# The geometric basis of mimetic spectral approximations

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Discretization Methods for Polygonal and Polyhedral Meshes

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# Relation between physical variables and geometric objects

## All physical variables are related to geometric objects

Mass $M$ is associated to a volume, $V$.

Average density $\bar{\rho} = \frac{M}{V}$. $\rho = \lim_{V \to 0} \frac{M}{V}$ mathematically justified, but physically questionable.

# Relation between physical variables and geometric objects

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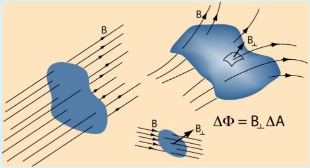
Flux $F$ is associated to a area, $A$.




Average flux density $F/A$. $\lim_{A\to 0}\frac{F}{A}$ mathematically justified, but physically questionable.

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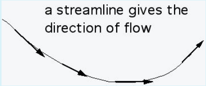
Velocity $F$ is associated to a curve, $C$.




The velocity, $\mathbf{v}$, can be measured by recording the position, $\mathbf{r}$, of a particle at two consecutive time instants, $t_1$ and $t_2$. These positions are related to the velocity by

$$\mathbf{r}(t_2) - \mathbf{r}(t_1) = \int_{t_1}^{t_2} \frac{d\mathbf{r}}{dt}\, dt = \int_{t_1}^{t_2} \mathbf{v}\, dt$$

This relation is exact and then we approximate 'the' velocity by

$$\mathbf{v} \approx \frac{\mathbf{r}(t_2) - \mathbf{r}(t_1)}{t_2 - t_1}$$

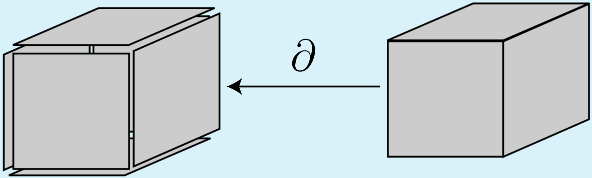
Letting $\Delta t = t_2 - t_1 \rightarrow 0$ to have the velocity at a time instant is physically not meaningful.

# Relation between geometric objects

## Boundary operator

The most important operator in mimetic methods is the boundary operator $\partial$

$$\partial \, : \, k\text{-dim} \longrightarrow (k-1)\text{-dim}$$



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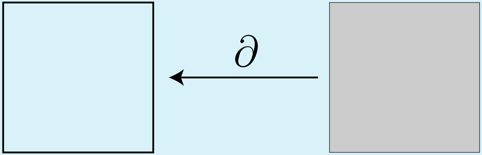

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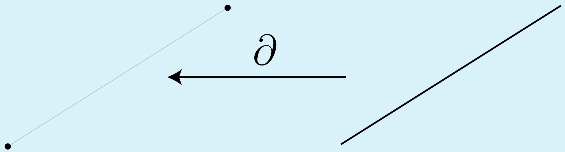
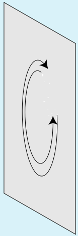


# Orientation and type of orientation

## Orientation and sense of orientation

Every geometric object can be oriented in two ways. For instance, in a surface we define a sense of rotation, either clockwise or counter clockwise



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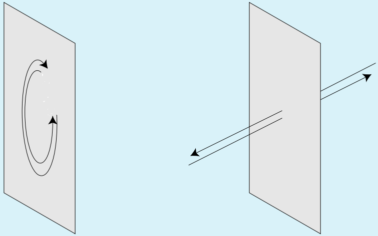
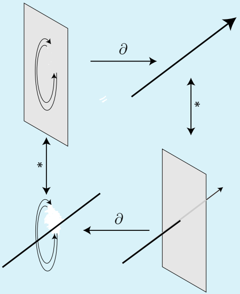


Furthermore, we distinguish between inner-orientation and outer-orientation

# Orientation and type of orientation

## $\partial$ and $\star\partial\star$

Let $\star$ denote the operator which switches between inner- and outer-orientation



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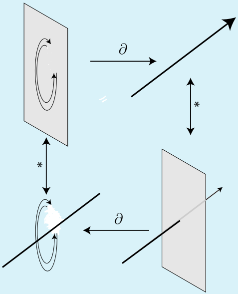


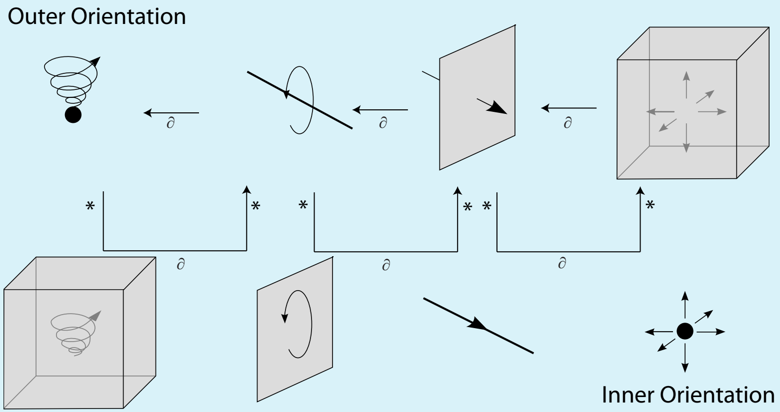

Then we have the operations:

$$\partial \; : \; k\text{-dim} \longrightarrow (k-1)\text{-dim} \qquad \star\partial\star \; : \; k\text{-dim} \longrightarrow (k+1)\text{-dim}$$

# Oriented dual cell complexes

## Double boundary complex

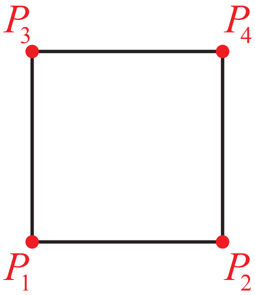
In 3D we have points, curves, surfaces and volumes



# Matrix representation of boundary operator

Set of points:

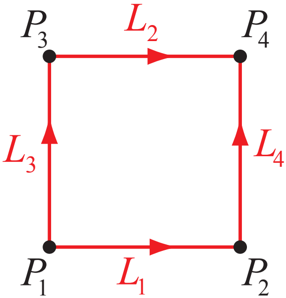
$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix}$$



