Mass and Averaging Procedures for Spherically Symmetric Metrics in General Relativity

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Abstract

In general relativity, mass cannot generally be defined as the volume integral over some density, as it is in Newtonian physics. This thesis investigates this question and how a spherically symmetric spacetime provides the tools to construct a volume integral expression for mass. However, the volume integral for this mass contains no 'curvature factor', which would normally be expected for volume integral in a general relativistic setting. The static and dust solutions, important special cases of spherically symmetric spacetimes, are reviewed to demonstrate this peculiar property. In addition, we comment on the implications of this mass definition for the averaging procedures in cosmology.

Conventions and Notation

Throughout the thesis, the following conventions and definitions are used:

- Mass and energy are used interchangeably because of $E = mc^2$.
- Units with $c = 1$, where c is the speed of light.
- $\kappa \equiv 8\pi G$, where G is the gravitational constant.
- The choice for the signature of the metric is $(- + ++)$.
- The Einstein summation convention is adopted.
- Greek indices, e.g. ν, are summed from 0 to 3, whereas Latin indices, e.g. i, are summed from 1 to 3.
- $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$
- Working in coordinate expressions for the mathematical objects like vectors: V^μ represents $V = V^\mu \partial_\mu$.
- The covariant derivative along a trajectory $x^\mu(\tau)$ is denoted as ∇_μ and the proper time covariant derivative is denoted as $\frac{D}{d\tau} \equiv u^\mu \nabla_\mu$, where u^μ is $\frac{dx^\mu}{d\tau}$.
- For brevity of notation, the dependencies of the functions are sometimes implicit: $A = A(r, t)$.
- Spherical coordinates $\{t, r, \theta, \phi\}$ are used.
- Generally, a $(0, 2)$-tensor is denoted as $A_{\mu\nu}$ and A_{tt} indicates that the two indices have explicitly been filled in with t.
- $\dot{A} \equiv \partial_t A(r, t)$ and $A' \equiv \partial_r A(r, t)$.

Finally, we note that the definitions for the Christoffel symbols, the Ricci tensor and the Einstein tensor are given in appendix A.

Contents

Chapter 1

Introduction

In Newtonian physics, mass is a well-defined quantity. The mass of a closed system is expressed as the volume integral over the mass density. For instance, the mass of a sphere, where the density $\rho(r)$ depends on the radius, is given as $\int_0^R 4\pi\rho(r)r^2 dr$. In the Euclidean space, we can interpret the volume element as the area of the sphere, $4\pi r^2$, multiplied with the physical distance between two spherical shells, dr. In general relativity, the spatial geometry can be curved, so that the physical distance between the successive shells is not necessarily equal to "dr". As a result, the volume element has to include a factor to account for the spatial curvature and is referred to as the proper volume element. Naively, one then attempts to express the mass as the integral over ρ with respect to this proper volume element. Nevertheless, this seemingly natural generalization fails for mass in general relativity.

Admittedly, this is the standard procedure for integration of all other scalars in general relativity [6, Page 90]. Without any issues, the theory also allows for the definition of the mass of an isolated system, as measured by an outside observer, for example through the Schwarzschild mass or Kepler's mass. However, in general, it does not admit a meaningful, covariant and unique definition for mass as a volume integral over some energy density. This is because the gravitational field itself contributes to the total energy of a system but cannot be captured by a stress-energy tensor that can source the Einstein equation [22, Page 466 - 468].

It is then a natural question to ask whether a certain set of assumptions or conditions could allow us to localize the mass of a system. In this thesis, we will study spherical symmetric spacetimes, because the assumption of spherical symmetry is also used in other areas of physics to simplify problems. The first study of these spacetimes in a cosmological context was done by Georges Lemaître [17], Richard Tolman [27] and Hermann Bondi [3]. It turns out that spherically symmetry is indeed a sufficient condition to define a mass as an integral over some density, which will be called m in this thesis. Surprisingly, the integral for m does not include a curvature factor. This begs the main question that will be studied by this thesis: Why does the total mass-energy m of a closed spherically symmetric system not incorporate the curvature?

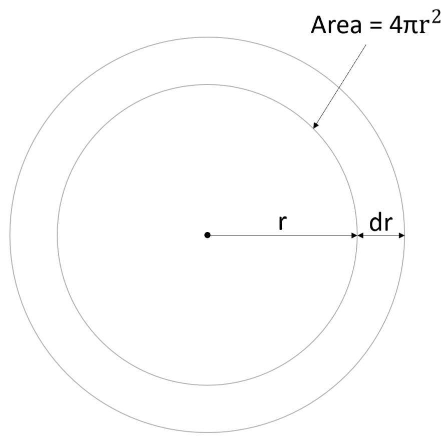

Figure 1.1: 2D plot of 3D spherical shells in Euclidean space. The area of the inner spherical shell is $4\pi r^2$ and the distance between the depicted shells is dr.

A related problem is the *averaging problem* in cosmology. In cosmology, the universe on a large enough scale can be accurately modeled through the well-known Friedmann-Lemaître-Robertson-Walker (FLRW) metric. [9] This metric assumes the universe to be homogeneous and isotropic. Homogeneity and isotropy mean[1] that all points in the universe are equivalent at a given point in time and that the universe looks the same from all directions. However, the real universe is inhomogeneous on small scales.[10]

When modeling the universe with the FLRW metric, we have implicitly assumed one of the two procedures that can be applied to an inhomogeneous universe: We can "average" the inhomogeneous metric to obtain FLRW and then plug the FLRW metric into the Einstein equation to describe the evolution of the universe. However, the possibility exists that the small inhomogeneities develop in such a way that they affect dynamics on larger scales. [24] The second, physically more accurate procedure would be to plug the inhomogeneous metric into the Einstein equation and apply the averaging afterwards. Also, due to the nonlinearity of Einstein's equation, these procedures are not at all equivalent: "$\langle G_{\mu\nu}(g_{\mu\nu})\rangle \neq G_{\mu\nu}(\langle g_{\mu\nu}\rangle)$", where the angled brackets indicate averages. The difference between them is referred to in the literature as *backreaction*. [15, Chapter 8]

Because averages of tensors over spacetime are also not well-defined in general relativity, the effect of back-reaction is typically studied through averaging

[1]The precise mathematical definitions can be found in [28, Page 91-93]

the evolution equations, which are equivalent to Einstein's equations. A subset of these evolution equations can entirely be expressed in terms of scalars, allowing for a well-defined averaging framework. The standard approach defines the average of a scalar as its proper volume integral divided by the total proper volume. [4] In spherically symmetric spacetimes, this average that includes the curvature factor is applied to ρ as well. In this context, it would perhaps be physically more reasonable to consider the integral without the curvature, which is more aligned with the definition of mass in spherically symmetric universes.

In chapter 2, we will give the general form of a spherically symmetric spacetime in a special coordinate system and we will look at three important special cases. One special case will be the static solutions, which is the simplest case where the problem of the missing curvature factor will become apparent. Chapter 3 then will detail a discussion of the definition of mass m. Consequently, chapter 4 will consider two averaging procedures of the scalar evolution equations: the standard averaging procedure and one that is more aligned with the definition of m. In chapter 5, we will make some final remarks. It is also important to highlight that reading this thesis requires the knowledge of General Relativity at the level of a first-year graduate course, see in particular the book *Spacetime and Geometry* by Sean Caroll in [6].

I would like to thank my supervisors Ana Achúcarro and Bas Janssens for their help and the many interesting discussions which demonstrated for me how difficult the problems this thesis tackles actually are. I am also grateful for the correspondence with Thomas Buchert and Roberto Sussman. Lastly, I would like to thank my parents for all their support.

Chapter 2

Spherically symmetric metrics

We suggestively express a spherically symmetric metric through the spherical coordinates (t, r, θ, φ). Because we do not assume anything about this coordinate system as of yet, we are allowed to do so without loss of generality. To define what a spherically symmetric metric is, we first have to define what a symmetry is in the context of a spacetime metric. A symmetry for a metric is called an isometry, which preserves the metric $g_{\mu\nu}$. Formally, this isometry in differential geometry is defined to be a diffeomorphism ϕ for which the pullback ϕ^* satisfies $\phi^* g_{\mu\nu} = g_{\mu\nu}$. A spherically symmetric spacetime can then be defined:

Definition. Following [28, Page 120], a spacetime is spherically symmetric if the mathematical group of its isometries includes a subgroup, which is isomorphic with the group of all rotations *SO(3)* and whose generic orbits[1] are given by the two-dimensional spheres S^2.

This definition of spherical symmetry imposes strong conditions on the form of the metric. The three standard rotational Killing vectors K^μ that generate *SO(3)* have to satisfy the Killing equation, $\nabla_{(\mu} K_{\nu)} = 0$. This leads to the most general, spherically symmetric metric in 4 dimensions:

$$ds^2 = \alpha(t,r)dt^2 + \beta(t,r)dtdr + \gamma(t,r)dr^2 + \delta(t,r)d\Omega^2, \tag{2.1}$$

where $d\Omega^2 \equiv d\theta^2 + sin^2\theta\, d\phi^2$ and the exact details can be found in [14, Page 82-86]. We notice that the invariance under rotations is reflected by the fact that it contains a multiple[2] of $d\Omega^2$, which should be the case as the metric on a unit sphere is canonically written as $d\Omega^2 = d\theta^2 + sin^2\theta\, d\phi^2$. Secondly, this is

[1] The set of points that can be reached by applying these rotations on a point on the spacetime manifold.

[2] The δ is invariant under the 'expected' transformations $t = g(t', r')$ and $r = h(t', r')$, but there exist pathological counterexamples, see the footnote in the book [14, Page 83 and 85].

reflected by the metric functions which are independent of θ and φ, except for the necessary $\sin^2(\theta)$ factor.

To simplify the metric (2.1), we will define *comoving* coordinates. By definition of comoving coordinates, the four-velocity U^μ of the fluid that we model through the energy-momentum tensor $T_{\mu\nu}$ only has a time component, i.e. U^μ will be proportional to $(1,0,0,0) \equiv \delta_0^\mu$ [3]. Therefore, the comoving coordinates follow the worldlines of the fluid. It must be emphasized that these coordinates are merely labels attached to the fluid and are therefore also called Lagrangian coordinates. In addition, we define coordinates that are *'synchronous'* as coordinates in which all the spacetime components of the metric g_{ti} are zero. It can be proved [14, Page 294] for a spherically symmetric spacetime that we can choose this coordinate system, simultaneously with the comoving coordinate system, such that g_{ti} vanish. The Lemaître-Tolman-Bondi metric for these synchronous-comoving coordinates is then given by:

$$ds^2 = -e^{C(r,t)}dt^2 + X^2(r,t)dr^2 + R^2(r,t)d\theta^2 + R^2(r,t)sin^2\theta\, d\phi^2, \qquad (2.2)$$

where $C(r,t)$, $X(r,t)$ and $R(r,t)$ are yet to be determined by the Einstein equation. Because the metric (2.2) should have Lorentzian signature, we can set $\alpha = e^{-C}$, $\gamma = X^2$ and $\delta = R^2$. With this form of the metric, the function R can be given physical meaning. For constant r and t, if you let $R(r,t)$ fully rotate by the angles θ and φ, it sweeps out a spherical shell of area $4\pi R^2$ by rotation. Then, R, referred to as the areal radius, is the physically meaningful, measurable radius of that sphere. We will assume in the following that $R(r=0)=0$ and that R is invertible.

Before substituting the metric (2.2) into the Einstein equation, we assume the following form for the energy-momentum tensor:

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \qquad (2.3)$$

which is the expression for a perfect fluid. u_μ is the covariant 4-velocity of the fluid, ρ is the energy density and P is the pressure. We have chosen the perfect fluid tensor [4], due to its rather simple dependence on only two new quantities, ρ and P. It is also chosen, because it is the most general $T_{\mu\nu}$ compatible with the homogeneity and isotropy assumptions for FLRW [28, Page 96]. As such, the perfect fluid tensor does not take into account any heat flux, radiation, viscosity or diffusion.

[3] These coordinates can always be defined by solving the equations $\frac{\partial x^\alpha}{\partial \tau}\frac{\partial x^i}{\partial x^\alpha} = 0$ [14, Page 294].

[4] A more general $T_{\mu\nu}$ for the spherically symmetric metric can meaningfully be allowed to contain anisotropies [15]

To compute the explicit matrix form of the energy-momentum tensor, the time-component of u^μ must be computed. The normalization of the 4-velocity gives us this component: $u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = -e^C (u^0)^2 = -1$, which results in the 4-velocity[5] $u^\mu = (e^{-\frac{C}{2}}, 0, 0, 0)$. Here, the positive root is chosen because the observer moves forward into the future. Lowering the index of u^μ gives $u_\mu = (-e^{\frac{C}{2}}, 0, 0, 0)$, so that the energy-momentum tensor is given as:

$$T_{\mu\nu} = \begin{pmatrix} \rho(r,t)e^{C(r,t)} & 0 & 0 & 0 \\ 0 & P(r,t)X^2(r,t) & 0 & 0 \\ 0 & 0 & P(r,t)R^2(r,t) & 0 \\ 0 & 0 & 0 & P(r,t)R^2(r,t)\sin^2(\theta) \end{pmatrix}$$

where it should be noted that because the metric only depends on r and t, by the Einstein equation, P and ρ may only depend on r and t as well. As such, these quantities are general enough to model spherically symmetric inhomogeneities.

Before we write down the equations resulting from the Einstein equation, there is something to be said about the Einstein equation itself. The elegant equation is written as:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.4}$$

Due to the metric compatibility ($\nabla^\mu g_{\mu\nu} = 0$), the symmetries of the Einstein equation $G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$ are still conserved with the addition of a term $\Lambda g_{\mu\nu}$ on the left-hand side of (2.4), where Λ is referred to as the cosmological constant. However, its physical meaning is more apparent if we shift it to the right-hand side. With the addition of Λ, the Einstein equation is then written as:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{2.5}$$

We can incorporate Λ in $T_{\mu\nu}$ by defining $\overline{T_{\mu\nu}} \equiv T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu}$ so that: $G_{\mu\nu} = 8\pi G \overline{T_{\mu\nu}}$. Positive Λ can then be seen as a negative pressure term which can cause the expansion of the universe. Λ is often associated with "vacuum energy", a form of dark energy, as it then also represents the energy density of an empty universe if we set $T_{\mu\nu}$ to zero.

Using appendix A for the Einstein tensor components as well as the energy-momentum tensor components, we can now construct these equations[6] as a result from the Einstein equation (2.5):

[5] As we will later see, for zero pressure, this means that in these coordinates the fluid is at rest.

