The Rogers-Ramanujan Identities Explored

From Ramanujan's Proof to Modern Bijective Approaches

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TU Delft

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Layman's Summary

Some mathematical formulas are more than just tools; they are like little puzzles that connect different worlds. The Rogers-Ramanujan identities are two such formulas. First discovered over 100 years ago, they have turned out to be incredibly deep and mysterious. Over time, they have appeared in many unexpected places, including the solution of puzzles involving whole numbers, the understanding of particle behavior in physics, and even in the study of symmetry and geometry.

This thesis explores four things. First, it explains the proof of the famous mathematician Ramanujan for these identities. Second, it looks at how these formulas can be understood through the idea of ”partitions”, ways of breaking numbers into smaller parts. Third, it gives a historical overview of how other mathematicians over the past century have tried to explain these identities in clever and creative ways. Finally, it offers a new explanation for one of the formulas using visual and logical steps, following the ideas of a modern mathematician named Igor Pak.

The goal of this work is to show that even a pair of formulas, written long ago, can still surprise us and lead to new discoveries.

Abstract

The Rogers-Ramanujan identities are among the most remarkable results in the theory of integer partitions and q-series. Discovered independently by Rogers, Ramanujan, and Schur, these identities have since appeared in a wide range of mathematical disciplines, including number theory, combinatorics, representation theory, and statistical mechanics. Despite their concise form, the identities have inspired dozens of proofs and interpretations, each revealing new layers of structure and insight.

This thesis investigates four core aspects of the Rogers-Ramanujan identities. First, it presents and explains Ramanujan's original analytic proof. Second, it explores the combinatorial interpretation of the identities in terms of integer partitions. Third, the thesis provides a chronological overview of key combinatorial bijective proofs, highlighting how these have evolved and deepened our understanding of the identities. Lastly, it presents Igor Pak's combinatorial proof of the first Rogers-Ramanujan identity.

By combining historical context with mathematical exposition, this work aims to clarify the richness of the Rogers-Ramanujan identities and to contribute to the ongoing effort to understand them more fully.

Key words: Rogers-Ramanujan identities, integer partitions, q-series, combinatorics, bijective proofs

1 Introduction

The Rogers-Ramanujan identities are among the most celebrated and mysterious results in the theory of integer partitions and q-series. First discovered by Leonard James Rogers in 1893 (Rogers, 1893) and later independently rediscovered by Srinivasa Ramanujan (Aiyangar, 2000) and Issai Schur, these identities have since captivated generations of mathematicians. Despite their analytic origin, they have proven remarkably deep, connecting seemingly unrelated fields such as representation theory (Andrews, 1984b), group theory (Fulman, 1997), statistical mechanics (Baxter, 1990), modular forms (Bringmann et al., 2007) and even knot theory (Armond and Dasbach, 2011).

The historical development of these identities is as fascinating as their mathematics. Rogers' original formulation in 1894 received little attention until Ramanujan, working in isolation in India, rediscovered the identities sometime before 1913. Without proof, he included them in his first letter to Hardy. Hardy and others initially failed to prove them, and they were published without proof in MacMahon's *Combinatory Analysis* in 1916. It was only in 1917 that Ramanujan found a proof. Around the same time, Ramanujan discovered Rogers' earlier publication and expressed deep admiration for his work. Meanwhile, Schur, isolated from England due to World War I, also rediscovered and proved the identities independently, introducing a combinatorial perspective still influential today (Sills, p. 54).

Since their rediscovery, the Rogers-Ramanujan identities have assumed a nearly legendary status in mathematics. Hardy once remarked: "None of the proofs of [the Rogers-Ramanujan identities] can be called 'simple' and 'straightforward' [...]; and no doubt it would be unreasonable to expect a really easy proof" (Hardy, as cited in Boulet & Pak, 2006, p. 1). Even today, with dozens of distinct proofs, each highlights different structural or combinatorial insights, and the depth and beauty of these identities remain a source of inspiration and challenge.

The identities have also achieved relevance beyond pure mathematics. In 1980, Baxter discovered a direct application of Rogers-Ramanujan type identities while solving the hard hexagon model in statistical mechanics, a breakthrough that sparked extensive interdisciplinary research (Baxter, 1990). As Campbell notes, this result "opened up a new world of research in both theoretical physics and partition-theoretic mathematics that is still evolving" (Campbell, 2024, p. 1).

It is no surprise, then, that entire volumes have been written in an attempt to introduce these identities. Yet even these works acknowledge their limitations. One such introduction opens with the statement: "Experts in the field will realize immediately upon picking up this volume that it cannot possibly be long enough to adequately discuss the multitude of subjects arising from the Rogers-Ramanujan identities... this volume is but an introduction [...] and an invitation to study further" (Sills, p. xvii).

This thesis aims to contribute to that ongoing invitation. It focuses on four main objectives:

1. To explain Ramanujan's own analytic proof of the Rogers-Ramanujan identities.
2. To develop the combinatorial interpretation of the identities through the theory of integer partitions.
3. To provide a chronological overview of key combinatorial bijective proofs of these identities, mapping the development of ideas over time.
4. To present a combinatorial proof of the first Rogers-Ramanujan identity by Igor Pak.

The structure of this thesis is as follows. Chapter 2 offers the necessary background: the precise statements of the identities, the notation used throughout, the various interpretations of the identities, and the Jacobi Triple Product Identity with proof. Chapter 3 presents Ramanujan's own proof and explains the main steps. In Chapter 4, we explore the combinatorial meaning of the identities by defining the relevant concepts in partition theory. Chapter 5 surveys the major bijective proofs in chronological order, focusing on how each contributed new insights into the identities. Finally, in Chapter 6, Igor Pak's bijective proof of the first identity is presented. The thesis ends with a conclusion and discussion of open questions and future directions.

2

An Introduction to the Rogers-Ramanujan Identities

This chapter begins with the formal statement of the Rogers-Ramanujan identities and introduces the necessary notation, including the q-Pochhammer symbol and Ramanujan's theta function. It explores several interpretations of the identities: analytically as q-series, in statistical mechanics via Baxter's hard hexagon model, in representation theory through affine Lie algebras, and in the theory of modular forms. The chapter then presents the Jacobi Triple Product Identity, which underpins many proofs of the Rogers-Ramanujan identities. To build up to this, the q-binomial theorem is proved, followed by key corollaries. The chapter concludes with a proof of the Jacobi identity.

2.1. The Identities

The Rogers-Ramanujan identities form a pair of infinite q-series identities. Each equates a sum side, composed of rational functions involving powers of q, with an infinite product. These identities hold for complex values of q with $|q| < 1$, ensuring convergence of the infinite expressions.

Identity 2.1 (The first Rogers-Ramanujan identity). For $|q| < 1$, the first Rogers-Ramanujan identity is:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}. \tag{2.1}$$

Identity 2.2 (The second Rogers-Ramanujan identity). For $|q| < 1$, the second Rogers-Ramanujan identity is:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+2})(1-q^{5k+3})}. \tag{2.2}$$

To illustrate the structure of these identities, we expand the sum and product sides. The first Rogers-Ramanujan Identity expanded is:

$$1 + \frac{q^1}{(1-q)} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \frac{q^{16}}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \cdots$$

$$= \frac{1}{(1-q)(1-q^4)} \cdot \frac{1}{(1-q^6)(1-q^9)} \cdot \frac{1}{(1-q^{11})(1-q^{14})} \cdot \frac{1}{(1-q^{16})(1-q^{19})} \cdot \cdots.$$

The second Rogers-Ramanujan Identity expanded is:

$$1+\frac{q^2}{(1-q)}+\frac{q^6}{(1-q)(1-q^2)}+\frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}+\frac{q^{20}}{(1-q)(1-q^2)(1-q^3)(1-q^4)}+\cdots$$
$$=\frac{1}{(1-q^2)(1-q^3)}\cdot\frac{1}{(1-q^7)(1-q^8)}\cdot\frac{1}{(1-q^{12})(1-q^{13})}\cdot\frac{1}{(1-q^{17})(1-q^{18})}\cdot\cdots.$$

2.2. Notation

Throughout this paper, we assume that $|q| < 1$ to ensure the convergence of all infinite series and products.

A central object in this work is the q-Pochhammer symbol, also known as the q-shifted factorial. It is defined as follows:

$$(a;q)_k = \begin{cases} \prod_{j=0}^{k-1}(1-aq^j) & \text{if } k>0, \\ 1 & \text{if } k=0, \\ \prod_{j=1}^{|k|}(1-aq^{-j})^{-1} & \text{if } k<0, \\ \prod_{j=0}^{\infty}(1-aq^j) & \text{if } k=\infty. \end{cases}$$

Using the q-Pochhammer symbol, the identities (2.1) and (2.2) can be rewritten in compact product form as:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}, \tag{2.3}$$

and the second,

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{2.4}$$

We also make use of the Ramanujan theta function, defined as:

$$f(a,b):=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}. \tag{2.5}$$

This function plays a key role in the analytic study of q-series and modular forms and appears frequently throughout the literature on the Rogers-Ramanujan identities.

2.3. Interpretations

2.3.1. Analytic: q-series

One of the most remarkable aspects of the Rogers-Ramanujan identities is their connection between an infinite sum and an infinite product. This duality is rare and intriguing, especially in analytic number theory.

In analytic contexts, the Rogers-Ramanujan identities are typically expressed as:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}, \tag{2.6}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+2})(1-q^{5k+3})}. \tag{2.7}$$

The sum sides of these identities can be interpreted as basic hypergeometric series, while the product sides are examples of Euler-type infinite products.

Rogers (and later others) discovered many series-product identities similar in form to the Rogers-Ramanujan identities, and as such are referred to as “identities of the Rogers-Ramanujan type” (Sills, 2005). Ramanujan recorded several Rogers-Ramanujan type identities in his Lost Notebook (Andrews and Berndt, 2005).

A breakthrough came during World War II when Bailey developed a systematic method to link q-hypergeometric series with Euler-type products in a pair of influential papers (Bailey, 1946)(Bailey, 1948). Bailey and Slater continued this work by classifying numerous single-sum Rogers-Ramanujan type identities. However, it was Andrews who extended the theory significantly by introducing elegant examples of multisum identities. His development of the Bailey chain method enabled the construction of entire infinite families of Rogers-Ramanujan type identities (Andrews, 1984a).

Today, research continues into new Bailey pairs and generalizations of Bailey chains. Many new identities discovered in this framework are still derivable from Bailey’s original principles.

2.3.2. The Hard Hexagon Model

The Hard Hexagon Model is a prominent example of an exactly solvable model in statistical mechanics. Introduced and solved by Rodney Baxter in 1980 (Baxter, 1990), it describes the behavior of particles on a two-dimensional hexagonal lattice, subject to strict exclusion rules. What makes this model extraordinary is its deep and unexpected connection to the Rogers-Ramanujan identities.

In solving the model, Baxter discovered that the partition function, an essential object in statistical mechanics, can be expressed in terms of q-series that match the Rogers-Ramanujan identities. This surprising appearance of combinatorial identities in a physical context provided a profound link between enumerative combinatorics, q-series, and phase transitions in physical systems. Baxter’s work not only earned him the 1980 Boltzmann Medal but also stimulated a vast body of research bridging physics and pure mathematics.

How the Model Works

The hard hexagon model considers a triangular (hexagonally tiled) lattice, where each site can be either occupied by a particle or left empty. The key constraint is a nearest-neighbor exclusion rule: no two adjacent sites may both be occupied. This mimics the behavior of hard-core particles, such as molecules that physically cannot overlap, as shown in Figure 2.1 The model introduces a parameter called fugacity (denoted z), which controls the probability of particle placement.

As z increases, the system transitions from a disordered phase (low density, approximately uniform) to an ordered phase (where particles prefer certain sublattices). This sharp transition at a critical value $z = z_c$ is an example of a phase transition, a fundamental concept in physics.

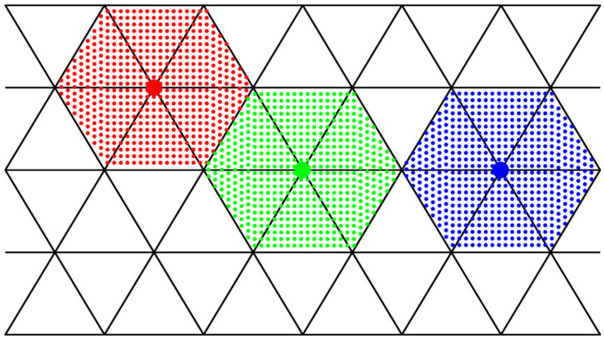

Figure 2.1: Particles on the triangular lattice. No two particles can be together or adjacent. Adjacent faces have been shaded to show how they form hard (i.e., non-overlapping) hexagons.

Baxter's Exact Solution and the Role of the Identities

Baxter solved the hard hexagon model exactly using the corner transfer matrix method, a powerful technique from integrable systems. His solution involved computing several thermodynamic quantities:

The average particle density ρ, the free energy per site κ, the density difference $R = \rho_1 - \rho_2$, where ρ_i refers to occupation density on specific sub-lattices.

He showed that these quantities could be expressed using infinite products and q-series. For certain regimes of fugacity, these expressions reduce to forms involving functions like:

$$G(x) = \prod_{n\geq 1} \frac{1}{(1-x^{5n-4})(1-x^{5n-1})}, \quad H(x) = \prod_{n\geq 1} \frac{1}{(1-x^{5n-3})(1-x^{5n-2})},$$

which are precisely the functions appearing in the Rogers-Ramanujan identities.

Baxter's solution revealed that deep combinatorial structures underlie physical systems. The fact that the Rogers-Ramanujan identities, originally discovered in a purely mathematical context, govern the behavior of a real physical model, illustrates their profound universality. Baxter's book *Exactly Solved Models in Statistical Mechanics* (Baxter, 1990), in which the solution is detailed, has become a classic, cited thousands of times across physics and mathematics.

This connection not only enriches our understanding of the Rogers-Ramanujan identities but also exemplifies the power of combinatorics and q-series methods in modeling real-world phenomena.

2.3.3. Other Interpretations

The Rogers-Ramanujan identities have found remarkable interpretations across various areas of mathematics and physics, far beyond their original analytic formulation.

One of the most striking developments occurred in the late 1970s, when Lepowsky, motivated by their work in the representation theory of Lie algebras, encountered Andrews' recent book *The Theory of Partitions* (Andrews, 1984b).

Lepowsky, together with Wilson, succeeded in giving the first Lie-theoretic (more precisely, vertex operator-theoretic) proofs of the identities. They showed that the sum sides of the two identities correspond in a precise way to the characters of the two inequivalent level 3 standard modules for the affine Kac-Moody Lie algebra $A_1^{(1)}$. This established a powerful bridge between partition identities and the representation theory of infinite-dimensional Lie algebras.

Encouraged by this breakthrough, they initiated further work on other affine Lie algebras. Their student, Capparelli, undertook the more complex case of $A_2^{(1)}$, which led to the discovery of new, deeper partition identities in the same spirit as the Rogers-Ramanujan identities.

Beyond Lie theory, the product sides of the identities are closely related to modular forms. The infinite products appearing in the identities can be interpreted as q-expansions of modular functions for congruence subgroups. This modular interpretation situates the identities within the arithmetic geometry of modular curves and the theory of theta functions. Setting $q = e^{2\pi i\tau}$, with $\mathrm{Im}(\tau) > 0$, it turns out that

$$q^{-1/60}G(q) \quad \text{and} \quad q^{11/60}H(q)$$

where $G(q)$ and $H(q)$ are the product sides of the first and second Rogers-Ramanujan, respectively, are modular functions of τ. This connection has revealed that the Rogers-Ramanujan identities reflect modular transformations in disguise.

Moreover, these identities appear in a variety of other mathematical contexts, such as knot theory (Armond and Dasbach, 2011) and group theory (Fulman, 1997).

2.4. The Jacobi Triple Product Identity

Among the most celebrated results in the theory of q-series and theta functions is the Jacobi Triple Product Identity. Discovered by Carl Gustav Jacob Jacobi, this identity reveals a remarkable equivalence between an infinite sum and an infinite product. It states that:

Identity 2.3 (The Jacobi Triple Product Identity). If $z \in \mathbb{C} \setminus \{0\}$, then

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + zq^{2n-1})(1 + z^{-1}q^{2n-1}).$$

This identity can be rewritten in such a way that we obtain the following:

Identity 2.4 (The Jacobi Triple Product Identity rewritten). If $a, b \in \mathbb{C} \setminus \{0\}$, then

$$f(a,b) = (-a;ab)_\infty\,(-b;ab)_\infty\,(ab;ab)_\infty,$$

where $f(a,b)$ is the Ramanujan theta function (2.5).

