The Transition from Geometry to Abstraction:

Illustration Practices in Linear Algebra Textbooks

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Abstract. Students often struggle to abandon geometric intuition when transitioning to formal abstraction in linear algebra. Prior textbook analyses documented a decline in visual support alongside this transition, but the communicative function of the remaining illustrations has not been examined in detail. This qualitative study investigates the role of textbook illustrations across treatments of Euclidean spaces ($\mathbb{R}^2$–$\mathbb{R}^3$), general $\mathbb{R}^n$, and abstract vector spaces. A thematic analysis was conducted through open coding of four undergraduate textbooks, informed by established typologies. The analysis reveals three shifts in function. First, illustrations systematically shed spatial characteristics to distance readers from geometric intuition. Second, the way illustrations coordinate different mathematical representations shifts from literal geometric mapping in $\mathbb{R}^2$–$\mathbb{R}^3$ to metaphorical proxies in abstract spaces. Third, illustrations shift from introducing new concepts to reinforcing established theory through repeated examples. These findings emphasize limitations of relying on geometric models for abstraction.

Keywords: Linear algebra · Illustrations · Mathematical abstraction · Vector spaces · Textbook analysis

1 Introduction

Visual representations are powerful tools for communicating mathematical ideas. In mathematics education, they support learning through problem solving and memory retention [12]. They additionally aid the abstraction process by coordinating visual and symbolic thinking [19] and giving abstract expressions a concrete, intuitive form [1]. However, concrete visual examples can also hinder the abstraction process by anchoring learners to specific instances, thereby limiting their ability to generalize mathematical concepts [8]. The role of illustrations in supporting abstraction thus appears to be both promising and problematic.

Linear algebra is a particularly relevant case for examining this tension as it requires a shift from concrete geometry to formal abstraction. To make this shift precise, Hillel distinguishes three modes of description in linear algebra: the geometric (spaces in $\mathbb{R}^2$ and $\mathbb{R}^3$), the algebraic (general $\mathbb{R}^n$), and the abstract (axiomatic vector spaces) [7]. From this perspective, illustrations seem naturally suited to support the geometric mode, but they become less directly applicable

as instruction moves toward general $\mathbb{R}^n$ and abstract spaces. Students generally have difficulty with this shift to abstraction in linear algebra [4], often struggling to abandon their reliance on geometric imagery [13].

In response to this difficulty, textbooks may reduce the number of illustrations in chapters on abstract spaces [5,9]. However, given that visual representations can also aid the abstraction process, it remains an open question whether minimizing them is the only effective response, or if they could instead be adapted to play a meaningful role in this transition.

That open question motivates the present study. Previous research primarily focused on the pedagogical function of the geometric model used in illustrations across different types of vector spaces. What remains relatively unexplored is the communicative function of the illustrations themselves in these spaces. This study therefore examines illustrations in the vector space chapters of linear algebra textbooks to analyze their communicative role across modes of description. Understanding this role may provide valuable insights into how future instructional materials can better facilitate students' transition to abstraction. Specifically, the central research question is: **What is the role of illustrations across treatments of $\mathbb{R}^2$–$\mathbb{R}^3$, general $\mathbb{R}^n$, and abstract vector spaces within linear algebra textbooks?**

This study answers that question through a qualitative thematic analysis of four university-level textbooks using an open coding approach. The analysis revealed three main themes describing the role of illustrations: (1) they weaken spatial characteristics in the transition to abstraction, (2) they serve to coordinate multiple representations, shifting from literal to proxy visual models, and (3) they contribute to theory architecture, with geometric illustrations frequently defining new concepts, whereas abstract illustrations reinforce established ones. The remainder of this paper reviews the existing literature, details the methodology, elaborates on these three themes, and discusses their implications for mathematics education.

2 Literature Review

The challenge of teaching abstraction in linear algebra has prompted various research efforts. Hillel [7] introduced a framework that distinguishes between three coexisting "modes of description" used in linear algebra instruction: the geometric, the algebraic, and the abstract. The geometric mode concerns 2- and 3-dimensional spaces and uses objects such as points, lines, and planes. The algebraic mode centers on concepts in $\mathbb{R}^n$ and its associated language of n-tuples and matrices. The abstract mode uses the language of general formalized theory, including vector spaces defined axiomatically. Hillel argues that navigating these modes is crucial for learning linear algebra, yet often difficult for students. The present study utilizes Hillel's work to categorize the mathematical *context* of illustrations (Section 3.3), providing a structured way to observe how the roles of illustrations change as textbooks transition from geometric to abstract language.

