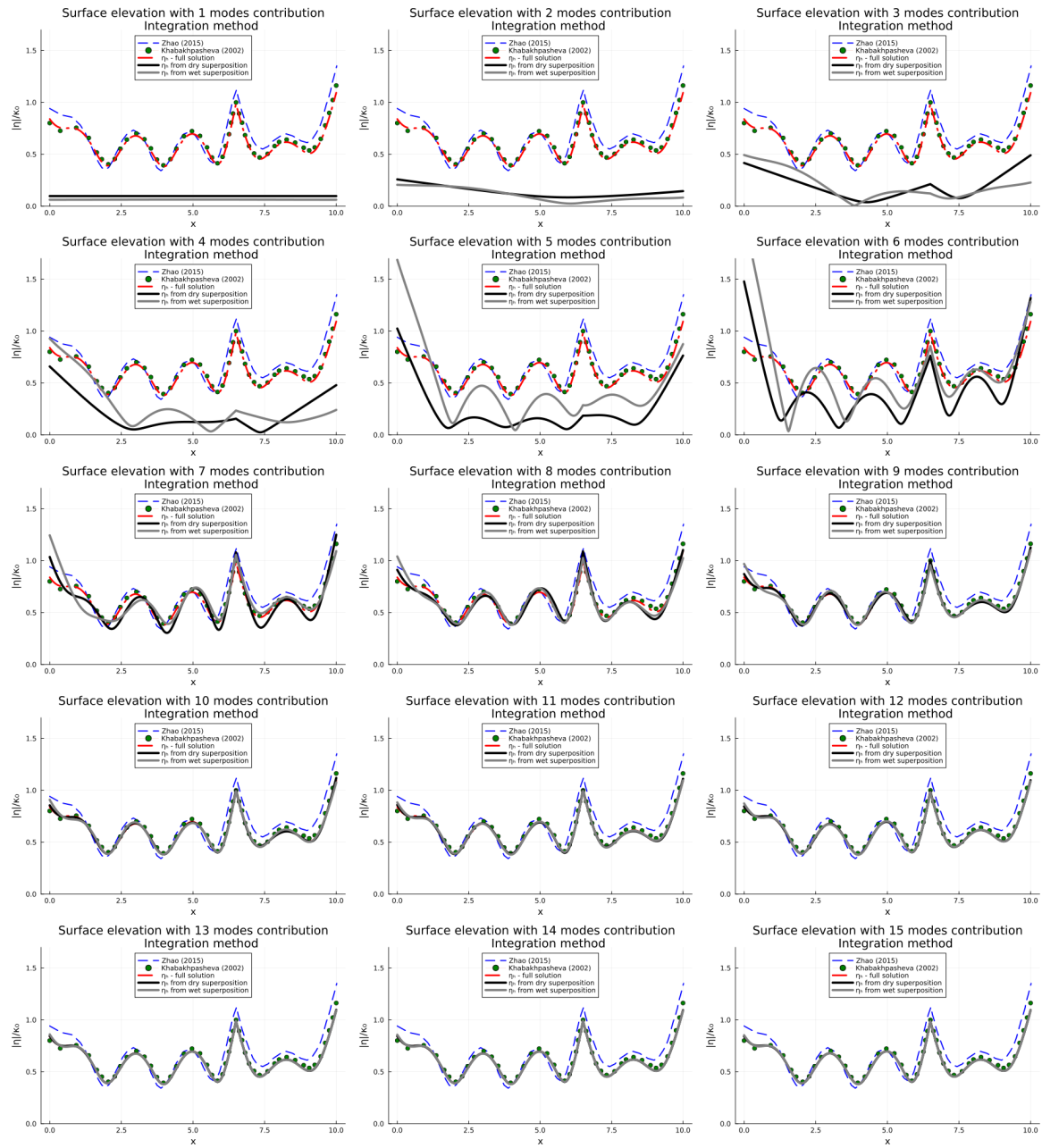


Figure B.8: Integration Method

B.5. Case 5: Floating Membrane

B.5.1. Matrix Method

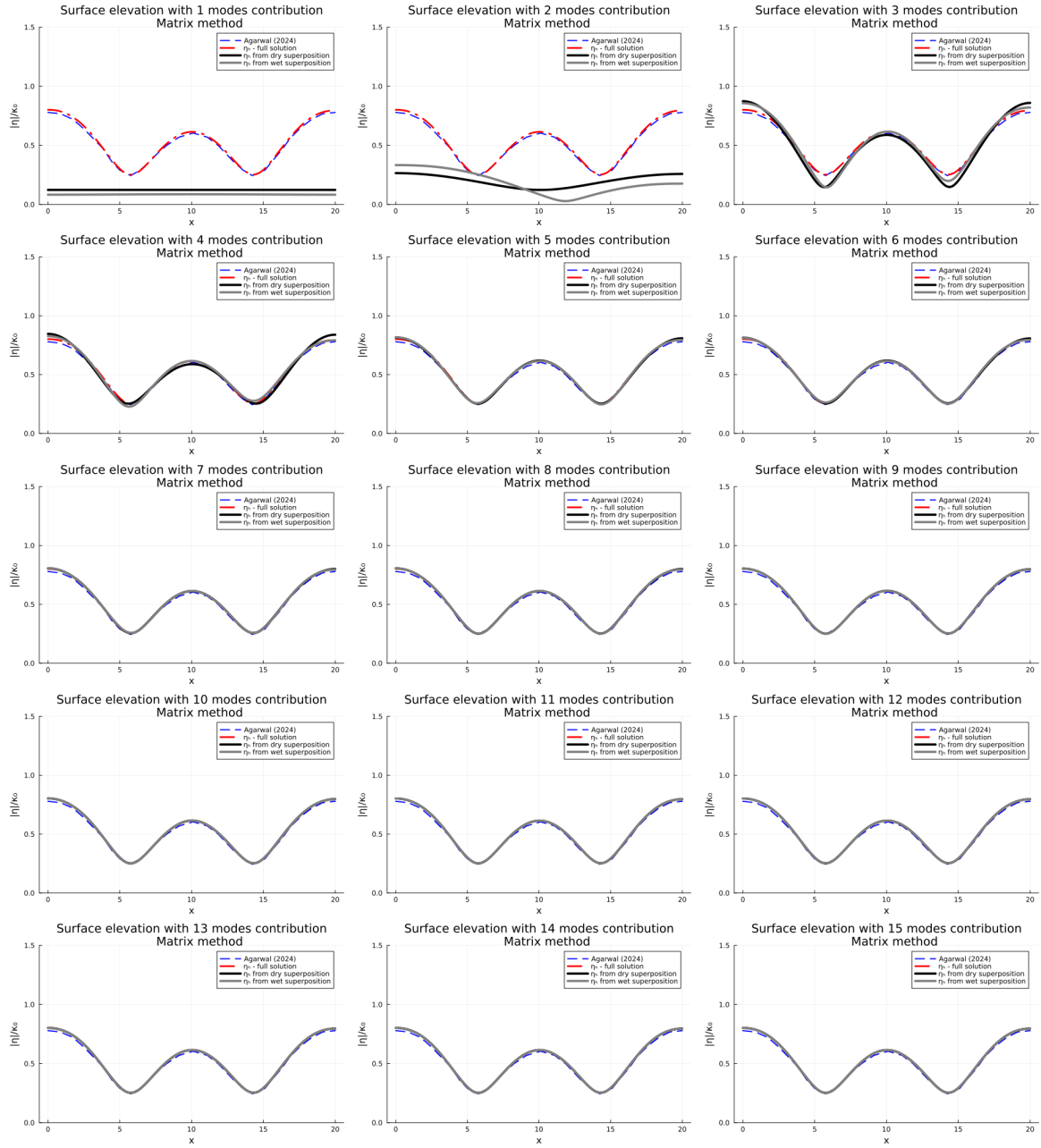


Figure B.9: Matrix Method

B.5.2. Integration Method

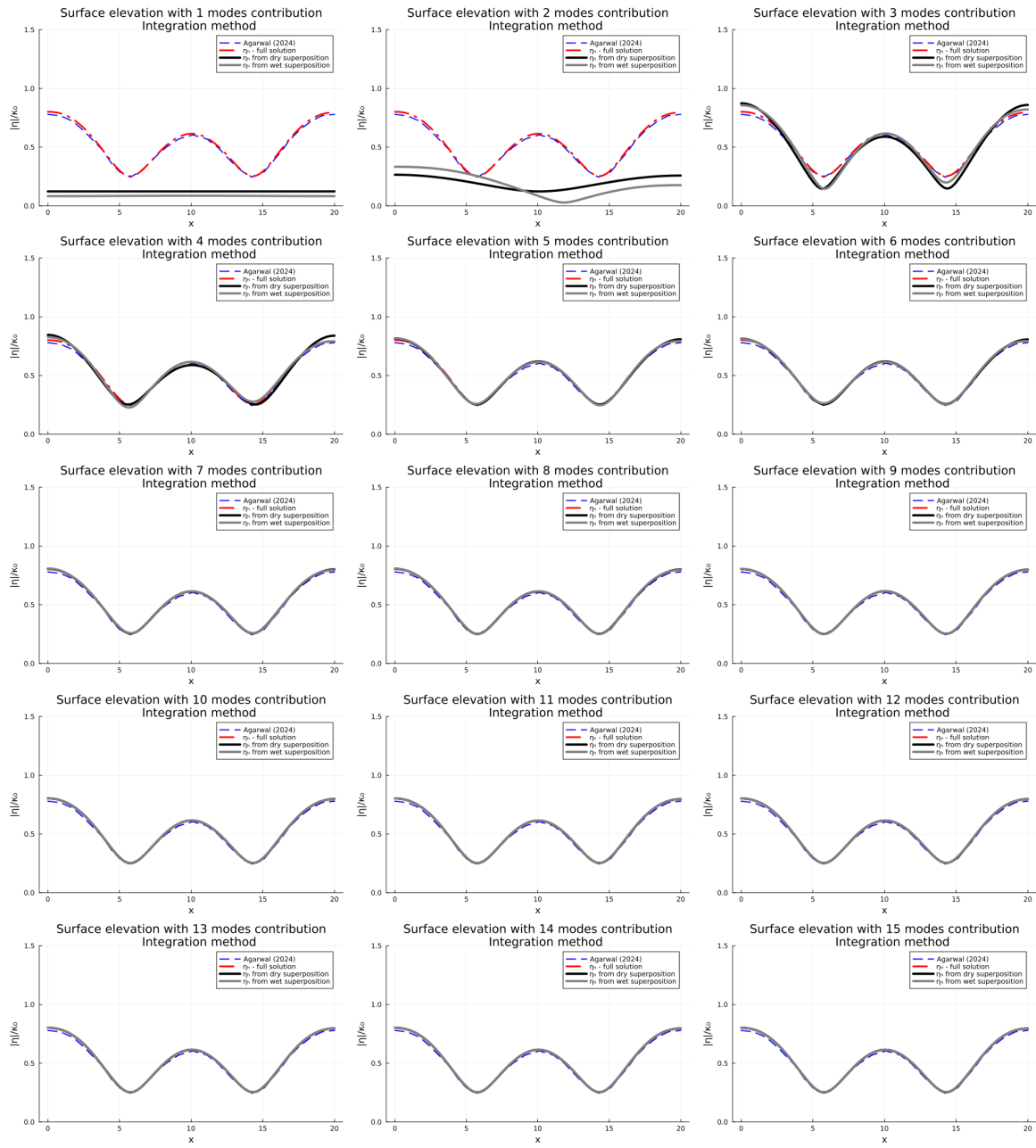


Figure B.10: Integration Method

C

Added Mass and Added Damping Contribution

This appendix provides added mass and added damping contribution plots for Case 1 (Section 4.1) and Case 5 (Section 4.5).

- C.1 Case 1: Free-end floating beam in high frequency
- C.2 Case 5: Floating membrane

