A posteriori uncertainty quantification of PIV-derived pressure fields

Iliass Azijli¹, Andrea Sciacchitano¹, Daniele Ragni¹, Artur Palha^{1,2} and Richard P. Dwight¹

I.Azijli@tudelft.nl

¹ Aerodynamics, Wind-Energy, Flight Performance and Propulsion Department, Faculty of Aerospace Engineering, Delft University of Technology, Delft, The Netherlands

² Department of Mechanical Engineering, Eindhoven University of Technology, Eindhoven, The Netherlands

ABSTRACT

The present work proposes a methodology for the *a posteriori* quantification of the uncertainty of pressure data retrieved from PIV measurements. It relies upon the Bayesian framework, where the posterior distribution (probability distribution of the true velocity, given the PIV measurements) is obtained from the prior distribution (prior knowledge of the velocity, e.g., within a certain bound or divergence-free) and the distribution representing the PIV measurement uncertainty. Once the posterior covariance matrix of the velocity is known, it is propagated through the discretized Poisson equation for pressure. Numerical assessment of the proposed method on a steady Lamb Oseen vortex shows excellent agreement with Monte Carlo simulations, while linear uncertainty propagation underestimates the uncertainty of the pressure by up to 30%. The method is finally applied to an experimental test case of a turbulent boundary layer in air, obtained using time-resolved tomographic PIV. The pressure reconstructed from tomographic PIV data is compared to a microphone measurement conducted simultaneously at the wall to determine the actual error of the former. The comparison between actual error and estimated uncertainty shows the accuracy of the proposed method for uncertainty quantification of pressure data from tomographic PIV experiments: for a 95% confidence level, 93% of the data points fall within the estimated uncertainty bound with the Eulerian approach, and 90% with the Eulerian approach. When using the prior knowledge that the velocity field should be divergence-free, these values are 98% with the Eulerian approach. Also, enforcing the divergence-free constructed pressure fields than the Eulerian approach. Also, enforcing the divergence-free constraint is found to result in a more accurate reconstructed pressure field. Both observations also follow from the uncertainty quantification, through a decrease in the estimated uncertainty.

1. Introduction

Particle image velocimetry (PIV) is nowadays recognized as a reliable tool for the determination of the instantaneous static pressure in two or three dimensional flow fields [19]. The main advantage of the PIV-based pressure reconstruction with respect to conventional pressure probes (e.g., pressure transducers or microphones) is the instantaneous 2D or 3D pressure *field* determination, without the need of an expensive design and manufacturing of models where densely distributed sensors are installed [21]. PIV-based pressure measurements are made possible for applications where probes cannot be installed, such as micro aerial vehicles. Additionally, they convey a quantitative visualization of the flow structures responsible for the aerodynamic forces acting on the model. Industrial tests typically require information on pressure fluctuations, rather than on velocity, especially when dealing with fatigue studies and aeroacoustic noise measurements [10, 20].

The computation of the pressure field relies upon the PIV measurement of the velocity and acceleration and is conventionally performed by spatial integration of the pressure gradient or by solution of the Poisson equation, provided appropriate boundary conditions. The formulation of the problem in terms of the Poisson equation is often employed because it avoids accumulation of errors in the measurement domain [19]. Applying the divergence operator to the momentum equation and assuming incompressible flow, the following Poisson equation for pressure follows:

$$\nabla^2 p = -\rho \nabla \cdot (\mathbf{u} \cdot \nabla) \mathbf{u}. \tag{1}$$

To solve (1), appropriate boundary conditions on the pressure (Dirichlet) or its spatial gradient (Neumann) must be specified. It is important to remark that, although no temporal derivative appears in (1), the boundary conditions are time dependent and require the computation of the flow acceleration at the border of the measurement domain. In practice, (1) is discretized:

$$A\mathbf{p} = \mathbf{f},\tag{2}$$

and then solved. A is a discrete representation of the Laplacian operator and **f** will be referred to as the 'source term'.

When solving (2) to determine the static pressure field, several sources contribute to the overall uncertainty. According to van Oudheusden [19], the most relevant factors that affect the accuracy of the reconstructed pressure field are: (i) the accuracy of the underlying velocity data; (ii) the spatial and temporal resolutions, which affect the accuracy of velocity derivatives; (iii) the approach used to evaluate the velocity material acceleration (Eulerian or Lagrangian approach); (iv) the boundary conditions; (v) the pressure-gradient integration procedure;

(vi) for PIV-based pressure reconstruction of planar PIV data, application of a 2D model to a 3D flow. As a result, the uncertainty of the static pressure is a complex function of the parameters above. Quantifying the uncertainty of the pressure measurement is of primary importance for the determination of a confidence interval where the true pressure value lies. Furthermore, it constitutes the basis for estimating the uncertainty of the aerodynamic forces (e.g., lift and drag) acting on a model. Knowing the uncertainty of these forces is relevant especially for the optimization of the model geometry, where the shape that minimizes the aerodynamic drag or maximizes the dynamical stability is sought. Several works have focused on evaluating the contribution of the above mentioned factors to the error of the reconstructed pressure. Van Oudheusden [19] derived an analytical expression of the error of the material acceleration for both Eulerian and Lagrangian approaches. In both approaches, the laser pulse separation Δt has opposite effects on truncation and random error: increasing Δt increases the truncation error, but decreases the relative random error. Furthermore, the error of the material acceleration is typically smaller when it is calculated by the Lagrangian approach. Violato et al [26] introduced a criterion for the selection of the minimum and maximum Δt that accounts for the maximum allowed error on the material acceleration and the effect of the out-of-plane displacement of the particles. The authors found that the Lagrangian approach features a random error about 1.5 times smaller than that estimated for the Eulerian approach. The difference between the two approaches is less pronounced when looking at the resulting pressure field, due to the error suppression during the pressure gradient integration. Several schemes for the integration of the pressure gradient have been proposed, including the space-marching integration [2], the omni-directional integration [15] and the Poisson equation for pressure [9]. De Kat and van Oudheusden [12] conducted a comparative assessment of pressure integration methods. Although their main outcome was that the integration method has minor impact and only the space marching approach is clearly inferior, their conclusions were based on statistical analysis and not on the instantaneous pressure uncertainty. In fact, no quantitative information about the uncertainty of computed pressure fields was provided in these early studies. Charonko et al [5] reported that the error of the velocity field has a major influence on the accuracy of the reconstructed pressure. In their work, the pressure result became 'unusable' as soon as very small levels of random error were introduced. Extensive data post-processing was required to reduce the velocity error and make accurate pressure reconstructions feasible. However, Murai et al [18] reported that a fairly simple smoothing of the velocity data is sufficient for accurate pressure results. Charonko et al [5] also investigated the effect of off-axis measurement and out-of-plane velocity. They found that schemes performing well in 2D conditions continued to do so for misalignments up to 30 degrees.

