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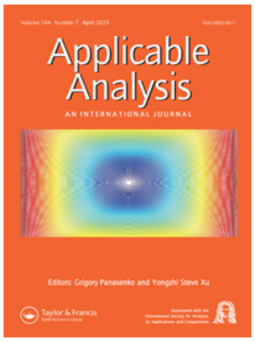

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Parameter-dependent fractional boundary value problems: analysis and approximation of solutions

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ABSTRACT

We study a parameter-dependent non-linear fractional differential equation, subject to Dirichlet boundary conditions. Using the fixed point theory, we restrict the parameter values to secure the existence and uniqueness of solutions, and analyse the monotonicity behaviour of the solutions. Additionally, we apply a numerical-analytic technique, coupled with the lower and upper solutions method, to construct a sequence of approximations to the boundary value problem and give conditions for its monotonicity. The theoretical results are confirmed by an example of the Antarctic Circumpolar Current equation in the fractional setting.

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1. Introduction

Fractional boundary value problems (FBVPs) have gained considerable popularity due to recent developments in the theory of fractional calculus, as well as their wide range of applications in mathematics, engineering and the natural sciences. Fractional operators give models an extra degree of freedom in the form of the fractional derivative order, and are able to capture memory and non-local effects, which are exhibited by many processes, see [1–5] and the references therein. Some applications of fractional differential equations (FDEs) include viscoelasticity [6, 7], porous media flow [8–10] and anomalous diffusion [11, 12].

Modelling complex real-world phenomena requires the use of non-linear FDEs, whose exact solutions are unavailable, which has prompted the development of approximation techniques. One such technique is the monotone iterative method, which has been combined with the lower and upper solutions method to establish the existence and uniqueness of solutions for fractional initial and boundary value problems, and to construct monotone sequences of approximations to their solutions [13–16]. Another method for approximating solutions of FBVPs is the numerical-analytic technique, which has been applied to FDEs, subject to various boundary conditions [17–20]. In this work, we show how combining the numerical-analytic technique with the lower and upper solutions method can improve the efficiency of the former.

We consider a parameter-dependent non-linear FDE of the Caputo type, subject to Dirichlet boundary conditions, where the parameter value controls the effect of the non-linear term and determines the monotonicity of the right-hand side function in the equation. The present paper consists of

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six sections. Section 2 contains the definitions and preliminary results, which will be used throughout the paper. In Section 3 we apply fixed point theory to determine the range of parameter values, which guarantees the existence and uniqueness of solutions to the studied problem. A sequence of approximate solutions is constructed using the numerical-analytic technique and its monotonicity behaviour is analysed. In particular, we show that for an FDE with a decreasing right-hand side function, the numerical-analytic technique produces a monotone sequence. In Section 4 we apply the lower and upper solutions method to the case when the right-hand side function is increasing to construct an alternating sequence of approximations. We demonstrate how the lower and upper solutions method can be used in this case to simplify the form of the sequence, resulting from the numerical-analytic technique and to thereby reduce the computational time. Our results are applied to a fractional order problem, which in the case of the second derivative models the Antarctic Circumpolar Current in Section 5. Section 6 presents a summary of our conclusions.

2. Definitions and auxiliary statements

Throughout this paper we use the following definitions and preliminary results:

Definition 2.1: Let $n-1<p<n$ for some $n\in\mathbb{Z}_+$. Then the Riemann–Liouville fractional integral of order p is given by (see [1], Def. 2.88)

$$ {}_aI_t^p f(t) := \frac{1}{\Gamma(p)}\int_a^t (t-s)^{p-1} f(s)\,\mathrm{d}s. $$

Definition 2.2: Let $n-1<p<n$ for some $n\in\mathbb{Z}_+$ and $f(t):(0,\infty)\to\mathbb{R}$. Then the left Caputo fractional derivative of $f(t)$ of order p is given by

$$ {}_a^CD_t^p f(t) := \frac{1}{\Gamma(n-p)}\int_a^t (t-s)^{n-p-1} f^{(n)}(s)\,\mathrm{d}s. \tag{1} $$

When $p=n$, (1) reduces to the ordinary derivative of order n (see [1], Def. 2.138). When $p\in(1,2)$, as in (6), the Caputo derivative reads

$$ {}_a^CD_t^p f(t) := \frac{1}{\Gamma(2-p)}\int_a^t (t-s)^{1-p} f''(s)\,\mathrm{d}s. $$

The following Lemma gives the relationship between the Caputo fractional derivative and the fractional integral.

Lemma 2.1 ([2]): (i) *Let $p\in(n-1,n)$, $f(t)\in L_\infty(0,b)$ or $f(t)\in C[0,b]$. Then*

$$ ({}_a^CD_t^p\,{}_aI_t^p)f(t) = f(t). $$

(ii) *Let $p\in(n-1,n)$, $f(t)\in AC^n[0,b]$ or $f(t)\in C^n[0,b]$. Then*

$$ ({}_aI_t^p\,{}_a^CD_t^p)f(t) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k. $$

Lemma 2.2 ([18]): *If $f(t)$ is a continuous function on $t \in [a,b]$, then the following estimate*

$$\frac{1}{\Gamma(p)}\left|\int_a^t (t-s)^{p-1}f(s)\,\mathrm{d}s - \left(\frac{t-a}{b-a}\right)^p \int_a^b (b-s)^{p-1}f(s)\,\mathrm{d}s\right| \le \alpha_1(t)\max_{a\le t\le b}|f(t)| \tag{2}$$

holds for all $t \in [a,b]$, where

$$\alpha_1(t) := \frac{2(t-a)^p}{\Gamma(p+1)}\left(\frac{b-t}{b-a}\right)^p.$$

Lemma 2.3 ([18]): *Let $\{\alpha_n(\cdot)\}_{n\ge1}$ be a sequence of continuous functions on $t \in [a,b]$, given by*

$$\alpha_n(t) := \frac{1}{\Gamma(p)}\left[\int_a^t \left[(t-s)^{p-1} - \left(\frac{t-a}{b-a}\right)^p (b-s)^{p-1}\right]\alpha_{n-1}(s)\,\mathrm{d}s + \left(\frac{t-a}{b-a}\right)^p \int_t^b (b-s)^{p-1}\alpha_{n-1}(s)\,\mathrm{d}s\right],$$

where

$$\alpha_0(t) := 1,$$
$$\alpha_1(t) := \frac{2(t-a)^p}{\Gamma(p+1)}\left(\frac{b-t}{b-a}\right)^p.$$

Then the estimate

$$\alpha_{n+1}(t) \le \frac{(b-a)^{np}\alpha_1(t)}{2^{[n(2p-1)]}[\Gamma(p+1)]^n} \le \frac{(b-a)^{(n+1)p}}{2^{[(n+1)(2p-1)]}[\Gamma(p+1)]^{n+1}} \tag{3}$$

holds for $n \in \mathbb{Z}_0^+$.

For the proofs of Lemmas 2.2 and 2.3 we refer to [18].
Consider a FBVP of the form

$$\begin{aligned} {}_0^C D_t^p u(t) &= f(t,u(t)), \quad t \in [0,1], \\ u(0) &= \alpha_0, \quad u(1) = \alpha_1, \end{aligned} \tag{4}$$

where $p \in (1,2]$, $u(t) : [0,1] \to D \subset \mathbb{R}$, $f(t,u(t)) : [0,1] \times D \to \mathbb{R}$.

Definition 2.3 ([13]): A function $v(t) \in C^2([0,1],\mathbb{R})$ is called a lower solution of the BVP (4) of type I if it satisfies

$$\begin{aligned} {}_0^C D_t^p v(t) &> f(t,v(t)), \quad t \in [0,1], \\ v(0) &\le \alpha_0, \quad v(1) \le \alpha_1. \end{aligned}$$

If a function $w(t) \in C^2([0,1],\mathbb{R})$ satisfies the reversed inequalities, it is called an upper solution of the BVP (4) of type I.