[6] These equations can also be derived much more simply using exterior differential forms.[22, Page 355-357]

$$G_{tt} = \frac{2\dot{X}\dot{R}}{XR} + \frac{\dot{R}^2}{R^2} + e^C\left(\frac{1}{R^2} - \frac{2R''}{X^2R} - \frac{(R')^2}{X^2R^2} + \frac{2X'R'}{X^3R}\right) \tag{2.6a}$$
$$= 8\pi G \cdot \rho e^C + \Lambda e^C,$$
$$G_{rt} = 2\frac{\dot{X}R'}{XR} + \frac{\dot{R}C'}{R} - 2\frac{\dot{R}'}{R} = 0, \tag{2.6b}$$
$$G_{rr} = \frac{(R')^2}{R^2} + \frac{R'C'}{R} - \frac{X^2}{R^2} - e^{-C}X^2\left(\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \frac{\dot{R}\dot{C}}{R}\right) \tag{2.6c}$$
$$= 8\pi G \cdot PX^2 - \Lambda X^2,$$
$$G_{\theta\theta} = \frac{G_{\varphi\varphi}}{\sin^2(\theta)} = \frac{R^2}{X^2}\left(\frac{R''}{R} + \frac{C''}{2} + \frac{(C')^2}{4} - \frac{X'R'}{XR} + \frac{R'C'}{2R} - \frac{X'C'}{2X}\right) \tag{2.6d}$$
$$+ e^{-C}R^2\left(-\frac{\ddot{X}}{X} - \frac{\ddot{R}}{R} - \frac{\dot{X}\dot{R}}{XR} + \frac{\dot{X}\dot{C}}{2X} + \frac{\dot{R}\dot{C}}{2R}\right)$$
$$= 8\pi G \cdot PR^2 - \Lambda R^2,$$

where all the other components of the Einstein equation are identically zero. From equations (2.6) and the conservation of the energy-momentum tensor ($\nabla_\mu T^{\mu\nu} = 0$), we can derive the following set of equations:

$$m(r,t) = \frac{1}{2G}\left[R - \frac{(R')^2R}{X^2} - \frac{1}{3}\Lambda R^3 + e^{-C}R(\dot{R})^2\right], \tag{2.7a}$$
$$\frac{\partial}{\partial r}m(r,t) = 4\pi\rho(r,t)R^2R', \tag{2.7b}$$
$$\frac{\partial}{\partial t}m(r,t) = -4\pi P(r,t)R^2\dot{R}, \tag{2.7c}$$
$$e^{-\frac{C}{2}}\partial_t(e^{-\frac{C}{2}}\dot{R}) = -(4\pi GP(r,t)R + \frac{Gm}{R^2}) + \frac{C'R'}{2X^2} + \frac{1}{3}\Lambda R, \tag{2.7d}$$
$$\dot{R}' = \frac{\dot{X}}{X}R' + \frac{\dot{R}C'}{2}, \tag{2.7e}$$
$$0 = \partial_t\rho + (\frac{\dot{X}}{X} + \frac{2\dot{R}}{R})(\rho + P), \tag{2.7f}$$
$$0 = \partial_r P + \frac{1}{2}C'(P + \rho), \tag{2.7g}$$

where equations (2.7a) - (2.7e) are derived in appendix B and equations (2.7f) and (2.7g) in appendix C. Equation (2.7e) essentially encapsulates the information about our use of the comoving coordinate system [5]. Equations (2.6) and equations (2.7) were derived in a similar fashion as in [14, Chapter 18].

With an equation of state, which for example relates ρ to P, or another condition to close the system of equations (2.7) for integration, the six unknown functions R, X, ρ, P, C and m can be determined. The additional constraint is needed due to the introduction of an additional function as a result of integration. The equations you would need, in addition to the equations (2.7), for a general coordinate system are given in [5].

2.0.1 Physical Interpretation

The integral form of equation (2.7b) is analogous to the Newtonian integral:

$$m(r,t) = 4\pi \int_0^r \rho R^2 R' dr = 4\pi \int_0^R \rho s^2 ds, \tag{2.8}$$

where it must be stressed that in the Newtonian case the areal radius and 'the proper distance radius' are equal, but in the general relativistic setting the s can only be interpreted as the areal radius. Equation (2.8) may encourage us to interpret m in (2.7a) as a mass. In that case, equation (2.7c) indicates that the rate of change of mass is equal to the power transferred by the pressure to the boundary, which is a dynamical spherical shell with radius R [22, Exercise 32.7 and Box 23.1]. Finally, both equations (2.7b) and (2.7c) give the mandatory result of $m = 0$ when both $\rho = P = 0$. This interpretation of mass will be considered in more detail in chapter 3.

Equation (2.7d) is the relativistic version of the classical law of gravitation [5]. To see this, we will have to adjust the equation slightly. Because t is just a label, we would like to consider the physically meaningful proper time and the derivative taking this proper time into account. To this end, we introduce the proper time derivative $\frac{d}{d\tau} \equiv e^{-\frac{C}{2}} \frac{\partial}{\partial t}$, which is the proper time in the frame of the fluid, and the physical radial velocity $\frac{dR}{d\tau} \equiv e^{-\frac{C}{2}} \dot{R}$ to obtain the following equation:

$$\frac{d^2 R}{d\tau^2} = -(4\pi G P(r,t) R + \frac{Gm}{R^2}) + \frac{C' R'}{2X^2} + \frac{1}{3}\Lambda R, \tag{2.9}$$

where the left-hand side is proper acceleration, the first term in the parenthesis a pressure term and the second term in the parenthesis the gravitational acceleration. The other terms are the ones that are the 'relativistic' terms and do not have a nonrelativistic analog.

Equation (2.7f) can be seen as the first law of thermodynamics without any heat (transfer) [26]. To understand this, we consider the proper spatial infinitesimal volume $dV = \sqrt{|g_{ij}|} d^3x = XR^2 sin(\theta) dr d\theta d\varphi$, where $|\ .\ |$ denotes the determinant, and we differentiate with respect to t: $d\dot{V} = (\dot{X}R^2 + 2XR\dot{R}) dr d\theta d\varphi = (\frac{\dot{X}}{X} + 2\frac{\dot{R}}{R}) dV$. Now, equation (2.7f) is identical to the equation:

$$\partial_t(\rho dV) + P \partial_t(dV) = 0, \tag{2.10}$$

which can also be written as the thermodynamic law: $dE = -PdV$.

Equation (2.7g) is referred to as the Euler equation and reduces to the following equation in the Newtonian limit[7]:

$$\rho \partial_r V = -\partial_r P, \tag{2.11}$$

which describes the equilibrium between the gradient of the pressure and gravity [22, Page 601-602]. It represents the Newtonian view on the hydrostatic equilibrium, as we will see in the section about static solutions.

In the following sections, we will simplify equations (2.7) and solve for the metric functions for three special cases. In section 2.1, we will look at an isotropic and homogeneous universe, which will result in the FLRW metric and the two well-known Friedmann equations. Its inclusion is based on its ubiquity in the literature, its simplicity and accuracy in modeling the universe on a large scale. In subsection (2.1.1), we will look at subcase of FLRW to show a specific coordinate choice can give expansion. It will also be used later in this thesis to showcase the difference between two definitions for an average. The second case we will look at, will be the static solutions in section 2.2, which can be used to model realistic massive objects in space and the potential collapse thereof. Lastly, the Lemaître-Tolman-Bondi (LTB) metric with zero pressure will be studied in section 2.3, because they produce analytic solutions and the differential equations in its context will be needed for the averaging. To summarize and visualize this, a tree with the various cases is added in 2.1.

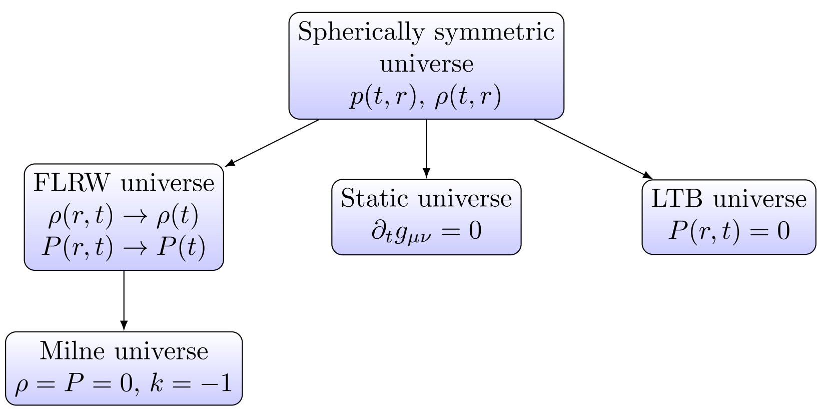

Figure 2.1: This tree shows the different special cases of the spherically symmetric spacetimes considered with their additional assumptions.

[7] In the Newtonian limit, $g_{tt} \approx -1 - C$, where $V \equiv \frac{C}{2}$ is the Newtonian potential, as the metric becomes almost time-independent and approaches the Minkowski metric, and $P << \rho$ as the velocities are small ($v << c$)

2.1 Isotropic & Homogeneous Universe: FLRW

We take the universe to be homogeneous everywhere. That has the consequence that the energy density and pressure only depend on time and not on the coordinate r anymore. It should be remarked that this model cannot model stars and voids, for instance, because the energy density is necessarily equal everywhere. The following derivation closely follows the calculations in the paper [23]. With the assumptions, equations (2.7g) and (2.7f) reduce to:

$$\partial_t \rho + (\frac{\dot{X}}{X} + \frac{2\dot{R}}{R})(\rho + P) = 0, \tag{2.12}$$

$$\frac{1}{2}C'(P + \rho) = 0. \tag{2.13}$$

Equation (2.13) gives $C' = 0$, assuming $\rho + P \neq 0$. Renaming the time-coordinate t as $\int e^{-C(\bar{t})} d\bar{t}$ then gives $C = 0$ in this new coordinate system. In general relativity, coordinates do not carry any physical meaning until they are interpreted. Until then, they are merely labels for the physical system and we are allowed scale or rename as we like. Equation (2.7e) now reduces to:

$$\dot{R}' = \frac{\dot{X}R'}{X}, \tag{2.14}$$

with the solution being $X(r,t) = h(r) \cdot R'(r,t)$, where $h(r)$ is a function of only the r-coordinate. Equation (2.7b) gives the following expression for m, because ρ is independent of r:

$$m(r,t) = 4\pi \int_0^r \rho(t) R^2 R' dr = \frac{4\pi}{3}\rho(t) R^3. \tag{2.15}$$

At this point, we take the derivative of expression (2.15) with respect to t and set it equal to the right-hand side of equation (2.7c):

$$\frac{4\pi}{3}\dot{\rho}R^3 + 4\pi\rho R^2 \dot{R} = -4\pi P(r) R^2 \dot{R} \tag{2.16}$$

Multiplying by $\frac{3}{4\pi R^3}$ gives the equation:

$$\dot{\rho} + 3\frac{\dot{R}}{R}(\rho + P) = 0 \tag{2.17}$$

Comparing with (2.12) gives $\frac{\dot{X}}{X} = \frac{\dot{R}}{R}$ and if we combine this with (2.14), we can write down the following equation for R:

$$\frac{\dot{R}}{R} = \frac{\dot{R}'}{R'}. \tag{2.18}$$

This is solved by the separable function $R(r,t) = g(r)S(t)$. Because R appears only in the rr-component of the metric (2.2), we can write $g(r) = r \cdot \gamma(r)$ and by the same argument with which we set $C = 0$, rename the r coordinate, so that we can set $R = rS(t)$. Then, we substitute both the expression for m in (2.15), $X = h(r)R'$ and $R = rS(t)$ into (2.7a) to obtain the equation:

$$\frac{4\pi}{3}\rho(t)r^3S^3(t) = \frac{1}{2G}\Big[rS(t) - \frac{S^3(t)r}{h^2(r)S^2(t)} + r^3S(t)(\dot{S}(t))^2 - \frac{1}{3}\Lambda r^3S^3(t)\Big]. \tag{2.19}$$

Rewriting this equation gives:

$$\delta(r) \equiv \frac{1}{h^2(r)} = 1 - K(t)r^2, \tag{2.20}$$

where $K(t) = \frac{8\pi G\rho S^2(t)}{3} + \frac{1}{3}\Lambda S^2(t) - (\dot{S}(t))^2$. Because δ is a function of r only, K(t) cannot depend on t, meaning that K is constant. It is important to know that we should consider three cases for the constant K: K is positive, K = 0 and K is negative. Later on, we will rescale K to k, such that k can only be 1, 0 or -1. k, then, is associated with the curvature of space and the different values correspond to different geometries of the universe. $k = +1$ is associated with the geometry of a 3-sphere, $k = 0$ is associated with flat Euclidean space and $k = -1$ is associated with hyperbolic geometry [8] [18].

With $X^2 = h^2(R')^2 = \frac{S^2(t)}{1-Kr^2}$ and $R^2 = r^2S^2(t)$, the metric can now be written in the form:

$$ds^2 = -dt^2 + S^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2d\theta^2 + r^2\sin^2(\theta)d\varphi^2\right), \tag{2.21}$$

which can be written in the standard FLRW form by defining $a^2(t) = \frac{S^2(t)}{|K|}$, $(\overline{r})^2 = |K|r^2$ and $k = sign(K)$, so that k is only allowed to be -1, 0 or +1:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2(\theta)d\varphi^2\right), \tag{2.22}$$

where $\overline{r}$ was renamed r again. The first Friedmann equation immediately follows from the expression for K:

$$K = \frac{8G\pi\rho S^2(t)}{3} + \frac{1}{3}\Lambda S^2(t) - (\dot{S}(t))^2, \tag{2.23}$$

or in standard form:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{2.24}$$

The second Friedmann equation is found by taking the derivative of (2.23) with respect to time and substituting (2.17) into that resulting equation, which eventually gives the second Friedmann equation:

[8] $k = +1$ is also associated with a 'closed' universe, as its spherical geometry is finite in the sense that parallel lines eventually have to meet. For $k = 0$, the parallel lines stay parallel and for $k = -1$, the parallel lines diverge, such that it is called an 'open', infinite universe.

$$\frac{\ddot{S}}{S} = \frac{\Lambda}{3} - \frac{8\pi G}{6}(\rho + 3P), \tag{2.25}$$

or in standard form:

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3P). \tag{2.26}$$

The two Friedmann equations describe the rate and acceleration of the expansion of the universe, given by $a(t)$.

2.1.1 Milne Universe

We consider the special case of the FLRW universe with $k = -1$, the case associated with negative spatial curvature, and $\rho = P = 0$.. This results in the differential equation from the first Friedmann equation (2.24):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{a^2}, \tag{2.27}$$

That gives $a(t) = t$, with the integration constant set to zero, so that the scale factor vanishes for $t = 0$. We remark that 2.27 explains why we are forced to choose $k = -1$, because $(\dot{a})^2 = -1$ has no real solutions. Thus, the metric for this special case of the FLRW universe, which is called the Milne universe, is given by:

$$ds^2 = -dt^2 + t^2[\frac{dr^2}{1+r^2} + r^2(d\theta^2 + sin^2(\theta)d\phi^2)] \tag{2.28}$$

or, equivalently with $r^2 = \sinh^2(\chi)$:

$$ds^2 = -dt^2 + t^2 d\chi^2 + t^2 sinh(\chi)^2 d\Omega^2 \tag{2.29}$$

Einstein's equation implies that the space should be flat without a source term. Indeed, we can also deduce metric (2.28) by rewriting the Minkowski metric in polar coordinates, which is given by:

$$ds^2 = -dT^2 + dR^2 + R^2 d\Omega^2. \tag{2.30}$$

If we assume $T = t\cosh(\chi)$ and $R = t\sinh(\chi)$ with $t > 0$, the metric (2.28) is again obtained. This transformation of variables only holds if it is assumed that $T > 0$, as $\cosh(\chi) > 0$, and $T^2 > R^2$, because $T^2 - R^2 = t^2 \ > \ 0$. This means that Milne coordinates are an incomplete chart of the Minkowski space, as they only cover a quarter of the entire space. In particular, they are only valid within the future light cone.

