Nonlinear Fractional Boundary Value Problems: Existence Results and Approximation of Solutions

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DOI
10.4233/uuid:3c618b5c-cb6f-4336-83ed-355f5ec331c4

Publication date
2025

Document Version
Final published version

Citation (APA)
Pantova, D. H. (2025). *Nonlinear Fractional Boundary Value Problems: Existence Results and Approximation of Solutions*. [Dissertation (TU Delft), Delft University of Technology]. https://doi.org/10.4233/uuid:3c618b5c-cb6f-4336-83ed-355f5ec331c4

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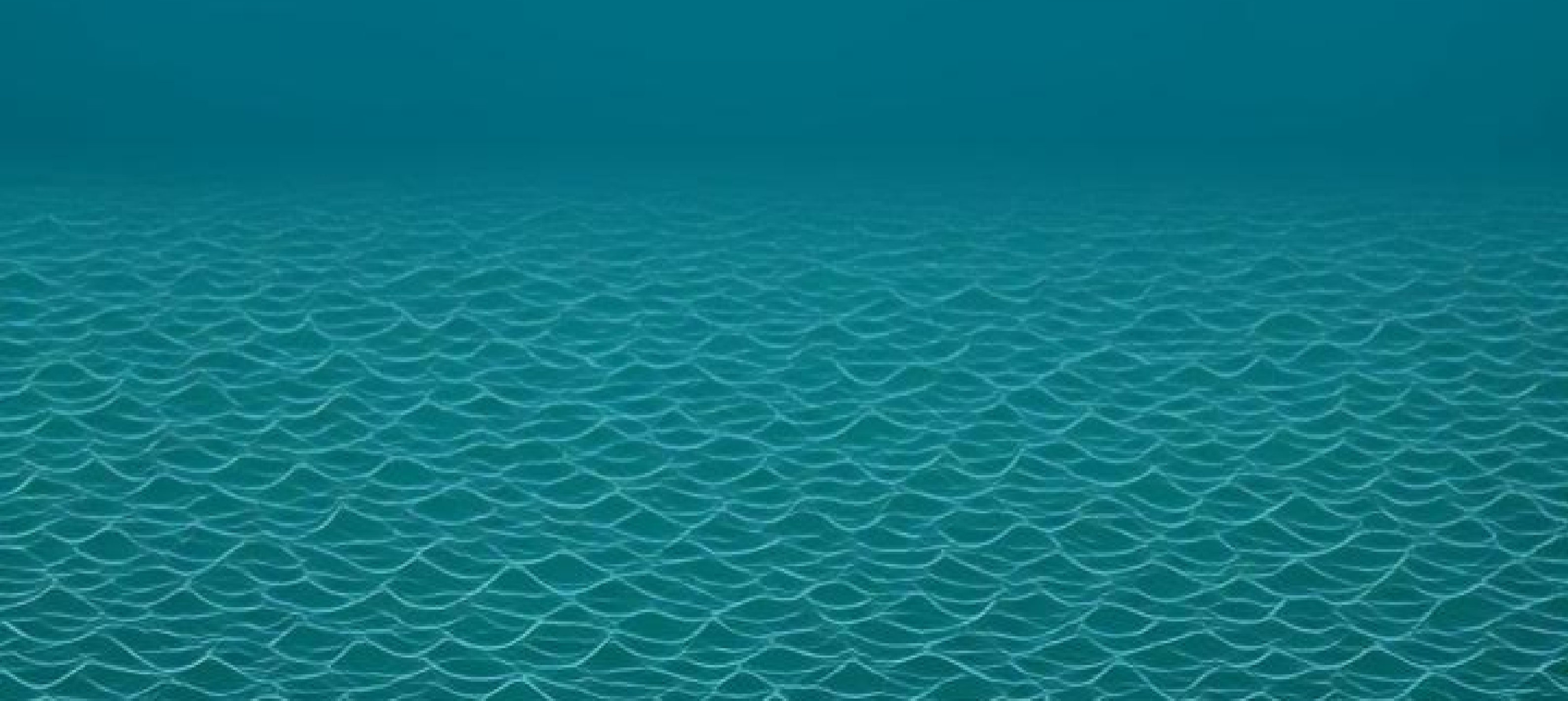
Dona Pantova



Nonlinear Fractional Boundary Value Problems: Existence Results and Approximation of Solutions

Dissertation

for the purpose of obtaining the degree of doctor
at Delft University of Technology
by the authority of the Rector Magnificus, prof. dr. ir. T.H.J.J. van der Hagen,
chair of the Board for Doctorates
to be defended publicly on
Wednesday, 10 December, 2025 at 10:00 o'clock

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Keywords: Caputo fractional differential equations, Numerical-analytic methods, Constructive approximations, Parametrization

Printed by: Proefschriften

Cover by: Dona Pantova

ISBN 000-00-0000-000-0

An electronic copy of this dissertation is available at https://repository.tudelft.nl/.

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1

INTRODUCTION

1.1. GENERAL INTRODUCTION

1.1.1. A BRIEF HISTORY OF FRACTIONAL CALCULUS

Fractional calculus generalizes the concepts of integrals and derivatives from integer to arbitrary orders. The first known mention of a derivative of non-integer order dates back to a correspondence between Marquis de L'Hospital and Gottfried Leibniz [1, 2]. In a letter from 1695, Leibniz, who introduced the notation for the integer order derivative $d^n y/dx^n$, was asked by L'Hospital, *'What if n is $1/2$?'*. To this, Leibniz replied, *'This is an apparent paradox, from which, one day, useful consequences will be drawn.'*

Although there were subsequent mentions of derivatives of fractional order during the following century, the first example of fractional calculus operators arising in the context of a physics problem did not appear until 1823, when Niels Henrik Abel published his paper *Opløsning af et par opgaver ved hjælp af bestemte integrale* [3]. In it, Abel set out to give an analytical solution of the tautochrone problem, which poses the following question: Suppose that a bead of mass m starts sliding down a thin frictionless wire from point $P(x^*, y^*)$ under the force of gravity, with initial velocity zero (see Figure 1.1). What is the shape of the curve $\psi(y) = x$ such that the slide time $t(y^*)$ - the time it takes for the bead to reach the lowest point on the wire, $y = 0$ - is independent of the bead's initial position?

Since the wire is assumed to be frictionless, the law of energy conservation can be applied. Denoting the arc length by $\phi(t)$ and applying the conservation law yields a separable equation, which can be integrated to solve for the slide time. The expression for the time $t(y^*)$ it takes for the bead to travel from $y = y^*$ to $y = 0$ is given by

$$t(y^*) = \int_0^{y^*} \phi(y)(y^* - y)^{-1/2} dy. \tag{1.1.1}$$

This is known as Abel's equation of the first kind, and its solution, obtained using the Laplace transform, gives the equation of the curve in the tautochrone problem:

$$\phi(y) = \frac{1}{\pi} \frac{d}{dy^*} \int_0^{y^*} t(y^*)(y^* - y)^{-1/2} dy. \tag{1.1.2}$$

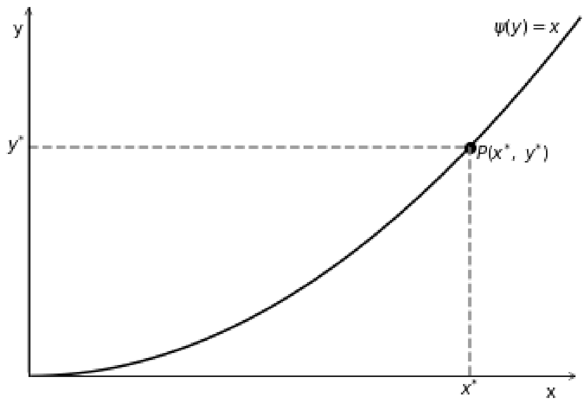


Figure 1.1: A possible curve $\psi(y) = x$ for the bead in the tautochrone problem.

As shown in Chapter 2, Equations (1.1.1) and (1.1.2) coincide with the Riemann-Liouville fractional integral D^{-p} and derivative D^p of order $p = 1/2$ of the functions $\phi(t)$ and $t(y^*)$, respectively, differing only by a constant factor. Moreover, the equations establish the inverse relationship between the two operators:

$$t(y^*) = \sqrt{\pi} D^{-1/2} \phi(y),$$
$$\phi(y) = \frac{1}{\pi} D^{1/2} t(y^*).$$

This suggests that the fundamental theorem of calculus extends to the fractional case. Although Abel did not pursue these ideas further, his work on the tautochrone problem marks the first appearance of fractional calculus in the mathematical description of a physical phenomenon.

His elegant solution likely inspired the first major effort toward a formal definition of a derivative of fractional order, made by Joseph Liouville in 1834. Subsequently, Riemann published a paper in which he built upon Liouville's work and proposed an alternative definition of a fractional integration operator [1, 2]. Since both definitions presented certain difficulties, the pursuit of mathematically rigorous definitions of fractional calculus operators continued and ultimately led to the explicit formulation of the central problem:

Define an operator D^p, such that, for every function $f(z)$, $g(z) = D^p f(z)$ satisfies the following criteria:

1. If $f(z)$ is an analytic function of z, then $g(z)$ is also an analytic function.
2. When p is an integer, the operator D^p must agree with the corresponding derivative of integer order. If $p = -n$ is a negative integer, D^{-p} must agree with the n-fold integral.
3. The operator must be linear.
4. When $p = 0$, the operator must leave the function unchanged.
5. The law of exponents must hold for integration of arbitrary order, i.e.,

$$D^{-p} D^{-q} f(z) = D^{-p-q} f(z).$$

The works of Sonin, Letnikov, and Laurent [4–6] eventually led to the first rigorous definitions of a fractional integral and derivative, satisfying the above-mentioned criteria. These operators are now known as the Riemann-Liouville fractional integral and derivative.

Today, a wide variety of definitions for fractional derivatives exist, differing in the methods used to derive them, the properties they satisfy, and their applicability. Some examples of fractional derivatives include the Grunwald-Letnikov, Hadamard, Weyl, Riesz, and variable-order fractional derivatives, [7–11], as well as some more recent formulations, such as the Caputo-Fabrizio, the Atangana-Beleanu, and the Sun-Hao-Zhang-Baleanu operators [12–14].

1.1.2. WHY FRACTIONAL OPERATORS?

Having established the theoretical framework for fractional calculus, it is natural to ask where these concepts find practical application. Integer-order differential equations have long served as a fundamental mathematical tool for modeling phenomena across physics, engineering, and other sciences. However, their local nature poses a limitation on their applicability. Many processes of interest to scientists and engineers exhibit memory effects or non-local behavior, meaning that their present state depends on the history of their previous states. This makes descriptions based on integer-order models insufficient, since they assume instantaneous responses and local interactions, neglecting the system's dependence on its past states. Fractional differential equations (FDEs), by contrast, provide a natural framework for modeling such processes, as they incorporate long-range interactions and memory effects into the governing equations. FDEs can account for the system's historical states, making them better suited to describe hereditary and non-local phenomena. Unlike delay equations which are suitable for systems with known sharp lags, a fractional calculus framework can deal with processes with continuous memory, which has proven valuable in various fields. Two classic examples of FDE applications are *viscoelasticity* and *anomalous diffusion*. In both cases, fractional operators not only provide a more accurate mathematical framework but also naturally emerge from fundamental physical principles, as discussed below.

A key example of *memory-dependent behavior* is observed in viscoelastic materials, which exhibit a combination of the stress-strain ($\sigma - \epsilon$) relationship characteristic of both ideal elastic (Hookean solids) and viscous (Newtonian fluids) elements. The deformation of a Hookean solid is proportional to the strain applied to it, i.e. $\sigma = k\epsilon$, where k is a material constant. This property is sometimes referred to as having *full memory*. On the other hand, the deformation of a Newtonian fluid is proportional to the rate of change of the strain, i.e. $\sigma = \eta\dot{\epsilon}$, where η is the fluid viscosity, which is known as having *no memory*. Viscoelastic materials are intermediate, that is, they exhibit *fading memory*, meaning that they only retain recent parts of their history. This is illustrated in Figure 1.2 which shows the stress response to a constant strain (a) and the strain response to a stress step (b) of an elastic material ($p = 0$, red lines), a viscous material ($p = 1$, blue lines) and viscoelastic materials with varying values of p ($p = 0.1, ..., 0.9$, grey lines). The orange, yellow and green lines correspond to $p = 0.3, 0.5, 0.7$. It can be seen from the figure that the grey lines, corresponding to fractional values of p, are intermediate between the $p = 0$ (elastic) and $p = 1$ (viscous) cases.

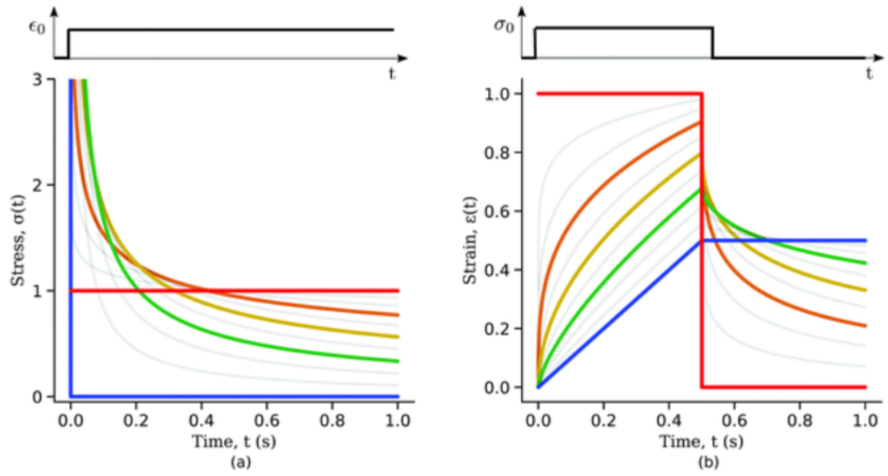


Figure 1.2: Stress $\sigma(t)$ (a) and strain $\epsilon(t)$ (b) responses of elastic (red), viscous (blue) and viscoelastic (grey) elements. Reproduced from [15].

While classical models attempt to represent viscoelastic materials as combinations of elastic and viscous elements, they fail to capture the memory-dependent behavior observed in real materials. These models result in stress-strain responses described by exponential functions, which cannot account for the power-law relationships exhibited by viscoelastic materials in experiments dating back to the 1920s [16]. This discrepancy prompted the development of new modeling approaches using operators of fractional order, pioneered by Gemant and Scott-Blair [17–19]. Since fractional derivatives of orders between 0 and 1 can be seen as interpolating between the identity operator and the first order derivative, it seems natural to employ them in the description of viscoelasticity. In the simplest case, this involves introducing elements in the models where the deformation is proportional to the fractional derivative, i.e., $\sigma = c_p D^p \epsilon$, where c_p is a parameter that lacks a clear physical interpretation in this context. Fractional order models have since gained popularity in the study of viscoelastic materials, as they offer a better fit to experimental data compared to classical models. Figure 1.3 shows an example of the measured stress responses (dots) of four different materials and the predicted values obtained with classical linear models (blue solid lines) and fractional models (red dashed lines). The schematic in each plot illustrates the number of elastic (k_i), viscous (η_i), and fractional (c_β, β) elements included in the corresponding model, which in turn determines the number of parameters to be fitted. In the classical model shown in panel (a), there are two elastic and one viscous elements. Panels (b) through (d) show classical models with three elastic and two viscous elements each. For the fractional models, panel (a) includes one fractional and one elastic element; panel (b) includes one fractional and one viscous element; and panels (c) and (d) each include one fractional, one elastic, and one viscous element. It is clear from the plot in panel (a) that the fractional model is in better agreement with the collected data than the classical one, even though both models use three parameters. The performance of the two models for the other three materials is comparable, but the fractional models use fewer parameters, improving efficiency.

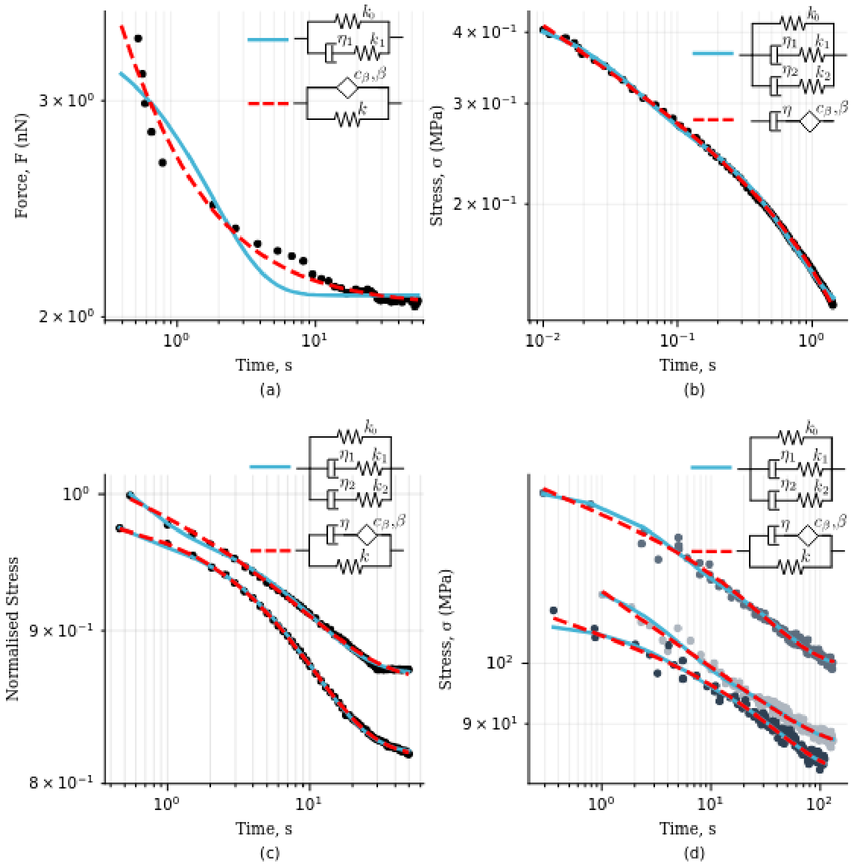


Figure 1.3: Stress responses fitted with a classical and a fractional model for (a) zonal articular chondrocytes, (b) tomato mesocarp cells, (c) PCL/bio-active glass and (d) collagen fibrils. Reproduced from [15].

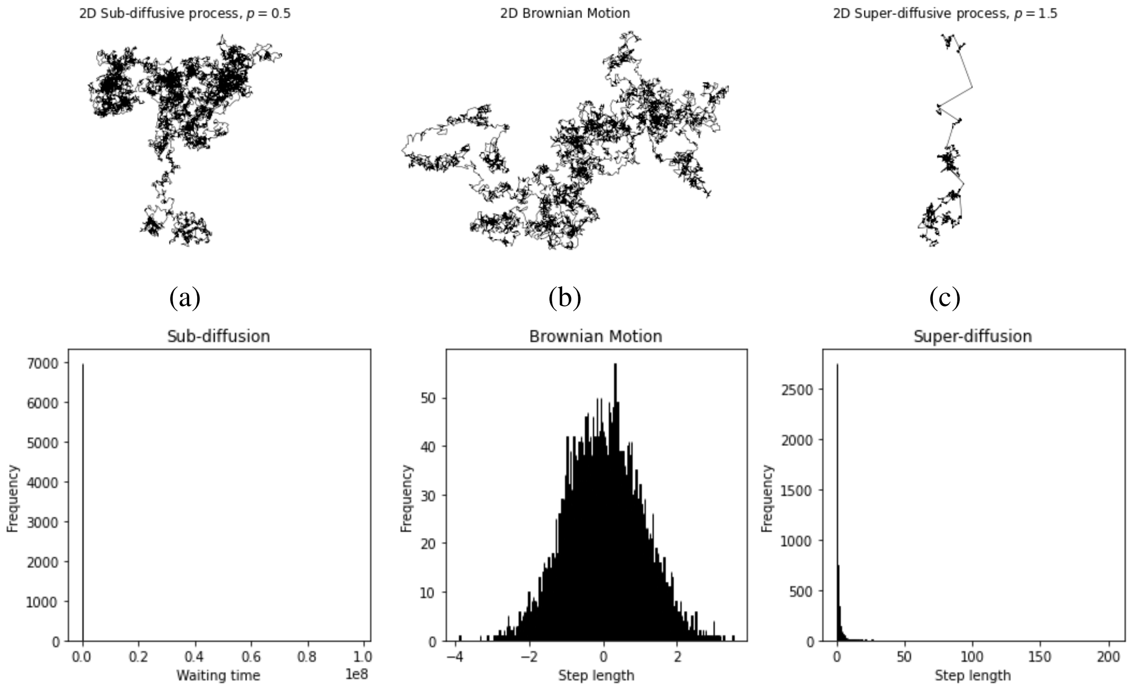
In addition to the empirical evidence supporting the utility of using fractional derivatives in the study of viscoelasticity, a fractional order model can be derived from physical principles. According to the Boltzmann principle of superposition, the stress response of a viscoelastic material is proportional to the cumulative effect of its past deformations, or in mathematical terms,

$$\sigma(t) = \int_0^t G(t-s)\dot{\epsilon}(s)\, ds,$$

where $G(t)$ is a memory kernel and $\dot{\epsilon}(s)$ is the shear rate [20]. When $G(t)$ takes the form of a power law, the stress-strain relationship is expressed by the Caputo fractional derivative. The decaying kernel of the fractional derivative accounts for the *fading* memory of the material, and the derivative order serves as an adjustable parameter which depends on the specific material. Experiments confirm the utility of fractional order models in the study of viscoelasticity, spanning polymer science, structural materials, bioengineering applications, and more [21–28].

Beyond viscoelasticity, *anomalous diffusion* constitutes another class of phenomena where fractional derivatives arise naturally due to their capacity to capture memory effects and non-local interactions. Anomalous diffusion refers to a process where the mean square displacement of a particle, $\langle x^2 \rangle$, is not linearly proportional to the elapsed time, as in standard diffusion, but instead obeys a power law relationship, i.e. $\langle x^2 \rangle \propto t^p$. When $p < 1$, the process is slower than standard diffusion and is termed *sub-diffusion*; when $p > 1$, it is faster and is called *super-diffusion*. The probability density functions associated with anomalous diffusion deviate from the Gaussian distributions typical of Brownian motion, usually displaying heavy tails or skewness. Figure 1.4 illustrates two-dimensional trajectories for three cases: (a) a sub-diffusive process with $p = 0.5$, (b) Brownian motion with $p = 1$, and (c) a super-diffusive process with $p = 1.5$. In the super-diffusive case, the particle step sizes follow a Pareto distribution with exponent $p = 1.5$, allowing for large jumps with higher probability than in the Gaussian case. Conversely, in the sub - diffusive case, long waiting times - also Pareto-distributed with exponent $p = 0.5$ - slow the particle's motion. Panel (d) of Figure 1.4 presents histograms showing the statistical features of these processes: the distribution of waiting times for sub-diffusion (left), and the distributions of step sizes for Brownian motion (middle) and super-diffusion (right). As expected, Brownian motion displays a Gaussian distribution, whereas both sub- and super-diffusive cases exhibit skewed, heavy-tailed distributions, confirming their departure from classical behavior.

Integer-order diffusion equations fail to describe these anomalous processes accurately because they lack the ability to account for long-range temporal or spatial dependencies. In such contexts, fractional-order operators provide a more suitable framework. Sub-diffusive processes can be modeled with time-fractional derivatives, which capture the system's history, through long waiting times. Similarly, space-fractional derivatives capture the occurrence of large, rare jumps characteristic of super-diffusion. When both phenomena are present - i.e., long waiting times and long jumps - a combination of time- and space-fractional derivatives is required.



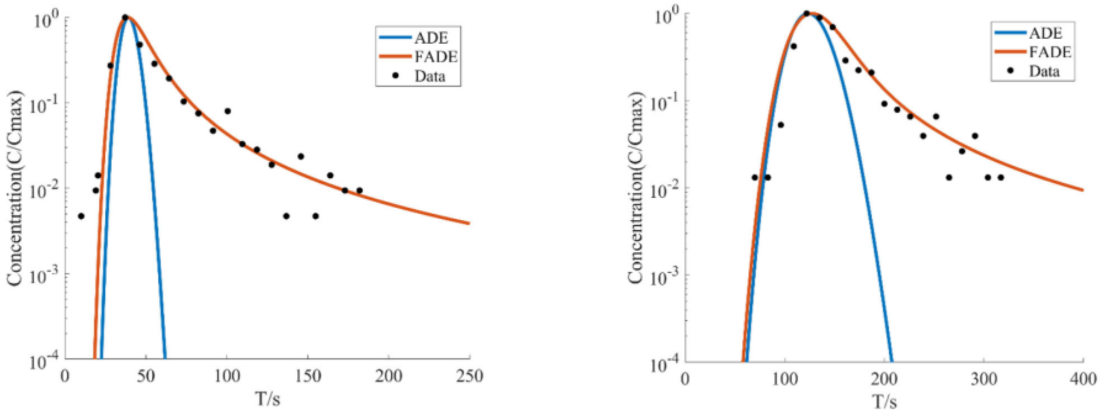
(d) Distributions of the waiting times (left) and step sizes of normal (middle) and anomalous (right) diffusion processes.

Figure 1.4: 2D trajectories, waiting time and step size distributions for sub- and super-diffusion and Brownian motion. All walks are drawn for 7000 steps.

One practical application of fractional models is in the study of solute transport in fractured porous media, a problem relevant to groundwater contamination. Due to the heterogeneous and complex nature of such environments, classical advection-diffusion equations (ADEs) often fail to provide accurate predictions. In [29], the authors employ a time-fractional ADE to model solute concentrations in various fracture geometries, comparing the results against experimental measurements and classical ADE predictions. Figure 1.5 displays the comparison for two cases involving different flow velocities and fracture widths. The fractional model (solid red lines) shows significantly better agreement with empirical data (dots) than the classical ADE (blue lines).

The time and space fractional derivatives appear as the scaling limits of the governing equations of stochastic processes, such as continuous random walks and Levy flights, see for instance [30, 31]. Although referred to as ‘anomalous’, these processes are, in fact, quite ubiquitous; the authors of [32] made the claim that *anomalous is the new normal*. Examples of such processes include diffusion of particles within cell membranes, transport phenomena in plasma and heterogeneous media, fluid flows in porous media, turbulent flows, the fluctuations of financial markets, etc. [33–39].

Fractional calculus operators have also been applied to various other fields, including control theory, electric circuits, capacitor theory, biology, wave propagation through complex media, see for instance [7, 40–42] and the references therein.



(a) (b)

Figure 1.5: Solute concentrations in fractures with crack sized and flow velocities 4cm, 20.36mm/s (a) and 9cm, 6.16mm/s. Reproduced from [29].

1.1.3. FRACTIONAL BOUNDARY VALUE PROBLEMS WITH 'SPECIAL TYPE' BOUNDARY CONDITIONS

In many scientific and engineering fields, modeling processes require solving boundary value problems (BVPs), which provide mathematical descriptions of the behavior of systems under specific conditions. Besides the traditional boundary conditions used in physical modeling - such as periodic, anti-periodic, Dirichlet, and Neumann conditions - particular attention has been given by both applied scientists and pure mathematicians to what we refer to as 'special type' constraints. These include non-local or integral restrictions, multi-point and nonlinear conditions, as well as parameter-dependent and asymptotic constraints. Some examples of where such conditions find applications include problems involving diffusion (e.g., specifying total mass), wave propagation and fluid dynamics (e.g., requiring specific long-term behavior) [43–48].

These specialized boundary conditions can arise in systems where the underlying dynamics exhibit non-local interactions or memory effects - features that are not well-captured by classical, integer-order models. As a result, there has been a growing interest in fractional boundary value problems (FBVPs), which generalize classical BVPs by incorporating fractional-order differential operators and thus provide a more suitable framework for describing such systems. FBVPs are studied both for their relevance in certain scientific and engineering applications and for their intrinsic theoretical interest in other cases [49–52].

With the increased interest in FBVPs, considerable effort has been devoted to both the theoretical analysis and the numerical computation of their solutions. Foundational results for fractional differential equations (FDEs) under various boundary conditions have been established using tools such as fixed-point theorems, variational methods, and monotone operator theory [53–56]. Since many real-world systems are governed by nonlinear FDEs - whose exact solutions are often unattainable - there is a strong demand for efficient and reliable approximation techniques. A review of the literature reveals several well-established methods applicable to the fractional setting, including series expansion methods, the Grünwald-Letnikov approach, and both direct and indirect schemes, such as Adams-type approximations and quadrature-based techniques [7, 57]. However, the non-local nature of fractional derivatives poses significant challenges for both analysis and computation. Fundamental questions regarding the existence, uniqueness, and stability of solutions require sophisticated mathematical tools and techniques. Moreover, many of the existing approximation methods rely on prior knowledge of initial values or conditions for the solution - information that is not always available in practical applications. This limitation highlights the ongoing need for more flexible and robust approaches to solving FBVPs. One such approach is the *numerical-analytic technique*, which offers a key advantage over the aforementioned solvers. The method was first developed for the study of periodic BVPs for ODEs [58], and since its inception has been extended to non-periodic BVPs with various boundary conditions, as well as to FBVPs [59–63]. It incorporates complex boundary constraints through appropriate parametrization, enabling the derivation of closed-form approximate solutions governed by a set of numerically computed parameters. The following subsection provides a brief overview of this method.

1.1.4. THE NUMERICAL-ANALYTIC METHOD

Numerical-analytic methods combine the analytical representation of approximate solutions to BVPs with the numerical computation of parameters which govern the behavior of those solutions. The technique discussed here is based on an iterative scheme, which allows us to derive a sequence of approximate solutions in analytic form and use its properties to show the solvability of the BVP under consideration. The boundary conditions are incorporated through a suitable parametrization, which grants the method its flexibility, since it allows it to handle various types of boundary constraints, as well as the case when the initial conditions are unknown. The sequence of approximate solutions depends on the artificially introduced parameters whose values are unknown and thus have to be calculated numerically by solving an algebraic or transcendental equation, i.e. a so-called determining equation, at each iteration. The constructed sequence is used to prove the existence and uniqueness of solutions to the BVP under certain conditions.

In more concrete terms, consider the FBVP

$$ {}_0^C D_t^p u(t) = f(t, u(t)), \tag{1.1.3} $$
$$ u(0) = u_1,\ u(T) = u_2, \tag{1.1.4} $$

where ${}_0^C D_t^p$ denotes the Caputo fractional derivative of order p, defined in (2.1.9); $p \in (1, 2]$ and $t \in [0, T]$. Let $u'(0) := \chi$, and consider it as a parameter whose value will be calculated. Equation (1.1.3) is modified by adding a term $\Delta(\chi)$ to the right-hand side function:

$$ {}_0^C D_t^p u(t) = f(t, u(t)) + \Delta(\chi). \tag{1.1.5} $$

The Riemann-Liouville fractional integral is applied to both sides of the equation to obtain

$$u(t;\chi) = u_1 + \chi t + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, u(s;\chi)) ds + \frac{t^p}{\Gamma(p+1)} \Delta(\chi),$$

and the term $\Delta(\chi)$ is determined using the boundary condition at $t = T$. With this, we obtain the integral equation for the solution of the modified FBVP (1.1.5), (1.1.4)

$$u(t;\chi) = u_1 + \chi t + (u_2 - u_1 - \chi T) \left(\frac{t}{T}\right)^p$$
$$+ \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f(s, u(s;\chi)) - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s, u(s;\chi)) ds \right].$$

The sequence of approximation is derived based on the equation above:

$$u_0(t;\chi) = u_1 + \chi t + (u_2 - u_1 - \chi T) \left(\frac{t}{T}\right)^p$$
$$u_m(t;\chi) = u_0(t;\chi) + \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f(s, u_{m-1}(s;\chi)) ds \right.$$
$$\left. - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s, u_{m-1}(s;\chi)) ds \right].$$

It can be shown that the constructed sequence of functions is continuous, satisfies the given boundary conditions, and converges uniformly to a limit function, which is the unique solution of the original FBVP (1.1.3), (1.1.4), provided that the term $\Delta(\chi)$ satisfies

$$\Delta(\chi) = 0.$$

In practice, the values of χ are calculated by solving an approximate equation

$$\Delta_m(\chi) = 0$$

at each step, where $\Delta_m(\chi)$ depends on $u_m(t;\chi)$.

The derivation of the integral equation and approximating sequence, as well as how they are used to show the existence of unique solutions to the FBVP under consideration, will be discussed in more detail in Chapters 3 and 4.

There remains a gap in the theory of nonlinear FBVPs, subject to 'special type' boundary conditions, as well as in the development of effective approximation methods for their solutions. In particular, the systematic analysis of FBVPs with integral constraints, parameter-dependent and asymptotic boundary conditions has not yet been fully established. Moreover, the monotonicity behavior of solutions to FBVPs remains largely unexplored. The numerical-analytic technique is a suitable tool for addressing these challenges. The aim of the present thesis is to contribute to bridging these gaps.

1.2. OUTLINE OF THE THESIS

This thesis is dedicated to the development of constructive methods for approximation of solutions to nonlinear BVPs in the fractional setting, subject to different types of boundary conditions. The main problems considered are:

(P1) Solvability analysis and approximation of solutions of the fractional BVPs with two-point and integral boundary conditions;

(P2) Constructive approximations and monotonicity behavior of solutions of the fractional BVPs with parameter-dependent right-hand side;

(P3) Solvability analysis and approximation of solutions of the fractional BVPs with parameter-dependent boundary conditions and a boundary condition at infinity.

For all problems (**P1**) - (**P3**), the setting consists of a coupled system of n fractional ordinary differential equations of the Caputo type (possibly of mixed order) with a nonlinear, non-autonomous right-hand side function, subject to Dirichlet, integral, (**P1**), parameter-dependent, (**P2**), and asymptotic, (**P3**), boundary conditions. The numerical-analytic method, originally developed for ordinary differential equations of the integer order, is extended to the fractional setting. This extension allows us to establish a theoretical framework for the existence and uniqueness of solutions to FBVPs and to construct sequences of approximations in analytic form. We introduce parametrization techniques to simplify the given constraints, enabling their incorporation into the approximating sequences. Moreover, the convergence speed of the sequence is enhanced by an interval splitting method, which decomposes the original FBVPs into smaller, so-called *model-type* problems. We further refine the interval-splitting method by considering a FBVP defined on a domain of unknown arbitrary length. Conditions are provided for the existence of bounded solutions to a FBVP defined on a semi-infinite domain and subject to asymptotic boundary conditions. We also investigate the existence of solutions to a FBVP with a parameter-dependent right-hand side function and study the monotonicity behavior of the constructed sequence of approximations. All theoretical results are supported by illustrative examples demonstrating the effectiveness of the proposed techniques.

The main part of the thesis is organized in three chapters. Below, we briefly summarize the contents of each chapter.

Chapter 2 introduces the definitions and properties of the relevant fractional calculus operators, as well as some more general auxiliary results, which are used throughout the thesis.

Chapter 3 is concerned with the study of non-linear FBVPs, subject to Dirichlet boundary conditions, that is, it aims to tackle the first part of (**P1**) (FBVPs with Dirichlet conditions) and (**P2**). In Section 3.1 we use a numerical-analytic technique to construct a sequence of successive approximations to the solution of a system of FDEs, subject to Dirichlet boundary conditions. We prove the uniform convergence of the sequence of approximations to a limit function, which is the unique solution to the BVP under consideration, and give necessary and sufficient conditions for the existence of solutions. The obtained theoretical results are confirmed by a model example. Section 3.2 deals with a parameter-dependent non-linear FDE, subject to Dirichlet boundary conditions. Using fixed point theory, we restrict the parameter values to secure the existence and uniqueness of solutions, and analyze the monotonicity behavior of the solutions. Additionally, we apply a numerical-analytic technique, coupled with the lower and upper solutions method, to construct a sequence of approximations to the BVP and give conditions for its monotonicity. The theoretical results are confirmed by an example of the Antarctic Circumpolar Current equation in the fractional setting.

Chapter 4 deals with FBVPs with more complex boundary constraints, defined on intervals on which we cannot guarantee the convergence of the numerical-analytic technique. It addresses the second part of (**P1**) (FBVPs with integral constraints) and (**P3**). In Section 4.1 we study a system of non-linear FDEs, subject to integral boundary conditions. We use a parametrization technique and a dichotomy-type approach to reduce the original problem to two *model-type* FBVPs with linear two-point boundary conditions. A numerical-analytic technique is applied to analytically construct approximate solutions to the *model-type* problems. The behavior of these approximate solutions is governed by a set of parameters, whose values are obtained by numerically solving a system of algebraic equations. The obtained results are confirmed by an example of the fractional order problem which in the case of the second order differential equation models the Antarctic Circumpolar Current. In Section 4.2 we study a non-linear fractional differential equation, defined on a finite and infinite intervals. In the finite interval setting, we attach initial conditions and parameter-dependent boundary conditions to the problem. We apply a dichotomy approach, coupled with the numerical-analytic method to analyze the problem and to construct a sequence of approximations. Additionally, we study the existence of bounded solutions in the case when the FDE is defined on the half-axis and is subject to asymptotic conditions. Our theoretical results are applied to the Arctic gyre equation in the fractional setting on a finite interval.

Chapter 5 summarizes the main contributions of the thesis and addresses the research questions (**P1**)–(**P3**). It highlights the development and application of the numerical-analytic method to FBVPs, and discusses how the results address the core objectives of the study. In addition, the chapter outlines potential directions for future work, including extending the method to more complex systems and exploring its applicability to partial FDEs, and pursuing a deeper theoretical understanding of the qualitative behavior of fractional-order systems.

2

DEFINITIONS AND AUXILIARY STATEMENTS

In this chapter we give some definitions and properties of the fractional calculus operators, as well as some preliminary statements, which are used throughout this thesis.

2.1. THE FRACTIONAL CALCULUS OPERATORS AND SOME OF THEIR PROPERTIES

In the following chapters we analyse FBVPs of the Caputo type, where the fractional Caputo derivative is defined on a finite interval or on the half-axis. We introduce the definitions of the Riemann-Liouville fractional integral and the Caputo fractional derivative. The following fractional operator definitions are valid for complex, as well as real orders of integration and differentiation, $p \in \mathbb{C}$. In this work we only consider fractional operators of the real order, hence we restrict ourselves to the $p \in \mathbb{R}^+$ case. In particular, we assume that the order of the operators p is such that $n-1 < p < n$ for some $n \in \mathbb{Z}^+$. The notation ${}_aI_t^p$ is used for the fractional integral operator of order p instead of writing it as the fractional derivative of negative order.

We begin by defining some function spaces and special functions, closely related to fractional calculus. These following definitions and properties can be found in Chapters 1 and 2 of [42], unless otherwise indicated.

Let $AC([a,b])$ be the space of functions which are absolutely continuous on the interval $[a,b]$. A function $u(t)$ belongs to the space $AC^n([a,b])$ if it has absolutely continuous derivatives on $[a,b]$ up to order $n-1$, i.e.

$$AC^n([a,b]) := \{u(t) : [a,b] \to \mathbb{R} : \ u^{(n-1)}(t) \in AC([a,b])\}.$$

Equivalently, $u(t) \in AC^n([a,b])$ if $u(t) \in C^{n-1}([a,b])$ and $u^{(n)} \in L^1([a,b])$, where $L^1([a,b])$ denotes the space of Lebesgue integrable functions.

The weighted space of functions $C_\gamma[a,b]$, for $0 \le \gamma < 1$, consists of functions $u(t)$ on $(a,b]$, such that $(t-a)^\gamma u(t) \in C[a,b]$, where $C[a,b]$ denotes the space of continuous functions on $[a,b]$, i.e.

$$C_\gamma[a,b] := \{u(t) : [a,b] \to \mathbb{R} : \ (t-a)^\gamma u(t) \in C[a,b]\}.$$

The case $\gamma = 0$ corresponds to $C[a,b]$.

The Banach space of functions $C^n_\gamma[a,b]$, for $n \in \mathbb{N}$, contains functions $u(t)$ on $(a,b]$, which are continuously differentiable on $[a,b]$ up to order $n-1$, and have derivative of order n in the space of weighted functions $C_\gamma[a,b]$, i.e.

$$C^n_\gamma[a,b] := \{u(t) : [a,b] \to \mathbb{R} \ \ u(t) \in C^{n-1}[a,b] \ \text{ and } \ u^{(n)}(t) \in C_\gamma[a,b]\}.$$

Definition 2.1.1. *The Gamma function $\Gamma(z)$ for $\Re(z) > 0$ is defined by the Euler integral of the second kind, as follows*

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt. \tag{2.1.1}$$

Property 1. *The Gamma function, defined in (2.1.1) satisfies the following functional equation*

$$\Gamma(z+1) = z\Gamma(z). \tag{2.1.2}$$

Since $\Gamma(1) = 1$, and (2.1.2) holds, it is clear that the Gamma function is a generalization of the factorial. In particular, when $z \in \mathbb{Z}$, we have

$$\Gamma(z+1) = z!.$$

Definition 2.1.2. *The Beta function $B(z_1, z_2)$ for $\Re(z_1) > 0$, $\Re(z_2) > 0$ is defined by the Euler integral of the first kind*

$$B(z_1, z_2) := \int_0^1 t^{z_1-1}(1-t)^{z_2-1} dt. \tag{2.1.3}$$

Property 2. *The Beta function, (2.1.3) can be written in terms of the Gamma function via the following formula*

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}. \tag{2.1.4}$$

The left-sided Riemann-Liouville fractional integral and derivative of $u(t)$ on a finite interval are defined as follows:

Definition 2.1.3. *The left-sided Riemann-Liouville fractional integral of $u(t)$ of order p on a finite interval, as defined in [7], is given by*

$$_aI_t^p u(t) := \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} u(s) ds, \quad t > a. \tag{2.1.5}$$

When $p = n \in \mathbb{Z}^+$, the integral in (2.1.5) reduces to the $n-$th repeated integral:

$$
{}_aI_t^n u(t) := \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}u(s)ds.
$$

Definition 2.1.4. *The left-sided Riemann-Liouville fractional derivative of $u(t)$ of order $n-1 \leq p < n$ on a finite interval, as defined in [7], is given by*

$$
{}_aD_t^p u(t) := \frac{1}{\Gamma(n-p)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-p-1}u(s)ds. \tag{2.1.6}
$$

From (2.1.5) and (2.1.6) it is clear that

$$
{}_aD_t^p u(t) = \left(\frac{d}{dt}\right)^n ({}_aI_t^{n-p}u(t)).
$$

Thus, if $p = n-1$, we see that the Riemann-Liouville fractional derivative coincides with the integer-order derivative, [7]

$$
\begin{aligned}
{}_aD_t^{n-1}u(t) =& \left(\frac{d}{dt}\right)^n ({}_aI_t^1 u(t)) \\
=& \left(\frac{d}{dt}\right)^{n-1} u(t).
\end{aligned}
$$

Similarly to the operators in (2.1.5) and (2.1.6), the right-sided Riemann-Liouville fractional integral and derivative are defined as:

Definition 2.1.5. *[7] The right-sided Riemann-Liouville fractional integral of $u(t)$ of order p on a finite interval is given by*

$$
{}_tI_b^p u(t) := \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}u(s)ds, \quad t < b. \tag{2.1.7}
$$

Definition 2.1.6. *[7] The right-sided Riemann-Liouville fractional derivative of $u(t)$ of order $n-1 \leq p < n$ on a finite interval is given by*

$$
{}_tD_b^p u(t) := \frac{1}{\Gamma(n-p)}\left(\frac{d}{dt}\right)^n \int_t^b (s-t)^{n-p-1}u(s)ds. \tag{2.1.8}
$$

When $u(t)$ describes a process evolving in time, and t is the present moment, the left-sided fractional integral (2.1.5) is an operator which acts on all of the past states of $u(t)$ $(a \leq s \leq t)$, while the right-sided integral (2.1.7) acts on its future states $(t \leq s \leq b)$. The same holds for the left- and right-sided fractional derivatives. Since the current state of a dynamical system is expected to depend on its past states, rather than its future ones, which are unknown, we will only consider left-sided fractional integral and derivatives. However, both definitions are mathematically equally valid.

One of the most commonly used fractional derivatives in the modeling of physical processes is the Caputo derivative. It was first introduced in [64], where Caputo studied a linear model of dissipation, and was independently formulated two years later by M. M. Djrbashian and A. B. Nersesian, see [65, 66]. The operator is defined as follows:

Definition 2.1.7. *[7] The left-sided Caputo fractional derivative of $u(t)$ of order $n-1 < p < n$ on a finite interval is given by*

$$ {}_a^C D_t^p u(t) := \frac{1}{\Gamma(n-p)} \int_a^t (t-s)^{n-p-1} u^{(n)}(s) ds, \tag{2.1.9} $$

where $u^{(n)}(t)$ denotes the $n-$th order classical derivative of $u(t)$.

In particular, when $0 < p < 1$, the Caputo fractional derivative reads

$$ {}_a^C D_t^p u(t) := \frac{1}{\Gamma(1-p)} \int_a^t \frac{u'(s)}{(t-s)^p} ds. $$

In the following chapters, the derivative is considered for the range $n-1 < p < n$, but it is worth pointing out that the method under consideration is also valid when $p = n$, see [58].

From a practical perspective, the main advantage of the Caputo formulation over the Riemann-Liouville one is that the former allows for prescribing integer order initial and boundary conditions to FDEs, as will be further explained in the present chapter. This broadens the applicability of FDEs, since it enables the incorporation of initial and boundary conditions with a clear physical interpretation.

The definitions of the left-sided fractional integral and Caputo derivative, given in (2.1.5) and (2.1.9) on a finite interval can be extended to the half-axis as follows:

Definition 2.1.8. *[42] The left-sided Riemann-Liouville fractional integral of $u(t)$ of order p on the half-axis $\mathbb{R}^+$ is given by*

$$ I_-^p u(t) := \frac{1}{\Gamma(p)} \int_t^\infty (s-t)^{p-1} u(s) ds. \tag{2.1.10} $$

Definition 2.1.9. *[42] The left-sided Caputo fractional derivative of $u(t)$ of order p on the half-axis $\mathbb{R}^+$ is given by*

$$ {}^C D_-^p u(t) := \frac{(-1)^n}{\Gamma(n-p)} \int_t^\infty (s-t)^{n-p-1} u^{(n)}(s) ds. \tag{2.1.11} $$

Similarly, the right-sided fractional integral and Caputo derivative can also be defined on the half-axis.

From the definitions in (2.1.9) and (2.1.11), it is clear that the Caputo fractional derivative of a function $u(t)$ can be expressed in terms of the Riemann-Liouville fractional integral of the integer-order derivative of $u(t)$, namely,

$$ {}_a^C D_t^p u(t) = ({}_a I_t^{n-p} u^{(n)})(t) \tag{2.1.12} $$

in the finite interval case, and

$$ {}^C D_-^p u(t) = (I_-^{n-p} u^{(n)})(t) \tag{2.1.13} $$

in the case of the half-axis.

Computing derivatives of fractional order is more complicated than in the case of ordinary derivatives, however, in some cases we can give general formulae for the Caputo fractional derivatives. We will demonstrate this in the following example, where we derive an expression for the Caputo derivatives of the power function.

Example 2.1.1. *The Caputo fractional derivative of order p, $n-1 < p < n$, of a power function $f(t) = t^\alpha$, $\alpha \in \mathbb{R}$, is given by*

$$ {}_a^C D_t^p (t^\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-p+1)} t^{\alpha-p} \tag{2.1.14}$$

for $\alpha > n-1$, and it is 0 otherwise.

When $\alpha > n-1$, we use the fact that

$$\frac{d^n}{dt^n}(t^\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)} t^{\alpha-n}$$

and the substitution $s = \mu t$, $0 < \mu \le 1$ in definition (2.1.9) to obtain

$$\begin{aligned} {}_a^C D_t^p (t^\alpha) =& \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)} \int_0^t (t-s)^{n-p-1} s^{\alpha-n} ds \\ =& \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)} t^{\alpha-p} \int_0^1 (1-\mu)^{n-p-1} \mu^{\alpha-n} d\mu. \end{aligned}$$

From the definition of the Beta function, (2.1.3) and its relationship to the Gamma function, (2.1.4), it follows that

$$\begin{aligned} {}_a^C D_t^p (t^\alpha) =& \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)} t^{\alpha-p} B(\alpha-n+1, n-p) \\ =& \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)} \frac{\Gamma(\alpha-n+1)\Gamma(n-p)}{\Gamma(\alpha-p+1)} t^{\alpha-p} \\ =& \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-p+1)} t^{\alpha-p}. \end{aligned}$$

If $\alpha \le n-1$ and $\alpha \in \mathbb{Z}^+$, then

$$\frac{d^n}{dt^n}(t^\alpha) = 0,$$

which implies that ${}_a^C D_t^p (t^\alpha) = 0$. Otherwise, if $\alpha \le n-1$ and $\alpha \notin \mathbb{Z}^+$, the Caputo derivative does not exist.

It is worth noting that from this it immediately follows that the Caputo derivative of a constant is equal to 0. This is not true for all other fractional derivatives, which in some cases makes the Caputo derivative preferable to use in models.