# Matrix representation of boundary operator

Set of lines:

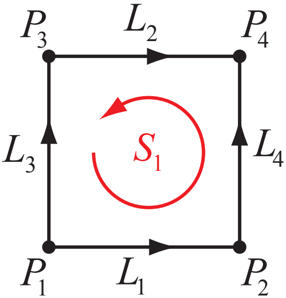
$$\begin{pmatrix} \partial L_1 \\ \partial L_2 \\ \partial L_3 \\ \partial L_4 \end{pmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix}$$



# Matrix representation of boundary operator

Surface:

$$\partial\partial S_1 = \begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix} \begin{pmatrix}\partial L_1 \\ \partial L_2 \\ \partial L_3 \\ \partial L_4\end{pmatrix}$$



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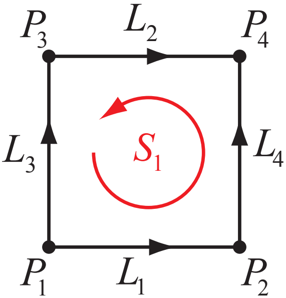
$$= \underbrace{\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{=\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix}$$



# Final remarks geometric objects

**Topological vs metric-dependent operations**

The boundary operator $\partial$ is topological operator, $\star$ operator is metric-dependent.

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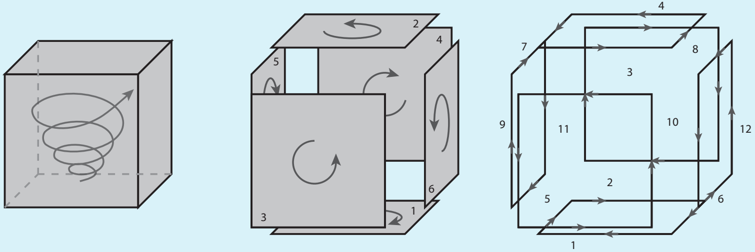
## Topological vs metric-dependent operations

The boundary operator $\partial$ is topological operator, $\star$ operator is metric-dependent.

## Nilpotency of $\partial$ and $\star\partial\star$

Application of the boundary operator twice always yields the empty set: $\partial \circ \partial \equiv 0$




$c$ $\partial c$ $\partial\partial c$

# Assigning a value to geometric objects

## $k$-chains and $k$-cochains

A basic $k$-dimensional object will be called a $k$-cell, $\tau_k$. A collection of oriented $k$-cells is called a $k$-chain, $c_k$. The space of all $k$-chains will be denoted by $C_k$
The operation which assigns a value to a physical quantity associated with a geometric object is called a $k$-cochain, $c^k$:

$$c^k \,:\, C_k \longrightarrow \mathbb{R} \quad \Longleftrightarrow \quad \left\langle c^k, c_k \right\rangle \in \mathbb{R}$$

# Assigning a value to geometric objects

## $k$-cochains and integration

$$c^k : C_k \longrightarrow \mathbb{R} \quad \Longleftrightarrow \quad \left\langle c^k, c_k \right\rangle \in \mathbb{R}$$

In the continuous setting in 3D this should be compared to

$$k = 0\,,\ \text{point}\ :\ f(P)\ ,$$

$$k = 1\,,\ \text{curve}\ :\ \int_C a(x,y,z)\,\mathrm{d}x + b(x,y,z)\,\mathrm{d}y + c(x,y,z)\,\mathrm{d}z\ ,$$

$$k = 2\,,\ \text{surface}\ :\ \int_S P(x,y,z)\,\mathrm{d}y\mathrm{d}z + Q(x,y,z)\,\mathrm{d}z\mathrm{d}x + R(x,y,z)\,\mathrm{d}x\mathrm{d}y\ ,$$

$$k = 3\,,\ \text{volume}\ :\ \int_V \rho(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\ .$$

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The expression underneath the integral sign is called a differential $k$-form, $a^{(k)}$.

$$\left\langle a^{(k)}, \Omega_k \right\rangle := \int_{\Omega_k} a^{(k)} \in \mathbb{R}$$

# Assigning a value to geometric objects

## $k$-cochains and integration

Both integration of differential forms and duality pairing between cochains and chains is a metric-free operation

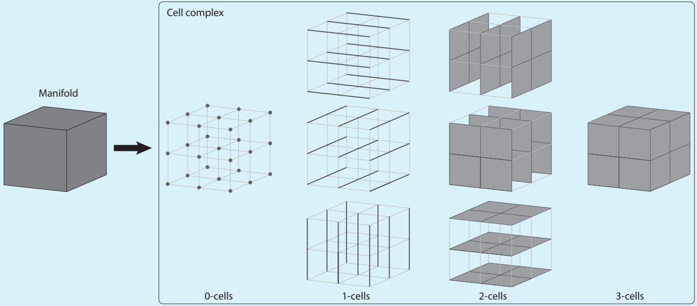
$$\left\langle c^k, c_k \right\rangle \in \mathbb{R} \quad \Longleftrightarrow \quad \left\langle a^{(k)}, \Omega_k \right\rangle \in \mathbb{R}$$

$\langle c^k, c_k \rangle$ is to be considered as discrete integration.

# Cell complex ⇔ computational grid

## Topological mesh

If we glue volumes, surfaces, lines and points together we obtain a so-called cell-complex.