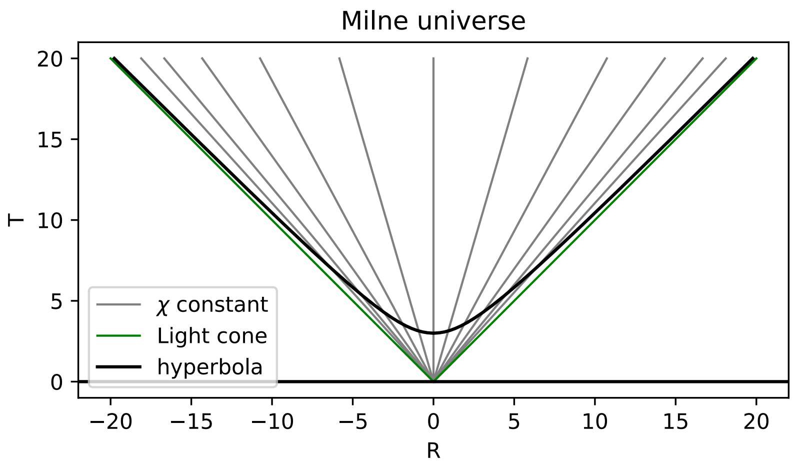

Figure 2.2: Milne's universe plotted in 2D with the Minkowski coordinates R and T, with spherical coordinates θ and φ left out. The black hyperbola is the spatial hypersurface for proper time $t = 3$. The green lines give the future light cone and the grey lines give the constant χ worldlines for comoving observers. Although $R \geq 0$, the plot also includes negative values for R to better illustrate the hyperbolic geometry.

For FLRW in general, a comoving observer, who "moves with the coordinates", is a reasonable choice for an observer, because those observers see the universe as isotropic and homogeneous. Then, for comoving observers in Milne, where the spatial coordinates χ, θ and φ are necessarily fixed, the proper time is given by t, which is immediately clear from (2.29). Because $T^2 - R^2 = t^2$, the spatial 3D-surfaces with constant proper time t are given by hyperboloids in Minkowski space for comoving observers. In figure 2.2, the Minkowski space and therein the Milne universe are plotted in 2D. The black line depicts one hyperbola, representing the $t = 3$-surface of 'simultaneity' for the comoving observers[9] The inertial worldlines for the observers are given for constant χ, because $v = \tanh(\chi)$. An observer fixed at the origin accordingly sees all of these observers moving away at constant speed. We can conclude that the apparent expansion $a(t)$ in the metric (2.29) is due to the choice of coordinates. This underlines the fact that coordinates in general relativity are just labels and carry no physical meaning until they are properly interpreted. The Milne universe will be used in section 4.5 to apply our averaging procedures.

[9]The usual constant T-surface is only the surface of simultaneity for an fixed observer at the origin of the Minkowski diagram.

2.2 Static Universe: TOV-equation

For the spacetime 2.2 to be static[10], it has to be stationary, i.e. the metric components are all independent of time now:

$$ds^2 = -e^{C(r)}dt^2 + X^2(r)dr^2 + R^2(r)d\theta^2 + R^2(r)sin^2\theta\, d\phi^2, \tag{2.31}$$

With the same trick of relabeling coordinates[11] as for the FLRW solution, the metric can be written without loss of generality:

$$ds^2 = -e^{C(r)}dt^2 + X^2(r)dr^2 + r^2d\theta^2 + r^2sin^2\theta\, d\phi^2. \tag{2.32}$$

Because $G_{\mu\nu}$ depends on the metric, it is also time-independent. The Einstein equation then immediately implies that the pressure and the energy density are time-independent. With metric (2.32), the quantity m in (2.7a) becomes:

$$m(r) = \frac{1}{2G}\Big[r - \frac{r}{X^2}\Big], \tag{2.33}$$

or equivalently for the metric component X^2:

$$X^2 = \frac{1}{1 - \frac{2Gm}{r}}. \tag{2.34}$$

Equations (2.7b) and (2.7c) reduce to the independent equation for m in terms of ρ:

$$m(r) = \int_0^r 4\pi\rho(s)s^2 ds, \tag{2.35}$$

Notice that this expression is different from what expression you normally expect in general relativity:

$$\overline{m}(r) = \int_0^r 4\pi\rho(s)s^2 X ds, \tag{2.36}$$

where the integral is now taken over the proper volume element $dV_p = 4\pi r^2 X(r)dr$. This apparent issue will be discussed in more detail in chapter 3. Equations (2.7f) and (2.7g) lead to one non-trivial equation:

$$-\partial_r P(r) = (\rho + P)\frac{C'}{2} \tag{2.37}$$

The dynamical equation (2.7d) becomes in rewritten form:

$$\Big(4\pi GP(r)r + \frac{Gm(r)}{r^2}\Big) \cdot \frac{1}{1 - \frac{2Gm}{r}} = \frac{C'}{2}, \tag{2.38}$$

[10]The definition of a static spacetime is that it must have zero rotation and be stationary. The first condition is already satisfied [14, Chapter 18].

[11]The exact details and a similar derivation of the TOV equation can be found in [6, 194-195]

where Λ is again set to zero. Substituting this equation into (2.37) gives the Tolman-Oppenheimer-Volkoff (TOV) equation[12]:

$$\frac{dP(r)}{dr} = -\frac{1}{2}\frac{\big(\rho(r)+P(r)\big)\big(Gm(r)+4\pi r^3 P(r)\big)}{r\big(r-2Gm(r)\big)}. \tag{2.40}$$

The TOV-equation, which is a balance equation between pressure and gravity, can be used to model neutron stars and white dwarfs. It can also be used to give an estimate for limitations on some characteristics of such bodies before collapse due to gravitation is imminent. As mentioned for the general LTB metric, we have to provide an equation of state or another condition to be able to solve this further. To model a star and simplify the problem even more, we posit the assumption that the fluid[13] within the star is incompressible and, in fact, that ρ is constant:

$$\rho(r) = \begin{cases} \rho_0, & \text{if } r < R_0 \\ 0, & \text{if } r > R_0, \end{cases} \tag{2.41}$$

where ρ_0 and R_0 are constants. From (2.35) we write down the mass:

$$m(r) = \begin{cases} \frac{\kappa}{3}\rho_0 r^3, & \text{if } r < R_0 \\ \frac{\kappa}{3}\rho_0 R_0^3, & \text{if } r > R_0, \end{cases} \tag{2.42}$$

which means the mass is constant from outside the star. Solving the differential equation[14] (2.40) gives the pressure:

$$P(r) = \rho_0 \left[\frac{R_0\sqrt{R_0-2GM} - \sqrt{R_0^3-2GMr^2}}{\sqrt{R_0^3-2GMr^2} - 3R_0\sqrt{R_0-2GM}}\right] \tag{2.43}$$

From plugging this in (2.37), we determine e^c for $r < R_0$:

$$e^{C(r)} = \left(\frac{3}{2}\left(1-\frac{2GM}{R_0}\right)^{1/2} - \frac{1}{2}\left(1-\frac{2GMr^2}{R_0^3}\right)^{1/2}\right)^2, \tag{2.44}$$

so that the full metric for that region is known. The metric reduces to the Schwarzschild metric for $r \geq R_0$ with the Schwarzschild mass equal to $\frac{\kappa}{3}\rho_0 R_0^3$.

[12]Note that for $P << \rho$ and $m(r) << r$, TOV will reduce to the classical equation of hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2} \tag{2.39}$$

[13]The TOV equation is a hydrostatic equation, which warrants the use of the word "fluid".
[14]All the details for integration can be found in the slides of the reference [20].

2.3 Dust Universe: Lemaître-Tolman-Bondi

We take the pressure to be zero, which is the case for an idealized pressureless fluid called *dust*. Because the pressure is zero, equation (2.7g) immediately implies that $C' = 0$. Like in the case of FLRW, we can then set $C = 0$. Then the energy-momentum tensor (2.3) reduces to $T_{\mu\nu} = \rho u_\mu u_\nu$, which can be written in matrix form:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.45}$$

At this point, we can simplify the equations (2.7) describing the LTB model. The differential "mass" equations (2.7c) and (2.7b) reduce to:

$$\frac{\partial}{\partial t} m(r,t) = 0, \tag{2.46}$$

$$\frac{\partial}{\partial r} m(r,t) = 4\pi\rho(r,t)R^2R', \tag{2.47}$$

where the first equation motivates us to write m(r,t) = m(r), which means that the quantity m is conserved over time. Then, we take $C = 0$ and $P = 0$ in the dynamical equation (2.7d) and multiply by $2\dot{R}$ to obtain:

$$2\ddot{R}\dot{R} = -\frac{2Gm}{R^2}\dot{R} + \frac{2}{3}\Lambda R\dot{R} \tag{2.48}$$

This is similar to taking the partial derivative with respect to time of a given expression:

$$\frac{\partial}{\partial t}\left((\dot{R})^2 - \frac{2Gm}{R} - \frac{1}{3}\Lambda R^2\right) = 0. \tag{2.49}$$

Integrating and rewriting this equation results in:

$$(\dot{R})^2 = E(r) + \frac{2Gm}{R} + \frac{1}{3}\Lambda R^2, \tag{2.50}$$

where E(r) is an arbitrary function of integration that depends solely on r. We can also find an expression for X^2 under the conditions that the pressure is zero (and C is equal to zero). The quantity $\frac{2Gm}{R}$, using equation (2.7a), becomes:

$$\frac{2Gm}{R} = 1 - \frac{(R')^2}{X^2} - \frac{1}{3}\Lambda R^2 + (\dot{R})^2 \tag{2.51}$$

Using the dynamical equation (2.50), we can express X^2 as the following:

$$X^2 = \frac{(R')^2}{1 + E(r)} \tag{2.52}$$

Now, the LTB metric for dust can be written in the following standard form:

$$ds^2 = -dt^2 + \frac{(R'(r,t))^2}{1+E(r)}dr^2 + R^2(r,t)(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.53}$$

where it must be remarked that $E \geq -1$ for all r as it must respect the signature[15]. Please note that the dust FLRW metric is a special case of this LTB metric if we set $R(r,t) = ra(t)$ and $E(r) = -kr^2$. For $\Lambda = 0$, equations (2.50) and (2.47) yield the following system of differential equations for the LTB metric (2.53):

$$(\dot{R}(r,t))^2 = E(r) + \frac{2Gm(r)}{R(r,t)}, \tag{2.54}$$

$$\frac{m'(r)}{R^2(r,t)R'(r,t)} = 4\pi\rho(r,t). \tag{2.55}$$

From Newtonian physics, these equations seem to be all familiar; there, the mass can be calculated by integrating the density over a sphere and E is the energy in the equation for the conservation of energy for a spherical shell ($\frac{1}{2}(\dot{r})^2 - \frac{Gm}{r} = E$). In Newtonian physics, the latter describes an object in a central potential with examples such as the Kepler problem or, even at smaller scales, the Coulomb potential. This motivates the interpretation of m as the mass and E as the energy of the physical shells of size R. Because $E(r)$ and $m(r)$ are the functions resulting from integration and we have derived two equations, we can solve this. The solutions of this differential equation are (see the explicit computations in appendix D) split into three cases for the function $E(r)$:

For $E(r) > 0$:

$$\frac{Gm(r)}{E(r)^{\frac{3}{2}}}(\sinh(z) - z) = (t - t_0(r)), \tag{2.56}$$

$$R(r,t) = \frac{2Gm(r)}{E(r)}\sinh^2(u) = \frac{Gm(r)}{E(r)}(\cosh(z) - 1) \tag{2.57}$$

For $E(r) = 0$:

$$R(r,t) = \left[\frac{9}{2}Gm(r)(t - t_0(r))^2\right]^{\frac{1}{3}}, \tag{2.58}$$

For $E(r) < 0$:

$$\frac{Gm(r)}{(-E(r))^{\frac{3}{2}}}(z - \sin(z)) = (t - t_0(r)), \tag{2.59}$$

$$R(r,t) = \frac{2Gm(r)}{K}\sin^2(\frac{z}{2}) = \frac{Gm(r)}{-E(r)}(1 - \cos(z)). \tag{2.60}$$

[15] The case that $E = -1$ is allowed under certain conditions [14, Section 18.10].

Here $t_0(r)$ is the function of integration arising from solving the differential equations. It is called the 'bang-time function', representing the time where the Big-Bang happened, and it is allowed to depend on r. The name is motivated by the choice that $R(t_0(r), r) = 0$ during the integration. From the equations, we can also determine the interval in which the parametric coordinate z is permitted to vary. For $E < 0$, at start-time $t = t_0(r)$, $z = 0$ and $R(r, t) = 0$. From there, the term $1 - \cos(z)$ lets R increase until a maximum and then decrease until $z = 2\pi$, where $R = 0$ again. The formula tells us that R will increase again if we increase z, but physically we have hit a singularity. Indeed, the solutions were derived from (2.54), which does not hold at $R = 0$ [16]. From (2.55), we can see that as the physical radius becomes zero, the density will have to diverge to infinity. The singularity $R = 0$ is therefore called shell-focusing. As a consequence, z is only allowed to range between 0 and 2π. From similar reasons for the case of $E > 0$, it can be deduced that z is allowed to range between 0 and ∞.

Another singularity in the LTB-model is given by $R' = 0$, which can be seen from (2.55). Both $R = 0$ and $R' = 0$ are situations where the LTB-model does not hold anymore. Both are curvature singularities, which is confirmed by the coordinate invariant Ricci scalar (A.6). Provided that $R \neq 0$, $R' = 0$ is called a shell crossing, because R is the same for different values of r due to the definition of the derivative. $R' = 0$ is seen as a 'weak singularity', because a shell-crossing will not 'crush' an object in a volume [13]. In the real universe, the presence of a gradient in pressure also prevents shells from crossing each other. This could be interpreted as a limitation of the LTB-model in accurately modeling the real universe.