This identity plays a fundamental role in the development of partition theory, modular forms, and the theory of mock theta functions. Almost every known proof of the Rogers-Ramanujan identities depends directly on the Jacobi Triple Product Identity or one of its generalizations (Soodak, 2018. Its ability to transform q series into infinite product expressions is crucial for revealing the underlying structure of these identities, making it an indispensable tool in their derivation and analysis.

This section is dedicated to proving the Jacobi Triple Product Identity. This proof will not only enhance our understanding of this identity, but will also introduce calculation techniques that will be used more in Sections 3 and beyond, such as calculations involving the q-Pochhammer symbol or infinite sums.

Firstly, the q-binomial theorem will be presented. Using different substitutions in this theorem, 3 corollaries will be obtained. These corollaries are essential tools for the proof of the Jacobi triple product identity. Finally, we show that Identity 2.3 is equivalent to Identity 2.4.

2.4.1. The q-binomial theorem

In this section, the q-binomial theorem will be proved.

Lemma 2.4.1 (The q-binomial theorem)**.** *If* $|t|, |q| < 1$, *we have the following result,*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} t^n = \frac{(at;q)_\infty}{(t;q)_\infty}.$$

Proof. Note that proving this lemma is equivalent to showing that

$$1 + \sum_{n=1}^{\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} t^n = \prod_{n=0}^{\infty} \frac{1-atq^n}{1-tq^n}.$$

First, define the function

$$F(t) := \prod_{n=0}^{\infty} \frac{1-atq^n}{1-tq^n}.$$

Next, consider

$$(1-t)F(t) = (1-t)\prod_{n=0}^{\infty} \frac{1-atq^n}{1-tq^n}.$$

Separating the first term of the product, we get

$$(1-t)F(t) = \frac{(1-t)(1-at)}{(1-t)} \prod_{n=1}^{\infty} \frac{1-atq^n}{1-tq^n} = (1-at)\prod_{n=1}^{\infty} \frac{1-atq^n}{1-tq^n}.$$

Re-indexing the infinite product,

$$(1-at)\prod_{n=1}^{\infty} \frac{1-atq^n}{1-tq^n} = (1-at)\prod_{n=0}^{\infty} \frac{1-a(tq)q^n}{1-(tq)q^n} = (1-at)F(tq).$$

Thus, the functional equation

$$(1-t)F(t) = (1-at)F(tq)$$

holds.

Now, we will try to to write $F(t)$ as a power series expansion. In other words, we try to find the values of A_n in the following expression:

$$F(t) = \sum_{n=0}^{\infty} A_n t^n.$$

Note that $F(0) = 1$, so $A_0 = 1$.

Substituting the power series expansion in the functional equation, we find

$$(1-t)\sum_{n=0}^{\infty} A_n t^n = (1-at)\sum_{n=0}^{\infty} A_n (tq)^n = (1-at)\sum_{n=0}^{\infty} A_n q^n t^n.$$

Expanding and comparing coefficients yields

$$\sum_{n=0}^{\infty} A_n t^n - \sum_{n=0}^{\infty} A_n t^{n+1} = \sum_{n=0}^{\infty} A_n q^n t^n - \sum_{n=0}^{\infty} A_n a q^n t^{n+1}.$$

Rearranging,

$$\sum_{n=0}^{\infty} A_n(1-q^n)t^n = \sum_{n=0}^{\infty} A_n(1-aq^n)t^{n+1}.$$

Note that the term $n=0$ of the infinite sum on the left-hand side does not have a contribution. Shifting the index on the right side to align powers of t,

$$\sum_{n=1}^{\infty} A_n(1-q^n)t^n = \sum_{n=1}^{\infty} A_{n-1}(1-aq^{n-1})t^n.$$

Equating coefficients of t^n for $n \geq 1$ gives the recurrence

$$A_n(1-q^n) = A_{n-1}(1-aq^{n-1}),$$

with initial condition $A_0 = 1$.

Solving this recurrence,

$$A_n = A_{n-1}\frac{(1-aq^{n-1})}{(1-q^n)} = A_{n-2}\frac{(1-aq^{n-1})(1-aq^{n-2})}{(1-q^n)(1-q^{n-1})}$$

$$= \cdots = A_0\prod_{k=1}^{n}\frac{1-aq^{k-1}}{1-q^k} = \frac{(a;q)_n}{(q;q)_n}.$$

Hence,

$$F(t) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}t^n,$$

which concludes the proof. □

2.4.2. Corollaries

Corollary 2.4.2. *This corollary is the special case of the q-binomial theorem with $a=0$, resulting in the identity*

$$\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty}.$$

Proof. The result follows immediately by setting $a=0$ in Lemma 2.4.1. □

Corollary 2.4.3. *The following two identities arise from specific substitutions in the q-binomial theorem:*

(a)

$$(-zq;q^2)_\infty = \sum_{m=0}^{\infty}\frac{z^m q^{m^2}}{(q^2;q^2)_m},$$

(b)

$$(q^{2m+2};q^2)_\infty = \sum_{r=0}^{\infty}\frac{(-1)^r q^{r^2+2rm+r}}{(q^2;q^2)_r}.$$

Proof. Starting from Lemma 2.4.1, replace a with $\frac{a}{b}$ and t with bt,

$$1+\sum_{n=1}^{\infty}\frac{(1-\frac{a}{b})(1-\frac{a}{b}q)\cdots(1-\frac{a}{b}q^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)}b^nt^n=\prod_{n=0}^{\infty}\frac{1-\frac{a}{b}btq^n}{1-btq^n}.$$

Simplifying,

$$1+\sum_{n=1}^{\infty}\frac{(b-a)(b-aq)\cdots(b-aq^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)}t^n=\prod_{n=0}^{\infty}\frac{1-atq^n}{1-btq^n}.$$

Now, set $b=0$ and $a=-1$:

$$1+\sum_{n=0}^{\infty}\frac{1\cdot q\cdot q^2\cdots q^{n-1}}{(1-q)(1-q^2)\cdots(1-q^n)}t^n=\prod_{n=0}^{\infty}(1+tq^n).$$

Since

$$1+2+3+\cdots+(n-1)=\frac{n(n-1)}{2},$$

this becomes

$$1+\sum_{n=1}^{\infty}\frac{q^{\frac{n(n-1)}{2}}}{(q)_n}t^n=\prod_{n=0}^{\infty}(1+tq^n).$$

Hence,

$$\sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}}{(q)_n}t^n=\prod_{n=0}^{\infty}(1+tq^n). \tag{2.8}$$

Performing substitutions in (2.8) yields the two identities.

To obtain the first identity, replace q by q^2 and t by zq:

$$\prod_{n=0}^{\infty}\left(1+(zq)q^{2n}\right)=\sum_{m=0}^{\infty}\frac{z^mq^mq^{m^2-m}}{(q^2;q^2)_m}.$$

Rewriting,

$$(-zq;q^2)_\infty=\sum_{m=0}^{\infty}\frac{z^mq^{m^2}}{(q^2;q^2)_m}.$$

To obtain the second identity, replace q by q^2 and t by $-q^{2m+2}$:

$$\prod_{n=0}^{\infty}\left(1-q^{2m+2}q^{2n}\right)=\sum_{r=0}^{\infty}\frac{(-q^{2m+2})^rq^{r(r-1)}}{(q^2;q^2)_r}.$$

Simplifying,

$$\prod_{n=0}^{\infty}\left(1-q^{2n+2m+2}\right)=\sum_{r=0}^{\infty}\frac{(-1)^rq^{r^2-r+2mr+2r}}{(q^2;q^2)_r},$$

$$(q^{2m+2};q^2)_\infty=\sum_{r=0}^{\infty}\frac{(-1)^rq^{r^2+2rm+r}}{(q^2;q^2)_r}.$$ □

Corollary 2.4.4. *This corollary is the special case of Lemma 2.4.2 resulting in the identity:*

$$\sum_{r=0}^{\infty}\frac{(-qz^{-1})^r}{(q^2;q^2)_r}=\frac{1}{(-qz^{-1};q^2)_\infty}.$$

Proof. The result follows by replacing q by q^2 and then t by $-qz^{-1}$ in Lemma 2.4.2. □

2.4.3. Proof of the Jacobi Triple Product Identity

In this section, firstly, the proof of the Jacobi Triple Product Identity as shown in 2.3 will be presented. Using Corollaries 2.4.2, 2.4.3, 2.4.4, and some calculations, we obtain the identity. Subsequently, we will show that Identity 2.3 is equivalent to Identity 2.4.

Proof. Starting from Corollary 2.4.3a, we have

$$\begin{aligned}\prod_{n=0}^{\infty}\left(1+zq^{2n+1}\right) &= \sum_{m=0}^{\infty}\frac{z^m q^{m^2}}{(q^2;q^2)_m}\\ &= \sum_{m=0}^{\infty}\frac{z^m q^{m^2}(q^{2m+2};q^2)_\infty}{(q^2;q^2)_\infty},\end{aligned}$$

using the fact that the following is equal:

$$\frac{1}{(q^2;q^2)_m} = \frac{(q^{2m+2};q^2)_\infty}{(q^2;q^2)_\infty}.$$

Note that $(q^{2m+2};q^2)_\infty = 0$ for $m < 0$, so we can extend the sum over all integers m:

$$\prod_{n=0}^{\infty}\left(1+zq^{2n+1}\right) = \frac{1}{(q^2;q^2)_\infty}\sum_{m=-\infty}^{\infty} z^m q^{m^2}(q^{2m+2};q^2)_\infty.$$

By Corollary 2.4.3b, we substitute the expansion of $(q^{2m+2};q^2)_\infty$:

$$\begin{aligned}\prod_{n=0}^{\infty}\left(1+zq^{2n+1}\right) &= \frac{1}{(q^2;q^2)_\infty}\sum_{m=-\infty}^{\infty} z^m q^{m^2}\sum_{r=0}^{\infty}\frac{(-1)^r q^{r^2+2mr+r}}{(q^2;q^2)_r}\\ &= \frac{1}{(q^2;q^2)_\infty}\sum_{m=-\infty}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r\, q^{r^2+2mr+m^2}\, z^{m+r}\, q^r\, z^{-r}}{(q^2;q^2)_r}.\end{aligned}$$

Rewrite the double sum by factoring terms and re-indexing:

$$\begin{aligned}\prod_{n=0}^{\infty}\left(1+zq^{2n+1}\right) &= \frac{1}{(q^2;q^2)_\infty}\sum_{r=0}^{\infty}\frac{(-1)^r z^{-r} q^r}{(q^2;q^2)_r}\sum_{m=-\infty}^{\infty} q^{(m+r)^2} z^{m+r}\\ &= \frac{1}{(q^2;q^2)_\infty}\left(\sum_{r=0}^{\infty}\frac{(-1)^r z^{-r} q^r}{(q^2;q^2)_r}\right)\left(\sum_{m=-\infty}^{\infty} q^{m^2} z^m\right).\end{aligned}$$

In the last step, we re-indexed m.

By Corollary 2.4.4 the first sum simplifies to

$$\sum_{r=0}^{\infty}\frac{(-1)^r z^{-r} q^r}{(q^2;q^2)_r} = \frac{1}{(-qz^{-1};q^2)_\infty}.$$

Therefore,

$$\prod_{n=0}^{\infty}\left(1+zq^{2n+1}\right) = \frac{1}{(q^2;q^2)_\infty}\cdot\frac{1}{(-qz^{-1};q^2)_\infty}\sum_{m=-\infty}^{\infty} q^{m^2} z^m,$$

which can be rearranged to

$$\sum_{m=-\infty}^{\infty} q^{m^2} z^m = \prod_{n=0}^{\infty} (1 + zq^{2n+1})(q^2; q^2)_\infty (-qz^{-1}; q^2)_\infty.$$

Expressing the infinite products in standard form,

$$\sum_{m=-\infty}^{\infty} q^{m^2} z^m = \prod_{n=0}^{\infty} (1 + zq^{2n+1})(1 - q^{2n+2})(1 + z^{-1} q^{2n+1}),$$

and re-indexing $n \to n - 1$ gives

$$\sum_{m=-\infty}^{\infty} q^{m^2} z^m = \prod_{n=1}^{\infty} (1 + zq^{2n-1})(1 - q^{2n})(1 + z^{-1} q^{2n-1}),$$

which is the Jacobi Triple Product Identity as shown in Identity 2.3. □

We now prove the Jacobi Triple Product Identity, as shown in Identity 2.4, using Identity 2.3.

Proof. Recall Identity 2.3:

$$\sum_{m=-\infty}^{\infty} q^{m^2} z^m = \prod_{n=1}^{\infty} (1 + zq^{2n-1})(1 - q^{2n})(1 + z^{-1} q^{2n-1}).$$

Substitute $z = (ab^{-1})^{\frac{1}{2}}$ and $q = (ab)^{\frac{1}{2}}$, into the identity:

$$\sum_{m=-\infty}^{\infty} ((ab)^{\frac{1}{2}})^{m^2} ((ab^{-1})^{\frac{1}{2}})^m =$$

$$\prod_{n=1}^{\infty} (1 + ((ab^{-1})^{\frac{1}{2}})((ab)^{\frac{1}{2}})^{2n+1})(1 - ((ab)^{\frac{1}{2}})^{2n+2})(1 + ((ab^{-1})^{\frac{1}{2}})^{-1}((ab)^{\frac{1}{2}})^{2n+1})$$

Rewriting the left-hand side where $f(a; b)$ is the Ramanujan theta function (2.5).

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} ((ab)^{\frac{1}{2}})^{m^2} ((ab^{-1})^{\frac{1}{2}})^m &= \sum_{m=-\infty}^{\infty} (ab)^{\frac{m^2}{2}} (ab^{-1})^{\frac{m}{2}} \\
&= \sum_{m=-\infty}^{\infty} a^{\frac{m^2}{2}} \cdot b^{\frac{m^2}{2}} \cdot a^{\frac{m}{2}} \cdot b^{\frac{-m}{2}} \\
&= \sum_{m=-\infty}^{\infty} a^{\frac{m(m+1)}{2}} b^{\frac{m(m-1)}{2}} \\
&= f(a; b).
\end{aligned}$$

Now, let's rewrite the right-hand side:

$$\begin{aligned}
&\prod_{n=1}^{\infty} (1 + ((ab^{-1})^{\frac{1}{2}})((ab)^{\frac{1}{2}})^{2n-1}) \cdot (1 - ((ab)^{\frac{1}{2}})^{2n}) \cdot (1 + ((ab^{-1})^{\frac{1}{2}})^{-1}((ab)^{\frac{1}{2}})^{2n-1}) \\
&= \prod_{n=1}^{\infty} (1 + a^n b^{n-1})(1 - a^{n-1} b^n)(1 + (ab)^n) \\
&= (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty
\end{aligned}$$

Combining these results, we conclude that:

$$f(a;b) = (-a;ab)_\infty\,(-b;ab)_\infty\,(ab;ab)_\infty.$$

This completes the proof. □

3

Proof Ramanujan

In this chapter, we present the proof of the Rogers-Ramanujan identities (2.1) and (2.2), as discovered by Ramanujan himself. This proof is copied in ”An Invitation to the Rogers-Ramanujan Identities” (Sills, 2017). However, this proof contains some minor errors. We corrected these using insights from the original proof (Aiyangar, 2000).

We begin by establishing a series of lemmas. The first four concern the Ramanujan theta function and an auxiliary function $G(z)$, which will be defined below. We then show that $G(z)$ satisfies a certain q-difference equation. Using this, we define a new function $H(z)$, and derive two representations of it: one using its definition, and one via its power series expansion. Evaluating $H(z)$ at specific values of z will lead us directly to the Rogers-Ramanujan identities.