# The Mother of all equations

## The coboundary operator

Duality pairing between chains and cochains allows us to define the adjoint of the boundary operator $\delta$

$$\left\langle \delta c^k, c_{k+1} \right\rangle := \left\langle c^k, \partial c_{k+1} \right\rangle$$

The coboundary operator maps $k$-cochains into $(k+1)$-cochains:

$$\delta \, : \, C^k \longrightarrow C^{k+1}$$

$$\partial \circ \partial \equiv 0 \quad \Longleftrightarrow \quad \delta \circ \delta \equiv 0$$

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$$\left\langle \delta c^k, c_{k+1} \right\rangle := \left\langle c^k, \partial c_{k+1} \right\rangle$$

Let $C$ be an arbitrary curve going from the point $A$ to the point $B$

$$k = 0 \;:\; \int_C \text{grad}\, \phi \, \mathrm{d}\vec{s} = \int_{\partial C} \phi = \phi(B) - \phi(A)$$

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## The coboundary operator

$$\left\langle \delta c^k, c_{k+1} \right\rangle := \left\langle c^k, \partial c_{k+1} \right\rangle$$

Let $S$ be a surface bounded by $\partial S$ then

$$k = 1 \ : \int_S \text{curl}\, \vec{A}\, \mathrm{d}\vec{S} = \int_{\partial S} \vec{A} \cdot \mathrm{d}\vec{s}$$

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## The coboundary operator

$$\left\langle \delta c^k, c_{k+1} \right\rangle := \left\langle c^k, \partial c_{k+1} \right\rangle$$

Let $V$ be a volume, bounded by $\partial V$ then

$$\int_V \text{div}\, \vec{F} \mathrm{d}V = \int_{\partial V} \vec{F} \cdot \mathrm{d}\vec{S}$$

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Duality pairing and the boundary operator DEFINE the coboundary operator!
I.e. grad, curl and div are defined through the topological relations and are therefore coordinate-free and metric-free.

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If we choose basis functions for our numerical method, the basis functions should cancel from the equations. There cannot be an explicit dependence on the basis functions. The same topological relations hold for low order methods and high order methods.

# The Mother of all equations

## The coboundary operator

$$\left\langle \delta c^k, c_{k+1} \right\rangle := \left\langle c^k, \partial c_{k+1} \right\rangle$$

At the continuous level, in terms of differential forms, this relation is given by the generalized Stokes Theorem

$$\int_{\Omega_{k+1}} \mathrm{d}\omega^{(k)} := \int_{\partial\Omega_{k+1}} \omega^{(k)}$$

# The 'Hodge-$\star$' operator

## The 'Hodge-$\star$' operator

Remember that $\star$ was the operator which switches between inner- and outer orientation. We can also write down a formal adjoint of this operation

$$\left\langle \star c^k, c_{n-k} \right\rangle := \left\langle c^{n-k}, \star c_k \right\rangle$$

The $\star$ operator applied to $k$-dimensional geometric objects turns them into $(n-k)$-dimensional geometric objects with the other type of orientation.
The $\star$ operator applied to $k$-cochains turns them into $(n-k)$-cochains acting on geometric objects of the other orientation.

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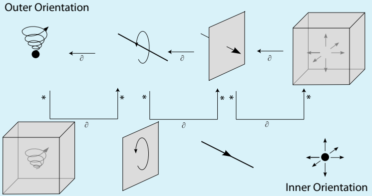
The $\star$ operator applied to $k$-cochains turns them into $(n-k)$-cochains acting on geometric objects of the other orientation. The $\star$ operator is metric-dependent and can therefore not be described in purely topological terms

# The ugly stepmother

## $\delta^\star = \star\delta\star$

Recall that

$$\star\partial\star \; : \; C_k \longrightarrow C_{k+1}$$




So the formal adjoint of $\star\partial\star$ would be

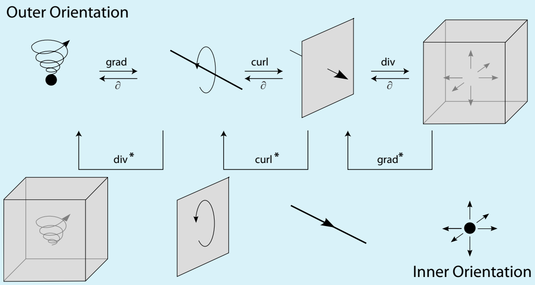
$$\left\langle \delta^\star c^k, c_{k-1} \right\rangle := \left\langle \star\delta\star c^k, c_{k-1} \right\rangle = \left\langle c^k, \star\partial\star c_{k-1} \right\rangle$$

# The ugly stepmother

## $\delta^\star$ and grad, curl and div

$\delta^\star$ also represents the grad, curl and div

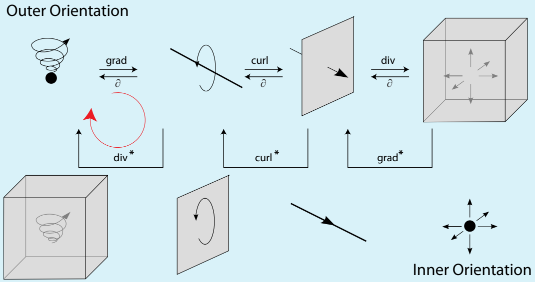
$$\delta^\star : C^k \longrightarrow C^{k-1}$$




Note that in contrast to $\delta$, $\delta^\star$ is a metric-dependent version of grad, curl and div and can therefore NOT be the same as the topological grad, curl and div. We will make this difference explicit by grad*, curl* and div*.

# Laplace-Hodge operator

## Laplace-Hodge operator




The scalar Laplace operator acting on outward oriented points is given by

$$-\text{div}^* \, \text{grad}\phi$$

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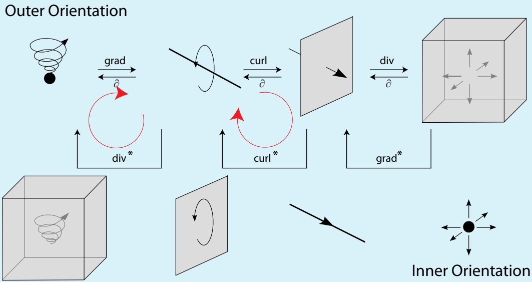


The vector Laplace operator acting on outward oriented lines is given by

$$\left[-\text{grad}\,\text{div}^* + \text{curl}^*\,\text{curl}\right] \vec{A}$$

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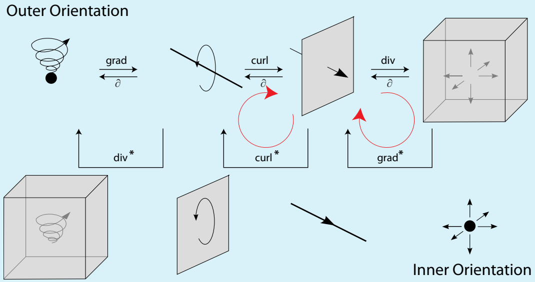