C.1. Case 1

C.1.1. Contribution for dry mode

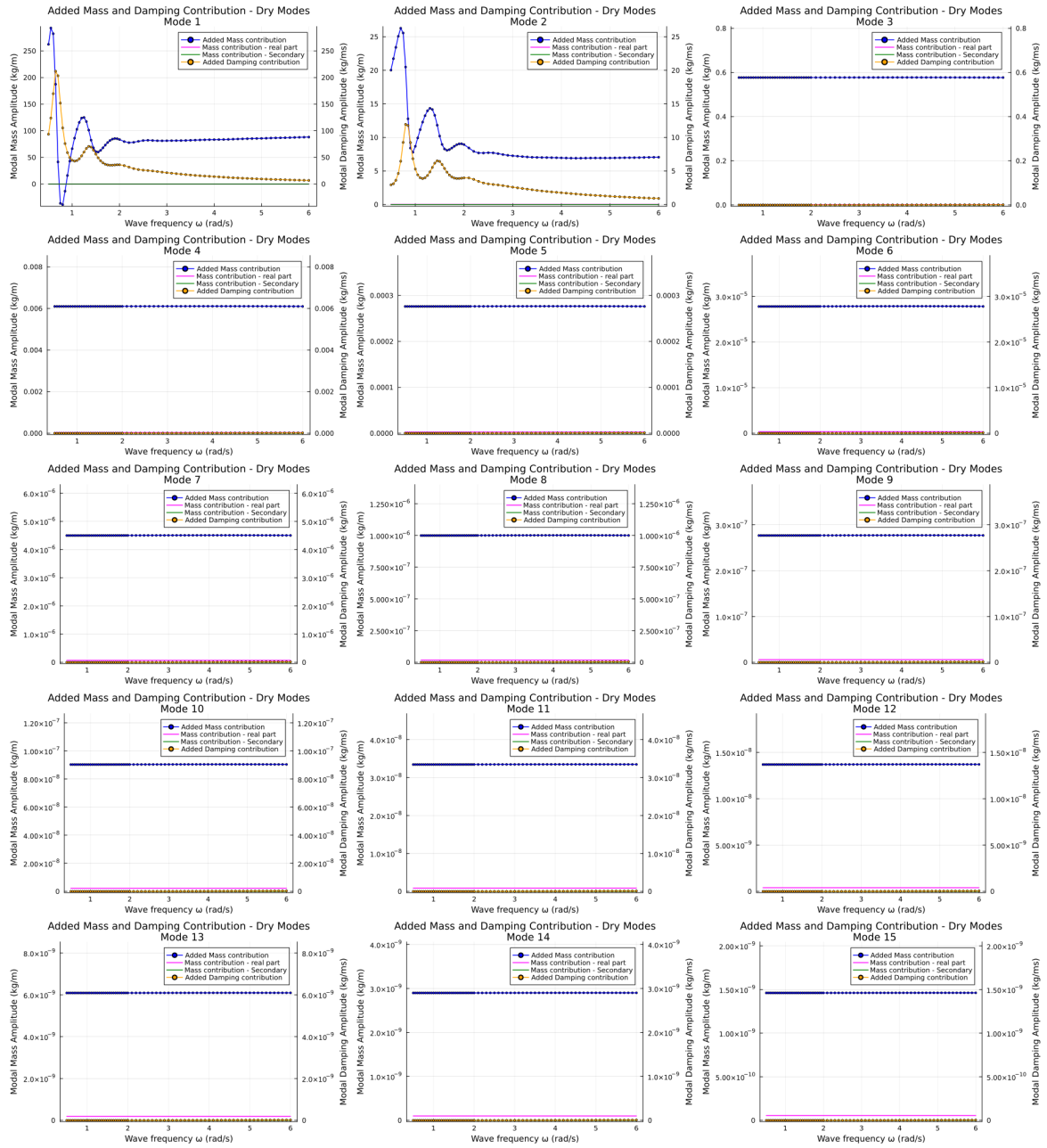


Figure C.1: Added mass and damping contribution - Dry modes