Definition 2.4 ([14]): A function $v(t) \in C^2([0,1],\mathbb{R})$ is called a lower solution of the BVP (4) of type II if it satisfies

$$\begin{aligned} {}_0^C D_t^p v(t) &< f(t,u(t)), \quad t \in [0,1], \\ v(0) \leq \alpha_0, &\quad v(1) \leq \alpha_1. \end{aligned}$$

If a function $w(t) \in C^2([0,1],\mathbb{R})$ satisfies the reversed inequalities, it is called an upper solution of the BVP (4) of type II.

Lemma 2.4 ([13], Positivity Result): *Let $z(t) \in C^2([0,1],\mathbb{R})$ and $r(t) < 0$, $t \in [0,1]$, bounded. If $z(t)$ satisfies the inequality*

$$\begin{aligned} {}_0^C D_t^p z(t) + r(t)z(t) &\leq 0, \quad t \in (0,1), \\ z(0),\ z(1) &\geq 0, \end{aligned}$$

then $z(t) \geq 0, \forall t \in [0,1]$.

Lemma 2.5: *Let $z(t) \in C^2([0,1],\mathbb{R})$. If $z(t)$ satisfies conditions*

$$\begin{aligned} {}_0^C D_t^p z(t) &> 0, \quad t \in (0,1), \\ z(0),\ z(1) &\leq 0, \end{aligned} \tag{5}$$

then $z(t) < 0$ for $t \in (0,1)$.

Proof: Let $z(t) \in C^2([0,1],\mathbb{R})$ be such that it satisfies (5), and assume for the sake of contradiction that $z(t) \geq 0$ for (at least one) $t \in (0,1)$. Then $z(t)$ attains a local maximum at some $t_0 \in (0,1)$, thus the Caputo derivative of $z(t)$ is non-positive at t_0, i.e. ${}_0^C D_t^p z(t_0) \leq 0$, by Theorem 2.1 in [15]. This is in contradiction with (5), thus, $z(t) < 0$ for $t \in [0,1]$. ■

3. Analysis of the parameter-dependent FBVP

In this section we present the general form of the FBVP under consideration. We give sufficient conditions on the right-hand side function for the existence and uniqueness of solutions to the studied problem and apply the numerical-analytic technique to construct a sequence of approximate solutions. Moreover, we analyse the monotonicity behaviour of the resulting sequence.

3.1. Problem setting and solvability analysis

We consider a parameter-dependent Caputo FBVP of the form

$$\begin{aligned} {}_0^C D_t^p u(t) + \lambda a(t) F(u(t)) &= b(t), \quad t \in [0,1], \\ u(0) &= \alpha_0, \\ u(1) &= \alpha_1, \end{aligned} \tag{6}$$

where $p \in (1,2]$, $\lambda \in \mathbb{R}$, $u : [0,1] \to D \subset \mathbb{R}$, $u \in C^2([0,1],\mathbb{R})$, and D is a closed and bounded domain. The right-hand side parameter λ and functions $a(t)$ and $b(t)$ are such that

$$\begin{aligned} |\lambda| &:= \Lambda, \\ A &:= \sup_{t\in[0,1]} |a(t)|, \\ B &:= \sup_{t\in[0,1]} |b(t)|. \end{aligned} \tag{7}$$

We assume that function $F(u(t)) : [0,1] \times D \to \mathbb{R}$ is (generally) non-linear, bounded and Lipschitz continuous, i.e.

$$|F(u(t))| \leq M,$$
$$|F(u_1(t)) - F(u_2(t))| \leq K|u_1(t) - u_2(t)| \tag{8}$$

hold for all $t \in [0,1]$, $u_1(t), u_2(t) \in D$, where $M, K \in \mathbb{R}$ are constants.

We aim to determine the values of the right-hand side parameter λ for which there exists a unique solution to FBVP (6). In addition, we analyse the monotonicity behaviour of the solution and construct a sequence of approximations using the numerical-analytic technique and the lower and upper solutions method.

Note that FBVP (6) is equivalent to the following integral equation:

$$\begin{aligned} u(t,\lambda;\chi) &= \alpha_0 + \chi t + t^p(\alpha_1 - \alpha_0 - \chi) \\ &\quad + \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi)) + b(s)]\,\mathrm{d}s \right. \\ &\quad \left. - t^p \int_0^1 (1-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi)) + b(s)]\,\mathrm{d}s\right], \end{aligned} \tag{9}$$

where $\chi := u'(0)$, see [21] for details.

Denote the operator associated with (9) by $\mathcal{H}$:

$$\begin{aligned} (\mathcal{H}u)(t) &:= \alpha_0 + \chi t + t^p(\alpha_1 - \alpha_0 - \chi) \\ &\quad + \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi)) + b(s)]\,\mathrm{d}s \right. \\ &\quad \left. - t^p \int_0^1 (1-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi)) + b(s)]\,\mathrm{d}s\right]. \end{aligned}$$

In the following theorem we give conditions on the parameter λ, for which the integral equation (9) has a unique solution.

Theorem 3.1: *If $u \in B_r$, where $B_r := \{u \in C^2([0,1],\mathbb{R}) : |u(t)| \leq r\}$ with*

$$r > \frac{2^{2p-1}\Gamma(p+1)U + \Lambda A N + B}{2^{2p-1}\Gamma(p+1) - \Lambda A K},$$
$$U := \max_{0 \leq t \leq 1} \left|\alpha_0 + \chi t + t^p(\alpha_1 - \alpha_0 - \chi)\right|,$$
$$N := \sup_{t \in [0,1]} |F(0)|,$$

and Λ satisfies the following inequality

$$\Lambda < \frac{2^{2p-1}\Gamma(p+1)}{AK}, \tag{10}$$

then $\mathcal{H}$ is a contraction operator, and therefore, the integral equation (9) has a unique solution in $C^2([0,1])$.

Proof: From the Lipschitz condition (8) it follows that

$$\begin{aligned}|F(u(t))| &= |F(u(t)) + F(0) - F(0)| \\ &\leq K|u(t)| + |F(0)| \leq K|u(t)| + N.\end{aligned}$$

For $u \in B_r$ we have

$$\begin{aligned}|(\mathcal{H}u)(t)| \leq{}& \left|\alpha_0 + \chi t + t^p(\alpha_1 - \alpha_0 - \chi)\right| \\ &+ \frac{1}{\Gamma(p)}\left|\int_0^t (t-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi))]\,\mathrm{d}s\right. \\ &\left.-t^p\int_0^1 (1-s)^{p-1}[-\lambda a(s)F(u(s,\lambda,\chi))\,\mathrm{d}s\right| \\ &+ \frac{1}{\Gamma(p)}\left|\int_0^t (t-s)^{p-1}b(s)\,\mathrm{d}s - t^p\int_0^1 (1-s)^{p-1}b(s)\,\mathrm{d}s\right|.\end{aligned}$$

Using inequality (2) from Lemma 2.2 yields

$$\begin{aligned}|(\mathcal{H}u)(t)| &\leq U + \alpha_1(t)\max_{0\leq t\leq 1}|-\lambda a(t)F(u(t,\lambda;\chi))| + \alpha_1(t)\max_{0\leq t\leq 1}|b(t)| \\ &\leq U + (\Lambda A\max_{0\leq t\leq 1}|F(u(t,\lambda,\chi))| + B)\alpha_1(t) \\ &\leq U + [\Lambda A(K|u(t)| + N) + B]\alpha_1(t) \\ &\leq U + [\Lambda A(Kr + N) + B]\alpha_1(t).\end{aligned}$$

Applying estimate (3) in Lemma 2.3 with $n = 0$ yields

$$\begin{aligned}|(\mathcal{H}u)(t)| &\leq U + [\Lambda A(Kr+N) + B]\alpha_1(t) \\ &\leq U + \frac{\Lambda A(Kr+N)+B}{2^{2p-1}\Gamma(p+1)} \leq r,\end{aligned}$$

i.e. if $u \in B_r$, then $\mathcal{H}u \subset B_r$.