The vector Laplace operator acting on outward oriented surfaces is given by

$$[\text{curl}\,\text{curl}^* - \text{grad}^*\,\text{div}]\,\vec{F}$$

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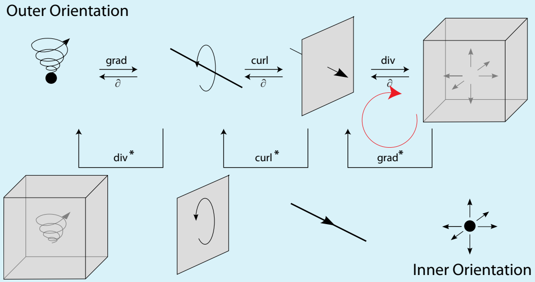


The vector Laplace operator acting on outward oriented volumes is given by

$$-\mathrm{div}\,\mathrm{grad}^{*}\rho$$

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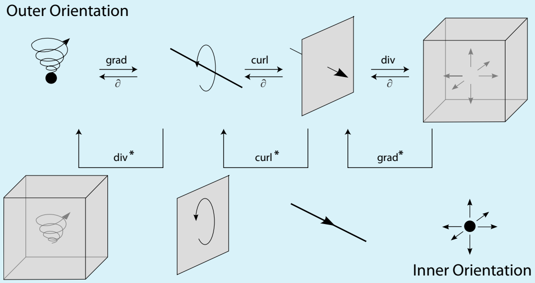


On contractible domains the geometric structure given above is called the double DeRham complex

# Metric

## Metric

How do we discretize the metric-dependent part?

# Metric

## Reduction

Let the reduction operator be defined by

$$\mathcal{R} \,:\, \Lambda^k(\Omega) \longrightarrow C^k(\Omega)$$

$$\left\langle \left( \mathcal{R} a^{(k)} \right), \tau_k \right\rangle := \int_{\tau_k} a^{(k)}$$

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$$\mathcal{R}\mathrm{d} = \delta\mathcal{R}$$

$$\begin{array}{ccc} \Lambda^k & \xrightarrow{\mathrm{d}} & \Lambda^{k+1} \\ \downarrow \mathcal{R} & & \downarrow \mathcal{R} \\ C^k & \xrightarrow{\delta} & C^{k+1} \end{array}$$

# Metric

## Reconstruction

The reconstruction operator needs to satisfy

$$\mathcal{I} : C^k(\Omega) \longrightarrow \Lambda_h^k(\Omega) \subset \Lambda^k(\Omega)$$

$$\mathrm{d}\mathcal{I} = \mathcal{I}\delta \qquad \text{and} \qquad \mathcal{R} \circ \mathcal{I} \equiv \mathbb{I}$$

$$\begin{array}{ccc} C^k & \xrightarrow{\delta} & C^{k+1} \\ \big\downarrow \mathcal{I} & & \big\downarrow \mathcal{I} \\ \Lambda^k & \xrightarrow{\mathrm{d}} & \Lambda^{k+1} \end{array}$$

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Spectral element basis functions which satisfy these relations are called mimetic spectral elements

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## Discretization ⇔ projection

We define the projection operator as

$$\pi := \mathcal{I} \circ \mathcal{R}$$

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The commutation relations ensure that

$$\mathrm{d}\pi = \mathrm{d}\mathcal{I}\mathcal{R} = \underline{\mathcal{I}\delta\mathcal{R}} = \mathcal{I}\mathcal{R}\mathrm{d} = \pi\mathrm{d}$$

$$\begin{array}{ccc} \Lambda^k & \xrightarrow{\mathrm{d}} & \Lambda^{k+1} \\ \downarrow \pi & & \downarrow \pi \\ \Lambda^k_h & \xrightarrow{\mathrm{d}} & \Lambda^{k+1}_h \end{array}$$

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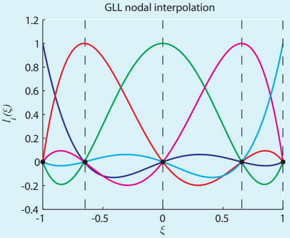
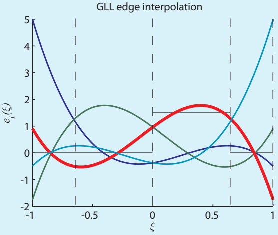
NOTE: This only holds for the topological grad, curl and div! NOT for grad$^*$, curl$^*$ or div$^*$.

# Mimetic spectral elements

## Basis functions 1D



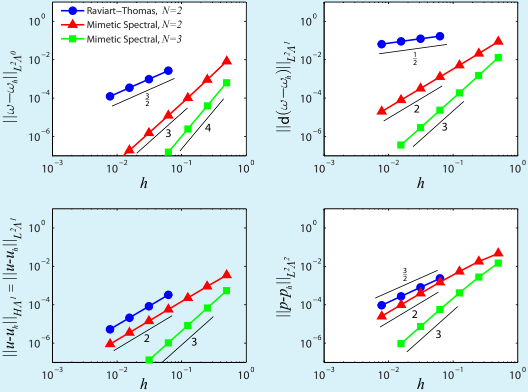

In 1D we only have points and line segments, so we use

$$\text{nodal Lagrange interpolation :} \quad h_i(x_j) = \delta_{ij}$$

$$\text{Edge interpolation :} \int_{x_{j-1}}^{x_j} e_i(x) = \delta_{i,j} \ , \quad e_i(x) = -\sum_{k=0}^{i-1} \mathrm{d}h_k(x)$$

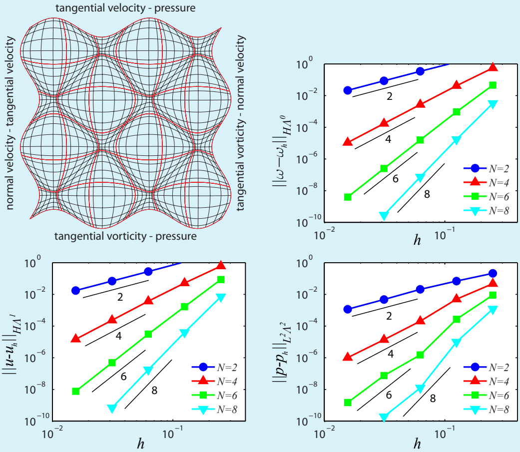
# Comparison with higher order RT-elements