In particular, the Caputo derivatives of $f(t) = t^2$ of orders $p = 1/2$ and $p = 3/2$ are given by

$$
\begin{aligned}
{}_a^C D_t^{1/2}(t^2) =& \frac{2}{\Gamma(2.5)} t^{3/2}, \\
{}_a^C D_t^{3/2}(t^2) =& \frac{2}{\Gamma(1.5)} t^{1/2}.
\end{aligned}
$$

Figure 2.1 shows plots of the Caputo derivatives of $f(t)$ for various orders, $p \in [0, 1]$ in the left panel and $p \in [1, 2]$ in the right panel. Clearly, as p approaches n, the Caputo derivative converges towards the ordinary derivative of order n.

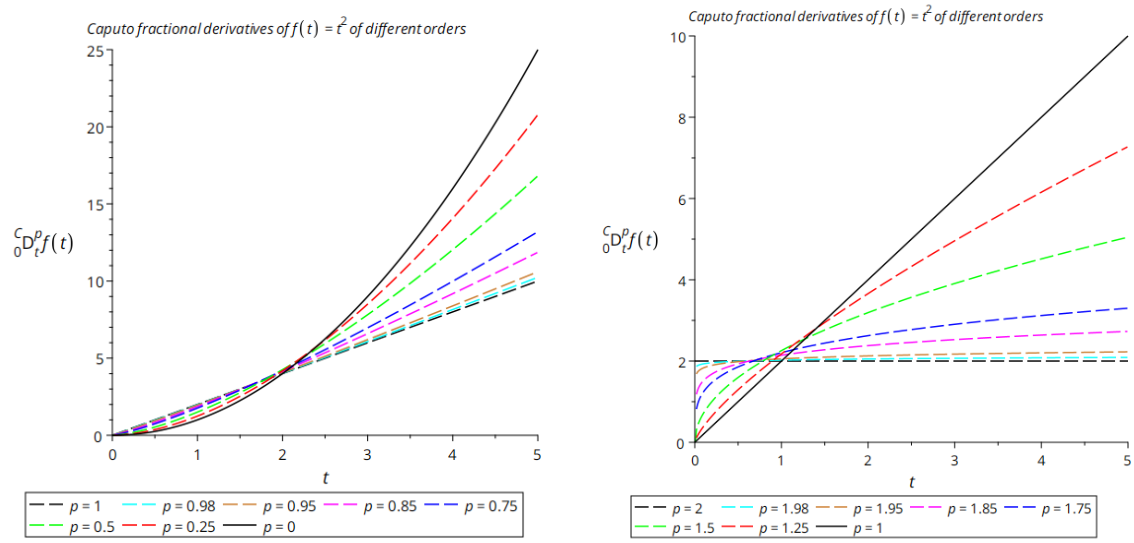


Figure 2.1: Plots of the Caputo fractional derivatives of the function $f(t) = t^2$ for different values of the derivative order $p \in [0, 1]$ (left) and $p \in [1, 2]$ (right). (Color online)

Next, we give some results on fundamental properties of the Riemann-Liouville fractional integral and the Caputo fractional derivative, such as existence, continuity, and the relationship between the two operators, which will be relevant in further chapters.

The following two theorems give conditions for the existence and continuity of the Caputo fractional derivative. For more details and proofs of the statements, we refer to Theorems 2.2 and 2.3 in [42].

Theorem 2.1.1. *Let $u(t) \in AC^n([a, b])$. Then the left-sided Caputo fractional derivative ${}_a^C D_t^p u(t)$ exists almost everywhere on $[a, b]$.*

Theorem 2.1.2. *Let $u(t) \in C^n([a, b])$. Then the left-sided Caputo fractional derivative ${}_a^C D_t^p u(t)$ is continuous on $[a, b]$, i.e. ${}_a^C D_t^p u(t) \in C([a, b])$.*

Some of the basic properties of the Caputo fractional derivative, such as linearity, how it commutes with integer order derivatives, and its Laplace transform, which can be found in [7], are stated below.

Property 3. *The Caputo fractional derivative is a linear operator, i.e. for any $a, b \in \mathbb{R}$*

$$ {}_a^C D_t^p (au(t) + bv(t)) = a {}_a^C D_t^p u(t) + b {}_a^C D_t^p v(t). \tag{2.1.15}$$

Property 4. *For the Caputo fractional derivative, and $m \in \mathbb{Z}$, it holds that*

$$ {}_a^C D_t^p \Big(\frac{d^m}{dt^m} u(t) \Big) = \frac{d^m}{dt^m} ({}_a^C D_t^p u(t))) \tag{2.1.16}$$

Property 5. *The Laplace transform of the Caputo fractional derivative is given by*

$$ \mathcal{L}\{ {}_a^C D_t^p u(t)\} = s^p U(s) - \sum_{k=0}^{n-1} s^{p-k-1} u^{(k)}(0). \tag{2.1.17}$$

The Caputo differential operator has the advantage over other fractional operators that applying the Laplace transform to it yields an expression containing the initial value of the function, $u(0)$ and its integer order derivatives, which have clear interpretations. This is not the case for the Riemann-Liouville fractional derivative, but can be preferable for modeling physical processes, where the initial conditions are given in terms of the function value and its classical derivative.

Theorem 2.1.3 combines Theorems 2.1 and 2.2 in [67] to state the extremum principle for the Caputo fractional derivative. A proof of the statement can be found in [67].

Theorem 2.1.3. *Let a function $u \in C^2(0,1) \cap C[0,1]$ attain its maximum (minimum) over the interval $[0,1]$ at the point $t_0 \in (0,1]$. Then the Caputo derivative of the function u is non-positive (non-negative) at the point t_0 for any p, ${}_a^C D_t^p u(t_0) \leq 0$ (${}_a^C D_t^p u(t_0) \geq 0$), $1 < p \leq 2$.*

The semigroup property holds for the Riemann-Liouville fractional integral, defined both on a finite interval and on the half-axis. The following two results can be found in Lemmas 2.3 and 2.19 in [42].

Lemma 2.1.1. *(i) Let $u(t) \in L_p(a,b)$. Then the relation*

$$ ({}_a I_t^q \, {}_a I_t^r u)(t) = {}_a I_t^{q+r} u(t). \tag{2.1.18}$$

holds almost everywhere on $[a,b]$. If $q + r > 1$, then (2.1.18) holds everywhere on $[a,b]$.

(ii) Let $q, r > 0$, $p \geq 1$, and $q + r < 1/p$. Then if $u(t) \in L_p(\mathbb{R}^n)$, the following semigroup property holds:

$$ (I_-^q I_-^r u)(t) = I_-^{q+r} u(t). \tag{2.1.19}$$

Lemmas 2.1 and 2.2 in [42] state and prove the relationships between the Riemann-Liouville fractional integral and the Caputo fractional derivative operators, defined on a finite interval. The results of these two lemmas are summarized below, in the case of left-sided integrals and derivatives.

Lemma 2.1.2. *(i) Let $u(t) \in L_\infty(a,b)$ or $u(t) \in C[a,b]$. Then*

$$({}_a^C D_t^p \, {}_a I_t^p) u(t) = u(t). \tag{2.1.20}$$

(ii) Let $u(t) \in AC^n[a,b]$ or $u(t) \in C^n[a,b]$. Then

$$({}_a I_t^p \, {}_a^C D_t^p) u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}(t-a)^k. \tag{2.1.21}$$

Finally, we state and prove the relationship between the Riemann-Liouville integral and the Caputo derivative in the case when they are defined on the half-axis.

Lemma 2.1.3. *Let $u(t) \in C^n(\mathbb{R}^+)$ and $p \in (n-1,n)$. Then*

$$(I_-^p \, {}^C D_-^p u)(t) = u(t) + \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{k!} \lim_{\zeta \to \infty} u^{(k)}(\zeta)(\zeta - t)^k. \tag{2.1.22}$$

Proof. We will prove the lemma using mathematical induction. First note that from (2.1.13) and the semigroup property in (2.1.18) it follows that

$$(I_-^p \, {}^C D_-^p u)(t) = (-1)^n (I_-^p I_-^{n-p} d^{(n)} u)(t) = (-1)^n (I_-^n d^{(n)} u)(t). \tag{2.1.23}$$

Let $n = 1$, i.e. $p \in (0,1)$. Then, according to (2.1.23) and the definition in (2.1.8), we have

$$\begin{aligned}
(I_-^p \, {}^C D_-^p u)(t) &= (-1)(I_-^1 u')(t) \\
&= \frac{-1}{\Gamma(1)} \int_t^\infty u'(s)ds = -\lim_{\zeta \to \infty} \int_t^\zeta u'(s)ds \\
&= -\lim_{\zeta \to \infty} [u(\zeta) - u(t)] = u(t) - \lim_{\zeta \to \infty} u(\zeta),
\end{aligned}$$

that is, (2.1.22) holds for $n = 1$. Now assume that (2.1.22) holds for some $p \in (n-1,n)$, that is,

$$\begin{aligned}
(I_-^p \, {}^C D_-^p u)(t) &= \frac{(-1)^n}{\Gamma(n)} \lim_{\zeta \to \infty} \int_t^\zeta (s-t)^{n-1} u^{(n)}(s)ds \\
&= u(t) + \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{k!} \lim_{\zeta \to \infty} u^{(k)}(\zeta)(\zeta - t)^k,
\end{aligned} \tag{2.1.24}$$

and consider $(I_-^q \, {}^C D_-^q u)(t)$ for some $q \in (m-1,m)$, where $m = n+1$. Applying integration by parts and the induction hypothesis (2.1.24) yields

$$
\begin{aligned}
(I_{-}^{q}\,{}^{C}D_{-}^{q}u)(t) &= \frac{(-1)^m}{\Gamma(m)} \lim_{\zeta\to\infty} \int_t^{\zeta} (s-t)^{m-1} u^{(m)}(s)ds \\
&= \frac{(-1)^{n+1}}{\Gamma(n+1)} \lim_{\zeta\to\infty} u^{(n)}(\zeta)(\zeta-t)^n + \frac{(-1)^n}{\Gamma(n)} \lim_{\zeta\to\infty} \int_t^{\zeta} (s-t)^{n-1} u^{(n)}(s)ds \\
&= \frac{(-1)^{n+1}}{\Gamma(n+1)} \lim_{\zeta\to\infty} u^{(n)}(\zeta)(\zeta-t)^n + u(t) + \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{k!} \lim_{\zeta\to\infty} u^{(k)}(\zeta)(\zeta-t)^k \\
&= u(t) + \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k!} \lim_{\zeta\to\infty} u^{(k)}(\zeta)(\zeta-t)^k,
\end{aligned}
$$

that is, (2.1.22) holds for $n = 1$, and, assuming that it holds for arbitrary n, we have shown that it also holds for $n+1$. Hence, the relation in (2.1.22) is true for all $n \geq 1$. □

2.2. PRELIMINARY RESULTS

This section contains some definitions and general results, which are used throughout the following chapters.

We first give some definitions, related to metric spaces, and state the well-known Mean Value Theorem. All of these can be found in any standard analysis book, see for example [68], and will be relevant in the following chapters.

Definition 2.2.1. *Let $M(X, d)$ be a metric space and $x \in X$. Then the open ball of radius r of x in M is defined as*

$$B(x, r) := \{y \in X : d(x, y) < r\}. \tag{2.2.1}$$

Definition 2.2.2. *A metric space $M(X, d)$ is complete if every Cauchy sequence converges in it.*

Definition 2.2.3. *A Banach space is a normed linear space which is complete, i.e. every Cauchy sequence converges to a limit in the space. The space of continuous real-valued functions $C([a, b], \mathbb{R}^n)$ is a Banach space with respect to the supremum norm.*

Theorem 2.2.1. *(Mean Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be a function, such that f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$, such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The definition of the spectral radius of a matrix, which can be found in [69], is referred to repeatedly in the following chapters, hence, we state it here:

Definition 2.2.4. *Let $Q \in \mathbb{R}^{N\times N}$ be a matrix with eigenvalues $\lambda_1, ..., \lambda_N$. Then the spectral radius of Q, $r(Q)$, is defined as*

$$r(Q) = \max\{|\lambda_1|, ..., |\lambda_N|\}. \tag{2.2.2}$$

In Sections 3.1 and 4.2 we use tools from topological degree theory in order to study the solvability of the FBVPs under consideration. Therefore, here we introduce the definitions and theorems from the field which are used in these sections.

Definition 2.2.5. *[70] (Homotopic vector fields) Let X be a topological space, and V_1 and V_2 two vector fields on X. V_1 and V_2 are homotopic if there exists a continuous map $P : X \times [0, 1] \to X$, such that $P(x, \theta)$ is a vector field on X, $P(x, 0) = V_1$, and $P(x, 1) = V_2$ for all $x \in X$.*

Definition 2.2.6. *[71] (Regular and singular values) A value $p = f(x)$ is called regular if the Jacobian of f $J_f(x) \neq 0$ for all $x \in f^{-1}(p) := \{y \in \bar{\Omega} : f(y) = p\}$. If for some $x \in f^{-1}(p)$, $J_f(x) = 0$, then p is called a singular value.*

Definition 2.2.7. *[71] (Brouwer degree) Let $f \in C(\bar{\Omega})$ with p a regular value, such that $p \notin \partial\Omega$, where $\bar{\Omega}$ is the closure of the bounded open set $\Omega \in \mathbb{R}^n$, and $\partial\Omega$ is its boundary. Then we define the Brouwer degree as*

$$deg(f, \Omega, p) = \sum_{x\in f^{-1}(p)} sign(J_f(x)),$$

where $deg(f, \Omega, p) = 0$ if $f^{-1}(p) = \emptyset$.

Theorem 2.2.2. *[71] (Brouwer fixed point theorem) Let $H \subset \mathbb{R}^n$ be a closed bounded convex subset. If $f \in C(H, H)$, then there exists a point $x^0 \in H$, such that $f(x^0) = x^0$.*

Theorem 2.2.3. *[71] Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and $f : \bar{\Omega} \to \mathbb{R}^n$ a continuous mapping. If $p \notin f(\partial\Omega)$, then there exists an integer $deg(f, \Omega, p)$, satisfying the following properties:*

1. (Solvability) If $deg(f, \Omega, p) \neq 0$, then $f(x) = p$ has a solution in Ω;

2. (Homotopy) If $f_t(x) : [0, 1] \times \bar{\Omega} \to \mathbb{R}^n$ is continuous and $p \notin \cup_{t\in[0,1]} f_t(\partial\Omega)$, then $deg(f_t, \Omega, p)$ does not depend on $t \in [0, 1]$.

Theorem 2.2.4. *[70] (Banach fixed point theorem) Let $M(X, d)$ be a complete metric space and $f : X \to X$ be a contraction, i.e. there exists a constant $0 \leq c < 1$, such that, for $x, x' \in X$,*

$$d(f(x), f(x')) \leq cd(x, x').$$

Then f has a unique fixed point, that is, there exists a unique element $x \in X$, such that $f(x) = x$.

Definition 2.2.8. *[72] Let $H \subset \mathbb{R}^n$ be a non-empty set. For any pair of functions*

$$f_j = col(f_{j,1}(x), ..., f_{j,n}(x)) : H \to \mathbb{R}^n, \quad j = 1, 2$$

the following statement holds

$$f_1 \triangleright_H f_2$$

if and only if there exists a function $k : H \to \{1, 2, ..., n\}$, such that

$$f_{1,k(x)} > f_{2,k(x)},$$

for all $x \in H$. It means that at least one of the components of $f_1(x)$ is less than the appropriate component of $f_2(x)$ in every point in H.

The study of FBVPs, presented in the following chapters, relies upon the reduction of the original problem to an equivalent initial value problem (IVP), and the connection between the IVP and a Volterra integral equation. The following theorem establishes the equivalence between a fractional IVP of the Caputo type and the corresponding integral equation.

Consider the non-linear fractional differential equation

$$ {}_a^C D_t^p u(t) = f(t, u(t)), \tag{2.2.3}$$

defined on the finite interval $[a, b]$, of order p, where $n - 1 < p < n$, with initial conditions

$$u^{(k)}(a) = b_k, \quad k = 0, ..., n - 1. \tag{2.2.4}$$

Theorem 2.2.5. *[42] Let p be such that $n - 1 < p < n$. Let G be an open set in $\mathbb{R}$ and let $f : (a, b] \times G \to \mathbb{R}$ be a function such that, for any $u \in G$, $f[t, u] \in C_\gamma[a, b]$ with $0 \le \gamma < 1$ and $\gamma \le p$. Let $r = n$ for $p \in \mathbb{N}$ and $r = n - 1$ for $p \notin \mathbb{N}$. If $u(t) \in C^r[a, b]$, then $u(t)$ satisfies the relations (2.2.3) and (2.2.4) if, and only if, $u(t)$ satisfies the Volterra integral equation (2.2.5):*

$$u(t) = \sum_{k=0}^{n-1} \frac{u^{(k)}(a)(t-a)^k}{k!} + \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} f(s, u(s)) ds. \tag{2.2.5}$$

The proof of the theorem can be found in Theorem 3.24, [42].

The results in the next two lemmas are used extensively throughout this thesis.

Lemma 2.2.1. *[73] If $f(t)$ is a continuous function on $t \in [t_1, t_2]$, then for all $t \in [t_1, t_2]$, the following estimate*

$$\begin{aligned} \frac{1}{\Gamma(p)} \left| \int_a^t (t-s)^{p-1} f(s) ds - \Big(\frac{t - t_1}{t_2 - t_1}\Big)^p \int_{t_1}^{t_2} (t_2 - s)^{p-1} f(s) ds \right| \\ \le \alpha_1(t) \max_{t_1 \le t \le t_2} |f(t)|, \end{aligned} \tag{2.2.6}$$

holds, where

$$\alpha_1(t) := \frac{2(t-t_1)^p}{\Gamma(p+1)}\Big(\frac{t_2-t}{t_2-t_1}\Big)^p. \tag{2.2.7}$$

Lemma 2.2.2. *[73] Let $\{\alpha_m(\cdot)\}_{m\geq 1}$ be a sequence of continuous functions on $t \in [a,b]$, given by*

$$\begin{aligned}\alpha_m(t) :=& \frac{1}{\Gamma(p)}\Bigg[\int_a^t \Big[(t-s)^{p-1} - \Big(\frac{t-a}{b-a}\Big)^p (b-s)^{p-1}\Big]\alpha_{m-1}(s)ds \\ &+ \Big(\frac{t-a}{b-a}\Big)^p \int_t^b (b-s)^{p-1}\alpha_{m-1}(s)ds\Bigg],\end{aligned}$$

where

$$\begin{aligned}&\alpha_0(t) := 1, \\ &\alpha_1(t) := \frac{2(t-a)^p}{\Gamma(p+1)}\Big(\frac{b-t}{b-a}\Big)^p.\end{aligned}$$

Then the estimate

$$\alpha_{m+1}(t) \leq \frac{(b-a)^{mp}\alpha_1(t)}{2^{[m(2p-1)]}[\Gamma(p+1)]^m} \leq \frac{(b-a)^{(m+1)p}}{2^{[(m+1)(2p-1)]}[\Gamma(p+1)]^{m+1}} \tag{2.2.8}$$

holds for $m \in \mathbb{Z}_0^+$.

The proofs of Lemmas 2.2.1 and 2.2.2 can be found in [73].

3

FRACTIONAL BVPS WITH DIRICHLET BOUNDARY CONDITIONS

This chapter focuses on the analysis of FBVPs subject to Dirichlet boundary conditions, and the construction of sequences of approximations to their solutions. Dirichlet boundary conditions are ubiquitous in many different applications due to their simple and intuitive physical interpretation. Moreover, they are mathematically easier to handle compared to other types of boundary constraints. In fact, as will be discussed in Chapter 4, one approach to dealing with boundary conditions of more complex forms is to reduce them to the Dirichlet-type. Therefore, understanding BVPs with Dirichlet conditions is foundational for the study of FBVPs subject to constraints of a more general form.

In the following two sections, we investigate the existence of solutions to FBVPs with a general non-linear right-hand side (Section 3.1) and with a parameter-dependent right-hand side function (Section 3.2). In both cases, we restrict the fractional derivative order to $p \in (1, 2]$, though all of the presented results can be easily extended to other orders. The numerical-analytic technique is adopted to construct sequences of approximating functions in closed form and is used in combination with the upper and lower solutions method in case of parameter-dependent FBVPs. The methods are implemented in *Maple* and applied to some illustrative examples.

3.1. THE FRACTIONAL BVP WITH DIRICHLET TYPE BOUNDARY CONDITIONS

We study a system of FDEs with Dirichlet boundary conditions. The original FBVP is reduced to an equivalent IVP. Using the connection between the solution of the fractional initial value problem (FIVP) and the integral equation, established in Theorem 2.2.5, we construct a sequence of functions, depending on a vector-parameter, which is found as a root of the so-called determining system of algebraic equations. We prove the uniform convergence of the sequence to a limit function, and show the relationship between the limit function and the original FBVP. Finally, we prove two results on the necessary and sufficient conditions for the existence of solutions of the FBVP. The obtained theoretical results and the effectiveness of the developed technique are confirmed with two examples. First, we apply the method to a non-linear equation with a known exact solution. This allows us to explicitly calculate the error between the terms of the constructed sequence and the exact solution. The technique is also applied to the gyre equation for the Antarctic Circumpolar Current considered in the fractional setting (for more details about the mathematical model of the Antarctic Circumpolar Current we refer to [46, 75–77]).

3.1.1. PROBLEM SETTING AND SEQUENCE DERIVATION

We consider a fractional differential system (FDS) of the general form

$$ {}_0^C D_t^p u(t) = f(t, u(t)) \tag{3.1.1} $$

for some $p \in (1, 2)$, and subject to the non-homogeneous Dirichlet boundary conditions

$$ u(0) = \alpha_1, \quad u(T) = \alpha_2. \tag{3.1.2} $$

Here ${}_0^C D_t^p(\cdot)$ is the Caputo fractional derivative with lower limit at 0, defined in (2.1.9), and $t \in [0, T]$.

The functions $u(t) := (u_1(t), ..., u_n(t))$ and $f(t, u(t)) := (f_1(t, u(t)), ..., f_n(t, u(t)))$ are vector-valued and the boundary conditions in (3.1.2) are given n-dimensional vectors.

In (3.1.1) we take the Caputo derivative of the same order, p, of each component of $u(t)$. However, it is worth noting that the method we use to analyse FBVP (3.1.1), (3.1.2) is also applicable in the more general case, where we allow Caputo fractional derivatives of different orders to act on the different components of $u(t)$.

It is assumed that the vector-valued function $u : [0, T] \to D$, which is the solution to the FBVP (3.1.1), (3.1.2), belongs to the space of continuously differentiable functions, $C^1([0, T], D)$, defined on the closed and bounded domain $D \subset \mathbb{R}^n$. Moreover, we assume that the right-hand side function in (3.1.1), $f : G \to \mathbb{R}^n$, is continuous, non-autonomous, and generally non-linear in $u(t)$, where the domain of $f(t, u(t))$ is

This section is based on the paper [74].

denoted by $G := [0, T] \times D$.

We aim to find a solution of the FDS (3.1.1), $u \in C^1([0,T], D)$, which satisfies the Dirichlet boundary conditions (3.1.2). In order to do so, we construct a sequence of approximating functions, which satisfy the differential system in (3.1.1), and the boundary conditions in (3.1.2), and show that this sequence converges uniformly to the exact solution of the FBVP (3.1.1), (3.1.2).

The numerical-analytic technique, outlined in Section 1.1.4, is used to derive the sequence of approximating functions. We want to transform the FBVP (3.1.1), (3.1.2) into an equivalent IVP, and to this end, we add a perturbation term Δ to the right-hand side of the system in (3.1.1). Since the term Δ is independent of t, applying the fractional integral (2.1.5) to both sides of the system and (2.1.21) in Lemma 2.1.2 yields

$$\begin{aligned}
({}_aI_t^p\, {}^C_aD_t^p u)(t) =& ({}_aI_t^p(f(t,u(t)) + \Delta) \\
\implies u(t) =& \alpha_1 + \chi_1 t + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}[f(s,u(s)) + \Delta]ds \\
=& \alpha_1 + \chi_1 t + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s,u(s))ds + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}\Delta ds.
\end{aligned}$$

Computing the second integral gives us the following integral equation for $u(t)$:

$$u(t) = \alpha_1 + \chi_1 t + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} f(s,u(s))ds + \frac{t^p\Delta}{\Gamma(p+1)}, \tag{3.1.3}$$

where $u(0) = \alpha_1$ is given in (3.1.2), and the value of the first derivative of the solution at $t = 0$, $u'(0)$, is denoted by

$$u'(0) := \chi_1.$$

Here χ_1 is an unknown vector parameter to be computed. The term Δ will be constructed in such a way to ensure that $u(t)$ in (3.1.3) satisfies both boundary conditions in the original FBVP (3.1.1), (3.1.2).

Let us impose the second boundary condition in (3.1.2):

$$u(T) = \alpha_1 + \chi_1 T + \frac{1}{\Gamma(p)}\int_0^T (T-s)^{p-1} f(s,u(s))ds + \frac{T^p\Delta}{\Gamma(p+1)} = \alpha_2,$$

which allows us to obtain the following expression for the perturbation term Δ

$$\Delta(\chi_1) = \Gamma(p+1)\frac{(\alpha_2 - \alpha_1 - \chi_1 T)}{T^p} - \frac{p}{T^p}\int_0^T (T-s)^{p-1} f(s,u(s))ds.$$

Plugging this back into Equation (3.1.3) yields the following integral equation

$$\begin{aligned}
u(t) =& \alpha_1 + \chi_1 t + (\alpha_2 - \alpha_1 - \chi_1 T)\left(\frac{t}{T}\right)^p + \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1} f(s,u(s))ds \\
&- \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s,u(s))ds\Bigg].
\end{aligned} \tag{3.1.4}$$

Based on this, we connect the FBVP (3.1.1), (3.1.2) to the following parametrized sequence of functions $\{u_m(\cdot;\chi_1)\}_{m\in\mathbb{Z}_0^+}$, $\mathbb{Z}_0^+ = \{0,1,2,...\}$, given by the iterative formula:

$$
\begin{aligned}
u_0(t;\chi_1) :=& \alpha_1 + \chi_1 t + (\alpha_2 - \alpha_1 - \chi_1 T)\left(\frac{t}{T}\right)^p, \\
u_m(t;\chi_1) :=& u_0(t,\chi_1) + \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1} f(s, u_{m-1}(s;\chi_1))ds \\
&- \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s, u_{m-1}(s;\chi_1))ds\Bigg],
\end{aligned} \tag{3.1.5}
$$

where $t \in [0,T]$, $u_0(t;\chi_1) \in D$, and $\chi_1 \in \Omega \subset \mathbb{R}$ is the value of the first derivative of $u(t)$ at $t = 0$, i.e. $u'(0) = \chi_1$.

We claim that under certain conditions on the right-hand side function in system (3.1.1), and the set of initial values $\alpha_1 \in D$, the approximating sequence in (3.1.5) converges uniformly to the exact unique solution of the integral equation (3.1.4). The necessary conditions for convergence and the proof of our claim are given in the following subsection.

3.1.2. SEQUENCE CONVERGENCE

In order to ensure the uniform convergence of sequence (3.1.5) to the unique solution of the integral equation (3.1.4), we assume that the following conditions hold for the FBVP (3.1.1), (3.1.2):

(i) The function $f(t,u(t))$ in system (3.1.1) is bounded by a constant vector $M = col(M_1, M_2, ..., M_n) \in \mathbb{R}^n$, i.e.

$$|f(t,u(t))| \le M, \tag{3.1.6}$$

for $t \in [0,T]$, $u \in D$.

(ii) The function $f(t,u(t))$ satisfies a Lipschitz condition with a non-negative real matrix $K = (k_{ij})_{i,j=1}^n$, i.e., the following inequality

$$|f(t,u_1) - f(t,u_2)| \le K|u_1 - u_2| \tag{3.1.7}$$

holds for $t \in [0,T]$, $u_1, u_2 \in D$.

Note that the operations $|\cdot|$, $=$, $\le$, max, *etc. between matrices and vectors are understood componentwise.*

(iii) The set

$$D_\beta := \{\alpha_1 \in D : \{|u - \alpha_1| \le \beta,\ u \in \mathbb{R}^n\} \subset D\} \tag{3.1.8}$$

is non-empty, where

$$\alpha_1 = u(0),$$

and

$$\beta = \frac{MT^p}{2^{2p-1}\Gamma(p+1)}. \tag{3.1.9}$$

(iv) The spectral radius $r(Q)$ of the matrix

$$Q := \frac{KT^p}{2^{2p-1}\Gamma(p+1)}, \tag{3.1.10}$$

defined in Def. 2.2.4, satisfies the inequality

$$r(Q) < 1. \tag{3.1.11}$$

In the following theorem we show that the terms in sequence (3.1.5) satisfy the boundary conditions, given in (3.1.2), and form a Cauchy sequence in the Banach space $C^1([0,T],D)$, which converges uniformly to a limit function, denoted by $u_\infty(t,\chi_1)$. This limit function also satisfies the boundary conditions in (3.1.2), and is the unique solution to the integral equation (3.1.4). Moreover, we establish an equivalence between the integral equation and a Cauchy problem for a modified system of FDEs. Finally, we give an estimate for the error between the terms of the sequence in (3.1.5) and the exact solution, $u_\infty(t;\chi_1)$.

Theorem 3.1.1. *Assume that conditions (3.1.6)-(3.1.11) hold for the FBVP (3.1.1), (3.1.2). Then for all fixed $\chi_1 \in \Omega$, it holds:*

1. Functions of the sequence (3.1.5) are continuous and satisfy Dirichlet boundary conditions

$$u_m(0;\chi_1) = \alpha_1, \quad u_m(T;\chi_1) = \alpha_2.$$

2. The sequence of functions (3.1.5) for $t \in [0,T]$ converges uniformly as $m \to \infty$ to the limit function

$$u_\infty(t;\chi_1) = \lim_{m\to\infty} u_m(t;\chi_1). \tag{3.1.12}$$

3. The limit function satisfies boundary conditions

$$u_\infty(0;\chi_1) = \alpha_1, \quad u_\infty(T;\chi_1) = \alpha_2.$$

4. The limit function (3.1.12) is a unique solution to the integral equation

$$\begin{aligned} u(t) =& \alpha_1 + \chi_1 t + (\alpha_2 - \alpha_1 - \chi_1 T)\left(\frac{t}{T}\right)^p + \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1} f(s,u(s))ds \\ &- \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s,u(s))ds\Bigg], \end{aligned} \tag{3.1.13}$$

i.e. it is a unique solution on $t \in [0,T]$ of the Cauchy problem for the modified system of FDEs:

$$\begin{aligned} {}_0^C D_t^p u(t) &= f(t,u(t)) + \Delta(\chi_1) \\ u(0) &= \alpha_1, \\ u'(0) &= \chi_1, \end{aligned} \tag{3.1.14}$$

where $\Delta : \Omega \to \mathbb{R}^n$ is a mapping defined by

$$\Delta(\chi_1) := \frac{(\alpha_2 - \alpha_1 - \chi_1 T)\Gamma(p+1)}{T^p} - \frac{p}{T^p}\int_0^T (T-s)^{p-1} f(s, u_\infty(s;\chi_1))ds. \tag{3.1.15}$$

5. *The following error estimate holds:*

$$|u_\infty(t;\chi_1) - u_m(t;\chi_1)| \leq \frac{T^p}{2^{2p-1}\Gamma(p+1)} Q^m (I_n - Q)^{-1} M, \tag{3.1.16}$$

where M and Q are defined by (3.1.6) and (3.1.10), and I_n is a unit $n \times n$ matrix.

Proof. (1) This follows directly since the sequence of functions (3.1.5) is constructed in such a way that it satisfies the Dirichlet boundary conditions (3.1.2) for $m \geq 0$.

(2) Now we prove that functions (3.1.5) form a Cauchy sequence in the Banach space $C([0,T], D)$. We first show that for an arbitrary point $(t,\chi_1) \in [0,T] \times \Omega$, the terms of the sequence (3.1.5) are contained in the domain D, i.e. $u_m(t;\chi_1) \in D, \forall m \geq 0$.

For $m = 1$ we have

$$\begin{aligned} u_1(t;\chi_1) =& \alpha_1 + \chi_1 t + (\alpha_2 - \alpha_1 - \chi_1 T)\left(\frac{t}{T}\right)^p \\ &+ \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} f(s, u_0(s;\chi_1))ds \right. \\ &\left. - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s, u_0(s;\chi_1))ds\right], \end{aligned}$$

hence,

$$\begin{aligned} |u_1(t;\chi_1) - u_0(t;\chi_1)| =& \left| \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} f(s, u_0(s;\chi_1))ds \right.\right. \\ &\left.\left. - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s, u_0(s;\chi_1))ds\right]\right| \\ \leq& \alpha_1(t) \max_{0\leq t\leq T} |f(t,u_0)| \leq \alpha_1(t) M \\ \leq& \frac{T^p M}{2^{2p-1}\Gamma(p+1)} = \beta, \end{aligned} \tag{3.1.17}$$

where we used the estimates in (2.2.6) and (2.2.8). This shows that, given an arbitrary point $(t,\chi_1) \in [0,T] \times \Omega$, the first term of the sequence belongs to the domain D, i.e.

$u_1(t, \chi_1) \in D$.

Similarly, by the principle of mathematical induction, for $m > 1$

$$
\begin{aligned}
|u_m(t;\chi_1) - u_0(t;\chi_1)| =& \Bigg| \frac{1}{\Gamma(p)} \Bigg[\int_0^t (t-s)^{p-1} f(s, u_{m-1}(s;\chi_1)) ds \\
& - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s, u_{m-1}(s;\chi_1)) ds \Bigg] \Bigg| \\
\leq & \alpha_1(t) \max_{0 \leq t \leq T} |f(t, u_{m-1}(s;\chi_1))| \\
\leq & \frac{T^p M}{2^{2p-1}\Gamma(p+1)} = \beta,
\end{aligned}
$$

which proves that for any arbitrary point $(t, \chi_1) \in [0, T] \times \Omega$, $u_m(t;\chi_1) \in D$ for all $m \geq 0$. That is, all terms of the iterative sequence (3.1.5) are contained in the closed and bounded domain D.

Now we will prove that the estimate

$$
|u_m(t;\chi_1) - u_{m-1}(t;\chi_1)| \leq K^{m-1} M \alpha_m(t) \leq Q^{m-1} M \alpha_1(t) \tag{3.1.18}
$$

holds for $m \geq 1$, where Q is defined in (3.1.10). When $m = 1$, (3.1.18) follows directly from (3.1.17). Assume (3.1.18) holds, and consider

$$
\begin{aligned}
& |u_{m+1}(t;\chi_1) - u_m(t;\chi_1)| \\
=& \Bigg| \frac{1}{\Gamma(p)} \Bigg[\int_0^t \left[(t-s)^{p-1} - \left(\frac{t}{T}\right)^p (T-s)^{p-1} \right] [f(s, u_m(s;\chi_1)) \\
& - f(s, u_{m-1}(s;\chi_1))] ds \\
& - \left(\frac{t}{T}\right)^p \int_t^T (T-s)^{p-1} [f(s, u_m(s;\chi_1)) - f(s, u_{m-1}(s;\chi_1))] ds \Bigg] \Bigg| \\
\leq & \frac{1}{\Gamma(p)} \Bigg[\int_0^t \left[(t-s)^{p-1} - \left(\frac{t}{T}\right)^p (T-s)^{p-1} \right] |f(s, u_m(s;\chi_1)) \\
& - f(s, u_{m-1}(s;\chi_1))| ds \\
& + \left(\frac{t}{T}\right)^p \int_t^T (T-s)^{p-1} |f(s, u_m(s;\chi_1)) - f(s, u_{m-1}(s;\chi_1))| ds \Bigg] \\
\leq & \frac{K}{\Gamma(p)} \Bigg[\int_0^t \left[(t-s)^{p-1} - \left(\frac{t}{T}\right)^p (T-s)^{p-1} \right] |u_m(s;\chi_1) - u_{m-1}(s;\chi_1)| ds \\
& + \left(\frac{t}{T}\right)^p \int_t^T (T-s)^{p-1} |u_m(s;\chi_1) - u_{m-1}(s;\chi_1)| ds \Bigg],
\end{aligned}
$$

where in the last step we used the Lipschitz continuity of $f(t, u(t))$, (3.1.7). By the induction hypothesis, and using the estimate in (2.2.8), we obtain

$$
\begin{aligned}
|u_{m+1}(t;\chi_1)-u_m(t;\chi_1)| \leq & K^m M\frac{1}{\Gamma(p)}\left\{\int_0^t\left[(t-s)^{p-1}-\left(\frac{t}{T}\right)^p(T-s)^{p-1}\right]\alpha_m(s)ds\right. \\
& \left.+\left(\frac{t}{T}\right)^p\int_t^T(T-s)^{p-1}\alpha_m(s)ds\right\} \\
= & K^m M\alpha_{m+1}(t)\leq K^m M\frac{T^{mp}\alpha_1(t)}{2^{[m(2p-1)]}[\Gamma(p+1)]^m} \\
= & Q^m M\alpha_1(t),
\end{aligned}
$$

for all $t\in[0,T]$, $u_0\in D$. That is, (3.1.18) holds for all $m\geq 1$.

In view of (3.1.18), we get the estimate

$$
\begin{aligned}
& |u_{m+j}(t;\chi_1)-u_m(t;\chi_1)| \\
& \quad \leq |u_{m+j}(t;\chi_1)-u_{m+j-1}(t;\chi_1)|+|u_{m+j-1}(t;\chi_1)-u_{m+j-2}(t;\chi_1)| \\
& \qquad +|u_{m+j-2}(t;\chi_1)-u_{m+j-3}(t;\chi_1)|+\ldots+|u_{m+1}(t;\chi_1)-u_m(t;\chi_1)| \\
& \quad =\sum_{k=1}^{j}|u_{m+k}(t;\chi_1)-u_{m+k-1}(t;\chi_1)|\leq\sum_{k=1}^{j}K^{m+k-1}M\alpha_{m+k}(t) \\
& \quad \leq\sum_{k=1}^{j}\frac{K^{m+k-1}T^{p(m+k-1)}M\alpha_1(t)}{2^{(m+k-1)(2p-1)}[\Gamma(p+1)]^{m+k-1}} \\
& \quad =\sum_{k=0}^{j-1}Q^{m+k}M\alpha_1(t)=Q^m\sum_{k=0}^{j-1}Q^kM\alpha_1(t).
\end{aligned}
$$

Since $r(Q)<1$, it holds that

$$
\lim_{n\to\infty}\sum_{k=0}^{n}Q^k\leq(I_n-Q)^{-1}\quad\text{and}\quad\lim_{m\to\infty}Q^m=O_n,
$$

where O_n denotes the $n\times n$ matrix of zeros.

Passing to the limit as $j\to\infty$ in the inequality

$$
|u_{m+j}(t;\chi_1)-u_m(t;\chi_1)|\leq Q^m\sum_{k=0}^{j-1}Q^kM\alpha_1(t),
$$

we obtain the estimate in (3.1.16). As $m\to\infty$, $|u_\infty(t;\chi_1)-u_m(t;\chi_1)|\to 0$, since $Q^m\to 0$. Thus, the sequence of functions in (3.1.5) converges uniformly to the limit function $u_\infty(t;\chi_1)$ in the domain $[0,T]\times D$, according to the Cauchy criteria.

(3) Since $u_\infty(t;\chi_1)$ is the limit of a sequence of functions (3.1.5), all of which satisfy boundary conditions (3.1.2), $u_\infty(t;\chi_1)$ also satisfies them. Passing in (3.1.5) to the limit as $m\to\infty$, we get that the function $u_\infty(t;\chi_1)$ is a solution to the integral equation

(3.1.13).

(4) Next, we show that the integral equation (3.1.13) has a unique continuous solution. To see that $u_\infty(t;\chi_1)$ is a solution to the integral equation (3.1.13), we pass to the limit as $m \to \infty$ in the sequence (3.1.5):

$$\begin{aligned}\lim_{m\to\infty} u_m(t;\chi_1) = \lim_{m\to\infty} \Bigg\{ &\alpha_1 + \chi_1 t + (\alpha_2 - \alpha_1 - \chi_1 T)\left(\frac{t}{T}\right)^p \\ &+ \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} f(s, u_{m-1}(s;\chi_1))ds \right.\\ &\left. - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s, u_{m-1}(s;\chi_1))ds\right]\Bigg\} \\ =&\alpha_1 + \chi_1 t + (\alpha_2 - \alpha_1 - \chi_1 T)\left(\frac{t}{T}\right)^p \\ &+ \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} \lim_{m\to\infty} f(s, u_{m-1}(s;\chi_1))ds \right.\\ &\left. - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} \lim_{m\to\infty} f(s, u_{m-1}(s;\chi_1))ds\right]\end{aligned}$$

On the left hand side we have

$$\lim_{m\to\infty} u_m(t;\chi_1) = u(t),$$

and on the right hand side we obtain

$$\begin{aligned}&\alpha_1 + \chi_1 t + (\alpha_2 - \alpha_1 - \chi_1 T)\left(\frac{t}{T}\right)^p + \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} \lim_{m\to\infty} f(s, u_{m-1}(s;\chi_1))ds \right.\\ &\left. - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} \lim_{m\to\infty} f(s, u_{m-1}(s;\chi_1))ds\right] \\ =&\alpha_1 + \chi_1 t + (\alpha_2 - \alpha_1 - \chi_1 T)\left(\frac{t}{T}\right)^p + \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} f(s, u(s))ds \right.\\ &\left. - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s, u(s))ds\right].\end{aligned}$$

Hence, the limit function $u_\infty(t;\chi_1)$ is a solution to the integral equation (3.1.13).

Now suppose $u_1(t)$ and $u_2(t)$ are two solutions of Equation (3.1.13), that is,

$$\begin{aligned}u_1(t) =&\alpha_1 + \chi_1 t + (\alpha_2 - \alpha_1 - \chi_1 T)\left(\frac{t}{T}\right)^p \\ &+ \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} f(s, u_1(s))ds - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s, u_1(s))ds\right],\end{aligned}$$

$$u_2(t) = \alpha_1 + \chi_1 t + (\alpha_2 - \alpha_1 - \chi_1 T)\left(\frac{t}{T}\right)^p + \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} f(s,u_2(s))ds - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s,u_2(s))ds\right].$$

We will show that it must hold that $u_1(t) = u_2(t)$. Consider

$$\begin{aligned}
|u_1(t)-u_2(t)| &\leq \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1}|f(s,u_1(s)) - f(s,u_2(s))|ds \right.\\
&\quad \left. + \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1}|f(s,u_1(s)) - f(s,u_2(s))|ds\right]\\
&\leq \frac{K}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1}|u_1(s) - u_2(s)|ds \right.\\
&\quad \left. + \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1}|u_1(s) - u_2(s)|ds\right]\\
&\leq \frac{K}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1}ds + \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1}ds\right] \max_{0\leq s\leq T}|u_1(s) - u_2(s)|\\
&= K\alpha_1(t) \max_{0\leq s\leq T}|u_1(s) - u_2(s)|\\
&\leq \frac{KT^p}{2^{2p-1}\Gamma(p+1)} \max_{0\leq s\leq T}|u_1(s) - u_2(s)| = Q \max_{0\leq s\leq T}|u_1(s) - u_2(s)|.
\end{aligned}$$

The inequality

$$|u_1(t) - u_2(t)| \leq Q \max_{0\leq s\leq T}|u_1(s) - u_2(s)|$$

holds for all $t \in [0,T]$, thus, taking the maximum over t on both sides yields

$$\max_{0\leq t\leq T}|u_1(t) - u_2(t)| \leq Q \max_{0\leq t\leq T}|u_1(t) - u_2(t)|,$$

which implies

$$\max_{0\leq t\leq T}|u_1(t) - u_2(t)| = 0,$$

since $Q < 1$. Thus, $u_1(t) = u_2(t)$ for all $t \in [0,T]$. Moreover, the FIVP (3.1.14) is equivalent to the integral equation

$$\begin{aligned} u(t) =& \alpha_1 + \chi_1 t + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1}[f(s,u(s)) + \Delta(\chi_1)]ds \\ =& \alpha_1 + \chi_1 t + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s,u(s))ds + \frac{t^p \Delta(\chi_1)}{\Gamma(p+1)} \\ =& \alpha_1 + \chi_1 t + \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f(s,u(s))ds \right. \\ & \left. - \left(\frac{t}{T}\right)^p \int_0^T (T-s)^{p-1} f(s,u_\infty(s;\chi_1))ds \right], \end{aligned} \tag{3.1.19}$$

where the perturbation $\Delta(\chi_1)$ is given by (3.1.15). Comparing (3.1.13) and (3.1.19) and recalling that $u_\infty(t;\chi_1)$ is the unique continuous solution of (3.1.13), it follows that $u(t) = u_\infty(t;\chi_1)$ in (3.1.19), i.e. $u_\infty(t;\chi_1)$ is the unique continuous solution of (3.1.14). This completes the proof. □

It has been shown that the sequence of functions (3.1.5) converges uniformly to the unique solution of the integral equation (3.1.13), or equivalently, of the Cauchy problem (3.1.14). Next, we establish a connection between the solution of the Cauchy problem for the modified system in (3.1.14) and the original FBVP (3.1.1), (3.1.2).

3.1.3. CONNECTION OF THE LIMIT FUNCTION TO THE ORIGINAL FBVP

Let us consider a Cauchy problem of the following form

$$ {}_0^C D_t^p u(t) = f(t,u(t)) + \mu, \quad t \in [0,T], \tag{3.1.20a}$$

$$ u(0) = \alpha_1, \tag{3.1.20b}$$

$$ u'(0) = \chi_1, \tag{3.1.20c}$$

where $\mu \in \mathbb{R}^n$ is referred to as a control parameter, $\alpha_1 \in D_\beta$ and $\chi_1 \in \Omega$.

In the following two theorems we establish the connection between the Cauchy problem (3.1.20) and the original FBVP (3.1.1), (3.1.2). We show that the solution to (3.1.20) also satisfies the second boundary condition in (3.1.2) if and only if the control parameter μ is given by (3.1.15). Furthermore, because the equation in (3.1.20) is perturbed by the Δ term, in order for the solution to (3.1.20) to coincide with that of the original FBVP (3.1.1), (3.1.2), it is necessary and sufficient for the unknown parameter χ_1 in (3.1.15) to be such that the perturbation term satisfies $\Delta(\chi_1) = 0$.

Theorem 3.1.2. *Let $\chi_1 \in \Omega$, $\mu \in \mathbb{R}^n$ be given vectors. Assume that all conditions of Theorem 3.1.1 are satisfied for the FDS (3.1.1). Then the solution $u = u(\cdot;\chi_1,\mu)$ of the Cauchy problem (3.1.20) also satisfies boundary conditions (3.1.2) if and only if*

$$\mu = \Delta(\chi_1), \tag{3.1.21}$$

where $\Delta(\chi_1)$ is given by (3.1.15), and in this case

$$u(t;\chi_1,\mu) = u_\infty(t;\chi_1) \ \text{ for } \ t \in [0,T], \tag{3.1.22}$$

where $u_\infty(t;\chi_1)$ is the limit function, defined in (3.1.12).

Proof. First note that the existence and uniqueness of the solution to the FIVP (3.1.20) on $t \in [0, T]$ and its continuous dependence on χ_1 and μ follow from the theory in [42].

Sufficiency. Suppose that

$$\mu = \Delta(\chi_1).$$

By Theorem 3.1.1, it follows that the limit function $u_\infty(t; \chi_1)$ of the sequence (3.1.5) is a unique solution to the equation in (3.1.20), which satisfies boundary conditions (3.1.2). Moreover, the limit function $u_\infty(t; \chi_1)$ also satisfies the initial conditions (3.1.20b), (3.1.20c). Thus, it is the unique solution to the Cauchy problem (3.1.20) for $\mu = \Delta(\chi_1)$, and $u(t; \chi_1, \mu) = u_\infty(t; \chi_1)$ holds. This also means that the equality in (3.1.22) takes place.