Now we consider $u, v \in C^2([0,1],\mathbb{R})$ and apply estimate (2) again:

$$\begin{aligned}|(\mathcal{H}u)(t) - (\mathcal{H}v)(t)| &= \frac{1}{\Gamma(p)}\left|\int_0^t (t-s)^{p-1}(-\lambda a(s))[F(u(s,\lambda;\chi)) - F(v(s,\lambda;chi))]\,\mathrm{d}s\right. \\ &\quad\left.- t^p\int_0^1 (1-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi)) - F(v(s,\lambda;\chi))]\,\mathrm{d}s\right| \\ &\leq \alpha_1(t)\max_{0\leq t\leq 1}|-\lambda a(s)[F(u(s,\lambda;\chi)) - F(v(s,\lambda;\chi))]| \\ &\leq \alpha_1(t)\Lambda AK\max_{0\leq t\leq 1}|u(s,\lambda;\chi) - v(s,\lambda;\chi)| \\ &\leq \frac{\Lambda AK}{2^{2p-1}\Gamma(p+1)}||u-v|| \\ \Longrightarrow\ ||\mathcal{H}u - \mathcal{H}v|| &\leq \frac{\Lambda AK}{2^{2p-1}\Gamma(p+1)}||u-v|| \leq ||u-v||.\end{aligned}$$

From (10) it follows that

$$\Lambda\frac{AK}{2^{2p-1}\Gamma(p+1)} < \frac{2^{2p-1}\Gamma(p+1)}{AK}\frac{AK}{2^{2p-1}\Gamma(p+1)} < 1,$$

i.e. $\mathcal{H}$ is a contraction. Therefore, by the Banach fixed point theorem, (9) has a unique solution in $C^2([0,1])$. ■

3.2. Successive approximations method

The integral representation of the exact solution of FBVP (6) is given in (9), however, a difficulty of its application arises, since the quantity under the integral depends on $u(t,\lambda;\chi)$, whose explicit form is unknown. To overcome this, we construct a sequence of approximations, which converges uniformly to the exact solution. We set the value of the right-hand side parameter to $\lambda = \bar{\lambda}$, such that condition (10) is satisfied, and associate with BVP (6) the following sequence

$$\begin{aligned} u_0(t,\bar{\lambda};\chi) &= \alpha_0 + \chi t + t^p\left(\alpha_1 - \alpha_0 - \chi\right) \\ u_n(t,\bar{\lambda};\chi) &= u_0(t,\bar{\lambda};\chi) + \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1}[-\bar{\lambda}a(s)F(u_{n-1}(s,\bar{\lambda};\chi)) + b(s)]\,\mathrm{d}s\right. \\ &\qquad \left. - t^p\int_0^1 (1-s)^{p-1}[-\bar{\lambda}a(s)F(u_{n-1}(s,\bar{\lambda};\chi)) + b(s)]\,\mathrm{d}s\right], \end{aligned} \tag{11}$$

$n \in \mathbb{N}, t \in [0,1]$.

Additionally, we assume that the set

$$D_\beta := \{\alpha_0 \in D : B(\alpha_0 + \chi t + t^p\left(\alpha_1 - \alpha_0 - \chi\right), \beta) \subset D\},$$

is non-empty, where

$$\beta = \frac{\bar{\lambda}AM + B}{2^{2p-1}\Gamma(p+1)}. \tag{12}$$

Then the following theorem holds.

Theorem 3.2: *Provided that for all $\chi \in \Omega$ and $t \in [0,1]$ conditions (10) and (12) are satisfied,*

(1) *The terms of the sequence (11) are continuous and satisfy boundary conditions*

$$\begin{aligned} u_n(0,\bar{\lambda},\chi) &= \alpha_0, \\ u_n(1,\bar{\lambda},\chi) &= \alpha_1 \end{aligned}$$

for $n \in \mathbb{N}$.

(2) *The sequence (11) converges uniformly to the limit function*

$$u_\infty(t,\bar{\lambda};\chi) = \lim_{n\to\infty} u_n(t,\bar{\lambda};\chi). \tag{13}$$

(3) *The limit function (13) satisfies boundary conditions $u_\infty(0,\bar{\lambda};\chi) = \alpha_0$, $u_\infty(1,\bar{\lambda};\chi) = \alpha$.*

(4) *The limit function (13) is the unique solution to the integral equation (9)*

$$\begin{aligned} u(t,\bar{\lambda};\chi) &= \alpha_0 + \chi t + t^p\left(\alpha_1 - \alpha_0 - \chi\right) \\ &\quad + \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1}[-\bar{\lambda}a(s)F(u(s,\bar{\lambda};\chi)) + b(s)]\,\mathrm{d}s\right. \\ &\quad \left. - t^p\int_0^1 (1-s)^{p-1}[-\bar{\lambda}a(s)F(u(s,\bar{\lambda};\chi)) + b(s)]\,\mathrm{d}s\right], \end{aligned} \tag{14}$$

i.e. it is the unique solution to the perturbed Cauchy problem

$$\begin{aligned} {}_0^C D_t^p u(t) &= -\bar{\lambda} a(t) F(u(t)) + b(t) + \Delta(\chi), \quad p \in (1,2], \\ u(0) &= \alpha_0, \\ u'(0) &= \chi, \end{aligned} \tag{15}$$

where $\Delta : \Omega \to \mathbb{R}$ is a mapping defined by

$$\Delta(\chi) := \Gamma(p+1)(\alpha_1 - \alpha_0 - \chi) - p \int_0^1 (1-s)^{p-1} [\bar{\lambda} a(s) F(u(s, \bar{\lambda}; \chi)) + b(s)] \, \mathrm{d}s. \tag{16}$$

(5) *The following error estimate holds*

$$|u_\infty(t, \bar{\lambda}; \chi) - u_n(t, \bar{\lambda}; \chi)| \leq \frac{Q^n(\bar{\lambda} AM + B)}{2^{2p-1}\Gamma(p+1)} \frac{1}{1-Q}, \tag{17}$$

where

$$Q := \frac{\bar{\lambda} AK}{2^{2p-1}\Gamma(p+1)},$$

and A, K are defined in (7) and (8).

To establish the connection between the solution to the IVP (15) and the original BVP (6), consider the Cauchy problem

$$\begin{aligned} {}_0^C D_t^p u(t) &= f(t, u(t)) + \mu, \quad t \in [0, T], \\ u(0) &= \alpha_0, \\ u'(0) &= \chi, \end{aligned} \tag{18}$$

where $\mu \in \mathbb{R}$ we will call a control parameter, $\alpha_0 \in D_\beta$ and $\chi \in \Omega$.

Theorem 3.3: *Let $\chi \in \Omega$, $\mu \in \mathbb{R}$ be given. Assume that all conditions of Theorem 3.2 are satisfied for the FBVP (6). Then the solution $u = u(\cdot, \bar{\lambda}; \chi, \mu)$ of the IVP (18) also satisfies the boundary conditions in (6) if and only if*

$$\mu = \Delta(\chi),$$

where $\Delta(\chi)$ is given by (16), and in this case

$$u(t, \bar{\lambda}; \chi, \mu) = u_\infty(t, \bar{\lambda}; \chi) \quad \text{for } t \in [0, 1].$$

Theorem 3.4: *Let the original BVP (6) satisfy conditions (12) and (10). Then $u_\infty(\cdot, \bar{\lambda}; \chi^*)$ is a solution to the BVP (6) if and only if the point χ^* is a solution to the determining equation*

$$\Delta(\chi^*) = 0,$$

where Δ is given by (16).

For the proofs of Theorems 3.2 – 3.4 in the case of a parameter-independent FBVP we refer to [21].