In turbulent flows, the PIV-based pressure reconstruction presents two major challenges: the small magnitude of the pressure fluctuations and the broad frequency range. Ghaemi et al [8] conducted time-resolved PIV (TR-PIV) measurements to investigate the pressure fluctuations in a turbulent boundary layer at $Re_{\theta} = 2400$. The agreement between the PIV-based pressure and that measured by a surface microphone was evaluated via the cross-correlation coefficient, which was equal to 0.6. The probability density functions of the two signals agreed up to 3 kHz for tomographic PIV data and up to 1 kHz for planar data. Similarly, Pröbsting et al [20] carried out the PIV-based pressure reconstruction in a turbulent boundary layer at $Re_{\theta} = 730$. The data were validated by comparison with a surface microphone and with DNS simulation data. Also in this case, the PIV-based pressure signal showed fair agreement with the microphone signal, but the cross-correlation coefficient between the two did not exceed 0.6. Despite the relatively high correlation coefficient achieved, those studies opened several doubts on the accuracy of the PIV-based pressure reconstruction. Koschatzky et al [14] investigated the acoustic emissions of the flow over a rectangular cavity. The fluctuating pressure fields computed from TR-PIV were used as input for the prediction of the sound emission. The authors found that the hydrodynamic pressure computed from the PIV data had a frequency spectrum comparable to that of the direct pressure measurement at the cavity walls. The main frequency tone and the first harmonic were correctly captured, although the amplitude was significantly underestimated. At locations where the pressure fluctuations were weak, the PIV measurement uncertainty strongly affected the accuracy of the reconstructed pressure.

From the discussion above, it emerges that PIV-based pressure data are often highly inaccurate due to the several error sources that contribute to the total error. As a result, quantifying the uncertainty of the reconstructed pressure is of primary importance to define the confidence interval of the measurement. Furthermore, the *a posteriori* uncertainty quantification could be used to optimize the experimental and processing parameters, along with the pressure reconstruction approach, to maximize the precision of the pressure field. The present work discusses the mathematical framework for the uncertainty propagation from the velocity field to the pressure field. It is formulated from a Bayesian perspective, to allow the measurement uncertainty of the velocity field from PIV to be combined naturally with any prior knowledge [32]. Therefore, it is suitable for the quantification of the pressure random uncertainty, which stems from the random uncertainty of the velocity. A number of *a posteriori* approaches have recently been proposed to quantify the PIV measurement uncertainty [6, 24, 25, 30]. The present framework can also propagate the uncertainty of the velocity spatial and temporal derivatives, when information on the latter is available. Conversely, the uncertainty due to the application of a 2D model to a 3D flow is not tackled here.

Section 2 discusses the methodology proposed. Section 3 presents a numerical assessment with an analytical test case, where the proposed method is compared with Monte Carlo simulations and linear uncertainty propagation. Section 4 applies the method to the turbulent boundary layer experiment done by Pröbsting et al [20], where microphone measurements are available to assess the PIV-derived pressure field and the usefulness of the confidence intervals obtained from the present method.

2. Methodology

As mentioned in the introduction, the method is formulated from a Bayesian framework, since it allows the measurement uncertainty of the velocity field from PIV to be combined naturally with any prior knowledge. One can identify three distinct steps in this process. First, we characterize uncertainty in the velocity field. We express it as a stochastic process, acknowledging our uncertainty of the *true* field. With *n* spatial locations of interest $\mathbf{x} \in \mathbb{R}^{d \cdot n}$, we define a vector $\mathbf{u} \in \mathbb{R}^{d \cdot n}$, describing the *d* dimensional velocity components at these locations. For planar PIV, d = 2, and for tomographic PIV, d = 3. The uncertainty in the velocity field is expressed through a probability distribution

 $\rho_0(\mathbf{u})$, known as the prior, because the measurements have not been included yet. Prior information that may be incorporated can be as simple as a range. For example, one sets a wind tunnel to 5 m/s to investigate a laminar boundary layer and expects the velocity to be between 0 and 7 m/s. More advanced prior information that can be incorporated is smoothness of the velocity field, that may be expressed in a number of shape functions, from linear to radial basis functions. Finally, for incompressible flows, it is also possible to enforce the constraint that the velocity field should be divergence-free (solenoidal), as dictated by the mass conservation equation [1]. The second step is concerned with the measurements. We define the vector $\mathbf{u}_{\text{PIV}} \in \mathbb{R}^{d \cdot m}$, since there are *m* measurements¹. The measurements are expressed as a conditional distribution, $\rho(\mathbf{u}_{\text{PIV}}|\mathbf{u})$, i.e., the measurements are conditioned on the true state. This is known as the likelihood. With the final step, we are interested in the distribution of the true state given the measurements, so $\rho(\mathbf{u}|\mathbf{u}_{\text{PIV}})$. This is known as the posterior distribution. Following Bayes Rule, the prior knowledge and the measurements can be combined to obtain the posterior:

$$\rho(\mathbf{u}|\mathbf{u}_{\text{PIV}}) \propto \rho(\mathbf{u}_{\text{PIV}}|\mathbf{u})\rho_0(\mathbf{u}). \tag{3}$$

As general as it is, (3) is not straightforward to use for general probability density functions. Therefore, in the present paper we will assume that they are Gaussian. Assuming a Gaussian process may seem like a limiting assumption, but in fact this leads to a general interpolation framework for arbitrary fields [22]. It is most well known in the field of geostatistics, where it is referred to as *Kriging* and used to reconstruct smooth fields from scattered observations [16].