3.1. The Product Forms via the Theta Function

Lemma 3.1.1. *Let $f(-q^2; -q^3)$ be the Ramanujan theta function, then the product side of the first Rogers-Ramanujan identities can be rewritten as follows,*

$$\frac{f(-q^2; -q^3)}{(q;q)_\infty} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}. \tag{3.1}$$

Proof. Applying Jacobi's triple product identity, (2.4),

$$f(a;b) = (-a;ab)_\infty\,(-b;ab)_\infty\,(ab;ab)_\infty.$$

Evaluating the left-hand side of Identity 3.1, we find:

$$\begin{aligned}
\frac{f(-q^2;-q^3)}{(q;q)_\infty} &= \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty}{(q;q)_\infty} \\
&= \frac{\prod_{k=0}^{\infty}(1-q^{5k+2}) \prod_{k=0}^{\infty}(1-q^{5k+3}) \prod_{k=1}^{\infty}(1-q^{5k})}{\prod_{k=1}^{\infty}(1-q^k)} \\
&= \prod_{k=1}^{\infty} \frac{(1-q^{5k-3})(1-q^{5k-2})(1-q^{5k})}{(1-q^k)} \\
&= \prod_{k=1}^{\infty} \frac{1}{(1-q^{5k-4})(1-q^{5k-1})} \\
&= \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}. \qquad \square
\end{aligned}$$

Lemma 3.1.2. *Let $f(-q;-q^4)$ be the Ramanujan theta function, then the product side of the second Rogers-Ramanujan identities can be rewritten as follows,*

$$\frac{f(-q;-q^4)}{(q;q)_\infty} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+2})(1-q^{5k+3})}. \tag{3.2}$$

Proof. Similarly as the previous proof, for Identity 3.2, we have

$$\begin{aligned}
\frac{f(-q;-q^4)}{(q;q)_\infty} &= \frac{(q;q^5)_\infty (q^4;q^5)_\infty (q^5;q^5)_\infty}{(q;q)_\infty} \\
&= \frac{\prod_{k=0}^{\infty}(1-q^{5k+1}) \prod_{k=0}^{\infty}(1-q^{5k+4}) \prod_{k=1}^{\infty}(1-q^{5k})}{\prod_{k=1}^{\infty}(1-q^k)} \\
&= \prod_{k=1}^{\infty} \frac{(1-q^{5k-4})(1-q^{5k-1})(1-q^{5k})}{(1-q^k)} \\
&= \prod_{k=1}^{\infty} \frac{1}{(1-q^{5k-4})(1-q^{5k-1})} \\
&= \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+2})(1-q^{5k+3})}. \qquad \square
\end{aligned}$$

3.2. The Function $G(z)$ and its Properties

For $z \in \mathbb{C}$, define the function

$$G(z) := G(z;q) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m} q^{\frac{m(5m-1)}{2}} (1-zq^{2m})(z;q)_m}{(1-z)(q;q)_m}.$$

In this section, the following identities will be proved,

$$G(1) = f(-q^2; -q^3) \quad \text{and} \quad (1-q)G(q) = f(-q; -q^4),$$

Furthermore, it will be shown that $G(z)$ satisfies the q-difference equation as stated in Lemma 3.2.1.

Identity 3.1. The following is true:

$$G(1) = f(-q^2; -q^3).$$

Proof. Setting $z = 1$ in $G(z)$ gives us the following equation:

$$G(1) = 1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{m(5m-1)}{2}} (1 - q^{2m}) \frac{(q;q)_{m-1}}{(q;q)_m}.$$

Using the fact that $(1 - q^{2m}) = (1 + q^m)(1 - q^m)$ we have,

$$G(1) = 1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{m(5m-1)}{2}} (1 + q^m).$$

Splitting $(1 + q^m)$ results into two infinite sums, and re-indexing $m \to -m$ the sum corresponding to q^m, gives us:

$$G(1) = 1 + \sum_{m=1}^{\infty} (-1)^m q^{\frac{m(5m-1)}{2}} + \sum_{m=-\infty}^{-1} (-1)^m q^{\frac{m(5m-1)}{2}}.$$

Finally, combining the sums,

$$G(1) = \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{m(5m-1)}{2}}.$$

Note that this infinite sum exactly equals $f(-q^2; -q^3)$. □

Identity 3.2. The following is true:

$$(1-q)G(q) = f(-q; -q^4).$$

Proof. Writing $(1-q)G(q)$ using the definition and splitting the $m = 0$-term, gives

$$(1-q)G(q) = 1 - q + \sum_{m=1}^{\infty} (1 - q^{2m+1}) q^{m(5m+3)/2} (-1)^m.$$

Splitting $(1 - q^{2m+1})$, results in two infinite sums,

$$(1-q)G(q) = 1 - q + \sum_{m=1}^{\infty} q^{m(5m+3)/2} (-1)^m - q \sum_{m=1}^{\infty} q^{m(5m+7)/2} (-1)^m.$$

Re-indexing $m \to -m - 1$ in the second infinite sum,

$$(1-q)G(q) = 1 - q + \sum_{m=1}^{\infty} q^{m(5m+3)/2} (-1)^m + \sum_{m=-\infty}^{-2} q^{m(5m+3)/2} (-1)^m,$$

combining the infinite sums,

$$(1-q)G(q) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(5m+3)/2} = f(-q; -q^4).$$ □

Lemma 3.2.1. *$G(z)$ satisfies the following q-difference equation:*

$$G(z) = (1 - zq)G(zq) + zq(1 - zq)(1 - zq^2)G(zq^2).$$

Proof. Note that proving the lemma is equivalent to proving the following,

$$\frac{G(z)}{(1 - zq)} - G(zq) = zq(1 - zq^2)G(zq^2).$$

Express $G(zq)$ as:

$$\begin{aligned} G(zq) &= \sum_{r=0}^{\infty} (-1)^r (zq)^{2r} q^{\frac{1}{2}r(5r-1)} (1 - zq^{2r+1}) \frac{(zq;q)_r}{(1 - zq)(q;q)_r} \\ &= \sum_{r=0}^{\infty} (-1)^r z^{2r} q^{\frac{1}{2}r(5r+1)} \frac{(zq;q)_r}{(1 - zq)(q;q)_r} \cdot q^r (1 - zq^{2r+1}). \end{aligned}$$

Rewrite $G(z)$ as:

$$G(z) = \sum_{r=0}^{\infty} (-1)^r z^{2r} q^{\frac{1}{2}r(5r+1)} \frac{(z;q)_{r+1}}{(1 - z)(q;q)_r} \cdot (1 - z^2 q^{4r+2}).$$

(The detailed computations are provided in the Appendix (A.1))

Divide $G(z)$ by $(1 - zq)$:

$$\frac{G(z)}{(1 - zq)} = \sum_{r=0}^{\infty} (-1)^r z^{2r} q^{\frac{1}{2}r(5r+1)} \frac{(zq;q)_r}{(1 - zq)(q;q)_r} \cdot (1 - z^2 q^{4r+2}).$$

Subtract $G(zq)$, to obtain:

$$\begin{aligned} \frac{G(z)}{(1 - zq)} - G(zq) &= \sum_{r=0}^{\infty} (-1)^r z^{2r} q^{\frac{1}{2}r(5r+1)} \frac{(zq;q)_r}{(1 - zq)(q;q)_r} \cdot ((1 - z^2 q^{4r+2}) - q^r (1 - zq^{2r+1})) \\ &= zq(1 - zq^2) \sum_{r=0}^{\infty} (-1)^r (zq^2)^{2r} q^{\frac{1}{2}r(5r-1)} (1 - zq^{2r+2}) \frac{(zq^2;q)_r}{(1 - zq^2)(q;q)_r} \\ &= zq(1 - zq^2)G(zq^2). \end{aligned}$$

(The detailed computations are provided in the Appendix (A.2).) □

3.3. The Function $H(z)$ and the Power Series Identity

Define

$$H(z) := H(z;q) = \frac{G(z)}{(zq;q)_\infty}. \tag{3.3}$$

Then the q-difference equation for $G(z)$ translates to:

Lemma 3.3.1. *The q-difference equation for $H(q)$ is*

$$H(z) = H(zq) + zqH(zq^2).$$

Proof. Starting from the q-difference equation of $G(z)$ as shown in Lemma 3.2.1 , and dividing it by $(zq;q)_\infty$, we have the following:

$$\frac{G(z)}{(zq;q)_\infty} = \frac{(1-zq)G(zq)}{(zq;q)_\infty} + zq\frac{(1-zq)(1-zq^2)G(zq^2)}{(zq;q)_\infty}.$$

Simplifying,

$$\frac{G(z)}{(zq;q)_\infty} = \frac{G(zq)}{(zq^2;q)_\infty} + zq\frac{G(zq^2)}{(zq^3;q)_\infty}.$$

By definition, this is equivalent to

$$H(z) = H(zq) + zqH(zq^2).$$ □

Lemma 3.3.2. *$H(z)$ can be rewritten using a power series expansion as follows,*

$$H(z) = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n}.$$

Proof. Assume the power series expansion:

$$H(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we seek the power series coefficients a_n of $H(z)$.

Recall that in power series, the following is true: $0^0 = 1$. So in this case, we see that $H(0) = \sum_{n=0}^{\infty} a_n 0^n$ only has a contribution if $n = 0$. Since

$$H(0) = \frac{G(0)}{(0\cdot q;q)} = 1,$$

we know that $a_0 = 1$.

Using the recurrence from the functional equation:

$$\begin{aligned} H(z) &= H(zq) + zqH(zq^2) \\ \sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{\infty} a_n q^n z^n + zq\sum_{n=0}^{\infty} a_n q^{2n} z^n \\ &= \sum_{n=0}^{\infty} a_n q^n z^n + \sum_{n=0}^{\infty} a_n q^{2n+1} z^{n+1} \\ &= \sum_{n=0}^{\infty} a_n q^n z^n + \sum_{n=1}^{\infty} a_{n-1} q^{2n-1} z^n. \end{aligned}$$

Comparing coefficients, for $n \geq 1$, gives the recurrence

$$a_n = a_n q^n + a_{n-1}q^{2n-1} \quad \Rightarrow \quad (1-q^n)a_n = q^{2n-1}a_{n-1}.$$

Solving recursively:

$$a_n = \frac{q^{2n-1}}{(1-q^n)}a_{n-1} = \frac{q^{2n-1}}{(1-q^n)}\frac{q^{2n-3}}{(1-q^{n-1})}a_{n-2} = \cdots = \frac{q^{(2n-1)+(2n-3)+\cdots+1}}{(q;q)_n}a_0 = \frac{q^{n^2}}{(q;q)_n},$$

since $(2n-1)+(2n-3)+\cdots+1 = n^2$.

Thus,

$$H(z) = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n}.$$ □

3.4. Deriving the Rogers-Ramanujan Identities

Finally, we prove the Rogers-Ramanujan Identities as stated in (2.1) and (2.2).

Proof. Recall that, using the definition of $H(z)$ from Equation (3.3) and Lemma 3.3.2, the function $H(z)$ can be expressed in two different forms:

$$\frac{G(z)}{(zq;q)_\infty} = H(z) = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q;q)_n}.$$

Evaluating at $z = 1$ and $z = q$ yield the first identity,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = H(1) = \frac{G(1)}{(q;q)_\infty} = \frac{f(-q^2,-q^3)}{(q;q)_\infty},$$

and the second identity:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = H(q) = \frac{G(q)}{(q^2;q)_\infty} = \frac{f(-q,-q^4)}{(1-q)(q^2;q)_\infty} = \frac{f(-q,-q^4)}{(q;q)_\infty}.$$

These are precisely the Rogers-Ramanujan identities. □

4

Combinatorial Interpretations

4.1. Definitions

In this section, we introduce the necessary terminology and structures used throughout this chapter and later chapters. We begin by defining integer partitions, their graphical representations via Young diagrams, and their conjugates. Building on this, we introduce Durfee rectangles and describe how they divide partitions into structured subcomponents, denoted α, β, and γ, which are crucial in defining the $(2, m)$-rank of a partition. We then formalize several important sets of partitions, including the full set of partitions of an integer n, the subset of Rogers-Ramanujan partitions, and sets defined by fixed rank values. Finally, we describe classes of partitions with specific constraints on their parts, such as difference conditions or congruence relations modulo 5, which will play a central role in later combinatorial interpretations of the Rogers-Ramanujan identities.

4.1.1. Partitions

Definition 4.1.1 (Integer Partition)**.** *A partition λ of a positive integer n is a finite sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ of positive integers, where*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0,$$

and

$$n = \sum_{i=1}^{\ell(\lambda)} \lambda_i.$$

We define $\lambda_1, \lambda_2 \ldots, \lambda_{\ell(\lambda)}$ as the parts of λ, and $\ell(\lambda)$ as the number of parts of λ or the length of λ. By convention, define $\lambda_j = 0$ for all $j > \ell(\lambda)$. We denote by $e(\lambda)$ the smallest part, so $e(\lambda) = \lambda_{\ell(\lambda)}$.

The partition λ is represented graphically by its Young diagram $[\lambda]$ as illustrated in Figure 4.1. The Young diagram of λ consists of a total of n blocks in $\ell(\lambda)$ left-justified rows, and the number of blocks in the row i is equal to λ_i.

Definition 4.1.2 (The conjugate partition)**.** *The conjugate partition λ' is obtained by reflecting $[\lambda]$ along the main diagonal. Thus, by interchanging the rows and columns of the Young diagram of λ (see Figure 4.1).*

Mathematically, this means

$$\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1}),$$

where

$$\lambda'_i = \#\{j : \lambda_j \geq i\}.$$

λ λ'

Figure 4.1: Partition $\lambda = (5, 5, 4, 1)$ and conjugate partition $\lambda' = (4, 3, 3, 3, 2)$

Definition 4.1.3 (First m-Durfee rectangle)**.** *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ be a partition and m be a non-negative integer. The first m-Durfee rectangle is defined as the rectangle with height $s_m(\lambda)$, and width $s_m(\lambda) - m$, where*

$$s_m(\lambda) = \max\{n \in \mathbb{N} \mid n - m \leq \lambda_n\}.$$

Graphically, an m-Durfee rectangle is the rectangle of maximal area that fits in the upper left corner of the Young diagram $[\lambda]$, such that its height minus its width equals m. Width 0 is allowed, but height 0 is not.

Definition 4.1.4 (Second m-Durfee rectangle)**.** *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ be a partition and m a nonnegative integer. The second m-Durfee rectangle is defined as the rectangle with height $t_m(\lambda)$, and width $t_m(\lambda) - m$, where*

$$t_m(\lambda) = \max\{n \in \mathbb{N} \mid n - m \leq \lambda_{s_m(\lambda)+n}\}.$$

Graphically, it is the rectangle of maximal area that fits in the diagram $[\lambda]$ below the first m-Durfee rectangle, whose height minus its width is m. Thus, the first m-Durfee rectangle of the partition $(\lambda_{s_m(\lambda)}, \dots, \lambda_{\ell(\lambda)})$. Width 0 is allowed, but height 0 is not.

Definition 4.1.5 (Partitions α, β, and γ)**.** *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ be a partition and m a nonnegative integer. For $i \in \{1, \dots, s_m(\lambda)\}$, define the partition α as,*

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{s_m(\lambda)}) \quad \text{where } \alpha_i = \lambda_i - s_m(\lambda) + m.$$

For $j \in \{1, \dots, s_m(\lambda)\}$, define the partition β as,

$$\beta = (\beta_1, \beta_2, \dots, \beta_{t_m(\lambda)}) \quad \text{where } \beta_j = \lambda_{j+s_m(\lambda)} - t_m(\lambda) + m.$$

For $k \in \{1, \dots, \ell(\lambda) - s_m(\lambda) - t_m(\lambda)\}$, define the partition γ as,

$$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_{\ell(\lambda)-s_m(\lambda)-t_m(\lambda)}) \quad \text{where } \gamma_k = \lambda_{k+s_m(\lambda)+t_m(\lambda)}.$$

Graphically, α, β and γ are the three partitions located to the right, in the middle of and below the first and second m-Durfee rectangle, respectively (see Figures 4.2 and 4.3). If $m > 0$ and we have an m-Durfee rectangle with width 0, as shown in Figure 3, then γ must be the empty partition.