Necessity. Now we show that the parameter value in (3.1.21) is unique. Suppose that there exists another parameter $\bar{\mu}$, such that the solution $\bar{u}(t; \chi_1, \bar{\mu})$ to the FIVP

$$\begin{aligned} {}_0^C D_t^p u(t) &= f(t, u(t)) + \bar{\mu}, \quad t \in [0, T], \\ u(0) &= \alpha_1, \\ u'(0) &= \chi_1, \end{aligned}$$

also satisfies the boundary conditions in (3.1.2). Then, according to ([42], Cor. 3.24), the function $\bar{u}(t; \chi_1, \bar{\mu})$ is also a continuous solution to the integral equation

$$\bar{u}(t; \chi_1, \bar{\mu}) = \alpha_1 + \chi_1 t + \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} f(s, \bar{u}(s; \chi_1, \bar{\mu})) ds + \frac{t^p \bar{\mu}}{\Gamma(p + 1)}. \tag{3.1.23}$$

Moreover, $\bar{u}(t, \chi_1, \bar{\mu})$ satisfies the boundary conditions in (3.1.2) and the initial condition (3.1.20c), that is,

$$\begin{aligned} \bar{u}(0; \chi_1, \bar{\mu}) &= \alpha_1, \\ \bar{u}(T; \chi_1, \bar{\mu}) &= \alpha_2, \\ \bar{u}'(0) &= \chi_1. \end{aligned}$$

Substituting this into equation (3.1.23) for $t = T$, we obtain

$$\bar{\mu} = \frac{(\alpha_2 - \alpha_1 - \chi_1 T)\Gamma(p + 1)}{T^p} - \frac{p}{T^p} \int_0^T (T - s)^{p-1} f(s, u(s)) ds. \tag{3.1.24}$$

Plugging (3.1.24) into (3.1.23) yields

$$\begin{aligned} \bar{u}(t; \chi_1, \bar{\mu}) = &\alpha_1 + \chi_1 t + (\alpha_2 - \alpha_1 - \chi_1 T) \left(\frac{t}{T}\right)^p + \frac{1}{\Gamma(p)} \Bigg[\int_0^t (t - s)^{p-1} f(s, u(s)) ds \\ &- \left(\frac{t}{T}\right)^p \int_0^T (T - s)^{p-1} f(s, u(s)) ds \Bigg]. \end{aligned} \tag{3.1.25}$$

Since $\alpha_1 \in D_\beta$, according to the integral equation (3.1.25) and the definition of D_β, it can be shown that $\bar{u}(t; \chi_1, \bar{\mu}) \in D$. Moreover, since Equations (3.1.13) and (3.1.25) are equivalent, it follows from part 4 of Theorem 3.1.1 that $\bar{u}(t; \chi_1, \bar{\mu}) = u_\infty(t; \chi_1)$ and $\mu = \Delta(\chi_1)$. This completes the proof. □

Theorem 3.1.3. *Let the original BVP (3.1.1), (3.1.2) satisfy conditions (3.1.6)-(3.1.11). Then $u_\infty(\cdot;\chi_1^*)$ is a solution to the FDS (3.1.1) with boundary conditions (3.1.2) if and only if the point χ_1^* is a solution to the determining equation*

$$\Delta(\chi_1^*) = 0, \tag{3.1.26}$$

where Δ is given by (3.1.15).

Proof. The conditions of Theorem 3.1.1 hold, thus we can apply Theorem 3.1.2 and note that the perturbed system in (3.1.14) coincides with the original FDS (3.1.1) if and only if the vector of parameters χ_1^* satisfies the determining equation (3.1.26). That is, $u_\infty(\cdot;\chi_1^*)$ is a solution to the FBVP (3.1.1), (3.1.2) if and only if (3.1.26) holds.

□

With this, we have shown the connection between the Cauchy problem (3.1.14) and the original FBVP (3.1.1), (3.1.2). Theorem 3.1.3 gives necessary and sufficient conditions for the solvability of FBVP (3.1.1), (3.1.2), and the construction of its solution. However, a difficulty in its application arises from the fact that the explicit form of the exact function $\Delta(\chi_1)$ is unknown. To overcome this, in practice we solve an approximate determining equation,

$$\Delta_m(\chi_1) = 0, \tag{3.1.27}$$

which depends only on the m-th term of the sequence in (3.1.5), and is thus known explicitly. In particular, the approximate determining function $\Delta_m : \Omega \to \mathbb{R}^n$ is given by

$$\Delta_m(\chi_1) := \frac{(\alpha_2 - \alpha_1 - \chi_1 T)\Gamma(p+1)}{T^p} - \frac{p}{T^p}\int_0^T (T-s)^{p-1} f(s, u_m(s;\chi_1))ds. \tag{3.1.28}$$

In the following subsection we deal with the solvability of the FBVP (3.1.1), (3.1.2).

3.1.4. SOLVABILITY ANALYSIS

To establish conditions for the solvability of FBVP (3.1.1), (3.1.2), we use topological degree theory to show the existence of a vector parameter $\chi_1 \in \Omega$, which determines the solution to the FBVP (3.1.1), (3.1.2) (Lemma 3.1.1, Theorem 3.1.4). We give a bound on the approximate determining function (3.1.27), which is required for the solvability of our problem (Lemma 3.1.3, Theorem 3.1.5). This provides the basis for a search algorithm for the vector parameter $\chi_1 \in \Omega$, see Remark 3.1.1. In addition, we prove two results which estimate the distance between two limit functions $u_\infty(t;\chi_1^0)$ and $u_\infty(t;\chi_1^1)$ for two different vectors $\chi_1^0, \chi_1^1 \in \Omega$ (Lemma 3.1.2) and the deviation between the approximate and exact solutions of FBVP (3.1.1), (3.1.2) (Theorem 3.1.6).

We begin by estimating the difference between the exact and approximate determining functions, (3.1.15) and (3.1.28). This will be used along with the Brouwer topological degree theory in order to show the existence of $\chi_1 \in \Omega$, which defines the solution of (3.1.1), (3.1.2).

Lemma 3.1.1. *Suppose the conditions of Theorem 3.1.1 are satisfied. Then for arbitrary $m \geq 1$ and $\chi_1 \in \Omega$ for the exact and approximate determining functions $\Delta : \Omega \to \mathbb{R}^n$ and $\Delta_m : \Omega \to \mathbb{R}^n$, defined by (3.1.15) and (3.1.28), respectively, the inequality*

$$|\Delta(\chi_1) - \Delta_m(\chi_1)| \leq Q^m M(I_n - Q)^{-1} \tag{3.1.29}$$

holds, where M, K and Q are given in (3.1.6), (3.1.7), and (3.1.10).

Proof. Let us fix an arbitrary $\chi_1 \in \Omega$. Then by virtue of the Lipschitz condition (3.1.7) and the estimates in (2.2.8) and (3.1.16), we have

$$\begin{aligned}|\Delta(\chi_1) - \Delta_m(\chi_1)| =& \Bigg| - \frac{p}{T^p}\int_0^T (T-s)^{p-1} f(s, u_\infty(s;\chi_1))ds \\ &+ \frac{p}{T^p}\int_0^T (T-s)^{p-1} f(s, u_m(s;\chi_1))ds \Bigg| \\ \leq& \frac{p}{T^p}\int_0^T (T-s)^{p-1} |f(s, u_\infty(s;\chi_1)) - f(s, u_m(s;\chi_1))|ds \\ \leq& \frac{pK}{T^p}\int_0^T (T-s)^{p-1} |u_\infty(s;\chi_1) - u_m(s;\chi_1)|ds \\ \leq& Q^m M(I_n - Q)^{-1}.\end{aligned}$$

The obtained estimate proves the lemma. □

On the basis of the exact and approximate determining equations (3.1.26) and (3.1.27), let us introduce the mappings $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ and $\Phi_m : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$\Phi(\chi_1) := \frac{(\alpha_2 - \alpha_1 - \chi_1 T)\Gamma(p+1)}{T^p} - \frac{p}{T^p}\int_0^T (T-s)^{p-1} f(s, u_\infty(s;\chi_1))ds, \tag{3.1.30a}$$

$$\Phi_m(\chi_1) := \frac{(\alpha_2 - \alpha_1 - \chi_1 T)\Gamma(p+1)}{T^p} - \frac{p}{T^p}\int_0^T (T-s)^{p-1} f(s, u_m(s;\chi_1))ds. \tag{3.1.30b}$$

Theorem 3.1.4. *Suppose the conditions of Theorem 3.1.1 hold, and one can find an $m \geq 1$ and a set Ω, such that*

$$\Phi_m \rhd_{\partial\Omega} Q^m M(I_n - Q)^{-1}, \tag{3.1.31}$$

where the relation $\rhd$ is given in Definition 2.2.8, and $\partial\Omega$ denotes the boundary of the domain Ω.

If the Brouwer degree of the mapping Φ_m satisfies

$$deg(\Phi_m, \Omega, 0) \neq 0, \tag{3.1.32}$$

then there exists a vector of parameters $\chi_1^* \in \Omega$, such that

$$u_\infty(t) = u_\infty(t; \chi_1^*) = \lim_{m\to\infty} u_m(t; \chi_1^*) \tag{3.1.33}$$

is a solution to the FBVP (3.1.1), (3.1.2) satisfying

$$u'_\infty(0) = \chi_1^* \in \Omega. \tag{3.1.34}$$

Proof. We first show that the vector fields Φ and Φ_m are homotopic. Let us introduce the family of vector mappings

$$P(\theta, \chi_1) = \Phi_m(\chi_1) + \theta[\Phi(\chi_1) - \Phi_m(\chi_1)], \quad \chi_1 \in \partial\Omega, \quad \theta \in [0, 1]. \tag{3.1.35}$$

Then $P(\theta, \chi_1)$ is continuous for all $\chi_1 \in \partial\Omega, \ \theta \in [0, 1]$. We have

$$P(0, \chi_1) = \Phi_m(\chi_1), \quad P(1, \chi_1) = \Phi(\chi_1)$$

and for any $\chi_1 \in \Omega$,

$$\begin{aligned} |P(\theta, \chi_1)| =& |\Phi_m(\chi_1) + \theta[\Phi(\chi_1) - \Phi_m(\chi_1)]| \\ \geq& |\Phi_m(\chi_1)| - |\Phi(\chi_1) - \Phi_m(\chi_1)|. \end{aligned} \tag{3.1.36}$$

From the other side, by virtue of (3.1.30a), (3.1.30b) we have

$$|\Phi(\chi_1) - \Phi_m(\chi_1)| \leq Q^m M (I_n - Q)^{-1}. \tag{3.1.37}$$

From (3.1.31), (3.1.36), and (3.1.37) it follows that

$$|P(\theta, \chi_1)| \triangleright_{\partial\Omega} 0, \quad \theta \in [0, 1],$$

which means that $P(\theta, \chi_1) \neq 0$ for all $\theta \in [0, 1]$ and $\chi_1 \in \Omega$, i.e. the mappings (3.1.35) are non-degenerate, and thus the vector fields Φ and Φ_m are homotopic. Since relation (3.1.32) holds and the Brouwer degree is preserved under homotopies, it follows that

$$\deg(\Phi, \Omega, 0) = \deg(\Phi_m, \Omega, 0) \neq 0.$$

This implies that there exists $\chi_1^* \in \Omega$ such that $\Phi(\chi_1^*) = 0$ by the classical topological result in [70].

Hence, the vector of parameters χ_1^* satisfies the determining equation (3.1.26).

By Theorem 3.1.3 it follows that the function defined in (3.1.33) is a solution to the original FBVP with the Dirichlet boundary conditions (3.1.1), (3.1.2) and satisfies the initial condition (3.1.34). □

The following lemma gives the closeness of the limit functions $u_\infty(t; \chi_1^0)$ and $u_\infty(t; \chi_1^1)$ for two different vectors $\chi_1^0, \chi_1^1 \in \Omega$.

Lemma 3.1.2. *Suppose the conditions of Theorem 3.1.1 are satisfied. Then the limit function $u_\infty(t; \chi_1)$ satisfies the Lipschitz-type condition of the form*

$$|u_\infty(t; \chi_1^0) - u_\infty(t; \chi_1^1)| \leq \left[R + \alpha_1(t) R (I_n - Q)^{-1}\right] |\chi_1^0 - \chi_1^1|, \tag{3.1.38}$$

where

$$R := \sup_{t\in[0,T]} \left| t - T\left(\frac{t}{T}\right)^p \right|. \tag{3.1.39}$$

Proof. Using (3.1.5) for $m = 1$, we find that

$$
\begin{aligned}
&|u_1(t;\chi_1^0) - u_1(t;\chi_1^1)| \le |\chi_1^0 - \chi_1^1|R \\
&\quad + \frac{1}{\Gamma(p)} \int_0^t \left[(t-s)^{p-1} - (T-s)^{p-1}\left(\frac{t}{T}\right)^p \right] |f(s,u_0(s;\chi_1^0)) - f(s,u_0(s;\chi_1^1))| ds \\
&\quad + \frac{1}{\Gamma(p)} \left(\frac{t}{T}\right)^p \int_t^T (T-s)^{p-1} |f(s,u_0(s;\chi_1^0)) - f(s,u_0(s;\chi_1^1))| ds \\
&\le |\chi_1^0 - \chi_1^1|R \\
&\quad + \frac{K}{\Gamma(p)} \int_0^t \left[(t-s)^{p-1} - (T-s)^{p-1}\left(\frac{t}{T}\right)^p \right] |u_0(s;\chi_1^0) - u(s;\chi_1^1)| ds \\
&\quad + \frac{K}{\Gamma(p)} \left(\frac{t}{T}\right)^p \int_t^T (T-s)^{p-1} |u_0(s;\chi_1^0) - u(s;\chi_1^1)| ds \\
&\le |\chi_1^0 - \chi_1^1|R + \frac{KR}{\Gamma(p)}|\chi_1^0 - \chi_1^1| \int_0^t \left[(t-s)^{p-1} - (T-s)^{p-1}\left(\frac{t}{T}\right)^p \right] ds \\
&\quad + \frac{KR}{\Gamma(p)}|\chi_1^0 - \chi_1^1| \left(\frac{t}{T}\right)^p \int_t^T (T-s)^{p-1} ds \\
&= |\chi_1^0 - \chi_1^1|R + KR\alpha_1(t)|\chi_1^0 - \chi_1^1|
\end{aligned}
$$

holds for all $t \in [0, T]$, where the matrix K and vector R are defined in (3.1.7) and (3.1.39), and the function $\alpha_1(t)$ is defined in (2.2.7). Analogously, for $m = 2$ we find

$$
\begin{aligned}
&|u_2(t;\chi_1^0) - u_2(t;\chi_1^1)| \le |\chi_1^0 - \chi_1^1|R \\
&\quad + \frac{K}{\Gamma(p)} \int_0^t \left[(t-s)^{p-1} - (T-s)^{p-1}\left(\frac{t}{T}\right)^p \right] |u_1(t;\chi_1^0) - u_1(t;\chi_1^1)| ds \\
&\quad + \frac{K}{\Gamma(p)} \left(\frac{t}{T}\right)^p \int_t^T (T-s)^{p-1} |u_1(t;\chi_1^0) - u_1(t;\chi_1^1)| ds \\
&= [R + KR\alpha_1(t) + K^2\alpha_2(t)]|\chi_1^0 - \chi_1^1|.
\end{aligned}
$$

By induction we get:

$$
\begin{aligned}
|u_m(t;\chi_1^0) - u_m(t;\chi_1^1)| &\le \left[R + \sum_{i=1}^{m-1} K^i R\alpha_i(t) + K^m\alpha_m(t) \right] |\chi_1^0 - \chi_1^1| \\
&\le \left[R + \sum_{i=1}^{m-1} Q^i R\alpha_1(t) + Q^m \right] |\chi_1^0 - \chi_1^1| \\
&\le \left[R + R\alpha_1(t)(I_n - Q)^{-1} + Q^m \right] |\chi_1^0 - \chi_1^1|,
\end{aligned}
$$

and passing to the limit $m \to \infty$ in the inequality above yields

$$
|u_\infty(t;\chi_1^0) - u_\infty(t;\chi_1^1)| \le \left[R + \alpha_1(t)R(I_n - Q)^{-1} \right] |\chi_1^0 - \chi_1^1|,
$$

as required. □

Next, we show the continuous dependence of the determining function (3.1.15) on the vector parameter χ_1. This will be used to establish an upper bound for (3.1.15) required for the existence of a vector $\chi_1 \in \Omega$, which determines the solution $u(t;\chi_1)$ of (3.1.1), (3.1.2).

Lemma 3.1.3. *Suppose the conditions of Theorem 3.1.1 hold. Then the function $\Delta : \Omega \to \mathbb{R}^n$ satisfies the following estimate:*

$$|\Delta(\chi_1^0) - \Delta(\chi_1^1)| \leq \frac{\Gamma(p+1)}{T^{p-1}}|\chi_1^0 - \chi_1^1| + \Big(KR + QR(I_n - Q)^{-1}\Big)|\chi_1^0 - \chi_1^1|. \quad (3.1.40)$$

Proof. From (3.1.15) we have

$$\begin{aligned}\Delta(\chi_1^0) - \Delta(\chi_1^1) =& \frac{\Gamma(p+1)}{T^{p-1}}(\chi_1^1 - \chi_1^0) \\ &+ \frac{p}{T^p}\int_0^T (T-s)^{p-1}[f(s,u_\infty(s;\chi_1^1)) - f(s,u_\infty(s;\chi_1^0))]ds.\end{aligned}$$

Applying (3.1.7) and (3.1.38) yields

$$\begin{aligned}|\Delta(\chi_1^0) - \Delta(\chi_1^1)| \leq& \frac{\Gamma(p+1)}{T^{p-1}}|\chi_1^0 - \chi_1^1| + \frac{pK}{T^p}\int_0^T (T-s)^{p-1}|u_\infty(s;\chi_1^0) - u_\infty(s;\chi_1^1)|ds \\ \leq& \frac{\Gamma(p+1)}{T^{p-1}}|\chi_1^0 - \chi_1^1| + \Big(KR + QR(I_n - Q)^{-1}\Big)|\chi_1^0 - \chi_1^1|,\end{aligned}$$

as required. □

Theorem 3.1.5. *Suppose the conditions of Theorem 3.1.1 are satisfied. Then in order for the domain Ω to contain a point $\chi_1 = \chi_1^*$, which determines the value of the first derivative, $u'(0;\chi_1^*)$, of the solution $u(t;\chi_1)$ of the BVP (3.1.1), (3.1.2) at $t = 0$, it is necessary that for all $m \geq 1$, $\tilde{\chi}_1 \in \Omega$, the following inequality holds:*

$$|\Delta_m(\tilde{\chi}_1)| \leq \sup_{\chi_1 \in \Omega}\Big[KR + \frac{QR}{1-Q} + \frac{\Gamma(p+1)}{T^{p-1}}\Big]|\chi_1 - \tilde{\chi}_1| + \frac{Q^m M}{1-Q}.$$

Proof. Assume that the determining function $\Delta(\chi_1)$ vanishes at $\chi_1 = \chi_1^*$, i.e. $\Delta(\chi_1^*) = 0$. Then, according to Theorem 3.1.3, the initial value of the first derivative of the solution of FBVP (3.1.1), (3.1.2), is given by $u'(0) = \chi_1^*$.

Let us apply Lemma 3.1.2, where $\chi_1^0 = \tilde{\chi}_1$ and $\chi_1^1 = \chi_1^*$, to the difference $|\Delta(\tilde{\chi}_1) - \Delta(\chi_1^*)|$. Then

$$|\Delta(\tilde{\chi}_1) - \Delta(\chi_1^*)| = |\Delta(\tilde{\chi}_1)| \leq \Big[KR + \frac{QR}{1-Q} + \frac{\Gamma(p+1)}{T^{p-1}}\Big]|\tilde{\chi}_1 - \chi_1^*|.$$

By Lemma 3.1.1, it follows that

$$|\Delta(\tilde{\chi}_1) - \Delta_m(\tilde{\chi}_1)| \leq \frac{Q^m M}{1-Q},$$

thus,

$$
\begin{aligned}
|\Delta_m(\tilde{\chi}_1)| \leq & |\Delta(\tilde{\chi}_1)| + |\Delta_m(\tilde{\chi}_1) - \Delta(\tilde{\chi}_1)| \\
\leq & \Big[KR + \frac{QR}{1-Q} + \frac{\Gamma(p+1)}{T^{p-1}}\Big]|\tilde{\chi}_1 - \chi_1^*| + \frac{Q^m M}{1-Q} \\
\leq & \sup_{\chi_1 \in \Omega} \Big[KR + \frac{QR}{1-Q} + \frac{\Gamma(p+1)}{T^{p-1}}\Big]|\tilde{\chi}_1 - \chi_1| + \frac{Q^m M}{1-Q}.
\end{aligned}
$$

This proves the theorem. □

Remark 3.1.1. *On the basis of Theorem 3.1.5, we can establish an algorithm of approximate search for the point χ_1^*, which defines the solution $u(\cdot)$ of the original FBVP (3.1.1), (3.1.2). Let us represent the open set $\Omega \subset \mathbb{R}^n$ as the finite union of subsets Ω_i:*

$$
\Omega = \cup_{i=1}^{N} \Omega_i. \tag{3.1.41}
$$

In each subset Ω_i, we pick a point $\tilde{\chi}_1^i$ and calculate the approximate solution $u_m(t, \tilde{\chi}_1^i)$ using the recurrence formula (3.1.5). Then we find the value of the determining function $\Delta_m(\tilde{\chi}_1^i)$, according to (3.1.26), and exclude from (3.1.41) subsets Ω_i for which the inequality does not hold. According to Theorem 3.1.5, these subsets cannot contain a point χ_1^* that determines the solution $u(\cdot)$. The remaining subsets $\Omega_{i_1}, ..., \Omega_{i_s}$ form a set $\Omega_{m,N}$, such that only $\tilde{\chi}_1 \in \Omega_{m,N}$ can determine $u(\cdot)$.
As $N, m \to \infty$, the set $\Omega_{m,N}$ "follows" the set Ω^*, which may contain a value χ_1^* and defines a solution to the FBVP (3.1.1), (3.1.2). Each point $\tilde{\chi}_1$ can be seen as an approximation of χ_1^*, which determines solution of the FBVP (3.1.1), (3.1.2). It is clear that

$$
|\tilde{\chi}_1 - \chi_1^*| \leq \sup_{\chi_1 \in \Omega_{m,N}} |\tilde{\chi}_1 - \chi_1|,
$$

and the function $u_m(t, \tilde{\chi}_1)$, calculated using the iterative formula (3.1.5), can be seen as an approximate solution to the FBVP (3.1.1), (3.1.2).

Finally, we estimate the deviation between the exact solution to the FBVP (3.1.1), (3.1.2), $u_\infty(t; \chi_1^*)$, and its approximate solution $u_m(t; \tilde{\chi}_1)$.

Theorem 3.1.6. *Suppose the conditions of Theorem 3.1.1 are satisfied and a point χ_1^*, defined in the set Ω, is the solution of the exact determining equation (3.1.26), and $\tilde{\chi}_1$ is an arbitrary point in the set $\Omega_{m,N}$. Then the following estimate holds:*

$$
\begin{aligned}
|u_\infty(t; \chi_1^*) - u_m(t; \tilde{\chi}_1)| \leq & Q^m M (I_n - Q)^{-1} \alpha_1(t) \\
& + \sup_{\tilde{\chi}_1 \in \Omega_{m,N}} \Big(R + R\alpha_1(t)(I_n - Q)^{-1} + Q^m\Big)|\chi_1^* - \tilde{\chi}_1|.
\end{aligned}
$$

Proof. Let us use the following inequality:

$$
|u_\infty(t; \chi_1^*) - u_m(t; \tilde{\chi}_1)| \leq |u_\infty(t; \chi_1^*) - u_m(t; \chi_1^*)| + |u_m(t; \chi_1^*) - u_m(t; \tilde{\chi}_1)|.
$$

According to the estimate in (3.1.16), we have

$$|u_\infty(t;\chi_1^*) - u_m(t;\chi_1^*)| \leq Q^m(I_n - Q)^{-1} M\alpha_1(t).$$

Moreover, from the estimate in Lemma 3.1.2, it follows that

$$|u_m(t;\chi_1^*) - u_m(t;\tilde{\chi}_1)| \leq \Big(R + R\alpha_1(t)(I_n - Q)^{-1} + Q^m\Big)|\chi_1^* - \tilde{\chi}_1|.$$

Therefore, we find

$$\begin{aligned} |u_\infty(t;\chi_1^*) - u_m(t;\tilde{\chi}_1)| \leq & \frac{Q^m}{I_n - Q} M\alpha_1(t) + \Big(R + \frac{R\alpha_1(t)}{I_n - Q} + Q^m\Big)|\chi_1^* - \tilde{\chi}_1| \\ \leq & \frac{Q^m}{I_n - Q} M\alpha_1(t) + \sup_{\tilde{\chi}_1 \in \Omega_{m,N}} \Big(R + \frac{R\alpha_1(t)}{I_n - Q} + Q^m\Big)|\chi_1^* - \tilde{\chi}_1|, \end{aligned}$$

as required. □

This completes our analysis of the solvability of the FBVP (3.1.1), (3.1.2).

Next, we apply the theory discussed so far to two example problems.

3.1.5. EXAMPLES

In this subsection, we use the numerical-analytic technique presented in Sections 3.1.1-3.1.3 to construct approximate solutions to two model examples. For the sake of simplicity of computation, we consider only one dimensional problems, where $u : [0, T] \to D \in \mathbb{R}$. We apply the technique to a non-linear FDE, whose exact solution is known, and to the Antarctic Circumpolar Current equation in the fractional setting.

EXAMPLE WITH KNOWN EXACT SOLUTION

Let us consider the FBVP

$$\begin{aligned} {}_0^C D_t^{\frac{3}{2}} u(t) =& u^2(t) - \frac{1}{4}(t^4 + 2t^3 + t^2) + \frac{\sqrt{t}}{\Gamma(3/2)} := (f(t, u(t))), \\ u(0) =& 0, \quad u(1) = 1, \end{aligned} \tag{3.1.42}$$

defined on $t \in [0, 1]$ for $p = 3/2$. It is easy to verify that the exact solution to this problem is given by

$$u(t) = \frac{t^2}{2} + \frac{t}{2}, \tag{3.1.43}$$

and

$$\chi_1 := u'(0) = \frac{1}{2}.$$

In all of the examples throughout this thesis, the set D, containing the solution, is determined aposteriori, based on the computed approximation terms, so as to ensure that the non-emptiness of the set of initial values D_β is satisfied. In the present example, we calculated

$$D := \{u : -0.22 \leq u \leq 1.22\}.$$

Since $u : [0, 1] \to D \subset \mathbb{R}$, the constant vectors M and β and matrices K and Q, defined by (3.1.6), (3.1.9), (3.1.7), and (3.1.10), respectively, are now scalars, given by

$$M = \frac{2}{\sqrt{\pi}}, \quad \beta = \frac{2}{3\pi}, \quad K = 2, \quad Q = \frac{2}{3\sqrt{\pi}}.$$

The domain of initial values D_β, defined in (3.1.8), contains the given initial value $u(0) = 0$, since $u(0) \in D$ and

$$\{|u| \leq \beta\} \subset D.$$

That is, the condition of non-emptiness of D_β is satisfied and the function $f(t, u(t))$ is bounded and Lipschitz continuous on the interval $t \in [0, 1]$. Since now Q is a scalar, the convergence condition in (3.1.11) becomes $Q < 1$, and is satisfied, thus we can apply the technique described in 3.1.1 to construct approximate solutions to (3.1.42).

For the FBVP (3.1.42), the approximate determining equation (3.1.27) reads

$$\Delta_m(\chi_1) = \frac{(1-\chi_1)\sqrt{\pi}}{2} + \int_0^1 (1-s)^{1/2} f(s, u_m(s; \chi_1))ds = 0, \tag{3.1.44}$$

and the sequence of approximations (3.1.5) takes the form

$$\begin{aligned} u_0(t; \chi_1) =& \chi_1 t + (1-\chi_1)t^{3/2}. \\ u_m(t; \chi_1) =& u_0(t, \chi_1) + \frac{1}{\Gamma(3/2)} \int_0^t (t-s)^{1/2} f(s, u_{m-1}(s; \chi_1))ds \\ & - \frac{1}{\Gamma(3/2)} t^{3/2} \int_0^1 (1-s)^{1/2} f(s, u_{m-1}(s; \chi_1))ds, \end{aligned} \tag{3.1.45}$$

where $m \in \mathbb{Z}^+$, $t \in [0, 1]$.

In order to obtain the approximate value of the parameter $\chi_1 \in \Omega := [0.4, 0.6]$, Equation (3.1.44) is solved at each iteration step.

At the initial step $m = 0$, the zero-th approximation $u_0(t, \chi_1)$, as given in (3.1.45), is substituted into the expression for $\Delta_0(\chi_1)$, which yields

$$\Delta_0(\chi_1) = \frac{(1-\chi_{1,0})\sqrt{\pi}}{2} + \int_0^1 (1-s)^{1/2} f(s, u_0(s; \chi_1))ds,$$

where

$$f(s, u_0(s; \chi_1)) = u_0^2(s; \chi_1) - \frac{1}{4}(s^4 + 2s^3 + s^2) + \frac{\sqrt{s}}{\Gamma(3/2)}.$$

The approximate determining equation

$$\Delta_0(\chi_1) = 0 \tag{3.1.46}$$

is solved numerically to obtain $\chi_{1,0} = 0.481$.

Thus, the initial approximation to the solution of BVP (3.1.42) is given by

$$u_0(t;\chi_{1,0}) = 0.4813t + 0.5187t^{3/2}.$$

At the next step, $m = 1$, the expression for $u_0(t,\chi_1)$ is used to construct the first approximation:

$$\begin{aligned} u_1(t;\chi_1) =& \chi_1 t + (1-\chi_1)t^{3/2} + \frac{1}{\Gamma(3/2)}\int_0^t (t-s)^{1/2} f(s,u_0(s;\chi_1))ds \\ &- \frac{1}{\Gamma(3/2)}t^{3/2}\int_0^1 (1-s)^{1/2} f(s,u_0(s;\chi_1))ds, \end{aligned}$$

which is substituted into $\Delta_1(\chi_1)$:

$$\Delta_1(\chi_1) = \frac{(1-\chi_{1,1})\sqrt{\pi}}{2} + \int_0^1 (1-s)^{1/2} f(s,u_1(s;\chi_1))ds = 0.$$

The approximate determining equation

$$\Delta_1(\chi_1) = 0$$

is solved again to find $\chi_{1,1} = 0.501$. With the obtained value for $\chi_{1,1}$, the first approximation becomes

$$\begin{aligned} u_1(t;\chi_{1,1}) =& 0.5013t + 0.4987t^{3/2} + \frac{1}{\Gamma(3/2)}\int_0^t (t-s)^{1/2} f(s,u_0(s;\chi_{1,1}))ds \\ &- \frac{1}{\Gamma(3/2)}t^{3/2}\int_0^1 (1-s)^{1/2} f(s,u_0(s;\chi_{1,1}))ds, \end{aligned}$$

where

$$f(s,u_0(s;\chi_{1,1})) = (0.5013s + 0.4987s^{3/2})^2 - \frac{1}{4}(s^4 + 2s^3 + s^2) + \frac{\sqrt{s}}{\Gamma(3/2)}.$$

Next, $u_1(t;\chi_1)$ is used to construct $u_2(t;\chi_1)$:

$$\begin{aligned} u_2(t;\chi_1) =& \chi_1 t + (1-\chi_1)t^{3/2} + \frac{1}{\Gamma(3/2)}\int_0^t (t-s)^{1/2} f(s,u_1(s;\chi_1))ds \\ &- \frac{1}{\Gamma(3/2)}t^{3/2}\int_0^1 (1-s)^{1/2} f(s,u_1(s;\chi_1))ds, \end{aligned}$$

which is substituted into $\Delta_2(\chi_1)$ and the approximate determining equation $\Delta_2(\chi_1) = 0$ is solved to obtain $\chi_{1,2} = 0.499$. This value is substituted into the expression for $u_2(t;\chi_1)$:

$$\begin{aligned} u_2(t;\chi_{1,2}) =& 0.4999t + 0.5001t^{3/2} + \frac{1}{\Gamma(3/2)} \int_0^t (t-s)^{1/2} f(s, u_1(s;\chi_{1,2}))ds \\ &- \frac{1}{\Gamma(3/2)} t^{3/2} \int_0^1 (1-s)^{1/2} f(s, u_1(s;\chi_{1,2}))ds. \end{aligned}$$

The same process is repeated at each consecutive step. Plots of the approximating functions $u_m(t;\chi_{1,m})$ for $m = 0,\ 1,\ 2$ and the exact solution to FBVP (3.1.42) are shown in the left panel of Figure 3.1a.

We also verify the theoretical error bound given in (3.1.16). For our example we calculate

$$E_m := |u_\infty(t;\chi_1) - u_m(t;\chi_{1,m})|$$

for $m = 0,\ 1,\ 2$, and compared it to

$$\tilde{E}_m := \max_{0 \le t \le 1} |u_\infty(t;\chi_1) - u_m(t;\chi_{1,m})|,$$

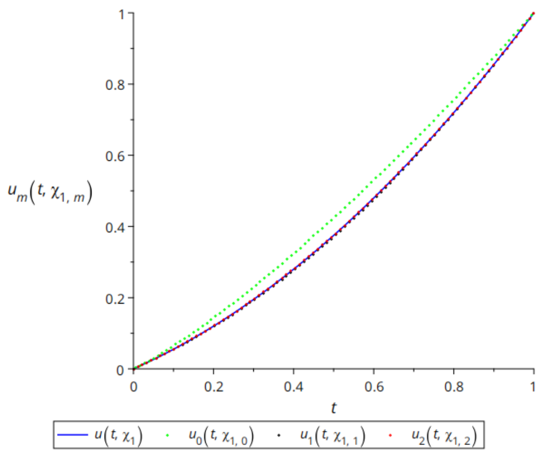
obtained directly from the differences between the exact solution and the computed approximations. Figure 3.1b shows plots of the calculated error between the exact solution and each approximating function.

The approximate parameter values $\chi_{1,m}$ are shown in Table 3.1, along with the theoretical error bounds E_m and the computed maximum error $\tilde{E}_m$ for $m = 0,\ 1,\ 2$.

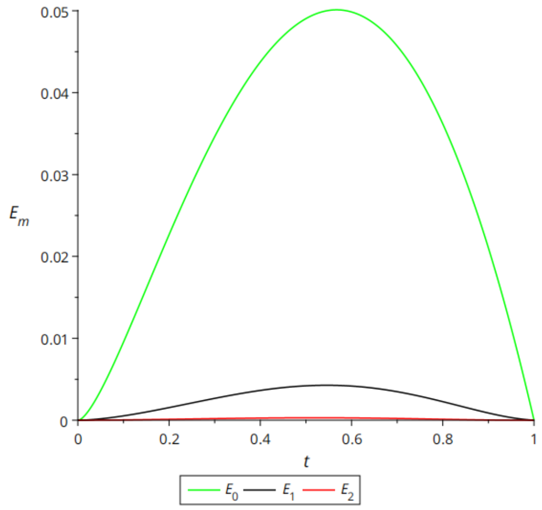
Table 3.1: Approximate parameter values $\chi_{1,m}$, theoretical error bounds E_m and computed maximum error $\tilde{E}_m$ for $m = 0,\ 1,\ 2$ for FBVP (3.1.42).

m	$\chi_{1,m}$	E_m	$\tilde{E}_m$
0	0.481	0.1324	0.0501
1	0.501	0.0498	0.0043
2	0.499	0.0187	0.0003

From the values of $\chi_{1,m}$, shown in Table 3.1, it is clear that the approximate parameter values approach the exact value $\chi_1 = 0.5$. Moreover, the calculated maximum error $\tilde{E}_m$ is at least an order of magnitude smaller than the predicted value at each iteration. The plots in Figure 3.1a show that the sequence of approximations tends to the exact solution. At iteration $m = 2$ we see a good overlap between the approximate solution $u_2(t;\chi_{1,2})$ (dotted red line) and the exact solution $u(t)$ (solid blue line). Figure 3.1b shows that the error between the exact solution and the constructed approximation terms decreases with each consecutive term, as expected.



(a) Plots of the approximate solutions $u_m(t; \chi_{1,m})$ for $m = 0, 1, 2$ (dotted lines) and the exact solution to FBVP (3.1.42), given in (3.1.43) (solid line).



(b) Plots of and the computed error functions $\tilde{E}_m = |u_\infty(t; \chi_1) - u_m(t; \chi_{1,m})|$ for $m = 0, 1, 2$.

Figure 3.1: Approximate solutions and computed error for FBVP (3.1.42).

EXAMPLE WITH UNKNOWN EXACT SOLUTION

Motivated by [76], we consider a BVP for a fractional differential equation of the form

$$ {}_0^C D_t^{\frac{3}{2}} u(t) = a(t)F(u(t)) + b(t) \ (:= f(t, u(t))), \tag{3.1.47} $$

subject to the Dirichlet boundary conditions

$$ u(0) = 1, \ \ u(1) = 2. \tag{3.1.48} $$

Here

$$ a(t) := \frac{-2e^t}{(1+e^t)^2}, \quad b(t) := -\frac{2\omega e^t(1-e^t)}{(1+e^t)^3}, $$

and $F(u(t))$ is taken as a linear function of $u(t)$:

$$ F(u(t)) = u(t), $$

i.e. the right-hand side function in Equation (3.1.47) becomes

$$ f(t, u(t)) = \frac{-2e^t}{(1+e^t)^2} u(t) - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3}. \tag{3.1.49} $$

If the fractional order derivative is replaced by the second order derivative, Equation (3.1.47) becomes the equation used for modeling the flow of the Antarctic Circumpolar Current. In this context ω is a scalar which corresponds to the dimensionless Coriolis parameter being equal to 4649.56.

The calculated set D for the FBVP (3.1.47), (3.1.48) is

$$D := \{u : |u| \leq 100\}, \quad t \in [0, 1].$$

We have

$$M = 844.11, \quad K = \frac{1}{2}, \quad \beta = \frac{1}{3\sqrt{\pi}}, \quad Q = \frac{1}{6\sqrt{\pi}}.$$

Since $u(0) = 1 \in D$ and

$$\{|u - 1| \leq \beta\} \subset D,$$

the condition of nonemptiness of the set D_β is satisfied.

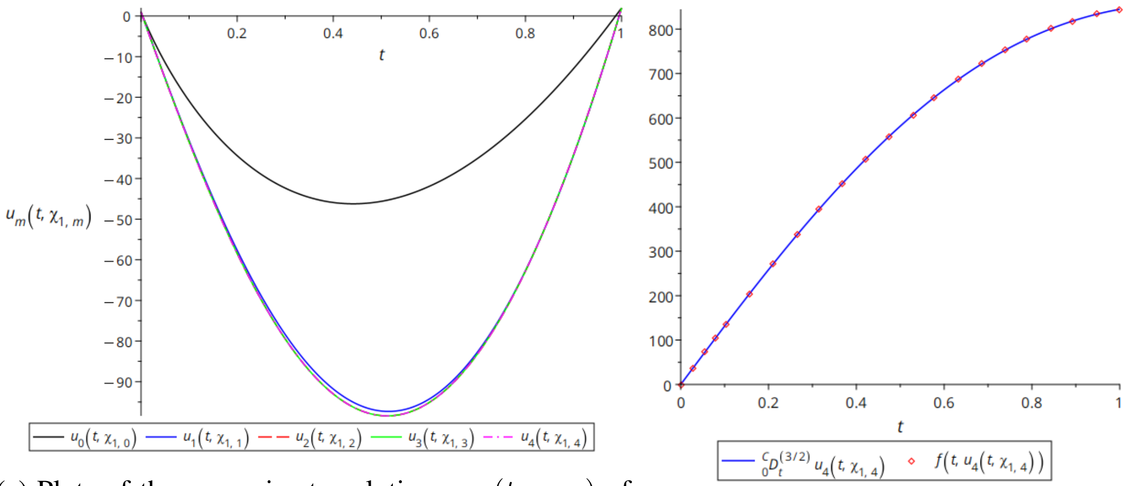
Since $Q < 1$, $f(t, u(t))$ is bounded and satisfies a Lipschitz condition with constant K, conditions (3.1.6)-(3.1.11) are satisfied. Hence, we can apply the numerical-analytic technique to the present problem. The procedure is the same as that explained in the previous example, and results in the approximate determining equation given in (3.1.44) and the sequence terms given in (3.1.45).

Solving the approximate determining equation (3.1.44) for $m = \{0, ..., 4\}$ yields the parameter values $\chi_{1,m}$ shown in Table 3.2.

Table 3.2: Approximate parameter values $\chi_{1,m}$ for $m = \{0, ..., 4\}$ for FBVP (3.1.47), (3.1.48).

m	$\chi_{1,m}$
0	-320.734
1	-332.105
2	-332.346
3	-332.347
4	-332.347

The plots of the first 5 approximations are shown in Figure 3.2a. The initial approximation in the sequence, $u_0(t; \chi_{1,0})$ (solid black line), is significantly different from the rest of the terms. Starting with the second term in the sequence, $u_1(t; \chi_{1,1})$ (solid blue line), the approximating functions are similar in shape and values, and from $u_2(t; \chi_{1,2})$ onward they nearly begin to overlap. This suggests that the calculated sequence converges towards the unknown exact solution to FBVP (3.1.47), (3.1.48).



(a) Plots of the approximate solutions $u_m(t; \chi_{1,m})$ of FBVP (3.1.47), (3.1.48) for $m = \{0, ..., 4\}$.

(b) Comparison plots between the left- (solid blue line) and right- (dotted red line) hand sides of FDE (3.1.47) with $u_4(t, \chi_{1,4})$.

Figure 3.2: Approximate solutions and comparison plots for FBVP (3.1.47), (3.1.48).

Since the exact solution to FBVP (3.1.47), (3.1.48) is not available, we can only compare the computed error between consecutive terms of the sequence to the theoretical estimate given in (3.1.18). We denote the maximum computed error between terms at iterations $m+1$ and m by $\tilde{E}_{m+1,m}(t)$ and calculate it as

$$\tilde{E}_{m+1,m}(t) = \max_{0 \le t \le 1} |u_{m+1}(t; \chi_{1,m+1}) - u_m(t; \chi_{1,m})|. \tag{3.1.50}$$

The values we obtained for $m = 0,\ 1,\ 2,\ 3$ over the interval $t \in [0, 1]$ are shown in Table 3.3, along with the theoretical estimates from (3.1.18), denoted by $E_{m+1,m}$. The computed values are at least an order of magnitude smaller than the theoretical error at each iteration, confirming that the estimate is satisfied.

Table 3.3: Theoretical error estimate between consecutive sequence terms, $E_{m+1,m}$, maximum computed error $\tilde{E}_{m+1,m}$ and maximum difference between the left- and right-hand sides of FDE (3.1.47) for $m = \{0, ...4\}$.

m	$E_{m+1,m}$	$\tilde{E}_{m+1,m}(t)$	δ_m
0	158.746	52.691	427.727
1	14.927	1.212	14.548
2	1.404	0.021	0.351
3	0.132	0.001	0.009
4	-	-	0.001

To verify how well the terms in the approximating sequence satisfy the original FDE (3.1.47), we calculate the Caputo derivatives of $u_m(t, \chi_{1,m})$ and compare them to the right-hand side functions $f(t, u_m(t, \chi_{1,m}))$ for $m = 0, ..., 4$. The maximum of the absolute value of the difference between the left- and right-hand sides for each term is denoted by

$$\delta_m := \max \left| {}_0^C D_t^{\frac{3}{2}} u_m(t;\chi_{1,m}) - f(t, u_m(t;\chi_{1,m}) \right|$$

and shown in the last column in Table 3.3. The values in the table show that as m increases, the approximating functions $u_m(t, \chi_{1,m})$ satisfy Equation (3.1.47) better. A plot of the left- and right-hand sides of Equation (3.1.47) with $u_4(t;\chi_{1,4})$ is shown in Figure 3.2b. We see a good overlap between the two sides of the equation. This suggests that already on the fourth iteration step we have an approximation which satisfies the FBVP (3.1.47), (3.1.48) well. If necessary, this process can be continued even further and a better precision of computations can be obtained.

3.2. PARAMETER-DEPENDENT FRACTIONAL BOUNDARY VALUE PROBLEMS: ANALYSIS AND APPROXIMATION OF SOLUTIONS

We study a parameter-dependent non-linear FDE of the Caputo type, subject to Dirichlet boundary conditions. The parameter in the right-hand side function of the equation controls the effect of the non-linear term and determines the monotonicity of the right-hand side. We begin by introducing two definitions of upper and lower solutions, and some of their properties, which will be used throughout the sub-section. We use the fixed point theory to analyse the solvability of the parameter-dependent FBVP. In particular, we use the Banach fixed point theory for determining the range of parameter values, which guarantees the existence and uniqueness of solutions to the studied problem. We construct a sequence of approximate solutions using the numerical-analytic technique and analyze its monotonicity behaviour. In particular, we show that for an FDE with a decreasing right-hand side function, the numerical-analytic technique produces a monotone sequence, whereas when the right-hand side function is increasing, the technique results in an alternating sequence. The upper and lower solutions method, combined with the numerical-analytic technique, is applied to the case when the right-hand side function is increasing. We demonstrate how the lower and upper solutions method can be used in this case to simplify the form of the sequence, resulting from the numerical-analytic technique and to thereby reduce the computational time. Our results are applied to a fractional order problem, which in the case of the second derivative models the Antarctic Circumpolar Current.

3.2.1. PRELIMINARY STATEMENTS

We begin by introducing the definitions of a type I and type II upper and lower solution to a FBVP, which will be used in the remainder of this sub-section. Here the domain of definition of the FBVP is restricted to $t \in [0,1]$ for simplicity, however, all the results which follow are applicable to domains of the more general form $t \in [a,b]$ for $0 \leq a < b$.

Consider a FBVP of the form

$$\begin{aligned} {}_0^C D_t^p u(t) &= f(t, u(t)), \quad t \in [0,1], \\ u(0) &= \alpha_0, \ \ u(1) = \alpha_1, \end{aligned} \tag{3.2.1}$$

where $p \in (1,2)$, $u(t) : [0,1] \to D \subset \mathbb{R}$, $f(t,u(t)) : [0,1] \times D \to \mathbb{R}$.

Definition 3.2.1. *[79] A function $v(t) \in C^1([0,1],\mathbb{R})$ is called a lower solution of the FBVP (3.2.1) of type I if it satisfies*

$$\begin{aligned} {}_0^C D_t^p v(t) &> f(t, v(t)), \ \ t \in [0,1], \\ v(0) &\leq \alpha_0, \ \ v(1) \leq \alpha_1. \end{aligned}$$

If a function $w(t) \in C^1([0,1],\mathbb{R})$ satisfies the reversed inequalities, it is called an upper solution of the FBVP (3.2.1) of type I.

This section is based on the paper [78].

Definition 3.2.2. *[80] A function $v(t) \in C^1([0,1],\mathbb{R})$ is called a lower solution of the FBVP (3.2.1) of type II if it satisfies*

$$\begin{aligned} {}_0^C D_t^p v(t) &< f(t,u(t)), \ \ t \in [0,1], \\ v(0) \leq \alpha_0, \ \ & v(1) \leq \alpha_1. \end{aligned}$$

If a function $w(t) \in C^1([0,1],\mathbb{R})$ satisfies the reversed inequalities, it is called an upper solution of the FBVP (3.2.1) of type II.

The following result gives conditions for the positivity of an upper solution of type I. For the proof of the statement we refer to [79].

Lemma 3.2.1. *(Positivity result) Let $z(t) \in C^1([0,1],\mathbb{R})$ and $r(t) < 0$, $t \in [0,1]$, bounded. If $z(t)$ satisfies the inequality*

$$\begin{aligned} {}_0^C D_t^p z(t) + r(t)z(t) \leq & 0, \quad t \in (0,1), \\ z(0), z(1) \geq & 0, \end{aligned}$$

then $z(t) \geq 0, \forall t \in [0,1]$.

Next, we state and prove a result on the negativity of a lower solution of type I.

Lemma 3.2.2. *Let $z(t) \in C^1([0,1],\mathbb{R})$. If $z(t)$ satisfies conditions*

$$\begin{aligned} & {}_0^C D_t^p z(t) > 0, \quad t \in (0,1), \\ & z(0), \ z(1) \leq 0, \end{aligned} \tag{3.2.2}$$

then $z(t) < 0$ for $t \in (0,1)$.

Proof. Let $z(t) \in C^1([0,1],\mathbb{R})$ be such that it satisfies (3.2.2), and assume for the sake of contradiction that $z(t) \geq 0$ for (at least one) $t \in (0,1)$. Then $z(t)$ attains a local maximum at some $t_0 \in (0,1)$, thus the Caputo derivative of $z(t)$ is non-positive at t_0, i.e. ${}_0^C D_t^p z(t_0) \leq 0$, by Theorem 2.1.3. This is in contradiction with (3.2.2), thus, $z(t) < 0$ for $t \in [0,1]$. □

3.2.2. PROBLEM SETTING AND SOLVABILITY

We consider a parameter-dependent FDE

$${}_0^C D_t^p u(t) + \lambda a(t) F(u(t)) = b(t) \tag{3.2.3}$$

with Dirichlet boundary conditions

$$u(0) = \alpha_0, \quad u(1) = \alpha_1, \tag{3.2.4}$$

where $p \in (1,2)$, and ${}_0^C D_t^p$ is the Caputo fractional derivative with lower limit at 0, defined in (2.1.9), and $t \in [0,1]$.

For the sake of simplicity, we restrict ourselves to the one-dimensional case $u : [0,1] \to D \subset \mathbb{R}$, however, the following analysis can be extended to a vector-valued setting $D \subset \mathbb{R}^n$.

We assume that the function $u(t)$ is continuously differentiable, i.e. $u \in C^1([0,1],\mathbb{R})$, and D is a closed and bounded domain. The boundary conditions $\alpha_0, \alpha_1 \in \mathbb{R}$ are given values, and $\lambda \in \mathbb{R}$ is an unknown parameter, which determines if the right-hand side function in the FDE is increasing or decreasing.

We denote the absolute value of the parameter λ and the suprema of the functions $a(t)$ and $b(t)$ by Λ, A, and B, respectively:

$$\begin{aligned} |\lambda| &:= \Lambda, \\ A &:= \sup_{t\in[0,1]} |a(t)|, \\ B &:= \sup_{t\in[0,1]} |b(t)|. \end{aligned} \tag{3.2.5}$$

The function $F(u(t)) : G \to \mathbb{R}$ is assumed to be (generally) non-linear, bounded and Lipschitz continuous, i.e.

$$|F(u(t))| \le M, \tag{3.2.6}$$

$$|F(u_1(t)) - F(u_2(t))| \le K|u_1(t) - u_2(t)| \tag{3.2.7}$$

hold for all $t \in [0,1]$, $u_1(t), u_2(t) \in D$, where M, $K \in \mathbb{R}^+$ are constants, and the domain of $F(u(t))$ is given by $G := [0,1] \times D$.