Focusing specifically on the geometric mode, Harel [6] developed a typology to describe the different pedagogical roles geometry plays across six linear algebra textbooks. He categorized these roles into seven distinct ways in which geometric models support student learning, which he terms "varieties". Geometric varieties can support the transition between geometry and abstraction, either by abstracting geometric cases into general theories (labeled "generalization"), or conversely by instantiating general concepts into concrete geometric examples ("reduction"). Harel warns that the latter may confine students to specific geometric instantiations if the general theory is not fully grasped. Additional varieties describe applying linear algebra to geometric problems ("application"), providing direct translations between symbolic and geometric modalities ("literal representation"), or conveying concepts conceptually rather than literally ("metaphorical representation"). The final two varieties involve curricula centered entirely on Euclidean spaces ("self-contained investigation"), and the use of geometry-based language for linear algebra concepts ("spatially-based terminology"). Harel's categories provide a useful framework for understanding the role of geometry in linear algebra instruction. However, his focus remains on the mathematical models rather than the illustrations used to convey them. As such, the present study adapts his typology to specifically code the *geometric role* of textbook illustrations, as detailed in Section 3.3.

Other work has focused more directly on the illustrations in textbooks. As part of a broader investigation into the use of geometry in university linear algebra, Gueudet-Chartier [5] conducted a textbook analysis of *Banchoff and Wermer's Linear Algebra Through Geometry*. She documented a two-step progression to the general theory: first, $\mathbb{R}^2$–$\mathbb{R}^3$ serves as a paradigmatic model for $\mathbb{R}^n$, which in turn is used as a paradigmatic model for abstract vector spaces. Her analysis revealed a shift in visual support alongside this progression: 92% of the illustrations were concentrated in $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$, with illustrations virtually disappearing in the second step. Combining this textbook analysis with classroom observations and interviews, she concluded that while geometric models in $\mathbb{R}^2$–$\mathbb{R}^3$ can be helpful for introducing $\mathbb{R}^n$, there is no evidence that they are similarly effective for abstract vector spaces. Ultimately, her analysis of these illustrations centers on (1) the pedagogical utility of the underlying geometric model and (2) the quantitative decrease in visual support. Although she makes passing visual observations (such as noting whether vectors are depicted as points or arrows), she does not systematically analyze or interpret the function of these choices.

Kaneko and Takato [9] extended her line of work by examining a broader set of linear algebra textbooks. Their survey similarly found illustrations common in Euclidean spaces but sparse in general $\mathbb{R}^n$ and abstract spaces. However, their textbook analysis primarily served to motivate their own project on 3D graphics software for teachers.

Together, these prior studies establish a foundation for analyzing textbooks across different mathematical spaces. Yet, they do not examine the communicative function of illustrations in detail, as it was not their primary focus. This study therefore shifts attention toward visual analysis of the illustrations themselves,

adapting Hillel's modes of description and Harel's typology to explore their communicative roles.

3 Methodology

This study analyzes a corpus of university-level linear algebra textbooks. To address the research question, thematic analysis through iterative open coding is used. Identifying an illustration's "role" is an inherently interpretive task, making thematic analysis suitable. Compared to content analysis, which typically identifies themes based on frequency of occurrence, thematic analysis judges a theme's importance by its relevance to the overall research question [17]. Because earlier research noted that the number of illustrations tends to decline in abstract contexts [5,9], a method independent of frequency is beneficial. By using open coding to let patterns emerge from the data, this method can effectively track thematic shifts across mathematical contexts. The remainder of this section discusses the corpus selection, the coding process, and the resulting codebook.

3.1 Sampling strategy, selection criteria, and textbook corpus

The textbook corpus was constructed using a sampling strategy combining purposive, criterion-based, and convenience sampling, as defined by Tenny et al. [15]. First, textbooks were deliberately chosen for their potential to provide rich and visual treatments of vector spaces in both Euclidean and abstract settings (*purposive sampling*). Second, textbooks were required to meet a set of criteria, such as level and scope of topics (*criterion sampling*). Finally, the corpus was limited to textbooks that were accessible as open-access resources or through institutional library access (*convenience sampling*).

Four criteria guided textbook selection. First, they had to be intended for use in tertiary-level teaching and learning. Second, to ensure contemporary relevance to current educational practices, the textbooks had to be actively used in syllabi from the 2025 or 2026 academic years at well-regarded universities. For the purposes of this study, a university was classified as well-regarded if it appeared in the top 1000 of the Times Higher Education (THE) World University Rankings 2026 [16], a widely used global ranking of universities. The most recent publication of textbooks with multiple editions was utilized where possible. Third, textbooks needed to make systematic use of illustrations throughout the volume. Fourth, each textbook had to devote substantial attention to vector space theory in both Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, their generalization to $\mathbb{R}^n$, and abstract vector spaces. At a minimum, chapters on subspaces, span, basis, and linear combinations were required.

The final corpus consists of four textbooks: *Linear Algebra and Its Applications* by Lay et al. [11]; *A First Course in Linear Algebra* by Kuttler [10]; *Introduction to Linear Algebra* by Strang [14]; and *Linear Algebra* by Cherney et al. [3]. For convenience, these will be referred to as the Lay-book, Kuttler-book, Strang-book and Cherney-book, respectively.

Table 1: Overview of most relevant code categories and corresponding codes.