Remark 3.1: Since the explicit form of the solution $u(t, \bar{\lambda}; \chi)$ is unknown, in practice, we compute the values of the parameter χ by solving the approximate determining equation

$$\Delta(\chi_n) = 0, \tag{19}$$

where

$$\Delta(\chi_n) := \Gamma(p+1)(\alpha_1 - \alpha_0 - \chi_n) - p\int_0^1 (1-s)^{p-1}[\bar{\lambda}a(s)F(u_n(s, \bar{\lambda}; \chi_n)) + b(s)]\,\mathrm{d}s$$

at each iteration step n.

3.3. Monotonicity of the approximating sequence

Let us denote the right-hand side function in the BVP (6) by $f(t, u(t, \bar{\lambda}; \chi); \bar{\lambda}) := -\bar{\lambda}a(t)F(u(t, \bar{\lambda}; \chi)) + b(t)$. The following two theorems give conditions for which the terms in (11) form a monotone or an alternating sequence respectively.

Theorem 3.5: *Consider the BVP (6) and the sequence of approximations (11), and assume that $f(t, u(t, \bar{\lambda}; \chi); \bar{\lambda})$ is strictly decreasing in $u(t, \bar{\lambda}; \chi)$, i.e.*

$$\frac{\partial f}{\partial u} < 0.$$

Then the following statements hold:

(S1) *If the initial approximation $u_0(t, \bar{\lambda}; \chi)$ is such that $u_0(t, \bar{\lambda}; \chi) < u_1(t, \bar{\lambda}; \chi)$, then the sequence $u_n(t, \bar{\lambda}; \chi)$ is well-ordered and increasing, i.e.*

$$u_{k-1}(t, \bar{\lambda}; \chi) < u_k(t, \bar{\lambda}; \chi), \quad \forall\, k \in \mathbb{N}.$$

(S2) *If the initial approximation $u_0(t, \bar{\lambda}; \chi)$ is such that $u_1(t, \bar{\lambda}; \chi) < u_0(t, \bar{\lambda}; \chi)$, then the sequence $u_n(t, \bar{\lambda}; \chi)$ is well-ordered and decreasing, i.e.*

$$u_k(t, \bar{\lambda}; \chi) < u_{k-1}(t, \bar{\lambda}; \chi), \quad \forall\, k \in \mathbb{N}.$$

Proof: The terms in the approximating sequence are obtained from the scheme

$$\begin{aligned} {}_0^C D_t^p u_n(t, \bar{\lambda}; \chi) &= f(t, u_{n-1}(t, \bar{\lambda}; \chi)), \\ u_n(0) = u(0), \quad u_n(1) &= u(1), \quad n \geq 1. \end{aligned}$$

(S1) Assume that $u_0(t, \bar{\lambda}; \chi) < u_1(t, \bar{\lambda}; \chi)$. Then

$$\begin{aligned} {}_0^C D_t^p [u_1(t, \bar{\lambda}; \chi) - u_2(t, \bar{\lambda}; \chi)] &= f(t, u_0(t, \bar{\lambda}; \chi); \bar{\lambda}) - f(t, u_1(t, \bar{\lambda}; \chi); \bar{\lambda}) > 0, \\ u_1(0, \bar{\lambda}; \chi) - u_2(0, \bar{\lambda}; \chi) &= 0, \\ u_1(1, \bar{\lambda}; \chi) - u_2(1, \bar{\lambda}; \chi) &= 0, \end{aligned}$$

hence, by Lemma 2.5, $u_1(t, \bar{\lambda}; \chi) < u_2(t, \bar{\lambda}; \chi)$. Assume the statement holds for $n = k$. Then, for $n = k+1$ we have

$$\begin{aligned} {}_0^C D_t^p [u_k(t, \bar{\lambda}; \chi) - u_{k+1}(t, \bar{\lambda}; \chi)] &= f(t, u_{k-1}(t, \bar{\lambda}; \chi); \bar{\lambda}) - f(t, u_k(t, \bar{\lambda}; \chi); \bar{\lambda}) > 0, \\ u_k(0, \bar{\lambda}; \chi) - u_{k+1}(0, \bar{\lambda}; \chi) &= 0, \\ u_k(1, \bar{\lambda}; \chi) - u_{k+1}(1, \bar{\lambda}; \chi) &= 0, \end{aligned}$$

hence, by Lemma 2.2, $u_k(t,\bar{\lambda};\chi) < u_{k+1}(t,\bar{\lambda};\chi)$. Therefore, the sequence $u_n(t,\bar{\lambda};\chi)$ is monotone and increasing.

The proof of (S2) follows the lines of the proof of (S1). ■

Theorem 3.6: *Consider the BVP (6), and the sequence of approximations (11), and assume that $f(t,u(t,\bar{\lambda};\chi);\bar{\lambda})$ is strictly increasing in $u(t,\bar{\lambda};\chi)$, i.e.*

$$\frac{\partial f}{\partial u} > 0.$$

Then the following statements hold:

(S1) *If the initial approximation $u_0(t,\bar{\lambda};\chi)$ is such that $u_1(t,\bar{\lambda};\chi) < u_0(t,\bar{\lambda};\chi)$, then the terms $u_n(t,\bar{\lambda};\chi)$, given by (11), form an alternating sequence, for which*

$$u_1(t,\bar{\lambda};\chi) < \cdots < u_{2n+1}(t,\bar{\lambda};\chi) < u_{2n}(t,\bar{\lambda};\chi) < \cdots < u_0(t,\bar{\lambda};\chi). \tag{20}$$

(S1) *If the initial approximation $u_0(t,\bar{\lambda};\chi)$ is such that $u_0(t,\bar{\lambda};\chi) < u_1(t,\bar{\lambda};\chi)$, then the terms $u_n(t,\bar{\lambda};\chi)$, given by (11), form an alternating sequence, for which*

$$u_0(t,\bar{\lambda};\chi) < \cdots < u_{2n}(t,\bar{\lambda};\chi) < u_{2n+1}(t,\bar{\lambda};\chi) < \cdots < u_1(t,\bar{\lambda};\chi). \tag{21}$$

Proof: (S1) Assume that $u_1(t,\bar{\lambda};\chi) < u_0(t,\bar{\lambda};\chi)$. Then

$$\begin{aligned}
{}_0^C D_t^p[u_1(t,\bar{\lambda};\chi) - u_2(t,\bar{\lambda};\chi)] &= f(t,u_0(t,\bar{\lambda};\chi);\bar{\lambda}) - f(t,u_1(t,\bar{\lambda};\chi);\bar{\lambda}) > 0,\\
u_1(0,\bar{\lambda};\chi) - u_2(0,\bar{\lambda};\chi) &= 0,\\
u_1(1,\bar{\lambda};\chi) - u_2(1,\bar{\lambda};\chi) &= 0,
\end{aligned}$$

hence, by Lemma 2.5, $u_1(t,\bar{\lambda};\chi) < u_2(t,\bar{\lambda};\chi)$. Assume the statement holds for $n = k$, that is, $u_{2k+1}(t,\bar{\lambda};\chi) < u_{2k}(t,\bar{\lambda};\chi)$ Then, for $n = k+1$ we have

$$\begin{aligned}
{}_0^C D_t^p[u_{2k+1}(t,\bar{\lambda};\chi) - u_{2(k+1)}(t,\bar{\lambda};\chi)] &= f(t,u_{2k}(t,\bar{\lambda};\chi);\bar{\lambda}) - f(t,u_{2k+1}(t,\bar{\lambda};\chi);\bar{\lambda}) > 0,\\
u_{2k+1}(0,\bar{\lambda};\chi) - u_{2(k+1)}(0,\bar{\lambda};\chi) &= 0,\\
u_{2k+1}(1,\bar{\lambda};\chi) - u_{2(k+1)}(1,\bar{\lambda};\chi) &= 0,
\end{aligned}$$

hence, by Lemma 2.5, $u_{2k+1}(t,\bar{\lambda};\chi) < u_{2(k+1)}(t,\bar{\lambda};\chi)$. Thus,

$$\begin{aligned}
{}_0^C D_t^p[u_{2k+3}(t) - u_{2(k+1)}(t)] &= f(t,u_{2(k+1)}(t,\bar{\lambda};\chi);\bar{\lambda}) - f(t,u_{2k+1}(t,\bar{\lambda};\chi);\bar{\lambda}) > 0,\\
(u_{2k+3} - u_{2(k+1)})(0) &= 0,\\
(u_{2k+3} - u_{2(k+1)})(1) &= 0,
\end{aligned}$$

which implies $u_{2k+3}(t,\bar{\lambda};\chi) < u_{2(k+1)}(t,\bar{\lambda};\chi)$, that is, the statement holds for $n = k+1$. Therefore, the sequence $u_n(t,\bar{\lambda};\chi)$ is alternating, i.e. (20) holds.