The following subsections explain in detail how the different components are defined and how ultimately the uncertainty is propagated from the velocity to the pressure field. Specifically, Subsection 2.1 explains how the measurement uncertainty model is defined, rendering the likelihood. Subsection 2.2 discusses how the prior on the velocity field can be defined. Subsection 2.3 then explains how the prior and likelihood can be combined to obtain the posterior on the velocity field. Once this is known, the posterior needs to be propagated through the source term **f** from (2), which is explained in Subsection 2.4. Subsection 2.5 explains how the discretization error that results from solving the discrete system (2) can be estimated and included in the framework. Finally, Subsection 2.6 discusses how the tuning parameters that result from the proposed method can be determined.

A one dimensional example is used to make the discussion clear. Extension to two and three dimensional problems is relatively straightforward, and will be explained in Sections 3 and 4, covering the synthetic test case and the experimental data, respectively. Consider the following one dimensional example:

$$\frac{d^2p}{dx^2} = 2u\frac{du}{dx}, \quad \forall x \in (0,1), \quad \text{with} \quad p(0) = 0, p(1) = \pi.$$
(4)

Eq. (4) is similar to the Poisson equation (1) in that the second derivative of the pressure is taken and the right hand side is nonlinear in *u*. The problem is completed with Dirichlet boundary conditions. If we take $u(x) = \sin(2\pi \cdot x)$, it can be verified that the pressure is $p(x) = \pi \cdot x - \frac{1}{8\pi} \sin(4\pi \cdot x)$.

2.1 The PIV measurement uncertainty model

The measurement uncertainty is assumed to come from a Gaussian distribution, where the mean represents the bias error and the standard deviation represents the random error. Once quantified, the bias error can be subtracted from the measurements. Therefore, the measurements $\mathbf{u}_{\text{PIV}} = H\mathbf{u} + \boldsymbol{\epsilon} \in \mathbb{R}^{d \cdot m}$ can be represented as a multivariate Gaussian distribution, where $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, R)$ is the measurement error of the velocity field, assumed to be Gaussian noise with zero mean and covariance matrix *R*. The matrix *H* maps the velocities from the *n* locations of interest to the *m* measurement locations. In the present work, the measurement locations are taken to be a subset of the locations of interest, therefore *H* would be a Boolean matrix. Otherwise, it would represent an arbitrary interpolation matrix (representing for example polynomial or radial basis function interpolation).

For the one dimensional example, we take six uniformly distributed measurement locations (m = 6). Measurement noise is included, with a variance of $\sigma^2 = 2.5 \cdot 10^{-5}$. It is assumed that the measurement uncertainty at the locations are uncorrelated to each other, therefore $R = \sigma^2 I$, where *I* is the identity matrix.

2.2 The prior on the velocity field

Before the PIV data has been acquired, we assume that the velocity field comes from a Gaussian process, $\mathbf{u} \sim \mathcal{N}(\mu_0, P)$ with μ_0 and P the prior mean and prior covariance matrix, respectively. We start with the prior mean $\mu_0 \in \mathbb{R}^{d,n}$. For simplicity, we assume each velocity component to be a constant field. This is acceptable since the complexity of the field that can be reconstructed is contained in the prior covariance used. For the three dimensional case, $\mu_{0,x} = \mu_{0,x}\mathbf{1}, \mu_{0,y} = \mu_{0,y}\mathbf{1}, \mu_{0,z} = \mu_{0,z}\mathbf{1}$. The actual values $\mu_{0,x}, \mu_{0,y}$, and $\mu_{0,z}$ can be set to the free stream conditions in the experiment. For example $\mu_{0,x} = u_{\infty}, \mu_{0,y} = 0, \mu_{0,z} = 0$ if the free stream is aligned with the *x*-direction.

Having discussed the prior mean, we now proceed to the prior covariance matrix *P*. Take two points \mathbf{x}^i and \mathbf{x}^j . Assuming a stationary process, the way their velocity vectors are correlated will only depend on their separation, $\mathbf{x}^i - \mathbf{x}^j$. This behavior can be represented using

¹We can either take *n* equal to or larger than *m*. When we take it equal to *m*, we will obtain the pressure field on the same mesh as the PIV field. When we take it larger than *m*, we attempt to obtain the pressure field on a finer mesh. Of course, an appropriate interpolation method needs to be used to obtain the velocity at the intermediate points. Section 2.2 describes in more detail how this is done in the present paper.



Figure 1: The one dimensional example. (a) True function evaluations (black dots), observations (red crosses), prior mean (blue line), prior variance scaled with τ^2 (black line); (b) posterior mean (blue line), posterior variance scaled with prior variance τ^2 (black line).

a covariance function $\phi(r^{ij})$, where r^{ij} represents the (scaled) distance between \mathbf{x}^i and \mathbf{x}^j :

$$\left(r^{ij}\right)^2 = \sum_{k=1}^d \left(\frac{x_k^i - x_k^j}{\theta_k}\right)^2,\tag{5}$$

where θ_k is the correlation length in the direction k. Note that the covariance function used should exhibit the following behavior: (i) $\phi(0) = 1$; (ii) $\lim_{r^{ij} \to \infty} \phi(r^{ij}) = 0$, i.e., at infinite distance the behavior of the two points is uncorrelated. Gaussians are popular covariance functions. In the present paper however, we use the Wendland function with smoothness C^4 :

$$\phi(r^{ij}) = \left(1 - r^{ij}\right)_{+}^{6} \left(\frac{35}{3} \left(r^{ij}\right)^{2} + 6r^{ij} + 1\right),\tag{6}$$

where + indicates that $(1 - r^{ij})_+^6 = 0$ for $r^{ij} \ge 1$. The Wendland function approximates a Gaussian function well. Due to its compact support, it results in a sparse covariance matrix, leading to savings in memory and computational time when inverting it. Note that the prior knowledge we have assumed is in terms of the covariance function we have chosen to model the correlation between points. This relation holds for velocity components separately. In other words, we assume that the different velocity components at a single location are uncorrelated. In that case, the prior covariance matrix $P \in \mathbb{R}^{d \cdot n \times d \cdot n}$ is a $d \times d$ block diagonal matrix, with elements P_k , k = 1, ..., d on the diagonal, where

$$P_k^{ij} = \tau_k^2 \phi\left(r^{ij}\right),\tag{7}$$

where τ_k^2 is the prior variance for velocity component k. The off-diagonal blocks are zero by assumption of uncorrelated velocity components. In reality, the velocity components *are* related through the mass conservation equation. For incompressible flows, it follows that the velocity field should be divergence-free. This prior knowledge can be included, and the resulting covariance matrix can be found in [1].