Category	**Code**	**Description**
Vector	Arrow	Vector depicted as an arrow
	Point	Vector depicted as a point
Graph	Present	Includes Cartesian axes
	Absent	No Cartesian axes shown
Dimension	2D	Illustration drawn in 2D
	3D	Illustration drawn in 3D
Context	$\mathbb{R}^2/\mathbb{R}^3$	Euclidean geometric context
	$\mathbb{R}^n$	General dimensional context
	Abstract	Abstract vector space context
Geometric role	Generalization	Specific geometric case visually inviting extension to general setting
	Reduction	General concept instantiated in a geometric case
	Literal representation	Direct translation between algebraic and visual forms
	Metaphorical representation	Visual aid without literally depicting the mathematical object
	Application	Applied situation used to ground the concept and demonstrate meaningfulness
Theory function	Concept-defining	Tied to initial introduction of a concept
	Concept-stabilizing	Revisits established concept to reinforce it
	Concept-extending	Broadens established concept to a new sub-concept

3.2 Coding process

The textbook material was collected into ATLAS.ti [2] and analyzed through open coding [18]. An illustration was coded when it functioned as a standalone visualization of one or more mathematical concepts. Illustrations were excluded when they were not directly used to support mathematical content, for example, when they served primarily decorative, historical, or biographical purposes. The surrounding context was also taken into account, including adjacent text, definitions, theorems, and proofs that clarified the role of the illustration. In cases where several images functioned together as a single analytical unit, they were coded both individually and as a group.

Coding proceeded iteratively, combining inductive and deductive elements. Inductively, initial codes were developed from the data itself with descriptive labels for each illustration. Deductively, the process was informed by both Hillel's

[7] modes of description and Harel's [6] typology of uses of geometry in linear algebra.

3.3 Codebook

The codebook consisted of several categories that captured both descriptive features and interpretive roles of the illustrations. The descriptive categories most relevant to the findings include *Vector*, *Graph*, and *Dimension*. These respectively describe how vectors are depicted, whether the illustration depicts Cartesian axes, and the dimensional setting of the illustration.

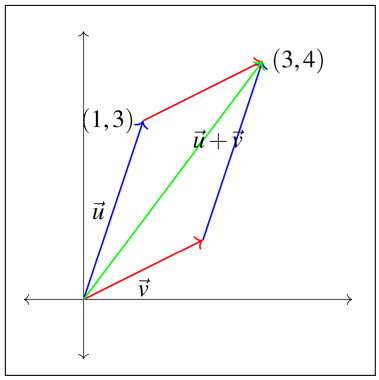
Interpretive categories were developed to capture the function of the illustrations. The category *Context* captures the mathematical setting of the illustration based on Hillel's [7] modes of description. The category *Geometric role* adapts Harel's [6] typology of geometric varieties. Because his framework describes the pedagogical function of geometric models broadly, it was modified here to specifically capture the communicative function of individual illustrations. The "self-contained investigation" and "spatially-based terminology" varieties were excluded as they describe overall course design and vocabulary, rather than visual elements. The remaining five varieties were adapted into distinct codes. Finally, the category *Theory function* describes how an illustration contributes to the structuring of the theory. Table 1 summarizes the categories and defines specific relevant codes.

4 Results

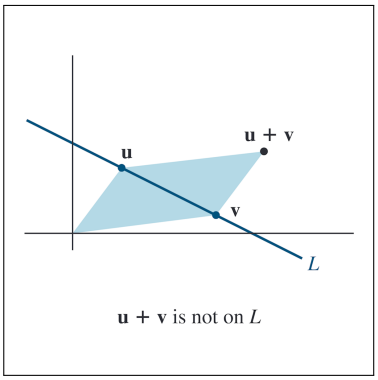
This chapter presents three themes that emerged from analyzing the coded illustrations. These themes highlight both the role illustrations play in treatments of vector spaces and how that role shifts across $\mathbb{R}^2$, $\mathbb{R}^3$, general $\mathbb{R}^n$, and abstract contexts. The first theme shows how illustrations deliberately shed spatial cues to support the move toward abstraction. The second examines how illustrations coordinate multiple representations, from literal geometric representations in Euclidean spaces to proxy visualizations in abstract settings. The third explores how illustrations help organize the theory itself, shifting from concept-defining uses in geometric contexts to concept-stabilizing uses in abstract ones.

4.1 De-spatialization as a bridge to abstraction

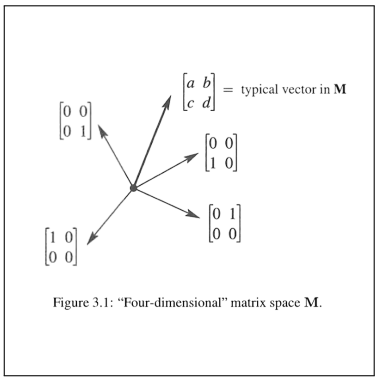
In the transition from spaces in $\mathbb{R}^2$–$\mathbb{R}^3$ to general $\mathbb{R}^n$ and abstract vector spaces, Euclidean characteristics are weakened to help the reader move away from geometric interpretation and towards reading vector spaces as general structures. We refer to this systematic reduction of the spatial framing of vectors as *de-spatialization*. Rather than being a simplification of the material, de-spatialization serves to reposition the reader's intuition. In the corpus, de-spatialization occurs by reducing the spatial framing of vectors and favouring lower-dimensional illustrations over visually complex three-dimensional ones.