C.1.2. Contribution for wet mode

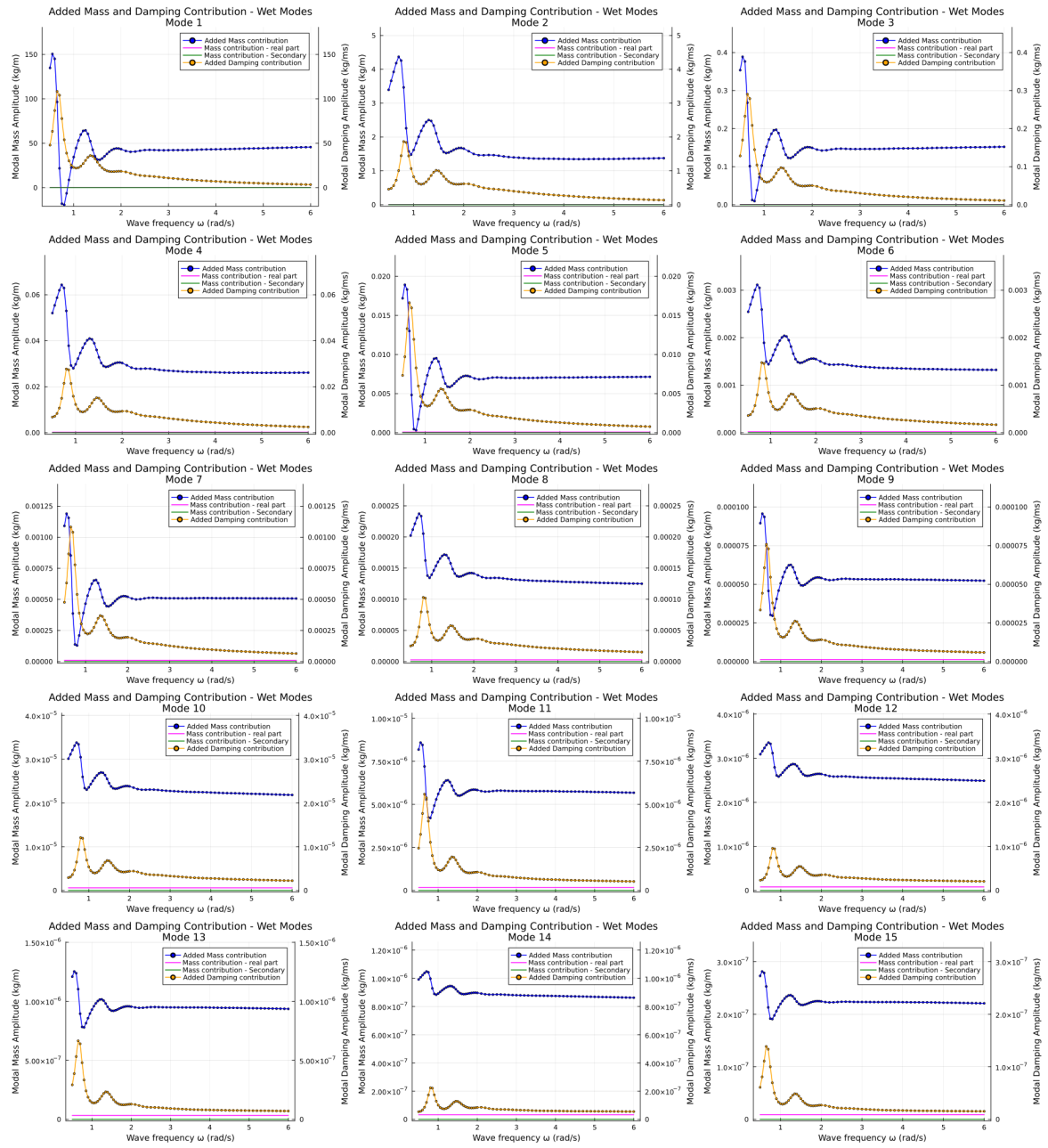


Figure C.2: Added mass and damping contribution - Wet modes

C.1.3. Modal force RAO