## Stokes problem



# Comparison with higher order RT-elements

## Stokes problem



# How to avoid grad*, curl* and div*

## Integration by parts

Finite element methods remove the metric-dependent vector operations through integration by parts

$$(\mathrm{d}a^k, b^{k+1}) = \mathrm{d}a^k \wedge \star b^{k+1} = (-1)^{k+1} a^k \wedge \mathrm{d} \star b^{k+1} =$$

$$a^k \wedge \star \mathrm{d}^* b^{k+1} = (a^k, \mathrm{d}^* b^{k+1})$$

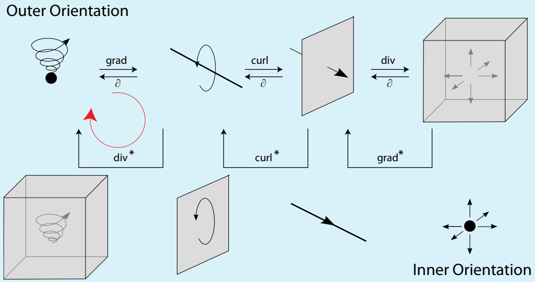
## Vector operations

In conventional vector operations this reads (without boundary)

$$(\text{grad}\phi, \vec{b}) = (\phi, -\text{div}^* \vec{b}) \;, \quad (\text{curl}\vec{a}, \vec{b}) = (\vec{a}, \text{curl}^* \vec{b}) \;, \quad (\text{div}\vec{a}, \phi) = (\vec{a}, -\text{grad}^* \phi)$$

# Laplace-Hodge operator

## Laplace-Hodge operator




The scalar Laplace operator acting on outward oriented points is given by

$$-\text{div}^* \, \text{grad}\phi = f$$

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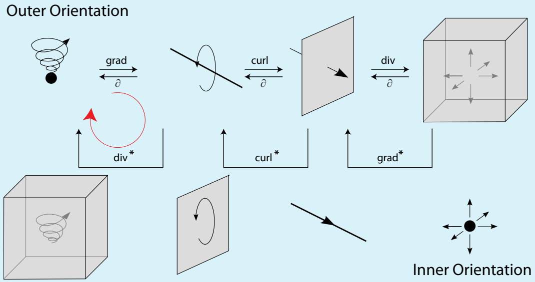


The scalar Laplace operator acting on outward oriented points is given by

$$(-\text{div}^* \, \text{grad}\phi, \psi) = (f, \psi)$$

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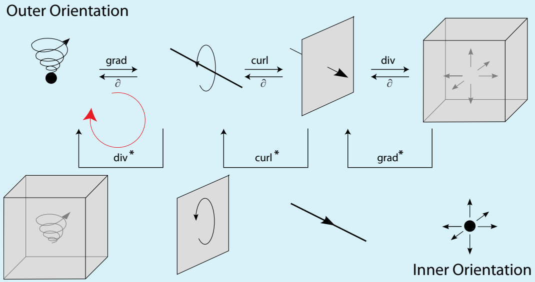


The scalar Laplace operator acting on outward oriented points is given by

$$(\text{grad}\phi, \text{grad}\psi) + b.i. = (f, \psi)$$

# Laplace-Hodge operator

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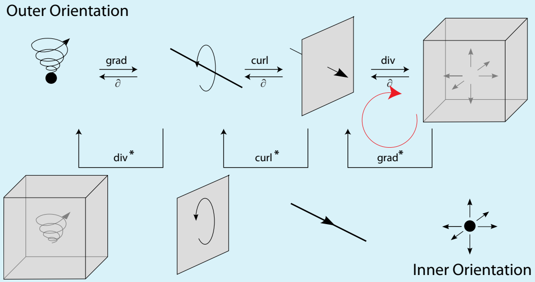


The vector Laplace operator acting on outward oriented volumes is given by

$$\mathrm{div}\,\mathrm{grad}^*\rho = f$$

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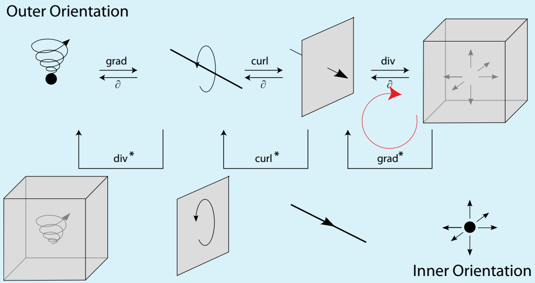


The vector Laplace operator acting on outward oriented volumes is given by

$$\vec{q} = \text{grad}^* \rho$$

$$\text{div}\vec{q} = f$$

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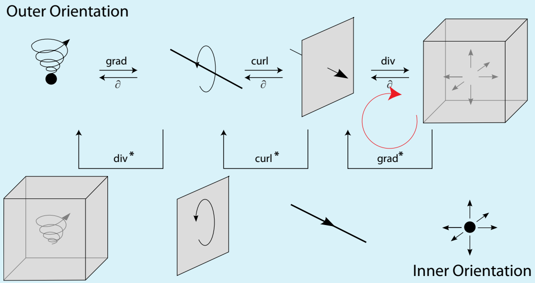