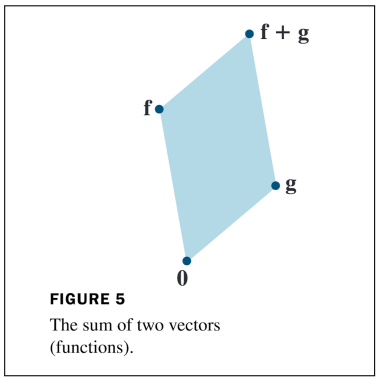
(a) $\mathbb{R}^2$ space



(b) $\mathbb{R}^n$ space



(c) Abstract space (matrices)



(d) Abstract space (functions)

Fig. 1: Spectrum of de-spatialization in vector space representations. Spatial characteristics are weakened by representing vectors as points and/or omitting Cartesian axes. Reproduced from Kuttler-book (a), Lay-book (b, c), and Strang-book (d).

Illustrations systematically strip away the spatial characteristics of vectors to support the transition to abstraction. In geometric treatments of $\mathbb{R}^2$ and $\mathbb{R}^3$, vectors are traditionally represented as directed arrows within a Cartesian coordinate system (frequently assigned the *Vector: Arrow* and *Graph: Present* codes). This depiction emphasizes inherently Euclidean properties, specifically magnitude and direction. Fig. 1a provides an example: vectors in $\mathbb{R}^2$ are depicted as arrows on Cartesian axes, linking the concept to a geometric model.

As textbooks transition to higher-dimensional spaces ($\mathbb{R}^n$) or abstract vector spaces, these spatial properties become less applicable. Illustrations for these spaces therefore frequently soften or remove the geometric meaning of vectors. A key strategy is to represent vectors as points rather than arrows. Unlike arrows, points do not visualize a length or direction, weakening the spatial meaning

of vectors. The illustration in Fig. 1b is an example of such weakening. In the coded illustrations, the *Arrow* code predominated over the *Point* code in $\mathbb{R}^2$ and $\mathbb{R}^3$, but this pattern did not continue in $\mathbb{R}^n$ or abstract contexts, where point representation appeared in roughly equal measure.

A more pronounced de-spatialization move involves the complete omission of a Cartesian coordinate graph. Because graphs function as explicit mathematical models of physical space, they feature prominently in treatments of $\mathbb{R}^2$ and $\mathbb{R}^3$. This spatial framing also persists in representations of $\mathbb{R}^n$, where the concept of space remains mathematically and visually relevant. In abstract contexts, however, graphs are occasionally deliberately omitted (captured by the *Graph: Absent* code) to visually detach concepts from a geometric background. Figs. 1c and 1d both omit axes: the former retains arrows, while the latter uses points, making it the most fully de-spatialized of the four. Without these spatial cues, the illustrations invite readers to view vector spaces as general structures, instead of solely as geometric objects in physical space.

These de-spatialization strategies are observable across the entire corpus, with Fig. 1 showing the full spectrum using examples from three different books. They can also serve as a progressive pedagogical tool within individual textbooks, as seen in the Lay-book. Chapter 1 introduces vectors in $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^n$ using spatial cues, such as Cartesian axes in Fig. 1b. When abstract spaces are formally introduced in Chapter 4, the accompanying illustrations frequently shed these spatial elements, as seen in Fig. 1d.

De-spatialization is further supported by a consistent preference for two-dimensional representations over three-dimensional ones when abstract contexts are involved. Out of the ten illustrations coded as *Context: Abstract*, all but one were drawn in 2D. A flattened field avoids some of the visual clutter associated with three-dimensional illustrations, such as perspective and foreshortening. Stripping away visual information may serve a pedagogical function by putting more emphasis on the core of the concept.

4.2 Coordination of multiple representations of vector spaces

In the coded material, a central function of illustrations is to coordinate multiple representations of the same vector space concept. Besides visualizing the theory in the body text, these images make relationships visible across mathematical forms that might otherwise appear unrelated. In the context of $\mathbb{R}^2$ and $\mathbb{R}^3$, this coordination typically provides literal links between algebraic notation and geometric space. In the context of $\mathbb{R}^n$, this coordination relies on semi-literal geometric stand-ins. In abstract contexts, coordination becomes more complex, frequently relying on proxy visualizations to bridge abstract structures with familiar domains.