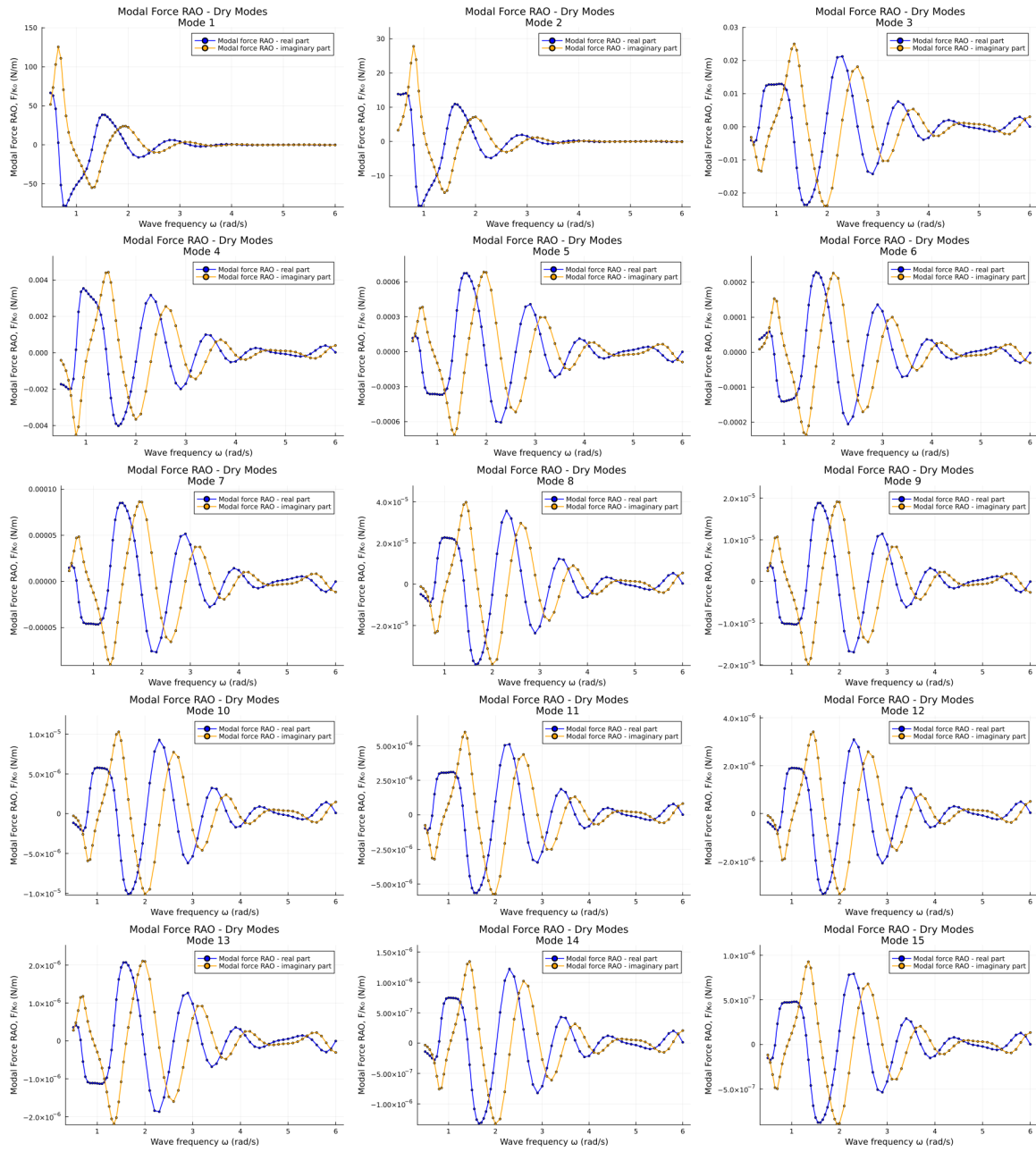


Figure C.3: Modal force RAO - Case 1 - Dry modes

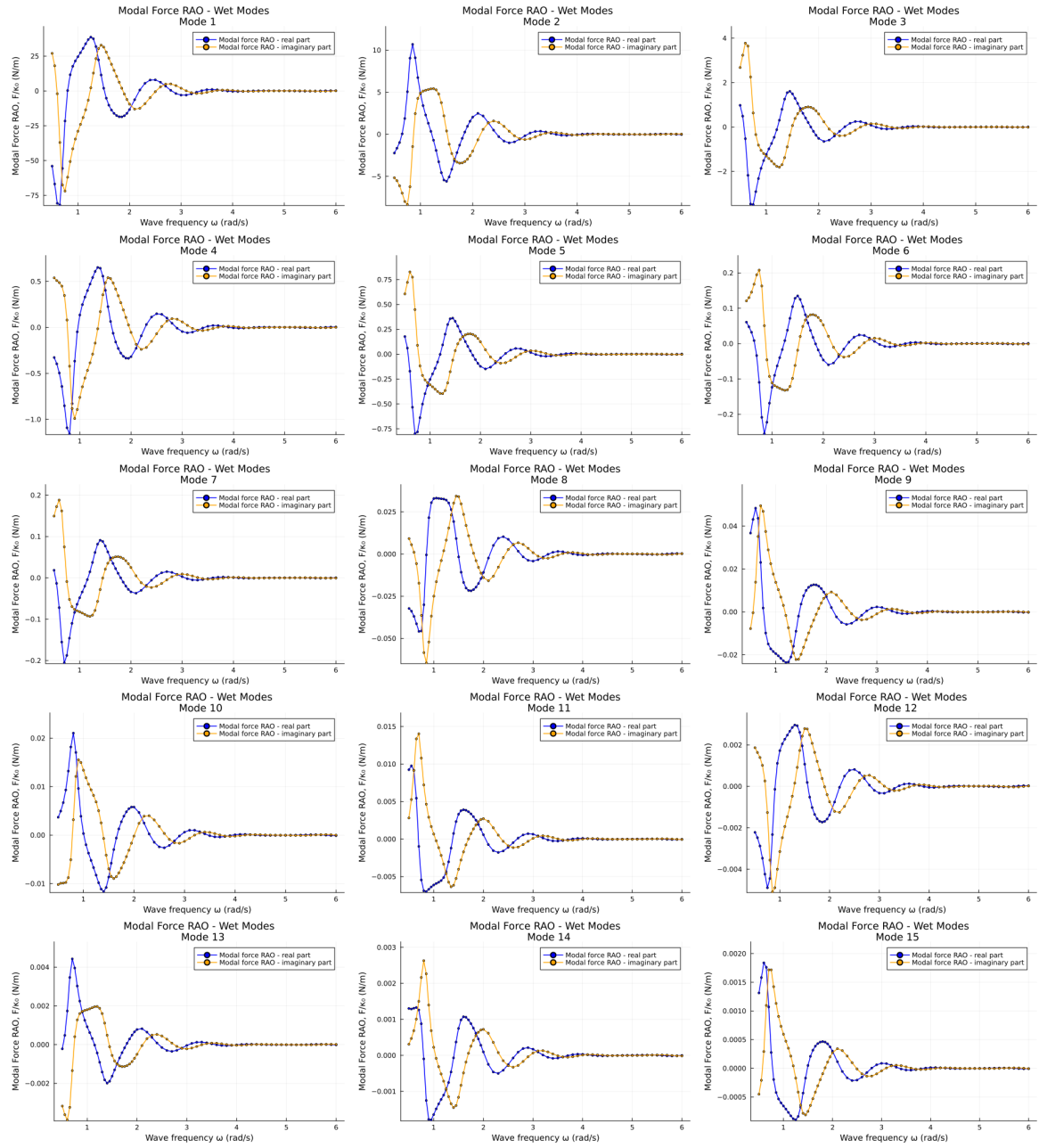


Figure C.4: Modal force RAO - Case 1 - Wet modes