We aim to derive a bound for the values of the parameter λ in the right-hand side of Equation (3.2.3), for which there exists a unique solution to FBVP (3.2.3), (3.2.4), $u(t) \in C^1([0,1],\mathbb{R})$. To this end, we re-write the FBVP (3.2.3), (3.2.4) in the form of an integral equation and show that the operator, associated with this integral equation, is a contraction when the parameter λ satisfies a given bound.

Using the procedure which was described in the previous section, we can transform FBVP (3.2.3), (3.2.4) into the following integral equation:

$$\begin{aligned} u(t,\lambda;\chi) =& \alpha_0 + \chi t + t^p\left(\alpha_1 - \alpha_0 - \chi\right) \\ &+ \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi)) + b(s)]ds \\ &- t^p \int_0^1 (1-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi)) + b(s)]ds\Bigg], \end{aligned} \tag{3.2.8}$$

where the first derivative of the solution $u(t)$ at $t = 0$ is denoted by $\chi := u'(0)$, which is a unknown parameter to be calculated later.

We use fixed point theory to show the existence of a unique solution to the integral equation (3.2.8). For this purpose, let us denote the operator associated with (3.2.8) by

$\mathcal{H}$:

$$
\begin{aligned}
(\mathcal{H}u)(t) :=& \alpha_0 + \chi t + t^p(\alpha_1 - \alpha_0 - \chi) \\
&+ \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi)) + b(s)]ds\right. \\
&\left. - t^p \int_0^1 (1-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi)) + b(s)]ds\right].
\end{aligned}
$$

In the following theorem we give conditions on the parameter λ, for which the integral equation (3.2.8) has a unique solution.

Theorem 3.2.1. *If $u \in B_r$, where $B_r := \{u \in C^1([0,1],\mathbb{R}) : |u(t)| \leq r\}$ with*

$$
\begin{aligned}
r &> \frac{2^{2p-1}\Gamma(p+1)U + \Lambda AN + B}{2^{2p-1}\Gamma(p+1) - \Lambda AK}, \\
U &:= \max_{0\leq t\leq 1} |\alpha_0 + \chi t + t^p(\alpha_1 - \alpha_0 - \chi)|, \\
N &:= \sup_{t\in[0,1]} |F(0)|,
\end{aligned}
$$

and Λ satisfies the following inequality

$$
\Lambda < \frac{2^{2p-1}\Gamma(p+1)}{AK}, \tag{3.2.9}
$$

then $\mathcal{H}$ is a contraction operator, and therefore, the integral equation (3.2.8) has a unique solution in $C^1([0,1],\mathbb{R})$.

Proof. From the Lipschitz condition (3.2.7) it follows that

$$
\begin{aligned}
|F(u(t))| =& |F(u(t)) + F(0) - F(0)| \\
\leq& K|u(t)| + |F(0)| \\
\leq& K|u(t)| + N.
\end{aligned}
$$

For $u \in B_r$ we have

$$
\begin{aligned}
|(\mathcal{H}u)(t)| \leq& |\alpha_0 + \chi t + t^p(\alpha_1 - \alpha_0 - \chi)| + \frac{1}{\Gamma(p)}\left|\int_0^t (t-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi))]ds\right. \\
&\left. - t^p \int_0^1 (1-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi))ds\right| \\
&+ \frac{1}{\Gamma(p)}\left|\int_0^t (t-s)^{p-1}b(s)ds - t^p\int_0^1 (1-s)^{p-1}b(s)ds\right|.
\end{aligned}
$$

Using inequality (2.2.6) from Lemma 2.2.1 yields

$$
\begin{aligned}
|(\mathcal{H}u)(t)| \leq& U + \alpha_1(t)\max_{0\leq t\leq 1}|-\lambda a(t)F(u(t,\lambda;\chi))| + \alpha_1(t)\max_{0\leq t\leq 1}|b(t)| \\
\leq& U + (\Lambda A \max_{0\leq t\leq 1}|F(u(t,\lambda;\chi))| + B)\alpha_1(t) \\
\leq& U + [\Lambda A(K|u(t)| + N) + B]\alpha_1(t) \\
\leq& U + [\Lambda A(Kr + N) + B]\alpha_1(t).
\end{aligned}
$$

Applying estimate (2.2.8) in Lemma 2.2.2 with $n = 0$ yields

$$
\begin{aligned}
|(\mathcal{H}u)(t)| \le & U + [\Lambda A(Kr + N) + B]\alpha_1(t) \\
\le & U + \frac{\Lambda A(Kr + N) + B}{2^{2p-1}\Gamma(p+1)} \le r,
\end{aligned}
$$

that is, if $u \in B_r$, then $\mathcal{H}u \subset B_r$.

Now we consider $u, v \in C^1([0,1], \mathbb{R})$ and apply estimate (2.2.6) again to obtain:

$$
\begin{aligned}
|(\mathcal{H}u)(t) - (\mathcal{H}v)(t)| = & \frac{1}{\Gamma(p)} \Big| \int_0^t (t-s)^{p-1}(-\lambda a(s))[F(u(s,\lambda;\chi)) - F(v(s,\lambda;\chi))]ds \\
& - t^p \int_0^1 (1-s)^{p-1}[-\lambda a(s)F(u(s,\lambda;\chi)) - F(v(s,\lambda;\chi))]ds \Big| \\
\le & \alpha_1(t) \max_{0 \le t \le 1} | - \lambda a(s)[F(u(s,\lambda;\chi)) - F(v(s,\lambda;\chi))]| \\
\le & \alpha_1(t) \Lambda A K \max_{0 \le t \le 1} |u(s,\lambda;\chi) - v(s,\lambda;\chi)| \\
\le & \frac{\Lambda A K}{2^{2p-1}\Gamma(p+1)} ||u - v||.
\end{aligned}
$$

Moreover, from (3.2.9) it follows that

$$
\Lambda \frac{AK}{2^{2p-1}\Gamma(p+1)} < \frac{2^{2p-1}\Gamma(p+1)}{AK} \frac{AK}{2^{2p-1}\Gamma(p+1)} < 1,
$$

which implies that

$$
||\mathcal{H}u - \mathcal{H}v|| \le ||u - v||,
$$

i.e. the operator $\mathcal{H}$ is a contraction. Therefore, by the Banach fixed point theorem (Theorem 2.2.4), the integral equation (3.2.8) has a unique solution in $C^1([0,1], \mathbb{R})$. □

3.2.3. SUCCESSIVE APPROXIMATIONS AND THEIR MONOTONICITY

In the previous sub-section we established the conditions for the parameter λ that guarantee the existence of a unique solution to Equation (3.2.8), which is the integral representation of the exact solution of FBVP (3.2.3), (3.2.4). However, a difficulty of its application arises, since the quantity under the integral depends on $u(t, \lambda; \chi)$, whose explicit form is unknown. To overcome this, we use the numerical-analytic technique to construct a sequence of approximations, which converges uniformly to the exact solution, similarly to what was done in Section 3.1.

Let the right-hand side parameter be set to a fixed value, $\lambda = \overline{\lambda}$, such that condition (3.2.9) is satisfied. Based on the integral representation, given in (3.2.8), we associate with FBVP (3.2.3), (3.2.4) the following parametrized sequence of functions

$\{u_m(\cdot,\overline{\lambda};\chi)\}_{m\in\mathbb{Z}_0^+}$, $\mathbb{Z}_0^+ := \{0,1,2,...\}$:

$$\begin{aligned} u_0(t,\overline{\lambda};\chi) =& \alpha_0 + \chi t + t^p\left(\alpha_1 - \alpha_0 - \chi\right) \\ u_m(t,\overline{\lambda};\chi) =& u_0(t,\overline{\lambda};\chi) \\ &+ \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1}[-\overline{\lambda}a(s)F(u_{m-1}(s,\overline{\lambda};\chi)) + b(s)]ds \right. \\ &\left. - t^p \int_0^1 (1-s)^{p-1}[-\overline{\lambda}a(s)F(u_{m-1}(s,\overline{\lambda};\chi)) + b(s)]ds\right], \end{aligned} \tag{3.2.10}$$

where $t\in[0,1]$, $u_0(t,\overline{\lambda};\chi)\in D$, and $\chi\in\Omega\subset\mathbb{R}$.

Additionally, we assume that the set of initial values

$$D_\beta := \{\alpha_0 \in D : B(\alpha_0 + \chi t + t^p\left(\alpha_1 - \alpha_0 - \chi\right), \beta) \subset D\}, \tag{3.2.11}$$

is non-empty, where

$$\alpha_0 = u(0),$$

$$\beta = \frac{\overline{\lambda}AM + B}{2^{2p-1}\Gamma(p+1)}. \tag{3.2.12}$$

The following theorem on the convergence of the sequence of functions (3.2.10) is analogous to Theorem 3.1.1 from Section 3.1.

Theorem 3.2.2. *Provided that for all $\chi\in\Omega$ and $t\in[0,1]$ conditions (3.2.9) and (3.2.12) are satisfied,*

1. The terms of the sequence (3.2.10) are continuous and satisfy boundary conditions

$$\begin{aligned} u_m(0,\overline{\lambda};\chi) =& \alpha_0, \\ u_m(1,\overline{\lambda};\chi) =& \alpha_1 \end{aligned}$$

for $m\in\mathbb{Z}_0^+$.

2. The sequence of functions (3.2.10) for $t\in[0,1]$ converges uniformly as $m\to\infty$ to the limit function

$$u_\infty(t,\overline{\lambda};\chi) = \lim_{m\to\infty} u_m(t,\overline{\lambda};\chi). \tag{3.2.13}$$

3. The limit function (3.2.13) satisfies boundary conditions

$$u_\infty(0,\overline{\lambda};\chi) = \alpha_0, \quad u_\infty(1,\overline{\lambda};\chi) = \alpha.$$

4. The limit function (3.2.13) is the unique solution to the integral equation

$$
\begin{aligned}
u(t,\overline{\lambda};\chi) =&\alpha_0 + \chi t + t^p\left(\alpha_1 - \alpha_0 - \chi\right)\\
&+ \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1}[-\overline{\lambda}a(s)F(u(s,\overline{\lambda};\chi)) + b(s)]ds\\
&- t^p\int_0^1 (1-s)^{p-1}[-\overline{\lambda}a(s)F(u(s,\overline{\lambda};\chi)) + b(s)]ds\Bigg],
\end{aligned}
\tag{3.2.14}
$$

i.e. it is the unique solution on $t\in[0,1]$ of the Cauchy problem for the modified FDE:

$$
\begin{aligned}
{}_0^C D_t^p u(t) &= -\overline{\lambda}a(t)F(u(t)) + b(t) + \Delta(\chi), \quad p\in(1,2],\\
u(0) &= \alpha_0,\\
u'(0) &= \chi,
\end{aligned}
\tag{3.2.15}
$$

where $\Delta:\Omega\to\mathbb{R}$ is a mapping defined by

$$
\Delta(\chi) := \Gamma(p+1)(\alpha_1-\alpha_0-\chi) \tag{3.2.16}
$$

$$
\qquad - p\int_0^1 (1-s)^{p-1}[\overline{\lambda}a(s)F(u(s,\overline{\lambda};\chi)) + b(s)]ds. \tag{3.2.17}
$$

5. The following error estimate holds

$$
|u_\infty(t,\overline{\lambda};\chi) - u_m(t,\overline{\lambda};\chi)| \le \frac{Q^m(\overline{\lambda}AM+B)}{2^{2p-1}\Gamma(p+1)}\frac{1}{1-Q}, \tag{3.2.18}
$$

where

$$
Q := \frac{\overline{\lambda}AK}{2^{2p-1}\Gamma(p+1)},
$$

and A, K are defined in (3.2.5) and (3.2.7).

Remark 3.2.1. *Note that one of the conditions, crucial to the convergence of sequence (3.1.5) in Theorem 3.1.1, was given by*

$$
r(Q) < 1,
$$

when $Q\in\mathbb{R}^n\times\mathbb{R}^n$, or equivalently,

$$
Q < 1
$$

in the one-dimensional case. This condition still holds in the present case, since $Q<1$ is implied by the inequality (3.2.9) satisfied by $\overline{\lambda}$.

As in the parameter-independent case, we establish the connection between the solution to the FIVP (3.2.15) and the original FBVP (3.2.3), (3.2.4) through the following two theorems.

Consider the Cauchy problem

$$\begin{aligned} &{}_0^C D_t^p u(t) = f(t, u(t)) + \mu, \ \ t \in [0, T], \\ &u(0) = \alpha_0, \\ &u'(0) = \chi, \end{aligned} \tag{3.2.19}$$

where $\mu \in \mathbb{R}$ we will call a control parameter, $\alpha_0 \in D_\beta$ and $\chi \in \Omega$.

Theorem 3.2.3. *Let $\chi \in \Omega$, $\mu \in \mathbb{R}$ be given. Assume that all conditions of Theorem 3.2.2 are satisfied for the FBVP (3.2.3), (3.2.4). Then the solution $u = u(\cdot, \overline{\lambda}; \chi, \mu)$ of the FIVP (3.2.19) also satisfies the boundary conditions in (3.2.4) if and only if*

$$\mu = \Delta(\chi),$$

where $\Delta(\chi)$ is given by (3.2.16), and in this case

$$u(t, \overline{\lambda}; \chi, \mu) = u_\infty(t, \overline{\lambda}; \chi) \ \text{ for } \ t \in [0, 1].$$

Theorem 3.2.4. *Let the original FBVP (3.2.3), (3.2.4) satisfy conditions (3.2.9) and (3.2.12). Then $u_\infty(\cdot, \overline{\lambda}; \chi^*)$ is a solution to the FBVP (3.2.3), (3.2.4) if and only if the point χ^* is a solution to the determining equation*

$$\Delta(\chi^*) = 0,$$

where Δ is given by (3.2.16).

The proofs of Theorems 3.2.2, 3.2.3, and 3.2.4 follow the lines of the proofs of Theorem 3.1.1, 3.1.2, and 3.1.3 from the previous section.

Remark 3.2.2. *Since the explicit form of the solution $u(t, \overline{\lambda}; \chi)$ is unknown, in practice, we compute the values of the parameter χ by solving the approximate determining equation*

$$\Delta_m(\chi) = 0, \tag{3.2.20}$$

where

$$\Delta_m(\chi) := \Gamma(p+1)(\alpha_1 - \alpha_0 - \chi) - p \int_0^1 (1-s)^{p-1} [\overline{\lambda} a(s) F(u_m(s, \overline{\lambda}; \chi)) + b(s)] ds$$

at each iteration step m.

Next, we study the monotonicity of the sequence of functions (3.2.10). Let us denote the right-hand side function in the FDE (3.2.3) by

$$f(t, u(t, \overline{\lambda}; \chi); \overline{\lambda}) := -\overline{\lambda} a(t) F(u(t, \overline{\lambda}; \chi)) + b(t).$$

The parameter $\overline{\lambda}$ determines if the right-hand side functions is decreasing or increasing with respect to $u(t, \overline{\lambda}; \chi)$, which in turn affects the behavior of the constructed approximating sequences as $m \to \infty$. The following two theorems give conditions for which the terms in (3.2.10) form a monotone or an alternating sequence, respectively. The case of an alternating sequence is of particular interest as it pertains to combining the numerical-analytic technique with the method of upper and lower solutions, as it will be seen later.

Theorem 3.2.5. *Consider the FBVP (3.2.3) and the sequence of approximations (3.2.10), and assume that $f(t, u(t, \overline{\lambda}; \chi); \overline{\lambda})$ is differentiable in $u(t, \overline{\lambda}; \chi)$ and its partial derivative with respect to $u(t, \overline{\lambda}; \chi)$ is strictly decreasing, i.e.*

$$\frac{\partial f}{\partial u} < 0.$$

Then the following statements hold:
(S1) If the initial approximation $u_0(t, \overline{\lambda}; \chi)$ is such that $u_0(t, \overline{\lambda}; \chi) < u_1(t, \overline{\lambda}; \chi)$, then the sequence $u_m(t, \overline{\lambda}; \chi)$ is well-ordered and increasing, i.e.

$$u_{k-1}(t, \overline{\lambda}; \chi) < u_k(t, \overline{\lambda}; \chi), \quad \forall k \in \mathbb{N}.$$

(S2) If the initial approximation $u_0(t, \overline{\lambda}; \chi)$ is such that $u_1(t, \overline{\lambda}; \chi) < u_0(t, \overline{\lambda}; \chi)$, then the sequence $u_m(t, \overline{\lambda}; \chi)$ is well-ordered and decreasing, i.e.

$$u_k(t, \overline{\lambda}; \chi) < u_{k-1}(t, \overline{\lambda}; \chi), \quad \forall k \in \mathbb{N}.$$

Proof. The terms in the approximating sequence are obtained from the scheme

$$\begin{aligned} &{}_0^C D_t^p u_m(t, \overline{\lambda}; \chi) = f(t, u_{m-1}(t, \overline{\lambda}; \chi)), \\ &u_m(0) = u(0), \quad u_m(1) = u(1), \ n \geq 1. \end{aligned}$$

(S1) Assume that $u_0(t, \overline{\lambda}; \chi) < u_1(t, \overline{\lambda}; \chi)$. Then

$$\begin{aligned} {}_0^C D_t^p[u_1(t, \overline{\lambda}; \chi) - u_2(t, \overline{\lambda}; \chi)] &= f(t, u_0(t, \overline{\lambda}; \chi); \overline{\lambda}) - f(t, u_1(t, \overline{\lambda}; \chi); \overline{\lambda}) > 0, \\ u_1(0, \overline{\lambda}; \chi) - u_2(0, \overline{\lambda}; \chi) &= 0, \\ u_1(1, \overline{\lambda}; \chi) - u_2(1, \overline{\lambda}; \chi) &= 0, \end{aligned}$$

hence, by Lemma 3.2.2, $u_1(t, \overline{\lambda}; \chi) < u_2(t, \overline{\lambda}; \chi)$. Assume the statement holds for $m = k$. Then, for $m = k + 1$ we have

$$\begin{aligned} {}_0^C D_t^p[u_k(t, \overline{\lambda}; \chi) - u_{k+1}(t, \overline{\lambda}; \chi)] &= f(t, u_{k-1}(t, \overline{\lambda}; \chi); \overline{\lambda}) - f(t, u_k(t, \overline{\lambda}; \chi); \overline{\lambda}) > 0, \\ u_k(0, \overline{\lambda}; \chi) - u_{k+1}(0, \overline{\lambda}; \chi) &= 0, \\ u_k(1, \overline{\lambda}; \chi) - u_{k+1}(1, \overline{\lambda}; \chi) &= 0, \end{aligned}$$

hence, by Lemma 3.2.2, $u_k(t, \overline{\lambda}; \chi) < u_{k+1}(t, \overline{\lambda}; \chi)$. Therefore, the sequence $u_m(t, \overline{\lambda}; \chi)$ is monotone and increasing.
The proof of (S2) follows the lines of the proof of (S1). □

Theorem 3.2.6. *Consider the FBVP (3.2.3), and the sequence of approximations (3.2.10), and assume that $f(t, u(t, \overline{\lambda}; \chi); \overline{\lambda})$ is differentiable in $u(t, \overline{\lambda}; \chi)$ and its partial derivative with respect to $u(t, \overline{\lambda}; \chi)$ is strictly increasing, i.e.*

$$\frac{\partial f}{\partial u} > 0.$$

Then the following statements hold:
(S1) If the initial approximation $u_0(t, \overline{\lambda}; \chi)$ is such that $u_1(t, \overline{\lambda}; \chi) < u_0(t, \overline{\lambda}; \chi)$, then the terms $u_m(t, \overline{\lambda}; \chi)$, given by (3.2.10), form an alternating sequence, for which

$$u_1(t, \overline{\lambda}; \chi) < \ldots < u_{2m+1}(t, \overline{\lambda}; \chi) < u_{2m}(t, \overline{\lambda}; \chi) < \ldots < u_0(t, \overline{\lambda}; \chi). \tag{3.2.21}$$

(S2) If the initial approximation $u_0(t, \overline{\lambda}; \chi)$ is such that $u_0(t, \overline{\lambda}; \chi) < u_1(t, \overline{\lambda}; \chi)$, then the terms $u_m(t, \overline{\lambda}; \chi)$, given by (3.2.10), form an alternating sequence, for which

$$u_0(t, \overline{\lambda}; \chi) < \ldots < u_{2m}(t, \overline{\lambda}; \chi) < u_{2m+1}(t, \overline{\lambda}; \chi) < \ldots < u_1(t, \overline{\lambda}; \chi). \tag{3.2.22}$$

Proof. (S1) Assume that $u_1(t, \overline{\lambda}; \chi) < u_0(t, \overline{\lambda}; \chi)$. Then

$$\begin{aligned} {}_0^C D_t^p[u_1(t, \overline{\lambda}; \chi) - u_2(t, \overline{\lambda}; \chi)] &= f(t, u_0(t, \overline{\lambda}; \chi); \overline{\lambda}) - f(t, u_1(t, \overline{\lambda}; \chi); \overline{\lambda}) > 0, \\ u_1(0, \overline{\lambda}; \chi) - u_2(0, \overline{\lambda}; \chi) &= 0, \\ u_1(1, \overline{\lambda}; \chi) - u_2(1, \overline{\lambda}; \chi) &= 0, \end{aligned}$$

hence, by Lemma 3.2.2, $u_1(t, \overline{\lambda}; \chi) < u_2(t, \overline{\lambda}; \chi)$. Assume the statement holds for $m = k$, that is, $u_{2k+1}(t, \overline{\lambda}; \chi) < u_{2k}(t, \overline{\lambda}; \chi)$ Then, for $m = k + 1$ we have

$$\begin{aligned} &{}_0^C D_t^p[u_{2k+1}(t, \overline{\lambda}; \chi) - u_{2(k+1)}(t, \overline{\lambda}; \chi)] \\ &\qquad = f(t, u_{2k}(t, \overline{\lambda}; \chi); \overline{\lambda}) - f(t, u_{2k+1}(t, \overline{\lambda}; \chi); \overline{\lambda}) > 0, \\ &u_{2k+1}(0, \overline{\lambda}; \chi) - u_{2(k+1)}(0, \overline{\lambda}; \chi) = 0, \\ &u_{2k+1}(1, \overline{\lambda}; \chi) - u_{2(k+1)}(1, \overline{\lambda}; \chi) = 0, \end{aligned}$$

hence, by Lemma 3.2.2, $u_{2k+1}(t, \overline{\lambda}; \chi) < u_{2(k+1)}(t, \overline{\lambda}; \chi)$. Thus,

$$\begin{aligned} &{}_0^C D_t^p[u_{2k+3}(t) - u_{2(k+1)}(t)] \\ &\qquad = f(t, u_{2(k+1)}(t, \overline{\lambda}; \chi); \overline{\lambda}) - f(t, u_{2k+1}(t, \overline{\lambda}; \chi); \overline{\lambda}) > 0, \\ &(u_{2k+3} - u_{2(k+1)})(0) = 0, \\ &(u_{2k+3} - u_{2(k+1)})(1) = 0, \end{aligned}$$

which implies $u_{2k+3}(t, \overline{\lambda}; \chi) < u_{2(k+1)}(t, \overline{\lambda}; \chi)$, that is, the statement holds for $m = k+1$. Therefore, the sequence $u_m(t, \overline{\lambda}; \chi)$ is alternating, i.e. (3.2.21) holds.
The proof of (2) follows the lines of the proof of (1). □

3.2.4. UPPER AND LOWER SOLUTIONS METHOD

The upper and lower solutions method is typically used to prove the existence of solutions to BVPs by bounding the exact solution between the upper and lower one. In some cases it is also used to construct approximations to the solutions of BVPs. In this sub-section, we describe how the numerical analytic technique can be combined with the upper and lower solutions method to construct approximating sequences to the solution of FBVP (3.2.3), (3.2.4). The idea consists of finding upper and lower solutions to FBVP (3.2.3), (3.2.4), and using them as starting points for the numerical-analytic iterative technique. This results in an alternating sequence of approximations, which 'traps' the exact solutions from above and below.

The following two theorems give the form of the alternating sequence, resulting from combining the numerical-analytic technique with the lower and upper solutions method, depending on how the lower and upper solutions are chosen.

Theorem 3.2.7. *Consider the FBVP (3.2.3). Assume that*

(i) v_0, w_0 $\in C^1([0,1],\mathbb{R})$ *are lower and upper solutions to the FBVP (3.2.3) of type I for* $t \in [0,1]$*;*

(ii) the right-hand side function $f(t, u(t, \overline{\lambda}; \chi); \overline{\lambda})$ *is an increasing function in* $u(t, \overline{\lambda}; \chi)$*;*

(iii) two sequences, $\{v_m(t)\}$ *and* $\{w_m(t)\}$*, are computed using the iterative scheme*

$$\begin{aligned} {}_0^C D_t^p v_{m+1}(t) =& f(t, v_m(t); \overline{\lambda}), \quad v_{m+1}(0) = u(0), \quad v_{m+1}(1) = u(1) \\ {}_0^C D_t^p w_{m+1}(t) =& f(t, w_m(t); \overline{\lambda}), \quad w_{m+1}(0) = u(0), \quad w_{m+1}(1) = u(1), \end{aligned} \tag{3.2.23}$$

for which

$$\begin{aligned} v_1(t) <& w_1(t), \\ w_2(t) <& v_2(t). \end{aligned} \tag{3.2.24}$$

Then,

(a) For $t \in [0,1]$ *it holds that*

$$v_0(t) < w_0(t).$$

(b) The terms computed using (3.2.23) form alternating sequences $\{v_{2m+1}(t), w_{2m+1}(t)\}$ *and* $\{w_{2m}(t), v_{2m}(t)\}$*, satisfying*

$$\begin{aligned} v_0(t) < v_1(t) < w_1(t) < \; \dots \; < v_{2m+1}(t) < w_{2m+1}(t) < u_\infty(t) < \\ < w_{2m}(t) < v_{2m}(t) < \; \dots \; < w_2(t) < v_2(t) < w_0(t) \end{aligned} \tag{3.2.25}$$

for $n \geq 0$*. Each term* $v_{m+1}(t)$*,* $w_{m+1}(t)$ *is computed from the corresponding integral equations:*

$$\begin{aligned} v_{m+1}(t) =& \alpha_0 + \eta t + (\alpha_1 - \alpha_0 - \eta) t^p + \frac{1}{\Gamma(p)} \Bigg[\int_0^t (t-s)^{p-1} f(s, v_m(s); \overline{\lambda}) ds \\ & + t^p \int_0^1 (1-s)^{p-1} f(s, v_m(s); \overline{\lambda}) ds \Bigg], \end{aligned} \tag{3.2.26}$$

$$w_{m+1}(t) = \alpha_0 + \zeta t + (\alpha_1 - \alpha_0 - \zeta)t^p + \frac{1}{\Gamma(p)}\left[\int_0^t (t-s)^{p-1} f(s, w_m(s); \overline{\lambda})ds \right.$$
$$\left. + t^p \int_0^1 (1-s)^{p-1} f(s, w_m(s); \overline{\lambda})ds \right], \tag{3.2.27}$$

where the values of the unknown parameters η *and* ζ *denote* $\eta := v'(0)$ *and* $\zeta := w'(0)$ *and are calculated by solving the determining equations*

$$\Delta(\eta) = 0, \tag{3.2.28}$$

$$\Delta(\zeta) = 0, \tag{3.2.29}$$

where

$$\Delta(\eta) = \Gamma(p+1)\,(\alpha_1 - \alpha_0 - \eta) - p\int_0^1 (1-s)^{p-1} f(s, v(s); \overline{\lambda})ds,$$

and

$$\Delta(\zeta) = \Gamma(p+1)\,(\alpha_1 - \alpha_0 - \zeta) - p\int_0^1 (1-s)^{p-1} f(s, w(s); \overline{\lambda})ds.$$

(c) Let $x_0(t) := v_0(t)$ *and* $\{x_m(t)\} := \{v_{2m+1}(t), w_{2m+1}(t)\}$ *for* $n \geq 0$*, that is,* $x_1(t) := v_1(t), x_3(t) := w_1(t), \ldots$ *and similarly,* $\{y_m(t)\} := \{v_{2m}(t), w_{2m}(t)\}$ *for* $n \geq 0$*, that is,* $y_0(t) := w_0(t), y_1(t) := v_2(t), y_3(t) := w_2(t), \ldots$ *. Then the sequences* $\{x_m(t)\}$ *and* $\{y_m(t)\}$ *converge uniformly to the limits* $x_\infty(t)$ *and* $y_\infty(t)$*, respectively, and* $x_\infty(t) < y_\infty(t)$*.*

(d) For the limit functions $x_\infty(t)$ *and* $y_\infty(t)$ *it holds that* $x_\infty(t) = y_\infty(t) = u_\infty(t)$*, where* $u_\infty(t)$ *is the unique solution to FBVP (3.2.3).*

Proof. (a) From Definition 3.2.1 of lower and upper solutions of type I, we have that

$${}_0^C D_t^p v_0(t) - f(t, v_0(t); \overline{\lambda}) > 0, \tag{3.2.30}$$

$${}_0^C D_t^p w_0(t) - f(t, w_0(t); \overline{\lambda}) < 0. \tag{3.2.31}$$

Subtracting (3.2.30) from (3.2.31) and using the Mean Value Theorem, we obtain

$${}_0^C D_t^p (w_0(t) - v_0(t)) - \frac{\partial f}{\partial u}(u^*)(w_0 - v_0) < 0,$$

where $u^* = \gamma v_0 + (1-\gamma)w_0$, $0 \leq \gamma \leq 1$. Since $f(t, u(t); \overline{\lambda})$ is an increasing function, $-\frac{\partial f}{\partial u}(u^*) < 0$. Moreover, $(w_0 - v_0)(0) \geq 0, (w_0 - v_0)(1) \geq 0$, thus, by Lemma 3.2.1 it follows that $w_0(t) > v_0(t)$.

(b) Let $z_0(t) = v_0(t) - v_1(t)$. Then

$$\begin{aligned} {}_0^C D_t^p z_0(t) &= {}_0^C D_t^p v_0(t) - f(t, v_0(t); \overline{\lambda}) > 0, \\ z_0(0) &\leq 0,\ z_0(1) \leq 0 \end{aligned}$$

thus, by Lemma 3.2.2, $v_0(t) < v_1(t)$.

Now let $z_1(t) = v_1(t) - v_2(t)$ and consider

$$\begin{aligned} {}_0^C D_t^p z_1(t) &= f(t, v_0(t); \overline{\lambda}) - f(t, v_1(t); \overline{\lambda}) < 0, \\ z_1(0) &\leq 0,\ z_1(1) \leq 0, \end{aligned}$$

where the first inequality follows from the fact that $f(t, u(t); \overline{\lambda})$ is increasing in $u(t)$. Thus, by Lemma 3.2.2, $v_1(t) < v_2(t)$.

Assume that $v_{2k+1}(t) < v_{2k}(t)$ for $k \geq 1$. We will show that it also holds for $k+1$. Consider $z_{2k+1}(t) = v_{2k+1}(t) - v_{2k+2}(t)$, for which

$$\begin{aligned} {}_0^C D_t^p z_{2k+1}(t) &= f(t, v_{2k}(t); \overline{\lambda}) - f(t, v_{2k+1}; \overline{\lambda}) > 0, \\ z_{2k+1}(0) &\leq 0,\ z_{2k+1}(1) \leq 0, \end{aligned}$$

thus, by Lemma 3.2.2, $v_{2k+1}(t) < v_{2k+2}(t)$.
Now let $z_{2k+3}(t) = v_{2k+3}(t) - v_{2k+2}(t)$ and consider

$$\begin{aligned} {}_0^C D_t^p z_{2k+3}(t) &= f(t, v_{2k+2}(t); \overline{\lambda}) - f(t, v_{2k+1}(t); \overline{\lambda}) > 0 \\ z_{2k+3}(0) &\leq 0,\ z_{2k+3}(1) \leq 0. \end{aligned}$$

Hence, $v_{2k+3}(t) < v_{2k+2}(t)$, which implies that $v_{2m+1}(t) < v_{2m}(t)$ holds for all $m \geq 1$.

Using the same method, we can show that $w_1(t) < w_0(t)$, $w_1(t) < w_2(t)$, and $w_{2n+1}(t) < w_{2n}(t)$ for $n \geq 0$.

From the assumptions and inequalities in (3.2.24), it follows that

$$v_0(t) < v_1(t) < w_1(t) < w_2(t) < v_2(t) < w_0(t).$$

Assume that $v_{2k+1}(t) < w_{2k+1}(t)$. We will show that this holds for $n = k+1$. Consider $z_{2k+2}(t) = w_{2k+2}(t) - v_{2k+2}(t)$:

$$\begin{aligned} {}_0^C D_t^p z_{2k+2}(t) &= f(t, w_{2k+1}(t); \overline{\lambda}) - f(t, v_{2k+1}(t); \overline{\lambda}) > 0 \\ z_{2k+2}(0) &\leq 0,\ z_{2k+2}(1) \leq 0, \end{aligned}$$

thus, by Lemma 3.2.2, $w_{2k+2}(t) < v_{2k+2}(t)$.
Now let $z_{2k+3}(t) = v_{2k+3}(t) - w_{2k+3}(t)$ and consider

$$\begin{aligned} {}_0^C D_t^p z_{2k+3}(t) &= f(t, v_{2k+2}(t); \overline{\lambda}) - f(t, w_{2k+2}(t); \overline{\lambda}) > 0 \\ z_{2k+3}(0) &\leq 0,\ z_{2k+3}(1) \leq 0, \end{aligned}$$

thus, $v_{2k+3}(t) < w_{2k+3}(t)$. This implies that $v_{2m+1}(t) < w_{2m+1}(t)$ holds for $m \geq 1$.

Similarly, we can show that $w_{2m}(t) < v_{2m}(t)$ for $n \geq 1$.

Thus far we have seen that for all $m \geq 1$, the following inequalities hold

$$\begin{aligned} &v_0(t) < v_{2m+1}(t) < v_{2m}(t), \\ &w_{2m+1}(t) < w_{2m}(t) < w_0(t), \\ &v_{2m+1}(t) < w_{2m+1}(t), \\ &w_{2m}(t) < v_{2m}(t). \end{aligned}$$

Combining these inequalities results in (3.2.25).

(c) The sequence $x_m(t)$ is a monotonically increasing sequence of continuous functions, bounded from above by $w_0(t)$, defined on the compact domain $[0,1]$, and the sequence $y_m(t)$ is a monotonically decreasing sequence of continuous functions, bounded from below by $v_0(t)$, defined on the compact domain $[0,1]$. Hence, $x_m(t)$ and $y_m(t)$ converge uniformly to their respective limits, $x_\infty(t)$ and $y_\infty(t)$. From part (a) we know that $x_m(t) < y_m(t)$ for all $m \geq 0$, thus $x_\infty(t) = \lim_{n\to\infty} x_m(t) < \lim_{m\to\infty} y_m(t) = y_\infty(t)$.

(d) Passing to the limit when $m \to \infty$ in the integral equations (3.2.26) and (3.2.27) yields

$$\begin{aligned} v_\infty(t) =& \alpha_0 + \eta t + (\alpha_1 - \alpha_0 - \eta)t^p + \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1} f(s, v_\infty(s); \overline{\lambda})ds \\ &+ t^p \int_0^1 (1-s)^{p-1} f(s, v_\infty(s); \overline{\lambda})ds\Bigg], \end{aligned}$$

$$\begin{aligned} w_\infty(t) =& \alpha_0 + \zeta t + (\alpha_1 - \alpha_0 - \zeta)t^p + \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1} f(s, w_\infty(s); \overline{\lambda})ds \\ &+ t^p \int_0^1 (1-s)^{p-1} f(s, w_\infty(s); \overline{\lambda})ds\Bigg], \end{aligned}$$

which are equivalent to Eq. (3.2.14). The limit function $u_\infty(t)$ is the unique solution to (3.2.14), thus, $v_\infty(t) = w_\infty(t) = u_\infty(t)$. □

Theorem 3.2.8. *Consider the FBVP (3.2.3). Assume that*

(i) v_0, w_0 $\in C^1([0,1], \mathbb{R})$ are lower and upper solutions to the FBVP (3.2.3) of type II with $v_0(t) < w_0(t)$ for $t \in [0,1]$.

(ii) the right-hand side function $f(t, u(t, \overline{\lambda}; \chi); \overline{\lambda})$ is an increasing function in $u(t, \overline{\lambda}; \chi)$.

(iii) two sequences, $\{v_n(t)\}$ and $\{w_n(t)\}$, are computed using the iterative scheme

$$
\begin{aligned}
{}_0^C D_t^p v_{m+1}(t) =& f(t, v_n(t); \overline{\lambda}), \quad v_{m+1}(0) = u(0), \quad v_{m+1}(1) = u(1)\\
{}_0^C D_t^p w_{m+1}(t) =& f(t, w_n(t); \overline{\lambda}), \quad w_{m+1}(0) = u(0), \quad w_{m+1}(1) = u(1),
\end{aligned}
$$

for which

$$
\begin{aligned}
v_0(t) <& w_1(t),\\
v_1(t) <& w_0(t).
\end{aligned}
\tag{3.2.32}
$$

Then,

(a) The terms computed using (3.2.32) form alternating sequences $\{v_{2m}(t), w_{2m+1}(t)\}$ and $\{v_{2m+1}(t), w_{2m}(t)\}$, satisfying

$$
\begin{aligned}
v_0(t) < w_1(t) < v_2(t) < \ \ldots\ < v_{2m}(t) < w_{2m+1}(t) < u_\infty(t) <\\
< v_{2m+1}(t) < w_{2m}(t) < \ \ldots\ < w_2(t) < v_1(t) < w_0(t)
\end{aligned}
$$

for $m \geq 0$. Each term $v_{m+1}(t)$, $w_{m+1}(t)$ is computed from the following integral equations:

$$
\begin{aligned}
v_{m+1}(t) =& \alpha_0 + \eta t + (\alpha_1 - \alpha_0 - \eta)t^p + \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1} f(s, v_m(s); \overline{\lambda}) ds\\
&+ t^p \int_0^1 (1-s)^{p-1} f(s, v_m(s); \overline{\lambda}) ds\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
w_{m+1}(t) =& \alpha_0 + \zeta t + (\alpha_1 - \alpha_0 - \zeta)t^p + \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1} f(s, w_m(s); \overline{\lambda}) ds\\
&+ t^p \int_0^1 (1-s)^{p-1} f(s, w_m(s); \overline{\lambda}) ds\Bigg],
\end{aligned}
$$

where the unknown parameters η and ζ denote $\eta := v'(0)$ and $\zeta := w'(0)$ and are calculated by solving the determining equations (3.2.28) and (3.2.29).

(b) Let $\{x_m(t)\} := \{v_{2m}(t), w_{2m+1}(t)\}$ and $\{y_m(t)\} := \{v_{2m+1}(t), w_{2m}(t)\}$ for $n \geq 0$. Then the sequences $\{x_m(t)\}$ and $\{y_m(t)\}$ converge uniformly to the limits $x_\infty(t)$ and $y_\infty(t)$, respectively, and $x_\infty(t) < y_\infty(t)$.

(c) For the limit functions $x(t)$ and $y(t)$ it holds that $x_\infty(t) = y_\infty(t) = u_\infty(t)$, where $u(t)$ is the unique solution to FBVP (3.2.3).

Proof. The proof of Theorem 3.2.8 follows the lines of the proof of Theorem 3.2.7. □

Remark 3.2.3. *As in the standard numerical-analytic technique, the values of parameters η and ζ are computed by solving the corresponding approximate determining equations*

$$
\Delta_m(\eta) = 0 \tag{3.2.33}
$$

and

$$\Delta_m(\zeta) = 0 \tag{3.2.34}$$

at each iteration step m, where

$$\Delta_m(\eta) = \Gamma(p+1)\,(\alpha_1 - \alpha_0 - \eta) - p\int_0^1 (1-s)^{p-1} f(s, v_m(s); \overline{\lambda})ds, \tag{3.2.35}$$

and

$$\Delta_m(\zeta) = \Gamma(p+1)\,(\alpha_1 - \alpha_0 - \zeta) - p\int_0^1 (1-s)^{p-1} f(s, w_m(s); \overline{\lambda})ds. \tag{3.2.36}$$

Remark 3.2.4. *It is worth emphasizing that the lower and upper solutions method can also be used to simplify the computations of the approximating sequence. In particular, we can construct a sequence $\tilde{u}_m(t, \overline{\lambda}; \chi)$, given by*

$$\begin{aligned} \tilde{u}_0(t) =& \frac{v_0(t) + w_0(t)}{2}, \\ \tilde{u}_m(t, \overline{\lambda}; \chi) =& \alpha_0 + \chi t + t^p(\alpha_1 - \alpha_0 - \chi) + \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1} f(s, \tilde{u}_{m-1}(s, \overline{\lambda}; \chi)ds \\ & - t^p \int_0^1 (1-s)^{p-1} f(s, \tilde{u}_{m-1}(s, \overline{\lambda}; \chi)ds\Bigg]. \end{aligned} \tag{3.2.37}$$

The statements of Theorem 3.2.3 hold for the new sequence $\tilde{u}_m(t, \overline{\lambda}; \chi)$ and the terms in it are of simpler form, which can lead to a reduction in the computational time.

3.2.5. EXAMPLES

In this sub-section, we use the theory presented in the previous sections to construct sequences of approximations to FBVPs of the form given in (3.2.3), (3.2.4). By varying the parameter λ, we examine cases where the right-hand side function is either decreasing or increasing. This leads to a monotone sequence of approximations in the former case and an alternating sequence in the latter. We also demonstrate how the lower and upper solutions method can be used to improve the efficiency of the numerical-analytic technique.

We consider a non-linear FDE of the form

$${}_0^C D_t^p u(t) = -\lambda a(t) F(u(t)) + b(t). \tag{3.2.38}$$

As in sub-section 3.1.5, we take the functions $a(t)$ and $b(t)$ to be given by

$$a(t) := \frac{-2e^t}{(1+e^t)^2}, \quad b(t) := -\frac{2\omega e^t(1-e^t)}{(1+e^t)^3}.$$

Now $F(u(t))$ is taken as a quadratic function of $u(t)$:

$$F(u(t)) = u^2(t).$$

With this, Equation (3.2.38) becomes

$$ {}_0^C D_t^p u(t) = \frac{2\lambda e^t}{(1+e^t)^2} u(t)^2 - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} \; (:= f(t, u(t); \lambda)), \tag{3.2.39}$$

where $t \in [0,1]$, $p = 1.98$ and ω is a scalar which in the context of the flow of the Antarctic Circumpolar Current is corresponding to the dimensionless Coriolis parameter being equal to 4649.56.

We associate with FDE (3.2.39) Dirichlet boundary conditions:

$$\begin{aligned} u(0) &= \alpha_0, \\ u(1) &= \alpha_1. \end{aligned} \tag{3.2.40}$$

Using the notations in (3.2.5), we have

$$A = 0.5, \quad B = 844.91,$$

and the value of the parameter λ should be chosen such that

$$\Lambda < \frac{2^{2p}\Gamma(p+1)}{K}$$

holds, where K is the Lipschitz constant in (3.2.7). We consider two cases of the FBVP (3.2.39), (3.2.40), when λ is positive and when it is negative, and use the numerical-analytic method to construct a monotone and an alternating sequence of approximate solutions.

Monotone Sequence

We first fix the parameter value to $\overline{\lambda} = 0.05$, and choose as boundary conditions

$$\alpha_0 = -1, \quad \alpha_1 = -1.5. \tag{3.2.41}$$

The set D for the FBVP (3.2.39), (3.2.41) is calculated to be

$$D := \{-105 \leq u(t) \leq 60\},$$

and using the notations in (3.2.6), (3.2.7), and (3.2.12) we compute

$$M = 845, \quad K = 10.5, \quad \beta = 56.7.$$

With this, we can ensure that the set of initial values D_β, defined in (3.2.11), is non-empty, thus we can apply the numerical-analytic method to construct approximations to FBVP (3.2.39), (3.2.41). Moreover, we know that

$$\frac{\partial}{\partial u} f(t, u(t); \overline{\lambda}) = \frac{0.2 e^t}{(1+e^t)^2} u(t) < 0$$

when $u(t) < 0$, i.e. the right-hand side in (3.2.39) is decreasing with respect to $u(t)$, hence we expect the resulting sequence to be monotone.

The sequence in (3.2.10) and the approximate determining equation (3.2.20) take the following forms:

$$\begin{aligned} u_0(t, \overline{\lambda}; \chi) =& -1 + \chi t + t^p(-0.5 - \chi), \\ u_m(t, \overline{\lambda}; \chi) =& -1 + \chi t + t^p(-0.5 - \chi) \\ &+ \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f(s, u_{m-1}(s, \overline{\lambda}; \chi), \overline{\lambda}) ds \right. \\ &\left. - t^p \int_0^1 (1-s)^{p-1} f(s, u_{m-1}(s, \overline{\lambda}; \chi), \overline{\lambda}) ds \right], \end{aligned}$$

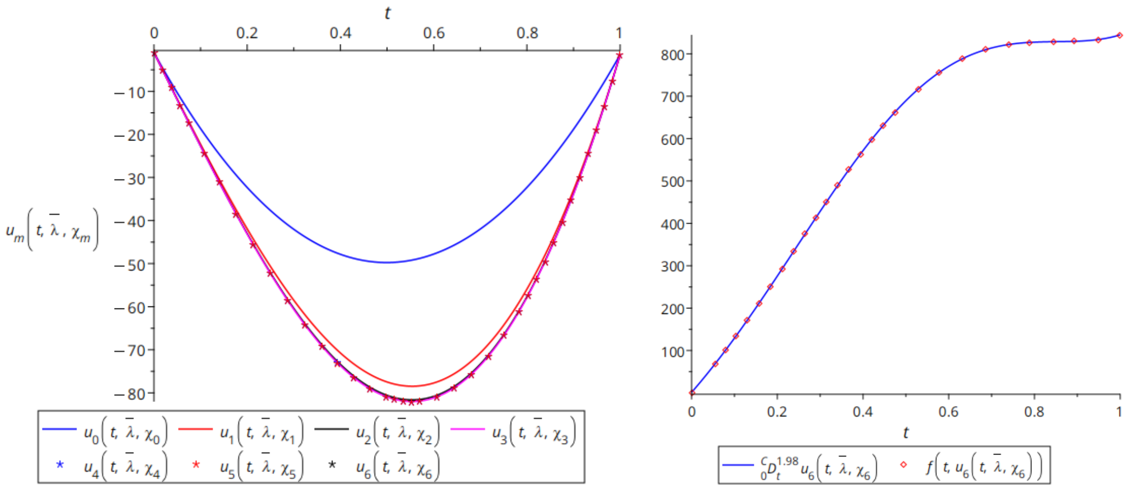
$$\Delta_m(\chi) = \Gamma(p+1)(-0.5-\chi) - p \int_0^1 (1-s)^{p-1} f(s, u_m(s, \overline{\lambda}; \chi), \overline{\lambda}) ds = 0. \quad (3.2.42)$$

The approximate parameter values χ_m, computed using (3.2.42), for $m = \{0, ..., 6\}$, are shown in Table 3.4.

Table 3.4: Approximate parameter values χ_m for FBVP (3.2.38), (3.2.41) with $\overline{\lambda} = 0.05$, $m = \{0, ..., 6\}$.

m	χ_m
0	-197.241
1	-216.956
2	-219.547
3	-219.852
4	-219.887
5	-219.890
6	-219.890

Figure 3.3a shows plots of the first 7 approximating terms which form a decreasing monotone sequence, as predicted by the theory in sub-section 3.2.3. The plots show that the sequence converges, as the terms begin to overlap for $m \geq 4$.



(a) Plots of the monotone sequence of approximate solutions $u_m(t, \lambda; \chi_m)$ to FBVP (3.2.38), (3.2.41) with $\overline{\lambda} = 0.05$, $m = \{0, ..., 6\}$.

(b) Comparison plots between the left- (solid blue line) and right- (dotted red line) hand sides of FDE (3.2.39) with $u_6(t, \overline{\lambda}; \chi_6)$, $\overline{\lambda} = 0.05$.

Figure 3.3: Approximate solutions and comparison plots for FBVP (3.2.38), (3.2.41) with $\overline{\lambda} = 0.05$.

The exact solution to FBVP (3.2.39), (3.2.40) is unknown; therefore, as in the previous section, we compare the left- and right-hand sides of Equation (3.2.39) with a few of the approximating functions plugged in to evaluate the performance of our approximation method. Table 3.5 shows the maximum of the absolute value of the difference between the left- and right- hand sides of the equation for $m = 0, 3, 6$ (denoted by δ_m, as in the previous section), which decreases with each successive term, as expected.

Table 3.5: Difference between the left- and right-hand sides of FDE (3.2.39) with monotone sequence terms $u_m(t; \chi_{1,m})$ for $m = 0, 3, 6, \overline{\lambda} = 0.05$

m	δ_m
0	472.4
3	0.638
6	0.014

Figure 3.3b shows a comparison of the right- and left-hand sides of Equation (3.2.39) with $u(t) = u_6(t, \lambda; \chi_6)$ plugged in. The two sides of the equation are in good agreement at the sixth iteration.