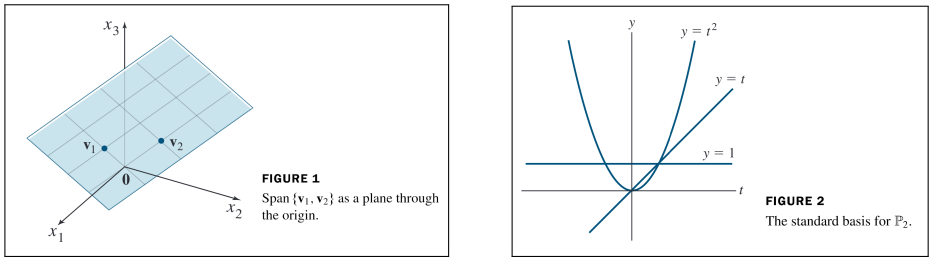
Literal switching between algebra and geometry in $\mathbb{R}^2$ and $\mathbb{R}^3$. In the geometric environments of $\mathbb{R}^2$ and $\mathbb{R}^3$, representation switching is explicit, relying on illustrations coded with *Geometric role: Literal representation*. Such illustrations provide a direct visualization of the concepts described in the text,

serving as a translation from algebraic notation into Euclidean geometric forms. For instance, a vector expressed algebraically as a coordinate tuple may be visually coordinated with a directed arrow (Fig. 1a). Similarly, the concept of vector addition is mapped to parallelograms (Figs. 1a, 5a, 5b), and the concept of a subspace is visualized by a line or plane passing through the origin within a larger space (Fig. 2a). None of these instances require metaphors or cognitive abstraction, as the illustration essentially provides a one-to-one mapping between representations.

Semi-literal switching in $\mathbb{R}^n$. Literal visualization of spaces in $\mathbb{R}^n$ is no longer possible for $n > 3$. While some illustrations may attempt to draw higher-dimensional spaces, like the "four-dimensional" matrix space in Fig. 1c, these are not literal representations (four orthogonal axes cannot be drawn in 2D). Rather than forcing complex n-dimensional drawings or abandoning geometry altogether, illustrations in the $\mathbb{R}^n$ context frequently employ lower-dimensional Euclidean diagrams ($\mathbb{R}^2$ or $\mathbb{R}^3$) as stand-ins for n-dimensional concepts.

The illustration in Fig. 2a, for example, accompanies the formal algebraic definition of a subspace in $\mathbb{R}^n$. The image itself may be lower-dimensional, showing a 2D plane within a 3D Cartesian coordinate system, but it does not restrict this geometry to $\mathbb{R}^3$. The textbook explicitly frames the illustration as "the standard way to visualize" the concept of subspaces in $\mathbb{R}^n$ [11, p. 179].

This case illustrates the geometric role of *Reduction*, which is common in illustrations in the $\mathbb{R}^n$ context. This reductive role makes the relationship between image and algebra only semi-literal: the geometric objects (axes, points, and planes) function as lower-dimensional stand-ins for $\mathbb{R}^n$, not literal representations. Generalizing these back out to $\mathbb{R}^n$ is left as an extra cognitive step for the reader. In sum, representation switching in $\mathbb{R}^n$ is less literal than in $\mathbb{R}^2$–$\mathbb{R}^3$, but is still reliant on Euclidean rules instead of metaphors.



(a) (b)

Fig. 2: (a) Illustration used alongside the definition of a subspace in $\mathbb{R}^n$. (b) Visualization of individual vectors rather than the abstract space itself. Reproduced from Lay-book.

Switching through proxy visualizations in abstract spaces. As textbooks transition to abstract vector spaces, literal geometric representations of the spaces themselves are generally no longer possible. Instead, representation switching in these settings relies on one of two distinct visual strategies to make non-geometric concepts visible.

The first strategy visualizes vectors making up the vector space, rather than the abstract space itself. For example, Fig. 2b accompanies an algebraic proof defining the standard basis for the abstract polynomial space $\mathbb{P}_n$. To ground this concept, the figure plots the basis vectors for $\mathbb{P}_2$ in a Cartesian plane. The illustration itself makes no attempt to visually prove that these elements span the space or are linearly independent. It instead gives the basis vectors a visual identity, while the structure of the abstract vector space remains unrepresented.

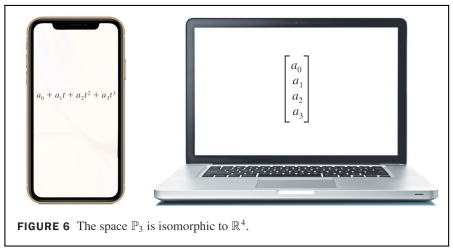


FIGURE 6 The space $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$.

(a)

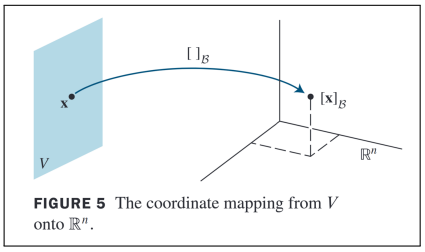


FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

(b)

Fig. 3: Two illustrations using metaphorical visualization. (a) A non-geometric everyday analogy. (b) A geometric plane used as a substitute model. Reproduced from Lay-book.

The second strategy involves the *Metaphorical representation* code, wherein textbooks assign a substitute visual model to represent the abstract space. This approach does not depict actual geometric objects and instead relies on analogies. An instance of such *Metaphorical representation* is provided in Fig. 3a, which illustrates an isomorphism between the polynomial space $\mathbb{P}_3$ and $\mathbb{R}^4$ using a non-geometric, everyday analogy of connected digital displays. A polynomial appears on the left screen and its corresponding coordinate vector on the right. The accompanying text leverages this metaphor to explain that the vectors may look different symbolically, but behave identically under vector operations. Using the screens as a proxy makes the concept of isomorphism visible without relying on geometric elements.