2.3.1 Physical interpretation of function $E(r)$

Next to the physical interpretation of energy, there is another view we could take on E. For this, we start by considering the three-dimensional t=constant-hypersurfaces (see appendix E) in the dust LTB metric (2.53). Subsequently, we can compute the *3D Riemann tensor* components for the spatial metric:

$$ds^2_{spatial} = \frac{1}{1 + E(R)} dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.61}$$

where we relabeled the coordinate r to R so that $R'(r)dr = dR$ and $E(r(R)) = E(R)$. This relabeling is necessary, as we can attribute physical meaning to R but not to r. Computation of all the nonzero components[14, Section 18.3]

[16] An attentive reader will raise the issue that $R = 0$ at $z = 0$, so the model already starts at a singularity, which is associated with the Big Bang. This is true, and therefore the model actually only describes the evolution from $t > t_0$.

gives:

$$R_{r\theta r\theta} = R_{r\varphi r\varphi} = \frac{-E'(R)}{2R}, \tag{2.62}$$

$$R_{\theta\varphi\theta\varphi} = -\frac{E}{R^2}, \tag{2.63}$$

where the prime now denotes differentiation with respect to R. If E is zero, then all the 3-Riemann curvature tensor components are zero. Because they will be zero in every coordinate system due to the linear nature of the transformation laws for tensors, the curvature is zero and every constant t surface will be flat space for $E = 0$. If $-\frac{E}{R^2}$ is constant, which we suggestively call k after the section on the FLRW metric, then $R_{\theta\varphi\theta\varphi}$ is also constant. Because $E' = -2kR$ in that case, both $R_{r\theta r\theta}$ and $R_{r\varphi r\varphi}$ will be constant as well and all the nonzero components will be equal to k. With $E = -kR^2$, the spatial metric can be rewritten:

$$ds^2_{spatial} = \frac{dR^2}{1 - kR^2} dR^2 + R^2(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{2.64}$$

which is similar to the spatial metric for FLRW (2.22) and is equal for $R(r) = r$. The quantity $-E$ therefore measures how curved the spatial hypersurfaces are, in the same way k does this for FLRW metric [18, Chapter 4] [28, Chapter 5]. The difference, however, is that the curvature is not constant and can depend on the position r in the universe. Because of this interpretation of E, the functions involving E, in particular $\frac{1}{\sqrt{1+E}}$, will be referred to as the curvature (factor) in the following.

Chapter 3

Misner-Sharp Mass

In general relativity, there exists no proper definition of the mass or energy in terms of an integral expression of a density. The covariant energy-momentum tensor, which acts as a source term for the curvature, does not incorporate the energy contribution of the gravitational field itself [1].

It is also not possible to construct a tensor which captures this gravitational field. This can be made precise by considering (an extended version of) Lovelock's theorem, which essentially tells you that no such a general tensor exists that both describes the gravitational field and respects the 'expected' properties, such as being symmetric and divergence-free. A proof of this can be found in [7]. It is important to note that the standard textbook argument related to the equivalence principle, e.g. in [22, Section 20.4], is not rigorous at all [2].

To emphasize, this does not mean that general relativity implies that the gravitational field does not exist. However, it does tell you that it is not meaningful to consider it in a local context. Globally, due to the contribution of the gravitational field, a neutron star, for instance, has a lower mass-energy than it would have if we counted the sum of its constituents. [22, Section 20.4]

Now, an attentive reader might point to the limiting case of the weak-field and slow motion, where general relativity reduces to Newtonian theory and where the total mass integral expression can be defined. Indeed, this gives the insight that, with specific conditions or symmetries for the metric, it could

[1] For this contribution, only "pseudotensors", objects that for their definition depend on the choice of coordinates, can be defined[22, Page 464-467]. In addition, those pseudotensors for the gravitational field are not uniquely defined.

[2] It goes as follows: "By the equivalence principle or the existence of Riemannian normal coordinates, it is always possible to choose a coordinate system for which in a point p in spacetime: $g_{\mu\nu}(p) = \eta_{\mu\nu}$ and $\partial_\sigma g_{\mu\nu}(p) = 0$. This implies that the Christoffel symbols are zero at p. Then, it is argued that the 'gravitational field tensor' will also be zero and by the general transformation laws for tensors, it will be zero in every coordinate system, which is inconsistent with the existence of the gravitational field." The second order derivative of the metric, however, is not zero. Therefore, this argument implicitly assumes that the gravitational field tensor cannot depend on higher order (larger than one) derivatives of the metric, but there is no clear physical reason given why this is the case.

be possible to construct a meaningful integral expression for the total mass-energy of a system in this spacetime. To start, we look back at the spherically symmetric and static universe of section 2.2. The calculations in both upcoming sections follow the reference [11].

3.1 m in spherically symmetric and static universe

The "mass"-like quantity in the static, spherically symmetric metric (2.32) is given as:

$$m(r) = 4\pi \int_0^r \rho(s)s^2 ds \tag{3.1}$$

Because of its form, we are motivated to consider m the total mass-energy. However, as mentioned earlier, one would expect to write the mass as the energy density integrated with respect to the proper volume element:

$$\overline{m} = 4\pi \int_0^r \frac{\rho(s)s^2}{\sqrt{1-\frac{2Gm(s)}{s}}} ds, \tag{3.2}$$

where the proper volume element is given by: $\sqrt{|g_{ij}|}\ d^3x = \left(1-\frac{2Gm(r)}{r}\right)^{-\frac{1}{2}} r^2 sin\theta\ drd\theta d\phi$. The number of baryons inside a star, for example, can be written as an integral expression of some density. And indeed, the correct way to calculate the total number of baryons within a radius r would be to use the proper volume:

$$N = 4\pi \int_0^r \frac{n(s)s^2}{\sqrt{1-\frac{2Gm(s)}{s}}}\ ds, \tag{3.3}$$

where $n(r)$ is the number density of baryons, given in the rest-frame of the fluid. To further motivate why the quantity m in the static universe can be seen as the total mass-energy, we must consider what happens in the Newtonian case. For a Newtonian universe, where curvature does not exist, $m(r)$ coincides with the definition of (3.1), because the areal radius is equal to the proper distance in that universe. We will also prove for the relativistic case that, in the Newtonian limit, the difference between $\overline{m}$ and $m(r)$ approximates the gravitational potential energy, further reinforcing the argument that this binding energy does not merely function as a bookkeeping device, but physically represents the binding of the constituents in the important Newtonian limit. The difference between m and $\overline{m}$ is:

$$m - \overline{m} = -\int_0^r \Big(\big(1-\frac{2Gm(s)}{s}\big)^{-\frac{1}{2}} - 1\Big)\rho \cdot 4\pi s^2 ds \tag{3.4}$$

The square root function in the integrand in the last expression can be expanded as $1 - \frac{Gm}{rc^2} + O((Gm)^2(c^2 r)^{-2})$, where c is restored a posteriori, so that the Newtonian limit $\frac{Gm}{rc^2} << 1$ gives:

$$m - \overline{m} \approx -\int_0^r \left(\frac{Gm}{s}\right)\rho \cdot 4\pi s^2 ds, \tag{3.5}$$

where $\rho 4\pi s^2 ds$ is the mass of a spherical shell. In classical theory, if a sphere is decomposed into a continuum of spherical shells and $m(r)$ is the mass of interior sphere, then the gravitational binding energy for shell with m_{shell} is given by the integral:

$$-\int \frac{Gm_{shell}(r)m_{interior}(r)}{r} dr, \tag{3.6}$$

which is equal to $m - \overline{m}$ in the Newtonian limit. The $m - \overline{m}$ is also called the gravitational mass defect, because of its resemblance with the well-known phenomenon in nuclear particle physics.

If we are, for instance, modeling a star, where $T_{\mu\nu}$ is zero outside of a given radius R, then outside of the ball with radius R, the mass will remain constant by equation (2.7b). By Birkhoff's theorem, which states that "Let the geometry of a given region of spacetime (1) be spherically symmetric, and (2) be a solution to the Einstein field equations in vacuum. Then that geometry is necessarily a piece of the Schwarzschild geometry" [22, Section 32.2], the total mass inside the ball will be the equal to the Schwarzschild mass. Also, in the limit where velocities are non-relativistic ($v << c$) and the gravitational potential is small $\frac{GM}{rc^2} << 1$, the Schwarzschild solution will reduce to Newton's law of gravitation. Under this law, the revolving bodies around m are governed by Kepler's third law of motion and the mass in that law is equal to m. Because masses of stars far away from us are actually determined using Kepler's law, this represents another way to operationally measure m. [22, Box 23.1]

To consider more arguments why m can be seen as mass, we will consider the general case of the LTB metric.

3.2 m in the LTB universe

With nonzero pressure and perfect fluid assumptions, equation (2.7b) essentially tells us the only way by which m can change over time. This change can be detected by locally measuring the power exerted by the pressure at the boundary R that moves with speed $\dot{R}$. [22, Box 23.1] We can also show that the 'LTB'-mass for the metric (2.53) can be decomposed in different contributions for energy in the zero pressure case. There, the 'mass' m[3] is written as:

$$m(r) = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \rho \; R^2 R' \sin(\theta) ds d\theta d\varphi. \tag{3.7}$$

We can also write this in a more Newtonian form:

$$m = \int_0^{2\pi} \int_0^{\pi} \int_0^{R_0} \rho \; R^2 \sin(\theta) dR d\theta d\varphi. \tag{3.8}$$

We express the integral in terms of the proper volume element for the spatial part of the metric (2.53), $\sqrt{det(g_{ij})} d^3x = R^2 R' \frac{1}{\sqrt{1+E}} \sin(\theta) dr d\theta d\varphi$:

$$m(r) = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \rho \; \sqrt{1+E} \sqrt{det(g_{ij})} \; ds d\theta d\varphi = \int_{D(r)} \rho \; \sqrt{1+E} \; dV_p \tag{3.9}$$

Using the dynamical equation (2.54), we obtain the following illuminating expression [4]:

$$m(r) = \int_{D(r)} \rho \; \sqrt{1 + (\dot{R})^2 + \frac{2Gm}{R}} \; dV_p \tag{3.10}$$

If we compare this to the "rest-mass" $\overline{m}$, which is given as:

$$\overline{m} = \int_{D(r)} \rho \; dV_p \; , \tag{3.11}$$

we can view m as the 'total energy' which incorporates kinetic and potential energetic contributions.

To see if the expression for the mass coincides with other definitions of mass-energy in the appropriate limit and to put it into greater context, we consider the definition of the Misner-Sharp mass.

[3] As expected, the mass is conserved over time, so that $m(r,t) = m(r)$

[4] Note that this is not a circular definition, but derived from two different expressions for $m(r)$ which are consistent with Einstein's equation.

3.3 Misner-Sharp mass

The definition of the Misner-Sharp mass [12][21] can be given as:

Definition. The Misner-Sharp mass, in the context of a spherically symmetric metric $g_{\mu\nu}$ can be defined as follows, where A is the areal radius, which multiplies the $d\Omega^2$ factor:

$$M_{MS} = \frac{A}{2G}\Big(1 - g^{\mu\nu}\nabla_\mu A \nabla_\nu A\Big), \tag{3.12}$$

From its form we can see that M_{MS} is a covariant quantity, if A can be considered a covariant quantity. For the LTB metric (2.2), the M_{ms} becomes:

$$M_{MS} = \frac{R}{2G}\Big(1 + e^{-C}(\dot{R})^2 - \frac{(R')^2}{X^2}\Big). \tag{3.13}$$

We notice from (2.7a) that m is equal to the Misner-Sharp mass, if we set $\Lambda = 0$[5], which means m is then also a coordinate invariant quantity and not just a result for a special choice of coordinates.

We notice, however, that while it is clear this how to measure this 'mass' from outside the spherically symmetric system, where Schwarzschild necessarily applies, there is no known, clear and coordinate invariant way to physically measure this mass from within the spherically symmetric system. Because of this reason, the Misner-Sharp mass is also referred to as *quasi-local mass*.

[5] The Misner-Sharp mass is not equal to m if $\Lambda \neq 0$.

Chapter 4

Averaging procedures

Real observations of cosmological parameters by physical instruments are typically coarse-grained, meaning that we do not observe the quantities from a point in the universe but the average, non-local effect. Further, the universe on small-scale is highly inhomogeneous, which means that there exists a need to study inhomogeneities that the idealized FLRW metric ignores.

As explained in the introduction (1), averaging in general relativity is not straight-forward, as opposed to (classical) statistical mechanics. This is due to the linear nature of averaging, whereas the Einstein Equations are nonlinear in the metric. This means that the Einstein tensor $G_{\mu\nu}$ is not commutative for the metric $g_{\mu\nu}$: "$\langle G_{\mu\nu}\rangle \neq G_{\mu\nu}(\langle g_{\mu\nu}\rangle)$". In fact, this even assumes that the averaging of tensors is well-defined, which is not the case. As of yet, there is no clear and covariant way of averaging tensors over spacetime in general relativity. If we were to define a tensor as something like:

$$\langle T_{\mu\nu}\rangle = \frac{1}{V}\int T_{\mu\nu}(x)\; d^4x, \tag{4.1}$$

the first issue that arises is that tensors in different points live in different tangent spaces and, therefore, they cannot be added trivially. To account for this, we can apply parallel transport, but then we encounter another issue: parallel transport in a curved space is necessarily path-dependent [16, Chapter 7]. What's more, the value of this average (4.1) will depend on the choice of coordinates.

Integration and averaging for scalar-valued functions, however, is covariant and well-defined, as these issues of different tangent spaces and parallel transport do not arise for them. Although Einstein's equation is in tensorial form, it can be equivalently written in a set of so-called evolution and constraint equations. Buchert proposed the systematic framework for applying averages to the evolutions equations that are only given in terms of scalar-valued functions. This is usually referred to as *Buchert's averaging* [4]. Before we apply the averaging, we will first derive the scalar equations we will apply the averaging to.

Also, from this point, we will only consider the LTB metric (2.53) and with the assumptions of zero pressure, comoving-synchronous coordinates and $\Lambda = 0$.

4.1 Evolution Equations for Covariant Scalars

While special relativity puts 'space' and 'time' on equal footing, the ADM-formalism, developed by R. Arnowitt, S Deser and C. Misner [1], decomposes 4D-spacetime into 3D-space and time to obtain a Hamiltonian formulation for general relativity. Using this split of spacetime, ten independent evolution and constraint equations can be derived [8]. These equations describe the evolution of spacetime in an equivalent manner as Einstein's equation. For the averaging, we will only consider the evolution equations with the scalar quantities θ, Σ^2, ${}^{(3)}\mathcal{R}$ and ρ. The first three scalar functions are computed for the LTB metric (2.53) in appendix (F):

$$\theta := \nabla_\mu u^\mu = \frac{2\dot{R}}{R} + \frac{\dot{R}'}{R'}, \tag{4.2}$$

where θ, the *expansion scalar*, is a measure of how much a spherical volume of test particles of a fluid changes in time [6, Appendix F]. The distortion of that sphere without any change in volume is then measured by Σ^2, the *shear scalar*, and is given by:

$$\Sigma^2 = \frac{1}{9}\Big(\frac{\dot{R}}{R} - \frac{\dot{R}'}{R'}\Big)^2 \tag{4.3}$$

Finally, ${}^{(3)}\mathcal{R}$ is the *spatial Ricci scalar* and in the LTB case, it is equal to:

$$ {}^{(3)}\mathcal{R} = -2\frac{(ER)'}{R^2R'}. \tag{4.4}$$

The spatial Ricci scalar is the spatial curvature scalar for the 3-dimensional spatial hypersurfaces[1], so it essentially gives intrinsic curvature of the 3D-space.