Definition 4.1.6 ((2,m)-rank)**.** *For a nonnegative integer m, we define the $(2, m)$-rank, $r_{2,m}(\lambda)$, of a partition λ by the formula:*

$$r_{2,m}(\lambda) := \beta_1 + \alpha_{s_m(\lambda)-t_m(\lambda)-\beta_1+1} - \gamma'_1.$$

4.1.2. Sets of partitions

Definition 4.1.7 (Set of all partitions of n)**.** *Denote by P_n the set of all partitions λ of n.*

$$P_n = \{\lambda : \quad \lambda \text{ is a partition of } n\}.$$

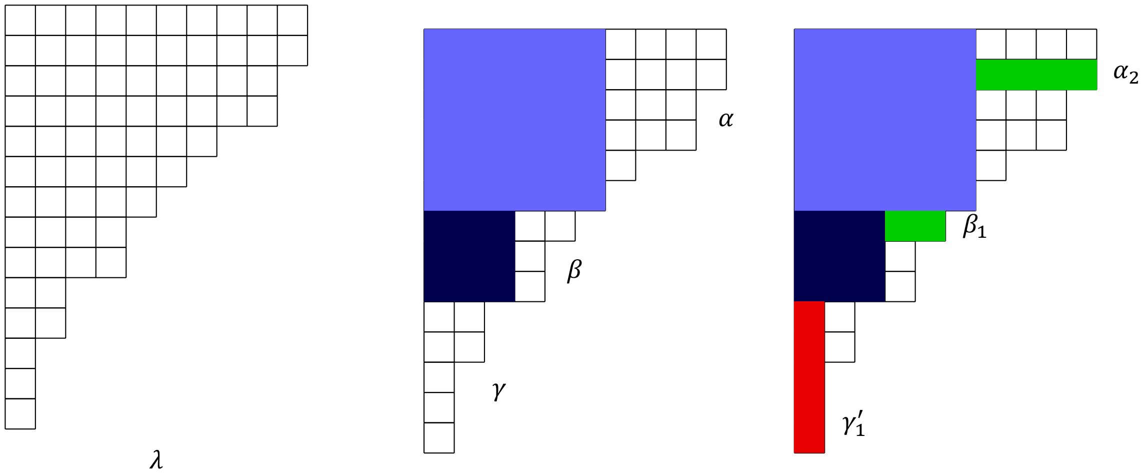

Figure 4.2: Partition $\lambda = (10, 10, 9, 9, 7, 6, 5, 4, 4, 2, 2, 1, 1, 1)$, the first 0-Durfee rectangle of height $s_0(\lambda) = 6$, and the second 0-Durfee rectangle of height $t_0(\lambda) = 3$. Here the remaining partitions are $\alpha = (4, 4, 3, 3, 1)$, $\beta = (2, 1, 1)$, and $\gamma = (2, 2, 1, 1, 1)$. In this case, the $(2, 0)$-rank is $r_{2,0}(\lambda) = \beta_1 + \alpha_2 - \gamma_1' = 2 + 4 - 5 = 1$.

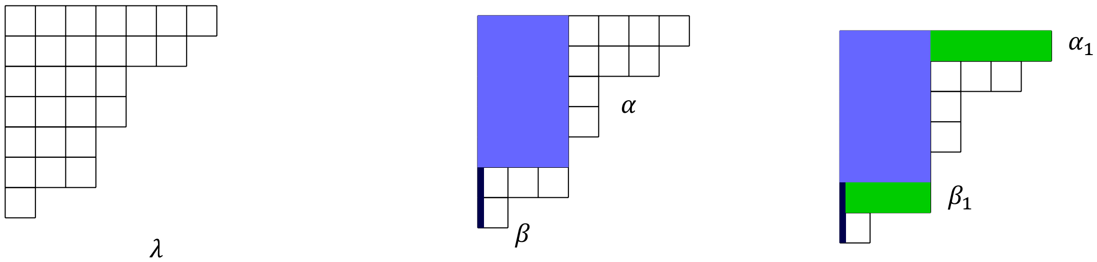

Figure 4.3: Partition $\lambda = (7, 6, 4, 4, 3, 3, 1)$, the first 2-Durfee rectangle of height $s_2(\lambda) = 5$ and width 3, and the second 2-Durfee rectangle of height $t_2(\lambda) = 2$ and width 0. Here the remaining partitions are $\alpha = (4, 3, 1, 1)$, $\beta = (3, 1)$, and γ which is empty. In this case, we have $(2, 2)$-rank $r_{2,2}(\lambda) = \beta_1 + \alpha_1 - \gamma_1' = 3 + 4 - 0 = 7$.

Let $p(n) = |P_n|$ be the partition number of n.

Let P be the union of P_n over all possible n:

$$P = \bigcup_{n=0}^{\infty} P_n.$$

Finally, Define the generating function $P(t)$ of the partition numbers $p(n)$ by:

$$P(t) := 1 + \sum_{n=1}^{\infty} p(n)t^n.$$

Remark. We define $p(0) := 1$. This definition is justified by considering the empty partition $\lambda = (\,)$ as the unique partition of 0, and the only partition of length zero. Hence, we can write $P(t)$ as,

$$P(t) = \sum_{n=0}^{\infty} p(n)t^n.$$

Further, we note that $p(n) = 0$ for all $n < 0$.

Definition 4.1.8 (Rogers-Ramanujan partitions)**.** *We say that λ is a Rogers-Ramanujan partition if $e(\lambda) \leq \ell(\lambda)$. Denote by Q_n the set of all Rogers-Ramanujan partitions of n.*

$$Q_n = \{\lambda \in P_n : \quad e(\lambda) \leq \ell(\lambda)\}.$$

Figure 4.4: The Rogers-Ramanujan partition $\lambda = (7,6,5,5)$, the first 0-Durfee rectangle of height $s_0(\lambda) = 5$ and width 5. Here the remaining partitions are $\alpha = (2,1)$, and γ which is empty. In this case, the rank and the second 0-Durfee are not defined.

Define $q(n) = |Q_n|$.

Let Q be the union of Q_n over all possible n:

$$Q = \bigcup_{n=0}^{\infty} Q_n.$$

Finally, define $Q(t)$ as follows:

$$Q(t) := 1 + \sum_{n=1}^{\infty} q(n)t^n.$$

Remark. Note that the $(2;0)$-rank is only defined for non-Rogers-Ramanujan partitions ($e(\lambda) > l(\lambda)$) because for Rogers-Ramanujan partitions the height of the first 0-Durfee rectangle will be $e(\lambda)$. However, as shown in Figure 4.4, the second 0-Durfee rectangle must have a height of 0, which is not allowed. Furthermore, in this case β_1 is not defined. The $(2;m)$-rank is defined for all partitions for all $m > 0$, since then an m-Durfee rectangle with height m and width 0 always exists. Again, see Figures 4.2 and 4.3 for examples.

Definition 4.1.9. *(Partitions with rank) Let $H_{n,m,r}$ be the set of partitions of n with $(2,m)$-rank r:*

$$H_{n,m,r} = \left\{\lambda \in P_n : \quad r_{2,m}(\lambda) = r\right\}.$$

Define $h(n,m,r) = |H_{n,m,r}|$.

Define $H_{n,m,\leq r}$ to be the set of partitions of n with $(2,m)$-rank smaller than r and $H_{n,m,\geq r}$ to be the set of partitions of n with $(2,m)$-rank greater than r :

$$H_{n,m,\leq r} = \left\{\lambda \in P_n : \quad r_{2,m}(\lambda) \leq r\right\},$$
$$H_{n,m,\geq r} = \left\{\lambda \in P_n : \quad r_{2,m}(\lambda) \geq r\right\}.$$

The following is apparent from the definition.

$$h(n,m,\leq r) + h(n,m,\geq r+1) = p(n), \quad \text{for } m > 0, \tag{4.1}$$

and

$$h(n,0,\leq r) + h(n,0,\geq r+1) = p(n) - q(n), \quad \text{for all } r \in \mathbb{Z}, n \geq 1. \tag{4.2}$$

Definition 4.1.10 (Partition numbers with restrictions)**.** *Here are four different partition numbers with certain restrictions on the parts of the partitions.*

a) The number of partitions such that the parts differ by at least 2 is denoted as follows,

$$p(n \mid \textit{diff } 2) := \#\left\{\ \lambda \in P_n : \quad \lambda_i - \lambda_{i+1} \geq 2 \textit{ for } i \in \{1,2,\dots,\ell(\lambda)-1\}\ \right\}.$$

b) The number of partitions of n such that the parts differ by at least 2 and where the smallest part is greater than or equal to 2 denoted as follows,

$$p(n \mid \textit{diff } 2, \geq 2) := \#\left\{\lambda \in P_n : \quad \lambda_{\ell(\lambda)} \geq 2 \quad \textit{and} \quad \lambda_i - \lambda_{i+1} \geq 2 \textit{ for } i \in \{1,2,\dots,\ell(\lambda)-1\}\ \right\}.$$

c) The number of partitions where all the parts are of the form 1 mod 5 or 4 mod 5 denoted as follows,

$$p(n \mid 1,4 \mod 5) := \#\{\lambda \in P_n : \quad \lambda_i \equiv 1 \text{ or } 4 \pmod 5 \quad \text{for } i \in \{1,2,\dots,\ell(\lambda)-1\}\}.$$

d) The number of partitions where all the parts are of the form 2 mod 5 or 3 mod 5 denoted as follows,

$$p(n \mid 2,3 \mod 5) := \#\{\lambda \in P_n : \quad \lambda_i \equiv 2 \text{ or } 3 \pmod 5 \quad \text{for } i \in \{1,2,\dots,\ell(\lambda)-1\}\}.$$

Remark. If there is only one part in the partition, then we say that all the parts differ by at least two. For example,

$$(8) \in \left\{\lambda \in P_8 : \quad \lambda_{\ell(\lambda)} \geq 2 \quad \text{and} \quad \lambda_i - \lambda_{i+1} \geq 2 \text{ for } i \in \{1,2,\dots,\ell(\lambda)-1\} \right\}.$$

Example 4.1.11. To understand the restrictions in the Definition 4.1.10 better, let's analyze the partitions of 6:

Let us examine if the partition (6) satisfies the following conditions:

- *All parts differ by at least 2:*

  The partition consists of only one part, 6. Since there's only one part, it trivially satisfies the condition of having parts differ by at least 2.

- *Smallest part* ≥ 2 *and all parts differ by at least 2:*

  This partition satisfies the previous condition, and the smallest part is 6, which is greater than or equal to 2, so it satisfies this condition.

- *Parts 1, 4 mod 5:*

  $6 \equiv 1 \pmod 5$, so it satisfies this condition.

- *Parts 2, 3 mod 5:*

  $6 \equiv 1 \pmod 5$, so this partition does *not* satisfy the condition, as 6 is neither congruent to 2 nor 3 modulo 5.

Let us examine if the partition (5,1) satisfies the following conditions:

- *All parts differ by at least 2:*

  The difference between the parts is $5 - 1 = 4$, which is greater than or equal to 2, so it satisfies the condition.

- *Smallest part* ≥ 2 *and all parts differ by at least 2:*

  The smallest part is 1, which is less than 2, so it *does not* satisfy this condition.

- *Parts 1, 4 mod 5:*

  $5 \equiv 0 \pmod 5$, and $1 \equiv 1 \pmod 5$. Since 5 is not congruent to 1 or 4 modulo 5, this partition does *not* satisfy this condition.

- *Parts 2, 3 mod 5:*

  $5 \equiv 0 \pmod 5$, and $1 \equiv 1 \pmod 5$. Neither part is congruent to 2 or 3 modulo 5, so this partition does *not* satisfy this condition either.

Let us examine if the partition (2, 2, 2) satisfies the following conditions:

- *All parts differ by at least 2:*

  The difference between the parts is $2 - 2 = 0$, which does *not* satisfy the condition that the difference between parts must be at least 2.

- *Smallest part ≥ 2 and all parts differ by at least 2:*

  This partition does *not* satisfy the previous condition. So it also does *not* satisfy this condition.

- *Parts 1, 4 mod 5:*

  $4 \equiv 4 \pmod 5$ and $2 \equiv 2 \pmod 5$. Since 2 is not congruent to 1 or 4 modulo 5, it does *not* satisfy this condition

- *Parts 2, 3 mod 5:*

  $2 \equiv 2 \pmod 5$. Since 2 is congruent to 2 modulo 5, it satisfies this condition.

As shown in Table 4.1, the partitions of 6 are categorized based on various conditions such as the difference between parts, the smallest part, and congruence modulo 5.

Description	**Partitions**	**# Partitions**
All partitions of 6	$(6), (5,1), (4,2), (4,1,1), (3,3), (3,2,1), (3,1,1), (2,2,2), (2,2,1,1), (2,1,1,1,1), (1,1,1,1,1,1)$	11
Partitions with diff at least 2	$(6), (5,1), (4,2)$	3
Partitions with diff at least 2, smallest part ≥ 2	$(6), (4,2)$	2
Parts 1,4 mod 5	$(6), (4,1,1), (1,1,1,1,1,1)$	3
Parts 2,3 mod 5	$(3,3), (2,2,2)$	2

Table 4.1: Partitions of 6 and their satisfaction with the given conditions: differences between parts, smallest part, and congruence modulo 5.

4.2. Rogers-Ramanujan Identities in terms of partitions

In this section, the Rogers-Ramanujan identities translated to partition theory will be stated. After that, an explanation will be provided on how this translation works using generating functions.

The Rogers-Ramanujan identities are often interpreted in the language of partition theory. The first identity can be expressed as the number of partitions of n such that the parts differ by at least 2 equals the number of partitions of n where all the parts are of the form 1 mod 5 or 4 mod 5:

$$p(n \mid \text{diff } 2) = p(n \mid 1, 4 \mod 5).$$

Similarly, the second Rogers-Ramanujan identity becomes the number of partitions of n such that the parts differ by at least 2 and where the smallest part is greater than or equal to 2, equals the number of partitions of n where all the parts are of the form 2 mod 5 or 3 mod 5:

$$p(n \mid \text{diff } 2, \geq 2) = p(n \mid 2, 3 \mod 5).$$

Before getting into the generating function for the restricted partitions stated above, we need to prove a couple of lemmas beforehand.

Lemma 4.2.1 (Euler's generating function for $p(n)$). *Let $P(q)$ be the generating function of partitions, then*

$$P(q) = \sum_{n=0}^{\infty} p(n) q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}. \tag{4.3}$$

where $p(n)$ denotes the number of partitions of n.

Proof. Consider the generating function of partitions:

$$P(q) = \sum_{n=0}^{\infty} p(n) q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots, \tag{4.4}$$

where $p(n)$ is the partition number of n. Note that the exponent of q can be expressed in exactly $p(n)$ distinct ways. For any fixed n, rewrite the exponent of q in every possible partition of n exactly once,

$$P(q) = 1 + q^1 + (q^2 + q^{1+1}) + (q^3 + q^{2+1} + q^{1+1+1}) + (q^4 + q^{3+1} + q^{2+1+1} + q^{2+2} + q^{1+1+1+1}) + \dots.$$

Furthermore, we can use an alternative representation of all partitions, ensuring that each appears exactly once as a power of q,

$$P(q) = (q^0 + q^1 + q^{1+1} + q^{1+1+1} + \dots)(q^0 + q^2 + q^{2+2} + \dots)(q^0 + q^3 + q^{3+3} + \dots)(q^0 + q^4 + \dots).$$

In this formulation, each term in the exponent of q describes the multiplicity of a given part size: the first term, $(q^0 + q^1 + q^{1+1} + q^{1+1+1} + \dots)$, indicates the number of ones in the partition, the second term indicates the number of twos, and so on.

Now using the geometric series, we have the following,

$$\begin{aligned} P(q) &= (1 + q + q^{1+1} + q^{1+1+1} + \dots)(1 + q^2 + q^{2+2} + \dots)(1 + q^3 + q^{3+3} + \dots)(1 + q^4 + \dots) && (4.5) \\ &= \left(\frac{1}{1-q}\right)\left(\frac{1}{1-q^2}\right)\left(\frac{1}{1-q^3}\right)\left(\frac{1}{1-q^4}\right)\cdots && (4.6) \\ &= \prod_{k=1}^{\infty} \frac{1}{1-q^k}. \end{aligned}$$

□

Lemma 4.2.2. *The number of partitions of n which have at most k parts equals the number of partitions of n into parts with the largest part less than or equal to k:*

$$\#\left\{ \text{ Partitions of } n \text{ into at most } k \text{ parts } \right\} = \#\left\{ \text{ Partitions of } n \text{ into parts with largest part } \leq k \right\}. \tag{4.7}$$

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n, and consider its conjugate partition λ'. As Figure 4.5 shows, if λ has at most k parts, then the largest part of λ' is less than or equal to k. Since every partition λ has a well-defined conjugate λ', we conclude that

$$\#\left\{ \text{ Partitions of } n \text{ into at most } k \text{ parts } \right\} \leq \#\left\{ \text{ Partitions of } n \text{ into parts with largest part } \leq k \right\}. \tag{4.8}$$

Similarly, by considering the conjugate of λ', namely $(\lambda')' = \lambda$, we see that this process is reversible. Therefore, we also have

$$\#\left\{ \text{ Partitions of } n \text{ into at most } k \text{ parts } \right\} \geq \#\left\{ \text{ Partitions of } n \text{ into parts with largest part } \leq k \right\}. \tag{4.9}$$

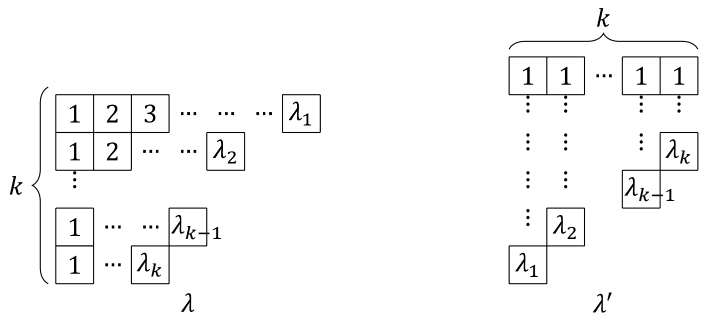

Figure 4.5: Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and with its conjugate partition λ' with labeled blocks.