The determination of the prior for the one dimensional example is illustrated now. We take n = 41, distributed uniformly over the interval. For the prior mean, we assume the zero vector, so $\mu_0 = 0$. For the prior variance, we take $\tau^2 = 0.48$ (Subsection 2.6 explains the choice for this value). The black dots in Figure 1a are the true function evaluations at the *n* locations of interest. The thick blue line represents the prior mean. The bottom plot of Figure 1a shows the prior variance, scaled with τ^2 . The Wendland function (6), with correlation length $\theta = 2$, was used (Subsection 2.6 explains the choice for this value).

2.3 The posterior on the velocity field

Having defined the prior and carried out the measurements, they can be combined to obtain the posterior distribution. The prior and likelihood are both Gaussian distributed, and since all operators defined are linear, the posterior is Gaussian distributed as well with mean and covariance given by [32]:

$$\mathbb{E}\left(\mathbf{u}|\mathbf{u}_{\mathrm{PIV}}\right) = \boldsymbol{\mu} + PH^{T}\left(R + HPH^{T}\right)^{-1}\left(\mathbf{u}_{\mathrm{PIV}} - H\boldsymbol{\mu}\right),\tag{8}$$

$$\Sigma(\mathbf{u}|\mathbf{u}_{\text{PIV}}) = P - PH^T \left(R + HPH^T\right)^{-1} P,\tag{9}$$

respectively. Note that since a continuous covariance kernel is used in *P*, the posterior can be evaluated at any point. Figure 1b shows the results for the one dimensional example. We can clearly see a decrease in the variance at the measurement locations and an increase in between. Indeed, the measurements are informative and the larger the distance from them, the larger the uncertainty in the reconstruction. Note that the variance is not decreasing to zero at the measurement locations, since we included measurement noise with a variance of $\sigma^2 = 2.5 \cdot 10^{-5}$.

In the present paper, we will also consider the situation where we construct the pressure field by completely relying on the PIV data, i.e., no prior knowledge is taken into account and the velocity is only assumed to be known at the locations coming from the measurements. Mathematically, this can be expressed by taking $P = \tau^2 I$, where $\tau = \infty$, and H = I. Eq.8 becomes

$$\mathbb{E}\left(\mathbf{u}|\mathbf{u}_{\text{PIV}}\right) = \boldsymbol{\mu} + \tau^2 I \left(R + \tau^2 I\right)^{-1} \left(\mathbf{u}_{\text{PIV}} - \boldsymbol{\mu}\right).$$
(10)

The term $\tau^2 I (R + \tau^2 I)^{-1}$ becomes *I* for infinitely large prior variance, so (10) reduces to

$$\mathbb{E}\left(\mathbf{u}|\mathbf{u}_{\mathrm{PIV}}\right) = \mathbf{u}_{\mathrm{PIV}}.\tag{11}$$

The posterior covariance (9) becomes

$$\Sigma(\mathbf{u}|\mathbf{u}_{\text{PIV}}) = \tau^2 I - \tau^2 I \left(R + \tau^2 I\right)^{-1} \tau^2 I.$$
(12)

With the prior variance approaching infinity, (12) can be rewritten to reveal

$$\Sigma(\mathbf{u}|\mathbf{u}_{\mathrm{PIV}}) = R. \tag{13}$$

In words, (11) and (13) state that the posterior mean and covariance are equal to the measured velocity field and measurement uncertainty, respectively. This makes sense intuitively, but it is important that it also follows naturally from the framework.

2.4 Propagating through the source term

The source term \mathbf{f} is the right-hand side of the discrete Poisson equation (2). The Laplacian operator, as given by the left-hand side of the equation, is a *linear* operator, so from the identities of the expected value and covariance it follows rather straightforwardly that:

$$\mathbb{E}(\mathbf{p}) = A^{-1}\mathbb{E}(\mathbf{f}), \tag{14}$$

$$\Sigma(\mathbf{p}) = A^{-1}\Sigma(\mathbf{f})A^{-T}.$$
(15)

Obtaining $\Sigma(\mathbf{f})$ on the other hand is not straightforward, since \mathbf{f} is nonlinear in velocity from the Navier-Stokes equations. The posterior distribution of the velocity (see (8) and (9)) needs to be propagated through \mathbf{f} to obtain $\mathbb{E}(\mathbf{f})$ and $\Sigma(\mathbf{f})$. Due to the nonlinearity, the resulting pressure will not be Gaussian distributed anymore.

Carrying out Monte Carlo simulations [17] is a popular method for uncertainty propagation. Briefly, random realizations of the posterior distribution $(\mathbf{u}|\mathbf{u}_{PIV})$ are generated, and for each realization the source term **f** is evaluated. From this one can construct $\mathbb{E}(\mathbf{f})$ and $\Sigma(\mathbf{f})$ and subsequently $\mathbb{E}(\mathbf{p})$ and $\Sigma(\mathbf{p})$ can be obtained from (14) and (15), respectively. To obtain a random realization of the posterior velocity field $(\mathbf{u}|\mathbf{u}_{PIV})$, one computes the Cholesky decomposition of $\Sigma(\mathbf{u}|\mathbf{u}_{PIV}) = LL^T$. Multiplying *L* with a vector containing uncorrelated random realizations from a standard Gaussian distribution gives the random realization with the correct covariance structure. This follows from the covariance identity as also expressed by (15).

Monte Carlo methods *approximate* the true statistics, only returning the exact statistics with infinitely many simulations. Interestingly, an exact uncertainty propagation is possible for the present problem. A closer look at the Navier-Stokes equations reveals that the nonlinear term contains a product of two variables. As a result, it is possible to calculate the expected value and covariance exactly. For generality, we define four variables a, b, c and d. For the expected value, one can use the definition of covariance:

$$\mathbb{E}(ab) = \mathbb{E}(a)\mathbb{E}(b) + \sigma(a,b), \tag{16}$$

where $\sigma(a,b)$ represents the covariance between *a* and *b*. For the covariance between two products of random variables *ab* and *cd*, Bohrnstedt and Goldberger [3] derived the following expression:

$$\sigma(ab,cd) = \mathbb{E}(a) \mathbb{E}(c) \sigma(b,d) + \dots$$

$$\mathbb{E}(a) \mathbb{E}(d) \sigma(b,c) + \dots$$

$$\mathbb{E}(b) \mathbb{E}(c) \sigma(a,d) + \dots$$

$$\mathbb{E}(b) \mathbb{E}(d) \sigma(a,c) + \dots$$

$$\sigma(a,c) \sigma(b,d) + \sigma(a,d) \sigma(b,c).$$
(17)

Eq. (16) is valid no matter what the distributions of a and b are, since it is simply a rewrite of the definition of covariance. Eq. (17) on the other hand is valid if the distributions are Gaussian. Indeed, this is what we have done in the present work. The source term however is not Gaussian distributed anymore.