In the special case of the (dust) FLRW metric (2.22), where $R = ra(t)$ and $E = -kr^2$, we note down the expressions of θ, ${}^{(3)}\mathcal{R}$ and Σ, as we will need those later:

$$\theta = \frac{2r\dot{a}}{ra} + \frac{\dot{a}}{a} = \frac{3\dot{a}}{a}, \tag{4.5a}$$

$$ {}^{(3)}\mathcal{R} = -2\frac{(-kr^2ra)'}{r^2a^2a} = \frac{6k}{a^2}, \tag{4.5b}$$

$$\Sigma = \frac{1}{3}\Big(\frac{r\dot{a}}{ra} - \frac{\dot{a}}{a}\Big) = 0. \tag{4.5c}$$

Using the "3+1 split of spacetime", described in appendix E, it is possible to

[1]For more information on this, please read appendix (E). By Gauß's equations, the 3-dimensional Ricci scalar can be related to its 4-dimensional equivalent.

derive the following evolution equations in the LTB universe for these scalars, which is done in the same appendix:

$$\frac{d\theta}{d\tau} = -\frac{\kappa\ \rho}{2} - 6\Sigma^2 - \frac{1}{3}\theta^2, \tag{4.6a}$$

$$\left(\frac{\theta}{3}\right)^2 = \frac{\kappa\ \rho}{3} - \frac{{}^{(3)}\mathcal{R}}{6} + \Sigma^2, \tag{4.6b}$$

$$\frac{d\rho}{d\tau} = -\theta\rho, \tag{4.6c}$$

where it should be stressed that there are more evolution and constraint equations and the equations (4.6) by themselves are incomplete, as they do not provide a closed system of equations. For the averaging procedures that will be applied, however, these equations are the ones that we will be focused on, because the rest of the equations are given in terms of non-scalar quantities and therefore, it is not clear how to average them.

Let us again consider the special case of dust FLRW; substituting the expressions (4.5) for the FLRW metric in the Raychaudhuri's equation (4.6a) and the Hamiltonian constraint equation (4.6b), where Σ^2 is zero, we can easily see that we have obtained the two Friedmann equations (2.24) and (2.26) with $p = \Lambda = 0$. Now, before we apply any averaging procedure, the definitions for the averages are given.

4.2 Definition of Averaging

For Buchert's way of averaging, a specific and simple foliation of spacetime, which we have seen earlier, is chosen. That is, the spacetime is decomposed into t=const three dimensional spacelike hypersurfaces, on which the coordinates are x^i and the metric $g_{\mu\nu}$ is restricted to the spatial part g_{ij}. Then, we define the spatial average used for Buchert's averaging of the evolution equations:

Definition. Following [4] [25], the Buchert's spatial average is defined for a smooth, integrable and scalar-valued function $A(t, x^i)$ on a compact domain $\mathbb{R} \times D$ which is comoving with the fluid:

$$\langle A\rangle(t) = \frac{\int_D d^3x\ \sqrt{g} \cdot A(t, x^i)}{\int_D d^3x\ \sqrt{g}} = \frac{1}{V_D(t)} \int_D d^3x\ \sqrt{g} \cdot A(t, x^i), \tag{4.7}$$

where $g = det(g_{ij})$ and t is held fixed during integration.

Note that this definition ensures that there is neither inward nor outward flux for the domain, D and that the domain may only change in time by expansion or contraction of the fluid itself. Because the LTB metric (2.53) is spherically symmetric, there is a strong motivation to choose the domain to be spherically symmetric as well: $D \equiv [0, r] \times S^2$, where S^2 is the unit sphere and $[0, r]$ is the closed interval between the center point 0 and the comoving label r. For the spatial part of the LTB metric (2.53), the proper volume element $\sqrt{g}d^3x = R^2R'\sin^2(\theta)\frac{1}{\sqrt{1+E}}drd\theta d\phi$, so that the average (4.7) can be written as:

$$\langle A\rangle(t) = \frac{\int_0^r drR^2R'\sin^2(\theta)\frac{1}{\sqrt{1+E}}\cdot A}{\int_0^r drR^2R'\sin^2(\theta)\frac{1}{\sqrt{1+E}}}, \tag{4.8}$$

where A does not depend on θ and φ due to spherical symmetry. It must be noted that r in the upper bound of the integrals on the right-hand side of (4.8) for $\langle A(t)\rangle$ is fixed beforehand. However, we can allow r to vary, which we define as $A_b = \langle A(t)\rangle(r)$. In the following, the dependence on the coordinates will be omitted.

An issue arises with physical interpretation when we try to reconcile the concept of the Misner-Sharp mass with how Buchert's averaging procedure is carried out for the energy density. Buchert's average for ρ can be computed as:

$$\langle\rho\rangle = \frac{\int_0^r drR^2R'\sin^2(\theta)\frac{1}{\sqrt{1+E}}\cdot \rho}{\int_0^r drR^2R'\sin^2(\theta)\frac{1}{\sqrt{1+E}}} = \frac{\overline{m}}{V}, \tag{4.9}$$

where $\overline{m}$ is the same quantity as in equation (3.11). While $\overline{m}$ can be seen as a 'rest mass', it would be more physically correct to use the 'active gravitating mass': the Misner-Sharp mass m. The integral for mass m, however, does not contain the spatial curvature factor $\frac{1}{\sqrt{1+E}}$:

$$m(r) = \int_0^r drR^2R'\sin^2(\theta)\cdot\rho \tag{4.10}$$

This motivates the definition of the *quasi-local average*:

$$A_q(t) = \frac{\int_0^r drR^2R'\cdot A(t,r)}{\int_0^r drR^2R'} = \frac{\int_0^r drR^2R'\cdot A(t,r)}{\frac{1}{3}R^3}, \tag{4.11}$$

where the t is held fixed again during the integration. (4.11) will be referred to as the Sussman's average in what follows, because the definition and the calculations in section 4.4 follow the paper [25] by R. Sussman. We assume here that r is not allowed to vary, such that A_q is only a function of t. Here, we can similarly define the Sussman's average for varying r as $\langle A\rangle_q(r,t)$. We also remark that this average (4.11) only makes sense for a spherical context, so that it cannot be defined for every possible spacetime like Buchert's average. Also, the Sussman's average is in a sense a weighted Buchert's average with the weight $w(r) = \sqrt{1+E(r)}$:

$$A_q = \frac{\int_0^r drR^2R'\sin^2(\theta)\frac{1}{\sqrt{1+E}}\cdot A\cdot w(r)}{\int_0^r drR^2R'\sin^2(\theta)\frac{1}{\sqrt{1+E}}\cdot w(r)}, \tag{4.12}$$

For both the average functions A_b and $\langle A \rangle$, the following commutation rule applies for the average and the derivative with respect to t:

$$\partial_t \langle A \rangle - \langle \partial_t A \rangle = \langle \Theta A \rangle - \langle \Theta \rangle \langle A \rangle \tag{4.13}$$

The average functions $\langle A \rangle$ satisfy the following equation as well, known for average distributions:

$$\langle (A - \langle A \rangle)^2 \rangle \; = \; \langle A^2 \rangle - \langle A \rangle^2. \tag{4.14}$$

Both identities (4.13) and (4.14) can be found in [4].

4.3 Buchert's Average Applied to the Evolution Equations

Now, we will apply the Buchert's average (4.7) to the evolution equations (4.6). In this, we completely follow the the paper [4] by Thomas Buchert. It should be noted that the general averaging procedure in (4.7) is defined for a general dust metric, so it also holds for the LTB metric. We will see that due to noncommutativity of the time derivative with the average, we will obtain FLRW-like evolution equations for averages as well as extra terms that are associated with backreaction. Applying the Buchert's average functional $\langle . \rangle$ to Raychaudhuri's equation (4.6a), the energy balance equation (4.6c) and the Hamiltonian constraint (4.6b), we obtain the following set of equations:

$$\langle \partial_t \Theta \rangle = -\frac{\langle \Theta^2 \rangle}{3} - \frac{\kappa}{2}\langle \rho \rangle - 6\langle \Sigma^2 \rangle, \tag{4.15a}$$

$$-\langle \rho \Theta \rangle = \langle \partial_t \rho \rangle, \tag{4.15b}$$

$$\frac{(\langle \Theta \rangle)^2}{9} = \frac{\kappa}{3}\langle \rho \rangle - \frac{\langle {}^{(3)}\mathcal{R} \rangle}{6} + \langle \Sigma^2 \rangle, \tag{4.15c}$$

Equation (4.13) helps to reduce the averaged evolution equation (4.15b):

$$-\langle \rho \Theta \rangle = \partial_t \langle \rho \rangle - \langle \rho \Theta \rangle + \langle \rho \rangle \langle \Theta \rangle \tag{4.16}$$

Some terms cancel, and the averaged energy balance is:

$$0 = \partial_t \langle \rho \rangle + \langle \rho \rangle \langle \Theta \rangle \tag{4.17}$$

We apply the same commutation rule (4.13) to the averaged Raychaudhuri's equation (4.15a):

$$\partial_t \langle \Theta \rangle - \langle \Theta^2 \rangle + (\langle \Theta \rangle)^2 = -\frac{\langle \Theta^2 \rangle}{3} - \frac{\kappa}{2}\langle \rho \rangle - 6\langle \Sigma^2 \rangle \tag{4.18}$$

Shuffling terms to the other sides of the equality, we are then left with:

$$\partial_t\langle\Theta\rangle = -\frac{(\langle\Theta\rangle)^2}{3} + \frac{2}{3}\Big(\langle\Theta^2\rangle - (\langle\Theta\rangle)^2\Big) - \frac{\kappa}{2}\langle\rho\rangle - \ 6\langle\Sigma^2\rangle. \tag{4.19}$$

We use equation (4.14) to rewrite the equation above as the averaged Raychaudhuri's equation:

$$\partial_t\langle\Theta\rangle = -\frac{(\langle\Theta\rangle)^2}{3} - \frac{\kappa}{2}\langle\rho\rangle + \ \mathcal{Q}, \tag{4.20}$$

where $\mathcal{Q} = \frac{2}{3}\langle(\Theta\ - \langle\Theta\rangle)^2\rangle - \ 6\langle\Sigma^2\rangle$, which is also called the back-reaction term. This backreaction term $\mathcal{Q}$, which we refer to in the introduction, measures how much the inhomogeneities in the expansion scalar and in the shear scalar affect the dynamics of our universe. $\mathcal{Q} > 0$ indicates that the expansion is more accelerated than in the dust FLRW universe. [4]

We apply the equation for averages (4.14) to the averaged Friedman equation (4.15c) to obtain:

$$\frac{(\langle\Theta\rangle)^2 - \langle(\Theta\ - \langle\Theta\rangle)^2\rangle}{9} = \frac{\kappa}{3}\langle\rho\rangle - \frac{\langle{}^{(3)}\mathcal{R}\rangle}{6} + \Sigma^2 \tag{4.21}$$

Using the expression for $\mathcal{Q}$ gives the averaged Hamiltonian constraint:

$$\frac{(\langle\Theta\rangle)^2}{9} = \frac{\kappa}{3}\langle\rho\rangle - \frac{\langle{}^{(3)}\mathcal{R}\rangle + \mathcal{Q}}{6}. \tag{4.22}$$

4.4 Sussman's average Applied to the Evolution Equations

To derive the evolution equations, we first compute the quasi-local average functionals of the functions ρ, θ and ${}^{(3)}\mathcal{R}$, which will be named ρ_q, θ_q and ${}^{(3)}\mathcal{R}_q$ respectively. As will become clear, $(\Sigma^2)_q$ is not needed in this derivation. Then, θ_q is derived as follows, where the expression for θ can be found in (4.2):

$$\begin{aligned}\theta_q &= \frac{\int_0^r dx\theta R^2 R'}{\int_0^r dx R^2 R'} = \frac{\int_0^r dx(\frac{2\dot{R}}{R} + \frac{\dot{R}'}{R'})R^2 R'}{\int_0^r dx R^2 R'} = \frac{\int_0^r dx(2\dot{R}RR' + \dot{R}'R^2)}{\int_0^r dx R^2 R'} \\ &= \frac{\int_0^r dx(\dot{R}R^2)'}{\int_0^r dx R^2 R'} = \frac{[\dot{R}R^2]_0^r}{[\frac{1}{3}R^3]_0^r} = 3\frac{\dot{R}}{R}\end{aligned} \tag{4.23}$$

Similarly, ${}^{(3)}\mathcal{R}_q$ is derived with the expression for ${}^{(3)}\mathcal{R}$ in (4.4):

$${}^{(3)}\mathcal{R}_q = \frac{\int_0^r dx({}^{(3)}\mathcal{R})R^2 R'}{\int_0^r dx R^2 R'} = -2\frac{\int_0^r dx(\frac{[ER]'}{R'R^2})R^2 R'}{\int_0^r dx R^2 R'} = -2\frac{[ER]_0^r}{[\frac{R^3}{3}]_0^r} = -6\frac{E}{R^2} \tag{4.24}$$

The energy density ρ, which is the motivation behind defining the Sussman's average, is given by:

$$\rho_q = \frac{\int_0^r dx \rho R^2 R'}{\int_0^r dx R^2 R'} = \frac{m(r)}{\frac{4\pi}{3} R^3}, \tag{4.25}$$

where we clearly used the equation (2.55), which characterizes the evolution of the LTB universe.

The following derivations in this entire section 4.4 follow the paper [25] by Roberto Sussman: Using equation (2.54), which also describes the evolution of the LTB metric, the quasi-locally averaged Hamiltonian constraint is derived. First, the equation is divided by R^2:

$$\frac{1}{9}\Big(\frac{3\dot{R}}{R}\Big)^2 = \frac{2Gm}{R^3} + \frac{1}{6}\frac{6E}{R^2} \tag{4.26}$$

With the expressions (4.23), (4.24), (4.25) for Θ_q, ${}^{(3)}\mathcal{R}_q$ and ρ_q respectively, this equation becomes the averaged Hamiltonian constraint:

$$\left(\frac{\Theta_q}{3}\right)^2 = \frac{\kappa}{3}\rho_q - \frac{{}^{(3)}\mathcal{R}_q}{6}, \tag{4.27}$$

To derive the averaged energy balance equation, we take the derivative of ρ_q:

$$\partial_t \rho_q = \frac{-m}{\frac{4\pi}{3} R^4} 3\dot{R} = -\Theta_q \rho_q \tag{4.28}$$

Finally, we derive the averaged Raychaudhuri's equation by taking the derivative of Θ_q:

$$\partial_t \Theta_q = 3\frac{\ddot{R}R - (\dot{R})^2}{R^2} = 3\frac{\frac{-Gm}{R} - (\dot{R})^2}{R^2}, \tag{4.29}$$

where in the last equality we have used equation (2.54), differentiated with respect to t, $\ddot{R} = -\frac{Gm}{R^2}$. Using the expressions for Θ_q and ρ_q, we obtain the averaged Raychaudhuri's equation:

$$\partial_t \Theta_q = -\frac{\Theta_q^2}{3} - \frac{\kappa}{2}\rho_q \tag{4.30}$$

Now, we notice that the equations (4.27), (4.30) and (4.28) are precisely the Hamiltonian constraint, Raychaudhuri's equation and the energy balance equation for the dust FLRW metric model[2]! This shows that this form of weighted averaging produces a metric where the inhomogeneities are averaged away. As a consequence, there is no backreaction term in the dust FLRW evolution equations and that is the reason why we did not need the quasi-local average of Σ^2.