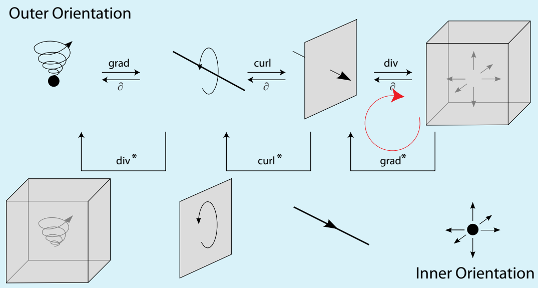

The vector Laplace operator acting on outward oriented volumes is given by

$$(\vec{q},\vec{p}) - (\text{grad}^*\rho,\vec{p}) = 0$$

$$(\text{div}\vec{q}, w) = (f, w)$$

# Laplace-Hodge operator

## Laplace-Hodge operator



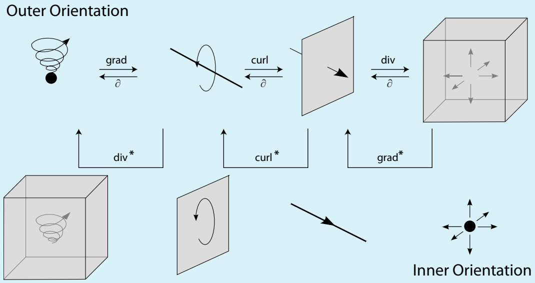

The vector Laplace operator acting on outward oriented volumes is given by

$$(\vec{q},\vec{p}) + (\rho, \text{div}\vec{p}) + b.i. = 0$$

$$(\text{div}\vec{q}, w) = (f, w)$$

# Laplace-Hodge operator

## Laplace-Hodge operator




The weak formulation (direct or mixed) is determined by the geometry which in turn is determined by the physics!

# Resonant Cavity problem (benchmark case)

## Eigenvalue problem (borrowed from our neighbors)

Maxwell equations with unit coefficients and zero force functions.

$$\nabla \times \left( \nabla \times \vec{E} \right) = \lambda \vec{E} \quad \text{on} \quad \Omega = [0, \pi]^2$$

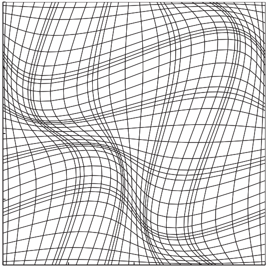
All eigenvalues are known integers: $\lambda = 1, 1, 2, 4, 4, 5, 5, 8, 9, 9, \ldots$

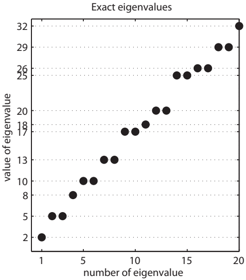
# Resonant Cavity problem (benchmark case)

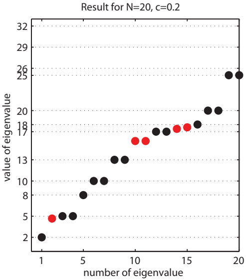
## Results

$$\Delta\vec{u} = \lambda\vec{u}, \quad \mathrm{div}\vec{u} = 0, \quad \Omega = [0,\pi]^2$$

Not solvable with standard FEM / SEM, see [Boffi, Acta Numerica 2010].





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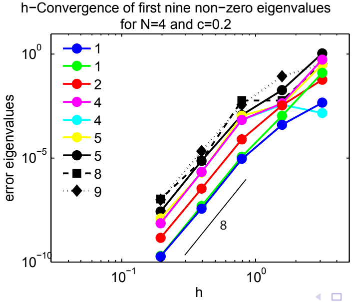
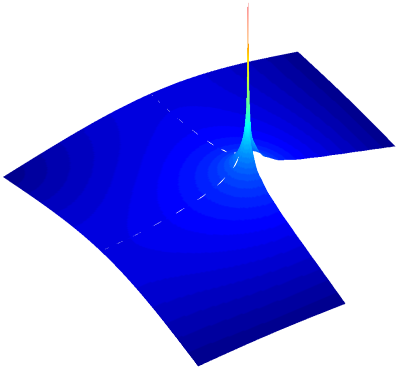
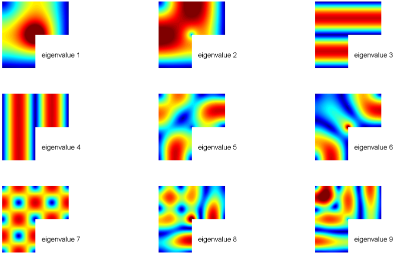