C.2. Case 5

C.2.1. Contribution for dry mode

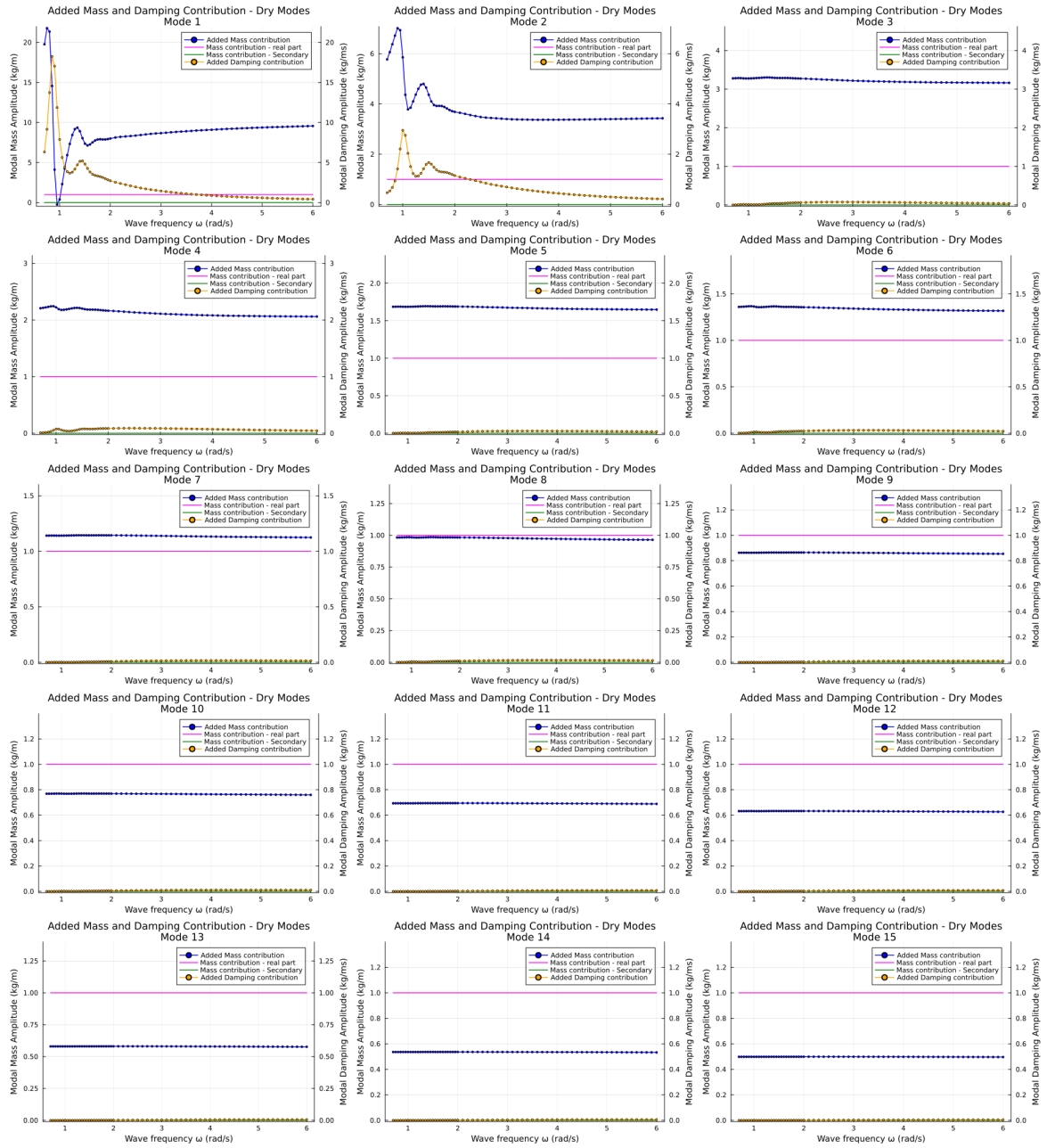


Figure C.5: Added mass and damping contribution - Dry modes