The proof of (2) follows the lines of the proof of (1). ■

4. Lower and upper solutions method

In this section, we describe how the lower/upper solutions method can be combined with the numerical-analytic technique, presented in Section 3, to construct approximating sequences to the solution of BVP (6).

The following two theorems give the form of the alternating sequence, resulting from combining the numerical-analytic technique with the lower and upper solutions method, depending on how the lower and upper solutions are chosen.

Theorem 4.1: *Consider the BVP (6). Assume that*

(i) v_0, w_0 $\in C^2([0,1],\mathbb{R})$ *are lower and upper solutions to the BVP (6) of type I for* $t \in [0,1]$.
(ii) *the right-hand side function* $f(t, u(t, \bar{\lambda}; \chi); \bar{\lambda})$ *is an increasing function in* $u(t, \bar{\lambda}; \chi)$.
(iii) *two sequences,* $\{v_n(t)\}$ *and* $\{w_n(t)\}$, *are computed using the iterative scheme*

$$\begin{aligned} {}_0^C D_t^p v_{n+1}(t) &= f(t, v_n(t); \bar{\lambda}), \quad v_{n+1}(0) = u(0), \quad v_{n+1}(1) = u(1) \\ {}_0^C D_t^p w_{n+1}(t) &= f(t, w_n(t); \bar{\lambda}), \quad w_{n+1}(0) = u(0), \quad w_{n+1}(1) = u(1), \end{aligned} \tag{22}$$

for which

$$\begin{aligned} v_1(t) &< w_1(t), \\ w_2(t) &< v_2(t). \end{aligned} \tag{23}$$

Then,

(a) (a)*For* $t \in [0,1]$ *it holds that*

$$v_0(t) < w_0(t).$$

(b) (b)*The terms computed using (22) form alternating sequences*$\{v_{2n+1}(t), w_{2n+1}(t)\}$ *and* $\{w_{2n}(t), v_{2n}(t)\}$, *satisfying*

$$\begin{aligned} &v_0(t) < v_1(t) < w_1(t) < \ldots < v_{2n+1}(t) < w_{2n+1}(t) < u_\infty(t) < \\ &\qquad < w_{2n}(t) < v_{2n}(t) < \ldots < w_2(t) < v_2(t) < w_0(t) \end{aligned} \tag{24}$$

for $n \geq 0$. *Each term* $v_{n+1}(t)$, $w_{n+1}(t)$ *is computed from the corresponding integral equations:*

$$\begin{aligned} v_{n+1}(t) = \alpha_0 + \eta t + (\alpha_1 - \alpha_0 - \eta)t^p + \frac{1}{\Gamma(p)} \Bigg[&\int_0^t (t-s)^{p-1} f(s, v_n(s); \bar{\lambda})\,\mathrm{d}s \\ &+ t^p \int_0^1 (1-s)^{p-1} f(s, v_n(s); \bar{\lambda})\,\mathrm{d}s \Bigg], \end{aligned} \tag{25}$$

$$\begin{aligned} w_{n+1}(t) = \alpha_0 + \zeta t + (\alpha_1 - \alpha_0 - \zeta)t^p + \frac{1}{\Gamma(p)} \Bigg[&\int_0^t (t-s)^{p-1} f(s, w_n(s); \bar{\lambda})\,\mathrm{d}s \\ &+ t^p \int_0^1 (1-s)^{p-1} f(s, w_n(s); \bar{\lambda})\,\mathrm{d}s \Bigg], \end{aligned} \tag{26}$$

where the unknown parameters η *and* ζ *denote* $\eta := v'(0)$ *and* $\zeta := w'(0)$ *and are computed by solving the determining equations*

$$\Delta(\eta) = 0, \tag{27}$$

$$\Delta(\zeta) = 0, \tag{28}$$

where

$$\Delta(\eta) = \Gamma(p+1)\,(\alpha_1 - \alpha_0 - \eta) - p \int_0^1 (1-s)^{p-1} f(s, v(s); \bar{\lambda})\,\mathrm{d}s.$$

$$\Delta(\zeta) = \Gamma(p+1)\,(\alpha_1 - \alpha_0 - \zeta) - p \int_0^1 (1-s)^{p-1} f(s, w(s); \bar{\lambda})\,\mathrm{d}s.$$

(c) (c)*Let* $x_0(t) := v_0(t)$ *and* $\{x_n(t)\} := \{v_{2n+1}(t), w_{2n+1}(t)\}$ *for* $n \geq 0$, *that is,* $x_1(t) := v_1(t), x_3(t) := w_1(t), \dots$ *and similarly,* $\{y_n(t)\} := \{v_{2n}(t), w_{2n}(t)\}$ *for* $n \geq 0$, *that is,* $y_0(t) := w_0(t), y_1(t) := v_2(t), y_3(t) := w_2(t), \dots$. *Then the sequences* $\{x_n(t)\}$ *and* $\{y_n(t)\}$ *converge uniformly to the limits* $x_\infty(t)$ *and* $y_\infty(t)$, *respectively, and* $x_\infty(t) < y_\infty(t)$.

(d) (d)*For the limit functions* $x_\infty(t)$ *and* $y_\infty(t)$ *it holds that* $x_\infty(t) = y_\infty(t) = u_\infty(t)$, *where* $u_\infty(t)$ *is the unique solution to BVP (6).*

Proof: (a) From Definition 2.3 of lower and upper solutions of type I, we have that

$$ {}_0^C D_t^p v_0(t) - f(t, v_0(t); \bar{\lambda}) > 0, \tag{29} $$

$$ {}_0^C D_t^p w_0(t) - f(t, w_0(t); \bar{\lambda}) < 0. \tag{30} $$

Subtracting (29) from (30) and using the Mean Value Theorem, we obtain

$$ {}_0^C D_t^p (w_0(t) - v_0(t)) - \frac{\partial f}{\partial u}(u^*)(w_0 - v_0) < 0, $$

where $u^* = \gamma v_0 + (1-\gamma) w_0$, $0 \leq \gamma \leq 1$. Since $f(t, u(t); \bar{\lambda})$ is an increasing function, $-\frac{\partial f}{\partial u}(u^*) < 0$. Moreover, $(w_0 - v_0)(0) \geq 0$, $(w_0 - v_0)(1) \geq 0$, thus, by Lemma 2.4 it follows that $w_0(t) > v_0(t)$.

(b) Let $z_0(t) = v_0(t) - v_1(t)$. Then

$$ {}_0^C D_t^p z_0(t) = {}_0^C D_t^p v_0(t) - f(t, v_0(t); \bar{\lambda}) > 0, $$
$$ z_0(0) \leq 0, \quad z_0(1) \leq 0 $$

thus, by Lemma 2.5, $v_0(t) < v_1(t)$.