Figures 2a and 2b show how the random velocity error in the one dimensional example propagates into the pressure. They show the standard deviation in the domain, calculated with the exact method and as comparison, Monte Carlo simulations using 10, 100 and 1000 realizations. The difference with the two figures is due to the accuracy of the pressure measurements at the boundaries. In Figure 2a, it is assumed that the pressure can be measured with perfect accuracy. In Figure 2b, we have taken standard deviations of $1 \cdot 10^{-4}$ and $5 \cdot 10^{-4}$ for the pressure measurements at the left and right boundaries, respectively.



Figure 2: The one dimensional example. Exact standard deviation of calculated pressure (black line), and Monte Carlo results using *N* realizations. (a) Perfect pressure measurements at boundaries, (b) assuming $\sigma_p = 1 \cdot 10^{-4}$ at x = 0 and $\sigma_p = 5 \cdot 10^{-4}$ at x = 1.

It is important to state that the exact method is only possible if the material acceleration is evaluated with respect to a stationary reference frame, i.e., the Eulerian approach. If the Lagrangian approach is used, products arise with more than two components. Brown and Alexander [4] derived general expressions for the covariance of a product of n random variables with a product of m random variables. However, these are not usable unless the terms containing products with three terms and higher are neglected, resulting in an approximation. Therefore, the Monte Carlo method is applied when the Lagrangian approach is used.

2.5 Calculating the pressure

Discretization errors arise as a result of solving the discrete Poisson equation (2). In numerical simulations, these errors can be reduced by solving on a finer mesh, i.e., increase n. We again use the one dimensional example as illustration. The red line in Figure 3 shows how reducing the mesh distance decreases the error of the calculated pressure, in the case of perfect measurements and observations available at all locations. A second order finite difference scheme was used to discretize the differential operators, therefore we notice second order convergence. However, for PIV the measurements are available at a subset of points, so one cannot arbitrarily decrease the error of the calculated pressure by refining the mesh. With Kriging, one can define n arbitrarily large. Recall from Figure 1b and (7) that the velocity can be evaluated anywhere in the domain. The blue line in Figure 3 shows what happens with the error of the calculated pressure as the mesh distance is reduced. At first, we notice a decrease in error with finer meshes. However, this trend plateaus out at a certain moment. These two regions are caused by the fact that, in essence, there are two competing error sources. At first, the error is dominated by the discretization of the continuous Poisson equation (1). An increase in the mesh resolution causes a decrease in this error. However, recall that the measurements were carried out at six uniformly distributed measurement points. This explains the plateau, caused by the interpolation or reconstruction error of Kriging. Really, this is the error caused by the limited spatial resolution of the measurement. As a side note, the magenta line shows what would happen in the absence of measurement noise. In this case we see the plateau more clearly, and the final error in calculated pressure is also lower. What makes Kriging an attractive reconstruction method compared to other methods, like polynomial interpolation or radial basis function interpolation, is that the reconstruction is accompanied by confidence intervals. Indeed, the reconstruction error is expressed in terms of the posterior covariance matrix (9). The influence of the limited resolution can therefore also be included in the quantification of the pressure uncertainty. So by solving the Poisson equation on a systematically refined mesh, one can reduce the discretization error until it is relatively negligible in the calculated pressure, thereby leaving the quantification of the error due to the limited measurement resolution. Notice that the posterior covariance matrix therefore includes both the effect of the measurement error and the limited resolution. Thanks to the Bayesian framework used, they have been combined in a consistent manner. The black line in Figure 3 shows the approximated error in the calculated pressure due to the limited resolution. We have used the posterior covariance matrix. Notice that we are interested in the results at high mesh density, where the approximation due to the discretization is negligible.



Figure 3: The one dimensional example. Root-mean-square error (rmse) of the reconstructed pressure as a function of mesh size h = 1/(n-1). When m = n, $\sigma = 0$, the observations are available at all points, without measurement noise; m = 6, $\sigma \neq 0$ represents six equally spaced observations, with $\sigma^2 = 2.5 \cdot 10^{-5}$; m = 6, $\sigma = 0$ represents six equally spaced observations, without measurement noise; 'approx' is the approximated error following from (15) when $\sigma^2 = 2.5 \cdot 10^{-5}$.

Notice that the approximated error gives a good approximation of the true error.

2.6 Parameter tuning

When discussing the prior in Subsection 2.2, it followed that there are two unknown parameters in Kriging, namely the correlation length θ (see (5)) and the prior variance τ (see (7)). In the most general case, they can have different values in the different coordinate directions. However, to save computational time, we will set them equal in all directions. Traditionally, the optimum values for these parameters can be found through maximum likelihood estimation [13]. In the present work however, we propose a more pragmatic approach, customized for the PIV setting. To find the optimum tuning parameters, we split the observations into two sets, namely a fitting set (\mathbf{u}_{PIV}) and validation set (\mathbf{u}_{PIV}). In the PIV setting, one can change overlaps for this. For example, going from 50% overlap to 75% doubles the number of velocity vectors. The velocity vectors from the 50% overlap case can be taken as the fitting set, and the extra vectors that result from switching to 75% can be taken as the validation set. We define two objective functions. Recall that in Kriging, the prior, likelihood and posterior have a Gaussian distribution. To ensure that the posterior distribution indeed has this property, we use the *three-sigma rule*; 68.27% of the predicted points are within one standard deviation, 95.45% are within two standard deviations; 99.73% are within three standard deviations. The first objective function *J* is therefore defined as follows:

$$J(\theta, \tau) = \left(\frac{N_1}{N} - 0.6827\right)^2 + \left(\frac{N_2}{N} - 0.9545\right)^2 + \left(\frac{N_3}{N} - 0.9973\right)^2,\tag{18}$$

where N is the total number of validation points and N_i is the number of points within *i* times the standard deviation, i.e., that fall inside the following interval:

$$\left| \frac{\check{\mathbf{u}}_{\mathrm{PIV}} - \mathbb{E}\left(\mathbf{u} | \mathbf{u}_{\mathrm{PIV}}\right)}{\operatorname{diag}\left(\sqrt{\Sigma\left(\mathbf{u} | \mathbf{u}_{\mathrm{PIV}}\right)}\right)} \right| < i.$$
(19)