C.2.2. Contribution for wet mode

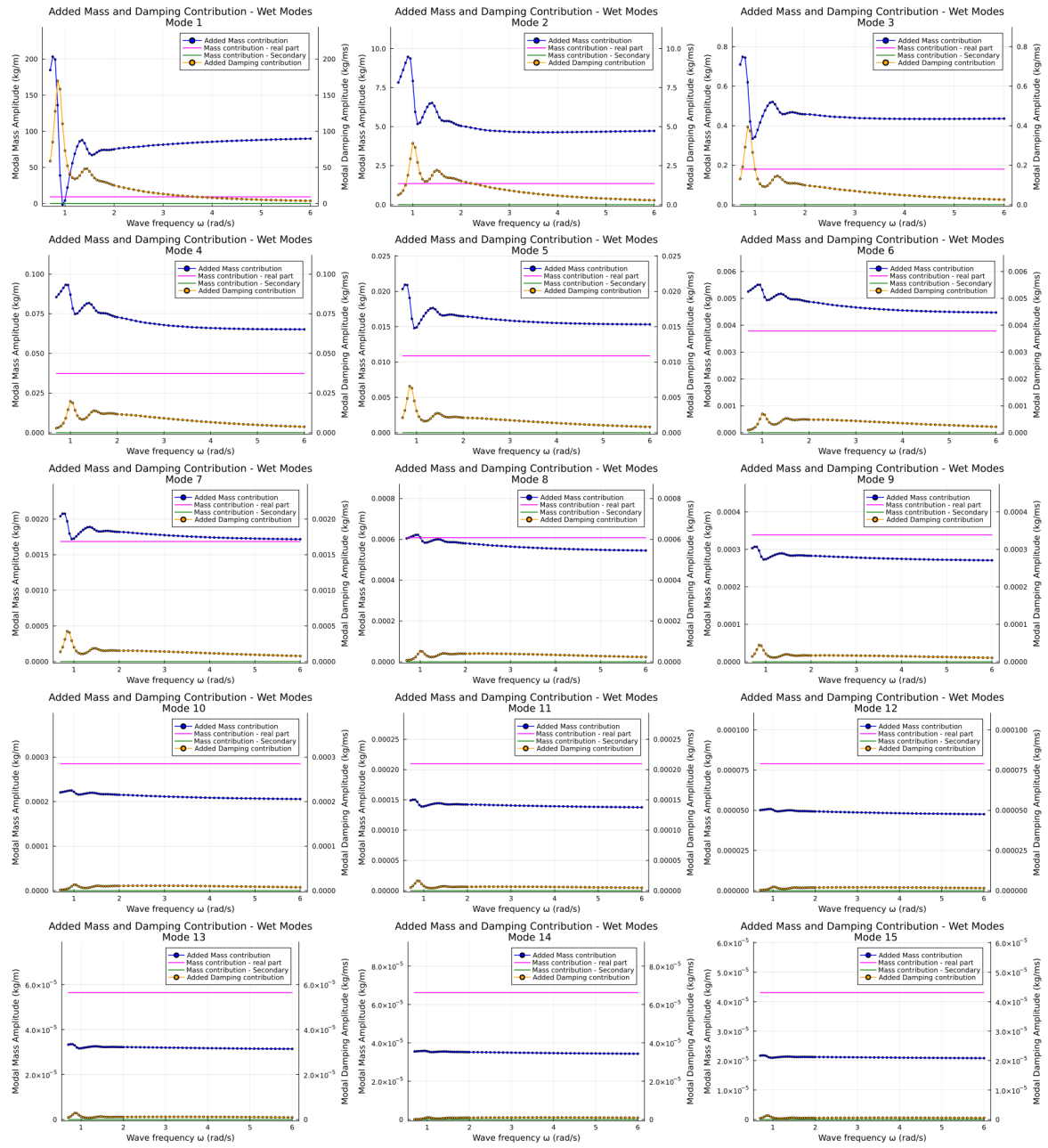


Figure C.6: Added mass and damping contribution - Wet modes

C.2.3. Modal force RAO