[2]Note that for FLRW the evolution equations are the equations (4.6) with $\Sigma^2 = 0$.

An advantage of this averaging procedure is that we have reduced the LTB universe into a background FLRW universe, on which the inhomogeneities evolve. This evolution is described exactly and it is encapsulated in the differences between the average and the quantity itself: $A - A_q$.

It must also be emphasized that the Sussman's average was not applied to the evolution equations (4.6), which was done for the Buchert's averaging framework. Here, the quasi-local averages for the evolution scalars in the evolution equations were computed and then the LTB differential equations (2.54) and (2.55) were used to derive the quasi-locally averaged evolution equations.

4.5 Buchert's and Sussman's Average Applied to Milne Universe

Before we apply the different averaging procedures, we summarize the three ways in which the metric for Milne's model is written with $d\Omega^2 = d\theta^2 + sin^2(\theta)d\phi^2$:

$$ds^2 = -dt^2 + t^2[\frac{dr^2}{1+r^2} + r^2 d\Omega^2], \tag{4.31}$$

$$ds^2 = -dt^2 + t^2 d\chi^2 + t^2 sinh(\chi)^2 d\Omega^2, \tag{4.32}$$

$$ds^2 = -dT^2 + dR^2 + R^2 d\Omega^2, \tag{4.33}$$

with $r^2 = \sinh^2(\chi)$ and $R = t\sinh(\chi)$ and $T = t\cosh(\chi)$: In this universe, the Sussman's average for a test function $A(r,t)$ will reduce to:

$$A_q = \frac{\int_0^r ds(st)^2 \cdot t \cdot A}{\int_0^r ds(st)^2 \cdot t} = \frac{\int_0^\psi d\chi \cosh(\chi)\sinh^2(\chi) \cdot t^3 \cdot A}{\int_0^\psi d\chi \cosh(\chi)\sinh^2(\chi) \cdot t^3} \tag{4.34}$$

because Milne is special case of LTB with $R(r,t) = rt$ and $E = r^2$. The coordinates for A_q are changed with $dR = t\cosh(\chi)d\chi$:

$$A_q = \frac{\int_0^{2\pi}\int_0^\pi \sin(\theta)d\theta d\varphi \cdot \int_0^{R_0} dR\ R^2 \cdot A}{\int_0^{2\pi}\int_0^\pi \sin(\theta)d\theta d\varphi \cdot \int_0^{R_0} dR\ R^2}, \tag{4.35}$$

where the angular volume elements are included. The Buchert's average can, for the sake of completeness, also be computed:

$$\langle A\rangle = \frac{\int_0^r ds(st)^2 \cdot \frac{t}{\sqrt{1+s^2}} \cdot A}{\int_0^r ds(st)^2 \cdot \frac{t}{\sqrt{1+s^2}}} = \frac{\int_0^\psi d\chi \cosh(\chi)\sinh^2(\chi) \cdot \frac{t^3}{\cosh(\chi)} \cdot A}{\int_0^\psi d\chi \cosh(\chi)\sinh^2(\chi) \cdot \frac{t^3}{\cosh(\chi)}} \tag{4.36}$$

After the coordinate change with the Minkowski metric in mind, this becomes:

$$\langle A\rangle = \frac{\int_0^{2\pi}\int_0^\pi \sin(\theta)d\theta d\varphi \cdot \int_0^{R_0} dR\ R^2\ \frac{t}{\sqrt{t^2+R^2}} \cdot A}{\int_0^{2\pi}\int_0^\pi \sin(\theta)d\theta d\varphi \cdot \int_0^{R_0} dR\ R^2\ \frac{t}{\sqrt{t^2+R^2}}}, \tag{4.37}$$

where it is emphasized again that t is constant. The volume element in Buchert's expression (4.37) underlines why it is the natural way to integrate; it aligns with the hyperboloid over which the average is taken. For Sussman's average, however, we notice that 'the curvature is indeed not taken into account', in the sense that the volume element in its definition is given by: $4\pi R^2 R' dr = 4\pi R^2 dR$. We remark that, while it is unclear yet what to make of this physically, the Milne universe provides the perfect testing ground for us to apply the averaging procedures.

Chapter 5

Discussion

Spherical symmetry is a key assumption that in the real universe does not necessarily hold everywhere. It should be stressed that the mass in the static universe or the LTB universe, written as an integral expression of ρ, is rather a miraculous fact than a general fact in general relativity. One could argue that the synchronous-comoving coordinates were used to derive (2.7) and that that choice is the reason for the integral expression of mass. And although, it is correct that the function, which is the factor that multiplies $d\Omega^2$, is not invariant under certain transformations (see footnote 2 on page 7 of this thesis), this function, also used to define the Misner-Sharp mass, is invariant under the 'expected' transformations. While it is not completely trivial and clear why it, in general, is referred to as a geometric invariant and the Misner-Sharp mass as an invariant quantity in the literature [11], we can at least expect M_{ms} to be invariant for these reasonable transformations. This would imply that spherical symmetry alone permits a good definition of mass without the need of the comoving-synchronous coordinate system.

It would be interesting to see, as the expression in (2.8) suggests, whether the integral expression is also manifestly covariant. At this point, we can only conclude that (2.8) was derived using the special comoving-synchronous coordinate system. It is, however, a very natural coordinate system to consider, because it 'follows' the fluid. For this very reason, it was also used to define the averages. Moreover, we emphasize the use of the comoving-synchronous coordinate system to obtain the various different Newtonian expressions throughout this thesis. The interpretation with respect to gravitational binding energy, for instance, was made after introduction of this special coordinate system. We also remark that, while chapter 3 explains why the curvature factor is missing by considering the Newtonian limit for $m - \overline{m}$, there is still no intuition in the general relativistic and general coordinate case.

Regardless of all the assumptions, it is also not clear if it is even possible to measure the Misner-Sharp mass, even in the static context, from within the system. There is a similar open question for the averages: Is it possible to measure the Sussman's or Buchert's average physically? Section 4.5 suggests that the physical meaning behind the averages can also be further investigated in the relatively simpler setting of the Milne universe.

Lastly, we remark on the Sussman's average. Although the average is mathematically consistent for the other scalars than ρ, it physically makes less sense to apply the Sussman's average to them. This is because the integral expressions containing these scalars are well-defined; Buchert's average seems to be the natural choice.

Appendix A

Derivation of Einstein tensor components for a spherically symmetric metric

If the metric is given by:

$$ds^2 = -e^{C(r,t)}dt^2 + X^2(r,t)dr^2 + R^2(r,t)(d\theta^2 + \sin^2\theta\, d\phi^2),$$

then the inverse metric will be given by:

$$g^{\mu\nu} = \begin{pmatrix} -e^{-C} & 0 & 0 & 0 \\ 0 & \frac{1}{X^2} & 0 & 0 \\ 0 & 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2 sin^2(\theta)} \end{pmatrix}$$

With this, the Christoffel symbols, $\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)$, can be written as the following:

$$\Gamma^t{}_{tt} = \frac{1}{2}\dot{C}, \qquad \Gamma^t{}_{tr} = \frac{1}{2}C', \qquad \Gamma^t{}_{rr} = \frac{X\dot{X}}{e^C},$$

$$\Gamma^t{}_{\theta\theta} = \frac{R\dot{R}}{e^C}, \qquad \Gamma^t{}_{\varphi\varphi} = \frac{R\dot{R}\sin^2\theta}{e^C}, \qquad \Gamma^r{}_{tt} = \frac{e^C C'}{2X^2},$$

$$\Gamma^r{}_{tr} = \frac{\dot{X}}{X}, \qquad \Gamma^r{}_{rr} = \frac{X'}{X}, \qquad \Gamma^r{}_{\theta\theta} = -\frac{RR'}{X^2},$$

$$\Gamma^r{}_{\varphi\varphi} = -\frac{RR'}{X^2}\sin^2\theta, \qquad \Gamma^\theta{}_{t\theta} = \frac{\dot{R}}{R}, \qquad \Gamma^\theta{}_{r\theta} = \frac{R'}{R},$$

$$\Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta, \qquad \Gamma^\varphi{}_{t\varphi} = \frac{\dot{R}}{R}, \qquad \Gamma^\varphi{}_{r\varphi} = \frac{R'}{R},$$

$$\Gamma^\varphi{}_{\theta\varphi} = \cot\theta.$$

Here the vanishing Christoffel symbols or those which are symmetric to the ones given are omitted. Using these Christoffel symbols, the non-vanishing Ricci tensor components, $R_{\mu\nu} = \partial_\rho \Gamma^\rho{}_{\mu\nu} - \partial_\nu \Gamma^\rho{}_{\mu\rho} + \Gamma^\rho{}_{\rho\lambda}\Gamma^\lambda{}_{\mu\nu} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\rho}$, can directly be given:

$$\begin{aligned} R_{tt} &= \frac{1}{2}\dot{C}\frac{\dot{X}}{X} + \dot{C}\frac{\dot{R}}{R} + \frac{e^C}{X^2}\left(\frac{1}{2}C'' + \frac{1}{4}(C')^2 - \frac{1}{2}C'\frac{X'}{X} + C'\frac{R'}{R}\right) \\ &\quad - \frac{\ddot{X}}{X} - 2\frac{\ddot{R}}{R}, \end{aligned} \tag{A.1}$$

$$R_{tr} = R_{rt} = 2\frac{\dot{X}R'}{XR} + \frac{\dot{R}C'}{R} - 2\frac{\dot{R}'}{R}, \tag{A.2}$$

$$\begin{aligned} R_{rr} &= e^{-C}\left(X\ddot{X} - \frac{1}{2}\dot{X}X\dot{C} + 2\frac{X\dot{X}\dot{R}}{R}\right) \\ &\quad - \left(\frac{2R''}{R} - \frac{X'C'}{2X} - \frac{2X'R'}{XR} + \frac{1}{2}C'' + \frac{1}{4}(C')^2\right), \end{aligned} \tag{A.3}$$

$$\begin{aligned} R_{\theta\theta} &= 1 + e^{-C}\left(\ddot{R}R + (\dot{R})^2 + R\dot{R}\frac{\dot{X}}{X} - \frac{1}{2}R\dot{R}\dot{C}\right) \\ &\quad - \frac{1}{X^2}\left(RR'' + (R')^2 + RR'\frac{1}{2}C' - RR'\frac{X'}{X}\right), \end{aligned} \tag{A.4}$$

$$R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta}. \tag{A.5}$$

As $R_{\varphi\varphi}$ is given by $R_{\theta\theta}\sin^2(\theta)$, we can compute the Ricci scalar, defined as $\mathcal{R} = g^{\mu\nu}R_{\mu\nu}$:

$$\mathcal{R} = -e^{-C}R_{tt} + \frac{1}{X^2}R_{rr} + \frac{2}{R^2}R_{\theta\theta} \tag{A.6}$$

$$\begin{aligned} &= e^{-C}\Big[\frac{2\ddot{X}}{X} + \frac{4\ddot{R}}{R} - \frac{\dot{C}\dot{X}}{X} - \frac{2\dot{R}\dot{C}}{R} + \frac{4\dot{R}\dot{X}}{RX} + \frac{2(\dot{R})^2}{R^2}\Big] \\ &+ \frac{1}{X^2}\Big[-\frac{4R''}{R} + \frac{4R'X'}{RX} - \frac{2(R')^2}{R^2}\Big] + \frac{2}{R^2} \\ &- \frac{C''}{X^2} + \frac{C'}{X^2}\Big[-\frac{2R'}{R} + \frac{X'}{X} - \frac{C'}{2}\Big]. \end{aligned} \tag{A.7}$$

Finally, the Einstein tensor components are given by, with $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$:

$$G_{tt} = \frac{2\dot{X}\dot{R}}{XR} + \frac{\dot{R}^2}{R^2} + e^C\left(\frac{1}{R^2} - \frac{2R''}{X^2R} - \frac{(R')^2}{X^2R^2} + \frac{2X'R'}{X^3R}\right), \tag{A.8}$$

$$G_{tr} = 2\frac{\dot{X}R'}{XR} + \frac{\dot{R}C'}{R} - 2\frac{\dot{R}'}{R}, \tag{A.9}$$

$$G_{rr} = \frac{(R')^2}{R^2} + \frac{R'C'}{R} - \frac{X^2}{R^2} - e^{-C}X^2\left(\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \frac{\dot{R}\dot{C}}{R}\right) \tag{A.10}$$

$$\begin{aligned} G_{\theta\theta} &= \frac{R^2}{X^2}\left(\frac{R''}{R} + \frac{C''}{2} + \frac{(C')^2}{4} - \frac{X'R'}{XR} + \frac{R'C'}{2R} - \frac{X'C'}{2X}\right) \\ &+ e^{-C}R^2\left(-\frac{\ddot{X}}{X} - \frac{\ddot{R}}{R} - \frac{\dot{X}\dot{R}}{XR} + \frac{\dot{X}\dot{C}}{2X} + \frac{\dot{R}\dot{C}}{2R}\right), \end{aligned} \tag{A.11}$$

$$G_{\varphi\varphi} = \sin^2\theta\, G_{\theta\theta}. \tag{A.12}$$

Appendix B

Equations that result from Einstein's equation

We follow the similar calculations in [14, Page 295 - 296].