Now let $z_1(t) = v_1(t) - v_2(t)$ and consider

$$ {}_0^C D_t^p z_1(t) = f(t, v_0(t); \bar{\lambda}) - f(t, v_1(t); \bar{\lambda}) < 0, $$
$$ z_1(0) \leq 0, \quad z_1(1) \leq 0, $$

where the first inequality follows from the fact that $f(t, u(t); \bar{\lambda})$ is increasing in $u(t)$. Thus, by Lemma 2.5, $v_1(t) < v_2(t)$.

Assume that $v_{2k+1}(t) < v_{2k}(t)$ for $k \geq 1$. We will show that it also holds for $k+1$. Consider $z_{2k+1}(t) = v_{2k+1}(t) - v_{2k+2}(t)$, for which

$$ {}_0^C D_t^p z_{2k+1}(t) = f(t, v_{2k}(t); \bar{\lambda}) - f(t, v_{2k+1}; \bar{\lambda}) > 0, $$
$$ z_{2k+1}(0) \leq 0, \quad z_{2k+1}(1) \leq 0, $$

thus, by Lemma 2.5, $v_{2k+1}(t) < v_{2k+2}(t)$. Now let $z_{2k+3}(t) = v_{2k+3}(t) - v_{2k+2}(t)$ and consider

$$ {}_0^C D_t^p z_{2k+3}(t) = f(t, v_{2k+2}(t); \bar{\lambda}) - f(t, v_{2k+1}(t); \bar{\lambda}) > 0 $$
$$ z_{2k+3}(0) \leq 0, \quad z_{2k+3}(1) \leq 0. $$

Hence, $v_{2k+3}(t) < v_{2k+2}(t)$, which implies that $v_{2n+1}(t) < v_{2n}(t)$ holds for all $n \geq 1$.

Using the same method, we can show that $w_1(t) < w_0(t)$, $w_1(t) < w_2(t)$, and $w_{2n+1}(t) < w_{2n}(t)$ for $n \geq 0$.

From the assumptions and inequalities in (23), it follows that

$$v_0(t) < v_1(t) < w_1(t) < w_2(t) < v_2(t) < w_0(t).$$

Assume that $v_{2k+1}(t) < w_{2k+1}(t)$. We will show that this holds for $n = k+1$. Consider $z_{2k+2}(t) = w_{2k+2}(t) - v_{2k+2}(t)$:

$$\begin{aligned} {}_0^C D_t^p z_{2k+2}(t) &= f(t, w_{2k+1}(t); \bar{\lambda}) - f(t, v_{2k+1}(t); \bar{\lambda}) > 0 \\ z_{2k+2}(0) &\le 0, \quad z_{2k+2}(1) \le 0, \end{aligned}$$

thus, by Lemma 2.5, $w_{2k+2}(t) < v_{2k+2}(t)$. Now let $z_{2k+3}(t) = v_{2k+3}(t) - w_{2k+3}(t)$ and consider

$$\begin{aligned} {}_0^C D_t^p z_{2k+3}(t) &= f(t, v_{2k+2}(t); \bar{\lambda}) - f(t, w_{2k+2}(t); \bar{\lambda}) > 0 \\ z_{2k+3}(0) &\le 0, \quad z_{2k+3}(1) \le 0, \end{aligned}$$

thus, $v_{2k+3}(t) < w_{2k+3}(t)$. This implies that $v_{2n+1}(t) < w_{2n+1}(t)$ holds for $n \ge 1$.

Similarly, we can show that $w_{2n}(t) < v_{2n}(t)$ for $n \ge 1$.

Thus far we have seen that

$$\begin{aligned} &v_0(t) < v_{2n+1}(t) < v_{2n}(t), \\ &w_{2n+1}(t) < w_{2n}(t) < w_0(t), \\ &v_{2n+1}(t) < w_{2n+1}(t), \\ &w_{2n}(t) < v_{2n}(t). \end{aligned}$$

Combining these inequalities results in (24).

(c) The sequence $x_n(t)$ is a monotonically increasing sequence of continuous functions, bounded from above by $w_0(t)$, defined on the compact domain $[0, 1]$, and the sequence $y_n(t)$ is a monotonically decreasing sequence of continuous functions, bounded from below by $v_0(t)$, defined on the compact domain $[0, 1]$. Hence, $x_n(t)$ and $y_n(t)$ converge uniformly to their respective limits, $x_\infty(t)$ and $y_\infty(t)$. From part (a) we know that $x_n(t) < y_n(t)$ for all $n \ge 0$, thus $x_\infty(t) = \lim_{n\to\infty} x_n(t) < \lim_{n\to\infty} y_n(t) = y_\infty(t)$.

(d) Passing to the limit when $n \to \infty$ in the integral equations (25) and (26) yields

$$\begin{aligned} v_\infty(t) &= \alpha_0 + \eta t + (\alpha_1 - \alpha_0 - \eta)t^p + \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f(s, v_\infty(s); \bar{\lambda})\, \mathrm{d}s \right. \\ &\quad \left. + t^p \int_0^1 (1-s)^{p-1} f(s, v_\infty(s); \bar{\lambda})\, \mathrm{d}s \right], \\ w_\infty(t) &= \alpha_0 + \zeta t + (\alpha_1 - \alpha_0 - \zeta)t^p + \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f(s, w_\infty(s); \bar{\lambda})\, \mathrm{d}s \right. \\ &\quad \left. + t^p \int_0^1 (1-s)^{p-1} f(s, w_\infty(s); \bar{\lambda})\, \mathrm{d}s \right], \end{aligned}$$

which are equivalent to equation (14). The limit function $u_\infty(t)$ is the unique solution to (14), thus, $v_\infty(t) = w_\infty(t) = u_\infty(t)$. ■

Theorem 4.2: *Consider the BVP (6). Assume that*

(i) $v_0, w_0 \in C^2([0, 1], \mathbb{R})$ *are lower and upper solutions to the BVP (6) of type II with* $v_0(t) < w_0(t)$ *for* $t \in [0, 1]$.

(ii) *the right-hand side function* $f(t, u(t, \bar{\lambda}; \chi); \bar{\lambda})$ *is an increasing function in* $u(t, \bar{\lambda}; \chi)$.
(iii) *two sequences,* $\{v_n(t)\}$ *and* $\{w_n(t)\}$*, are computed using the iterative scheme*

$$
\begin{aligned}
{}_0^C D_t^p v_{n+1}(t) &= f(t, v_n(t); \bar{\lambda}), \quad v_{n+1}(0) = u(0), \quad v_{n+1}(1) = u(1) \\
{}_0^C D_t^p w_{n+1}(t) &= f(t, w_n(t); \bar{\lambda}), \quad w_{n+1}(0) = u(0), \quad w_{n+1}(1) = u(1),
\end{aligned}
$$

for which

$$
\begin{aligned}
v_0(t) &< w_1(t), \\
v_1(t) &< w_0(t).
\end{aligned} \tag{31}
$$

Then,

(a) (a)*The terms computed using (31) form alternating sequences* $\{v_{2n}(t), w_{2n+1}(t)\}$ *and* $\{v_{2n+1}(t), w_{2n}(t)\}$*, satisfying*

$$
\begin{aligned}
v_0(t) < w_1(t) < v_2(t) < \ldots < v_{2n}(t) < w_{2n+1}(t) < u_\infty(t) < \\
< v_{2n+1}(t) < w_{2n}(t) < \ldots < w_2(t) < v_1(t) < w_0(t)
\end{aligned}
$$

for $n \geq 0$. *Each term* $v_{n+1}(t)$, $w_{n+1}(t)$ *is computed from the following integral equations:*

$$
\begin{aligned}
v_{n+1}(t) &= \alpha_0 + \eta t + (\alpha_1 - \alpha_0 - \eta) t^p + \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f(s, v_n(s); \bar{\lambda})\, \mathrm{d}s \right. \\
&\quad \left. + t^p \int_0^1 (1-s)^{p-1} f(s, v_n(s); \bar{\lambda})\, \mathrm{d}s \right], \\
w_{n+1}(t) &= \alpha_0 + \zeta t + (\alpha_1 - \alpha_0 - \zeta) t^p + \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f(s, w_n(s); \bar{\lambda})\, \mathrm{d}s \right. \\
&\quad \left. + t^p \int_0^1 (1-s)^{p-1} f(s, w_n(s); \bar{\lambda})\, \mathrm{d}s \right],
\end{aligned}
$$

where the unknown parameters η *and* ζ *denote* $\eta := v'(0)$ *and* $\zeta := w'(0)$ *and are computed by solving the determining equations (27) and (28).*

(b) (b)*Let* $\{x_n(t)\} := \{v_{2n}(t), w_{2n+1}(t)\}$ *and* $\{y_n(t)\} := \{v_{2n+1}(t), w_{2n}(t)\}$ *for* $n \geq 0$. *Then the sequences* $\{x_n(t)\}$ *and* $\{y_n(t)\}$ *converge uniformly to the limits* $x_\infty(t)$ *and* $y_\infty(t)$*, respectively, and* $x_\infty(t) < y_\infty(t)$.