The objective function *J* will be a discontinuous function, due to the use of integers (see (19). Therefore, the minimum value will be a region. The second objective function *G* represents the reconstruction accuracy, and is simply defined as a root-mean-square error with respect to the validation set. To find the optimum tuning parameters θ and τ , we first define the region containing the minimum *J*. In the second step, we look for the minimum *G* within this region. Figure 4 shows the process for the one dimensional example. Figure 4a shows $J(\theta, \tau)$, where the red contourlines contain the minimum *J*, which is equal to $8.9 \cdot 10^{-3}$. Figure 4b shows $G(\theta, \tau)$, where the white dot represents the minimum within the region enclosed by the contourlines, equal to $9.1 \cdot 10^{-3}$. Therefore, the optimum tuning parameters are $\theta = 2$ and $\tau = 0.69$. These values were used to define the prior of the one dimensional example in Subsection 2.2.



Figure 4: The one dimensional example. (a) $\log_{10} J(\theta, \tau)$. The red contour lines enclose the region containing J_{\min} . (b) $\log_{10} G(\theta, \tau)$. At the white dot, G_{\min} , provided that $J = J_{\min}$.

3. Numerical assessment

The proposed methodology for the uncertainty quantification of PIV-based pressure reconstruction is assessed by Monte Carlo simulations using 10,000 simulations. The considered flow field is a two dimensional incompressible Lamb Oseen vortex. In a polar reference system with radius *r* and azimuthal angle θ the analytical expressions of azimuthal velocity V_{θ} and radial velocity V_r are, respectively:

$$V_{\theta} = \frac{\Gamma}{2\pi r} \left(1 - \exp\left[-r^2/4\nu t\right] \right), \tag{20}$$

$$V_r = 0, \tag{21}$$

where Γ is the vortex circulation, ν is the kinematic viscosity ratio of the dynamic viscosity μ and the density ρ , and *t* is time. A steady simulation is conducted with t = 1 s. The values of dynamic viscosity and density are set to $\mu = 2.4 \cdot 10^{-7}$ Pa · s and $\rho = 1.2$ kg/m³, respectively, yielding a kinematic viscosity $\nu = 2.0 \cdot 10^{-7}$ m²/s. The vortex circulation is $\Gamma = 0.02$ m²/s. The instantaneous radius of the vortex is equal to $R = \sqrt{4\nu t} = 0.89$ mm. The analytical expression of the pressure field is obtained from integration of the Navier-Stokes equations in polar coordinates:

$$p = \frac{\rho\Gamma^2}{4\pi^2} \left[-\frac{1}{2r^2} + \frac{\exp\left[-r^2/4\nu t\right]}{r^2} - \frac{1}{2} \frac{\exp\left[-2r^2/4\nu t\right]}{r^2} + Ei\left(\frac{2r^2}{4\nu t}\right) \frac{1}{4\nu t} - Ei\left(\frac{r^2}{4\nu t}\right) \frac{1}{4\nu t} \right],$$
(22)

where Ei is the exponential integral function. The square domain has a size of 20 mm, and the center of the vortex coincides with the center of the domain at (x,y) = (0,0). The domain is divided into 50 grid points in both horizontal and vertical directions, resulting in a grid spacing $d = 0.4 \text{ mm} (d/R \approx 0.45)$. The analytical velocity and pressure field are illustrated in Figure 5; the pressure in the vortex core is $p_{\text{core}} = -9.78 \text{ Pa}$. To simulate noise in the velocity measurements, zero-mean white Gaussian noise is added to the two velocity components, with a standard deviation ranging from 0% to 30% of the maximum velocity magnitude V_{mag} : $R = U^2 I$, where $U = \alpha V_{\text{mag}}$. α is between 0 and 0.30. The pressure field is reconstructed from the velocity data using the Poisson equation approach with Dirichlet boundary conditions. The pressure at the boundary of the measurement domain is computed via the Bernoulli equation for incompressible flows.

For all values of noise level, the standard uncertainty of the reconstructed pressure is evaluated with two approaches: (i) the Bayesian framework presented in Section 2; (ii) linear uncertainty propagation [31]. In the latter approach, it is assumed that the two velocity components have the same uncertainty U and that the velocity errors are uncorrelated in space, which is true for the current simulation, but not in typical PIV experiments due to the finite interrogation window size. The expression of the uncertainty of the pressure evaluated by the linear uncertainty propagation reads as:

$$U_p = U \frac{\rho d}{\sqrt{6}} \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}.$$
(23)

The uncertainty results from both approaches are compared with the results obtained by Monte Carlo simulations. The results at the center of the vortex are shown in Figure 6: the pressure error increases linearly from 0.3% to 10% of p_{core} as the uncertainty of the velocity



Figure 5: (a) Exact velocity field of the Lamb Oseen vortex (in m/s); (b) exact pressure field (in Pa).

is increased. The Bayesian framework and linear uncertainty propagation reproduce the linear behavior of the uncertainty. However, the linear approach underestimates the uncertainty by about 30% with respect to the Monte Carlo results, whereas the agreement between the Bayesian framework and Monte Carlo results is excellent. For illustration purposes, we show the comparison between the standard deviation computed with the different methods. Figure 7a illustrates the standard deviation of the pressure error obtained via Monte Carlo simulations for the case where the standard deviation of the added noise is 15% of the velocity magnitude; the pressure error is below 1% outside the vortex core and increases up to 4% in the vortex center. The Bayesian framework for uncertainty propagation (Figure 7b) shows excellent agreement with the Monte Carlo results. The linear error propagation however (Figure 7c) significantly underestimates the uncertainty.



Figure 6: The Lamb Oseen vortex. Comparison between the standard deviation of the pressure error from Monte Carlo simulations, the Bayesian framework and linear uncertainty propagation, as a function of the uncertainty of the velocity.