B.1 "Mass" equations

We set $\kappa = 8\pi G$ and multiply equation (2.6a) with $\frac{R^2 R'}{e^C}$ to obtain:

$$\begin{aligned}\kappa\rho R^2 R' = {} & e^{-C}\frac{2RR'\dot{X}\dot{R}}{X} + e^{-C}R'\dot{R}^2 \\ & + \left(R' - \frac{2R''RR'}{X^2} - \frac{(R')^3}{X^2} + \frac{2X'(R')^2 R}{X^3} - \Lambda R^2 R'\right).\end{aligned} \tag{B.1}$$

This can be written as:

$$\begin{aligned}\kappa\rho R^2 R' = {} & e^{-C}\frac{2RR'\dot{X}\dot{R}}{X} - R\frac{\partial}{\partial r}(e^{-C}\dot{R}^2) \\ & + \frac{\partial}{\partial r}\left[R - \frac{(R')^2 R}{X^2} - \frac{1}{3}\Lambda R^3 + e^{-C}R(\dot{R})^2\right],\end{aligned} \tag{B.2}$$

where $e^{-C}(\dot{R})^2 R' = \frac{\partial}{\partial r}(e^{-C}R(\dot{R})^2) - R\frac{\partial}{\partial r}(e^{-C}(\dot{R})^2)$ was also used. Now, we multiply equation (2.6b) by $\dot{R}R^2 e^{-C}$ which gives:

$$e^{-C}\frac{2RR'\dot{X}\dot{R}}{X} + R(\dot{R})^2 C' - 2\dot{R}'R\dot{R} = 0 \tag{B.3}$$

We immediately recognize that, knowing $-R\frac{\partial}{\partial r}(e^{-C}(\dot{R})^2) = -e^{-C}RC'(\dot{R})^2 + 2e^{-C}R\dot{R}\dot{R}'$, the first two terms on the right-hand side of (B.2) are 0:

$$\kappa\rho R^2 R' = \frac{\partial}{\partial r}\left[R - \frac{(R')^2 R}{X^2} - \frac{1}{3}\Lambda R^3 + e^{-C}R(\dot{R})^2\right], \tag{B.4}$$

Similarly, we remember $\kappa = 8\pi G$ and multiply equation (2.6c) by $\frac{R^2\dot{R}}{X^2}$ to obtain:

$$\begin{aligned}\kappa P R^2 \dot{R} = {} & \frac{(R')^2\dot{R}}{X^2} + \frac{R'C'R\dot{R}}{X^2} - \dot{R} + \Lambda R^2\dot{R} \\ & - e^{-C}\left(2\ddot{R}R\dot{R} + \dot{R}^3 - \dot{R}^2 R\dot{C}\right)\end{aligned} \tag{B.5}$$

We again recognize that all the terms in the round brackets in equation (B.5) are the results of a derivation of a quantity with respect to t:

$$\begin{aligned}\kappa P R^2 \dot{R} = {} & \frac{(R')^2\dot{R}}{X^2} + \frac{R'C'R\dot{R}}{X^2} \\ & - \frac{\partial}{\partial t}\left[R - \frac{1}{3}\Lambda R^3 + e^{-C}(\dot{R})^2 R\right]\end{aligned} \tag{B.6}$$

Now, we multiply equation (2.6b) by $\frac{R'R^2}{X^2}$ which gives:

$$\frac{2R(R')^2\dot{X}}{X^3} + \frac{\dot{R}C'R'R}{X^2} - \frac{2\dot{R}'R\dot{R}}{X^2} = 0 \tag{B.7}$$

We immediately use this equation to rewrite equation (B.6):

$$\begin{aligned}\kappa P R^2 \dot{R} = {} & \frac{(R')^2\dot{R}}{X^2} + \frac{2\dot{R}'R\dot{R}}{X^2} - \frac{2R(R')^2\dot{X}}{X^3} \\ & - \frac{\partial}{\partial t}\left[R - \frac{1}{3}\Lambda R^3 + e^{-C}(\dot{R})^2 R\right]\end{aligned} \tag{B.8}$$

Recognizing the first three terms as a derivative of a quantity with respect to t gives the following equation:

$$-\kappa P R^2 \dot{R} = \frac{\partial}{\partial t}\left[R - \frac{(R')^2 R}{X^2} - \frac{1}{3}\Lambda R^3 + e^{-C}R(\dot{R})^2\right] \tag{B.9}$$

If we now define the mass in this context to be:

$$m(r,t) \equiv \frac{1}{2G}\left[R - \frac{(R')^2 R}{X^2} - \frac{1}{3}\Lambda R^3 + e^{-C}R(\dot{R})^2\right], \tag{B.10}$$

then equations (B.4) and (B.9) can be written as:

$$\frac{\partial}{\partial r}m(r,t) = 4\pi\rho(r,t)R^2R', \tag{B.11}$$

$$\frac{\partial}{\partial t}m(r,t) = 4\pi P(r,t)R^2\dot{R}. \tag{B.12}$$

(B.13)

B.2 Dynamical Equation

The quantity m can also be used to determine a dynamical equation for this spherically symmetric universe. If we take the derivative of m in equation (B.10) with respect to t and note that this is also given by (B.9), we obtain the following expression:

$$-4\pi P(r,t)R^2\dot{R} = \frac{\partial}{\partial t}m(r,t) \tag{B.14}$$

$$= \frac{1}{2G}\Big[\dot{R} - \frac{2R'\dot{R}'R}{X^2} - \frac{(R')^2\dot{R}}{X^2} + \frac{2(R')^2R\dot{X}}{X^3} - \frac{1}{3}\Lambda R^2\dot{R} \tag{B.15}$$

$$- \frac{2}{3}\Lambda R^2\dot{R} - \dot{C}e^{-C}R(\dot{R})^2 + e^{-C}(\dot{R})^3 + 2e^{-C}R\dot{R}\ddot{R}\Big]. \tag{B.16}$$

Recognizing the first, third, fifth and eighth terms as the quantity $\frac{m\dot{R}}{R}$ and seeing that $R\dot{R}e^{-C}(2\ddot{R} - \dot{C}\dot{R}) = 2R\dot{R}e^{-\frac{C}{2}}\partial_t(e^{-\frac{C}{2}}\dot{R})$, the equation becomes:

$$-4\pi P(r,t)R^2\dot{R} = \frac{m\dot{R}}{R} - \frac{1}{2G}\Big[\frac{2R'\dot{R}'R}{X^2} + \frac{2(R')^2R\dot{X}}{X^3} \tag{B.17}$$

$$- \frac{2}{3}\Lambda R^2\dot{R} + 2R\dot{R}e^{-\frac{C}{2}}\partial_t(e^{-\frac{C}{2}}\dot{R})\Big]. \tag{B.18}$$

Now, we multiply equation (2.6b) by $-\frac{R^2R'}{X^2}$ and rewrite it:

$$-2\frac{\dot{R}'RR'}{X^2} + -2\frac{(R')^2R\dot{X}}{X^3} = -\frac{\dot{R}C'R'R}{X^2} \tag{B.19}$$

Substituting this expression into equation (B.18) and dividing the equation by the factor $\frac{\dot{R}R}{G}$, the following rewritten dynamical equation is obtained:

$$e^{-\frac{C}{2}}\partial_t(e^{-\frac{C}{2}}\dot{R}) = -(4\pi GP(r,t)R + \frac{Gm}{R^2}) + \frac{C'R'}{2X^2} + \frac{1}{3}\Lambda R \tag{B.20}$$

B.3 G_{rt}-equation

Multiplying equation (2.6b) by $\frac{R}{2}$ and rewriting results in the following expression:

$$\dot{R}' = \frac{\dot{X}}{X}R' + \frac{\dot{R}C'}{2} \tag{B.21}$$

Appendix C

Equations that result from the Energy-Momentum Tensor

Using $T^{\mu\nu} = \mathrm{diag}(\rho e^{-C}, PX^{-2}, PR^{-2}, PR^{-2}\sin^{-2}(\theta))$, we see that the conservation law $\nabla_\mu T^{\mu\nu} = 0$, which is equivalent to $\partial_\mu T^{\mu\nu} + \Gamma^{\mu}_{\mu\lambda} T^{\lambda\nu} + \Gamma^{\nu}_{\mu\lambda} T^{\mu\lambda}$, gives:

$$
\begin{aligned}
v = t : 0 = \nabla_t T^{tt} &= \partial_t(\rho e^{-C}) + \Gamma^{\mu}_{\mu t} T^{tt} + \Gamma^{t}_{\mu\lambda} T^{\mu\lambda} \\
&= \partial_t(\rho e^{-C}) + (\frac{1}{2}\dot{C} + \frac{\dot{X}}{X} + \frac{2\dot{R}}{R})\rho e^{-C} \\
&+ \frac{1}{2}\dot{C}\rho e^{-C} + \frac{X\dot{X}}{e^C} PX^{-2} + 2\frac{R\dot{R}}{e^C} PR^{-2} \\
&= \partial_t(\rho e^{-C}) + \dot{C}\rho e^{-C} + e^{-C}(\frac{\dot{X}}{X} + \frac{2\dot{R}}{R})(\rho + P) \\
&= \partial_t(\rho) e^{-C} + e^{-C}(\frac{\dot{X}}{X} + \frac{2\dot{R}}{R})(\rho + P), \qquad \text{(C.1)} \\
v = r : 0 = \nabla_r T^{rr} &= \partial_r(\frac{P}{X^2}) + (\Gamma^{\mu}_{\mu r}) T^{rr} + \Gamma^{r}_{\mu\lambda} T^{\mu\lambda} \\
&= \partial_r(\frac{P}{X^2}) + (\frac{1}{2}C' + \frac{X'}{X} + 2\frac{R'}{R}) PX^{-2} \\
&+ \frac{e^C C'}{2X^2}\rho e^{-C} + \frac{X'}{X} PX^{-2} - 2\frac{RR'}{X^2} PR^{-2} \\
&= X^{-2}\partial_r(P) + \frac{1}{2}C' X^{-2}(P + \rho), \qquad \text{(C.2)} \\
v = \theta : 0 = \nabla_\theta T^{\theta\theta} &= \Gamma^{\mu}_{\mu\theta} T^{\theta\theta} + \Gamma^{\theta}_{\mu\lambda} T^{\mu\lambda} \\
&= (\cot(\theta)) PR^{-2} - \sin(\theta)\cos(\theta) PR^{-2}\sin^{-2}(\theta) = 0, \qquad \text{(C.3)} \\
v = \varphi : 0 = \nabla_\varphi T^{\varphi\varphi} &= \Gamma^{\mu}_{\mu\varphi} T^{\varphi\varphi} + \Gamma^{\varphi}_{\mu\lambda} T^{\mu\lambda} = 0 + 0 = 0. \qquad \text{(C.4)}
\end{aligned}
$$

This, therefore, gives two non-trivial equations:

$$\partial_t \rho + (\frac{\dot{X}}{X} + \frac{2\dot{R}}{R})(\rho + P) = 0, \tag{C.5}$$

$$\partial_r P + \frac{1}{2}C'(P + \rho) = 0. \tag{C.6}$$

It must be noted that these equations can also be obtained from the Einstein equations (2.4). For this reason and because of its complexity, we do not manipulate (2.6d) into another equation.

Appendix D

Solutions of the Differential Equation for LTB

We start from equation (2.54) in context of the LTB-metric (2.53):

$$(\dot{R}(t,r))^2 = E(r) + \frac{2Gm(r)}{R(t,r)} \tag{D.1}$$

We will solve this PDE independently for each shell labeled with r. Thus, we reduces this PDE to a non-linear first order ODE with m and E constant.

$$(\dot{R})^2 = E + \frac{2Gm}{R} \tag{D.2}$$

Take the square root of both sides:

$$\dot{R} = \pm\sqrt{E + \frac{2Gm}{R}} = \pm\frac{\sqrt{ER + 2Gm}}{\sqrt{R}}. \tag{D.3}$$

The method of separation of variables is applied to this equation with subsequent integration:

$$\int \frac{\sqrt{R}}{\sqrt{ER + 2Gm}}\, dR = \pm \int dt. \tag{D.4}$$

At this point, the distinction is made between case $E > 0$, $E = 0$, $E < 0$.

D.1 Case $E > 0$

The nominator and denominator are divided by $\sqrt{E}$:

$$\frac{1}{\sqrt{E}} \int \frac{\sqrt{R}}{\sqrt{R + \frac{2Gm}{E}}}\, dR = \pm \int dt. \tag{D.5}$$

For the left-hand side, the change of variables is used with $R = \frac{2Gm}{E}\sinh^2(u)$, where u is the new variable, so that $\sqrt{R + \frac{2Gm}{E}} = \sqrt{\frac{2Gm}{E}}\cosh(u)$:

$$\int \frac{\frac{\sqrt{2Gm}}{E}\sinh(u)}{\sqrt{\frac{2Gm}{E}}\cosh(u)} \cdot 2 \cdot \frac{2Gm}{E}\sinh(u)\cosh(u)\,du = \pm(t - C), \tag{D.6}$$

where C is the integration constant. This reduces to:

$$\frac{4Gm}{E^{\frac{3}{2}}}\int \sinh^2(u)\,du = \pm(t - C), \tag{D.7}$$

The change of variables, now with $z = 2u$ is performed:

$$\frac{2Gm}{E^{\frac{3}{2}}}\int \sinh^2(\frac{z}{2})\,dz = \frac{Gm}{E^{\frac{3}{2}}}\int \cosh(z) - 1\,dz = \pm(t - C). \tag{D.8}$$

Absorbing the other integration constant in C, the following indirect expression for the solution is obtained, where C is renamed to t_0 and it is noted that it can still depend on r:

$$\frac{Gm}{E^{\frac{3}{2}}}(\sinh(z) - z) = \pm(t - t_0(r)), \tag{D.9}$$

We substitute z into equation for $R(r,t)$ to obtain the system of parametric solutions:

$$\frac{Gm}{E^{\frac{3}{2}}}(\sinh(z) - z) = (t - t_0(r)), \tag{D.10}$$

$$R(r,t) = \frac{2Gm(r)}{E}\sinh^2(u) = \frac{Gm(r)}{E}(\cosh(z) - 1), \tag{D.11}$$

where the positive root was chosen because $\sinh(z) - z$ is positive for all $z \in (0,\infty)$ on the left-hand side, and the right-hand side must also be positive. The latter is the consequence of the fact that time proceeds from t_0, also called the bang time function, so that the difference $t - t_0(r)$ must be positive.

D.2 Case $E = 0$

With $E = 0$, the expression (D.4) becomes:

$$\int \frac{\sqrt{R}}{\sqrt{2Gm}}\,dr = \pm\int dt. \tag{D.12}$$

Integration results in the following expression:

$$\frac{2}{3}R^{3/2} = \pm\sqrt{2Gm}(t - t_0(r)), \tag{D.13}$$

and this gives the final expression:

$$R(r,t) = \left[\frac{9}{2}Gm(r)(t - t_0(r))^2\right]^{\frac{1}{3}}, \tag{D.14}$$

where the positive root was chosen because $R(r,t)$ is a physical radius.

D.3 Case $E < 0$

Define $K = -E > 0$ and substitute this into the equation (D.4):

$$\int \frac{\sqrt{R}}{\sqrt{-KR + 2Gm}}\, dR = \pm \int dt. \tag{D.15}$$

The nominator and denominator are divided by $\sqrt{K}$:

$$\frac{1}{\sqrt{K}} \int \frac{\sqrt{R}}{\sqrt{-R + \frac{2Gm}{K}}}\, dR = \pm \int dt. \tag{D.16}$$

For the left-hand side, the change of variables is used with $R = \frac{2Gm}{K}\sin^2(u)$, where u is the new variable, so that $\sqrt{R + \frac{2Gm}{E}} = \sqrt{\frac{2Gm}{K}}\cos(u)$:

$$\int \frac{\frac{\sqrt{2Gm}}{K}\sin(u)}{\sqrt{\frac{2Gm}{K}}\cos(u)} \cdot 2\frac{2Gm}{E}\sin(u)\cos(u)\, du = \pm(t - C), \tag{D.17}$$

where C is the integration constant. This reduces to:

$$\frac{4Gm}{K^{\frac{3}{2}}} \int \sin^2(u)\, du = \pm(t - C), \tag{D.18}$$

The change of variables, now with $z = 2u$ is performed:

$$\frac{2Gm}{K^{\frac{3}{2}}} \int \sin^2(\frac{z}{2})\, dz = \frac{Gm}{K^{\frac{3}{2}}} \int 1 - \cos(z)\, dz = \pm(t - C). \tag{D.19}$$

Absorbing the other integration constant in C, the following indirect expression for the solution is obtained, where C is renamed to t_0 and E is substituted:

$$\frac{Gm}{(-E)^{\frac{3}{2}}}(z - \sin(z)) = \pm(t - t_0(r)). \tag{D.20}$$

We substitute z and $E = -K$ into the equation for $R(r,t)$ to obtain the system of solutions of parametric form:

$$\frac{Gm}{(-E)^{\frac{3}{2}}}(z - \sin(z)) = (t - t_0(r)), \tag{D.21}$$

$$R(r,t) = \frac{2Gm(r)}{K}\sin^2(\frac{z}{2}) = -\frac{Gm(r)}{E}(1 - \cos(z)), \tag{D.22}$$

where the positive root was chosen because $z - \sin(z)$ is positive for all $z \in (0, 2\pi)$ on the left-hand side, and the right-hand side must also be positive.