Combining both inequalities, we conclude that the two sets are equinumerous. □

Lemma 4.2.3. *For $n_1, n_2, \ldots, n_k$ integers, we have,*

$$\sum_{0 \leq n_1 \leq n_2 \leq \cdots \leq n_k} q^{n_1 + n_2 + \cdots + n_k} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

Proof. Note that, per definition, the left-hand side of the lemma is the generating function of partitions of n into at most k parts,

$$\sum_{0 \leq n_1 \leq n_2 \leq \cdots \leq n_k} q^{n_1 + n_2 + \cdots + n_k}.$$

Moreover, the generating function of partitions with the largest part $\leq k$ can be defined as follows,

$$P_{\leq k}(q) := \sum_{n=0}^{\infty} p_{\leq k}(n) q^n,$$

where $p_{\leq k}(n)$ denotes the number of partitions with the largest part $\leq k$. Moreover, similar to Lemma 4.2.1, this can be rewritten as,

$$P(q) = (q^0 + q^1 + q^{1+1} + q^{1+1+1} + \dots)(q^0 + q^2 + q^{2+2} + \dots)(q^0 + q^3 + q^{3+3} + \dots) \cdots (q^0 + q^k + q^{k+k} \dots).$$

This results in the right-hand side of the lemma:

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

Since we know by Lemma 4.2.2 that

$$\#\{\text{ Partitions of } n \text{ into at most } k \text{ parts }\} = \#\{\text{ Partitions of } n \text{ into parts with largest part } \leq k \text{ }\},$$

the two corresponding generating function should be equal. This finishes the proof. □

4.2.1. Generating Functions

The following four lemmas illustrate how both Rogers-Ramanujan identities can be interpreted combinatorially in terms of integer partitions.

Lemma 4.2.4. *The product side of the first Rogers-Ramanujan identity,*

$$\prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}.$$

is the generating function of $p(n \mid 1, 4 \mod 5)$.

Proof. If we restrict ourselves to using only numbers of the form $5m+1$, where m is a positive integer, then we exclude factors corresponding to parts not congruent to 1 or 4 mod 5 from Equation (4.5).

For example, the number 2 is not of the required form, so the factor $(1+q^2+q^{2+2}+\dots)$ must be reduced to just (1), as all higher powers in this factor correspond to multiples of 2, which we now exclude.

Applying this process to all non-conforming terms yields the following expression:

$$\begin{aligned}\sum_{n=0}^{\infty} p(n \mid 1, 4 \mod 5) q^n &= (1 + q + q^{1+1} + q^{1+1+1} + \dots)(1 + \cancel{q}^2 + \cancel{q}^{2+2} + \cancel{\dots})(1 + \cancel{q}^3 + \cancel{q}^{3+3} + \cancel{\dots})(1 + q^4 + \dots) \\ &= \left(\frac{1}{1-q}\right)\left(\cancel{\frac{1}{1-q^2}}\right)\left(\cancel{\frac{1}{1-q^3}}\right)\left(\frac{1}{1-q^4}\right)\cdots \\ &= \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}.\end{aligned}$$

□

Lemma 4.2.5. *The product side of the second Rogers-Ramanujan identity,*

$$\prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+2})(1-q^{5k+3})},$$

is the generating function of $p(n \mid 2, 3 \mod 5)$.

Proof. This is the same procedure as in Lemma 4.2.4 we start with Equation (4.5) and remove all terms that are not of the form $5m-2$ or $5m-3$, where m is a positive integer:

$$\sum_{n=0}^{\infty} p(n \mid 2,3 \mod 5) q^n = (1+\cancel{q}+\cancel{q^{1+1}}+\cancel{q^{1+1+1}}+\cancel{\ldots})(1+q^2+q^{2+2}+\ldots)(1+q^3+q^{3+3}+\ldots)(1+\cancel{q^4}+\cancel{\ldots})$$

$$= \cancel{\left(\frac{1}{1-q}\right)}\left(\frac{1}{1-q^2}\right)\left(\frac{1}{1-q^3}\right)\cancel{\left(\frac{1}{1-q^4}\right)}\cdots$$

$$= \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+2})(1-q^{5k+3})}.$$

□

Lemma 4.2.6. *The sum side of the first Rogers-Ramanujan identity,*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

is the generating function of $p(n \mid \textit{diff}\ 2)$.

Proof. Let $H(q)$ denote the generating function for all partitions in which the parts differ by at least 2, $p(n \mid \text{diff } 2)$. Note that

$$\sum_{1\le\lambda_1\le\cdots\le\lambda_k} q^{\lambda_1+\cdots+\lambda_k} \quad \text{where} \quad \lambda_i \ge \lambda_{i-1}+2$$

enumerates all partitions with exactly k parts satisfying this difference condition. Therefore, if we sum over all $k \in \mathbb{N}$, we obtain a generating function for all partitions in which the parts differ by at least 2:

$$H(q) = \sum_{k>0} \sum_{1\le\lambda_1\le\cdots\le\lambda_k} q^{\lambda_1+\cdots+\lambda_k} \quad \text{where} \quad \lambda_i \ge \lambda_{i-1}+2$$

Now, define n_i as follows:

$$n_i = \lambda_{k-i+1} - 2\cdot i + 1.$$

We can rewrite λ_i's in terms of new variables n_i's as follows:

$$\begin{aligned}
\lambda_k \ge 1 \quad &\Rightarrow \quad \lambda_k = n_1 + 1\\
\lambda_{k-1} \ge 3 \quad &\Rightarrow \quad \lambda_{k-1} = n_2 + 3\\
\lambda_{k-2} \ge 5 \quad &\Rightarrow \quad \lambda_{k-2} = n_3 + 5\\
&\vdots\\
\lambda_1 \ge 2k-1 \quad &\Rightarrow \quad \lambda_1 = n_k + 2k - 1\\
0 \le n_1 \le n_2 &\le \cdots \le n_k.
\end{aligned}$$

So $H(q)$ becomes:

$$\begin{aligned}
H(q) &= \sum_{k\ge0} \sum_{0\le n_1\le n_2\le\cdots\le n_k} q^{n_1+n_2+\cdots+n_k+(1+3+5+\cdots+(2k-1))}\\
&= \sum_{k\ge0} \sum_{0\le n_1\le n_2\le\cdots\le n_k} q^{n_1+n_2+\cdots+n_k+k^2}\\
&= \sum_{k\ge0} q^{k^2} \sum_{0\le n_1\le n_2\le\cdots\le n_k} q^{n_1+n_2+\cdots+n_k}.
\end{aligned}$$

We used the fact that $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. Now using Lemma 4.2.3, we have

$$H(q) = \sum_{k \geq 0} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}. \qquad \square$$

Lemma 4.2.7. *The sum side of the second Rogers-Ramanujan identity,*

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

is the generating function of $p(n \mid \text{diff } 2, \geq 2)$.

Proof. Similar to the proof of Lemma 4.2.6, let $H_2(q)$ denote the generating function for all partitions in which the parts differ by at least 2 and the smallest part is 2 or greater, $p(n \mid \text{diff } 2, \geq 2)$. Note that

$$\sum_{2 \leq \lambda_1 \leq \cdots \leq \lambda_k} q^{\lambda_1 + \cdots + \lambda_k} \quad \text{where} \quad \lambda_i \geq \lambda_{i-1} + 2$$

enumerates all partitions with exactly k parts satisfying this difference condition. Therefore, if we sum over all $k \in \mathbb{N}$, we obtain a generating function for all partitions in which the parts differ by at least 2:

$$H_2(q) = \sum_{k>0} \sum_{1 \leq \lambda_1 \leq \cdots \leq \lambda_k} q^{\lambda_1 + \cdots + \lambda_k} \quad \text{where} \quad \lambda_i \geq \lambda_{i-1} + 2.$$

Now, define n_i as follows:

$$n_i = \lambda_{k-i+1} - 2 \cdot i.$$

We can rewrite λ_i's in terms of new variables n_i's as follows:

$$\begin{aligned} \lambda_k \geq 2 \quad &\Rightarrow \quad \lambda_k = n_1 + 2 \\ \lambda_{k-1} \geq 4 \quad &\Rightarrow \quad \lambda_{k-1} = n_2 + 4 \\ \lambda_{k-2} \geq 6 \quad &\Rightarrow \quad \lambda_{k-2} = n_3 + 6 \\ &\vdots \\ \lambda_1 \geq 2k \quad &\Rightarrow \quad \lambda_1 = n_k + 2k \\ 0 \leq n_1 &\leq n_2 \leq \cdots \leq n_k. \end{aligned}$$

So $H_2(q)$ becomes:

$$\begin{aligned} H_2(q) &= \sum_{k \geq 0} \sum_{0 \leq n_1 \leq n_2 \leq \cdots \leq n_k} q^{n_1 + n_2 + \cdots + n_k + (2+4+6+\cdots+2k)} \\ &= \sum_{k \geq 0} \sum_{0 \leq n_1 \leq n_2 \leq \cdots \leq n_k} q^{n_1 + n_2 + \cdots + n_k + k^2 + k} \\ &= \sum_{k \geq 0} q^{k(k+1)} \sum_{0 \leq n_1 \leq n_2 \leq \cdots \leq n_k} q^{n_1 + n_2 + \cdots + n_k}. \end{aligned}$$

We used the fact that $2 + 4 + 6 + \cdots + 2k = k + 1 + 3 + 5 + \cdots + (2k - 1) = k(k + 1)$. So the exponent becomes $n_1 + \cdots + n_k + k(k + 1)$. Now using Lemma 4.2.3, we have

$$H_2(q) = \sum_{k \geq 0} \frac{q^{k(k+1)}}{(1-q)(1-q^2)\cdots(1-q^k)}. \qquad \square$$

Theorem 4.2.8. *The Rogers-Ramanujan Identities imply that the following equalities hold:*

$$p(n \mid \textit{diff } 2) = p(n \mid 1, 4 \mod 5), \tag{4.10}$$

and

$$p(n \mid \textit{diff } 2, \geq 2) = p(n \mid 2, 3 \mod 5). \tag{4.11}$$

Proof. Using the lemmas 4.2.6, 4.2.4, and the Rogers-Ramanujan Identities, we know that

$$\begin{aligned}\sum_{n=0}^{\infty} p(n \mid \text{diff } 2) q^n &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} \\ &= \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})} \\ &= \sum_{n=0}^{\infty} p(n \mid 1, 4 \mod 5) q^n\end{aligned}$$

is true. Since there are two power series, the coefficients must be the same (A.2.1). Thus,

$$p(n \mid \text{diff } 2) = p(n \mid 1, 4 \mod 5)$$

The proof of (4.11) is similar. □

4.2.2. Examples for the first identity

Example 4.2.9 ($n = 5$)**.** First, we will show that the statement as stated in Theorem 4.2.8, is valid for $n = 5$. The complete set of integer partitions of 5 is:

$$(5), \quad (4,1), \quad (3,2), \quad (3,1,1), \quad (2,2,1), \quad (2,1,1,1), \quad (1,1,1,1,1)$$

The partitions of 5 where each part differs from the next by at least 2 are:

$$(5), \quad (4,1)$$

The partitions using only parts congruent to 1 or 4 mod 5 are:

$$(4,1), \quad (1,1,1,1,1)$$

So, in both cases, there are exactly 2 valid partitions. This confirms the combinatorial interpretation for the first Rogers-Ramanujan identity for $n = 5$.

Example 4.2.10 ($1 \leq n \leq 6$)**.** Continuing the procedure of the previous subsection for integers $1 \leq n \leq 6$, we get the following table.

n	a_n	diff 2	$1, 4 \mod 5$
1	1	1	1
2	1	2	$1+1$
3	1	3	$1+1+1$
4	2	$4,\ 3+1$	$4,\ 1+1+1+1$
5	2	$5,\ 4+1$	$4+1,\ 1+1+1+1+1$
6	3	$6,\ 5+1,\ 4+2$	$6,\ 4+1+1,\ 1+1+1+1+1+1$

Indeed, for $1 \leq n \leq 6$, $p(n \mid 1, 4 \mod 5) = p(n \mid \text{diff } 2)$.

5

Bijective Combinatorial Proofs of the Rogers-Ramanujan Identities

The Rogers-Ramanujan identities were first discovered by Rogers in 1894 and later independently by Ramanujan and Schur around 1917. While Rogers's original proof (and the 1919 Rogers-Ramanujan joint proof) were analytic, Schur provided a combinatorial interpretation of the identities in 1917 (Hardy, 1940). However, finding a direct bijective proof, a one-to-one correspondence between the partition sets as shown in Chapter 4, became a long-standing challenge in partition theory. For decades, proofs of the Rogers-Ramanujan identities relied on generating functions or complex analytic techniques (often invoking the Jacobi triple product), rather than purely combinatorial bijections Soodak, 2018. It was not until the 1980s that the first explicit bijective proof emerged in the literature.

This chapter surveys the development of bijective combinatorial proofs of the Rogers-Ramanujan identities in chronological order, focusing exclusively on peer-reviewed contributions. We identify the first bijective combinatorial proof and describe its method, then examine how later proofs evolved, noting the types of bijections used, involutive constructions, direct explicit maps, recursive arguments, partition bijections via Ferrers diagrams or lattice path correspondences, etc., and what new insights each approach offered.

5.1. Early Combinatorial Work (1917-1970): MacMahon and Schur

During the early 20th century, the Rogers-Ramanujan identities were known but lacked a direct combinatorial proof. They were therefore stated without proof in the second volume of MacMahon's *Combinatory analysis* in 1916. Shortly thereafter, Schur independently rediscovered the Rogers-Ramanujan identities in 1917 and published the first combinatorial proof. Schur's proof, written in German, did not construct a bijection explicitly, but it established the partition interpretation of the identities and proved the equality of the two partition counting functions combinatorially. In particular, Schur showed that the number of partitions of n into parts differing by at least 2 equals the number of partitions of n into parts $\equiv 1$ or $4 \pmod 5$, as stated in Chapter 4, thereby giving a combinatorial meaning to the Rogers-Ramanujan identities. (Hardy, 1940)

Following Schur's work, interest in combinatorial proofs of partition identities grew. Sylvester, Franklin, and others developed constructive methods for simpler partition identities in the late 19th century (Pak, 2002). However, for the Rogers-Ramanujan identities, no simple bijection was known for many decades. Mathematicians like Watson, Jackson, and later Slater and Gordon focused on analytic or generating function generalizations, rather than direct bijections. By the 1970s, the quest for a "constructive" proof of Rogers-Ramanujan had become a central problem in partition theory. As the combinatorial theory of partitions entered a "golden age" in the 1960s-70s with Andrews and others proving many identities by combinatorial means, the Rogers-Ramanujan identities remained a notable holdout without a known bijective proof. This set the stage for a breakthrough in the 1980s.

5.2. The First Bijective Proof: Garsia and Milne (1981)

A major milestone was achieved in 1981 when Garsia and Milne published the first explicit bijective proof of the Rogers-Ramanujan identities (Garsia and Milne, 1981). Garsia and Milne's work, entitled "A Rogers-Ramanujan Bijection," introduced a powerful general method called the involution principle to construct complicated bijections systematically. The involution principle is a combinatorial mechanism that takes two involutions (self-inverse mappings) on a common set and, under certain conditions, uses them to produce a bijection between the two sets of fixed points of those involutions. In their 51-page paper, Garsia and Milne carefully crafted a sequence of involutions and intermediate bijections (combining ideas from earlier involutions by Vahlen, Sylvester, and Schur) to ultimately establish a one-to-one correspondence between partitions counted by the Rogers-Ramanujan generating function's product side and those counted by its series side. This involutive bijection was the first of its kind for the Rogers-Ramanujan identities.