# Resonant cavity in L-shaped domain

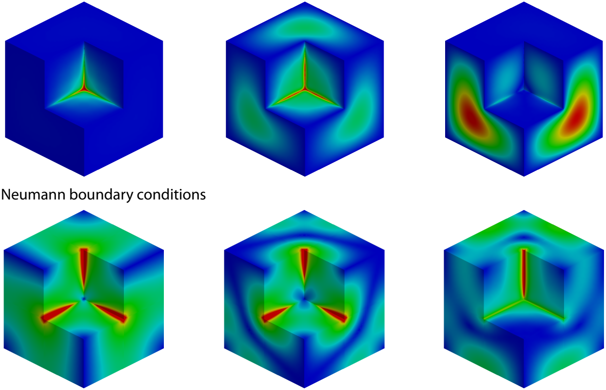




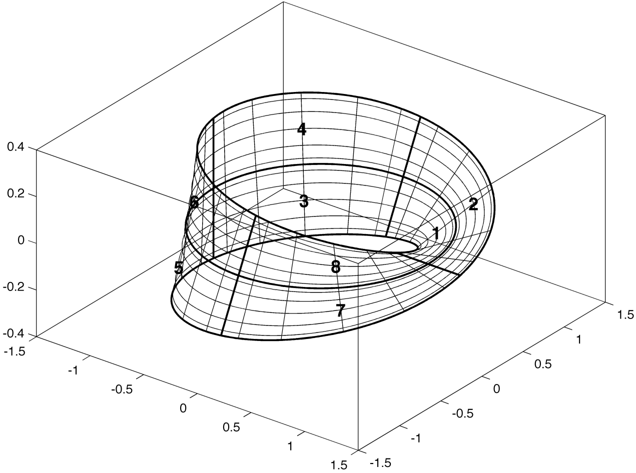
# Resonant cavity in L-shaped domain

Dirichlet boundary conditions

Neumann boundary conditions



# Eigenfunctions on the Möbius strip



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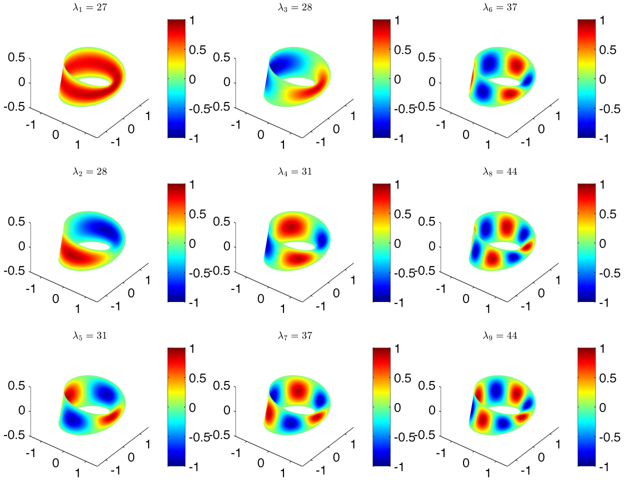
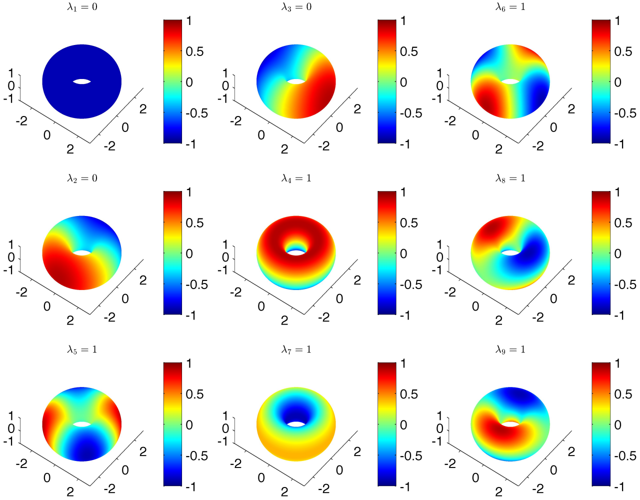
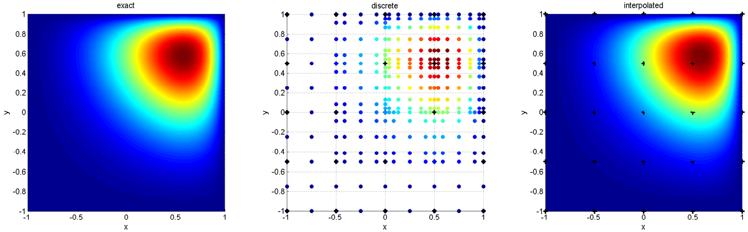


# Eigenfunctions on torus



# Adaptive grid refinement - 0-forms



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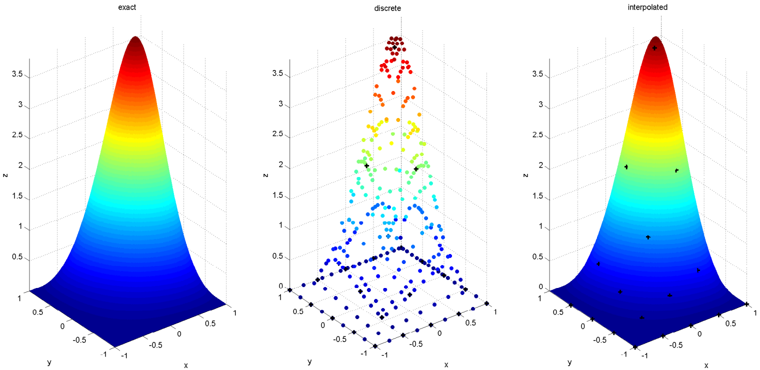
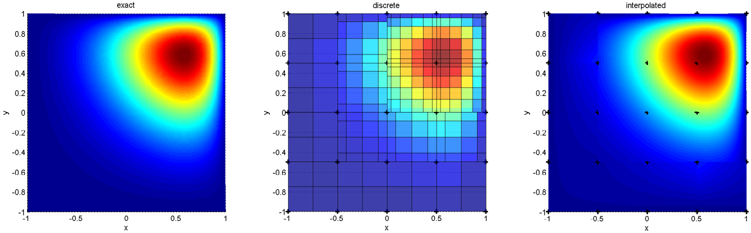


# Adaptive grid refinement - 2-forms



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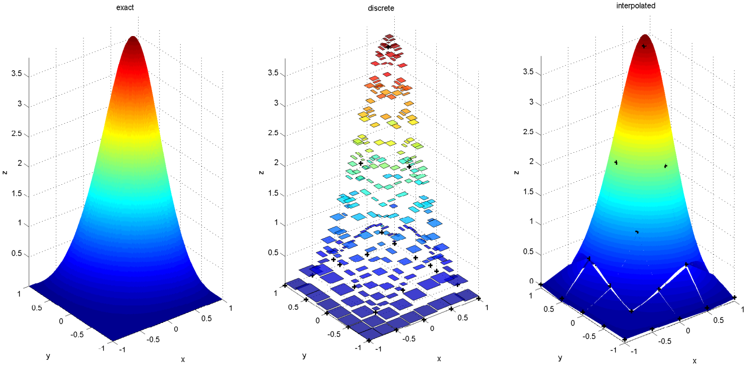
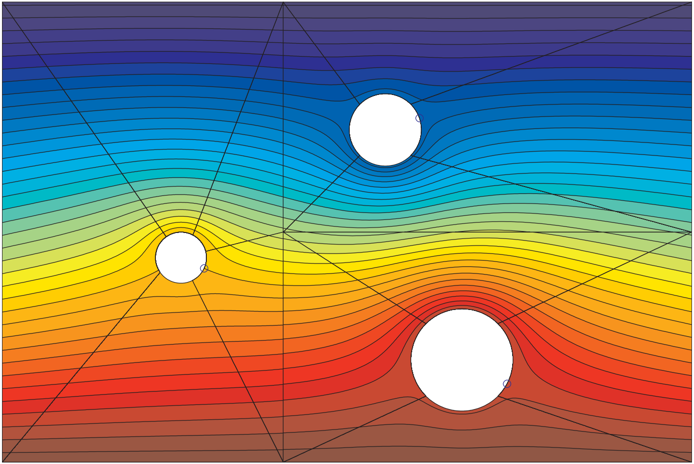