(c) (c)*For the limit functions* $x(t)$ *and* $y(t)$ *it holds that* $x_\infty(t) = y_\infty(t) = u_\infty(t)$*, where* $u(t)$ *is the unique solution to BVP (6).*

Proof: The proof of Theorem 4.2 follows the lines of the proof of Theorem 4.1. ■

Remark 4.1: As in the standard numerical-analytic technique, the values of parameters η and ζ are computed by solving the corresponding approximate determining equations (27) and (28) at each iteration step n.

Remark 4.2: It is worth emphasising that the lower and upper solutions method can also be used to simplify the computations of the approximating sequence. In particular, we can construct a sequence

$\tilde{u}_n(t, \bar{\lambda}; \chi)$, given by

$$\tilde{u}_0(t) = \frac{v_0(t) + w_0(t)}{2},$$
$$\tilde{u}_n(t, \bar{\lambda}; \chi) = \alpha_0 + \chi t + t^p(\alpha_1 - \alpha_0 - \chi) + \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f(s, \tilde{u}_{n-1}(s, \bar{\lambda}; \chi)\, ds - t^p \int_0^1 (1-s)^{p-1} f(s, \tilde{u}_{n-1}(s, \bar{\lambda}; \chi)\, ds \right]. \tag{32}$$

The statements of Theorem 3.3 hold for the new sequence $\tilde{u}_n(t, \bar{\lambda}; \chi)$ and the terms in it are of simpler form, which leads to a reduction in the computational time.

5. Model examples

Motivated by Marynets [22] we consider the non-linear FDE

$${}_0^C D_t^p u(t) = \frac{-2\lambda e^t}{(1+e^t)^2} u(t)^2 - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} \quad (:= f(t, u(t); \lambda)), \quad t \in [0, 1], \tag{33}$$

where $p = 1.98$ and ω is a scalar which in the context of the flow of the Antarctic Circumpolar Current is corresponding to the dimensionless Coriolis parameter being equal to 4649.56.

We attach to (33) Dirichlet boundary conditions

$$u(0) = \alpha, \quad u(1) = \beta, \tag{34}$$

and we will use the theory presented in the previous sections to construct sequences of approximations to FBVP (33), (34) for an increasing and decreasing right-hand side. We will also show how the lower and upper solutions method can be used to improve the efficiency of the numerical-analytic technique.

We define BVP (33), (34) on the domain

$$D := \{-112 \leq u(t) \leq 67\}.$$

Using notations (7) and (8) we conclude that for the given FDE in (33) the following relations hold:

$$A = 0.5, \quad B = 844.91, \quad M = 12530.56, \quad K = 223.88,$$

which means that we must choose a value for the parameter λ, such that

$$\Lambda < \frac{2^{2\cdot 1.98-1}\Gamma(2.98)}{0.5 \cdot 223.88} = 0.14.$$

5.1. Monotone sequence

We set $\bar{\lambda} = 0.05, \alpha = -1, \beta = -1.5$, and apply the numerical-analytic technique to construct approximations to BVP (33), (34). The values of the parameter χ_n, computed using (19), are shown in Table 1. The left panel in Figure 1 shows the first 5 approximations, which form a decreasing monotone sequence. The right panel of the same figure shows a comparison of the right- and left-hand sides of equation (33) with $u_4(t, \lambda; \chi)$ plugged in, which shows a good agreement between the two sides of the equation for the fourth approximation term.

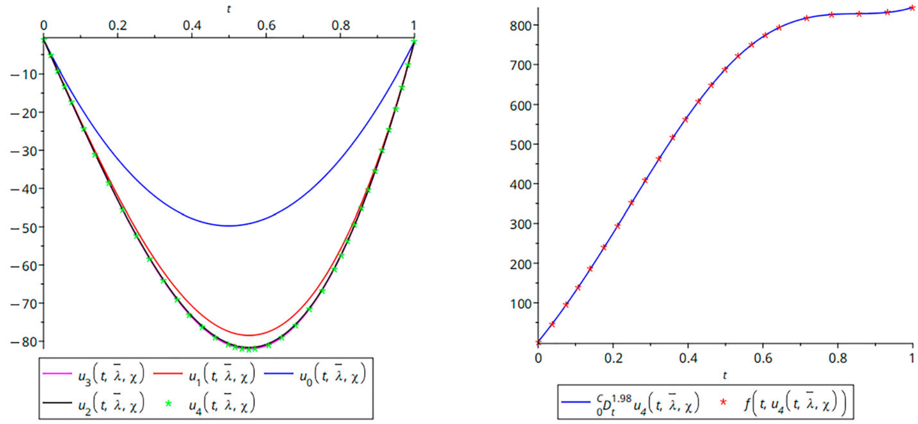

Figure 1. Plots of approximation terms $u_n(t, \lambda; \chi)$ for $n = 0, \ldots 4$ (left) and left- (blue solid line) and right- (red dotted line) hand sides of equation (33) with $u_4(t, \bar{\lambda}, \chi)$ (right).

Table 1. Numerically calculated parameter values for $n = 0, \ldots, 4$.

n	χ_n
0	−197.2411150
1	−216.9564132
2	−219.5465627
3	−219.8524774
4	−219.8865503

5.2. *Alternating sequence*

We consider BVP (33), (34) again and choose a negative parameter value in order to have an increasing right-hand side function. We set $\bar{\lambda} = -0.1$ and $\alpha = 1, \beta = 1.5$ and apply the numerical-analytic technique, as before. The computed values of χ_n for $n = 0, \ldots, 4$ are shown in Table 2. The terms of the alternating sequence are $u_n(t, \lambda; \chi)$ for $n = 0, \ldots, 4$ are shown in the left panel of Figure 2, and the left- and right- hand sides of equation (33) with $u_4(t, \lambda; \chi)$ are plotted in the right panel of the same figure. As before, we have good agreement between the left- and right-hand sides of the equation with the fourth approximation term.

Table 2. Numerically calculated parameter values for $n = 0, \ldots, 4$.

n	χ_n
0	−161.0107502
1	−145.4552893
2	−147.9224531
3	−147.5308218
4	−147.5895933

5.3. *Lower and upper solutions*

Let us now apply the lower/upper solutions method to construct approximate solutions to BVP (33), (34) with $\alpha = 1, \beta = 1.5$.