While groundbreaking, the Garsia-Milne bijective proof is notoriously complex (Soodak, 2018). The construction is not a simple "natural" mapping but rather a carefully engineered algorithm that applies involutions iteratively to redistribute partition parts according to certain rules. In essence, their proof mechanically translated an inclusion-exclusion argument into a combinatorial mapping. One involution in their construction was derived from the Jacobi triple product identity, which means the bijection implicitly uses analytic input. Nevertheless, the Garsia-Milne result was a triumph: it proved, for the first time, that a direct combinatorial bijection exists between the two classes of partitions in the Rogers-Ramanujan identities. This provided a new level of insight by not only confirming the numerical equality of the two sides, but actually pairing each partition on one side with a unique partition on the other.

The significance of Garsia and Milne's work was widely recognized. It demonstrated the power of the involution principle as a general combinatorial proof technique and spurred further research into bijections for Rogers-Ramanujan-type identities. However, combinatorialists also found the resulting bijection somewhat unsatisfying in terms of simplicity or transparency. The authors showed that one can mechanically construct bijections, but from a traditional point of view, the solution was indirect and hard to follow. As Pak later observed, the Garsia-Milne involution-method bijections were "not simpler than the analytic proofs" and did not immediately suggest new combinatorial interpretations or refinements beyond the identities themselves. In other words, the first bijective proof solved the problem in principle but left open the quest for a more elegant or natural bijection.

5.3. Subsequent Bijective Proofs and Improvements (1982-1989)

Following Garsia and Milne's pioneering work, other mathematicians quickly sought to simplify and extend combinatorial proofs of the Rogers-Ramanujan identities. In 1982, Bressoud and Zeilberger published what they called "A short Rogers-Ramanujan bijection" (Bressoud and Zeilberger, 1982), providing a much more concise bijective proof. Bressoud and Zeilberger's bijection, spanning only a few pages, still relied on the Garsia-Milne involution principle and the Jacobi triple product as an ingredient, but through clever use of involutions they managed to streamline the mapping considerably. This involutive bijection was of the same general character as Garsia-Milne's (and in fact was directly inspired by the latter), but it was less complex and more transparent in its execution. The shorter proof by Bressoud and Zeilberger demonstrated that the involution-based approach could be made more efficient. While the Bressoud-Zeilberger bijection was still not "simple" in an absolute sense, it represented a significant improvement in clarity and brevity over Garsia and Milne's original construction.

Around the same time, Andrews was interested in removing the remaining analytic crutches from the combinatorial proofs. Andrews posed the question of whether one could prove the Rogers-Ramanujan identities without invoking the Jacobi triple product (or any heavy analytic identity) at all, as that seemed to be the last vestige of non-combinatorial reasoning in these proofs. Zeilberger famously highlighted this challenge during a 1985 combinatorics colloquium, asking if a Rogers-Ramanujan proof could be found that avoids Jacobi's triple product entirely. In 1987, Andrews published a paper titled "The Rogers-Ramanujan identities without Jacobi's triple product," in which he presented new proofs of the identities not relying on the classical triple product lemma (Andrews, 1987). Andrews's approach was still not a direct bijection; it used a combinatorial generating function argument and a replacement for the Jacobi product. The importance of Andrews's 1987 work is that it paved the way for more purely combinatorial approaches, eliminating one major analytic tool, suggesting that a completely bijective proof might be within reach with more ingenuity. Andrews himself noted that removing Jacobi's identity was a step toward a truly direct combinatorial proof free of the involution principle's complexity. In the end, even Andrews's proof had to invoke a generalized product lemma, so the quest for a "pure" bijection continued (Soodak, 2018). Nevertheless, this effort reflected the community's drive to simplify the reasoning and seek a more natural explanation for the Rogers-Ramanujan partition identities.

The late 1980s saw further developments. In 1989, Bressoud and Zeilberger again collaborated to produce generalized Rogers-Ramanujan bijections (Bressoud and Zeilberger, 1989). In this paper, they extended the involution principle methods to construct bijections for the Andrews-Gordon identities, a family of partition identities that generalize the Rogers-Ramanujan identities to higher moduli and more parameters. This work showed that the techniques pioneered for Rogers-Ramanujan could be systematically applied to an infinite hierarchy of partition identities. The bijections in the generalized case were even more intricate, but they provided combinatorial insight into how the Rogers-Ramanujan phenomenon extends beyond modulus 5. In essence, the 1989 extensions clarified the partition structure behind the identities: they identified how to pair off partitions with difference conditions to partitions with higher-order congruence conditions using a network of involutions and bijections. These results also reinforced a key insight: although one could construct such bijections algorithmically, the resulting maps were often too complicated to yield a "natural" combinatorial story. As Joichi and Stanton remarked around that time, "the emphasis should be placed on combinatorially important proofs rather than just a proof" (Pak, 2002). In other words, by the end of the 1980s, combinatorialists had succeeded in proving the Rogers-Ramanujan identities bijectively and even generalizing them, but the next goal was to find better proofs: ones that are more elegant, transparent, or insightful.

5.4. New Perspectives in the 2000s: Direct Bijections and Structural Insights

The turn of the 21st century brought a renewed examination of Rogers-Ramanujan bijections, motivated by the desire for greater elegance and understanding. Pak, in particular, emerged as a central figure advocating for simpler bijective proofs. In a 2005 seminar talk, Pak pointed out that "a number of important partition identities (such as the Rogers-Ramanujan identities) do not seem to have a 'nice' bijective proof" and that finding such a proof remained an important open challenge. This sentiment echoed the general feeling in the combinatorics community: despite having technical bijections, the Rogers-Ramanujan identities were still lacking a conceptually natural bijection that one could describe without heavy machinery.

Pak not only surveyed the state of the art, but also contributed new ideas. Together with Boulet, Pak devised a fresh combinatorial proof of the Rogers-Ramanujan identities in 2005-2006 that attempted to minimize algebraic manipulation and make the bijections more direct (Boulet and Pak, 2006). The Boulet-Pak proof splits the first Rogers-Ramanujan identity into two symmetry statements about partitions and then proves each symmetry by an explicit bijection on partitions. They introduced a new statistic related to Dyson's partition rank and showed that this statistic exhibits two involutive symmetries. By composing the involutions associated with these symmetries, Boulet and Pak obtained a direct bijection between the two sides of the first Rogers-Ramanujan identity. In contrast to Garsia-Milne, their approach did not construct the bijection via a global inclusion-exclusion argument, but rather built it from two simpler bijections that were more “elementary and approachable”. The proof still used the Jacobi triple product at an initial step to set up the partition equations, but the heavy lifting in the combinatorial part was done by tangible combinatorial maps on partition diagrams. This was a novel blend of analytic and combinatorial reasoning: essentially, they reduced the Rogers-Ramanujan identities to two simpler combinatorial identities and then proved those bijectively. The Boulet-Pak bijection provided new insight by clarifying how certain partition statistics can naturally partition the set of all Rogers-Ramanujan partitions into symmetric classes. Many researchers viewed this as a step toward a more “natural” proof, even if Hardy’s century-old comment that no proof of Rogers-Ramanujan is truly simple remained apt (Hardy, 1940).

In addition to new proofs, modern work has placed the Rogers-Ramanujan identities in a broader combinatorial context. Connections to lattice path interpretations have been explored, wherein partitions with difference conditions are represented as lattice paths in a plane region and mapped to other lattice paths representing the modulus conditions. For example, some recent papers construct bijections between certain lattice path ensembles and Rogers-Ramanujan-type partitions to give combinatorial proofs of generalized identities (Hao and Shi, 2025; Marwah and Goyal, 2024). These approaches often use recursive or induction-based bijections: one inductively “builds” a bijection for partitions of n by extending a bijection for partitions of smaller integers, carefully adding or removing parts or steps in a path. Such recursive bijections can sometimes simplify the combinatorial interpretation, although for Rogers-Ramanujan itself the direct involution-based proofs still dominate the literature.

Despite all these advances, it remains generally accepted that no trivially simple bijection for the Rogers-Ramanujan identities is known. The existing bijective proofs are celebrated for solving the problem and illuminating various aspects of partition structure, but they are also seen as technically intricate. As Pak noted in 2008, the Rogers-Ramanujan identities still lack a “nice” bijection, and finding one stands as a benchmark test for combinatorial ingenuity. This ongoing quest reflects the high value placed on truly insightful bijective proofs in the field of combinatorics.

6

Combinatorial Proof

In this chapter, we present the combinatorial proof of Cilanne Boulet and Igor Pak for the first Rogers-Ramanujan identity (Boulet and Pak, 2006). We begin with a reformulation of the identity due to Schur and demonstrate how it is equivalent to the first Rogers-Ramanujan identity. Next, we derive a generating function for Rogers-Ramanujan partitions and study two symmetry theorems about these partitions. Finally, we use these results to carry out an algebraic argument involving generating functions, leading to Schur's identity and thus a combinatorial proof of the first Rogers-Ramanujan identity.

To avoid confusion between the partition number of Rogers-Ramanujan partitions, $q(n)$, and the variable q used in the Rogers-Ramanujan identities, we will replace the variable q in the identities with t throughout this chapter. Accordingly, we also assume in this chapter that $|t| < 1$.

6.1. Schur's identity

Theorem 6.1.1. *Proving Schur's identity is equivalent to proving the first Rogers-Ramanujan identity, where Schur's identity is the following:*

$$\left(1+\sum_{k=1}^{\infty}\frac{k^2}{(1-t)(1-t^2)\cdots(1-t^k)}\right)=\prod_{i=1}^{\infty}\frac{1}{(1-t^i)}\sum_{m=-\infty}^{\infty}(-1)^m t^{m(5m-1)/2}.$$

Proof. Recall the Jacobi triple product identity, as described in Identity 2.3:

$$\sum_{k=-\infty}^{\infty} z^k q^{\frac{k(k+1)}{2}}=\prod_{i=1}^{\infty}(1+zq^i)\prod_{j=0}^{\infty}(1+z^{-1}q^j)\prod_{i=1}^{\infty}(1-q^i).$$

We substitute $z=-t^{-2}$ and $q=t^5$. The left-hand side becomes:

$$\sum_{k=-\infty}^{\infty}(-t^{-2})^k(t^5)^{\frac{k(k+1)}{2}}=\sum_{k=-\infty}^{\infty}(-1)^k t^{\frac{k(5k+1)}{2}}=\sum_{m=-\infty}^{\infty}(-1)^m t^{\frac{m(5m-1)}{2}}$$

Where $m=-k$.

Now consider the right-hand side of the Jacobi identity:

$$\prod_{i=1}^{\infty}(1+-t^{-2}(t^5)^i)\prod_{j=0}^{\infty}(1+(-t^{-2})^{-1}(t^5)^j)\prod_{i=1}^{\infty}(1-(t^5)^i).$$

This simplifies to:

$$\prod_{i=1}^{\infty}(1-t^{5i-2})\prod_{j=0}^{\infty}(1-t^{5j+2})\prod_{i=1}^{\infty}(1-t^{5i}).$$

Re-indexing these products using $i \to i+1$, and writing as one infinite product, becomes the following,

$$\prod_{i=0}^{\infty}(1-t^{5i+2})(1-t^{5i+3})(1-t^{5i+5}).$$

Now multiply numerator and denominator by

$$\prod_{i=0}^{\infty}(1-t^{5i+1})(1-t^{5i+4}),$$

so that the product becomes:

$$\prod_{i=0}^{\infty}(1-t^{5i+2})(1-t^{5i+3})(1-t^{5i+5})\cdot\frac{\prod_{i=0}^{\infty}(1-t^{5i+1})(1-t^{5i+4})}{\prod_{j=0}^{\infty}(1-t^{5j+1})(1-t^{5j+4})}.$$

Simplifying this expression,

$$\frac{\prod_{i=1}^{\infty}(1-t^i)}{\prod_{j=0}^{\infty}(1-t^{5j+1})(1-t^{5j+4})}.$$

By equating the left- and right-hand sides of the Jacobi triple product identity and dividing both sides by $\prod_{i=1}^{\infty}(1-t^i)$, we find:

$$\prod_{r=0}^{\infty}\frac{1}{(1-t^{r+1})(1-t^{r+4})}=\sum_{m=-\infty}^{\infty}(-1)^m t^{\frac{m(m-1)}{2}}\prod_{i=1}^{\infty}\frac{1}{(1-t^i)}. \tag{6.1}$$

If Schur's identity,

$$\left(1+\sum_{k=1}^{\infty}\frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)}\right)=\prod_{i=1}^{\infty}\frac{1}{(1-t^i)}\sum_{m=-\infty}^{\infty}(-1)^m t^{m(5m-1)/2},$$

is proven and we rewrite the right hand side using 6.1, we have proven the first Rogers-Ramanujan Identity 2.6. □

6.2. Symmetries

Lemma 6.2.1. *The generating function of Rogers-Ramanujan partitions can be written as:*

$$Q(t) = 1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)}.$$

Proof. Similar to the proof of Lemma 4.2.6, let $Q(t)$ denote the generating function for all Rogers-Ramanujan partitions, $q(n)$. Note that

$$\sum_{1\le\lambda_1\le\cdots\le\lambda_k} t^{\lambda_1+\cdots+\lambda_k} \quad \text{where} \quad e(\lambda) \le \ell(\lambda)$$

enumerates all partitions with exactly k parts satisfying this condition. Therefore, if we sum over all $k \in \mathbb{N}$, we obtain a generating function for all partitions satisfying the Rogers-Ramanujan condition $e(\lambda) \le \ell(\lambda)$

$$Q(t) = \sum_{k>0} \sum_{1\le\lambda_1\le\cdots\le\lambda_k} t^{\lambda_1+\cdots+\lambda_k} \quad \text{where} \quad \lambda_i \ge \lambda_{i-1} + 2$$

Now, define n_i as follows:

$$n_i = \lambda_{k-i+1} - k$$

We can rewrite λ_i's in terms of new variables n_i's as follows:

$$\begin{aligned}
\lambda_k \ge k \quad &\Rightarrow \quad \lambda_k = n_1 + k \\
\lambda_{k-1} \ge k \quad &\Rightarrow \quad \lambda_{k-1} = n_2 + k \\
\lambda_{k-2} \ge k \quad &\Rightarrow \quad \lambda_{k-2} = n_3 + k \\
&\vdots \\
\lambda_1 \ge k \quad &\Rightarrow \quad \lambda_1 = n_k + k \\
0 \le n_1 \le n_2 &\le \cdots \le n_k
\end{aligned}$$

So $Q(t)$ becomes:

$$\begin{aligned}
Q(t) &= \sum_{k\ge0} \sum_{0\le n_1\le n_2\le\cdots\le n_k} t^{n_1+n_2+\cdots+n_k+(k+k+k+\cdots+k)} \\
&= \sum_{k\ge0} \sum_{0\le n_1\le n_2\le\cdots\le n_k} t^{n_1+n_2+\cdots+k^2} \\
&= \sum_{k\ge0} t^{k^2} \sum_{0\le n_1\le n_2\le\cdots\le n_k} t^{n_1+n_2+\cdots+n_k}
\end{aligned}$$

Now using Lemma 4.2.3, we have

$$Q(t) = \sum_{k\ge0} \frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)}$$ □

Now, we will prove two theorems that are the main ingredients of the proof.

Theorem 6.2.2. *Let $H_{n,m,r}$ be the set of partitions of n with $(2,m)$-rank r, and let $h(n,m,r) = |H_{n,m,r}|$. Then,*

$$h(n,0,r) = h(n,0,-r)$$

for $r \in \mathbb{Z}$.

To prove the first symmetry, we present an involution φ on $P \setminus Q$ which preserves the size of partitions as well as their 0-Durfee rectangles, but changes the sign of the rank:

$$\varphi : H_{n,0,r} \longrightarrow H_{n,0,-r}.$$

Let λ be a partition with two 0-Durfee rectangles and partitions α, β, and γ to the right of, in the middle of, and below the 0-Durfee rectangles. This map φ will preserve the 0-Durfee rectangles of λ whose sizes are denoted by

$$s = s_0(\lambda) \quad \text{and} \quad t = t_0(\lambda).$$

We will describe the action of φ as $\lambda \longmapsto \hat{\lambda}$ by first mapping (α, β, γ) to a 5-tuple of partitions $(\mu, \pi, \tau, \rho, \sigma)$, and subsequently mapping that 5-tuple to a different triple $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ which goes to the right of, in the middle of, and below the 0-Durfee rectangles in λ'.

The mapping from the partitions (α, β, γ) to the partitions $(\mu, \pi, \tau, \rho, \sigma)$, is as follows:

1. Let $\mu = \beta$.
2. Remove the following parts from $\alpha : \alpha_{s-t-\beta_j+j}$ for $1 \leq j \leq t$. Let ν be the partition comprising of parts removed from α, and π be the partition comprising the parts which were not removed.
3. For $1 \leq j \leq t$, let
$$k_j = \max\left\{k \leq s-t \,\middle|\, \gamma_j' - k \geq \pi_{s-t-k+1}\right\},$$
and let ρ be the partition with parts $\rho_j = k_j$ and σ be the partition with parts $\sigma_j = \gamma_j' - k_j$.