Appendix E

Derivation of Evolution Equations for dust

The evolution equations, equivalent to the Einstein equation, are for a general dust metric obtained by following the ADM formalism that decomposes spacetime. It does so by partitioning the spacetime manifold M into a collection of spacelike hypersurfaces[1], which are the 3D-submanifolds Σ_T where some 'time' function $T(t, x, y, z)$ on M is constant. The choice of this time function is not a priori fixed, due to the diffeomorphism invariance. This means that different observers can choose their time and those give different spatial hypersurfaces and this splitting is equivalent. However, because we are considering the zero pressure LTB universe with the metric (2.53), where t in this metric is equal to the proper time as measured by an comoving observer and global synchonisation is possible with this specific foliation of spacetime, we will consider the family of spatial hypersurfaces $\Sigma_t \cong \{x^1, x^2, x^3 : t = \text{constant}\}$, where $\Sigma_t \cong \{\text{t}\}\times S \cong S$, with S as the spatial manifold. While it is not the case for general hypersurfaces, the unit normal n^μ on constant-t hypersurfaces will simply be $(1, 0, 0, 0)$ in vector form[2]. We can define the projection tensor $P_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$, which projects a vector onto a spatial hypersurface with unit normal n^μ.

While till now the Riemann and Ricci tensors were used to consider curvature, the curvature they consider, is the *intrinsic* curvature. This is the curvature you would measure within the manifold itself. However, a cylinder for example has $R = 0$ and thus its intrinsic curvature vanishes. Obviously, we know that a 3D cylinder curves as an embedded object in a larger space, so this motivates the definition for *extrinsic* curvature, which is a measure of how much a submanifold curves as an embedding in a higher dimensional manifold. While the equivalent definitions can be found in [2], Caroll shows [6, Appendix D],

[1] This means that the three tangent vectors, forming the basis of the 3D-hypersurface, are spacelike.

[2] We ignore the minus case.

for a general metric, that the extrinsic curvature tensor $K_{\mu\nu}$ can be written as: $K_{\mu\nu} = \nabla_\mu u_\nu$, when the acceleration a^μ is zero.[3] Notice, that for the dust case, the unit normal and 4-velocity are equal, which validates the interchangable use of the two. We are allowed to express the extrinsic curvature into a trace part and a trace-free part, which could be easily checked by taking the trace of the tensors:

$$K_{\mu\nu} = \frac{1}{3}P^{\alpha\beta}K_{\alpha\beta}P_{\mu\nu} + \sigma_{\mu\nu}, \tag{E.1}$$

where $\sigma_{\mu\nu} \equiv K_{\mu\nu} - \frac{1}{3}P^{\alpha\beta}K_{\alpha\beta}P_{\mu\nu}$ represents the trace-free tensor. We define θ to be the trace of $K_{\mu\nu}$:

$$\theta \equiv P^{\alpha\beta}K_{\alpha\beta} = \nabla_\mu u^\mu, \tag{E.2}$$

where in the last equality it was used that the acceleration is zero.

E.1 Raychaudhuri's Equation

We take the directional covariant derivative of $K_{\mu\nu}$ with respect to the proper time of the fluid:

$$\frac{DK_{\mu\nu}}{d\tau} = u^\rho \nabla_\rho \nabla_\nu u_\mu \ . \tag{E.3}$$

To commute the two covariant derivatives ∇_ρ and ∇_ν, we remember that $[\nabla_\rho, \nabla_\nu]u_\mu = R^\alpha{}_{\mu\nu\rho}u_\alpha$, so that the equation becomes:

$$\frac{DK_{\mu\nu}}{d\tau} = -u^\rho R^\alpha{}_{\mu\nu\rho}u_\alpha + \nabla_\nu\Big[u^\rho \nabla_\rho u_\mu\Big] - \Big[\nabla_\nu u^\rho\Big]\Big[\nabla_\rho u_\mu\Big], \tag{E.4}$$

where the last two terms on the right-hand side are the result of the Leibniz rule for covariant derivatives. Now, we remember that $K_{\mu\nu} = \nabla_\mu u_\nu$ and that the acceleration is zero to rewrite the equation as:

$$\frac{DK_{\mu\nu}}{d\tau} = -u^\rho R^\alpha{}_{\mu\nu\rho}u_\alpha - K^\rho{}_\nu K_{\mu\rho}, \tag{E.5}$$

which can be written as:

$$\frac{d\theta}{d\tau} = -R_{\mu\nu}u^\mu u^\nu - \sigma^{\mu\nu}\sigma_{\mu\nu} - \frac{1}{3}\theta^2, \tag{E.6}$$

because we apply inverse metric $g^{\mu\nu}$ on both sides and we note that the trace of $K_{\mu\nu}$, θ, is a scalar for which the covariant derivative reduces to a normal one. Furthermore, we have used that $K^{\mu\nu}K_{\mu\nu} = \sigma^{\mu\nu}\sigma_{\mu\nu} + \frac{1}{3}\theta^2$, as $P^{\mu\nu}P_{\mu\nu} = 3$. Finally, dividing by 3 and using $\Sigma^2 \equiv \frac{1}{6}\sigma^{\mu\nu}\sigma_{\mu\nu}$, we write:

$$\frac{d\theta}{d\tau} = -\frac{\kappa\ \rho}{2} - 6\Sigma^2 - \frac{1}{3}\theta^2, \tag{E.7}$$

[3]This acceleration is defined as $a^\mu = n^\nu \nabla_\nu n^\mu$ and it is zero, because Γ^μ_{tt} are zero for dust LTB and $n^v = (1,0,0,0)$.

where the first term was the result of writing $R_{\mu\nu}u^\mu u^\nu = \frac{\kappa\rho}{2}$, with $\kappa = 8\pi G$, from Einstein's equation. Equation (E.7) is called the Raychaudhuri's equation. The derivation followed [6, Appendix F].

E.2 Hamiltonian Equation

Gauß's equation relates the 3-Ricci scalar to the 4-Ricci scalar of the spacetime manifold and, thus, the intrinsic curvature to the extrinsic curvature. When we apply the projection operator $P^{\sigma\nu}$ on the Gauß's equation, we obtain the following equation [6, Appendix D]:

$$^{(3)}\mathcal{R} = {}^{(4)}\mathcal{R} + 2R_{\mu\nu}u^\mu u^\nu - (g^{\mu\nu}K_{\mu\nu})^2 + K^{\mu\nu}K_{\mu\nu}\ . \tag{E.8}$$

We can rewrite the equation to:

$$\frac{^{(3)}\mathcal{R}}{6} = \frac{\mathcal{R}}{6} + \frac{1}{3}R_{\mu\nu}u^\mu u^\nu - \left(\frac{\theta}{3}\right)^2 + \Sigma^2\ . \tag{E.9}$$

Here, we have used that the trace of $K_{\mu\nu}$ is θ and we have defined $\Sigma^2 \equiv \frac{1}{6}\sigma^{\mu\nu}\sigma_{\mu\nu}$. The 4-Ricci scalar from this point is denoted as just $\mathcal{R}$. The Einstein equation, in a form that is multiplied by $u^\mu u^\nu$, expresses the first two terms on the right-hand side of (E.9) as $\frac{\kappa\rho}{3}$, where $\kappa = 8\pi G$, so that we obtain:

$$\left(\frac{\theta}{3}\right)^2 = \frac{\kappa\ \rho}{3} - \frac{^{(3)}\mathcal{R}}{6} + \Sigma^2\ . \tag{E.10}$$

Equation (E.10) is referred to as the Hamiltonian constraint or general Friedmann equation. The derivation followed [6, Appendix F][19, Page 29-32].

E.3 Energy Balance Equation

For the dust-LTB universe with zero pressure, equation (C.5) implies that:

$$\frac{d\rho}{d\tau} + \left(\frac{\dot{R}'}{\dot{R}} + 2\frac{\dot{R}}{R}\right)\rho = 0, \tag{E.11}$$

where we have used that $\frac{\dot{X}}{X} = \frac{\dot{R}'}{\dot{R}}$. As we will see, the factor that multiplies ρ is θ, but the equation can be in general derived for a dust universe by projecting onto the 4-velocity and using the normalization condition for the four-velocity. This is called the energy balance equation:

$$\partial_t\rho + \theta\rho = 0, \tag{E.12}$$

Appendix F

Derivation expressions for θ, ${}^{(3)}\mathcal{R}$ and Σ^2 in LTB

For the derivations, we assume the LTB universe, where the pressure is zero and comoving-synchronous coordinates are used, with metric (2.53), such that $g_{tt} = -1$. The relevant quantities are defined in (4.1) and (E).

F.1 Scalar θ in LTB

The expansion parameter can be calculated, using the expression for $\sqrt{-g}$, the fact that $u^\mu = (1, 0, 0, 0)$ and the fact that E is not dependent on t:

$$\theta = \nabla_\mu u^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \big(\sqrt{-g}\, u^\mu\big) = \frac{1}{R^2 R'}\, \partial_t \big(R^2 R'\big) \tag{F.1}$$

Because $\partial_t(R^2 R') = 2R\dot{R}\, R' + R^2 \dot{R}'$, we can compute the final expression as follows:

$$\theta = \frac{2\dot{R}}{R} + \frac{\dot{R}'}{R'} \tag{F.2}$$

F.2 Scalar ${}^{(3)}\mathcal{R}$ in LTB

We start with the contracted version of Gauß's equation:

$${}^{(3)}\mathcal{R} = {}^{(4)}\mathcal{R} + K_{\mu\nu} K^{\mu\nu} - \theta^2 + 2R_{\mu\nu} u^\mu u^\nu \tag{F.3}$$

Then, we compute the quantities on the left-hand side separately:

${}^{(4)}\mathcal{R}$ in (A.6) reduces to:

$$
\begin{aligned}
{}^{(4)}\mathcal{R} = &\Big[\frac{2\ddot{X}}{X} + \frac{4\ddot{R}}{R} + \frac{4\dot{R}\dot{X}}{RX} + \frac{2(\dot{R})^2}{R^2}\Big] \\
&+ \frac{1}{X^2}\Big[-\frac{4R''}{R} + \frac{4R'X'}{RX} - \frac{2(R')^2}{R^2}\Big] + \frac{2}{R^2}
\end{aligned}
\tag{F.4}
$$

For LTB, the nonzero $K_{\mu\nu} = \Gamma^t_{\mu\nu}$ are the components: $K_{rr} = \frac{R'\dot{R}'}{1+E}$, $K_{\theta\theta} = R\dot{R}$, $K_{\varphi\varphi} = R\dot{R}\sin^2(\theta)$. With $K^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}K_{\mu\nu}$, we can write down the nonzero components of $K^{\mu\nu}$: $K^{rr} = \frac{\dot{R}'(1+E)}{(R')^3}$, $K^{\theta\theta} = \frac{\dot{R}}{R^3}$, $K^{\varphi\varphi} = \frac{\dot{R}}{R^3\sin^2(\theta)}$. Then we contract the two to obtain:

$$
K_{\mu\nu}K^{\mu\nu} = \Big(\frac{\dot{R}'}{R'}\Big)^2 + 2\Big(\frac{\dot{R}}{R}\Big)^2
\tag{F.5}
$$

The last term in equation (F.3) can also be written more simply:

$$
2R_{\mu\nu}u^\mu u^\nu = 2R_{tt} = -\frac{\ddot{X}}{X} - 2\frac{\ddot{R}}{R}
\tag{F.6}
$$

Finally, we can compute the ${}^{(3)}\mathcal{R}$:

$$
\begin{aligned}
{}^{(3)}\mathcal{R} = &\left[\frac{2\ddot{X}}{X} + \frac{4\ddot{R}}{R} + \frac{4\dot{R}\dot{X}}{RX} + \frac{2(\dot{R})^2}{R^2}\right] \\
&+ \frac{1}{X^2}\left[-\frac{4R''}{R} + \frac{4R'X'}{RX} - \frac{2(R')^2}{R^2}\right] + \frac{2}{R^2} \\
&+ \frac{(\dot{R}')^2}{(R')^2} + \frac{2(\dot{R})^2}{R^2} - \left(\frac{\dot{R}'}{R'} + 2\frac{\dot{R}}{R}\right)^2 - 2\left[\frac{\ddot{X}}{X} + 2\frac{\ddot{R}}{R}\right] \\
= &\frac{1}{X^2}\Big(2\frac{X'R'}{XR} - 2\frac{R''}{R}\Big) + \frac{2}{R^2}\Big(-\frac{(R')^2}{X^2} - \frac{RR''}{X^2} + \frac{RR'X'}{X^3} + 1\Big), \\
= &\frac{1+E(r)}{(R')^2}\Big(2\frac{R''R'}{RR'} - \frac{E'(r)(R')^2}{RR'}\frac{1}{1+E(r)} - \frac{2R''}{R}\Big) \\
&+ \frac{2}{R^2}\Big(-(1+E(r)) - \frac{RR''}{(R')^2}(1+E(r)) + \\
&RR'\Big(\frac{1+E(r)}{(R')^3}R'' - \frac{R'E'(r)}{2(R')^3}\Big) + 1\Big) \\
= &-2\frac{E'(r)}{R'R} - 2\frac{E(r)}{R^2} = -2\frac{(ER)'}{R'R^2},
\end{aligned}
\tag{F.7}
$$

F.3 Scalar Σ^2 in LTB

From the derivation of Raychaudhuri's equation, we know that $\sigma^{\mu\nu}\sigma_{\mu\nu} = K^{\mu\nu}K_{\mu\nu} - \frac{1}{3}\theta^2$ in appendix (REF) and because we have already computed $K^{\mu\nu}K_{\mu\nu}$ in this appendix, we can compute Σ^2:

$$\begin{aligned}\Sigma^2 \equiv \frac{1}{6}\sigma^{\mu\nu}\sigma_{\mu\nu} &= \frac{1}{6}\Big(\frac{\dot{R}'}{R'}\Big)^2 + \frac{1}{6}\cdot 2\Big(\frac{\dot{R}}{R}\Big)^2 - \frac{1}{6}\cdot\frac{1}{3}\Big(\frac{2\dot{R}}{R} + \frac{\dot{R}'}{R'}\Big)^2 \\ &= \frac{1}{9}\Big[\frac{\dot{R}'}{R'} - \frac{\dot{R}}{R}\Big]^2 \end{aligned} \tag{F.8}$$

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