Alternating sequence

Let us now $\overline{\lambda} = -0.1$, and choose boundary conditions

$$\alpha = 1, \quad \beta = 1.5. \tag{3.2.43}$$

For FBVP (3.2.39), (3.2.43) we calculated that

$$D := \{-100 \leq u(t) \leq 25\},$$

and the constants in (3.2.6), (3.2.7), and (3.2.12) are calculated to be

$$M = 845, \quad K = 12, \quad \beta = 58.$$

As before, we find that the set D_β is non-empty, and our approximation method can be applied. Now we have

$$\frac{\partial}{\partial u} f(t, u(t); \overline{\lambda}) = \frac{-0.4e^t}{(1+e^t)^2} u(t) > 0$$

when $u(t) < 0$. Since the right-hand side function in (3.2.39) is increasing with respect to $u(t)$, the sequence we construct should be alternating.

After applying our approximation method to FBVP (3.2.39), (3.2.43), we obtain the following form of the sequence of approximations and the approximate determining equation:

$$\begin{aligned} u_0(t, \overline{\lambda}; \chi) =& 1 + \chi t + t^p(0.5 - \chi), \\ u_m(t, \overline{\lambda}; \chi) =& 1 + \chi t + t^p(0.5 - \chi) \\ &+ \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f(s, u_{m-1}(s, \overline{\lambda}; \chi), \overline{\lambda}) ds \right. \\ &\left. - t^p \int_0^1 (1-s)^{p-1} f(s, u_{m-1}(s, \overline{\lambda}; \chi), \overline{\lambda}) ds \right], \end{aligned}$$

$$\Delta_m(\chi) = \Gamma(p+1)(0.5 - \chi) - p \int_0^1 (1-s)^{p-1} f(s, u_m(s, \overline{\lambda}; \chi).\overline{\lambda}) ds = 0. \tag{3.2.44}$$

Solving Equation (3.2.44) yields the approximate parameter values χ_m, for $m = \{0, ..., 6\}$, shown in Table 3.6.

Table 3.6: Approximate parameter values χ_m for FBVP (3.2.38), (3.2.43) with $\overline{\lambda} = -0.1$, $m = \{0, ..., 6\}$.

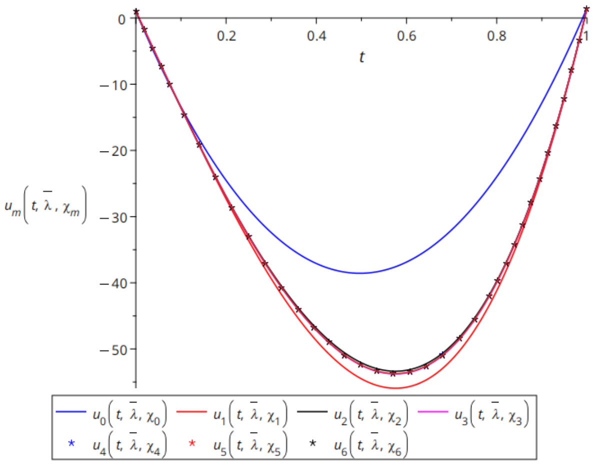
m	χ_m
0	−161.011
1	−145.455
2	−147.922
3	−147.531
4	−147.589
5	−147.581
6	−147.582

Plots of the first 7 approximating terms $u_m(t, \lambda; \chi)$, $m = 0, ..., 6$, are shown Figure 3.4a. They form an alternating sequence, as expected, which is most easily seen from the first 3 terms, plotted with the solid blue, red, and black lines. It is clear from the plots that the sequence converges, as the terms become visually indistinguishable for $m \geq 4$.

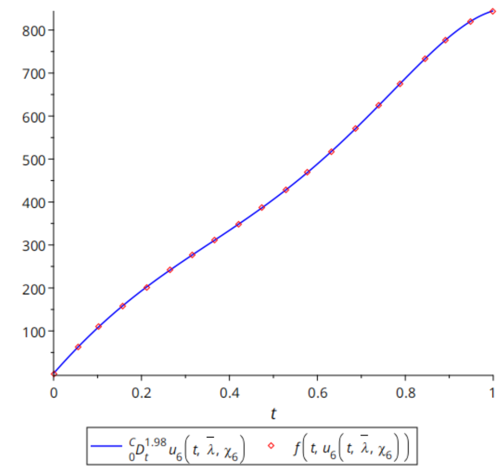
Table 3.7 shows the decreasing maximum differences between the left- and right-hand sides of Equation (3.2.39) for the $m = 0,\ 3,\ 6$ terms in the alternating sequence. A comparison plot of left- (solid blue line) and right- (dotted red line) hand sides of Equation (3.2.39) with $u(t) = u_6(t, \lambda; \chi_6)$ is shown in Figure 3.4b, where we see an overlap between the two sides of the equation, suggesting that the term $u_6(t, \lambda; \chi_6)$ satisfies the equation well.

Table 3.7: Difference between the left- and right-hand sides of FDE (3.2.39) with alternating sequence terms $u_m(t; \chi_{1,m})$ for $m = 0,\ 3,\ 6$, $\overline{\lambda} = -0.1$.

m	δ_m
0	533.2
3	1.462
6	0.005



(a) Plots of the alternating sequence of approximate solutions $u_m(t, \overline{\lambda}; \chi_m)$ to FBVP (3.2.38), (3.2.43) for $m = \{0, ..., 6\}$, $\overline{\lambda} = -0.1$.



(b) Comparison plots between the left- (solid blue line) and right- (dotted red line) hand sides of FDE (3.2.39) with $u_6(t, \overline{\lambda}, \chi_6)$, $\overline{\lambda} = -0.1$.

Figure 3.4: Approximate solutions and comparison plots for FBVP (3.2.38), (3.2.43) with $\overline{\lambda} = -0.1$.

Lower and upper solutions

Let us now apply the upper and lower solutions method, discusses in sub-section 3.2.4, to construct approximate solutions to FBVP (3.2.39), (3.2.43).

We choose the lower and upper solutions $v_0(t) = -148$, $w_0(t) = 10$, which satisfy the following differential inequalities:

$$\begin{aligned}{}_0^C D_t^{1.98} v_0(t) =& 0 > f(t, v_0(t); \lambda),\\ v_0(0) =& -148 < u(0), \ \ v_0(1) = -148 < u(1),\end{aligned}$$

$$\begin{aligned}{}_0^C D_t^{1.98} w_0(t) =& 0 < f(t, w_0(t); \lambda),\\ w_0(0) =& 10 > u(0), \ \ v_0(1) = 10 > u(1),\end{aligned}$$

that is, they are a lower and upper solution to FBVP (3.2.39), (3.2.43) of type I.

We implement the sequences, given in (3.2.26) and (3.2.27), which now read

$$\begin{aligned}v_{m+1}(t) =& 1 + \eta t + (0.5 - \eta)t^p + \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1} f(s, v_m(s); \overline{\lambda})ds\\ &+ t^p \int_0^1 (1-s)^{p-1} f(s, v_m(s); \overline{\lambda})ds\Bigg],\end{aligned}$$

$$\begin{aligned}w_{m+1}(t) =& 1 + \zeta t + (0.5 - \zeta)t^p + \frac{1}{\Gamma(p)}\Bigg[\int_0^t (t-s)^{p-1} f(s, w_m(s); \overline{\lambda})ds\\ &+ t^p \int_0^1 (1-s)^{p-1} f(s, w_m(s); \overline{\lambda})ds\Bigg].\end{aligned}$$

The parameter values η_m and ζ_m are calculated by solving the following approximate determining equations

$$\Delta_m(\eta) = \Gamma(p+1)(0.5 - \eta) - p\int_0^1 (1-s)^{p-1} f(t, v_m(t, \overline{\lambda}; \eta).\overline{\lambda})ds = 0, \tag{3.2.45}$$

$$\Delta_m(\zeta) = \Gamma(p+1)(0.5 - \zeta) - p\int_0^1 (1-s)^{p-1} f(t, w_m(t, \overline{\lambda}; \zeta).\overline{\lambda})ds = 0. \tag{3.2.46}$$

Table 3.8 shows the obtained approximate values for η_m and ζ_m for $m = \{0, ..., 6\}$.

Table 3.8: Approximate parameter values χ_m for FBVP (3.2.39), (3.2.43), obtained from the upper and lower solutions, $m = \{0, ..., 6\}$.

m	η_m	ζ_m
1	-140.473	-144.272
2	-148.827	-148.084
3	-147.390	-147.509
4	-147.611	-147.592
5	-147.578	-147.581
6	-147.611	-147.592

The calculated sequence terms $v_m(t,\overline{\lambda};\eta)$ and $w_m(t,\overline{\lambda};\zeta)$, for $m=\{0,...,6\}$ are plotted in Figure 3.5. The same figure also shows a plot of the alternating sequence term $u_6(t,\overline{\lambda};\chi)$, which is 'trapped' between the upper and lower solutions. The left- and right-hand sides of the FDE (3.2.39) are plotted in Figure 3.6 with $v_6(t,\overline{\lambda};\eta_6)$ (left panel) and $w_6(t,\overline{\lambda};\zeta_6)$ (right panel). We see a good agreement between the two sides of the equation in both cases. Table 3.9 shows the maximum differences between the left- and right-hand sides of the equation with the upper and lower solution terms $v_m(t;\chi_{1,m})$ and $w_m(t;\chi_{1,m})$, denoted by δ^v_m and δ^w_m, respectively, for $m=1,\,4,\,6$.

Table 3.9: Difference between the left- and right-hand sides of FDE (3.2.39) with upper and lower solution terms $v_m(t;\chi_{1,m})$ and $w_m(t;\chi_{1,m})$ for $m=1,\,4,\,6$.

m	δ^v_m	δ^w_m
1	147.4	79.8
4	0.828	0.306
6	0.087	0.032

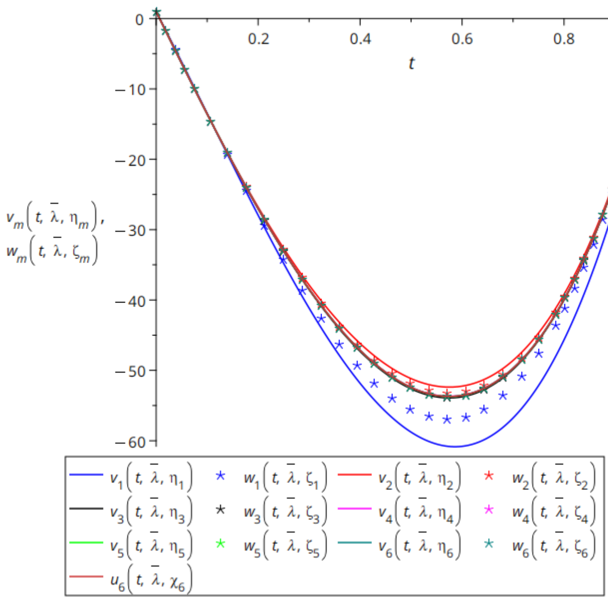


Figure 3.5: Plots of the upper and lower solutions $w_m(t,\overline{\lambda};\zeta_m)$ and $v_m(t,\overline{\lambda};\eta_m)$ to FBVP (3.2.39), (3.2.43), $m=\{0,...,6\}$, and the alternating sequence term $u_6(t,\overline{\lambda};\chi_6)$ (solid brown line).

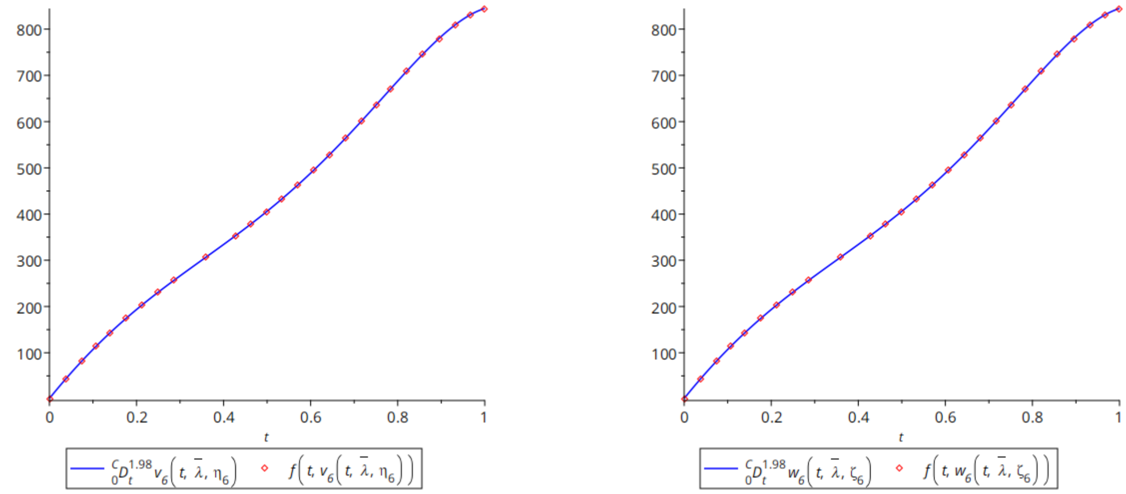


Figure 3.6: Comparison plots between the left- (solid blue line) and right- (dotted red line) hand sides of FDE (3.2.39) with the lower solution $v_6(t, \overline{\lambda}; \eta_6)$ (left panel) and the upper solution $w_6(t, \overline{\lambda}; \zeta_6)$ (right panel).

Lastly, we construct the sequence $\tilde{u}_m(t, \overline{\lambda}; \bar{\chi})$, given in (3.2.37), by taking

$$\tilde{u}_0(t) = \frac{v_0(t) + w_0(t)}{2} = \frac{-148 + 10}{2},$$

$$\tilde{u}_m(t, \overline{\lambda}; \bar{\chi}) = 1 + \chi t + t^p(0.5 - \chi) + \frac{1}{\Gamma(p)} \left[\int_0^t (t-s)^{p-1} f(s, \tilde{u}_{m-1}(s, \overline{\lambda}; \bar{\chi}) ds - t^p \int_0^1 (1-s)^{p-1} f(s, \tilde{u}_{m-1}(s, \overline{\lambda}; \bar{\chi}) ds \right].$$

The computed values of the parameter $\bar{\chi}_m$ are shown in Table 3.10. The value of $\bar{\chi}_6$ agrees with that of χ_6, calculated using the alternating sequence, to two decimal places (see Table 3.6).

Table 3.10: Approximate parameter values $\bar{\chi}_m$ for FBVP (3.2.39), (3.2.43), obtained from the sequence in (3.2.37), $m = \{0, ..., 6\}$

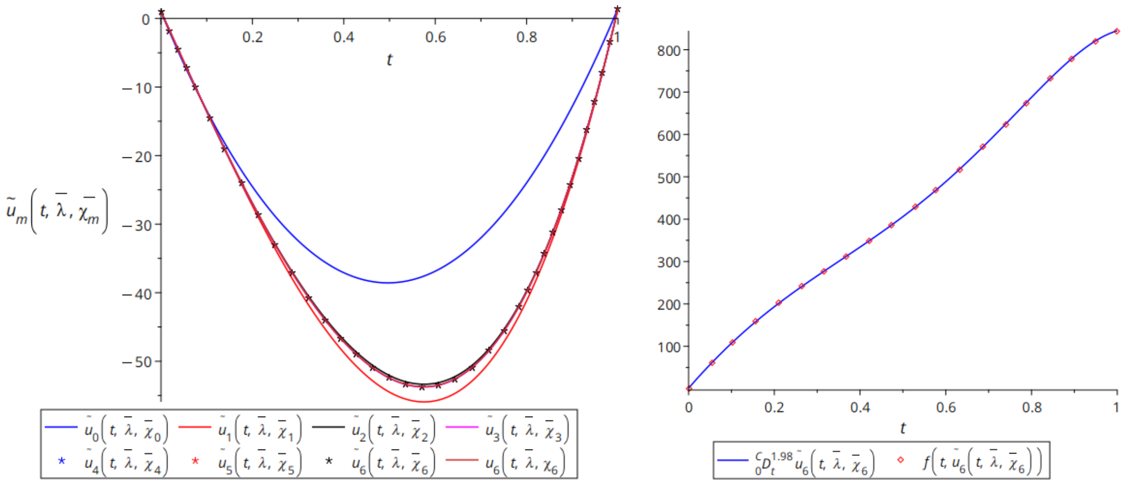
m	$\bar{\chi}_m$
0	−113.607
1	−137.367
2	−149.289
3	−147.327
4	−147.620
5	−147.476
6	−147.583

The maximum differences between the left- and right-hand sides of the equation with terms $\bar{u}_m(t; \tilde{\chi}_{1,m})$ for $m = 0, 3, 6$ are shown in Table 3.11. Figure 3.7b shows a comparison between the left- and right-hand sides of Equation (3.2.39) with $\bar{u}_6(t, \overline{\lambda}; \bar{\chi}_6)$ which are in good agreement.

Table 3.11: Difference between the left- and right-hand sides of FDE (3.2.39) with sequence terms $\bar{u}_m(t;\tilde{\chi}_{1,m})$ for $m = 0,\ 3,\ 6$.

m	δ_m
0	630.4
3	2.361
6	0.009

The plots of the sequence terms $\bar{u}_m(t,\overline{\lambda};\bar{\chi})$ for $m = \{0, ..., 6\}$ are shown in Figure 3.7a, along with the plot of $u_6(t,\overline{\lambda};\chi)$, computed using the alternating sequence for comparison. This verifies that the terms of the sequence $\bar{u}_m(t,\overline{\lambda};\bar{\chi})$ agree with the terms obtained from the standard numerical-analytic technique. Moreover, the recorded CPU time for calculating the first 7 values of $\bar{\chi}_m$, averaged over 5 calculations, was 129.43 seconds. In comparison, the CPU time for the calculation of the parameter values χ_m was 1017.05 seconds (also averaged over 5 calculations).



(a) Plots of the sequence of approximate solutions $\bar{u}_m(t,\overline{\lambda};\tilde{\chi}_m)$ to FBVP (3.2.39), (3.2.43), $m = \{0, ..., 6\}$, and the alternating sequence term $u_6(t,\overline{\lambda};\chi_6)$ (solid brown line).

(b) Comparison plots between the left- (solid blue line) and right- (dotted red line) hand sides of FDE (3.2.39) with $u_6(t,\overline{\lambda},\chi_6)$, $\overline{\lambda} = -0.1$.

Figure 3.7: Approximate solutions and comparison plots for FBVP (3.2.39, (3.2.43).

3.3. CONCLUSION

The numerical-analytic technique is applied in Section 3.1 to analyze a non-linear BVP of fractional order subject to two-point Dirichlet boundary conditions, and to construct approximations to its solutions. In this setting, one of the initial conditions is not specified and is instead treated as an unknown parameter to be determined. We use the derived sequence of approximating functions to establish conditions for the existence of solutions to the FBVP, with the unknown parameter values obtained through the numerical solution of a so-called determining equation. A detailed analysis of the problem's solvability is presented. The theoretical findings are supported by two illustrative examples: one with a known analytical solution, and another without a known solution, based on a fractional reformulation of the equation governing the Antarctic Circumpolar Current, where the classical second-order derivative is replaced by a fractional-order derivative. While the use of fractional derivatives in this context lacks direct physical interpretation, it serves as a demonstration of the method's effectiveness and convergence properties.

In Section 3.2 we investigated a FBVP with a parameter-dependent right-hand side and Dirichlet boundary conditions. Conditions on the parameter that guarantee the existence and uniqueness of solutions are established using fixed point theory. The numerical-analytic technique is used to construct a sequence of approximating functions, and its monotonicity properties are analyzed. When the right-hand side of the FDE is strictly decreasing, the approximating sequence is well-ordered; in contrast, a strictly increasing right-hand side yields an alternating sequence. In the latter case, the method of upper and lower solutions is used in conjunction with the numerical-analytic approach to construct approximating sequences and their uniform convergence to the exact solution is shown. This hybrid strategy also allows for simplification of the approximation terms, thereby reducing the computational effort. As before, the theoretical results were validated through a model example derived from the equation governing the motion of a gyre in the Southern Hemisphere. Several different values of the parameter in the right-hand side function are considered in order to showcase the monotonicity behavior of the resulting approximations.

The developed technique and existence results, discussed in this chapter, can be further extended and applied to more complex fractional BVPs. In particular, our analysis so far has been limited to FBVPs, subject to boundary conditions of the Dirichlet type with right-hand side functions satisfying a Lipschitz continuity condition on the interval of definition of the problem. In the next chapter we present techniques for dealing with FBVPs, subject to boundary conditions of more complex forms, and for extending the applicability of the numerical-analytic technique to a wider class of problems, defined on intervals of arbitrary length.

4

FRACTIONAL BVPS WITH SPECIAL TYPE BOUNDARY CONDITIONS

In this chapter we explore FBVPs of the Caputo type, subject to ‘special-type’ boundary conditions. In Section 4.1 we analyze and construct sequences of approximations of solutions to a system of FDEs, subject to an integral-type boundary condition. Here by the integral boundary condition one understands restrictions on a physical process over the whole interval of consideration, instead of looking only at the localized values. Such constraints are of physical relevance, for instance, when we aim to model non-local effects, or only aggregate data is available for some process. In Section 4.2, we study a fractional initial boundary value problems (FIBV)P, subject to asymptotic and three-point parameter-dependent boundary conditions. In the case of asymptotic boundary conditions, the problem is defined on a semi-infinite interval, i.e. $t \in [a, \infty)$, and the operator in the equation is the Caputo fractional derivative on the half-axis. In this setting, we are interested in the existence of solutions which satisfy the given long-term behaviour. When the FDE is subject to initial and three-point parameter-dependent boundary conditions, the problem is defined on a finite interval of unknown length, and the numerical-analytic technique is used to construct a sequence of approximations to its solution.

In both the integral and three-point boundary conditions cases, we use a parametrization method to reduce the given constraints to Dirichlet-type conditions. This allows us to apply the theory from the previous chapter to the simplified problems. Moreover, we apply an interval-splitting method, which extends the applicability of the numerical-analytic technique to problems, whose right-hand side functions do not satisfy a Lipschitz condition on the original interval of definition. The FBVP is reduced to ‘model-type’ problems, each defined on a smaller interval, and the approximation technique is applied to each of those problems. We establish the connection between the solutions to each of the ‘model-type’ problems, and the solution to the original FBVP. The theoretical results are confirmed with two examples. In Section 4.1, we apply the method to the equation for the Antarctic Circumpolar Current in the fractional setting. In Section 4.2, the technique is applied to an equation used for modeling Arctic gyres in the second order derivative case.

4.1. THE FRACTIONAL BVPS WITH INTEGRAL BOUNDARY CONDITIONS

We consider a system of FDEs of the Caputo type, subject to integral boundary conditions, and extend the numerical-analytic technique to the novel investigation of the existence and construction of its solutions. As it was shown in [49, 50, 82], phenomena, such as heat conduction, fluid flow and viscoelasticity, can be reduced to the study of such non-local problems. Here by the integral boundary conditions one understands restrictions on a physical process (e.g., a speed element of the fluid flow) over the whole interval of consideration, instead of looking only at the localized values. Most results in this direction disclose the qualitative analysis of the integral FBVPs and are based on the fractional Green's function and/or topological degree theory (see results in [83–90]). However, in the physical setting one is especially interested in the visualisation of solutions that gives a better understanding of their behavior.

The original FBVP is reduced to two 'model-type' problems with two-point linear boundary conditions by adapting a parametrization technique used for the reduction of non-linearities in boundary conditions (see for instance [63, 91, 92]), and a dichotomy-type approach, based on the methodology described in [93–96]. A sequence of approximate solutions to each of the 'model-type' FBVPs, which depends on vector-parameters, is constructed in analytic form. We prove the uniform convergence of the sequence to a limit function and show its connection to the original FBVP. The values of the unknown parameters are obtained by numerically solving a system of the so-called approximate determining equations at each iteration of the sequence. The obtained results are applied to an example of the gyre equation for the Antarctic Circumpolar Current, considered in the fractional setting, and subject to the integral boundary conditions, [75–77, 97].

The novel technique presented here has never been applied to the study of FBVPs with integral boundary conditions. It allows us to improve the convergence of the numerical-analytic technique and to sharpen the error estimates obtained in [60–62, 73, 98]. Additionally, the dichotomy-type approach enables application of the aforementioned method to a broader class of FDEs, in particular to those, where the right hand-side does not satisfy the Lipschitz condition on the original domain. As we will show later, this condition is essential in the application of the method studied. Together with other approximation techniques, used for solving systems of FDEs under different boundary restrictions, the approach presented here complements the fundamental study of non-linear FBVPs.

4.1.1. PROBLEM SETTING AND DECOMPOSITION TECHNIQUE

We begin by introducing the FBVP under consideration, and the techniques used for reducing it to 'model-type' problems, defined on smaller intervals and subject to boundary conditions of a simplified form. For this purpose, we use a parametrization

This section is based on the paper [81].

technique to transform the given integral boundary contraints into Dirichlet conditions, and an interval splitting method to re-define the FDS on smaller intervals. Both of these methods are explained in detail in the present sub-section.

We consider a system of Caputo FDEs

$$\begin{aligned} {}_a^C D_t^p u_1(t) &= f_1(t, u(t)), \\ {}_a^C D_t^q u_2(t) &= f_2(t, u(t)), \end{aligned} \tag{4.1.1}$$

defined on $t \in [a, b]$ for some $p, q \in (0, 1)$, subject to the integral boundary condition

$$Au(a) + \int_a^b P(s)u(s)ds + Cu(b) = d. \tag{4.1.2}$$

Here ${}_a^C D_t^p(\cdot)$ denotes the Caputo fractional derivative of order p with the lower limit at a, and $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \in \mathbb{R}^N$, where $N = n + m$, $u_1(t) \in \mathbb{R}^n$, $u_2(t) \in \mathbb{R}^m$, for some $n, m \in \mathbb{Z}^+$. In (4.1.2) A and C are given real $N \times N$ matrices, $P(s)$ is a given continuous $N \times N$-dimensional matrix function, and $d \in \mathbb{R}^N$ is a given vector. The values of the solution at the end points of the interval, $u(a)$ and $u(b)$, are unknown. Note that the FDEs in (4.1.1) may be of mixed order, i.e. we allow for the case when $p \neq q$.

We assume that the unknown vector-valued functions $u_1(t)$ and $u_2(t)$ in (4.1.1) are continuous, i.e. $u_1(t) \in C([a, b], D_1)$, $u_2(t) \in C([a, b], D_2)$, where $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^m$ are closed and bounded domains. The right-hand side functions $f_1 : G \to \mathbb{R}^n$ and $f_2 : G \to \mathbb{R}^m$ are continuous, non-automonous, and generally non-linear in $u(t)$. The domain G is given by $G := [a, b] \times D_1 \times D_2$.

The matrices $A, C \in L(\mathbb{R}^N)$ in (4.1.2) are such that A is arbitrary and C is a singular matrix of the form

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & \mathbf{0}_{N-r} \end{pmatrix}. \tag{4.1.3}$$

Assuming that the matrix C is singular represents the most general case and requires introducing an additional parameter, as will be shown. When C is invertible, a similar treatment applies but with one fewer parameter.

In (4.1.3) C_{11} is a non-singular $r \times r$ matrix for some $r \in \mathbb{Z}^+$, $r < N$, C_{12} is a $r \times (N - r)$ matrix, C_{21} is a $(N - r) \times r$ matrix and $\mathbf{0}_{N-r}$ denotes the $(N - r) \times (N - r)$ matrix of zeros. Note, that any matrix, containing the appropriate number of zeros, can be reduced to the given block form using row operations.

We aim to find a continuous solution $u(t)$ of the FDS (4.1.1) that satisfies integral boundary conditions (4.1.2) in the domain $D = D_1 \times D_2$.

One of the most efficient ways to deal with this task is to write (4.1.1), (4.1.2) in an equivalent integral form. In order to incorporate the integral boundary conditions

(4.1.2) we first need to simplify the original FBVP to one with linear boundary conditions. This is done by an appropriate parametrization technique which is presented below.

Parametrization of the integral boundary conditions

To replace (4.1.2) by linear two-point boundary conditions, we apply a 'freezing' technique, similar to ([91]-[92]). For this we introduce the following vector-parameters

$$\begin{aligned} z &= col(z_1, z_2, ..., z_N), \\ \lambda &= col(\lambda_1, \lambda_2, ..., \lambda_N), \\ \eta &= col(\underbrace{0, 0, ..., 0}_{r}, \eta_{r+1}, \eta_{r+2}, ..., \eta_N) \end{aligned}$$

by putting

$$\begin{aligned} z :=& u(a), \\ \lambda :=& \int_a^b P(s)u(s)ds, \\ \eta_i :=& u_i(b), \quad i = r+1, r+2, ..., N, \end{aligned} \tag{4.1.4}$$

where, as mentioned before, $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$.

Under the parametrization (4.1.4) the integral boundary conditions (4.1.2) are re-written as

$$Au(a) + C_1 u(b) = d(\eta, \lambda), \tag{4.1.5}$$

where

$$\begin{aligned} d(\eta, \lambda) &= d - \lambda + \eta, \\ C_1 &= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & \mathbf{1}_{N-r} \end{pmatrix}, \quad \det C_1 \neq 0 \end{aligned}$$

and $\mathbf{1}_{N-r}$ denotes the $(N-r) \times (N-r)$ unit matrix.

Remark 4.1.1. *Note, that parametrization (4.1.4) allows us not only to reduce the integral boundary conditions (4.1.2) to the two-point linear ones, but also to eliminate the singularity of the matrix C. As it will be seen in Section 3, this step is essential for constructing our iterative sequence.*

After applying the above-described parametrization, we are going to study the family of parametrized FBVPs with linear two-point boundary conditions (4.1.1), (4.1.5), instead of the original FBVP with integral boundary conditions (4.1.1), (4.1.2). To return back to the original BVP, the values of the parameters are chosen appropriately.

As it will be seen in Theorem 4.1.1, one of the crucial conditions for functions f_1 and f_2 in (4.1.1) to satisfy is the Lipschitz condition. If it fails to hold in the domain of consideration, then one cannot guarantee the uniform convergence of the successive approximations technique we are talking about in this paper.

But we can overcome this difficulty by splitting the original interval $[a, b]$ (and thus the given problem (4.1.1), (4.1.5)) in such a way, that the Lipschitz condition holds on these subintervals and the convergence is guaranteed. On the other hand, even if the Lipschitz condition was not violated for the original FDS (4.1.1), the problem splitting is still beneficial since it improves the speed of convergence of the method.

Interval halving

Let us denote the mid-point of the interval $[a, b]$ by

$$c := \frac{b-a}{2},$$

and the solution on each half of the interval by

$$u_1(t) = \begin{cases} x_1(t), & t \in [a, c] \\ y_1(t), & t \in [c, b] \end{cases} \qquad u_2(t) = \begin{cases} x_2(t), & t \in [a, c] \\ y_2(t), & t \in [c, b]. \end{cases}$$

With this, we split the parametrized BVP (4.1.1), (4.1.5) into two 'model-type' problems, similarly to ([93–96]), which read

$$\begin{gathered} \left.\begin{aligned} {}_a^C D_t^p x_1(t) &= f_1(t, x(t)), \quad f_1 \in \mathbb{R}^n, \\ {}_a^C D_t^q x_2(t) &= f_2(t, x(t)), \quad f_2 \in \mathbb{R}^m, \end{aligned}\right. \quad t \in [a, c], \quad p, q \in [0, 1] \\ x_1(a) = z_1, \quad x_1(c) = \alpha_1, \\ x_2(a) = z_2, \quad x_2(c) = \alpha_2, \end{gathered} \tag{4.1.6}$$

$$\begin{gathered} \left.\begin{aligned} {}_c^C D_t^p y_1(t) &= g_1(t, x(t), y(t)), \quad g_1 \in \mathbb{R}^n, \\ {}_c^C D_t^q y_2(t) &= g_2(t, x(t), y(t)), \quad g_2 \in \mathbb{R}^m, \end{aligned}\right. \quad t \in [c, b], \quad p, q \in [0, 1] \\ y_1(c) = \alpha_1, \quad y_1(b) = C_1^{-1}[d(\eta, \lambda) - Az]_1, \\ y_2(c) = \alpha_2, \quad y_2(b) = C_1^{-1}[d(\eta, \lambda) - Az]_2. \end{gathered} \tag{4.1.7}$$

Here $x(\cdot) := \begin{bmatrix} x_1(\cdot) \\ x_2(\cdot) \end{bmatrix}$, $y(\cdot) := \begin{bmatrix} y_1(\cdot) \\ y_2(\cdot) \end{bmatrix}$, and

$$g_1(t, x(t), y(t)) := f_1(t, y(t)) - \frac{1}{\Gamma(1-p)} \int_a^c (t-s)^{-p} x'(s) ds, \tag{4.1.8}$$

$$g_2(t, x(t), y(t)) := f_2(t, y(t)) - \frac{1}{\Gamma(1-q)} \int_a^c (t-s)^{-q} x'(s) ds. \tag{4.1.9}$$

Functions $x_1(t) : [a, c] \to D_1^x \subset \mathbb{R}^n$, $x_2(t) : [a, c] \to D_2^x \subset \mathbb{R}^m$, $y_1(t) : [c, b] \to D_1^y \subset \mathbb{R}^n$, $y_2(t) : [c, b] \to D_2^y \subset \mathbb{R}^m$ are continuous on their respective domains. Moreover, the domains D_i^x, D_i^y are such that $D_i^x \cup D_i^y = D_i$, $D_i^x \cap D_i^y = \emptyset$ $(i \in \{1, 2\})$.

The parameter λ in the boundary conditions of (4.1.7) is written in terms of new functions as

$$\lambda = \int_a^c P(s)x(s)ds + \int_c^b P(s)y(s)ds.$$

Remark 4.1.2. *Note, that in the original system (4.1.1) we considered the Caputo derivatives ${}_a^C D_t^p(\cdot)$ with the lower limit at a, thus at the left end of the interval $[a,b]$, where the independent variable t was defined. After the interval splitting the Caputo derivatives in (4.1.7) are already taken with the lower limit at the middle point c. Due to the non-local nature of the Caputo fractional derivative, the right-hand side functions in the system (4.1.7), defined on the second half of the interval, need to be appropriately adjusted using the definition of the Caputo derivative, Def. 2.1.7, as it was done in (4.1.8), (4.1.9).*

Remark 4.1.3. *Another important remark is that in the boundary conditions of (4.1.6), (4.1.7) we have introduced an additional parameter $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, which denotes the solution value at the mid-point of the interval $[a,b]$. In order for the solution to be continuous on the entire interval $[a,b]$ we require that*

$$x(c) = y(c) = \alpha.$$

It is also worth noting that in principle, the domain of definition can be split into any number of intervals, and they need not be of equal length.

In the next sub-section we present the constructed sequences of approximations to the solutions of FBVPs (4.1.6) and (4.1.7). We state two theorems on the uniform convergence of the sequences to the exact solutions of the corresponding Cauchy problems. One of the proofs is shown, and the other one is omitted, as it follows the same lines.

4.1.2. SUCCESSIVE APPROXIMATIONS

Let us consider each of the FBVPs (4.1.6) and (4.1.7) separately.

Assume that the BVP (4.1.6) satisfies the following conditions:

1a) Functions f_1 and f_2 are bounded, i.e. they satisfy the inequalities:

$$|f_1(t,x(t))| \le M_1^x, \qquad |f_2(t,x(t))| \le M_2^x, \tag{4.1.10}$$

for all $t \in [a,c]$, $x_i \in D_i^x$ $(i \in \{1,2\})$ and some non-negative constant vectors $M_1^x \in \mathbb{R}^n$, $M_2^x \in \mathbb{R}^m$.

2a) Functions f_1 and f_2 satisfy the Lipschitz conditions

$$\begin{aligned} |f_1(t,x_1^1,x_2^1) - f_1(t,x_1^2,x_2^2)| &\le K_{11}|x_1^1 - x_1^2| + K_{12}|x_2^1 - x_2^2|, \\ |f_2(t,x_1^1,x_2^1) - f_2(t,x_1^2,x_2^2)| &\le K_{21}|x_1^1 - x_1^2| + K_{22}|x_2^1 - x_2^2|, \end{aligned} \tag{4.1.11}$$

for all $t \in [a,c]$, $x_i^1, x_i^2 \in D_i^x$ $(i \in \{1,2\})$ and some non-negative constant matrices K_{lj}, $l,j \in \{1,2\}$.

3a) The sets

$$D_{\beta_1^x} := \{z_1 \in D_1^x : B(z_1 + 2^p(t-a)^p(c-a)^{-p}(\alpha_1 - z_1), \beta_1^x) \subset D_1^x \ \ \forall (t,\alpha_1) \in \Omega_1^x\}$$
$$D_{\beta_2^x} := \{z_2 \in D_2^x : B(z_2 + 2^q(t-a)^q(c-a)^{-q}(\alpha_2 - z_2), \beta_2^x) \subset D_2^x \ \ \forall (t,\alpha_2) \in \Omega_2^x\} \tag{4.1.12}$$

are non-empty, where

$$\beta_1^x = \frac{(c-a)^p M_1^x}{2^{2p-1}\Gamma(p+1)}, \quad \beta_2^x = \frac{(c-a)^q M_2^x}{2^{2q-1}\Gamma(q+1)}, \tag{4.1.13}$$

$$\Omega_1^x := [a,c] \times D_{\beta_1^y}, \quad \Omega_2^x := [a,c] \times D_{\beta_2^y}, \tag{4.1.14}$$

and the sets $D_{\beta_1^y}$ and $D_{\beta_2^y}$ are defined in (4.1.21). This means that there exist non-empty sets of initial conditions, for which the solutions remain within their corresponding domains.

4a) The spectral radius of the matrix

$${}^xQ := K\ {}^x\Gamma_{pq} \tag{4.1.15}$$

satisfies the inequality

$$r({}^xQ) < 1, \tag{4.1.16}$$

where

$${}^x\Gamma_{pq} := \max\left\{\frac{(c-a)^p}{2^{2p-1}\Gamma(p+1)}, \frac{(c-a)^q}{2^{2q-1}\Gamma(q+1)}\right\} \tag{4.1.17}$$

and

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}. \tag{4.1.18}$$

Similar conditions are assumed to hold in the case of the BVP (4.1.7):

1b) Functions g_1 and g_2 are bounded, i.e. they satisfy the inequalities:

$$|g_1(t,x(t),y(t))| \le M_1^y, \quad |g_2(t,x(t),y(t))| \le M_2^y, \tag{4.1.19}$$

for all $t \in [c,b]$, $y_i \in D_i^y$ $(i \in \{1,2\})$ and some non-negative constant vectors $M_1^y \in \mathbb{R}^n$, $M_2^y \in \mathbb{R}^m$.

2b) Functions g_1 and g_2 satisfy the Lipschitz conditions

$$|g_1(t,y_1^1,y_2^1) - g_1(t,y_1^2,y_2^2)| \le J_{11}|y_1^1 - y_1^2| + J_{12}|y_2^1 - y_2^2|,$$
$$|g_2(t,y_1^1,y_2^1) - g_2(t,y_1^2,y_2^2)| \le J_{21}|y_1^1 - y_1^2| + J_{22}|y_2^1 - y_2^2|, \tag{4.1.20}$$

for all $t \in [c,b]$, $y_i^1, y_i^2 \in D_i^y$ $(i \in \{1,2\})$, and some non-negative constant matrices J_{lj}, $l,j \in \{1,2\}$.

3b) The sets

$$\begin{aligned} D_{\beta_1^y} :=& \Big\{ \alpha_1 \in D_1^y : B\Big(\alpha_1 + \Big(\frac{t-c}{b-c}\Big)^p \{[C_1^{-1}(d(\lambda,\eta) - Az)]_1 - \alpha_1\}, \beta_1^y\Big) \subset D_1^y \\ & \forall (t, z_1, \lambda, \eta) \in \Omega_1^y \Big\}, \\ D_{\beta_2^y} :=& \Big\{ \alpha_2 \in D_2^y : B\Big(\alpha_2 + \Big(\frac{t-c}{b-c}\Big)^q \{[C_1^{-1}(d(\lambda,\eta) - Az)]_2 - \alpha_2\}, \beta_2^y\Big) \subset D_2^y \\ & \forall (t, z_2, \lambda, \eta) \in \Omega_2^y \Big\} \end{aligned} \tag{4.1.21}$$

are non-empty, where

$$\beta_1^y = \frac{(b-c)^p M_1^y}{2^{2p-1}\Gamma(p+1)}, \quad \beta_2^y = \frac{(b-c)^q M_2^y}{2^{2q-1}\Gamma(q+1)}, \tag{4.1.22}$$

$$\Omega_1^y := [c,b] \times D_{\beta_1^x} \times \mathcal{P} \times D_1 \times D_2, \quad \Omega_2^y := [c,b] \times D_{\beta_2^x} \times \mathcal{P} \times D_1 \times D_2, \tag{4.1.23}$$

$$\mathcal{P} := \left\{ \int_a^c P(s)x(s)ds + \int_c^b P(s)y(s)ds, \ x \in C([a,b], D^x), y \in C([a,b], D^y) \right\}, \tag{4.1.24}$$

with $D^x := D_1^x \times D_2^x$, $D^y := D_1^y \times D_2^y$, and the sets $D_{\beta_1^x}$ and $D_{\beta_2^x}$ being defined in (4.1.12).

4b) The spectral radius of the matrix

$$^y Q := J \, ^y\Gamma_{pq} \tag{4.1.25}$$

satisfies the inequality

$$r(^y Q) < 1, \tag{4.1.26}$$

where

$$^y\Gamma_{pq} := \max \left\{ \frac{(b-c)^p}{2^{2p-1}\Gamma(p+1)}, \frac{(b-c)^q}{2^{2q-1}\Gamma(q+1)} \right\}, \tag{4.1.27}$$

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}. \tag{4.1.28}$$

Let us now connect with the FBVPs (4.1.6), (4.1.7) sequences of functions $\{x_m\}$, $\{y_m\}$, given by the iterative formulas:

$$\begin{aligned} x_{1,0}(t;z,\alpha) =& z_1 + \Big(\frac{t-a}{c-a}\Big)^p (\alpha_1 - z_1), \\ x_{1,m}(t;z,\alpha) =& x_{1,0}(t;z,\alpha) + \frac{1}{\Gamma(p)} \Big[\int_a^t (t-s)^{p-1} f_1(s, x_{m-1}(s;z,\alpha))ds \\ & - \Big(\frac{t-a}{c-a}\Big)^p \int_a^c (c-s)^{p-1} f_1(s, x_{m-1}(s;z,\alpha))ds \Big], \end{aligned} \tag{4.1.29}$$

$$
\begin{aligned}
x_{2,0}(t;z,\alpha) =& z_2 + \left(\frac{t-a}{c-a}\right)^q (\alpha_2 - z_2),\\
x_{2,m}(t;z,\alpha) =& x_{2,0}(t;z,\alpha) + \frac{1}{\Gamma(q)}\Big[\int_a^t (t-s)^{q-1} f_2(s, x_{m-1}(s;z,\alpha))ds \\
&- \left(\frac{t-a}{c-a}\right)^q \int_a^c (c-s)^{q-1} f_2(s, x_{m-1}(s;z,\alpha))ds\Big],
\end{aligned}
\tag{4.1.30}
$$

for $m \in \mathbb{Z}^+$ and $t \in [a,c]$, and

$$
\begin{aligned}
y_{1,0}(t;z,\alpha,\lambda,\eta) &= \alpha_1 + \left(\frac{t-c}{b-c}\right)^p \{[C_1^{-1}(d(\lambda,\eta) - Az)]_1 - \alpha_1\},\\
y_{1,m}(t;z,\alpha,\lambda,\eta) &= y_{1,0}(t;z,\alpha,\lambda,\eta)\\
&+ \frac{1}{\Gamma(p)}\Big[\int_c^t (t-s)^{p-1} g_1(s, x_{m-1}(s;z,\alpha,\lambda,\eta), y_{m-1}(s;z,\alpha,\lambda,\eta))ds\\
&- \left(\frac{t-c}{b-c}\right)^p \int_c^b (b-s)^{p-1} g_1(s, x_{m-1}(s;z,\alpha,\lambda,\eta), y_{m-1}(s;z,\alpha,\lambda,\eta))ds\Big],
\end{aligned}
\tag{4.1.31}
$$

$$
\begin{aligned}
y_{2,0}(t;z,\alpha,\lambda,\eta) &= \alpha_2 + \left(\frac{t-c}{b-c}\right)^q \{[C_1^{-1}(d(\lambda,\eta) - Az)]_2 - \alpha_2\},\\
y_{2,m}(t;z,\alpha,\lambda,\eta) &= y_{2,0}(t;z,\alpha,\lambda,\eta)\\
&+ \frac{1}{\Gamma(q)}\Big[\int_c^t (t-s)^{q-1} g_2(s, x_{m-1}(s;z,\alpha,\lambda,\eta), y_{m-1}(s;z,\alpha,\lambda,\eta))ds\\
&- \left(\frac{t-c}{b-c}\right)^q \int_c^b (b-s)^{q-1} g_2(s, x_{m-1}(s;z,\alpha,\lambda,\eta), y_{m-1}(s;z,\alpha,\lambda,\eta))ds\Big]
\end{aligned}
\tag{4.1.32}
$$

for $m \in \mathbb{Z}^+$ and $t \in [c,b]$.

Note, that every function in the sequences (4.1.29)-(4.1.30) and (4.1.31)-(4.1.32) is constructed to satisfy the parametrized boundary conditions of the corresponding problems (4.1.6), (4.1.7).

Remark 4.1.4. *It follows from the definitions (4.1.12) of $D_{\beta_1^x}$ and $D_{\beta_2^x}$ that the values of $x_{1,0}(t;z,\alpha)$, $x_{2,0}(t;z,\alpha)$ in (4.1.29), (4.1.30) do not escape D_1^x and D_2^x respectively, for any $z_1 \in D_{\beta_1^x}$, $z_2 \in D_{\beta_2^x}$, $\alpha_1 \in D_{\beta_1^y}$, $\alpha_2 \in D_{\beta_2^y}$.*

Similar conclusion holds for the values of $y_{1,0}(t;z,\alpha,\lambda,\eta)$, $y_{2,0}(t;z,\alpha,\lambda,\eta)$, defined in (4.1.31), (4.1.32), with respect to the sets $D_{\beta_1^y}$ and $D_{\beta_2^y}$ of the form (4.1.21), for any $\alpha_1 \in D_{\beta_1^y}$, $\alpha_2 \in D_{\beta_2^y}$, $z_1 \in D_{\beta_1^x}$, $z_2 \in D_{\beta_2^x}$, $\lambda \in \mathcal{P}$, $\eta \in D_1 \times D_2$.

Next we will prove, that under conditions **1a)-4a)** and **1b)-4b)** the sequences of functions (4.1.29)-(4.1.30) and (4.1.31)-(4.1.32) converge uniformly to the corresponding limit functions.

Let us consider the FBVP (4.1.6).

Theorem 4.1.1. *Assume that the BVP (4.1.6) satisfies conditions 1a)-4a). Then for all fixed $z_1 \in D_{\beta_1^x}$, $z_2 \in D_{\beta_2^x}$, $\alpha_1 \in D_{\beta_1^y}$, $\alpha_2 \in D_{\beta_2^y}$ it holds:*

1. Functions of the sequences (4.1.29), (4.1.30) are continuous and satisfy the parametrized boundary conditions

$$x_{1,m}(a;z,\alpha) = z_1, \quad x_{1,m}(c;z,\alpha) = \alpha_1, \tag{4.1.33}$$

$$x_{2,m}(a;z,\alpha) = z_2, \quad x_{2,m}(c;z,\alpha) = \alpha_2. \tag{4.1.34}$$

2. The sequences of functions (4.1.29), (4.1.30) for $t \in [a,c]$ converge uniformly as $m \to \infty$ to the limit functions

$$x_{1,\infty}(t;z,\alpha) = \lim_{m\to\infty} x_{1,m}(t;z,\alpha), \tag{4.1.35}$$

$$x_{2,\infty}(t;z,\alpha) = \lim_{m\to\infty} x_{2,m}(t;z,\alpha). \tag{4.1.36}$$

3. The limit functions (4.1.35), (4.1.36) satisfy the parametrized boundary conditions

$$x_{1,\infty}(a;z,\alpha) = z_1, \quad x_{1,\infty}(c;z,\alpha) = \alpha_1, \tag{4.1.37}$$

$$x_{2,\infty}(a;z,\alpha) = z_2, \quad x_{2,\infty}(c;z,\alpha) = \alpha_2. \tag{4.1.38}$$

4. The limit functions (4.1.35), (4.1.36) are the unique continuous solutions to the integral equations

$$\begin{aligned} x_1(t) = z_1 + \Big(\frac{t-a}{c-a}\Big)^p(\alpha_1 - z_1) + \frac{1}{\Gamma(p)}\Big[\int_a^t (t-s)^{p-1} f_1(s,x(s))ds \\ -\Big(\frac{t-a}{c-a}\Big)^p \int_a^c (c-s)^{p-1} f_1(s,x(s))ds\Big], \end{aligned} \tag{4.1.39}$$

$$\begin{aligned} x_2(t) = z_2 + \Big(\frac{t-a}{c-a}\Big)^q(\alpha_2 - z_2) + \frac{1}{\Gamma(q)}\Big[\int_a^t (t-s)^{q-1} f_2(s,x(s))ds \\ -\Big(\frac{t-a}{c-a}\Big)^q \int_a^c (c-s)^{q-1} f_2(s,x(s))ds\Big], \end{aligned} \tag{4.1.40}$$

or equivalently, they are the unique continuous solutions to the Cauchy problems

$${}_a^C D_t^p x_1(t) = f_1(t,x(t)) + \Delta^{p_x}(z,\alpha), \;\; x_1(a) = z_1, \tag{4.1.41}$$

$${}_a^C D_t^q x_2(t) = f_2(t,x(t)) + \Delta^{q_x}(z,\alpha), \;\; x_2(a) = z_2, \tag{4.1.42}$$

where

$$\Delta^{p_x}(z,\alpha) = \frac{\Gamma(p+1)}{(c-a)^p}(\alpha_1 - z_1) - \frac{p}{(c-a)^p}\int_a^c (c-s)^{p-1} f_1(s,x(s))ds, \tag{4.1.43}$$

$$\Delta^{q_x}(z,\alpha) = \frac{\Gamma(q+1)}{(c-a)^q}(\alpha_2 - z_2) - \frac{q}{(c-a)^q}\int_a^c (c-s)^{q-1} f_2(s,x(s))ds. \tag{4.1.44}$$

5. The following error estimate holds

$$\begin{pmatrix} |x_{1,\infty}(t;z,\alpha) - x_{1,m}(t;z,\alpha)| \\ |x_{2,\infty}(t;z,\alpha) - x_{2,m}(t;z,\alpha)| \end{pmatrix} \leq {}^x\Gamma_{pq}{}^xQ^m(I - {}^xQ)^{-1}\begin{pmatrix} M_1^x \\ M_2^x \end{pmatrix}, \tag{4.1.45}$$

where $t \in [a,c]$, ${}^xQ$ is defined by (4.1.15), and I is a unit N-dimensional matrix.