The mapping from the partitions $(\mu, \pi, \tau, \rho, \sigma)$ to the partitions $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$, is as follows:

1. Let $\hat{\gamma}' = \nu + \mu$ be the sum of partitions, defined to have parts $\hat{\gamma}_j{}' = \nu_j + \mu_j$.
2. Let $\hat{\alpha} = \sigma \cup \tau$ be the union of partitions, defined as a union of parts in σ and π.
3. Let $\hat{\beta} = \rho$.

Figure 4 shows an example of φ and the relation between these two steps.

Remark. The key to understanding the map φ is the definition of k_j. By considering $k = 0$, we see that k_j is defined for all $1 \leq j \leq t$. Moreover, one can check that k_j is the unique integer k which satisfies

$$\pi_{s-t-k+1} \leq \gamma_j' - k \leq \pi_{s-t-k}. \tag{6.2}$$

(We do not consider the upper bound for $k = s - t$.) This characterization of k_j can also be taken as its definition. Equation (6.2) is used repeatedly in our proof of the next lemma.

Lemma 6.2.3. *The map φ defined above is an involution.*

Proof. We divide the proof into five parts, showing that:

1. ρ is a partition,
2. σ is a partition,
3. $\hat{\lambda} = \varphi(\lambda)$ is a partition,
4. φ^2 is the identity map
5. $r_{2,0}(\hat{\lambda}) = -r_{2,0}(\lambda)$.

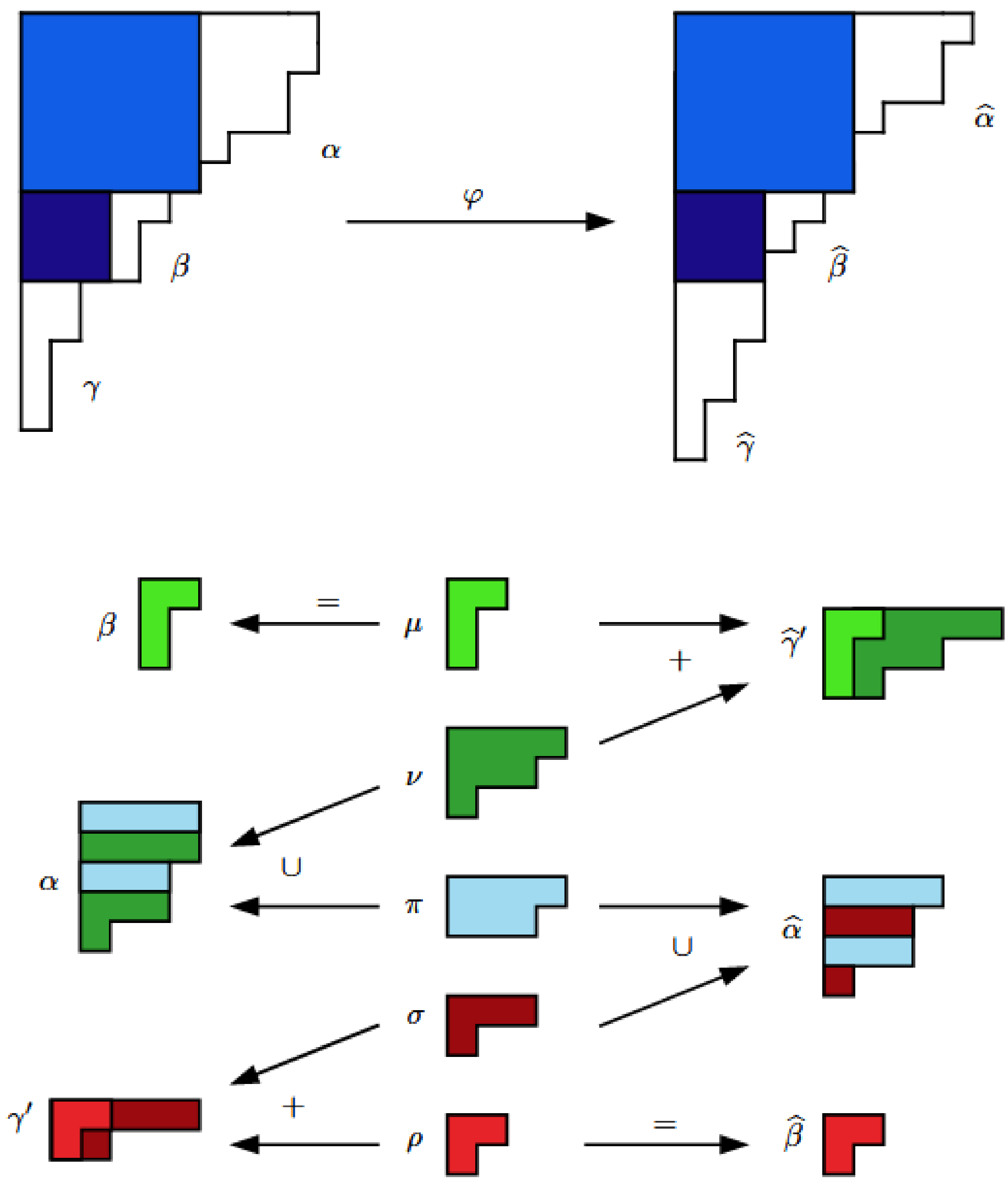

Figure 6.1: An example of the first symmetry involution $\varphi : \lambda \mapsto \hat{\lambda}$, where $\lambda \in H_{n,0,r}$ and $\hat{\lambda} \in H_{n,0,-r}$, for $n = 71$ and $r = 1$. The maps are defined by the following rules:

$$\beta = \mu, \quad \alpha = \nu \cup \pi, \quad \gamma' = \sigma + \rho,$$
$$\hat{\beta} = \rho, \quad \hat{\alpha} = \pi \cup \sigma, \quad \hat{\gamma}' = \mu + \nu.$$

Also,

$$\lambda = (10, 10, 9, 9, 7, 6, 5, 4, 4, 2, 2, 1, 1, 1), \quad \hat{\lambda} = (10, 9, 9, 7, 6, 6, 5, 4, 3, 3, 3, 2, 2, 1, 1).$$

(1) Let's first verify that ρ is a partition.

We consider the bounds (6.2) for indices j and $j+1$. Assuming $k_j \leq k_{j+1}$, we have

$$\pi_{s-t-k_j+1} + k_j \leq \pi_{s-t-k_j+1} + k_{j+1} \leq \gamma'_{j+1} \leq \gamma'_j \leq \pi_{s-t-k_j} + k_j.$$

Rewriting,

$$\pi_{s-t-k_j+1} \leq \gamma'_{j+1} - k_j \leq \pi_{s-t-k_j}.$$

Uniqueness therefore implies that $k_j = k_{j+1}$. Hence, $k_j \geq k_j + 1$. Thus, ρ is a partition.

(2) Now check that σ is a partition. Suppose $k_j > k_{j+1}$, then $s-t-k_j+1 \leq s-t-k_{j+1}$ must be true. Since π is a partition, we have,

$$\pi_{s-t-k_{j+1}} \leq \pi_{s-t-k_j+1}.$$

By (6.2), for j and $j+1$, we have,

$$\gamma'_j - k_j \geq \gamma'_{j+1} - k_{j+1}$$

If $k_j = k_{j+1}$, then we simply need to recall that γ' is a partition to see that

$$\gamma'_j - k_j \geq \gamma'_{j+1} - k_{j+1}.$$

Thus, σ is a partition.

(3) Since μ, ν, π are clearly partitions by definition, and we just estableshed that ρ and σ are partitions as well, it follows that $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ are also partitions. Furthermore, by their definitions, we see that μ, ν and σ have at most t parts, π has at most $s-t$, and ρ has at most t parts each of which is less than or equal to $s-t$. This implies that $\hat{\alpha}$ has at most s parts, $\hat{\beta}$ has at most t parts each of which is less than or equal to $s-t$, and $\hat{\gamma}'$ has parts at most t.

So, α, β, γ fit to the right, in the middle of, and below 0-Durfee rectangles of size s and t and so $\varphi(\lambda)$ is a partition. This confirms that $\varphi(\lambda)$ results in a valid partition.

(4) We apply φ twice to a non-Rogers-Ramanujan partition λ where α, β, γ appear to the right of, in the middle of, and below two 0-Durfee rectangles. As usual let $\mu, \nu, \pi, \rho, \sigma$ be the partitions occuring in the intermediate stage of the first application of φ to λ and let $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ be the partitions to the right of, in the middle of, and below the 0-Durfee rectangles of $\hat{\lambda} = \varphi(\lambda)$. Similarly, let $\hat{\mu}, \hat{\nu}, \hat{\pi}, \hat{\rho}, \hat{\sigma}$ be the partitions occurring in the intermediate stage of the second application of φ and let $\alpha*, \beta*, \gamma*$ be the partitions to the right of, in the middle of and below the 0-Durfee rectangles of $\varphi^2(\lambda) = \varphi(\hat{\lambda})$.

We need to establish several key facts. Observe that $\hat{\mu} = \hat{\beta} = \rho$. Also, from Equation (6.2) we have:

$$\pi_{s-t-k_j+1} \leq \gamma'_j - k_j = \sigma_j \leq \pi_{s-t-k_j}.$$

Because σ is a partition, it follows that $\hat{\alpha}_{s-t-k_j+1} = \sigma_j$. On the other hand, since $\hat{\beta}_j = \rho_j = k_j$, the map φ removes the parts $\hat{\alpha}_{s-t-k_j+1} = \sigma_j$ from $\hat{\alpha}$. Thus, $\hat{\nu} = \sigma$ and $\hat{\pi} = \pi$.

Define

$$\hat{k_j} = \max\left\{\hat{k} \leq s-t \mid \gamma'_j - k \geq \pi_{s-t-\hat{k}+1}\right\}.$$

By the remark, we know that $\hat{k_j}$ as defined above is the unique integer $\hat{k}$ which satisfies:

$$\hat{\pi}_{s-t-\hat{k}+1} \leq \hat{\gamma}_j{}' - \hat{k} \leq \pi_{s-t-\hat{k}}.$$

On the other hand, recall that $\hat{\gamma}_j{}' = \mu_j + \nu_j$ and $\beta_j = \mu_j$. This implies $\hat{\gamma}_j{}' - \beta_j = \nu_j$. Also, by the definition of ν, we have $\nu_j = \alpha_{s-t-\beta_j+j}$. Therefore, by the definition of π, we have:

$$\pi_{s-t-\beta_j+1} \leq \alpha_{s-t-\beta_j+j} = \nu_j = \hat{\gamma}_j{}' - \beta_j \leq \pi_{s-t-\beta_j}.$$

Since $\hat{\pi} = \pi$, by the uniqueness in, we have $\hat{k}_j = \beta_j = \mu_j$. This implies that $\hat{\rho} = \mu$ and $\hat{\sigma} = \nu$.

Finally, the second step of our bijection gives $\alpha^* = \nu \cup \pi = \alpha, \beta^* = \mu = \beta$, and $(\gamma^*)' = \rho + \sigma = \gamma'$. This implies that φ^2 is the identity map.

(5) Using the results from (4), we have:

$$r_{2,0}(\lambda) = \beta_1 + \alpha_{s-t-\beta_1+1} + \gamma_1' = \mu_1 + \nu_1 - \rho_1 - \sigma_1.$$

On the other hand,

$$r_{2,0}(\hat{\lambda}) = \hat{\beta_1} + \alpha_{s-\hat{t}-\hat{\beta_1}+1} - \hat{\gamma_1}' = \rho_1 + \sigma_1 - \mu_1 - \nu_1.$$

We conclude that $r_{2,0}(\hat{\lambda}) = -r_{2,0}(\lambda)$. □

Theorem 6.2.4. *Let $H_{n,m,r}$ be the set of partitions of n with $(2,m)$-rank r, and let $h(n,m,r) = |H_{n,m,r}|$. Then the following is true:*

$$h(n,m,\leq -r) = h(n-r-2m-2, m+2, \geq -r)$$

for $m,r > 0$ and for $m = 0$ and $r \geq 0$.

To prove the second symmetry, we present a bijection $\psi_{m,r}$:

$$\psi_{m,r} : H_{n,m,\leq -r} \longrightarrow H_{n-r-2m-2,m+2,\geq -r}.$$

This map will only be defined for $m,r > 0$ and for $m = 0$ and $r \geq 0$, and in both of these cases the first and second m-Durfee rectangles of a partition $\lambda \in H_{n,m,\leq r}$ have non-zero width. For $m = 0$, $(2,0)$-rank is only defined for partitions in $P \setminus Q$ which by definition have two 0-Durfee rectangles of non-zero width. For $m > 0$, since we have $r > 0$, a partition $\lambda \in H_{n,m,\leq -r}$ must have

$$r_{2,m}(\lambda) = \beta_1 + \alpha_{s_m(\lambda)-t_m(\lambda)-\beta_1+1} - \gamma_1' \leq -r < 0.$$

This forces $\gamma_j' > 0$ and so both m-Durfee rectangles must have non-zero width.

We describe the action of $\psi = \psi_{m,r}$ by giving the sizes of the Durfee rectangles of $\hat{\lambda} := \psi_{m,r}(\lambda) = \psi(\lambda)$ and the partitions $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ which go to the right of, in the middle of, and below those Durfee rectangles in $\hat{\lambda}$.

1. If λ has two m-Durfee rectangles of height

   $$s := s_m(\lambda) \quad \text{and} \quad t := t_m(\lambda),$$

   then $\hat{\lambda}$ has two $(m+2)$-Durfee rectangles of height

   $$s' := s_{m+2}(\hat{\lambda}) = s + 1 \quad \text{and} \quad t' := t_{m+2}(\hat{\lambda}) = t + 1.$$

2. Let

   $$k_1 = \max\{k \leq s - t \mid \gamma_1' - r - k \geq \alpha_{s-t-k+1}\}.$$

   Obtain $\hat{\alpha}$ from α by adding a new part of size $\gamma_1' - r - k_1$, $\hat{\beta}$ from β by adding a new part of size k_1, and $\hat{\gamma}$ from γ by removing its first column.

Figure 5 shows an example of the bijection $\psi = \psi_{m,r}$ and the relation between these two steps.

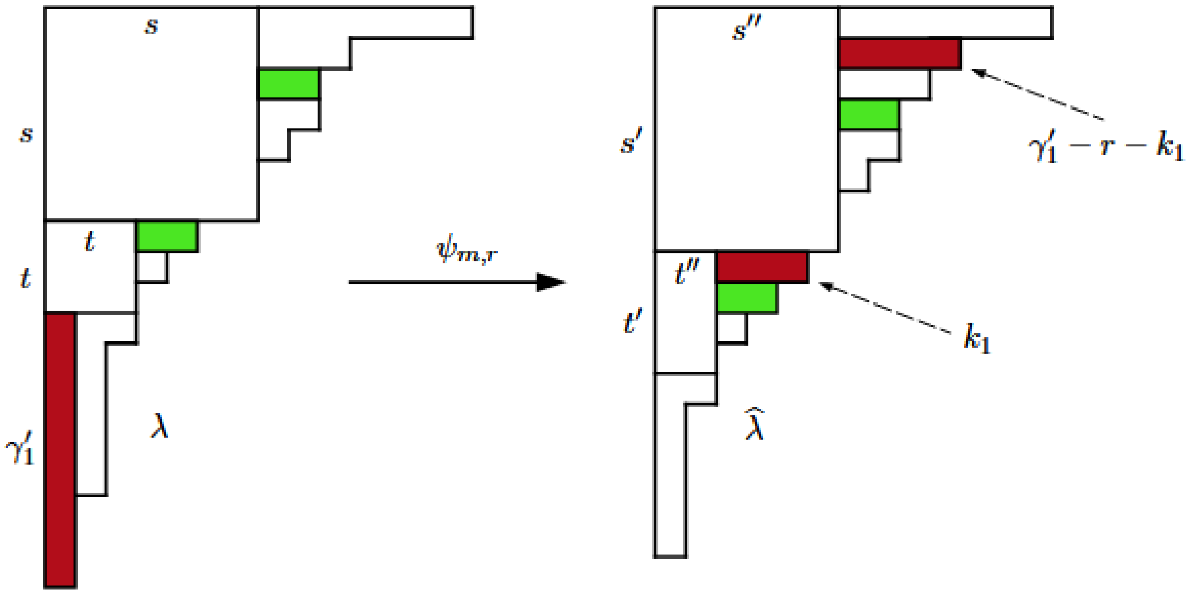

Figure 6.2: An example of the second symmetry bijection $\psi_{m,r} : \lambda \mapsto \hat{\lambda}$, where $\lambda \in H_{n,m,\leq -r}$, $\hat{\lambda} \in H_{n',m+2,\geq -r}$, for $m = 0$, $r = 2$, $n = 92$, and $n' = n - r - 2m - 2 = 88$. Here,$r_{2,0}(\lambda) = 2 + 2 - 9 = -5 \leq -2$, $r_{2,2}(\hat{\lambda}) = 3 + 4 - 6 = 1 \geq -2$, where

$$\lambda = (14, 10, 9, 9, 8, 7, 7, 5, 4, 3, 3, 2, 2, 2, 2, 2, 1, 1, 1),$$

$$\hat{\lambda} = (13, 10, 9, 8, 8, 7, 6, 6, 5, 4, 3, 2, 2, 1, 1, 1, 1, 1).$$

Also,

$$s = 7, \quad s' = s + 1 = 8, \quad s'' = s' - m - 2 = 6,$$

$$t = 3, \quad t' = 4, \quad t'' = 2, \quad \gamma_1' = 9, \quad k_1 = 3, \quad \gamma_1' - r - k_1 = 4.$$

Remark. As in the earlier Remark, by considering $k = \beta_1$, we see that k_1 is defined and indeed we have $k_1 \geq \beta_1$. Moreover, it follows from its definition that k_1 is the unique k such that

$$\alpha_{s-t-k+1} \leq \gamma_1' - r - k \leq \alpha_{s-t-k}.$$

If $k = s - t$ we do not consider the upper bound.