Figure 7: The Lamb Oseen vortex. Comparison between the standard deviation of the pressure error from Monte Carlo simulations (a), the Bayesian framework (b) and linear uncertainty propagation (c). The results are normalized with $|p_{core}|$ and in percentage. The noise level on the velocity: 15% of the maximum velocity magnitude.

4. Application to experimental data

The experimental test case considered is a fully developed turbulent boundary layer over a flat plate in air, obtained using time-resolved tomographic PIV [20]. The PIV acquisition frequency was 10 kHz, and 1500 instantaneous velocity fields were collected. The free stream velocity was 10 m/s, so assuming incompressible flow is an excellent approximation. Further details of the experimental parameters are reported in [20]. This experiment is particularly useful for the present paper since microphone measurements were conducted *simultaneously* with the tomo-PIV measurements. Specifically, a pinhole microphone located at the wall. Therefore, it is possible to assess the usefulnes of the proposed uncertainty propagation method. The assessment is done in the following way: the instantaneous error at time instant t_i is defined as the difference between the pressure from the microphone p_{mic} and the PIV-derived pressure p_{PIV} , evaluated at the microphone location \mathbf{x}_{mic} :

$$\delta(t_i) = p_{\text{mic}}(t_i) - p_{\text{PIV}}(\mathbf{x}_{\text{mic}}, t_i).$$
(24)

With the uncertainty propagation method described in the present paper, the standard deviation of the PIV-derived pressure at location \mathbf{x}_j and time instant t_i follows from the pressure covariance matrix (15):

$$\hat{\delta}\left(\mathbf{x}_{j},t_{i}\right) = \sqrt{\Sigma_{i}^{jj}}\left(\mathbf{p}\right). \tag{25}$$

 Σ_i^{jj} represents the *j*-th diagonal term of the covariance matrix at time instant t_i . When evaluated at the microphone location, we will denote (25) as $\hat{\delta}_{mic}$. With the expanded uncertainty $U = k\hat{\delta}_{mic}$, a certain percentage of $|\delta|$ should be smaller than U. Timmins et al [25] referred to this as the 'uncertainty effectiveness'. Assuming a Gaussian distribution, a 95% confidence level is obtained when k = 1.96 [7].

To carry out the uncertainty propagation, we start out with identifying the prior. Two cases are distinguished: (i) we completely rely on the PIV measurement and take no prior physical knowledge into account. Therefore, the posterior mean and covariance are simply the PIV field (11) and its uncertainty (13), respectively; (ii) realizing the flow is incompressible, we enforce the divergence-free constraint by applying a solenoidal filter [1]. The solenoidal filter is linear, therefore the posterior distribution will remain Gaussian distributed. Proceeding to the measurement uncertainty, the uncertainty of the PIV field is expressed through the observation error covariance matrix R. A number of methods are nowadays available to calculate a posteriori the uncertainty from PIV [6, 24, 25, 30]. However, since the particle images were not available, we have used a different method. Considering that the flow velocity should be divergence-free, we use the measured spurious velocity divergence as an estimator for the uncertainty in the velocity field. This has already been observed by others [23, 28]. In the present paper, we use the following approach: we apply the solenoidal filter to the PIV data \mathbf{u}_{PIV} , rendering an *analytically* divergence-free velocity field \mathbf{u}_{sol} . The difference between \mathbf{u}_{PIV} and \mathbf{u}_{sol} is an approximation of the true error. With 1500 samples, the variance of the error in time is taken as an approximation for the variance of the measurement uncertainty, i.e., the diagonal terms in the observation error covariance matrix R. Figure 8 shows the streamwise velocity before (top) and after (bottom) applying the solenoidal filter at one time instant. Figure 9 shows the standard uncertainty of each velocity component at the center plane. This method therefore returns a spatially varying measurement uncertainty for each velocity component. Notice how the uncertainty increases closer to the wall. The boundary layer data were processed with a $32 \times 16 \times 32$ interrogation window with 75% overlap. Due to the use of overlapping windows, the measurement error will be spatially correlated. Mathematically, the off-diagonal terms of the observation error covariance matrix R are unequal to zero. Wieneke and Sciacchitano [31] investigated how the spatial correlation of the measurement uncertainty depends on the overlap and interrogation window sizes using Monte Carlo simulations. From their analysis, they found correlation values for different grid spacings. Rather than directly inserting them into R, we have approximated these values by fitting the Wendland function (6) through them. This ensures that R will be numerically positive definite, thereby the matrix can be inverted [29].

The boundary conditions we apply are the same as done by Pröbsting et al [20], namely Dirichlet at the top boundary and Neumann at all other boundaries. We use both the Eulerian and the Lagrangian approaches to evaluate the material acceleration. According to Violato et al



Figure 8: The experimental boundary layer flow. Streamwise velocity in [m/s] before (top) and after (bottom) applying the solenoidal filter.

[27], the Lagrangian approach should result in a more accurate reconstructed pressure field, since the boundary layer flow is a convection dominated type of flow. Whether this is indeed true should not only follow from comparison with the microphone measurement, but is also expected to be represented by a smaller uncertainty in the reconstructed pressure. To propagate the uncertainties, we carry out Monte Carlo simulations using 100 realizations. This is the number suggested by Houtekamer and Mitchell [11] in the context of the Ensemble Kalman Filter, which is practically a Monte Carlo version of the Kalman Filter for nonlinear systems.

Table 1 shows a comparison of the standard deviation (std) of the true error δ and the standard deviation of the approximated error $\hat{\delta}_{mic}$, averaged over the 1500 fields in time. The results are shown for the four cases: with and without applying the solenoidal filter, using the Eulerian and Lagrangian approaches. The results are normalized with the free stream dynamic pressure q_{∞} . The first thing we note is that the true and approximated errors agree well. In addition, we observe that switching from the Eulerian to the Lagrangian approach indeed results in a decrease of the error in reconstructed pressure. The error is further decreased after applying the solenoidal filter.