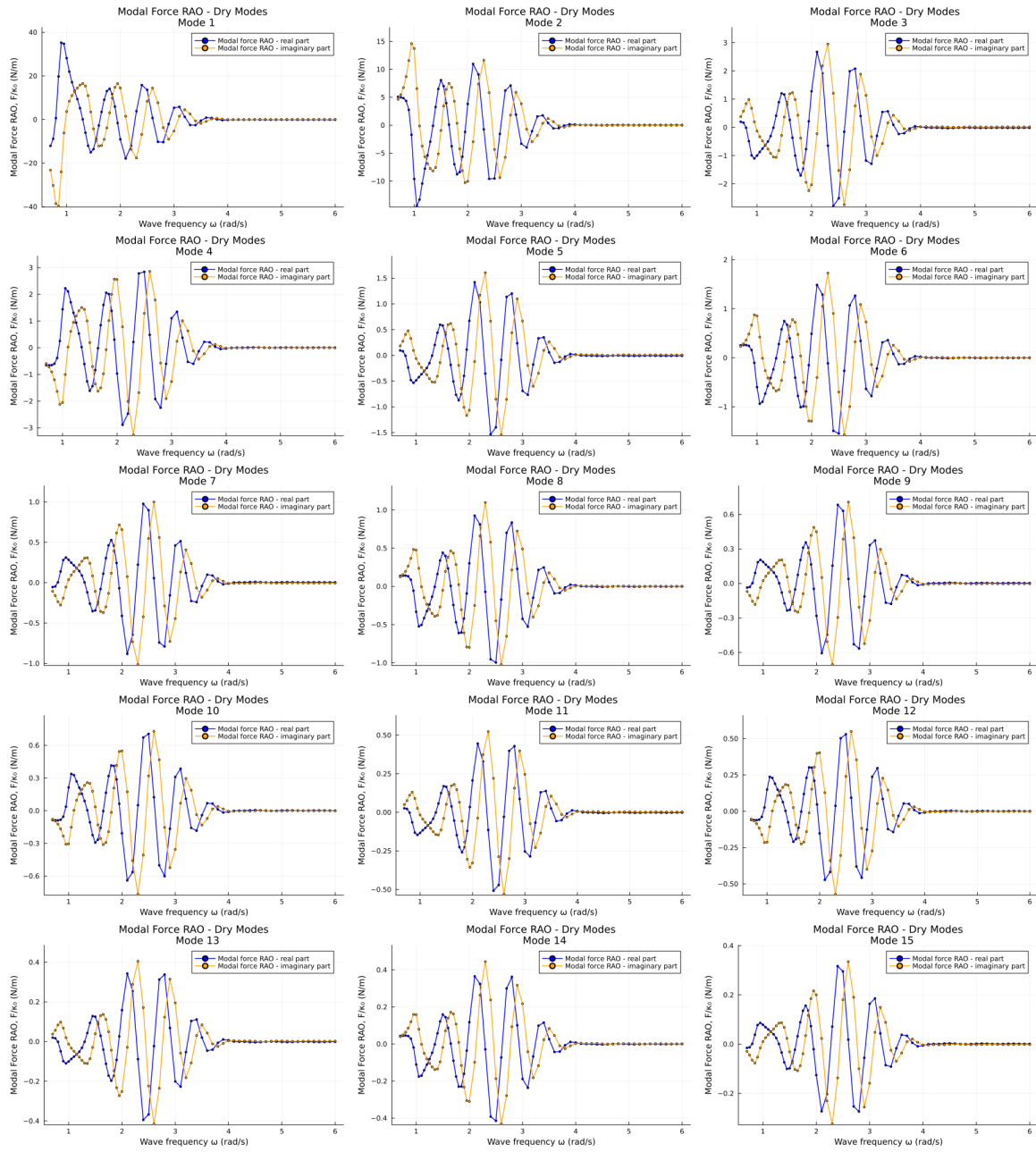


Figure C.7: Modal force RAO - Case 5 - Dry modes

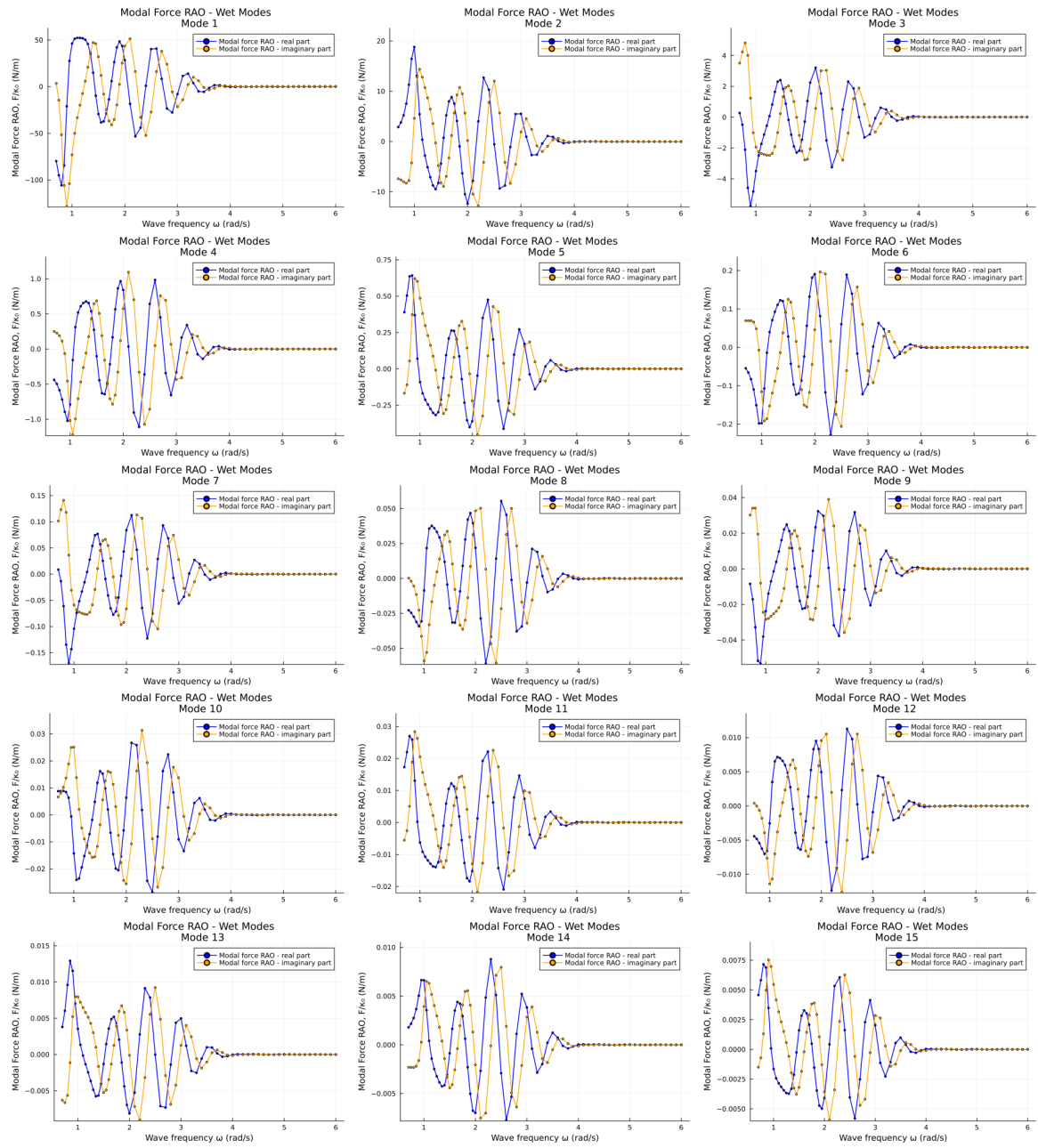


Figure C.8: Modal force RAO - Case 5 - Wet modes