# The use of harmonic forms in potential problems



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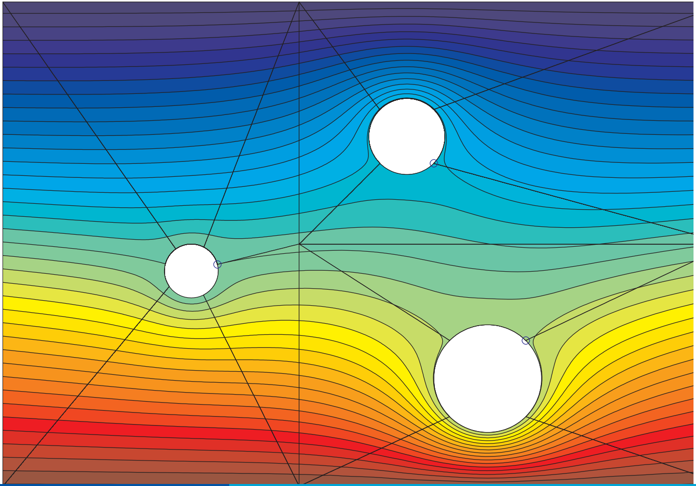

# Stokes problem

## Stokes problem

$$(\text{curl}\,\text{curl}^* - \text{grad}^*\,\text{div})\,u + \text{grad}^*\,p = f$$

$$\text{div}\,u = 0$$

Since $\text{div}\,u = 0$, we can remove this term from momentum. Introduce $\omega = \text{curl}^*\,u$.

# Stokes problem

## Stokes problem

$$\omega - \mathsf{curl}^* \, u = 0$$

$$\mathsf{curl} \, \omega + \mathsf{grad}^* \, p = f$$

$$\mathsf{div} \, u = 0$$

# Stokes problem

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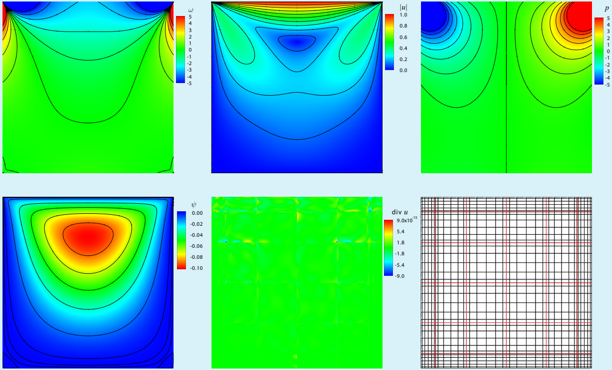

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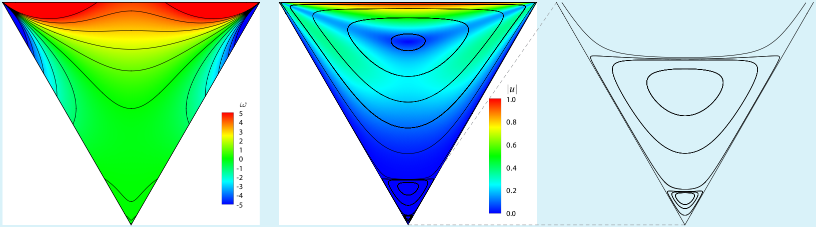

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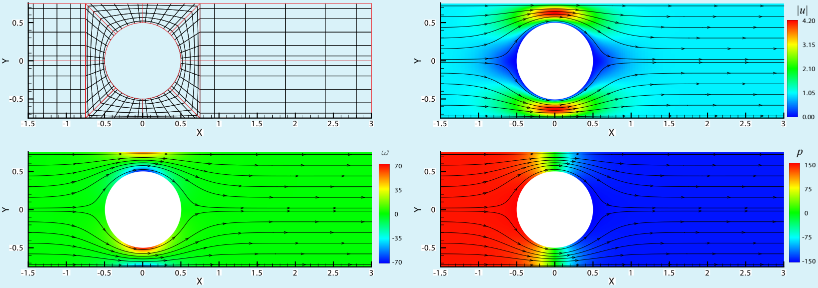

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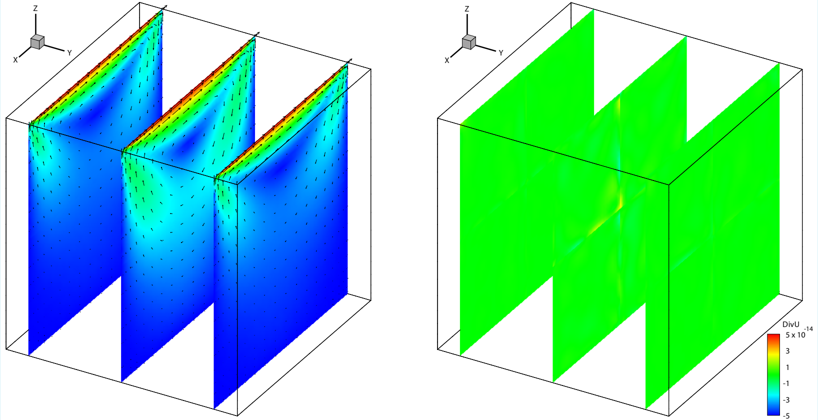

# Further reading

Further reading:

J. Kreeft, A. Palha, M. Gerritsma, *Mimetic Framework on curvilinear quadrilaterals of arbitrary order* http://arxiv.org/abs/1111.4304

J. Kreeft, M. Gerritsma, *Mixed mimetic spectral element method for Stokes flow: A pointwise divergence-free solution* http://arxiv.org/abs/1201.4409

J. Kreeft, M. Gerritsma, *A priori error estimates for compatible spectral discretization of the Stokes problem for all admissible boundary conditions* http://arxiv.org/abs/1206.2812

R.R. Hiemstra, R.H.M. Huijsmans, M. Gerritsma, *High order gradient, curl and divergence conforming spaces, with an application to NURBS-based IsoGeometric Analysis* http://arxiv.org/abs/1209.1793

http://mimeticspectral.com