Proof. The first statement follows directly from computations, since the sequences of functions (4.1.29), (4.1.30) are constructed in such a way that they satisfy the parametrized boundary conditions (4.1.33), (4.1.34).

Next, we show that $x_{1,m}(t;z,\alpha) \in D_1^x$, $x_{2,m}(t;z,\alpha) \in D_2^x$ for arbitrary $(t, z_1, \alpha_1) \in [a,c] \times D_{\beta_1^x} \times D_{\beta_1^y}$, $(t, z_2, \alpha_2) \in [a,c] \times D_{\beta_2^x} \times D_{\beta_2^y}$. By applying Lemma 2.2.1 to

$$\begin{aligned}
|x_{1,m}(t;z,\alpha) - & x_{1,0}(t;z,\alpha)| \\
&= \frac{1}{\Gamma(p)} \Bigg| \int_a^t \left[(t-s)^{p-1} - \left(\frac{t-a}{c-a}\right)^p (c-s)^{p-1}\right] f_1(s, x_{m-1}(s;z,\alpha))ds \\
&\quad - \left(\frac{t-a}{c-a}\right)^p \int_t^c (c-s)^{p-1} f_1(s, x_{m-1}(s;z,\alpha))ds \Bigg|
\end{aligned}$$

and

$$\begin{aligned}
|x_{2,m}(t;z,\alpha) - & x_{2,0}(t;z,\alpha)| \\
&= \frac{1}{\Gamma(q)} \Bigg| \int_a^t \left[(t-s)^{q-1} - \left(\frac{t-a}{c-a}\right)^q (c-s)^{q-1}\right] f_2(s, x_{m-1}(s;z,\alpha))ds \\
&\quad - \left(\frac{t-a}{c-a}\right)^q \int_t^c (c-s)^{q-1} f_2(s, x_{m-1}(s;z,\alpha))ds \Bigg|,
\end{aligned}$$

respectively, it follows that

$$\begin{aligned}
|x_{1,m}(t;z,\alpha) - x_{1,0}(t;z,\alpha)| &\leq {}^x\alpha_1^p(t) \max_{a \leq t \leq c} |f_1(t, x_{m-1}(t;z,\alpha))| \\
&= {}^x\alpha_1^p(t) M_1^x
\end{aligned}$$

$$\begin{aligned}
|x_{2,m}(t;z,\alpha) - x_{2,0}(t;z,\alpha)| &\leq {}^x\alpha_1^q(t) \max_{a \leq t \leq c} |f_2(t, x_{m-1}(t;z,\alpha))| \\
&= {}^x\alpha_1^q(t) M_2^x.
\end{aligned}$$

Applying Lemma 2.2.2 with $m = 1$ to the last two inequalities and using the definitions of β_1^x and β_2^x in (4.1.13) yields

$$\begin{aligned}
|x_{1,m}(t;z,\alpha) - x_{1,0}(t;z,\alpha)| &\leq \beta_1^x \\
|x_{2,m}(t;z,\alpha) - x_{2,0}(t;z,\alpha)| &\leq \beta_2^x.
\end{aligned}$$

Thus, we have shown that $x_{1,m}(t;z,\alpha) \in D_1^x$, $x_{2,m}(t,z,\alpha) \in D_2^x$ for arbitrary $(t, z_1, \alpha_1) \in [a,c] \times D_{\beta_1^x} \times D_{\beta_1^y}$, $(t, z_2, \alpha_2) \in [a,c] \times D_{\beta_2^x} \times D_{\beta_2^y}$.

Next, we set

$$\mu_1^x(t) := \alpha_1^p(t), \;\; \nu_1^x(t) := \alpha_1^q(t) \tag{4.1.46}$$

$$
\begin{aligned}
\mu_m^x(t) := \frac{1}{\Gamma(p)} \max \Bigg\{ & \int_a^t \left[(t-s)^{p-1} - \left(\frac{t-a}{c-a}\right)^p (c-s)^{p-1} \right] \mu_{m-1}^x(s) ds \\
& + \left(\frac{t-a}{c-a}\right)^p \int_t^c (c-s)^{p-1} \mu_{m-1}^x(s) ds, \\
& \int_a^t \left[(t-s)^{p-1} - \left(\frac{t-a}{c-a}\right)^p (c-s)^{p-1} \right] \nu_{m-1}^x(s) ds \\
& + \left(\frac{t-a}{c-a}\right)^p \int_t^c (c-s)^{p-1} \nu_{m-1}^x(s) ds \Bigg\},
\end{aligned} \tag{4.1.47}
$$

$$
\begin{aligned}
\nu_m^x(t) := \frac{1}{\Gamma(q)} \max \Bigg\{ & \int_a^t \left[(t-s)^{q-1} - \left(\frac{t-a}{c-a}\right)^q (c-s)^{q-1} \right] \nu_{m-1}^x(s) ds \\
& + \left(\frac{t-a}{c-a}\right)^q \int_t^c (c-s)^{q-1} \nu_{m-1}^x(s) ds, \\
& \int_a^t \left[(t-s)^{q-1} - \left(\frac{t-a}{c-a}\right)^q (c-s)^{q-1} \right] \mu_{m-1}^x(s) ds \\
& + \left(\frac{t-a}{c-a}\right)^q \int_t^c (c-s)^{q-1} \mu_{m-1}^x(s) ds \Bigg\},
\end{aligned} \tag{4.1.48}
$$

$\forall m \in \mathbb{Z}^+$, and use induction to show that

$$
\begin{aligned}
|x_{1,m}(t;z,\alpha) - x_{1,m-1}(t;z,\alpha)| &\leq (M_1^x)^m \mu_m^x(t), \\
|x_{2,m}(t;z,\alpha) - x_{2,m-1}(t;z,\alpha)| &\leq (M_2^x)^m \nu_m^x(t).
\end{aligned} \tag{4.1.49}
$$

When $m = 1$ it is clear from the previous calculations that (4.1.49) holds. Now assume (4.1.49) holds for some arbitrary $m > 1$ and consider

$$
\begin{aligned}
& |x_{1,m+1}(t;z,\alpha) - x_{1,m}(t;z,\alpha)| \\
\leq & \frac{1}{\Gamma(p)} \int_a^t \left[(t-s)^{p-1} - \left(\frac{t-a}{c-a}\right)^p (c-s)^{p-1} \right] |f_1(s; x_m(s,z,\alpha)) - f_1(s, x_{m-1}(s;z,\alpha))| ds \\
& - \left(\frac{t-a}{c-a}\right)^p \int_t^c (c-s)^{p-1} |f_1(s, x_m(s;z,\alpha)) - f_1(s, x_{m-1}(s;z,\alpha))| ds \\
\leq & \frac{K_{11}(M_1^x)^m}{\Gamma(p)} \Bigg\{ \int_a^t \left[(t-s)^{p-1} - \left(\frac{t-a}{c-a}\right)^p (c-s)^{p-1} \right] \mu_m^x(s) ds \\
& + \left(\frac{t-a}{c-a}\right)^p \int_t^c (c-s)^{p-1} \mu_m^x(s) ds \Bigg\} \\
& + \frac{K_{12}(M_2^x)^m}{\Gamma(p)} \Bigg\{ \int_a^t \left[(t-s)^{p-1} - \left(\frac{t-a}{c-a}\right)^p (c-s)^{p-1} \right] \nu_m^x(s) ds \\
& + \left(\frac{t-a}{c-a}\right)^p \int_t^c (c-s)^{p-1} \nu_m^x(s) ds \Bigg\},
\end{aligned}
$$

where we used the first Lipschitz condition in (4.1.11) and the induction hypothesis. Now applying definition (4.1.47) of $\mu_m^x(t)$ yields

$$\begin{aligned}|x_{1,m+1}(t;z,\alpha)-x_{1,m}(t;z,\alpha)| \leq &[K_{11}(M_1^x)^m+K_{12}(M_2^x)^m]\mu_{m+1}^x(t)\\ =&(M^x)^{m+1}\mu_{m+1}^x(t).\end{aligned}$$

Using the same reasoning to

$$|x_{2,m+1}(t;z,\alpha)-x_{2,m}(t;z,\alpha)|$$

yields the estimate in (4.1.49). Applying Lemma 2.2.2 to (4.1.49) and using definitions (4.1.15) and (4.1.17) gives

$$\begin{aligned}\begin{pmatrix}|x_{1,m+1}(t;z,\alpha)-x_{1,m}(t;z,\alpha)|\\|x_{2,m+1}(t;z,\alpha)-x_{2,m}(t;z,\alpha)|\end{pmatrix} &\leq {}^x\Gamma_{pq}^{m+1}\begin{pmatrix}(M_1^x)^{m+1}\\(M_2^x)^{m+1}\end{pmatrix}\\ &= {}^x\Gamma_{pq}^{m+1}K^m\begin{pmatrix}M_1^x\\M_2^x\end{pmatrix} = {}^x\Gamma_{pq}{}^xQ^m\begin{pmatrix}M_1^x\\M_2^x\end{pmatrix},\end{aligned}$$

which implies that

$$\begin{aligned}\begin{pmatrix}|x_{1,m+j}(t;z,\alpha)-x_{1,m}(t;z,\alpha)|\\|x_{2,m+j}(t;z,\alpha)-x_{2,m}(t;z,\alpha)|\end{pmatrix} &= \begin{pmatrix}\sum_{i=1}^{j}|x_{1,m+i}(t;z,\alpha)-x_{1,m+i-1}(t;z,\alpha)|\\\sum_{i=1}^{j}|x_{2,m+i}(t;z,\alpha)-x_{2,m+i-1}(t;z,\alpha)|\end{pmatrix}\\ &\leq {}^x\Gamma_{pq}{}^xQ^m\sum_{i=0}^{j-1}{}^xQ^i\begin{pmatrix}M_1^x\\M_2^x\end{pmatrix}.\end{aligned} \tag{4.1.50}$$

From (4.1.16) it follows that

$$\sum_{i=0}^{j-1}{}^xQ^i \leq (I-{}^xQ)^{-1}, \quad \lim_{m\to\infty}{}^xQ^m = O,$$

where O denotes the matrix of zeros. Passing in (4.1.50) to the limit as $j\to\infty$ we obtain the error estimate (4.1.45). Thus, the sequences of functions in (4.1.29), (4.1.30) converge uniformly to the limit functions (4.1.35), (4.1.36) in their domains $[a,c]\times D_{\beta_1^x}$ and $[a,c]\times D_{\beta_2^x}$.

The functions $x_{1,\infty}(t;z,\alpha)$ and $x_{2,\infty}(t;z,\alpha)$ are the limits to sequences of functions, all of which satisfy the boundary conditions (4.1.33), (4.1.34), therefore, the limit functions also satisfy the same boundary conditions.

To prove (4) we suppose $(x_1^1(t),x_1^2(t))$ and $(x_2^1(t),x_2^2(t))$ are two pairs of functions, both of which are solutions to the integral equations (4.1.39) and (4.1.40). Let

$$m_1 := \max_{a\leq t\leq c}|x_1^1(t)-x_1^2(t)|, \quad m_2 := \max_{a\leq t\leq c}|x_2^1(t)-x_2^2(t)|,$$

and consider

$$|x_1^1(t)-x_1^2(t)|$$

$$\leq \frac{K_{11}}{\Gamma(p)} \left\{ \int_a^t \left[(t-s)^{p-1} - \left(\frac{t-a}{c-a}\right)^p (c-s)^{p-1} \right] ds + \left(\frac{t-a}{c-a}\right)^p \int_t^c (c-s)^{p-1} ds \right\} m_1$$
$$+ \frac{K_{12}}{\Gamma(p)} \left\{ \int_a^t \left[(t-s)^{p-1} - \left(\frac{t-a}{c-a}\right)^p (c-s)^{p-1} \right] ds + \left(\frac{t-a}{c-a}\right)^p \int_t^c (c-s)^{p-1} ds \right\} m_2$$
$$\leq \frac{K_{11}}{\Gamma(p)} {}^x\Gamma_{pq} m_1 + \frac{K_{12}}{\Gamma(p)} {}^x\Gamma_{pq} m_2,$$

$$|x_2^1(t) - x_2^2(t)|$$
$$\leq \frac{K_{21}}{\Gamma(q)} \left\{ \int_a^t \left[(t-s)^{q-1} - \left(\frac{t-a}{c-a}\right)^q (c-s)^{q-1} \right] ds + \left(\frac{t-a}{c-a}\right)^q \int_t^c (c-s)^{q-1} ds \right\} m_1$$
$$+ \frac{K_{22}}{\Gamma(q)} \left\{ \int_a^t \left[(t-s)^{q-1} - \left(\frac{t-a}{c-a}\right)^q (c-s)^{q-1} \right] ds + \left(\frac{t-a}{c-a}\right)^q \int_t^c (c-s)^{q-1} ds \right\} m_2$$
$$\leq \frac{K_{21}}{\Gamma(q)} {}^x\Gamma_{pq} m_1 + \frac{K_{22}}{\Gamma(q)} {}^x\Gamma_{pq} m_2.$$

Thus,

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \leq {}^xQ \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

holds for all $t \in [a, c]$, and since $r({}^xQ) < 1$, $m_1 = m_2 = 0$, which implies that $x_1^1(t) = x_1^2(t)$ and $x_2^1(t) = x_2^2(t)$. Hence, $x_{1,\infty}(t; z, \alpha, \lambda)$ and $x_{2,\infty}(t; z, \alpha, \lambda)$ are the unique solutions to integral equations (4.1.39) and (4.1.40). Moreover, the Cauchy problems (4.1.41), (4.1.42) are equivalent to the integral equations

$$\begin{aligned}
x_1(t) =& z_1 + \frac{1}{\Gamma(p)} \int_a^t (t-s)^{p-1} [f_1(s, x(s)) + \Delta^{p_x}] ds \\
=& z_1 + \left(\frac{t-a}{c-a}\right)^p (\alpha_1 - z_1) + \frac{1}{\Gamma(p)} \Big[\int_a^t (t-s)^{p-1} f_1(s, x(s)) ds \\
& - \left(\frac{t-a}{c-a}\right)^p \int_a^c (c-s)^{p-1} f_1(s, x(s)) ds \Big] \\
x_2(t) =& z_2 + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} [f_2(s, x(s)) ds + \Delta^{q_x}] \\
=& z_2 + \left(\frac{t-a}{c-a}\right)^q (\alpha_2 - z_2) + \frac{1}{\Gamma(q)} \Big[\int_a^t (t-s)^{q-1} f_2(s, x(s)) ds \\
& - \left(\frac{t-a}{c-a}\right)^q \int_a^c (c-s)^{q-1} f_2(s, x(s)) ds \Big],
\end{aligned} \tag{4.1.51}$$

where Δ^{p_x} and Δ^{q_x} are given in (4.1.43) and (4.1.44). From comparing the integral equations in (4.1.51) to (4.1.39) and (4.1.40) and knowing that $x_{1,\infty}(t; z, \alpha)$ and $x_{2,\infty}(t; z, \alpha)$ are the unique continuous solutions to (4.1.39) and (4.1.40), it follows that they are also the unique continuous solutions to the Cauchy problems (4.1.41) and (4.1.42). This completes the proof.

□

A similar result holds for the second FBVP (4.1.7).

Theorem 4.1.2. *Assume that conditions 1b)-4b) for BVP (4.1.7) are true. Then for all fixed* $z_1 \in D_{\beta_1^x}$, $z_2 \in D_{\beta_2^x}$, $\alpha_1 \in D_{\beta_1^y}$, $\alpha_2 \in D_{\beta_2^y}$, $\lambda \in \mathcal{P}$, $\eta \in D_1 \times D_2$ *it holds:*

1. Functions of the sequences (4.1.31), (4.1.32) are continuous and satisfy the parametrized boundary conditions

$$y_{1,m}(c; z, \alpha, \lambda, \eta) = \alpha_1, \quad y_{1,m}(b; z, \alpha, \lambda, \eta) = [C_1^{-1}(d(\lambda, \eta) - Az)]_1$$

$$y_{2,m}(c; z, \alpha, \lambda, \eta) = \alpha_2, \quad y_{2,m}(b; z, \alpha, \lambda, \eta) = [C_1^{-1}(d(\lambda, \eta) - Az)]_2$$

2. The sequences of functions (4.1.31), (4.1.32) for $t \in [c, b]$ *converge uniformly as* $m \to \infty$ *to the limit functions*

$$y_{1,\infty}(t; z, \alpha, \lambda, \eta) = \lim_{m\to\infty} y_{1,m}(t; z, \alpha, \lambda, \eta) \tag{4.1.52}$$
$$y_{2,\infty}(t; z, \alpha, \lambda, \eta) = \lim_{m\to\infty} y_{2,m}(t; z, \alpha, \lambda, \eta). \tag{4.1.53}$$

3. The limit functions (4.1.52), (4.1.53) *satisfy the parametrized boundary conditions*

$$y_{1,\infty}(c; z, \alpha, \lambda, \eta) = \alpha_1, \quad y_{1,\infty}(b; z, \alpha, \lambda, \eta) = [C_1^{-1}(d(\lambda, \eta) - Az)]_1$$

$$y_{2,\infty}(c; z, \alpha, \lambda, \eta) = \alpha_2, \quad y_{2,\infty}(b; z, \alpha, \lambda, \eta) = [C_1^{-1}(d(\lambda, \eta) - Az)]_2.$$

4. The limit functions $y_{1,\infty}(t; z, \alpha, \lambda, \eta)$, $y_{2,\infty}(t; z, \alpha, \lambda, \eta)$ *are the unique continuous solutions to the integral equations*

$$\begin{aligned} y_1(t) =& \alpha_1 + \left(\frac{t-c}{b-c}\right)^p \{[C_1^{-1}(d(\lambda, \eta) - Az)]_1 - \alpha_1\} \\ &+ \frac{1}{\Gamma(p)} \Big[\int_c^t (t-s)^{p-1} g_1(s, x(s), y(s)) ds \\ &- \left(\frac{t-c}{b-c}\right)^p \int_c^b (b-s)^{p-1} g_1(s, x(s), y(s)) ds \Big] \end{aligned} \tag{4.1.54}$$

$$\begin{aligned} y_2(t) =& \alpha_2 + \left(\frac{t-c}{b-c}\right)^q \{[C_1^{-1}(d(\lambda, \eta) - Az)]_2 - \alpha_2\} \\ &+ \frac{1}{\Gamma(q)} \Big[\int_c^t (t-s)^{q-1} g_2(s, x(s), y(s)) ds \\ &- \left(\frac{t-c}{b-c}\right)^q \int_c^b (b-s)^{q-1} g_2(s, x(s), y(s)) ds \Big], \end{aligned} \tag{4.1.55}$$

or equivalently, they are the unique continuous solutions to the Cauchy problems

$${}_a^C D_t^p y_1(t) = g_1(t, x(t), y(t)) + \Delta^{p_y}(z, \alpha, \lambda, \eta), \;\; y_1(c) = \alpha_1 \tag{4.1.56}$$

$${}_a^C D_t^q y_2(t) = g_2(t, x(t), y(t)) + \Delta^{q_y}(z, \alpha, \lambda, \eta), \;\; y_2(c) = \alpha_2, \tag{4.1.57}$$

where

$$\Delta^{p_y}(z,\alpha,\lambda,\eta) = \frac{\Gamma(p+1)}{(b-c)^p}\{[C_1^{-1}(d(\lambda,\eta)-Az)]_1 - \alpha_1\} - \frac{p}{(b-c)^p}\int_c^b (b-s)^{p-1} g_1(s,x(s),y(s))ds \tag{4.1.58}$$

and

$$\Delta^{q_y}(z,\alpha,\lambda,\eta) = \frac{\Gamma(q+1)}{(b-c)^q}\{[C_1^{-1}(d(\lambda,\eta)-Az)]_2 - \alpha_2\} - \frac{q}{(b-c)^q}\int_c^b (b-s)^{q-1} g_2(s,x(s),y(s))ds. \tag{4.1.59}$$

5. The following error estimate holds

$$\begin{pmatrix} |y_{1,\infty}(t;z,\alpha,\lambda,\eta) - y_{1,m}(t;z,\alpha,\lambda,\eta)| \\ |y_{2,\infty}(t;z,\alpha,\lambda,\eta) - y_{2,m}(t;z,\alpha,\lambda,\eta)| \end{pmatrix} \leq {}^y\Gamma_{pq}\,{}^yQ^m(I - {}^yQ)^{-1}\begin{pmatrix} M_1^y \\ M_2^y \end{pmatrix}. \tag{4.1.60}$$

The outline of the proof is the same as in the Theorem 4.1.1, so we will leave for the reader.

Remark 4.1.5. *Theorems 4.1.1 and 4.1.2 guarantee that under the assumed conditions 1a)-4a) and 1b)-4b) the functions*

$$\begin{aligned} x_{1,\infty}(t;z,\alpha) &: \Omega_1^x \times D_{\beta_1^x} \to D_1^x, \\ x_{2,\infty}(t;z,\alpha) &: \Omega_2^x \times D_{\beta_2^x} \to D_2^x, \\ y_{1,\infty}(t;z,\alpha,\lambda,\eta) &: \Omega_1^y \times D_{\beta_1^y} \to D_1^y, \\ y_{2,\infty}(t;z,\alpha,\lambda,\eta) &: \Omega_2^y \times D_{\beta_2^y} \to D_2^y \end{aligned}$$

are well-defined for all sets of artificially introduced parameters $(z,\alpha) \in D_{\beta^x} \times D_{\beta^y}$ *and* $(\lambda,\eta) \in \mathcal{P} \times D_1 \times D_2$. *By putting*

$$u_{1,\infty}(t;z,\alpha,\lambda,\eta) = \begin{cases} x_{1,\infty}(t;z,\alpha), & t \in [a,c] \\ y_{1,\infty}(t;z,\alpha,\lambda,\eta), & t \in [c,b] \end{cases} \tag{4.1.61}$$

and

$$u_{2,\infty}(t;z,\alpha,\lambda,\eta) = \begin{cases} x_{2,\infty}(t;z,\alpha), & t \in [a,c] \\ y_{2,\infty}(t;z,\alpha,\lambda,\eta), & t \in [c,b] \end{cases} \tag{4.1.62}$$

we obtain the well-defined continuous functions $u_{1,\infty}(t;z,\alpha,\lambda,\eta)$ *and* $u_{2,\infty}(t;z,\alpha,\lambda,\eta)$, *which at* $t = c$ *coincide:*

$$\begin{aligned} u_{1,\infty}(c;z,\alpha,\lambda,\eta) &= x_{1,\infty}(c;z,\alpha) = y_{1,\infty}(c;z,\alpha,\lambda,\eta) = \alpha_1 \\ u_{2,\infty}(c;z,\alpha,\lambda,\eta) &= x_{2,\infty}(c;z,\alpha) = y_{2,\infty}(c;z,\alpha,\lambda,\eta) = \alpha_2. \end{aligned}$$

Next, we show the connection between the solutions of the Cauchy problems (4.1.41)-(4.1.42), (4.1.56)-(4.1.57) and the solutions to the 'model-type' FBVPs (4.1.6), (4.1.7).

4.1.3. RELATION TO THE ORIGINAL FBVP

Consider the FIVP with constant perturbation terms χ^{x_p}, χ^{y_p}, χ^{x_q}, and χ^{y_q}:

$$\begin{aligned} {}_0^C D_t^p x_1(t) &= f_1(t, x(t)) + \chi^{x_p}, \qquad t \in [a, c], \\ x_1(a) &= z_1, \end{aligned} \tag{4.1.63}$$

$$\begin{aligned} {}_0^C D_t^q x_2(t) &= f_2(t, x(t)) + \chi^{x_q}, \qquad t \in [a, c], \\ x_2(a) &= z_2, \end{aligned} \tag{4.1.64}$$

and

$$\begin{aligned} {}_0^C D_t^p y_1(t) &= g_1(t, x(t), y(t)) + \chi^{y_p}, \qquad t \in [c, b], \\ y_1(c) &= \alpha_1, \end{aligned} \tag{4.1.65}$$

$$\begin{aligned} {}_0^C D_t^q y_2(t) &= g_2(t, x(t), y(t)) + \chi^{y_q}, \qquad t \in [c, b], \\ y_2(c) &= \alpha_2, \end{aligned} \tag{4.1.66}$$

where $\chi^{x_p} = (\chi_1^{x_p}, \chi_2^{x_p}, ..., \chi_n^{x_p})^T$, $\chi^{y_p} = (\chi_1^{y_p}, \chi_2^{y_p}, ..., \chi_n^{y_p})^T \in \mathbb{R}^n$ and $\chi^{x_q} = (\chi_1^{x_q}, \chi_2^{x_q}, ..., \chi_m^{x_q})^T$, $\chi^{y_q} = (\chi_1^{y_q}, \chi_2^{y_q}, ..., \chi_m^{y_q})^T \in \mathbb{R}^m$ are referred to as control parameters.

Theorem 4.1.3. *Suppose $z \in D_{\beta^x}$, $\alpha \in D_{\beta^y}$, $\lambda \in \mathcal{P}$, $\eta \in D_1 \times D_2$ and assume the conditions of Theorem 4.1.1 hold.*

Then the solutions $x_1(\cdot; z, \alpha)$ and $x_2(\cdot; z, \alpha)$ of the FIVPs (4.1.63)-(4.1.64), satisfy conditions

$$x_1(c; z, \alpha) = \alpha_1, \quad x_2(c; z, \alpha) = \alpha_2, \tag{4.1.67}$$

i.e. they are solutions to the decomposed FBVPs (4.1.6) with parametrized boundary conditions on the subinterval $[a, c]$, if and only if the control parameters χ^{x_p} and χ^{x_q} in (4.1.63), (4.1.64) are given by

$$\chi^{x_p} := \frac{\Gamma(p+1)}{(c-a)^p}(\alpha_1 - z_1) - \frac{p}{(c-a)^p} \int_a^c (c-s)^{p-1} f_1(s, x_\infty(s)) ds, \tag{4.1.68}$$

$$\chi^{x_q} := \frac{\Gamma(q+1)}{(c-a)^q}(\alpha_2 - z_2) - \frac{q}{(c-a)^q} \int_a^c (c-s)^{q-1} f_2(s, x_\infty(s)) ds, \tag{4.1.69}$$

where $x_\infty(\cdot) = \begin{bmatrix} x_{1,\infty}(\cdot; z, \alpha) \\ x_{2,\infty}(\cdot; z, \alpha) \end{bmatrix}$ are the limit functions in (4.1.35) and (4.1.36).

Proof. Sufficiency: Suppose the control parameters in (4.1.63), (4.1.64) are given by (4.1.68) and (4.1.69) respectively. Then, according to Theorem 4.1.1, the limit functions (4.1.35), (4.1.36) of the sequences in (4.1.29), (4.1.30) are the unique solutions to BVP (4.1.6). That is, they satisfy the initial conditions in (4.1.63) and (4.1.64), which means that they are solutions to the Cauchy problems (4.1.63), (4.1.64) with χ^{x_p} and χ^{x_q}, defined as in (4.1.68) and (4.1.69). Thus, $x_1(\cdot; z, \alpha) = x_{1,\infty}(\cdot; z, \alpha)$ and $x_2(\cdot; z, \alpha) = x_{2,\infty}(\cdot; z, \alpha)$.

Necessity: Suppose that there exist control parameters $\bar{\chi}^{x_p}$ and $\bar{\chi}^{x_q}$, such that the functions $\bar{x}_1(t;z,\alpha)$ and $\bar{x}_2(t;z,\alpha)$ are solutions to the FIVPs (4.1.63), (4.1.64), which also satisfy conditions (4.1.67). Then $\bar{x}_1(t;z,\alpha)$ and $\bar{x}_2(t;z,\alpha)$ are continuous solutions to the integral equations

$$\begin{aligned}\bar{x}_1(t) &= z_1 + \frac{1}{\Gamma(p)}\int_a^t (t-s)^{p-1} f_1(s,\bar{x}(s))ds + \frac{(t-a)^p}{\Gamma(p+1)}\bar{\chi}^{x_p}\\ \bar{x}_2(t) &= z_2 + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f_2(s,\bar{x}(s))ds + \frac{(t-a)^q}{\Gamma(q+1)}\bar{\chi}^{x_q}\end{aligned} \tag{4.1.70}$$

Using conditions (4.1.67) in (4.1.70) and re-arranging the terms yields

$$\begin{aligned}\bar{\chi}^{x_p} &= \frac{\Gamma(p+1)}{(c-a)^p}(\alpha_1 - z_1) - \frac{p}{(c-a)^p}\int_a^c (c-s)^{p-1} f_1(s,\bar{x}(s))ds,\\ \bar{\chi}^{x_q} &= \frac{\Gamma(q+1)}{(c-a)^q}(\alpha_2 - z_2) - \frac{q}{(c-a)^q}\int_a^c (c-s)^{q-1} f_2(s,\bar{x}(s))ds.\end{aligned}$$

This implies that

$$\begin{aligned}\bar{x}_1(t) =& z_1 + \left(\frac{t-a}{c-a}\right)^p(\alpha_1 - z_1) + \frac{1}{\Gamma(p)}\Big[\int_a^t (t-s)^{p-1} f_1(s,\bar{x}(s))ds\\ &- \left(\frac{t-a}{c-a}\right)^p \int_a^c (c-s)^{p-1} f_1(s,\bar{x}(s))ds\Big],\end{aligned} \tag{4.1.71}$$

$$\begin{aligned}\bar{x}_2(t) =& z_2 + \left(\frac{t-a}{c-a}\right)^q(\alpha_2 - z_2) + \frac{1}{\Gamma(q)}\Big[\int_a^t (t-s)^{q-1} f_2(s,\bar{x}(s))ds\\ &- \left(\frac{t-a}{c-a}\right)^q \int_a^c (c-s)^{q-1} f_2(s,\bar{x}(s))ds\Big].\end{aligned} \tag{4.1.72}$$

Since $z_1 \in D_{\beta_1^x}$, $z_2 \in D_{\beta_2^x}$, according to the integral equations above and the definitions of the sets $D_{\beta_1^x}$ and $D_{\beta_2^x}$, it can be shown that $\bar{x}_1(t;z,\alpha) \in D_1^x$ and $\bar{x}_2(t;z,\alpha) \in D_2^x$. Equations (4.1.71) and (4.1.72) are equivalent to (4.1.39) and (4.1.40) respectively, hence, by part 4 of Theorem 4.1.1 it follows that $x_1(\cdot;z,\alpha) = x_{1,\infty}(\cdot;z,\alpha)$, $x_2(\cdot;z,\alpha) = x_{2,\infty}(\cdot;z,\alpha)$ and $\bar{\chi}^{x_p} = \chi^{x_p}$, $\bar{\chi}^{x_q} = \chi^{x_q}$, where χ^{x_p} and χ^{x_q} are given by (4.1.68) and (4.1.69), respectively. This completes the proof of the theorem.

□

A similar result holds for the FIVP (4.1.65), (4.1.66).

Theorem 4.1.4. *Suppose $z \in D_{\beta^x}$, $\alpha \in D_{\beta^y}$, $\lambda \in \mathcal{P}$, $\eta \in D_1 \times D_2$ and assume the conditions of Theorem 4.1.2 hold.*

Then the solutions $y_1(\cdot;z,\alpha,\lambda,\eta)$, $y_2(\cdot;z,\alpha,\lambda,\eta)$ of the FIVPs (4.1.65), (4.1.66) satisfy conditions

$$y_1(b;z,\alpha,\lambda,\eta) = [C_1^{-1}(d(\lambda,\eta) - Az)]_1, \;\; y_2(b;z,\alpha,\lambda,\eta) = [C_1^{-1}(d(\lambda,\eta) - Az)]_2,$$

i.e. they are solutions to the decomposed FBVPs (4.1.7) with parametrized boundary conditions on the subinterval $[c,b]$ if and only if the control parameters χ^{y_p}, χ^{y_q} in

(4.1.65), (4.1.66) are given by

$$\begin{aligned}\chi^{y_p} :=&\frac{\Gamma(p+1)}{(b-c)^p}\{[C_1^{-1}(d(\lambda,\eta)-Az)]_1-\alpha_1\}\\&-\frac{p}{(b-c)^p}\int_c^b(b-s)^{p-1}g_1(s,x_{\infty,}(s),y_\infty(s))ds,\end{aligned} \tag{4.1.73}$$

$$\begin{aligned}\chi^{y_q} :=&\frac{\Gamma(q+1)}{(b-c)^q}\{[C_1^{-1}(d(\lambda,\eta)-Az)]_2-\alpha_2\}\\&-\frac{q}{(b-c)^q}\int_c^b(b-s)^{q-1}g_2(s,x_{\infty,}(s),y_\infty(s))ds,\end{aligned} \tag{4.1.74}$$

where $y_\infty(\cdot;z,\alpha,\lambda,\eta)=\begin{bmatrix}y_{1,\infty}(\cdot;z,\alpha,\lambda,\eta)\\y_{2,\infty}(\cdot;z,\alpha,\lambda,\eta)\end{bmatrix}$ *are the limit functions in (4.1.52) and (4.1.53), respectively.*

Proof. The proof of Theorem 4.1.4 follows the lines of the proof of Theorem 4.1.3. □

Lastly, we relate the solutions of the 'model-type' FBVPs (4.1.6) and (4.1.7) to the solution of the original problem. The following theorem demonstrates the connection between the limit functions (4.1.61) and (4.1.62) and the solutions to the FBVP (4.1.1), (4.1.2).

Theorem 4.1.5. *Suppose the conditions of Theorem 4.1.1 and Theorem 4.1.2 hold. Then the functions* $u_{1,\infty}(\cdot;z,\alpha,\lambda,\eta)$ *and* $u_{2,\infty}(\cdot;z,\alpha,\lambda,\eta)$*, defined in (4.1.61) and (4.1.62) are continuous solutions to the original FBVP (4.1.1), (4.1.2), if and only if the following system of algebraic or transcendental equations is satisfied*

$$\begin{aligned}\Delta^{p_x}(z,\alpha)&=0,\\\Delta^{q_x}(z,\alpha)&=0,\\\Delta^{p_y}(z,\alpha,\lambda,\eta)&=0,\\\Delta^{q_y}(z,\alpha,\lambda,\eta)&=0,\\V(z,\alpha,\lambda,\eta)-\lambda&=0,\\y_i(b;z,\alpha,\lambda,\eta)-\eta_i&=0,\ \ i=p+1,...,N,\end{aligned} \tag{4.1.75}$$

where $\Delta^{p_x}(z,\alpha)$, $\Delta^{q_x}(z,\alpha)$, $\Delta^{p_y}(z,\alpha,\lambda,\eta)$, *and* $\Delta^{q_y}(z,\alpha,\lambda,\eta)$ *are given by (4.1.43), (4.1.44), (4.1.58), and (4.1.59), and* $V(z,\alpha,\lambda,\eta)$ *is defined as*

$$V(z,\alpha,\lambda,\eta):=\int_a^c P(s)x_\infty(s;z,\alpha)ds+\int_c^b P(s)y_\infty(s;z,\alpha,\lambda,\eta). \tag{4.1.76}$$

Proof. Since the conditions of Theorems 4.1.1 and 4.1.2 hold, we can apply Theorems 4.1.3 and 4.1.4. The perturbed IVPs (4.1.41)-(4.1.42) and (4.1.56)-(4.1.57) coincide with BVPs (4.1.6), (4.1.7) if and only if

$$\begin{aligned}\Delta^{p_x}(z,\alpha)&=0,\\\Delta^{q_x}(z,\alpha)&=0,\end{aligned}$$

$$\Delta^{p_y}(z,\alpha,\lambda,\eta)=0,$$
$$\Delta^{q_y}(z,\alpha,\lambda,\eta)=0.$$

Moreover, from the definition of λ in (4.1.4), it follows that in order for $x_1(\cdot;z,\alpha)$, $x_2(\cdot;z,\alpha)$, $y_1(\cdot;z,\alpha,\lambda,\eta)$, and $y_2(\cdot;z,\alpha,\lambda,\eta)$ to coincide with the solutions of (4.1.6) and (4.1.7), it must hold that

$$\begin{aligned}
&\int_a^c [P_{11}(s)x_{1,\infty}(s)+P_{12}(s)x_{2,\infty}(s)]ds\\
&\qquad\qquad+\int_c^b [P_{11}(s)y_{1,\infty}(s)+P_{12}(s)y_{2,\infty}(s)]ds-\lambda_1=0,\\
&\int_a^c [P_{21}(s)x_{1,\infty}(s)+P_{22}(s)x_{2,\infty}(s)]ds\\
&\qquad\qquad+\int_c^b [P_{21}(s)y_{1,\infty}(s)+P_{22}(s)y_{2,\infty}(s)]ds-\lambda_2=0,\\
&\qquad\qquad y_{i,\infty}(b;z,\alpha,\lambda,\eta)-\eta_i=0,
\end{aligned}$$

for $i=p+1,...,N$, where the notation $y_{i,\infty}(b;z,\alpha,\lambda,\eta)$ refers to the i-th component of the vector $y_\infty(b;z,\alpha,\lambda,\eta)$. Thus, $x_{1,\infty}(\cdot;z,\alpha)$, $x_{2,\infty}(\cdot;z,\alpha)$, $y_{1,\infty}(\cdot;z,\alpha,\lambda,\eta)$, and $y_{2,\infty}(\cdot;z,\alpha,\lambda,\eta)$ are the solutions of (4.1.6) and (4.1.7) if an only if the equations in (4.1.75) are satisfied. This completes the proof of the theorem. □

Remark 4.1.6. *Theorem 4.1.5 gives necessary and sufficient conditions on the solvability of the system of FBVPs (4.1.6), (4.1.7) and the construction of their solutions. However, a difficulty of its application arises from the fact that the explicit forms of the exact functions Δ^{p_x}, Δ^{q_x}, Δ^{p_y}, Δ^{q_y}, V, and $y(b;z,\alpha,\lambda,\eta)$ are unknown. In order to overcome this complication, in practice we solve an approximate system of determining equations*

$$\begin{aligned}
\Delta_m^{p_x}(z,\alpha)&=0,\\
\Delta_m^{q_x}(z,\alpha)&=0,\\
\Delta_m^{p_y}(z,\alpha,\lambda,\eta)&=0,\\
\Delta_m^{q_y}(z,\alpha,\lambda,\eta)&=0\\
V_m(z,\alpha,\lambda,\eta)&=0,\\
y_{m,i}(b;z,\alpha,\lambda,\eta)&=0,\ \ i=p+1,...,N,
\end{aligned}\tag{4.1.77}$$

that only depends on the m-th terms in the functional sequences (4.1.29)-(4.1.32), and can therefore be constructed explicitly. In particular, the equations in (4.1.77) at the m-th iteration are given by:

$$\begin{aligned}
\Delta_m^{p_x}(z,\alpha):=&\frac{\Gamma(p+1)}{(c-a)^p}(\alpha_1-z_1)-\frac{p}{(c-a)^p}\int_a^c (c-s)^{p-1}f_1(s,x_m(s))ds,\\
\Delta_m^{q_x}(z,\alpha):=&\frac{\Gamma(q+1)}{(c-a)^q}(\alpha_2-z_2)-\frac{q}{(c-a)^q}\int_a^c (c-s)^{q-1}f_2(s,x_m(s))ds,\\
\Delta_m^{p_y}(z,\alpha,\lambda,\eta):=&\frac{\Gamma(p+1)}{(b-c)^p}\{[C_1^{-1}(d(\lambda,\eta)-Az)]_1-\alpha_1\}\\
&-\frac{p}{(b-c)^p}\int_c^b (b-s)^{p-1}g_1(s,x_m(s),y_m(s))ds,
\end{aligned}$$

$$\Delta_m^{q_y}(z,\alpha,\lambda,\eta) := \frac{\Gamma(q+1)}{(b-c)^q}\{[C_1^{-1}(d(\lambda,\eta)-Az)]_2-\alpha_2\}$$
$$-\frac{q}{(b-c)^q}\int_c^b (b-s)^{q-1}g_2(s,x_m(s),y_m(s))ds, \tag{4.1.78}$$
$$V_m(z,\alpha,\lambda,\eta) := \int_a^c P(s)x_m(s;z,\alpha)ds + \int_c^b P(s)y_m(s;z,\alpha,\lambda,\eta).$$

In the last sub-section, we apply the theory presented thus far to an example problem.

4.1.4. EXAMPLE

Motivated by ([46], [99]) we consider a BVP for the non-linear fractional differential equation

$$ {}_0^C D_t^{\frac{3}{2}} u(t) = \frac{-2e^t}{(1+e^t)^2}\left[\frac{u(t)}{16}\right]^2 - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} \; (:= f(t,u(t))), \quad t\in[0,1], \tag{4.1.79}$$

subjected to the integral boundary conditions

$$\begin{aligned} u(0)+\dot{u}(0) &= -\int_0^1 u(s)d\zeta(s),\\ u(1)+\dot{u}(1) &= \int_0^1 u(s)d\eta(s), \end{aligned} \tag{4.1.80}$$

where $\zeta(t)$ and $\eta(t)$ are nondecreasing, right-continuous on $t\in[0,1)$ and left continuous at $t=1$, and $\int_0^1 u(s)d\zeta(s)$, $\int_0^1 u(s)d\eta(s)$ denote the Riemann-Stieltjes integrals of u with respect to $\zeta(t)$ and $\eta(t)$, [46]. For simplicity, we take $\zeta(t)=\eta(t)=t$, hence $d\zeta(t)=d\eta(t)=1$ and BCs (4.1.80) become

$$\begin{aligned} u(0)+\dot{u}(0) &= -\int_0^1 u(s)ds,\\ u(1)+\dot{u}(1) &= \int_0^1 u(s)ds. \end{aligned} \tag{4.1.81}$$

In (4.1.79) ω is a scalar which in the context of the flow of the Antarctic Circumpolar Current corresponds to the dimensionless Coriolis parameter being equal to 4649.56.

Equation (4.1.79) can be written as a system of a first order ODE and a FDE of order $q=1/2$ by letting

$$u_1(t) := u(t), \quad u_2(t) := \dot{u}(t) = \dot{u}_1(t). \tag{4.1.82}$$

Substituting (4.1.82) into (4.1.79) results in the following system

$$\begin{cases} \dot{u}_1(t) = u_2(t) \;\; (:= f_1(t,u(t)), \\ {}_0^C D_t^{\frac{1}{2}} u_2(t) = \frac{-2e^t}{(1+e^t)^2}\left[\frac{u_1(t)}{16}\right]^2 - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} \;\; (:= f_2(t,u(t)), \end{cases} \tag{4.1.83}$$

and the boundary conditions in (4.1.81) are transformed into

$$u_1(0)+u_2(0) = -\int_0^1 u_1(s)ds, \tag{4.1.84}$$

$$u_1(1) + u_2(1) = \int_0^1 u_1(s)ds. \tag{4.1.85}$$

We apply the parametrization technique, described in Section 4.1.1, by introducing

$$z_1 := u_1(0),\ \ z_2 := u_2(0),\ \ \lambda_1 := -\int_0^1 u_1(s)ds,$$
$$\lambda_2 := \int_0^1 u_1(s)ds,\ \eta := \begin{pmatrix} u_2(1) \\ 0 \end{pmatrix}.$$

With the given parametrization, boundary conditions (4.1.81) are re-written as

$$Az + C_1 u(1) = d(\eta, \lambda), \tag{4.1.86}$$

where

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ \ C_1 := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\ \ d(\eta, \lambda) := \eta + \lambda,\ \ u(1) := \begin{pmatrix} u_1(1) \\ u_2(1) \end{pmatrix}.$$

This allows us to express the values of $u_1(1)$ and $u_2(1)$ as

$$u_1(1) = [C_1^{-1}(d(\eta, \lambda) - Az)]_1 = -\eta_1 - \lambda_1 + z_1 + z_2 + \lambda_2,$$
$$u_2(1) = [C_1^{-1}(d(\eta, \lambda) - Az)]_2 = \eta_1 + \lambda_1 - z_1 - z_2.$$

The computed sets D_1 and D_2 are given by

$$D_1 :=\{u_1 : -855.04 \le u_1 \le 183.69\}, \qquad t \in [0, 1],$$
$$D_2 :=\{u_2 : -701.03 \le u_2 \le 1248.85\}, \qquad t \in [0, 1],$$

on which the right-hand side function $f(t, u_1(t), u_2(t)) = \begin{pmatrix} f_1(t, u_1(t), u_2(t)) \\ f_2(t, u_1(t), u_2(t)) \end{pmatrix}$ satisfies the Lipschitz condition with a constant matrix $\tilde{K} = \begin{pmatrix} 0 & 1 \\ 1.46 & 0 \end{pmatrix}$. The matrix Q has spectral radius $r(Q) \approx 1.36 > 1$. That is, condition 4a) is not satisfied, and hence the numerical-analytic technique cannot be used for constructing approximate solutions of system (4.1.83) on the whole interval $t \in [0, 1]$. Therefore, it is necessary to apply the interval halving technique.

Let $c = 1/2$ and

$$u_1(t) = \begin{cases} x_1(t), & t \in [0, 1/2] \\ y_1(t), & t \in [1/2, 1] \end{cases} \qquad u_2(t) = \begin{cases} x_2(t), & t \in [0, 1/2] \\ y_2(t), & t \in [1/2, 1] \end{cases}.$$

Introducing an additional parameter

$$\alpha := \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} x_1(1/2) \\ x_2(1/2) \end{pmatrix} = \begin{pmatrix} y_1(1/2) \\ y_2(1/2) \end{pmatrix}$$

allows us to decompose BVP (4.1.83), (4.1.86) into the following two BVPs

$$\begin{cases} \dot{x}_1(t) = x_2(t) := f_1(t, x(t)), \\ {}_0^C D_t^{\frac{1}{2}} x_2(t) = \frac{-2e^t}{(1+e^t)^2} \left[\frac{x_1(t)}{16}\right]^2 - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} := f_2(t, x(t)), \\ x_1(0) = z_1, \quad x_1(1/2) = \alpha_1, \\ x_2(0) = z_2, \quad x_2(1/2) = \alpha_2; \end{cases} \tag{4.1.87}$$

and

$$\begin{cases} \dot{y}_1(t) = y_2(t) := g_1(t, x(t), y(t)), \\ {}_{1/2}^C D_t^{\frac{1}{2}} y_2(t) = \frac{-2e^t}{(1+e^t)^2} \left[\frac{y_1(t)}{16}\right]^2 - \frac{2\omega e^t(1-e^t)}{(1+e^t)^3} - \frac{1}{\Gamma(1/2)} \int_0^{1/2} (s-t)^{-1/2} \dot{x}_2(t) dt, \\ \qquad := g_2(t, x(t), y(t)), \\ y_1(1/2) = \alpha_1, \quad y_1(1) = [C_1^{-1}(d(\eta, \lambda) - Az]_1, \\ y_2(1/2) = \alpha_2, \quad y_2(1) = [C_1^{-1}(d(\eta, \lambda) - Az]_2. \end{cases} \tag{4.1.88}$$

The adjustment in the right-hand side function $g(t, x(t), y(t))$ in the BVP (4.1.88) follows from the considerations presented in Remark 4.1.2.

Let BVPs (4.1.87), (4.1.88) be defined on the domains

$$\begin{aligned} D^x :=& \{(x_1, x_2) : -494 \leq x_1 \leq -365.74, \; -144.05 \leq x_2 \leq 318.15\}, \\ & t \in [0, 1/2], \\ D^y :=& \{(y_1, y_2) : -713.02 \leq y_1 \leq 87.35, \; -528.35 \leq y_2 \leq 1208.61\}, \\ & t \in [1/2, 1], \end{aligned}$$

respectively.