Table 1: Standard deviation of the true error, std (δ) and the standard deviation of the approximated error averaged over the 1500 fields in time, $\hat{\delta}_{\text{mic,avg}}$. The results are normalized with the free stream dynamic pressure q_{∞} .

	without solenoidal filter (Eulerian)	without solenoidal filter (Lagrangian)	with solenoidal filter (Eulerian)	with solenoidal filter (Lagrangian)
$\operatorname{std}(\delta)/q_{\infty}$	0.022	0.015	0.012	0.011
$\hat{\delta}_{ m mic,avg}/q_{\infty}$	0.021	0.013	0.015	0.010

Table 2 shows the results for the uncertainty effectiveness for 95% confidence level, $\xi_{95\%}$. In conclusion, the approximated error that follows from the uncertainty propagation method is indeed useful, given that the uncertainty effectiveness is close to the theoretical value. We used k = 1.96 for the coverage factor, which is the value to be used for a Gaussian. Though the velocity uncertainties we started out with were Gaussian, due to the nonlinearity of the Navier-Stokes equations the pressure need not be Gaussian distributed anymore. A Gaussian distribution is known to have skewness 0 and a kurtosis of 3. Since we used Monte Carlo simulations to propagate the uncertainties, we have the full probability distributions of the pressure. The skewness and kurtosis of the velocity at the microphone location were 0.034 and 3.02, respectively, since we used a *finite* number of random realizations, i.e., 100. Table 3 shows the resulting values for the pressure distributions, extracted from the estimated position of the microphone in the PIV field. There is indeed a departure from the Gaussian



Figure 9: Standard uncertainty in [m/s] for the streamwise (top), wall normal (middle) and spanwise (bottom) velocity components.

Table 2: The uncertainty effectiveness for 95% confidence level, $\xi_{95\%}$.

	without solenoidal filter	without solenoidal filter	with solenoidal filter	with solenoidal filter
	(Eulerian)	(Lagrangian)	(Eulerian)	(Lagrangian)
ξ95% (%)	93	90	98	94

observable, though not considerable. Therefore, it is acceptable to use k = 1.96.

Table 3: Skewness and kurtosis of the velocity and pressure distribution, extracted from the estimated position of the microphone in the PIV field.

	without solenoidal filter (Eulerian)	without solenoidal filter (Lagrangian)	with solenoidal filter (Eulerian)	with solenoidal filter (Lagrangian)
skewness (velocity)	0.034	0.034	0.034	0.034
skewness (pressure)	-0.20	-0.024	-0.15	0.15
kurtosis (velocity)	3.02	3.02	3.02	3.02
kurtosis (pressure)	2.93	2.44	2.59	2.89

The usefulness of the uncertainty propagation method will be further strengthened by investigating the spatial structure of the uncertainty field, even though a microphone measurement was only conducted at one point. In particular, one would naturally compare the reconstructed pressure at the bottom node of the PIV domain with the microphone, expecting the best possible correspondence (highest cross-correlation coefficient) with the microphone results since it is closest to it. However, as seen in Figure 10a, this turns out not to be the case. Instead, we observe a peak. Starting from the wall, the cross-correlation coefficient increases as we move away from the wall, reaching the peak and subsequently we observe a decrease. Originally, we expected to see a monotonic decrease. However, recall the measurement uncertainty of velocity (see Figure 9); the highest uncertainty occurs close to the wall. Therefore, one would expect this to propagate into the uncertainty of the pressure field. Indeed, this is confirmed by Figure 10b. Realizing that the highest uncertainty occurs closest to the wall and that the correlation with the microphone should decrease with greater distance, we understand the tradeoff that occurs, resulting in the peak as seen in Figure 10a. The expectation of the monotonic decrease in cross-correlation coefficient (and increase in root-mean-square error) with



Figure 10: (a) Cross-correlation coefficient ρ as a function of distance from the wall without (blue) and with the solenoidal filter (red) applied, using the Eulerian (solid) and Lagrangian (dashed) approaches; (b) standard uncertainty of the reconstructed pressure field as a function of the distance from the wall.

increasing distance from the microphone location, in the absence of measurement error, model error and discretization error can be shown as follows. We take the PIV-derived pressure field and choose a location at the bottom node where the microphone is located. The pressure signal is taken as the synthetic microphone measurement. Then, we calculate the cross-correlation coefficient and root-mean-square error of the signals at other locations with this node. Figure 11 shows the results. We see the expected monotonic behavior, i.e., a decrease in cross-correlation coefficient and increase in root-mean-square error with increasing distance from the microphone location. Of course, at the microphone location they are 1 and 0, respectively.



Figure 11: (a) Cross-correlation coefficient (top) and root-mean-square error (bottom) with respect to the pressure signal at the black dot; (b) cross-correlation coefficient ρ (left) and root-mean-square error (right) with respect to the pressure signal at the wall, as a function of wall-normal distance.

5. Conclusion

The present work proposed a methodology for the *a posteriori* quantification of the uncertainty of pressure data retrieved from PIV measurements. The approach relies upon the Bayesian framework, where the posterior distribution (probability distribution of the true velocity, given the PIV measurements) is obtained from the prior distribution (prior knowledge of the velocity, e.g., within a certain bound

or divergence-free) and the distribution representing the PIV measurement uncertainty. Once the covariance matrix of the velocity is known, it is propagated through the Poisson equation. Provided the velocity uncertainty is Gaussian and the Eulerian approach is used, the uncertainty can be calculated exactly. Otherwise, Monte Carlo simulations can be used. Numerical assessment of the method on a steady Lamb Oseen vortex showed excellent agreement of the propagated uncertainty with Monte Carlo simulations, while linear uncertainty propagation underestimates the uncertainty of the pressure by up to 30%. The method was finally applied to an experimental test case of a turbulent boundary layer in air, obtained using time-resolved tomographic PIV. The pressure reconstructed from tomographic PIV data was compared to the surface measurement conducted by a microphone to determine the actual error of the former. The PIV uncertainty was quantified by applying a solenoidal filter to remove spurious divergence. In addition, spatial correlation of the PIV uncertainty was included. It was found that close to the wall, the velocity uncertainty increases. This was observed to be propagated into the pressure field as well. The comparison between actual and approximated error showed the effectiveness of the proposed method for uncertainty quantification of pressure data from tomographic PIV experiments: for a 95% confidence level, 93% of the data points fell within the estimated uncertainty bound with the Eulerian approach, and 90% with the Lagrangian approach. When using the prior knowledge that the velocity field should be divergence-free, these values were 98% with the Eulerian approach and 94% with the Lagrangian approach. The Lagrangian approach resulted in more accurate reconstructed pressure fields than the Eulerian approach. Also, enforcing the divergence-free constraint was found to result in a more accurate reconstructed pressure field. Both observations also followed from the uncertainty quantification, through a decrease in the estimated uncertainty.

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