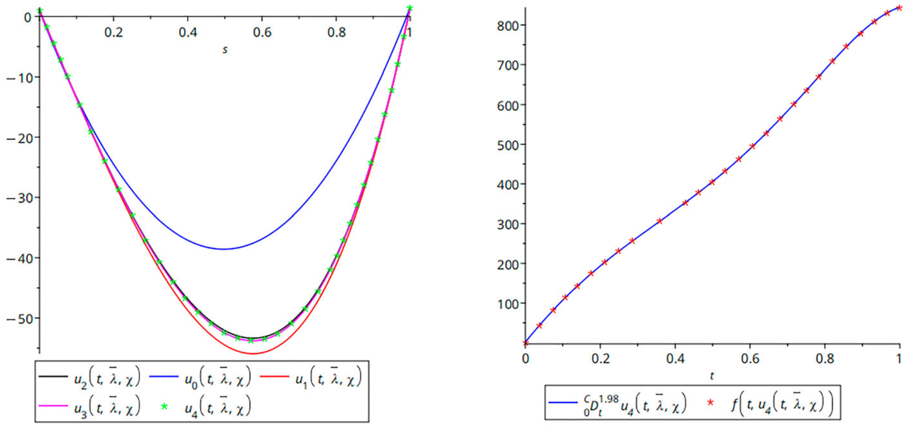

Figure 2. Plots of approximation terms $u_n(t, \bar{\lambda}; \chi)$ for $n = 0, \ldots, 4$ (left) and left- (blue solid line) and right- (red dotted line) hand sides of equation (33) with $u_4(t, \bar{\lambda}, \chi)$ (right).

We choose lower and upper solutions $v_0(t) = -148$, $w_0(t) = 10$, which satisfy the following differential inequalities:

$$
\begin{aligned}
{}_0^C D_t^{1.98} v_0(t) &= 0 > f(t, v_0(t); \lambda), \\
v_0(0) &= -148 < u(0), \quad v_0(1) = -148 < u(1), \\
{}_0^C D_t^{1.98} w_0(t) &= 0 < f(t, w_0(t); \lambda), \\
w_0(0) &= 10 > u(0), \quad v_0(1) = 10 > u(1),
\end{aligned}
$$

that is, they are upper and lower solutions of type I.

The parameter values shown in Table 3 are computed by solving the approximate determining equations for η and ζ. Implementing (25), (26) yields the sequence terms plotted in the left panel of Figure 3, which also shows a plot of $u_4(t, \bar{\lambda}; \chi)$, obtained with the numerical-analytic technique for comparison. A comparison of the left- and right-hand sides of the BVP with the term $w_4(t)$ plugged in is shown in the right panel of Figure 3.

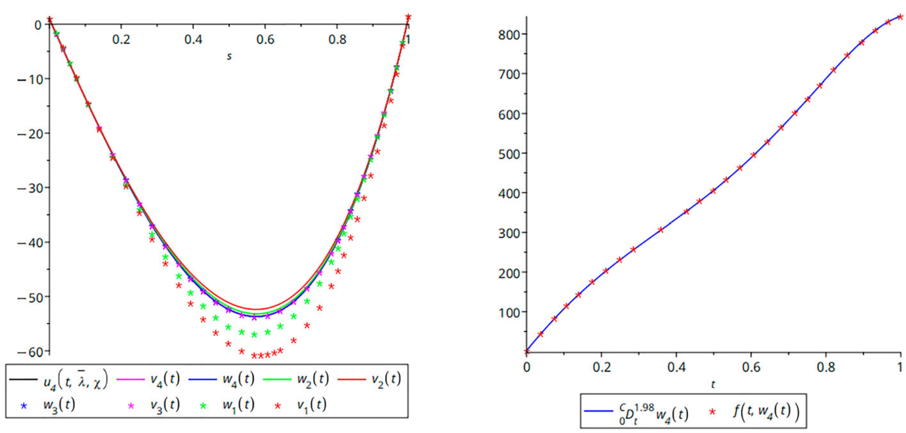

Figure 3. Plots of lower and upper solution sequence terms $v_n(t)$ and $w_n(t)$ for $n = 1, \ldots, 4$ (left) and left- (blue solid line) and right- (red dotted line) hand sides of equation (33) with $w_4(t)$ (right).

Table 3. Numerically calculated parameter values for $n = 0, \ldots, 4$.

n	η_n	ζ_n
1	−140.4727058	−144.2719451
2	−148.8266232	−148.0843496
3	−147.3904246	−147.5096299
4	−147.6109510	−147.5924731

Lastly, we construct the sequence $\tilde{u}_n(t, \bar{\lambda}; \bar{\chi})$, given in (32), by taking

$$\tilde{u}_0(t) = \frac{v_0(t) + w_0(t)}{2} = \frac{-148 + 10}{2},$$

$$\tilde{u}_n(t, \bar{\lambda}; \chi) = \alpha_0 + \chi t + t^p(\alpha_1 - \alpha_0 - \chi) + \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} f(s, \tilde{u}_{n-1}(s, \bar{\lambda}; \chi)\, \mathrm{d}s - t^p \int_0^1 (1-s)^{p-1} f(s, \tilde{u}_{n-1}(s, \bar{\lambda}; \chi)\, \mathrm{d}s\right].$$

The computes values of the parameter χ_n are shown in Table 4. The resulting plots for $n = 1, \ldots, 4$ are shown in the left panel of Figure 4, along with a plot of $u_4(t)$. The right panel of the same figure shows a comparison between the left- and right-hand sides of the BVP with $\tilde{u}_4(t)$. The recorded CPU time for calculating the values of χ_n for $n = 1, \ldots, 4$ was 79.8 s. In comparison, the CPU time for the calculation of the parameter values using the sequences (11) was 410.3 s.

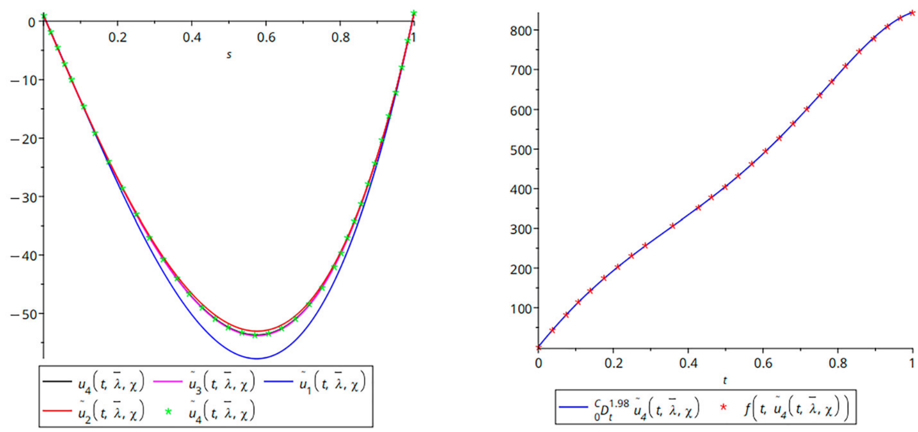

Figure 4. Plots of terms $\tilde{u}_n(t, \bar{\lambda}; \chi)$ for $n = 1, \ldots, 4$ (left) and left- (blue solid line) and right- (red dotted line) hand sides of equation (33) with $\tilde{u}_4(t, \bar{\lambda}; \chi)$ (right).

Table 4. Numerically calculated parameter values for $n = 0, \ldots, 4$.

n	χ_n
1	−143.4749133
2	−148.2377335
3	−147.4848912
4	−147.4848912

6. Conclusion

In this paper we studied a FBVP with a parameter-dependent right-hand side. We used fixed point theory to determine the values of the parameter for which there exists a unique solution to the BVP. The numerical-analytic technique was applied to construct a sequence of approximate solutions and their monotonicity behaviour was analysed. The approximation terms form a well-ordered sequence when the right-hand side in the FDE is strictly decreasing, whereas for a strictly increasing right-hand side, the approximating sequence is alternating. In the latter case, we applied the lower and upper solutions method in conjunction with the numerical-analytic technique to construct sequences of approximations, and proved their uniform convergence to the exact solution of the BVP. This approach can be used to simplify the terms of the approximating sequence and to therefore reduce the computational time. Our theoretical results are confirmed by a model example obtained from the equation modelling the motion of a gyre in the Southern hemisphere.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

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