The right-hand side functions $f(t, x_1(t), x_2(t))$ and $g(t, y_1(t), y_2(t))$ satisfy conditions 1a), 2a) and 1b), 2b), respectively, with

$$M^x = \begin{pmatrix} M_1^x \\ M_2^x \end{pmatrix} = \begin{pmatrix} 114.72 \\ 289.64 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ 1.46 & 0 \end{pmatrix},$$
$$M^y = \begin{pmatrix} M_1^y \\ M_2^y \end{pmatrix} = \begin{pmatrix} 586.75 \\ 779.43 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1.35 & 0 \end{pmatrix}.$$

The constants in (4.1.13), (4.1.22) and the spectral radii (4.1.16), (4.1.26) are calculated to be

$$\begin{aligned} \beta_1^x &= 28.68, & \beta_2^x &= 231.1, & r({}^xQ) &= 0.96, \\ \beta_1^y &= 293.38, & \beta_2^y &= 621.87, & r({}^yQ) &= 0.93. \end{aligned}$$

Since $r({}^xQ) < 1$ and $r({}^yQ) < 1$, the functions $f(t, x_1(t), x_2(t))$, $g(t, y_1(t), y_2(t))$ are bounded and satisfy Lipschitz conditions with constant matrices K and J respectively, conditions 1a)-4a) and 1b)-4b) are satisfied, therefore we can apply the numerical-analytic technique for constructing sequences of approximations of the solutions to BVPs (4.1.87) and (4.1.88).

Solving the system of approximate equations (4.1.78) at iterations $m = 0,\ 1,\ 2$ and applying Maple yields the following values of the artificially introduced parameters:

Table 4.1: Numerically calculated parameter values for $m = 0, 1, 2$

parameter	$m = 0$	$m = 1$	$m = 2$
$z_{1,m}$	-465.322	-378.252	-369.774
$z_{2,m}$	87.053	37.012	31.289
$\alpha_{1,m}$	-419.641	-372.998	-365.739
$\alpha_{2,m}$	93.512	115.168	114.717
$\lambda_{1,m}$	-378.269	-341.239	-338.484
$\lambda_{2,m}$	378.269	341.239	338.485
$\eta_{1,m}$	586.743	547.891	544.510

Plots of the first three iterates ($m = 0,\ 1,\ 2$) of the first and second components are shown in Figure 4.1. The continuity at $t = 1/2$ of each term in the approximating sequence is enforced with the introduction of the parameter α, as expected. In the initial approximation, differentiability fails at $t = 1/2$ and with each successive iteration the functions become smoother. To verify how well the approximations satisfy systems (4.1.87), (4.1.88), we computed the first derivatives of $x_{1,m}(t)$, $y_{1,m}(t)$ and the Caputo derivatives of $x_{2,m}(t)$, $y_{2,m}(t)$ for $m = 0,\ 1,\ 2$ and graphically compared them to the right-hand sides of the equations in (4.1.87), (4.1.88). These plots are shown in Figures 4.2 - 4.4. With each successive iteration, the computed approximations satisfy the equations more accurately. The larger error in the approximations at the initial iterations results in discontinuities in the left-hand sides of the equations. At $m = 2$ the equations are well satisfied, however, there is still an observed discontinuity at the midpoint. This likely arises due to the piecewise definitions of the approximating functions combined with the nonlocal properties of fractional derivatives. Although the approximations are constructed to ensure continuity of the functions at the midpoint, continuity of derivatives is not explicitly enforced. Furthermore, since fractional derivatives are non-local, the differing approximations on each side may lead to a mismatch in the computed fractional derivatives at $t = 1/2$. This discrepancy can be addressed by enforcing higher-order smoothness conditions or by continuing the iterative process to obtain approximations with higher precision.

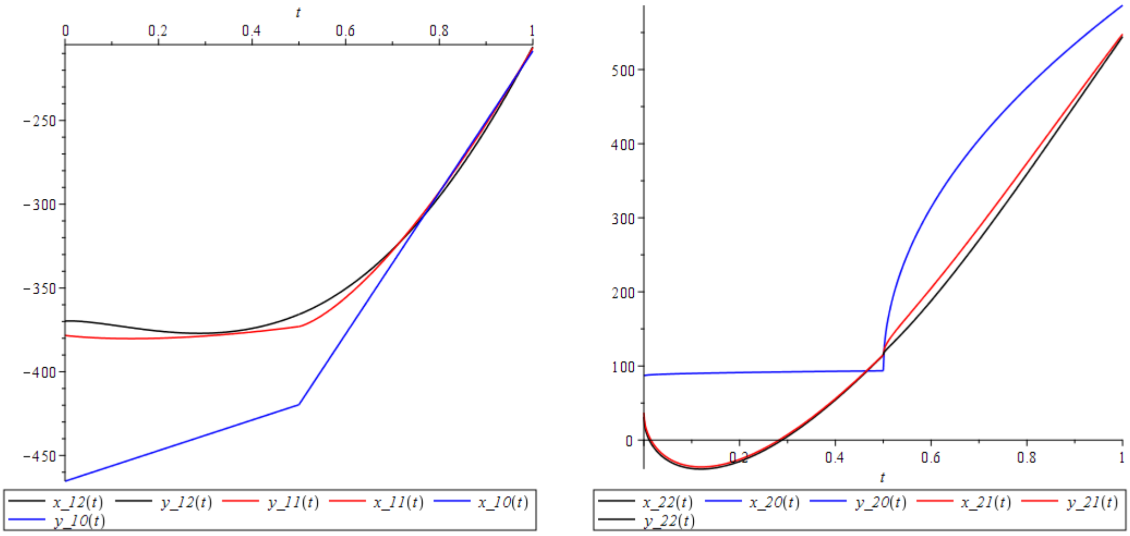


Figure 4.1: Plots of components $x_{1,m}(t)$, $y_{1,m}(t)$ (left) and $x_{2,m}(t)$, $y_{2,m}(t)$ (right) for $m = 0,\ 1,\ 2$ over $t \in [0, 1]$.

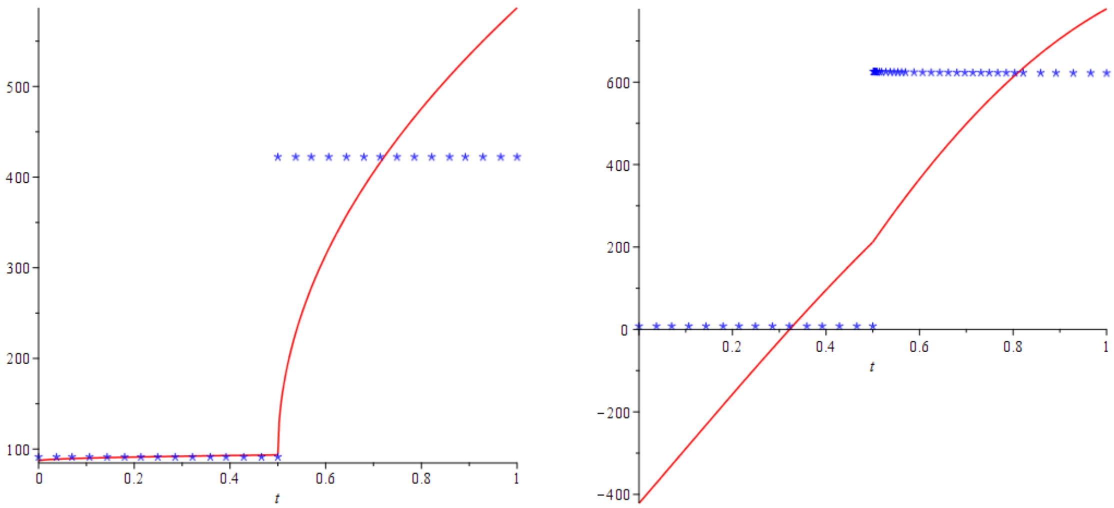


Figure 4.2: Left- (blue dotted lines) and right- (red solid lines) hand sides of system (4.1.83) for $m = 0$. The left panel shows plots for the first equation and the right panel shows plots for the second one.

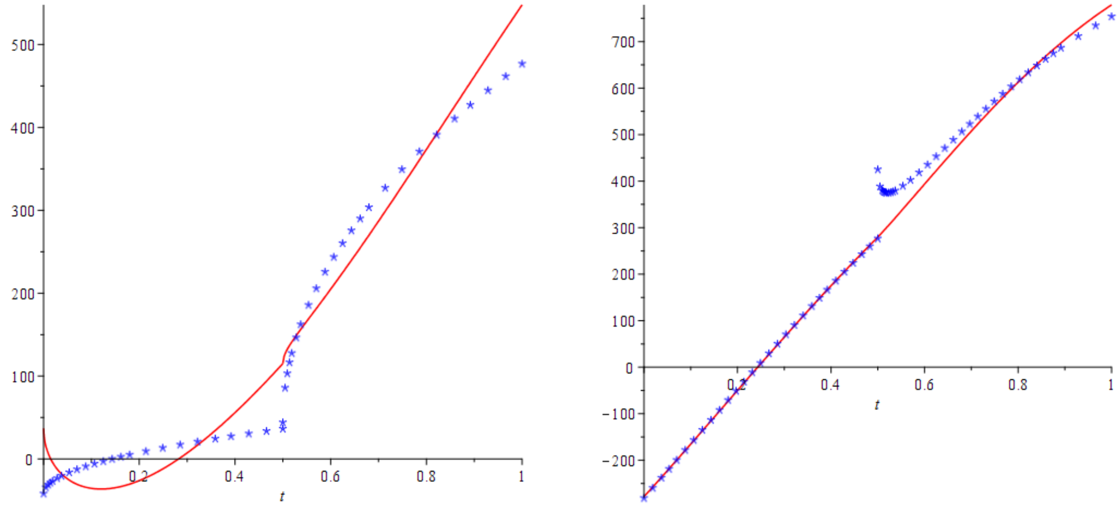


Figure 4.3: Left- (blue dotted lines) and right- (red solid lines) hand sides of system (4.1.83) for $m = 1$. The left panel shows plots for the first equation and the right panel shows plots for the second one.

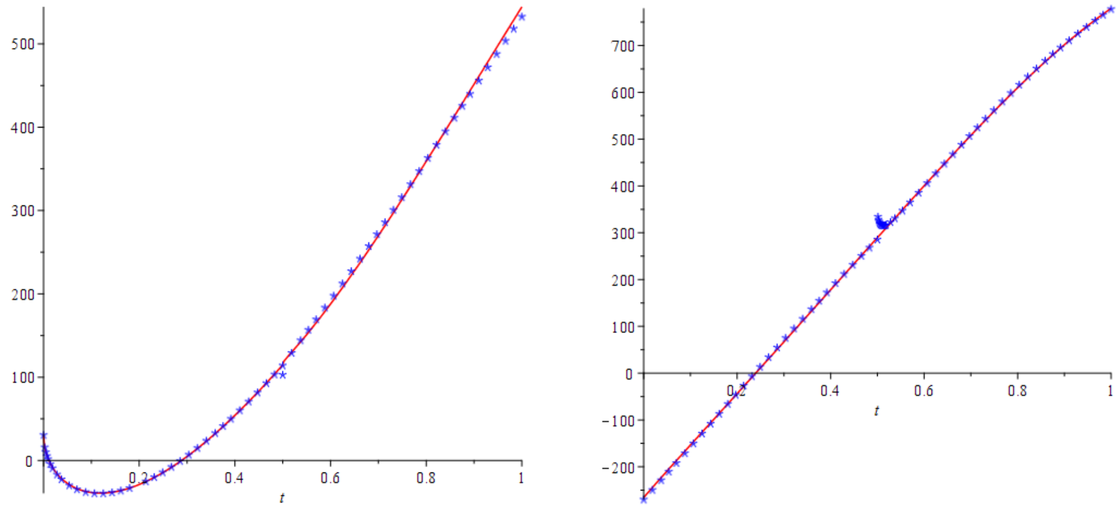


Figure 4.4: Left- (blue dotted lines) and right- (red solid lines) hand sides of system (4.1.83) for $m = 2$. The left panel shows plots for the first equation and the right panel shows plots for the second one.

4.2. THE FRACTIONAL BVP WITH PARAMETER-DEPENDENT AND ASYMPTOTIC CONDITIONS

We consider two different problem settings. First, we study a FIBVP of the Caputo type on a semi-finite domain. In particular, we study a nonlinear FDE of the general form

$$ {}_0^C D_t^p u(t) = f(t, u(t)), \quad t \in [t_0, \infty), \quad p \in (1, 2), $$

defined on the half-axis, and subject to asymptotic conditions of the form

$$ \lim_{t\to\infty} u(t) = \phi_0, \quad \lim_{t\to\infty} \{e^t u'(t)\} = 0. $$

We use fixed point theory to give some conditions for the existence of bounded solutions.

In the second setting, we consider the case when the FDE is restricted to a finite interval of unknown length λ, subject to a parameter-dependent boundary condition of the form

$$ Au(0) + Bu(\lambda) + Cu'(\lambda) = d $$

and initial conditions

$$ u(0) = \psi, \quad u'(0) = \chi_0. $$

We use an interval splitting method and the numerical-analytic technique to analyze the problem, and to construct a sequence of functions which converges to its exact solution. Finally, our method is applied to the Arctic gyre equation in the fractional setting to illustrate the validity of our results.

4.2.1. BOUNDED SOLUTIONS OF FDES WITH ASYMPTOTIC CONDITIONS

This sub-section is dedicated to the study of a FDE, defined on a semi-infinite domain, and subject to asymptotic conditions of integer order. We give conditions for the existence of bounded solutions to the FDE, satisfying the given asymptotic constraints.

Consider a FDE of the form

$$ {}^C D_-^p u(t) = f(t, u(t)), \; t \geq t_0, \tag{4.2.1} $$

where $p \in (1, 2)$, and ${}^C D_-^p(\cdot)$ denotes the Caputo fractional derivative on the half-axis, defined in Def. 2.1.9. Here $u(t) : [t_0, \infty) \to D \subset \mathbb{R}$ is a continuously differentiable function, and $f : G \to \mathbb{R}$ is continuous, non-autonomous and generally non-linear in $u(t)$, where $G := [t_0, \infty) \times D$.

Equation (4.2.1) is subject to the following asymptotic boundary conditions:

$$ \lim_{t\to\infty} u(t) = \phi_0, \quad \lim_{t\to\infty} \{e^t u'(t)\} = 0, \tag{4.2.2} $$

This section is based on the paper [100].

where $\phi_0 \in \mathbb{R}$ is a given scalar.

The next lemma shows the equivalence between the problem (4.2.1), (4.2.2) and the corresponding integral equation.

Lemma 4.2.1. *Let there exist positive constants k and $c > 1$, such that*

$$|f(t, u(t))| \leq ke^{-ct}. \tag{4.2.3}$$

Then problem (4.2.1), (4.2.2) is equivalent to the integral equation

$$u(t) = \phi_0 + \frac{1}{\Gamma(p)} \int_t^{\infty} (s-t)^{p-1} f(s, u(s)) ds. \tag{4.2.4}$$

Proof. Suppose that the right-hand side function in (4.2.1) is such that condition (4.2.3) holds. Then applying the fractional integral operator on the half-axis to both sides of equation (4.2.1), using the result of Lemma 2.1.3, and the conditions in (4.2.2) yields the integral equation in (4.2.4).

Conversely, starting from the integral equation in (4.2.4) and applying the Caputo fractional derivative on the half axis of order $q = p - 1$, we obtain

$$u'(t) = \frac{1}{\Gamma(q)} \int_t^{\infty} (s-t)^{q-1} f(s, u(s)) ds.$$

Since (4.2.3) holds, we have

$$|u'(t)| \leq \frac{1}{\Gamma(q)} \int_t^{\infty} (s-t)^{q-1} |f(s, u(s))| ds \tag{4.2.5}$$

$$\leq \frac{k}{\Gamma(q)} \int_t^{\infty} (s-t)^{q-1} e^{-cs} ds = \frac{k}{c^q} e^{-ct}. \tag{4.2.6}$$

Moreover,

$$|u(t)| \leq |\phi_0| + \frac{1}{\Gamma(q)} \int_t^{\infty} (s-t)^{p-1} |f(s, u(s))| ds \leq |\phi_0| + \frac{k}{c^p} e^{-ct}. \tag{4.2.7}$$

Inequalities (4.2.5) and (4.2.7) imply that asymptotic conditions (4.2.2) hold. Differentiating both sides of Equation (4.2.4) yields the FDE in (4.2.1). That is, when (4.2.3) holds, problem (4.2.1), (4.2.2) is equivalent to the integral equation (4.2.4).

□

Next, we state and prove a theorem, which ensures the existence of a unique solution of the integral equation (4.2.4), satisfying the asymptotic constraints in (4.2.2).

Theorem 4.2.1. *Assume that there exists a function $a : [t_0, \infty] \to \mathbb{R}^+$, such that*

$$\int_{t_0}^{\infty} s^{p-1} a(s) ds < \infty, \tag{4.2.8}$$

$$|f(t, u) - f(t, v)| \leq a(t)|u - v|, \;\; t \geq t_0, \;\; u, v, \in \mathbb{R}, \tag{4.2.9}$$

and the condition in (4.2.3) holds. Then for all $\phi_0 \in \mathbb{R}$, integral equation (4.2.4) has a unique continuous solution $u : [t_0, \infty] \to \mathbb{R}$ satisfying $\lim_{t\to\infty} u(t) = \phi_0$.

Proof. From (4.2.8) it follows that there exists a constant $T_0 \geq t_0$, such that

$$\int_{T_0}^{\infty} s^{p-1} a(s) ds < \Gamma(p).$$

On the Banach space X of continuous and bounded functions $u : [T_0, \infty) \to \mathbb{R}$, endowed with the norm $||u(t)|| := \sup_{t \geq T_0} |u(t)|$, define the operator

$$[F(u)](t) := \phi_0 + \frac{1}{\Gamma(p)} \int_t^{\infty} (s-t)^{p-1} f(s, u(s)) ds \tag{4.2.10}$$

for $t \geq T_0$. Then for $u \in X$ we have

$$\begin{aligned} |[F(u)](t)| &\leq |\phi_0| + \left| \frac{1}{\Gamma(p)} \int_t^{\infty} (s-t)^{p-1} f(s, u(s)) ds \right| \\ &\leq |\phi_0| + \int_t^{\infty} |I_-^q f(s, u(s))| ds < \infty, \end{aligned}$$

that is, $F : X \to X$. Now let $u, v \in X$ and consider

$$\begin{aligned} ||[F(u)] - [F(v)]|| &\leq \sup_{t \geq T_0} \frac{1}{\Gamma(p)} \int_t^{\infty} (s-t)^{p-1} |f(s,u) - f(s,v)| ds \\ &\leq \sup_{t \geq T_0} \frac{1}{\Gamma(p)} \int_t^{\infty} s^{p-1} a(s) |u-v| ds \\ &\leq ||u-v|| \sup_{t \geq T_0} \frac{1}{\Gamma(p)} \int_t^{\infty} s^{p-1} a(s) ds \\ &\leq ||u-v|| \frac{1}{\Gamma(p)} \int_{T_0}^{\infty} s^{p-1} a(s) ds < ||u-v||, \end{aligned}$$

which implies that the operator F is a contraction on X, hence it has a unique fixed point in X by the contraction principle, and this fixed point is the unique solution to Equation (4.2.4) on $t \in [T_0, \infty)$, [101].
If $T_0 = t_0$, we are done. If $t_0 < T_0$, $u(T_0)$ and $u'(T_0)$ are determined and the solution of (4.2.1) can be extended from $[T_0, \infty)$ to $[t_0, \infty)$. This is because, since $a(t)$ is positive and (4.2.8) holds, $a(t)$ is bounded for $t \geq t_0$, which implies that $f(t, u(t))$ is Lipschitz continuous in u. Hence (4.2.1) has a unique solution on $[T_0, \infty)$, which can be continued to $[t_0, \infty)$. Moreover, condition (4.2.3) on $f(t, u(t))$ prevents the blow up of solutions in finite time. Therefore, the solutions on $[T_0, \infty)$ can be extended to $[t_0, \infty)$. □

For the remainder of the section we restrict the domain of definition of Equation (4.2.1) to a finite interval.

4.2.2. APPROXIMATIONS TO THE PARAMETER-DEPENDENT FIBVP ON A FINITE INTERVAL

We study a FDE on a finite interval whose length is denoted by the unknown parameter λ, and can in principle be extended indefinitely, that is, we can let $\lambda \to \infty$. We attach parameter-dependent boundary conditions to the FDE and use the interval splitting method to re-define the original problem as a system of ‘model-type’ problems on smaller domains, similarly to Section 4.1. The numerical-analytic technique is applied to construct approximations to the solution of each problem. We establish a connection between the solutions to the ‘model’ problems and the original FIBVP, and give necessary and sufficient conditions for the existence of solutions (for details of the technique, see [58]).

PROBLEM SETTING AND INTERVAL SPLITTING

We consider the FIBVP for the FDE of the form:

$$ {}_0^C D_t^p u(t) = f(t, u(t)), \quad t \in J := [0, \lambda] \tag{4.2.11} $$

for some $p \in (1, 2)$, subjected to the parameter-dependent boundary conditions

$$ Au(0) + Bu(\lambda) + Cu'(\lambda) = d \tag{4.2.12} $$

and the initial conditions

$$ u(0) = \alpha_0, \;\; u'(0) = \chi_0. \tag{4.2.13} $$

Here ${}_0^C D_t^p(\cdot)$ is the Caputo derivative with lower limit at 0 (see Def. 2.1.7), $u : J \to D \subset \mathbb{R} \in C^1(J, \mathbb{R})$, $f : G \to \mathbb{R}$, $G = J \times D$, and D is a closed and bounded domain. The constants in the boundary and initial conditions $A, B, C, d, \alpha_0, \chi_0 \in \mathbb{R}$ are given scalars, and the end point of the interval J is an unknown parameter $\lambda \in \mathbb{R}$.

We aim to find a solution $u : J \to D$ of the FDE (4.2.11), which satisfies the parameter-dependent boundary conditions (4.2.12), and the given initial conditions (4.2.13) in the space $C^1(J, \mathbb{R})$.

For this purpose, we will construct a sequence of approximate solutions, and as it will be seen in Theorem 4.2.2, the convergence of this sequence is contingent upon the function $f(t, u(t))$ satisfying a Lipschitz condition on J. If this fails to hold, the uniform convergence of the sequence cannot be guaranteed. To deal with this difficulty, we will use a dichotomy type approach, similar to [81], but for a more general setting.

Let us decompose the interval $J = [0, \lambda]$ into N subintervals. Without loss of generality, let each subinterval have length λ/N, and denote them by $J_j = [\lambda_{j-1}, \lambda_j] := \Big[(j-1)\lambda/N, j\lambda/N\Big]$ for $j = 1, ..., N$. We denote the solution on each J_j by $u_j(t)$, and the values of $u_j(t)$ and $u_j'(t)$ at the end point of the subintervals in the following way:

$$ u_j(\lambda_j) = u_{j+1}(\lambda_j) = \alpha_j, \quad u_j'(\lambda_j) = u_{j+1}'(\lambda_j) = \chi_j, \quad j = 1, ..., N-1. \tag{4.2.14} $$

The solution $u(t)$ to FBVP (4.2.11)-(4.2.13) on the whole interval $t \in [0, \lambda]$ is defined by $u_j(t)$ piece-wise on each J_j. Thus, the boundary conditions (4.2.14) are chosen in such a way that they ensure that $u(t)$ is a continuous function. Here α_j and χ_j are unknown parameters to be calculated. Note that the boundary condition at the value of $u_N'(\lambda_N = \lambda)$ is given by $\chi_N = C^{-1}(d - A\alpha_0 - B\alpha_N)$.

With this, we split the FBVP (4.2.11)-(4.2.13) into N 'model-type' problems, which read

$$\begin{aligned} {}_{\lambda_{j-1}}^{\ C}D_t^p u_j(t) =& f(t,u_j(t)) - \frac{1}{\Gamma(n-p)} \sum_{i=0}^{j-1} \int_{\lambda_j}^{\lambda_{j+1}} (t-s)^{n-p-1} u_i^{(n)}(s) ds \\ :=& f_j(t,u_1(t),...,u_j(t)) = f_j(t,u(t)), \quad t \in J_j, \end{aligned} \tag{4.2.15}$$

$$\begin{aligned} u_j(\lambda_{j-1}) =& \alpha_{j-1}, \quad u_j(\lambda_j) = \alpha_j, \\ u_j'(\lambda_{j-1}) =& \chi_{j-1}, \quad u_j'(\lambda_j) = \chi_j, \quad j = 1,...,N. \end{aligned} \tag{4.2.16}$$

The functions $u_j : J_j \to D_j$ are continuous on D_j, and the domains D_j are such that $\cup_{j=1}^N D_j = D$.

Remark 4.2.1. *As in Section 4.1, due to the non-locality of the Caputo fractional derivative, after defining a new BVP (4.2.15), (4.2.16) on each subinterval, the right-hand side function had to be adjusted accordingly, since now the Caputo derivative is taken with lower limit λ_{j-1} on each J_j.*

Next, we give the form of the sequence of successive approximations to the solution of FBVP (4.2.15), (4.2.16) and a theorem on the convergence of the sequence. The proof of the theorem follows the lines of Theorem 4.1.1, so we omit it.

Successive Approximations

Let us consider FBVP (4.2.15), (4.2.16) for a fixed $j \in \{1,...,N\}$. We connect it with a sequence $\{u_j^m\}$, $m \in \mathbb{Z}_0^+$, for $t \in J_j$, $u_j^0(t;\alpha,\chi,\lambda) \in D_j$, given by

$$\begin{aligned} u_j^0(t;\alpha,\chi,\lambda) &= \alpha_{j-1} + \chi_{j-1}(t-\lambda_{j-1}) + \Big(\frac{t-\lambda_{j-1}}{\lambda_1}\Big)^p \Big(\alpha_j - \alpha_{j-1} - \chi_{j-1}\frac{\lambda}{N}\Big), \\ u_j^m(t;\alpha,\chi,\lambda) &= u_j^0(t;\alpha,\chi,\lambda) + \frac{1}{\Gamma(p)} \Bigg[\int_{\lambda_{j-1}}^t (t-s)^{p-1} f_j(s,u^{m-1}(s;\alpha,\chi,\lambda)) ds \\ &\quad - \Big(\frac{t-\lambda_{j-1}}{\lambda_1}\Big)^p \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s)^{p-1} f_j(s,u^{m-1}(s;\alpha,\chi,\lambda)) ds \Bigg]. \end{aligned} \tag{4.2.17}$$

Here λ_1 denotes the length of the each subinterval. The sequence above is derived by integrating the modified equation given in (4.2.15) and enforcing the boundary conditions (4.2.16).

Note that the approximating function $u^m(t;\alpha,\chi,\lambda)$ on the whole interval $t \in J$ is piece-wise given by

$$u^m(t;\alpha,\chi,\lambda) = u_j^m(t;\alpha,\chi,\lambda), \quad t \in J_j. \tag{4.2.18}$$

Assume that for the BVP (4.2.15), (4.2.16) the following conditions are satisfied:

(i) The function $f_j(t, u(t))$ is bounded:

$$|f_j(t, u(t))| \leq M_j, \tag{4.2.19}$$

for all $t \in J_j$, $u_j \in D_j$ and some non-negative integer M_j.

(ii) The function $f_j(t, u(t))$ is Lipschitz continuous in $u_j(t)$, i.e.

$$|f_j(t, u_1(t), ..., u_j^1(t)) - f_j(t, u_1(t), ..., u_j^2(t))| \leq K_j|u_j^1(t) - u_j^2(t)| \tag{4.2.20}$$

for all $t \in J_j$, $u_j^1, u_j^2 \in D_j$, and a non-negative Lipschitz constant K_j.

(iii) The set

$$D_{\beta_j} := \{\alpha_{j-1} \in D_j : B(u_j^0(t; \alpha, \chi, \lambda)), \beta_j\Big) \subset D_j \;\; \forall (t, \alpha_j, \chi_j, \lambda) \in J_j \times \Omega_j\} \tag{4.2.21}$$

is non-empty, where $\Omega_j = D_j \times X_j \times \Lambda$, $\chi_j \in X_j$, $\lambda \in \Lambda$, and

$$\beta_j = \frac{M_j \lambda_1^p}{2^{2p-1}\Gamma(p+1)}. \tag{4.2.22}$$

(iv) The inequality $Q_j < 1$ holds for Q_j, which is defined as

$$Q_j = \frac{K_j \lambda_1^p}{2^{2p-1}\Gamma(p+1)}. \tag{4.2.23}$$

The following theorem ensures that if conditions (i)-(iv) hold, for all $j \in \{1, ..., N\}$ there exists a limit function $u_j^\infty(t; \alpha, \chi, \lambda) : J_j \times \Omega_j \to D_j$, which is well defined for all artificially introduced parameters $(\alpha_j, \chi_j) \in D_j \times X_j$, and is the unique solution to the FBVP (4.2.15), (4.2.16) with corresponding index j. Moreover, letting

$$u_\infty(t; \alpha, \chi, \lambda) := u_j^\infty(t; \alpha, \chi, \lambda), \; t \in J_j \tag{4.2.24}$$

yields the well-defined continuous function $u_\infty(t; \alpha, \chi, \lambda)$, which satisfies the boundary and initial conditions in the original FIBVP (4.2.11)-(4.2.13):

$$\begin{aligned} u_\infty(0; \alpha, \chi, \lambda) &= u_1^\infty(0; \alpha, \chi, \lambda) = \alpha_0, \\ (u_\infty)'(0; \alpha, \chi, \lambda) &= (u_1^\infty)'(0; \alpha, \chi, \lambda) = \chi_0, \\ (u_\infty)'(\lambda; \alpha, \chi, \lambda) &= (u_N^\infty)'(\lambda; \alpha, \chi, \lambda) = C^{-1}(d - A\alpha_0 - B\alpha_N). \end{aligned}$$

Theorem 4.2.2. *Assume that the FBVP (4.2.15), (4.2.16) satisfies conditions (4.2.19)-(4.2.23). Then for all fixed $(\alpha_j, \chi_j, \lambda) \in \Omega_j$, it holds:*
1. Functions of the sequence (4.2.17) are continuous and satisfy the boundary condition

$$u_j^m(\lambda_{j-1}; \alpha, \chi, \lambda) = \alpha_{j-1}, \quad u_j^m(\lambda_j; \alpha, \chi, \lambda) = \alpha_j,$$

$$(u_j^m)'(\lambda_{j-1};\alpha,\chi,\lambda) = \chi_{j-1}, \quad (u_j^m)'(\lambda_j;\alpha,\chi,\lambda) = \chi_j.$$

2. The sequence of functions (4.2.17) for $t \in J_j$ converges uniformly as $m \to \infty$ to the limit function

$$u_j^\infty(t;\alpha,\chi,\lambda) = \lim_{m\to\infty} u_j^m(t;\alpha,\chi,\lambda). \tag{4.2.25}$$

3. The limit function satisfies the boundary conditions

$$\begin{aligned} u_j^\infty(\lambda_{j-1};\alpha,\chi,\lambda) &= \alpha_{j-1}, \quad u_j^\infty(\lambda_j;\alpha,\chi,\lambda) = \alpha_j, \\ (u_j^\infty)'(\lambda_{j-1};\alpha,\chi,\lambda) &= \chi_{j-1}, \quad (u_j^\infty)'(\lambda_j;\alpha,\chi,\lambda) = \chi_j. \end{aligned}$$

4. The limit function (4.2.25) is a unique solution to the integral equation

$$\begin{aligned} u_j(t) =& \alpha_{j-1} + \chi_{j-1}(t-\lambda_{j-1}) + \Big(\frac{t-\lambda_{j-1}}{\lambda_1}\Big)^p (\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1) \\ &+ \frac{1}{\Gamma(p)}\Bigg[\int_{\lambda_{j-1}}^t (t-s)^{p-1} f_j(s,u(s))ds \\ &- \Big(\frac{t-\lambda_{j-1}}{\lambda_1}\Big)^p \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s)^{p-1} f_j(s,u(s))ds\Bigg]. \end{aligned} \tag{4.2.26}$$

i.e. it is a unique solution on $t \in J_j$ of the Cauchy problem for the modified FDE:

$$\begin{aligned} {}_{\lambda_{j-1}}^{\;\;C}D_t^p u_j(t) &= f_j(t,u(t)) + \Delta_j(\alpha,\chi,\lambda), \\ u_j(\lambda_{j-1}) &= \alpha_{j-1}, \quad u_j'(\lambda_{j-1}) = \chi_{j-1}, \end{aligned} \tag{4.2.27}$$

where $\Delta_j(\alpha,\chi,\lambda) : \Omega \to \mathbb{R}$ is a mapping defined by

$$\Delta_j(\alpha,\chi,\lambda) = \frac{\Gamma(p+1)}{\lambda_1^p}(\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1) - \frac{p}{\lambda_1^p}\int_{\lambda_{j-1}}^{\lambda_j}\Big(\frac{j\lambda}{N} - s\Big)^{p-1} f_j(s,u(s))ds. \tag{4.2.28}$$

5. The following error estimate holds:

$$|u_j^\infty(t;\alpha,\chi,\lambda) - u_j^m(t;\alpha,\chi,\lambda)| \le \frac{\lambda_1^p}{2^{2p-1}\Gamma(p+1)}\frac{Q_j^m}{1-Q_j}M_j, \tag{4.2.29}$$

where $t \in J_j$, and M_j and Q_j are defined by (4.2.19) and (4.2.23).

Lastly, we establish the connection between the solution to the Cauchy problem (4.2.27) and the original FBVP (4.2.11)-(4.2.13).

Connection of the Limit Function to the FIBVP

First, we show the connection between (4.2.27) and the FBVP (4.2.15), (4.2.16).

Consider the Cauchy problem:

$$ {}_{\lambda_{j-1}}^{\ \ C}D_t^p u_j(t) = f_j(t, u(t)) + \mu_j, \ \ t \in J_j, \tag{4.2.30a} $$

$$ u_j(\lambda_{j-1}) = \alpha_{j-1}, \quad u_j'(\lambda_{j-1}) = \chi_{j-1}, , \tag{4.2.30b} $$

where $\mu_j \in \mathbb{R}$ is referred to as a control parameter.

The following result holds.

Theorem 4.2.3. *Suppose $\alpha_{j-1} \in D_{\beta_j}$, $(\alpha_j, \chi_j, \lambda) \in \Omega_j$ and assume the conditions of Theorem 4.2.2 hold. Then the solution $u_j(\cdot, \alpha, \chi, \lambda; \mu_j)$ of the Cauchy problem (4.2.30) also satisfies the boundary conditions in (4.2.16) if and only if*

$$ \mu_j = \Delta_j(\alpha, \chi, \lambda), \tag{4.2.31} $$

where $\Delta_j(\alpha, \chi, \lambda)$ is given by (4.2.28), and in this case

$$ u_j(t, \alpha, \chi, \lambda; \mu_j) = u_j^{\infty}(t; \alpha, \chi, \lambda). \tag{4.2.32} $$

For a proof of Theorem 4.2.3 we refer to [81].

The following theorem gives the connection between the limit function, defined in (4.2.24), and the solution to the original FBVP (4.2.11)-(4.2.13).

Theorem 4.2.4. *Let the FBVP (4.2.15)-(4.2.16) satisfy conditions (4.2.19)-(4.2.23). Then $u_j^{\infty}(t; \alpha^*, \chi^*, \lambda^*)$ is a solution to (4.2.15)-(4.2.16) if and only if the triple $(\alpha^*, \chi^*, \lambda^*)$ is a solution to the determining system*

$$ \begin{cases} \Delta_j(\alpha^*, \chi^*, \lambda^*) = 0, \\ V_j(\alpha^*, \chi^*, \lambda^*) = 0, \end{cases} \tag{4.2.33} $$

where $\Delta_j(\alpha, \chi, \lambda)$ is given in (4.2.28) and $V_j : \Omega \to \mathbb{R}$ is a mapping, defined by

$$ V_j(\alpha, \chi, \lambda) = \frac{d}{dt} u_j(\lambda_j; \alpha, \chi, \lambda) - \chi_j, \ \ \textit{for} \ \ j = 1, ..., N. \tag{4.2.34} $$

Proof. First, we note that the second equation in the determining system (4.2.33) is derived from the continuity of the solution $u(t)$ on J. The boundary conditions in (4.2.16) prescribe the derivative value of each $u_j(t)$ at the left-end point of the subinterval J_j. Equation (4.2.34) requires the derivative of $u_j(t)$ at the right end of the interval J_j to be equal to the derivative of $u_{j+1}(t)$ at the same point, therefore ensuring that the solution $u(t)$ is continuous.

Now, since the conditions of Theorem 4.2.2 hold, we can apply Theorem 2.2.3 and note that the perturbed equation in (4.2.27) coincides with the original FDE in (4.2.15), and the solution $u_\infty(t; \alpha^*, \chi^*, \lambda^*)$ satisfies the parameter-dependent boundary conditions in (4.2.16) if and only if the pair $(\alpha^*, \chi^*, \lambda^*)$ satisfies (4.2.33). That is, $u_\infty(t; \alpha^*, \chi^*, \lambda^*)$ is a solution to FIBVP (4.2.15), (4.2.16) if and only if (4.2.33) holds. □

Remark 4.2.2. *Theorem 4.2.4 gives necessary and sufficient conditions for the solvability of the FBVP (4.2.15), (4.2.16) and the construction of its solutions. However, a difficulty in its application arises from the fact that explicit forms of the exact functions $\Delta(\alpha, \chi, \lambda)$ and $V(\alpha, \chi, \lambda)$ are unknown. To overcome this difficulty, in practice we solve an approximate determining system*

$$\begin{cases} \Delta_j^m(\alpha, \chi, \lambda) = 0, \\ V_j^m(\alpha, \chi, \lambda) = 0, \end{cases} \tag{4.2.35}$$

which depends only on the $(m-1)$-th and m-th terms of the sequence (4.2.17), and can thus be constructed explicitly. In particular, the approximate functions $\Delta_j^m : \Omega \to \mathbb{R}$ and $V_j^m : \Omega \to \mathbb{R}$ are given by

$$\begin{aligned} \Delta_j^m(\alpha, \chi, \lambda) =& \frac{\Gamma(p+1)}{\lambda_1^p}(\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1) \\ &- \frac{p}{\lambda_1^p} \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s)^{p-1} f_j(s, u^m(s; \alpha, \chi, \lambda)) ds \end{aligned} \tag{4.2.36}$$

and

$$\begin{aligned} V_j^0(\alpha, \chi, \lambda) &= \frac{d}{dt} u_j^0(\lambda_j; \alpha, \chi, \lambda) - \chi_j = \lambda_j \chi_{j-1} + \frac{p}{\lambda_1^p}(\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1), \\ V_j^m(\alpha, \chi, \lambda) &= \frac{d}{dt} u_j^m(\lambda_j; \alpha, \chi, \lambda) - \chi_j = \lambda_j \chi_{j-1} + \frac{p}{\lambda_1^p}(\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1), \\ &+ \frac{1}{\Gamma(p)} \Bigg[(p-1) \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s)^{p-2} f_j(s, u^m(s; \alpha, \chi, \lambda)) ds \\ &- \frac{p}{\lambda_1} \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s)^{p-1} f_j(s, u^m(s; \alpha, \chi, \lambda)) ds \Bigg]. \end{aligned} \tag{4.2.37}$$

4.2.3. SOLVABILITY ANALYSIS

In this section we analyze the solvability of the 'model-type' problems (4.2.15), (4.2.16), similarly to sub-section 3.1.5. We give necessary and sufficient conditions for the existence of a set of parameters $(\alpha^*, \chi^*, \lambda^*)$, which is a solution to the determining system and governs the behaviour of the exact solution to each FBVP (4.2.15), (4.2.16).

Sufficient conditions

Lemma 4.2.2. *Suppose the conditions of Theorem 4.2.2 are satisfied. Then for arbitrary $m \geq 1$ and $(\alpha, \chi, \lambda) \in \Omega$ for the exact and approximate determining functions $\Delta_j : \Omega \to \mathbb{R}$, $\Delta_j^m : \Omega \to \mathbb{R}$, $V_j : \Omega \to \mathbb{R}$, and $V_j^m : \Omega \to \mathbb{R}$, defined by (4.2.28), (4.2.36),* *(4.2.34), and (4.2.37), respectively, the following inequalities hold:*

$$
\begin{aligned}
|\Delta_j(\alpha,\chi,\lambda) - \Delta_j^m(\alpha,\chi,\lambda)| &\le \frac{Q_j^m M_j}{1-Q_j},\\
|V_j(\alpha,\chi,\lambda) - V_j^m(\alpha,\chi,\lambda)| &\le 2\frac{Q_j^m}{1-Q_j} M_j \lambda_1^{p-1},
\end{aligned}
\tag{4.2.38}
$$

where M_j, K_j, and Q_j are given in (4.2.19),(4.2.20), and (4.2.23).

Proof. Fix an arbitrary pair $(\alpha,\chi,\lambda) \in \Omega$. Then, by virtue of the Lipschitz condition (4.2.20) and the estimates (4.2.23) and (4.2.29), we have

$$
\begin{aligned}
|\Delta_j(\alpha,\chi,\lambda) - \Delta_j^m(\alpha,\chi,\lambda)| \le & \frac{K_j}{\lambda_1^p} \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s)^{p-1} |u_j^m(s;\alpha,\chi,\lambda) - u_j^\infty(s;\alpha,\chi,\lambda)| ds\\
\le & \frac{K_j p}{\lambda_1^p} \frac{\lambda_1^p}{2^{2p-1}\Gamma(p+1)} \frac{Q_j^m}{1-Q_j} M_j \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s)^{p-1} ds\\
\le & \frac{Q_j^m}{1-Q_j} M_j.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|V_j(\alpha,\chi,\lambda) - & V_j^m(\alpha,\chi,\lambda)|\\
\le & \frac{K_j}{\Gamma(p)} \Bigg[(p-1) \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s))^{p-1} |u_j^m(s;\alpha,\chi,\lambda) - u_j^\infty(s;\alpha,\chi,\lambda)| ds\\
& + \frac{p}{\lambda_1} \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s)^{p-1} |u_j^m(s;\alpha,\chi,\lambda) - u_j^\infty(s;\alpha,\chi,\lambda)| ds \Bigg]\\
\le & \frac{K_j}{\Gamma(p)} \frac{\lambda_1^p}{2^{2p-1}\Gamma(p+1)} \frac{Q_j^m}{1-Q_j} M_j \Bigg[(p-1) \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s)^{p-1} ds\\
& + \frac{p}{\lambda_1} \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s)^{p-1} ds \Bigg] \le 2\frac{Q_j^m}{1-Q_j} M_j \lambda_1^{p-1}.
\end{aligned}
$$

This proves the lemma. □

On the basis of the exact and approximate determining functions (4.2.28), (4.2.36), (4.2.34), and (4.2.37), we introduce the mappings $\Phi : \mathbb{R} \to \mathbb{R}$ and $\Phi_m : \mathbb{R} \to \mathbb{R}$, defined by

$$
\Phi_j(\alpha,\chi,\lambda) := \begin{pmatrix} \Phi_j^1(\alpha,\chi,\lambda) \\ \Phi_j^2(\alpha,\chi,\lambda) \end{pmatrix}, \tag{4.2.39}
$$

$$
\Phi_j^m(\alpha,\chi,\lambda) := \begin{pmatrix} \Phi_j^{1,m}(\alpha,\chi,\lambda) \\ \Phi_j^{2,m}(\alpha,\chi,\lambda) \end{pmatrix}, \tag{4.2.40}
$$

with $\Phi_j^1(\alpha,\chi,\lambda)$, $\Phi_j^2(\alpha,\chi,\lambda)$, $\Phi_j^{1,m}(\alpha,\chi,\lambda)$, and $\Phi^{2,m}(\alpha,\chi,\lambda)$, defined as $\Delta_j(\alpha,\chi,\lambda)$, $V_j(\alpha,\chi,\lambda)$, $\Delta_j^m(\alpha,\chi,\lambda)$, and $V_j^m(\alpha,\chi,\lambda)$, respectively.

The following results hold.

Theorem 4.2.5. *Suppose the conditions of Theorem 4.2.2 hold, and one can find an $m \geq 1$ and a set $\Omega \subset \mathbb{R}$, such that the following relation is true:*

$$|\Phi_j^m| \rhd_{\partial\Omega_j} \begin{pmatrix} Q_j^m M_j(1-Q_j)^{-1} \\ 2Q_j^m M_j(1-Q_j)^{-1}\lambda_1^p \end{pmatrix}, \tag{4.2.41}$$

where $\partial\Omega_j$ is the boundary of the set Ω_j, and the definition of the relation $\rhd$ is given in Definition 2.2.8. If the Brouwer degree of the mapping Φ_m satisfies

$$deg(\Phi_j^m, \Omega_j, 0) \neq 0, \tag{4.2.42}$$

then there exists a triple $(\alpha_j^*, \chi_j^*, \lambda^*) \in \Omega_j$, such that

$$u_j^*(t) = u_j^*(t; \alpha_j^*, \chi_j^*, \lambda^*) = \lim_{m\to\infty} u_j^m(t; \alpha_j^*, \chi_j^*, \lambda^*) \tag{4.2.43}$$

is the solution to the non-linear FIBVP (4.2.15)-(4.2.16), defined on $J^* := [\lambda_{j-1}^*, \lambda_j^*]$, which satisfies

$$u_j^*(\lambda_j^*) = \alpha_j^*. \tag{4.2.44}$$

Proof. We first show that the vector fields Φ and Φ_m are homotopic. Let us introduce the family of vector mappings for $\theta \in [0,1]$

$$P(\theta, \alpha, \chi, \lambda) = \Phi_j^m(\alpha, \chi, \lambda) + \theta[\Phi_j(\alpha, \chi, \lambda) - \Phi_j^m(\alpha, \chi, \lambda)], \ \ (\alpha, \chi, \lambda) \in \partial\Omega, \tag{4.2.45}$$

Then $P(\theta, \alpha, \chi, \lambda)$ is continuous for all $(\alpha, \chi, \lambda) \in \partial\Omega, \ \theta \in [0,1]$. We have

$$P(0, \alpha, \chi, \lambda) = \Phi_j^m(\alpha, \chi, \lambda), \ \ P(1, \alpha, \chi, \lambda) = \Phi_j(\alpha, \chi, \lambda)$$

and for any $(\alpha, \chi, \lambda) \in \partial\Omega$,

$$\begin{aligned} |P(\theta, \alpha, \chi, \lambda)| =& |\Phi_j^m(\alpha, \chi, \lambda) + \theta[\Phi_j(\alpha, \chi, \lambda) - \Phi_j^m(\alpha, \chi, \lambda)]| \\ \geq& |\Phi_j^m(\alpha, \chi, \lambda)| - |\Phi_j(\alpha, \chi, \lambda) - \Phi_j^m(\alpha, \chi, \lambda)|. \end{aligned} \tag{4.2.46}$$

From the other side, by virtue of (4.2.39), (4.2.40) and the relations in (4.2.38), we have

$$|\Phi_j(\alpha, \chi, \lambda) - \Phi_j^m(\alpha, \chi, \lambda)| \leq \begin{pmatrix} Q_j^m M_j(1-Q_j)^{-1} \\ 2Q_j^m M_j(1-Q_j)^{-1}\lambda_1^p, \end{pmatrix}. \tag{4.2.47}$$

From (4.2.41), (4.2.46), and (4.2.47) it follows that

$$|P(\theta, \alpha, \chi, \lambda)| \rhd_{\partial\Omega} 0, \ \ \theta \in [0,1],$$

which means that $P(\theta, \alpha, \chi, \lambda) \neq 0$ for all $\theta \in [0,1]$ and $(\alpha, \chi, \lambda) \in \Omega$, i.e. the mappings (4.2.45) are non-degenerate, and thus the vector fields Φ_j and Φ_j^m are homotopic. Since relation (4.2.42) holds and the Brouwer degree is preserved under homotopies, it follows that

$$\deg(\Phi_j, \Omega, 0) = \deg(\Phi_j^m, \Omega, 0) \neq 0,$$

which implies that there exists $(\alpha_j^*, \chi_j^*, \lambda^*) \in \Omega$ such that $\Phi_j(\alpha_j^*, \chi_j^*, \lambda^*) = 0$ by the classical topological result in [70].

Hence, the triple $(\alpha_j^*, \chi_j^*, \lambda^*)$ satisfies the determining system (4.2.33).