Lemma 6.2.5. *The map $\psi_{m,r}$ defined above is a bijection.*

Proof. Our proof has four parts:

1. We prove that $\hat{\lambda} = \psi_{m,r}(\lambda)$ is a partition,
2. We prove that the size of $\hat{\lambda}$ is $n - r - 2m - 2$,
3. We prove that $r_{2,m+2}(\hat{\lambda}) \geq -r$,
4. We present the inverse map ψ^{-1}.

(1) Note that $\hat{\lambda}$ is a partition since λ has m-Durfee rectangles of non-zero width, $\hat{\lambda}$ may have $(m+2)$-Durfee rectangles of width $s-1$ and $t-1$. Furthermore, the partitions $\hat{\alpha}$ and $\hat{\beta}$ have parts of size at most $s-t$ and $t+1$, respectively, while the partitions $\hat{\beta}$ and $\hat{\gamma}$ have parts of size at most $s-t$ and $t-1$, respectively. This means that they fit to the right of, in the middle of, and below the two $(m+2)$-Durfee rectangles of $\hat{\lambda}$.

(2) To prove that the above construction gives a partition $\hat{\lambda}$ of $n - r - 2m - 2$, note that the sum of the sizes of the rows added to α and β is r less than the size of the column. removed from γ, and that both the first and second $(m+2)$-Durfee rectangles of $\hat{\lambda}$ have size $m+1$ less than the size of the corresponding m-Durfee rectangle of λ.

(3) By Remark 6.2, the part we inserted into β will be the largest part of the resulting partition, i.e., $\hat{\beta}_1 = k_1$. By equation **??**, we have:

$$\alpha_{s-t-k_1+1} \leq \gamma_1' - r - k_1 \leq \alpha_{s-t-k_1}.$$

Therefore, we must have:

$$\hat{\alpha}_{s'-t'-\hat{\beta}_1+1} = \hat{\alpha}_{s-t-k_1+1} = \gamma_j' - r - k_1.$$

Indeed, we have chosen k_1 in the unique way so that the rows we insert into α and β are $\hat{\alpha}_{s'-t'-\hat{\beta}_1+1}$ and $\hat{\beta}_1$ respectively.

Having determined $\hat{\alpha}_{s'-t'-\hat{\beta}_1+1}$ and $\hat{\beta}_1$ allows us to bound the $(2, m+2)$-rank of $\hat{\lambda}$:

$$r_{2,m+2}(\hat{\lambda}) = \hat{\alpha}_{s'-t'-\hat{\beta}_1+1} + \hat{\beta}_1 - \hat{\gamma}_1' = \gamma_1' - r - k_1 + k_1 - \hat{\gamma}_1' \leq -r,$$

where the last inequality follows since $\hat{\gamma}_1'$ is the size of the second column of γ, whereas γ_1' is the size of the first column of γ.

(4) The above characterization of k_1 also shows us that to recover α, β, and γ from $\hat{\alpha}, \hat{\beta}$, and $\hat{\gamma}$, we remove part $\hat{\alpha}_{s'-t'-\hat{\beta}_1+1}$ from $\hat{\alpha}$, remove part $\hat{\beta}_1$ from $\hat{\beta}$, and add a column of height $\hat{\alpha}_{s'-t'-\hat{\beta}_1+1} + \hat{\beta}_1 + r$ to $\hat{\gamma}$. Since we can also easily recover the sizes of the previous m-Durfee rectangles, we conclude that ψ is a bijection between the desired sets. □

6.3. The algebraic part

Let us continue to prove Schur's identity. For every $j \geq 0$ let

$$a_j = h(n - jr - 2jm - j(5j-1)/2, m + 2j, \leq -r - j),$$

and

$$b_j = h(n - jr - 2jm - j(5j-1)/2, m + 2j, \geq -r - j + 1).$$

Note that by definition the following is true:

$$h(n, m, \leq -r) = a_0 = b_1.$$

By Theorem 6.2.2, we see that $a_j = b_{j+1}$. So using the fact that $a_j - b_{j+1} = 0$, we have,

$$h(n, m, \leq -r) = b_1 + (a_1 - b_2) - (a_2 - b_3) + (a_3 - b_4) - \dots,$$

or equivalently,

$$h(n, m, \leq -r) = (b_1 + a_1) - (b_2 + a_2) + (b_3 + a_3) - (b_4 + a_4) + \dots.$$

By Theorem 6.2.4, $a_j + b_j = p(n - jr - 2jm - j(5j-1)/2)$, for all $r, j > 0$ holds. Applying this, gives,

$$\begin{aligned} h(n, m, \leq -r) &= p(n - r - 2m - 2) - p(n - 2r - 4m - 9) + p(n - 3r - 6m - 21) - \dots \\ &= \sum_{j=1}^{\infty} (-1)^{j-1} p(n - jr - 2jm - j(5j-1)/2). \end{aligned}$$

In terms of the generating functions, we have:

$$H_{m,\leq r}(t) := \sum_{n=1}^{\infty} h(n, m, \leq -r) t^n,$$

this gives for $m, r > 0$ and for $m = 0$ and $r \geq 0$:

$$H_{m,\leq r}(t) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{j-1} p(n - jr - 2jm - j(5j-1)/2) t^n.$$

Re-indexing $n \to n = jr + 2jm + j(5j-1)/2$ gives,

$$\begin{aligned} H_{m,\leq r}(t) &= \sum_{n=-jr-2jm-j(5j-1)/2+1}^{\infty} \sum_{j=1}^{\infty} (-1)^{j-1} p(n) t^{n+jr+2jm+j(5j-1)/2} \\ &= \sum_{n=1}^{\infty} p(n) t^n \sum_{j=1}^{\infty} (-1)^{j-1} t^{jr+2jm+j(5j-1)/2}. \end{aligned}$$

Using Lemma 4.2.1, we see that

$$H_{m,\leq r}(t) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \sum_{j=1}^{\infty} (-1)^{j-1} t^{jr+2jm+j(5j-1)/2}.$$

In particular, we have:

$$H_{0,\leq 0}(t) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \sum_{j=1}^{\infty} (-1)^{j-1} t^{j(5j-1)/2},$$

and

$$H_{0,\le -1}(t) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \sum_{j=1}^{\infty} (-1)^{j-1} t^{j(5j+1)/2}.$$

From the first symmetry equation and (4.2) we have:

$$H_{0,\le 0}(t) + H_{0,\le -1}(t) = H_{0,\le 0}(t) + H_{0,\ge 1}(t) = P(t) - Q(t).$$

We conclude:

$$\prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \left(\sum_{j=1}^{\infty} (-1)^{j-1} t^{\frac{j(5j-1)}{2}} + \sum_{j=1}^{\infty} (-1)^{j-1} t^{\frac{j(5j+1)}{2}} \right)$$
$$= \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} - \left(1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)} \right),$$

which implies Schur's identity and completes the proof of the first Rogers-Ramanujan identity.

7

Conclusion

The Rogers-Ramanujan identities are a well-known and intriguing subject in the theory of integer partitions and q-series. Since their initial appearance in Rogers' 1894 paper and their later rediscoveries by Ramanujan and Schur, they have attracted sustained interest due to their surprising connections across various areas of mathematics, including combinatorics, number theory, statistical mechanics, representation theory, and more. This thesis set out to explore both their historical and mathematical depth by focusing on four central aspects: Ramanujan's original analytic proof, the combinatorial interpretation via integer partitions, a chronological overview of bijective combinatorial proofs, and the combinatorial proof of Igor Pak of the first identity.

In explaining Ramanujan's own proof, we gained insight into the analytical elegance and technical complexity involved. The subsequent development of a combinatorial framework allowed for a more intuitive grasp of the identities through partition theory, revealing the structural beauty underlying the series-product equivalences. Furthermore, the historical overview of bijective combinatorial proofs served to highlight how the mathematical community has continually sought, and continues to seek, more natural, elegant, and insightful explanations for these identities. Finally, by reworking the second identity in a Pak-style bijective manner, this thesis attempted to fill a small but meaningful gap in the literature, offering a fresh perspective on a classical problem.

The continued fascination with the Rogers-Ramanujan identities is not merely a reflection of their aesthetic appeal, but of their uncanny ability to resurface in areas far removed from their original context. Whether in the classification of Lie algebras, the behavior of physical models in statistical mechanics, or the structure of modular forms, these identities remain an active source of inquiry and inspiration. This suggests that, rather than being a solved or closed chapter of mathematics, the Rogers-Ramanujan identities may still hold secrets yet to be discovered, awaiting new minds and new methods to reveal them.

In that spirit, this thesis does not aim to conclude the discussion, but rather to extend the invitation echoed by Sills and others: to study further, and to be drawn into the rich and still-unfolding story of the Rogers-Ramanujan identities.

8 Discussion

While much progress has been made in understanding the Rogers-Ramanujan identities, several important questions remain open. First, Pak's bijective proof of the first identity offers a compelling and intuitive approach, but a comparable, purely combinatorial proof for the second identity remains elusive. Developing such a proof could fill a gap in the literature.

Second, the many bijections developed over the past century, each with its own style and insight, lack a unified visual or structural framework. Creating a coherent way to visualize these bijections, perhaps through diagrams, videos, or interactive models, may offer new understanding and reveal connections previously unnoticed.

Finally, the central challenge persists: to find a truly simple and natural explanation for the combinatorial interpretation of the Rogers-Ramanujan identities. Such an explanation, if discovered, would not only deepen our insight into these identities but also serve as a powerful tool in the broader study of partition theory and q-series.

A

Appendix

A.1. Calculations

We can rewrite $G(z)$ as:

$$G(z) = 1 + \sum_{r=1}^{\infty} (-1)^r z^{2r} q^{\frac{1}{2}r(5r-1)} \cdot \frac{(z;q)_r}{(1-z)(q;q)_r}(1-q^r)$$
$$+ \sum_{r=1}^{\infty} (-1)^r z^{2r} q^{\frac{1}{2}r(5r-1)} \cdot \frac{(z;q)_r}{(1-z)(q;q)_r} \cdot q^r(1-zq^r)$$

$$= 1 - \sum_{r=0}^{\infty} (-1)^r z^{2(r+1)} q^{\frac{1}{2}(r+1)(5(r+1)-1)} \cdot \frac{(z;q)_{r+1}}{(1-z)(q;q)_r}$$
$$+ \sum_{r=1}^{\infty} (-1)^r z^{2r} q^{\frac{1}{2}r(5r+1)} \cdot \frac{(z;q)_{r+1}}{(1-z)(q;q)_r}$$

$$= 1 - z^2q^2 - \sum_{r=1}^{\infty} (-1)^r z^{2r} q^{\frac{1}{2}r(5r+1)} \cdot \frac{(z;q)_{r+1}}{(1-z)(q;q)_r} \cdot z^2 q^{4r+2}$$
$$+ \sum_{r=1}^{\infty} (-1)^r z^{2r} q^{\frac{1}{2}r(5r+1)} \cdot \frac{(z;q)_{r+1}}{(1-z)(q;q)_r}.$$

This results in the following:

$$G(z) = \sum_{r=0}^{\infty} (-1)^r z^{2r} q^{\frac{1}{2}r(5r+1)} \frac{(z;q)_{r+1}}{(1-z)(q;q)_r} \cdot (1 - z^2 q^{4r+2}). \tag{A.1}$$

Calculation $\frac{G(z)}{(1-zq)} - G(zq) = zq(1-zq^2)G(zq^2)$:

$$\begin{aligned}
\frac{G(z)}{(1-zq)} - G(zq) &= \sum_{r=0}^{\infty}(-1)^r z^{2r} q^{\frac{1}{2}r(5r+1)} \frac{(zq;q)_r}{(1-zq)(q;q)_r} \cdot ((1-z^2q^{4r+2}) - q^r(1-zq^{2r+1})) \\
&= \sum_{r=0}^{\infty}(-1)^r z^{2r} q^{\frac{1}{2}r(5r+1)} \frac{(zq;q)_r}{(1-zq)(q;q)_r} \cdot (1-q^r + zq^{3r+1}(1-zq^{r+1})) \\
&= \sum_{r=0}^{\infty}(-1)^r z^{2r} q^{\frac{1}{2}r(5r+1)} \frac{(zq;q)_r}{(1-zq)(q;q)_r} \cdot (1-q^r) \\
&+ \sum_{r=0}^{\infty}(-1)^r z^{2r} q^{\frac{1}{2}r(5r+1)} \frac{(zq;q)_r}{(1-zq)(q;q)_r} zq^{3r+1}(1-zq^{r+1}) \\
&= -\sum_{r=0}^{\infty}(-1)^r z^{2(r+1)} q^{\frac{1}{2}(r+1)(5(r+1)+1)} \frac{(zq;q)_{r+1}}{(1-zq)(q;q)_r} \\
&+ \sum_{r=0}^{\infty}(-1)^r z^{2r} q^{\frac{1}{2}r(5r+7)} \frac{(zq;q)_{r+1}}{(1-zq)(q;q)_r} zq \\
&= -\sum_{r=0}^{\infty}(-1)^r z^{2r} q^{\frac{1}{2}r(5r+7)} \frac{(zq;q)_{r+1}}{(1-zq)(q;q)_r} \cdot z^2 q^{2r+3} \\
&+ \sum_{r=0}^{\infty}(-1)^r z^{2r} q^{\frac{1}{2}r(5r+7)} \frac{(zq;q)_{r+1}}{(1-zq)(q;q)_r} zq \\
&= \sum_{r=0}^{\infty}(-1)^r z^{2r} q^{\frac{1}{2}r(5r+7)} \frac{(zq;q)_{r+1}}{(1-zq)(q;q)_r} (zq - \cdot z^2 q^{2r+3}) \\
&= (1-zq^2)\sum_{r=0}^{\infty}(-1)^r z^{2r} q^{\frac{1}{2}r(5r+7)} \frac{(zq^2;q)_r}{(1-zq^2)(q;q)_r} (zq - \cdot z^2 q^{2r+3}) \\
&= zq(1-zq^2)\sum_{r=0}^{\infty}(-1)^r (zq^2)^{2r} q^{\frac{1}{2}r(5r-1)} (1-zq^{2r+2}) \frac{(zq^2;q)_r}{(1-zq^2)(q;q)_r}.
\end{aligned}$$

Finally, we have,

$$\frac{G(z)}{(1-zq)} - G(zq) = zq(1-zq^2)G(zq^2). \tag{A.2}$$

A.2. Power Series

Theorem A.2.1. *Let $G(q)$ and $H(q)$ be power series,*

$$G(q) := \sum_{n=0}^{\infty} a(n)q^n,$$

$$H(q) := \sum_{n=0}^{\infty} b(n)q^n.$$

If the power series equals each other for all $|q| < R$ *for some* $R > 0$*,*

$$G(q) = H(q),$$

then we know that $a_n = b_n$ *for all nonnegative integers* n*.*

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