Other substitute models may still leverage the reader's pre-existing geometric intuition, such as in Fig. 3b. This illustration visualizes a coordinate mapping from an abstract space V onto $\mathbb{R}^n$ by assigning a visual stand-in to each component. A metaphorical plane-like shape represents V, a Cartesian graph represents $\mathbb{R}^n$, and the arrow between them represents the mapping function. The representation for V draws on a familiar geometric image, without giving the space a concrete geometric identity.

Although such substitute models may offer a visual reference for abstract concepts, they also introduce a pedagogical tension. Since the visual models are analogies rather than literal mappings, readers must actively abstract away certain elements of the proxies. For instance, the reader must recognize that the abstract space V in Fig. 3b is not literally equivalent to the geometric plane used to depict it, and that the mapping arrow does not belong to the underlying mathematical structure. This makes the link between symbolic and visual representations appear fundamentally weaker than in Euclidean contexts, requiring more effort from the reader.

It should be noted that not all illustrations in abstract contexts rely on metaphors. One unique exception occurred in Fig. 6a because the abstract vector space was itself comprised of geometric objects, namely the space of all directed arrows in 3D. That illustration reverts to *Literal representation* by visualizing the associativity axiom using geometric arrows.

4.3 Contribution to theory architecture through definition and stabilization

Beyond visualizing mathematical concepts, illustrations contribute to the organization of the theory itself. In the coding framework, illustrations were categorized according to their role in this conceptual development as either *Concept-defining*, *Concept-stabilizing*, or *Concept-extending*. Analyzing the distribution of these codes reveals a clear shift across spaces: whereas geometric contexts utilize concept-defining illustrations frequently, abstract contexts abandon that role almost entirely, relying predominantly on concept-stabilizing ones.

Prominence of concept-defining illustrations in geometric contexts. In treatments of $\mathbb{R}^2$ and $\mathbb{R}^3$, illustrations frequently function as entry points to new concepts within a familiar geometric setting. As an example, Fig. 1a offers an initial visual model of vector addition. These illustrations function as the text's dominant visual reference for the concept, which may potentially shape the reader's understanding of subsequent theoretical developments.

Notably, many of these *Concept-defining* illustrations were simultaneously coded with *Geometric role: Reduction*. The illustration in Fig. 4, for instance, introduces the general definition of a subspace by reducing it to a plane through the origin in 3D space. As discussed in Section 2, Harel warns that his "reduction" variety may confine student understanding to the specific geometric instantiation. In the context of this study, when an illustration is both concept-defining and reductive, the first mental image associated with the concept may be particularly narrow. Such illustrations may therefore confine the concept too closely to the geometric setting in which it is introduced.

Illustrations are rarely concept-defining in abstract spaces. One possible explanation is that abstract spaces are too diverse to be introduced through a single shared visual model. Consequently, unlike in geometric contexts, abstract vector spaces may not develop a dominant visual reference point in the reader's mind.

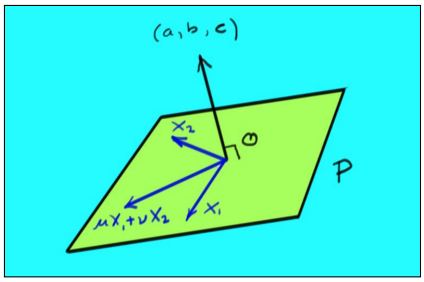
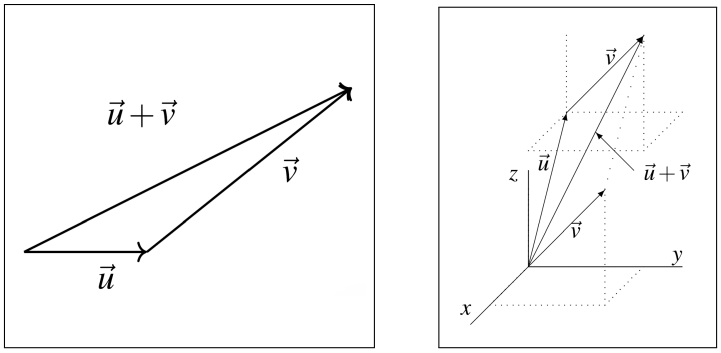


Fig. 4: Concept-defining illustration reducing to $\mathbb{R}^3$. Reproduced from Cherney-book.

Reliance on concept-stabilizing illustrations in abstract spaces. Once a concept has been introduced, subsequent illustrations can take on a *Concept-stabilizing* role. Their function within the text is not to introduce new theory, but to reinforce recognition of an already established concept. Illustrations in geometric contexts usually achieve stabilization by showing an established concept in a different setting. For example, after introducing vector addition through the 2D illustration in Fig. 5a, a subsequent 3D illustration in Fig. 5b stabilizes the concept by presenting it in a different geometric environment.