By Theorem 4.2.4 it follows that the function defined in (4.2.43) is a solution to the FBVP (4.2.15),(4.2.16) and satisfies (4.2.44). □

Lemma 4.2.3. *Suppose that the conditions of Theorem 4.2.2 are satisfied for a FIBVP (4.2.15)-(4.2.16) with parameter-dependent boundary conditions. Then, for arbitrary pairs*

$$(\alpha', \chi', \lambda'),\ (\alpha'', \chi'', \lambda'') \in \Omega,$$

the limit functions $u_j^\infty(t;\alpha',\chi',\lambda')$ *and* $u_j^\infty(t;\alpha'',\chi'',\lambda'')$ *of the sequences* $u_j^m(t;\alpha',\chi',\lambda')$ *and* $u_j^m(t;\alpha'',\chi'',\lambda'')$ *of the form (4.2.17) satisfy the inequality*

$$|u_j^\infty(t;\alpha',\chi',\lambda') - u_j^\infty(t;\alpha'',\chi'',\lambda'')| \leq \frac{1}{1-Q_j}\Big(L_j + \frac{4M_j(\lambda_1')^p}{\Gamma(p+1)}\Big), \tag{4.2.48}$$

where Q *is defined in (4.2.23), and*

$$\begin{aligned} L_j :=& L_j(\alpha'_{j-1,j}, \chi'_{j-1}, \lambda', \alpha''_{j-1,j}, \chi''_{j-1}, \lambda'') \\ =& |\alpha'_{j-1} - \alpha''_{j-1}| + \lambda'|\chi'_{j-1} - \chi''_{j-1}| \\ &+ \gamma^p(|\alpha'_j - \alpha'_{j-1} - \chi'_{j-1}\lambda'_1| + |\alpha''_j - \alpha''_{j-1} - \chi''_{j-1}\lambda''_1|), \\ \lambda_{max} :=& \max(\lambda', \lambda''). \end{aligned}$$

Proof. Let us first estimate the difference $|u_j^m(t;\alpha',\chi',\lambda') - u_j^m(t;\alpha'',\chi'',\lambda'')|$. Consider first $m = 0$:

$$\begin{aligned} |u_j^0(t;\alpha',\chi',\lambda') - u_j^0(t;\alpha',\chi',\lambda'')| \leq& |\alpha'_{j-1} - \alpha''_{j-1}| + t|\chi'_{j-1} - \chi''_{j-1}| \\ &+ \Big(\frac{t}{\lambda'\lambda''_1}\Big)^p[|\alpha'_j - \alpha'_{j-1} - \chi'_{j-1}\lambda'_1|(\lambda'')^p + |\alpha''_j - \alpha''_{j-1} - \chi''_{j-1}\lambda''_1|(\lambda')^p]. \end{aligned}$$

Assume without loss of generality that $\lambda_{max} = \lambda'$. Then

$$\begin{aligned} |u_j^0(t;\alpha',\chi',\lambda') - u_j^0(t;\alpha'',\chi'',\lambda'')| \leq& |\alpha'_{j-1} - \alpha''_{j-1}| + \lambda'|\chi'_{j-1} - \chi''_{j-1}| \\ &+ \Big(\frac{\lambda'}{\lambda''_1}\Big)^p(|\alpha'_j - \alpha'_{j-1} - \chi'_{j-1}\lambda'_1| + |\alpha''_j - \alpha''_{j-1} - \chi''_{j-1}\lambda''_1|) = L_j. \end{aligned} \tag{4.2.49}$$

Next, using (4.2.19), (4.2.20), (4.2.49), and the results of Lemmas 1 and 2 in [73], we obtain for $m = 1$:

$$\begin{aligned} |u_j^1(t;\alpha',\chi',\lambda') - u_j^1(t;\alpha'',\chi'',\lambda'')| \leq& |u_j^0(t;\alpha',\chi',\lambda') - u_j^0(t;\alpha'',\chi'',\lambda'')| \\ &+\frac{K_j}{\Gamma(p)}\int_{\lambda'_{j-1}}^t (t-s)^{p-1}|u^0(s;\alpha',\chi',\lambda') - u^0(s;\alpha'',\chi'',\lambda'')|ds \\ &-\Big(\frac{t-\lambda'_{j-1}}{\lambda'_1}\Big)^p\int_{\lambda'_{j-1}}^{\lambda'_{j-1}}(\lambda'_j - s)^{p-1}|u^0(s;\alpha',\chi',\lambda') - u^0(s;\alpha'',\chi'',\lambda'')|ds \\ &+\frac{M_j}{\Gamma(p)}\Bigg[\Big(\frac{t-\lambda'_{j-1}}{\lambda''_1}\Big)^p\int_{\lambda''_{j-1}}^{\lambda''_j}(\lambda''_j - s)^{p-1}ds \\ &+\Big(\frac{\lambda''_{j-1} - t}{\lambda'_1}\Big)^p\int_{\lambda'_{j-1}}^{\lambda'_j}(\lambda'_j - s)^{p-1}ds + \int_{\lambda''_{j-1}}^{\lambda'_{j-1}}(t-s)^{p-1}ds\Bigg] \\ \leq& L_j + L_jK_j\frac{(\lambda'_1)^p}{2^{2p-1}\Gamma(p+1)} + \frac{4M_j(\lambda'_1)^p}{\Gamma(p+1)} = L_j + L_jQ_j + \frac{4M_j(\lambda'_1)^p}{\Gamma(p+1)}. \end{aligned}$$

We will use induction to show that the following estimate holds for m:

$$|u_j^m(t;\alpha',\chi',\lambda') - u_j^m(t;\alpha'',\chi'',\lambda'')| \le L_j \sum_{i=0}^{m} Q_j^i + \frac{4M_j(\lambda_1')^p}{\Gamma(p+1)} \sum_{i=0}^{m-1} Q_j^i. \tag{4.2.50}$$

Assume that (4.2.50) holds for $m-1$, i.e.

$$|u_j^{m-1}(t;\alpha',\chi',\lambda') - u_j^{m-1}(t;\alpha'',\chi'',\lambda'')| \le L_j \sum_{i=0}^{m-1} Q_j^i + \frac{4M_j(\lambda_1')^p}{\Gamma(p+1)} \sum_{i=0}^{m-2} Q_j^i,$$

and consider

$$\begin{aligned}
&|u_j^m(t;\alpha',\chi',\lambda') - u_j^m(t;\alpha'',\chi'',\lambda'')| \le |u_j^0(t;\alpha',\chi',\lambda') - u_j^0(t;\alpha'',\chi'',\lambda'')| \\
&\quad + \frac{K_j}{\Gamma(p)} \int_{\lambda_{j-1}'}^{t} (t-s)^{p-1} |u^{m-1}(s;\alpha',\chi',\lambda') - u^{m-1}(s;\alpha'',\chi'',\lambda'')| ds \\
&\quad - \Big(\frac{t-\lambda_{j-1}'}{\lambda_1'}\Big)^p \int_{\lambda_{j-1}'}^{\lambda_j'} (\lambda_j' - s)^{p-1} |u^{m-1}(s;\alpha',\chi',\lambda') - u^{m-1}(s;\alpha'',\chi'',\lambda'')| ds \\
&\quad + \frac{M_j}{\Gamma(p)} \Bigg[\Big(\frac{t-\lambda_{j-1}''}{\lambda_1''}\Big)^p \int_{\lambda_{j-1}''}^{\lambda_j''} (\lambda_j'' - s)^{p-1} ds \\
&\quad + \Big(\frac{t-\lambda_{j-1}'}{\lambda_1'}\Big)^p \int_{\lambda_{j-1}'}^{\lambda_j'} (\lambda_j' - s)^{p-1} ds + \int_{\lambda_{j-1}''}^{\lambda_{j-1}'} (t-s)^{p-1} ds \Bigg] \\
&\le L_j + \Bigg[L_j \sum_{i=0}^{m-1} Q_j^i + \frac{4M_j(\lambda_1')^p)}{\Gamma(p+1)} \sum_{i=0}^{m-2} Q_j^i \Bigg] K_j \frac{(\lambda_1')^p}{2^{2p-1}\Gamma(p+1)} + \frac{4M_j(\lambda')^p)}{\Gamma(p+1)N^p} \\
&= L_j \sum_{i=0}^{m} Q_j^i + \frac{4M_j(\lambda_1')^p}{\Gamma(p+1)} \sum_{i=0}^{m-1} Q_j^i,
\end{aligned}$$

that is, (4.2.50) holds. Passing to the limit $m \to \infty$ in (4.2.50) and using (4.2.23) yields (4.2.48), as required.

□

Necessary conditions

Lemma 4.2.4. *Suppose the conditions of Theorem 4.2.2 are satisfied. Then the functions $\Delta : \Omega \to \mathbb{R}$ and $V : \Omega \to \mathbb{R}$ satisfy the following estimates for arbitrary pairs (α',χ',λ'), $(\alpha'',\chi'',\lambda'') \in \Omega$:*

$$\begin{aligned}
|\Delta_j(\alpha',\chi',\lambda') - \Delta_j(\alpha'',\chi'',\lambda'')| \le &\frac{\Gamma(p+1)}{(\lambda_1'')^p} (|\alpha_{j-1}' - \alpha_j' - \chi_{j-1}'\lambda_1'| \\
&+ |\alpha_{j-1}'' - \alpha_j'' - \chi_{j-1}''\lambda_1''|) \\
&+ \frac{2K_j}{1-Q_j} \Big(L_j + \frac{4M_j(\lambda_1')^p}{\Gamma(p+1)} \Big) + 2M_j
\end{aligned} \tag{4.2.51}$$

and

$$
\begin{aligned}
|V_j(\alpha',\chi',\lambda')) - V_j(\alpha'',\chi'',\lambda'')| \le & \frac{j}{N}|\lambda'\chi'_{j-1} - \lambda''\chi''_{j-1}| \\
& + \frac{p}{\lambda''_1}(|\alpha'_j - \alpha'_{j-1} - \chi'_{j-1}\lambda'_1| + |\alpha''_j - \alpha''_{j-1} - \chi''_{j-1}\lambda''_1|) \\
& + \frac{4K_j(\lambda'_1)^{p-1}}{1-Q_j}\Big(L_j + \frac{4M_j(\lambda'_1)^p}{\Gamma(p+1)}\Big) + 4(\lambda'_1)^{p-1}M_j
\end{aligned} \tag{4.2.52}
$$

Proof. By virtue of the definition of $\Delta(z,\lambda)$ in (4.2.28), the boundedness and Lipschitz-continuity of $f(t,u(t))$ (4.2.19), (4.2.20), and the estimate in Lemma 4.2.3, we obtain

$$
\begin{aligned}
& |\Delta_j(\alpha',\chi',\lambda') - \Delta_j(\alpha'',\chi'',\lambda'')| \\
& \le \Gamma(p+1)\left[\frac{|\alpha'_{j-1} - \alpha'_j - \chi'_{j-1}\lambda'_1|}{(\lambda'_1)^p} + \frac{|\alpha''_{j-1} - \alpha''_j - \chi''_{j-1}\lambda''_1|}{(\lambda''_1)^p}\right] \\
& + p\Bigg[\left|\frac{1}{(\lambda'_1)^p}\int_{\lambda'_{j-1}}^{\lambda'_j}(\lambda'_j - s)^{p-1}[f_j(s,u_j^\infty(s;\alpha',\chi',\lambda')) - f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))]ds\right| \\
& \quad + \left|\frac{1}{(\lambda''_1)^p}\int_{\lambda''_{j-1}}^{\lambda''_j}(\lambda''_j - s)^{p-1}[f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda'')) - f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))]ds\right| \\
& \quad + \left|\frac{1}{(\lambda'_1)^p}\int_{\lambda'_{j-1}}^{\lambda'_j}(\lambda'_j - s)^{p-1}f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))ds\right| \\
& \quad + \left|\frac{1}{(\lambda'_1)^p}\int_{\lambda''_{j-1}}^{\lambda''_j}(\lambda''_j - s)^{p-1}f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))ds\right|\Bigg] \\
& \quad \le \frac{\Gamma(p+1)}{(\lambda''_1)^p}(|\alpha'_{j-1} - \alpha'_j - \chi'_{j-1}\lambda'_1| + |\alpha''_{j-1} - \alpha''_j - \chi''_{j-1}\lambda''_1|) \\
& \quad + \frac{2K_j}{1-Q_j}\Big(L_j + \frac{4M_j(\lambda'_1)^p}{\Gamma(p+1)}\Big) + 2M_j,
\end{aligned}
$$

as required in (4.2.51).

Now, from the definition of $V(\alpha,\chi,\lambda)$, (4.2.34), we derive:

$$
\begin{aligned}
& |V_j(\alpha',\chi',\lambda') - V_j(\alpha'',\chi'',\lambda'')| \\
& \le |j\lambda'_1\chi'_{j-1} + \frac{p}{\lambda'_1}(\alpha'_j - \alpha'_{j-1} - \chi'_{j-1}\lambda'_1) - j\lambda''_2\chi''_{j-1} - \frac{p}{\lambda''_1}(\alpha''_j - \alpha''_{j-1} - \chi''_{j-1}\lambda''_1)| \\
& \quad + \frac{1}{\Gamma(p)}\Bigg[(p-1)\left|\int_{\lambda'_{j-1}}^{\lambda'_j}(\lambda'_j - s)^{p-2}[f_j(s,u_j^\infty(s;\alpha',\chi',\lambda')) - f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))]\right| \\
& \quad + \frac{p}{\lambda'_1}\left|\int_{\lambda'_{j-1}}^{\lambda'_j}(\lambda'_j - s)^{p-2}[f_j(s,u_j^\infty(s;\alpha',\chi',\lambda')) - f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))]\right|
\end{aligned}
$$

$$
\begin{aligned}
&+ (p-1)\left| \int_{\lambda''_{j-1}}^{\lambda''_j} (\lambda''_j - s)^{p-2}[f_j(s, u_j^\infty(s;\alpha'',\chi'',\lambda'')) - f_j(s, u_j^\infty(s;\alpha',\chi',\lambda'))] \right| \\
&+ \frac{p}{\lambda''_1}\left| \int_{\lambda''_{j-1}}^{\lambda''_j} (\lambda''_j - s)^{p-2}[f_j(s, u_j^\infty(s;\alpha'',\chi'',\lambda'')) - f_j(s, u_j^\infty(s;\alpha',\chi',\lambda'))] \right| \\
&+ (p-1)\left| \int_{\lambda'_{j-1}}^{\lambda'_j} (\lambda'_j - s)^{p-2} f_j(s, u_j^\infty(s;\alpha',\chi',\lambda'))ds \right| \\
&+ \frac{p}{\lambda'_1}\left| \int_{\lambda'_{j-1}}^{\lambda'_j} (\lambda'_j - s)^{p-2}[f_j(s, u_j^\infty(s;\alpha',\chi',\lambda'))ds \right| \\
&+ (p-1)\left| \int_{\lambda''_{j-1}}^{\lambda''_j} (\lambda''_j - s)^{p-2} f_j(s, u_j^\infty(s;\alpha'',\chi'',\lambda''))ds \right| \\
&+ \frac{p}{\lambda''_1}\left| \int_{\lambda''_{j-1}}^{\lambda''_j} (\lambda''_j - s)^{p-2}[f_j(s, u_j^\infty(s;\alpha'',\chi'',\lambda''))ds \right| \Bigg] \\
&\le \frac{j}{N}|\lambda'\chi'_{j-1} - \lambda''\chi''_{j-1}| + \frac{p}{\lambda''_1}(|\alpha'_j - \alpha'_{j-1} - \chi'_{j-1}\lambda'_1| + |\alpha''_j - \alpha''_{j-1} - \chi''_{j-1}\lambda''_1|) \\
&+ \frac{4K_j(\lambda'_1)^{p-1}}{1-Q_j}\Big(L_j + \frac{4M_j(\lambda'_1)^p}{\Gamma(p+1)}\Big) + 4(\lambda'_1)^{p-1}M_j
\end{aligned}
$$

This proves the lemma. □

Theorem 4.2.6. *Suppose the conditions of Theorem 4.2.2 are satisfied. Then in order for the domain Ω to contain a pair of parameters $(\alpha^*, \chi^*, \lambda^*)$, it is necessary that for all $m \ge 1$, $(\tilde\alpha, \tilde\chi, \tilde\lambda) \in \Omega$. the following inequalities hold:*

$$
\begin{aligned}
|\Delta_j^m(\tilde\alpha,\tilde\chi,\tilde\lambda)| \le \sup_{(\alpha,\chi,\lambda)\in\Omega} &\Bigg\{ \frac{\Gamma(p+1)}{\lambda_1^p}(|\tilde\alpha_j - \tilde\alpha_{j-1} - \tilde\chi_{j-1}\tilde\lambda_1| + |\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1|) \\
&+ \frac{2K_j}{1-Q_j}\Big(\tilde L_{j,2} + \frac{4M_j(\tilde\lambda_1)^p}{\Gamma(p+1)}\Big)\Bigg\} + 2M_j + \frac{Q_j^m M_j}{1-Q_j},
\end{aligned}
\tag{4.2.53}
$$

$$
\begin{aligned}
|V_j^m(\tilde\alpha,\tilde\chi,\tilde\lambda)| \le \sup_{(\alpha,\chi,\lambda)\in\Omega} &\Bigg\{ \frac{j}{N}|\tilde\lambda\tilde\chi_{j-1} - \lambda\chi_{j-1}| \\
&+ \frac{p}{\lambda_1}(|\tilde\alpha_j - \tilde\alpha_{j-1} - \tilde\chi_{j-1}\tilde\lambda_1| + |\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1|) \\
&+ \frac{4K_j(\tilde\lambda_1)^{p-1}}{1-Q_j}\Big(\tilde L_j + \frac{4M_j(\tilde\lambda_1)^p}{\Gamma(p+1)}\Big) + 4(\tilde\lambda_1)^{p-1}M_j\Bigg\} \\
&+ 2\frac{Q_j^m}{1-Q_j}M_j(\tilde\lambda_1)^{p-1},
\end{aligned}
\tag{4.2.54}
$$

where

$$
\tilde L_j := L_j(\tilde\alpha_{j-1,j}, \tilde\chi_{j-1}, \tilde\lambda, \alpha^*_{j-1,j}, \chi^*_{j-1}, \lambda^*),
$$

$$\tilde{L}_{j,2} := L_j(\tilde{\alpha}_{j-1,j}, \tilde{\chi}_{j-1}, \tilde{\lambda}, \alpha_{j-1,j}, \chi_{j-1}, \lambda).$$

Proof. Assume that the determining functions vanish at $\alpha = \alpha^*, \chi = \chi^*, \lambda = \lambda^*$, that is, $\Delta_j(\alpha^*, \chi^*, \lambda^*) = 0$ and $V_j(\alpha^*, \chi^*, \lambda^*) = 0$. Applying Lemma 4.2.4 with $(\alpha', \chi', \lambda') = (\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda})$ and $(\alpha'', \chi'', \lambda'') = (\alpha^*, \chi^*, \lambda^*)$ yields

$$\begin{aligned}
|\Delta_j(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda}) - \Delta_j(\alpha^*, \chi^*, \lambda^*)| &= |\Delta_j(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda})| \\
&\leq \frac{\Gamma(p+1)}{(\lambda_1^*)^p}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| \\
&\quad + |\alpha_j^* - \alpha_{j-1}^* - \chi_{j-1}^*\lambda_1^*|) + \frac{2K_j}{1-Q_j}\left(\tilde{L}_j + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) + 2M_j.
\end{aligned}$$

From Lemma 4.2.2 we know that

$$|\Delta_j(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda}) - \Delta_j^m(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda})| \leq \frac{Q_j^m M_j}{1-Q_j}.$$

Hence,

$$\begin{aligned}
|\Delta_j^m(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda})| \leq& |\Delta_j(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda})| + |\Delta_j^m(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda}) - \Delta_j(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda})| \\
\leq& \frac{\Gamma(p+1)}{(\lambda_1^*)^p}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| + |\alpha_j^* - \alpha_{j-1}^* - \chi_{j-1}^*\lambda_1^*|) \\
&+ \frac{2K_j}{1-Q_j}\left(\tilde{L}_j + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) + 2M_j + \frac{Q_j^m M_j}{1-Q_j} \\
\leq& \sup_{(\alpha,\chi,\lambda)\in\Omega} \left\{ \frac{\Gamma(p+1)}{\lambda_1}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| + |\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1|) \right. \\
&\left. + \frac{2K_j}{1-Q_j}\left(\tilde{L}_{j,2} + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) \right\} + 2M_j + \frac{Q_j^m M_j}{1-Q_j},
\end{aligned}$$

as stated in (4.2.53). Applying again Lemma 4.2.4, now to $V_j(\alpha, \chi, \lambda)$, we have

$$\begin{aligned}
|V_j(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda}) - V_j(\alpha^*, \chi^*, \lambda^*)| =& |V_j(\tilde{\alpha}, \tilde{\chi}\tilde{\lambda})| \\
\leq& \frac{j}{N}|\tilde{\lambda}\tilde{\chi}_{j-1} - \lambda^*\chi_{j-1}^*| + \frac{p}{\lambda_1^*}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| \\
&+ |\alpha_j^* - \alpha_{j-1}^* - \chi_{j-1}^*\lambda_1^*|) \\
&+ \frac{4K_j(\tilde{\lambda}_1)^{p-1}}{1-Q_j}\left(\tilde{L}_j + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) + 4(\tilde{\lambda})^{p-1}M_j
\end{aligned}$$

From Lemma 4.2.2 we know that

$$|V_j(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda}) - V_j^m(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda})| \leq 2\frac{Q_j^m}{1-Q_j}M_j(\tilde{\lambda})^{p-1},$$

thus, combining the two yields

$$|V_j^m(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda})| \leq |V_j(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda})| + |V_j^m(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda}) - V_j(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda})|$$

$$
\begin{aligned}
&\leq \frac{j}{N}|\tilde{\lambda}\tilde{\chi}_{j-1} - \lambda^*\chi^*_{j-1}| + \frac{p}{\lambda^*_1}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| + |\alpha^*_j - \alpha^*_{j-1} - \chi^*_{j-1}\lambda^*_1|) \\
&\quad + \frac{4K_j(\tilde{\lambda}_1)^{p-1}}{1-Q_j}\left(\tilde{L}_j + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) + 4(\tilde{\lambda}_1)^{p-1}M_j + 2\frac{Q_j^m}{1-Q_j}M_j(\tilde{\lambda}_1)^{p-1} \\
&\leq \sup_{(\alpha,\chi,\lambda)\in\Omega}\Bigg\{\frac{j}{N}|\tilde{\lambda}\tilde{\chi}_{j-1} - \lambda\chi_{j-1}| + \frac{p}{\lambda_1}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| + |\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1|) \\
&\quad + \frac{4K_j(\tilde{\lambda}_1)^{p-1}}{1-Q_j}\left(\tilde{L}_j + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) + 4(\tilde{\lambda}_1)^{p-1}M_j\Bigg\} + 2\frac{Q_j^m}{1-Q_j}M_j(\tilde{\lambda}_1)^{p-1}.
\end{aligned}
$$

This proves the theorem. □

The algorithm of approximate search for the set of $2N$ parameters $(\alpha^*_1, ..., \alpha^*_N, \chi^*_1, ..., \chi^*_{N-1}, \lambda^*)$, which define the solution $u(\cdot)$ of the FBVP (4.2.11)-(4.2.13) follows the line of the algorithm described in Remark 3.1.1.

Theorem 4.2.7. *Suppose the conditions of Theorem 4.2.2 hold and the pair $(\alpha^*, \chi^*, \lambda^*) \in \Omega$ is a solution to the exact determining system (4.2.33), and $(\tilde{\alpha}, \tilde{\chi}, \tilde{\lambda})$ is an arbitrary point in $\Omega_{m,N}$. Then the following estimate holds*

$$
\begin{aligned}
|u_j^\infty(t;\alpha^*,\chi^*,\lambda^*) - u_j^m(t;\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| &\leq \frac{(\lambda_1^*)^p}{2^{2p-1}\Gamma(p+1)}\frac{Q_j^m}{1-Q_j}M_j \\
&+ \sup_{(\alpha,\chi,\lambda)\in\Omega_{m,N}}\left[\tilde{L}_j\sum_{i=0}^{m}Q_j^i + \frac{4M_j(\lambda_1)^p)}{\Gamma(p+1)}\sum_{i=0}^{m-1}Q_j^i\right].
\end{aligned}
\tag{4.2.55}
$$

Proof. From the estimates in (4.2.29) and (4.2.50) we have

$$
\begin{aligned}
|u_j^\infty(t;\alpha^*,\chi^*,\lambda^*) - u_j^m(t;\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| &\leq |u_j^\infty(t;\alpha^*,\chi^*,\lambda^*) - u_j^m(t;\alpha^*,\chi^*,\lambda^*)| \\
&\quad + |u_j^m(t;\alpha^*,\chi^*,\lambda^*) - u_j^m(t;\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| \\
&\leq \frac{(\lambda_1^*)^p}{2^{2p-1}\Gamma(p+1)}\frac{Q_j^m}{1-Qj}M_j + L_j\sum_{i=0}^{m}Q_j^i + \frac{4M_j(\lambda_1')^p}{\Gamma(p+1)}\sum_{i=0}^{m-1}Q_j^i \\
&\leq \frac{(\lambda_1^*)^p}{2^{2p-1}\Gamma(p+1)}\frac{Q_j^m}{1-Q_j}M_j \\
&\quad + \sup_{(\alpha,\chi,\lambda)\in\Omega_{m,N}}\left[\tilde{L}_j\sum_{i=0}^{m}Q_j^i + \frac{4M_j(\lambda_1)^p}{\Gamma(p+1)}\sum_{i=0}^{m-1}Q_j^i\right].
\end{aligned}
$$

This proves the theorem. □

4.2.4. EXAMPLE

In this section we apply the numerical-analytic method from Section 4.2.2 to a model example in a finite interval setting.

Consider the BVP

$$
{}_0^C D_t^p u(t) = \frac{1}{(\cosh t)^2}F(u(t)) - \frac{2\omega\sinh t}{(\cosh t)^3} \quad (:= f(t,u(t))), \quad t\in[0,\lambda],
\tag{4.2.56}
$$

where the right end point of the interval of definition, λ, is unknown. We take $p = 3/2$ and parameter-dependent boundary conditions

$$\begin{aligned} u(0) &= 1000, \quad u'(0) = 1500, \\ u(0) + u(\lambda) &+ u'(\lambda) = 1000, \end{aligned} \tag{4.2.57}$$

i.e. $A = B = C = 1$, $d = 1000$, $\alpha_0 = 1000$, $\chi_0 = 1500$, and ω is given.

In the case of the second order derivative, Equation (4.2.56), coupled with asymptotic conditions of the type (4.2.2), is derived as a mathematical model of Arctic gyres with a vanishing azimuthal velocity and oceanic vorticity $F(u(t)) = \sin(u(t))/10$. Then $\omega = 4649.56$ is taken as the dimensionless Coriolis parameter, [47].

For simplicity of computations we construct an approximating sequence directly on the entire interval $[0, \lambda]$, however, in principle it is possible to apply the interval splitting method.

We calculated the set D for FIBVP (4.2.56), (4.2.57) to be

$$D := \{-29073.12 \leq u(t) \leq 33187.76\}, \quad t \in [0, \lambda],$$

on which the right-hand side function $f(t, u(t))$ satisfies the Lipschitz condition (4.2.20) with constant $K = 0.1$.

We denote the value of $u(t)$ at $t = \lambda$ by α_1, i.e.

$$u(\lambda) = \alpha_1,$$

which allows us to express the remaining unknown boundary value, $u'(\lambda)$, in terms of the given boundary conditions and the parameter α_1:

$$u'(\lambda) = d - \alpha_0 - \alpha_1 - \chi_0.$$

Implementing (4.2.17) and solving the corresponding system of approximate determining equations for 6 iterations yields the parameter values shown in Table 4.2.

Table 4.2: Computed parameter values of α_1^m and λ_m for $m = 0, ..., 5$.

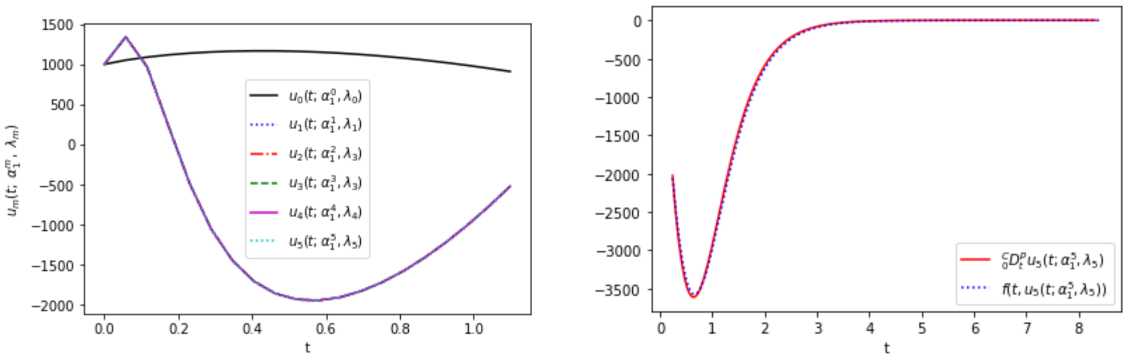
m	α_1^m	λ_m
0	909.162	0.856
1	−546.101	8.602
2	−546.101	8.602
3	−546.101	8.602
4	−546.101	8.602
5	−546.101	8.602

It is clear from these calculations that the values we obtain for α_1^m and λ_m converge. With the computed value of $\lambda \approx 8.6$ we find that $Q \approx 0.47$, i.e. the inequality in (4.2.23) is satisfied, which guarantees the convergence of the approximating sequence. Plots of the first 6 terms of the sequence are shown in the left panel of Figure 4.5a.

Figure 4.5b shows a comparison between the left- and right-hand sides of Equation (4.2.56) with $u_5(t;\alpha_1^5,\lambda_5)$ plugged in. From this comparison we see that the left- and right-hand sides of Equation (4.2.56) are in good agreement for $m=5$. The error between the last two iterations, defined as

$$E = |u_5(t;\alpha_1^5,\lambda_5) - u_4(t;\alpha_1^4,\lambda_4)|, \tag{4.2.58}$$

is $E = 1.7\times 10^{-9}$. If necessary, the iteration process can be continued until the desired precision of computation is obtained.



(a) Plots of the approximate solutions $u_m(t;\alpha_1^m,\lambda_m)$ of FBVP (4.2.56), (4.2.57) for $m=\{0,...,5\}$.

(b) Comparison plots between the left- (solid red line) and right- (dotted blue line) hand sides of FDE (4.2.56) with $u_5(t,\alpha_1^5,\lambda_5)$.

Figure 4.5: Approximate solutions and comparison plots for FBVP (4.2.56), (4.2.57).

4.3. CONCLUSION

The parametrization technique and dichotomy approach introduced in Sections 4.1 and 4.2 build on the results from Chapter 3, expanding the applicability of the numerical-analytic method to FBVPs with more complex boundary conditions and problems defined on intervals of arbitrary length. Through an appropriately chosen parametrization, integral boundary conditions of a general form and multi-point boundary conditions - potentially involving problems defined on intervals on unknown lengths - can be reduced to two-point linear conditions. This reformulation enables their integration into the construction of a closed form approximating sequence. The method is applicable to settings where the initial conditions of the system are unknown and generalizes naturally to accommodate a wide variety of boundary condition types.

The dichotomy approach further broadens the scope of applicability of the numerical-analytic method by decomposing the original FBVP into a ‘model-type’ sub-problems, each defined over a smaller sub-interval. Since the convergence of the technique is dependent on the length of the domain, this interval-splitting framework makes the method applicable to problems where convergence cannot be guaranteed over the full interval. The interval can be partitioned into an arbitrary number of subintervals — as many as necessary to ensure convergence—as detailed in Section 4.2. Moreover, this strategy enables the examination of solution behavior over extended intervals, facilitating long-term analysis. The approach also leads to improved convergence rates, as evidenced by the derived error estimates. Combined with an appropriate parametrization, the numerical-analytic method demonstrates notable flexibility and robustness in handling a broad spectrum of boundary condition formulations.

The theoretical results are supported by illustrative examples involving the equations governing the Antarctic Circumpolar Current and the Arctic gyre, considered here in a fractional-order setting. The use of fractional derivatives in these examples does not carry a direct physical interpretation; they serve to illustrate the applicability of the proposed method. In the absence of analytical solutions, convergence is verified by ensuring that the approximate solutions satisfy the original equations at each iteration step.

5
CONCLUSION

The modeling of complex systems and processes often requires solving non-linear problems, for which exact solutions are unattainable. This motivates the development of methods for constructing approximate solutions. The numerical-analytic method discussed in this thesis provides a tool for both the qualitative analysis of the existence and uniqueness in FBVPs, and for the construction of closed form approximations. Since the approximating sequences are derived analytically, the method requires only the numerical solution of a system of algebraic equations. The technique accommodates various types of boundary constraints without requiring pre-knowledge of the initial conditions of the system. Moreover, the range of its applicability can be extended to a wide class of problems using a dichotomy-type approach.

In the present work we first address the problem of solvability analysis and approximation of solutions of FBVPs with two-point and integral boundary conditions (**P1**). The numerical analytic technique is extended to the fractional case for a system of non-linear Caputo ODEs, subject to the Dirichlet boundary conditions, constructing a sequence of approximating functions. We prove the uniform convergence of the sequence to a limit function, which is the exact solution to the IVP for the modified system of equations. Additionally, we establish necessary and sufficient conditions for the limit function to satisfy the original BVP.

To analyze the solvability of the problem, we employ topological degree theory and prove the existence of a vector parameter governing the behavior of the FBVP solutions, along with a bound on the approximate determining function. The technique is applied to the equation modelling the motion of a gyre in the Southern hemisphere in the fractional setting. The approximate determining equation is solved numerically to obtain values of the unknown parameter, which are used to calculate the terms of the sequence. To verify the validity of the constructed approximations, we have checked how well they satisfy the original FDE.

The developed technique and existence results are further extended and applied to more complex problem settings. In particular, we introduce a novel approach for constructing approximate solutions to systems of non-linear FDEs of a mixed real order, subject to integral boundary constraints. The novelty consists in extending the applicability of the studied method to systems of FDEs with this special class of non-local constraints. By employing a parametrization technique, the given boundary restrictions are transformed into two-point linear boundary conditions, which can be easily incorporated in the approximating sequence.

A dichotomy-type approach is then applied to transform the original BVP into two ‘model-type’ BVPs, each defined on an interval with half the length of the original problem. This reduces the error estimate of the method, or can be applied to problems for which the approximation technique does not converge on the entire interval. The numerical-analytic technique is used for constructing sequences of approximations and analyzing the existence and uniqueness of solutions to the modified BVPs. The connection between the original BVP and the two ‘model-type’ problems is established.

Next, we study FBVPs with parameter-dependent right-hand side and focus on the construction of approximations to their solutions and the analysis of their monotonicity behavior (**P2**). Here, the parameter in the right-hand side governs the effect of the non-linearity and determines the monotonicity of the function. Fixed point theory is used to determine a range of values of the right-hand side parameter for which there exists a unique solution to the BVP. The numerical-analytic technique is applied to construct a sequence of approximate solutions, and their monotonicity behavior is investigated. The approximation terms form a well-ordered sequence when the right-hand side in the FDE is strictly decreasing, whereas for a strictly increasing function, the approximating sequence is alternating. In the latter case, we apply the lower and upper solutions method in conjunction with the numerical-analytic technique to construct sequences of approximations, and prove their uniform convergence to the exact solution of the FBVP. This approach can be used to simplify the terms of the approximating sequence and to therefore reduce the computational time.

Lastly, we consider a FBVP, subject to a parameter-dependent boundary condition, defined on an interval of unknown length, and a FBVP with a boundary condition at infinity (**P3**). We establish conditions for the existence of bounded solutions to FBVPs, defined on the half axis with asymptotic constraints. The numerical-analytic method is adopted to enable the analysis of solutions to FBVPs with parameter-dependent boundary conditions over intervals of arbitrary length, and the construction of approximations to their solutions. The dichotomy type approach, presented in the setting of integral boundary constraints, is generalized to accommodate problems defined on domains of arbitrary length. Moreover, it allows for extending the interval of definition of the problem, making it well-suited for investigating the long-term or asymptotic behavior of solutions to BVPs of fractional order.

Future work may include stability and/or bifurcation analysis of the solutions to non-linear FBVPs. The developed numerical-analytic techniques may be further extended to more general classes of FBVPs, including those with variable-order derivatives, and to FDEs containing more than one derivative term. Additionally, exploring the application of these methods to systems governed by fractional PDEs could provide further insights into complex physical and engineering phenomena. In particular, the numerical-analytic method can be adapted for fractional PDEs through techniques such as semi-discretization, where the spatial variables are discretized while the resulting system of fractional ordinary differential equations in time is treated using the existing framework. This approach preserves the advantages of the method while extending its applicability to a broader range of systems.

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SUMMARY

Fractional differential equations (FDEs) have attracted considerable attention in recent years due to their ability to model complex dynamical systems characterized by memory and hereditary effects—features that are inadequately captured by classical integer-order differential equations. The distinguishing aspect of FDEs lies in their use of non-local operators, which inherently account for the history of a process. This makes them particularly effective in describing a wide range of phenomena, including anomalous diffusion, viscoelastic behavior in materials, and control systems with long-term dependencies. Such properties have led to the successful application of FDEs across various disciplines, including physics, biology, engineering, and finance. Their increasing relevance in modeling real-world processes, ranging from transport in porous media to the analysis of complex financial systems, has spurred significant progress in both the theoretical analysis and numerical treatment of these equations. Consequently, a comprehensive understanding of the qualitative behavior of solutions, particularly in relation to boundary conditions, remains a central focus of ongoing research.

In this thesis we study the existence and uniqueness of solutions to non-linear FBVPs, subject to different types of boundary conditions. We use the numerical-analytic method to construct approximation to their solutions and analyze their solvability. The technique, originally developed for the study of periodic boundary value problems of the integer order, is adapted to the fractional setting with Dirichlet, integral, and multi-point boundary constraints. It enables the construction of approximations in closed form and the incorporation of various types of boundary conditions through a suitable parametrization. The applicability of the numerical-analytic method is extended to a larger class of problem by an interval-splitting method, which also improves the speed of convergence. For FBVPs with parameter-dependent right-hand side functions, we study the monotonicity of the resulting sequence and apply the upper and lower solutions method in conjunction with the numerical-analytic technique. Additionally, we explore the existence of bounded solutions for FBVPs defined on infinite domains.

In **Chapter 1**, we state the general problem setting and the primary objectives of this thesis. We provide a brief introduction to fractional calculus, explain the significance of studying boundary value problems of fractional order, and give an outline of the numerical-analytic technique.

In **Chapter 2** we have collected some definitions and auxiliary statements which are used throughout the remaining chapters of the thesis.

Chapter 3 deals with the solvability analysis and approximation of solutions to FBVPs, subject to Dirichlet boundary conditions. In Section 3.1.1 we use a numerical-analytic

technique to construct a sequence of successive approximations to the solution of a system of fractional differential equations, subject to Dirichlet boundary conditions. We prove the uniform convergence of the sequence of approximations to a limit function, which is the unique solution to the boundary value problem under consideration, and give necessary and sufficient conditions for the existence of solutions. The obtained theoretical results are confirmed by two model examples: one where the exact solution to the FBVP is available, and one derived from the equation used for modelling the Antarctic Circumpolar Current. In Section 3.2 we consider a parameter-dependent non-linear fractional differential equation, subject to Dirichlet boundary conditions. Using the fixed point theory, we restrict the parameter values to secure the existence and uniqueness of solutions, and analyze the monotonicity behavior of the solutions. Additionally, we apply a numerical-analytic technique, coupled with the lower and upper solutions method, to construct a sequence of approximations to the boundary value problem and give conditions for its monotonicity. The theoretical results are confirmed by an example of the equation in the fractional setting.

Chapter 4 is dedicated to the analysis and constructive approximations of solutions to FBVPs with the 'special type' boundary conditions. In Section 4.1 we study a system of non-linear fractional differential equations, subject to integral boundary conditions. We use a parametrization technique and a dichotomy-type approach to reduce the original problem to two *model-type* fractional boundary value problems with linear two-point boundary conditions. A numerical-analytic technique is applied to analytically construct approximate solutions to the *model-type* problems. The behavior of these approximate solutions is governed by a set of parameters, whose values are obtained by numerically solving a system of algebraic equations. The obtained results are confirmed by an example of the fractional order problem that in the case of the second order differential equation models the Antarctic Circumpolar Current. In Section 4.2 we study a non-linear fractional differential equation, defined on a finite and infinite interval. In the finite interval setting, we attach initial conditions and parameter-dependent boundary conditions to the problem. We apply a dichotomy approach, coupled with the numerical-analytic method to analyse the problem and to construct a sequence of approximations. Additionally, we study the existence of bounded solutions in the case when the fractional differential equation is defined on the half-axis and is subject to asymptotic conditions. Our theoretical results are applied to the Arctic gyre equation in the fractional setting on a finite interval.

Chapter 5 summarizes the main contributions of the thesis. It highlights the development and application of the numerical-analytic method to FBVPs, and discusses how the results address the core objectives of the study. In addition, the chapter outlines potential directions for future work, including extending the method to more complex systems and exploring its applicability to partial FDEs, and pursuing a deeper theoretical understanding of the qualitative behavior of fractional-order systems.

SAMENVATTING

Fractionele differentiaalvergelijkingen (FDV's) hebben de afgelopen jaren veel aandacht getrokken vanwege hun vermogen om complexe dynamische systemen te modelleren die worden gekenmerkt door geheugen- en erfelijkheidseffecten—eigenschappen die onvoldoende worden vastgelegd door klassieke differentiaalvergelijkingen van gehele orde. Het onderscheidende aspect van FDV's ligt in hun gebruik van niet-lokale operatoren, die inherent rekening houden met de geschiedenis van een proces. Dit maakt ze bijzonder effectief in het beschrijven van een breed scala aan fenomenen, waaronder anomale diffusie, visco-elastisch gedrag in materialen, en regelsystemen met langdurige afhankelijkheden. Dergelijke eigenschappen hebben geleid tot succesvolle toepassingen van FDV's in diverse disciplines, waaronder de natuurkunde, biologie, techniek en financiën. Hun toenemende relevantie bij het modelleren van reële processen, variërend van transport in poreuze media tot de analyse van complexe financiële systemen, heeft aanzienlijke vooruitgang gestimuleerd in zowel de theoretische analyse als de numerieke behandeling van deze vergelijkingen. Daarom blijft een grondig begrip van het kwalitatieve gedrag van oplossingen, met name in relatie tot randvoorwaarden, een centraal aandachtspunt in lopend onderzoek.

In deze scriptie bestuderen we het bestaan en de uniciteit van oplossingen van niet-lineaire FRWP's (fractionele randwaardeproblemen), onderworpen aan verschillende soorten randvoorwaarden. We gebruiken de numeriek-analytische methode om benaderingen van hun oplossingen te construeren en analyseren hun oplosbaarheid. Deze techniek, oorspronkelijk ontwikkeld voor de studie van periodieke randwaardeproblemen van gehele orde, wordt aangepast aan de fractionele context met Dirichlet-, integraal- en multipunt-randvoorwaarden. Ze maakt het mogelijk om benaderingen in gesloten vorm te construeren en diverse soorten randvoorwaarden op te nemen via een geschikte parametrisatie. De toepasbaarheid van de numeriek-analytische methode wordt uitgebreid naar een grotere klasse van problemen door een interval-splitsingsmethode, die tevens de convergentiesnelheid verbetert. Voor FRWP's met parameterafhankelijke functies aan de rechterkant bestuderen we de monotonie van de resulterende rij en passen we de methode van boven- en onderoplossingen toe in combinatie met de numeriek-analytische techniek. Daarnaast onderzoeken we het bestaan van begrensde oplossingen voor FRWP's gedefinieerd op oneindige domeinen.

Hoofdstuk 1 beschrijft de algemene probleemstelling en de primaire doelstellingen van deze scriptie. We geven een korte introductie tot de fractionele calculus, leggen het belang uit van het bestuderen van randwaardeproblemen van fractionele orde, en geven een overzicht van de numeriek-analytische techniek.

In **Hoofdstuk 2** hebben we enkele definities en hulpstellingen verzameld die worden gebruikt in de resterende hoofdstukken van de scriptie.

Hoofdstuk 3 behandelt de oplosbaarheidsanalyse en de benadering van oplossingen van FRWP's, onderworpen aan Dirichlet-randvoorwaarden. In Sectie 3.1.1 gebruiken we een numeriek-analytische techniek om een rij opeenvolgende benaderingen te construeren voor de oplossing van een systeem van fractionele differentiaalvergelijkingen, onderworpen aan Dirichlet-randvoorwaarden. We bewijzen de uniforme convergentie van de benaderingsrij naar een limietfunctie, die de unieke oplossing vormt van het beschouwde randwaardeprobleem, en geven noodzakelijke en voldoende voorwaarden voor het bestaan van oplossingen. De verkregen theoretische resultaten worden bevestigd door twee modelexemplaren: één waarin de exacte oplossing van de FRWP beschikbaar is, en één afgeleid van de vergelijking die wordt gebruikt voor het modelleren van de Antarctische Circumpolaire Stroming. In Sectie 3.2 beschouwen we een parameterafhankelijke niet-lineaire fractionele differentiaalvergelijking, onderworpen aan Dirichlet-randvoorwaarden. Met behulp van de dekpunttheorie beperken we de parameterwaarden om het bestaan en de uniciteit van oplossingen te garanderen, en analyseren we het monotoniegedrag van de oplossingen. Daarnaast passen we een numeriek-analytische techniek toe, gekoppeld aan de methode van boven- en onderoplossingen, om een benaderingsrij van het randwaardeprobleem te construeren en geven we voorwaarden voor zijn monotonie. De theoretische resultaten worden bevestigd aan de hand van een voorbeeld van de vergelijking in de fractionele context.

Hoofdstuk 4 is gewijd aan de analyse en constructieve benaderingen van oplossingen van FRWP's met randvoorwaarden van 'speciaal type'. In Sectie 4.1 bestuderen we een systeem van niet-lineaire fractionele differentiaalvergelijkingen, onderworpen aan integraalrandvoorwaarde. We gebruiken een parametrisatietechniek en een dichotomie- procedure om het oorspronkelijke probleem te reduceren tot twee modelproblemenmet lineaire tweepuntsrandvoorwaarden. Een numeriek-analytische techniek wordt toegepast om analytisch benaderingen van de oplossing te construeren voor deze modelproblemen. Het gedrag van deze benaderingen van de oplossing wordt gestuurd door een reeks parameters, waarvan de waarden worden verkregen door het numeriek oplossen van een systeem van algebraïsche vergelijkingen. De verkregen resultaten worden bevestigd door een voorbeeld van het fractionele orde probleem dat, in het geval van een tweede-orde differentiaalvergelijking, de Antarctische Circumpolaire Stroming modelleert. In Sectie 4.2 bestuderen we een niet-lineaire fractionele differentiaalvergelijking, gedefinieerd op een eindig en oneindig interval. In het geval van het eindige interval voegen we beginvoorwaarden en parameterafhankelijke randvoorwaarden toe aan het probleem. We passen een dichotomie-procedure toe, gecombineerd met de numeriek-analytische methode, om het probleem te analyseren en een benaderingsrij te construeren. Daarnaast bestuderen we het bestaan van begrensde oplossingen in het geval waarin de fractionele differentiaalvergelijking is gedefinieerd op de halve as en is onderworpen aan asymptotische randvoorwaarden. Onze theoretische resultaten worden toegepast op de vergelijking van de Arctische gyre in de fractionele context op een eindig interval.

Hoofdstuk 5 vat de belangrijkste bijdragen van de scriptie samen. Het benadrukt de ontwikkeling en toepassing van de numeriek-analytische methode op FRWP's en bespreekt hoe de resultaten bijdragen aan het bereiken van de kerndoelen van het onderzoek. Daarnaast schetst het hoofdstuk mogelijke richtingen voor toekomstig werk, waaronder het uitbreiden van de methode naar meer complexe systemen en het verkennen van de toepasbaarheid ervan op partiële FDV's, en het nastreven van een diepgaander theoretisch inzicht in het kwalitatieve gedrag van systemen van fractionele orde.

ACKNOWLEDGEMENTS

I would like to hereby express my gratitude to all the people who have contributed to the completion of this thesis.

I would like to thank my supervisory team, Dr. Kateryna Marynets, Prof. dr. Henk Schuttelaars and Dr. Johan Dubbeldam, for giving me the opportunity to pursue a PhD at TU Delft and for overseeing this work.

I would also like to thank all members of the doctoral committee for dedicating their time and expertise to evaluate my work.

I am grateful to my fellow PhD candidates, office mates, and colleagues, all of whom contributed to creating an enjoyable working environment. A special word of thanks goes to Mr. Joffrey Wallaart for all the technical support and the many other forms of assistance over the years.

Finally, I would like to express my gratitude to the people in my personal life - my family and friends - for their unwavering belief in me and their unconditional support.

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LIST OF PUBLICATIONS

1. K. Marynets and D. Pantova. “BVP with the Dirichlet type boundary conditions”. In: *Differ. Equ. Dyn. Syst.* 32 (2022), pp. 1047–1066. DOI: 10.1007/s12591-022-00613-y

2. K. Marynets and D. Pantova. “Parameter-dependent fractional boundary value problems: analysis and approximation of solutions”. In: *J. Appl. Anal.* (2024), pp. 1–20. DOI: 10.1080/00036811.2024.2426222

3. K. Marynets and D. Pantova. “Successive approximations and interval halving for fractional BVPs with integral boundary conditions”. In: *J. Comput. Appl. Math.* 436 (2024). DOI: 10.1016/j.cam.2023.115361

4. K. Marynets and D. Pantova. “Fractional boundary value problems with parameter-dependent and asymptotic conditions,” in: *Fractal Fract.* 9 (2025). DOI: 10.3390/fractalfract9070462