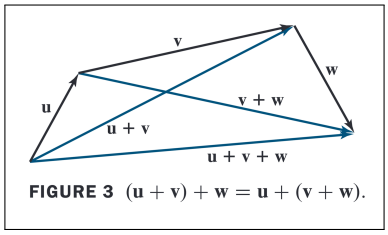
(a) $\mathbb{R}^2$ (b) $\mathbb{R}^3$

Fig. 5: Illustrations presenting vector addition geometrically. Reproduced from Kuttler-book.

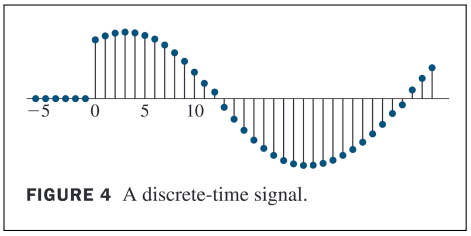
This stabilizing function becomes prominent in abstract contexts. More than half of the illustrations coded as *Abstract* were also coded as *Concept-stabilizing*. Because the diversity of abstract spaces hinders the use of a single introductory visual model, the textbooks in the selected corpus instead rely on a variety of distinct instantiations to stabilize the general theory. For instance, after defining the axioms of a general vector space, the Lay-book provides examples of a space

of directed arrows alongside a space of sequences. The accompanying illustrations in Fig. 6a and Fig. 6b stabilize the concept by demonstrating its application across different settings, without establishing a shared visual model.

Finally, although some illustrations function to extend introduced concepts to new sub-concepts, such as extending vector addition to vector subtraction, this role did not yield a clear cross-context pattern in the material.



(a) Abstract space (arrows)



(b) Abstract space (sequences)

Fig. 6: Concept-stabilizing illustrations without a shared visual model. (a) Illustration of the associativity axiom in a vector space of directed arrows. (b) Illustration of an application of $\mathbb{S}$, the space of all doubly infinite sequences of numbers. Reproduced from Lay-book.

5 Discussion

Overall, the three themes identified in the results suggest that illustrations shift from literal geometric models in Euclidean spaces to de-spatialized, metaphorical proxies that serve a concept-stabilizing role. This section interprets the effect of these findings in relation to the existing work done by Harel and Gueudet-Chartier to understand the pedagogical implications. Finally, it addresses some limitations of the present study and proposes directions for further research.

5.1 Extending metaphorical representation beyond geometry

The findings of this study can be interpreted further through Harel's [6] notion of "metaphorical representation". He describes this variety as a geometric sketch accompanying the communication of a general concept. Fig. 3b matches this description well, as it gives a visible form to the concept of coordinate mapping without presenting a literal image. Harel notes that illustrations intended as non-literal may still be perceived by students as actual geometric objects. In that sense, the plane-like depiction of V would fall within what Harel terms "students' geometry", even though it is not meant to define the space as a geometric plane. This mirrors the tension identified in Section 4.2, where proxy illustrations may provide a visual anchor for abstract concepts but require readers to actively abstract away non-mathematical features. This may suggest that metaphorical

representation offers a trade-off: it can support visualization of abstraction at the cost of increased interpretive effort.

The present findings also highlight visual strategies that fall outside the scope of Harel's geometry-focused framework. The metaphorical visualization in Fig. 3a discards geometric models entirely and instead utilizes an everyday analogy. This may be significant, because earlier research has noted that students often struggle to move beyond their geometric intuition in linear algebra [13]. Non-geometric metaphors may therefore offer a way of supporting visualization while loosening dependence on Euclidean space. However, this possibility remains tentative, since this study does not examine how students actually interpret such illustrations.

5.2 The risk of concept-defining, reductive illustrations

Section 4.3 indicates that illustrations coded as both *Reduction* and *Concept-defining* risk linking the concept too closely to a specific geometric instantiation. This further resonates with Gueudet-Chartier's [5] point that students may remain captured in a geometric context instead of transitioning to the general theory. This does not necessarily mean that concept-defining reductive illustrations should be avoided altogether, but it suggests they might benefit from more careful handling. One mitigation could be to include surrounding text explicitly stating the limitations of the geometric case.

5.3 The limits of geometry in abstract spaces

The deliberate de-spatialization and shift towards metaphors as opposed to literal geometry in abstract contexts aligns with Gueudet-Chartier's [5] argument that there is no clear evidence that using geometric models can effectively introduce abstraction. However, as discussed in Section 4.2, a noteworthy exception arises when an abstract vector space is composed of geometric objects, such as the space of directed arrows in Fig. 6a. Whether such examples support the abstraction process, or also risk confining the concept to geometric settings, remains unclear.

Gueudet-Chartier additionally argues for the need for a figural model specific to linear algebra that moves beyond geometric models. From this perspective, non-geometric proxy models for abstract spaces, such as the one in Fig. 3a, may be tentative emergent examples of such a model.

5.4 Limitations and further research

Several limitations of the present study's scope and methodology warrant acknowledgment. First and foremost, this study analyzes the intended communicative function of textbook illustrations, but it does not investigate how these are actually interpreted by students. Some identified points, such as the risk of readers taking metaphorical sketches literally, therefore remain interpretive rather than empirically verified. A natural next step of research would be to investigate actual student interpretation of these illustrations. Interviews or classroom observations

could reveal whether students experience the pedagogical tensions presented in this study.

Second, the scope of the study is strictly limited to chapters on vector spaces and closely related sub-topics (subspaces, span, basis, linear combinations, and coordinate systems). Other parts of linear algebra, including highly visual ones like orthogonality, linear transformations, and eigenvectors, may yield different communicative functions of illustrations. On a related note, the textbooks in the analyzed corpus are all of American origin. This is worth considering because previous research has noted that linear algebra teaching varies across different countries [5]. As a result, the communicative functions identified here may not be broadly applicable to textbooks in different parts of the world. Further research could address this by expanding the visual analysis to other linear algebra topics and textbooks in other countries to examine if the same patterns appear.

A final suggestion concerns the development of visual models specific to linear algebra, a need previously brought up by Gueudet-Chartier [5]. Investigating the potential of non-geometric proxies (such as Fig. 3a) as a foundation for non-geometric models could yield new pedagogical tools to support students in the transition to abstraction.

6 Conclusion

This study investigated the communicative role of illustrations in university-level linear algebra textbooks, specifically focusing on how visual support changes across treatments of $\mathbb{R}^2$–$\mathbb{R}^3$, general $\mathbb{R}^n$, and abstract vector spaces. A thematic analysis was conducted on a corpus of four widely used undergraduate textbooks. Through iterative open coding informed by typologies from Hillel and Harel, illustrations in each of the three contexts were analyzed for their visual features, geometric roles, and theoretical functions.

The analysis revealed three themes that demonstrate how the function of illustrations shifts across the different settings. First, textbooks systematically strip away spatial characteristics in illustrations to bridge the gap to abstraction. Second, illustrations coordinate multiple mathematical representations, though this coordination becomes less literal as spaces grow more abstract. Third, illustrations play a structural role, in geometric spaces often by introducing concepts, and in abstract spaces by reinforcing existing ones.

These findings align with and expand upon earlier work. The reliance on metaphorical representation allows for the visualization of abstract concepts, but demands higher interpretive effort from students. Moreover, when concept-defining illustrations are also reductive, they risk confining student understanding to a specific geometric case rather than the general theory. This ultimately highlights a limitation of relying on geometric models to teach abstract concepts, pointing to the need for future research analyzing alternative, non-geometric models specific to linear algebra.

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A Responsible Research

This section provides a critical reflection on the transparency, reproducibility, and integrity of the study, while also disclosing potential researcher biases and clarifying the use of copyrighted materials.

A.1 Study reproducibility and codebook availability

To support the transparency and reproducibility of this research, the document collection process is documented in the methodology section, which outlines the sampling strategy and selection criteria used. This clarifies the rationale for including these specific textbooks. Additionally, the codes most important to the themes are included in the same section.

At the same time, the reproducibility of the research is limited in several ways. For one, the full codebook is not publicly available, which limits readers' ability to trace how all specific codes were applied to the material. This reduces the transparency of the coding process and makes it difficult for other researchers to reproduce the study in detail. Publishing the full codebook in an external repository would mitigate this, but it was beyond the scope of the present study.

A further limitation is that all coding was conducted by me alone. Because qualitative analysis involves interpretation, having only one coder can increase the influence of the researcher's subjectivity. Involving a second researcher who independently coded the material could have strengthened the study by allowing for an assessment of inter-coder reliability. Since this was not part of the research design, the findings should be interpreted with this limitation in mind.

A.2 Positionality statement

Because this study relies on qualitative analysis, the interpretation of the material inherently involves subjectivity. For the sake of transparency and research integrity, it is therefore important to disclose my relevant experiences and preferences to contextualize the methodological and analytical choices made.

One potential source of bias relates to the selection of textbooks. In my own study of linear algebra, I preferred learning material with substantial use of visualizations. This preference may have influenced the formulation of selection criteria, which included "systematic use of illustrations throughout the book" as a requirement. Consequently, textbooks with fewer illustrations may have been discarded more readily, even if they had other relevant qualities that would have benefitted the study.

A second form of bias concerns the inclusion of Lay et al.'s [11] *Linear Algebra and Its Applications*. Because this textbook was used during my undergraduate studies, I already had a positive impression of its illustrations. Although this familiarity was not the sole reason for its inclusion in the study, it may have made me more inclined to choose it. Moreover, the prior exposure to the visualizations may have introduced a favorable bias towards the illustrations in Lay et al. during the coding phase.

A.3 Copyright and licensing of the source materials

This study uses four linear algebra undergraduate textbooks as source materials. The books by Kuttler [10] and Cherney et al. [3] are open-access publications. Lay et al.'s [11] and Strang's [14] textbooks were accessed through the TU Delft Library, which provides students with access to all materials in its database for academic use. The present study used the textbooks for analysis, and selected illustrations were only included as examples to support the discussion of the identified themes.