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**DOI**
10.1016/j.arcontrol.2018.04.004

**Publication date**
2018

**Document Version**
Final published version

**Published in**
Annual Reviews in Control

**Citation (APA)**
Komenda, J., Lahaye, S., Boimond, J. L., & van den Boom, T. (2018). Max-plus algebra in the history of discrete event systems. *Annual Reviews in Control*, *45*, 240-249. https://doi.org/10.1016/j.arcontrol.2018.04.004

Annual Reviews in Control 45 (2018) 240–249

Contents lists available at ScienceDirect

Annual Reviews in Control

journal homepage: www.elsevier.com/locate/arcontrol

Tutorial Article

# Max-plus algebra in the history of discrete event systems☆

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A R T I C L E I N F O

*Article history:*
Received 12 January 2018
Revised 21 March 2018
Accepted 3 April 2018
Available online 13 April 2018

*Keywords:*
Discrete-event system
Max-plus algebra
History

A B S T R A C T

This paper is a survey of the history of max-plus algebra and its role in the field of discrete event systems during the last three decades. It is based on the perspective of the authors but it covers a large variety of topics, where max-plus algebra plays a key role.



**Contents**



## 1. Emergence of max-plus approach. A system theory tailored for synchronization

This paper summarizes the history of max-plus algebra within the field of discrete event systems. It is based on brief survey of the role of max-plus algebra in the field of discrete event systems that appeared in Komenda, Lahaye, Boimond, and van den Boom (2017), but extended in several directions. In particular, there is a section, where computational aspects are discussed together with results about max-plus algebra from the computer science literature.

The emergence of a system theory for classes of discrete event systems (DES), in which max-plus algebra and similar algebraic tools play a central role, dates from the early 1980's. We emphasize that the idempotent semiring (also called dioid) of extended real numbers $(\mathbb{R}\cup\{-\infty\}, \max, +)$ is commonly called max-plus algebra, while it is not formally an algebra in the strictly mathematic sense.

Its inspiration stems certainly from the following observation: synchronization, which is a very non smooth and nonlinear phenomenon with regard to "usual" system theory, can be modeled by linear equations in particular algebraic structures such as max-plus algebra and other idempotent semiring structures (Cohen, Dubois, Quadrat, & Viot, 1983; Cuninghame-Green, 1979).

Two important features characterize this approach often called *max-plus linear system theory*:

- most of the contributions have used as a guideline the "classical" linear system theory;

☆ The research was supported by GAČR grant S15-2532 and by RVO 67985840.

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https://doi.org/10.1016/j.arcontrol.2018.04.004
1367-5788/

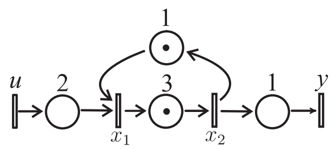


**Fig. 1.** A timed event graph.

- it is turned towards DES performance related issues (as opposed to logical aspects considered in other approaches such as automata and formal language theory) by including timing aspects in DES description.

A consideration has significantly contributed to the promotion and the scope definition of the approach: a class of ordinary[1] Petri nets, namely the timed event graphs (TEGs) has been identified to capture the class of stationary[2] max-plus linear systems (Cohen, Dubois, Quadrat, & Viot, 1985) and subsequent publications by Max Plus team.[3] TEGs are timed Petri nets in which each place has a single input transition and a single output transition. A single output transition means that no conflict is considered for the tokens consumption in the place, in other words, the attention is restricted to DES in which all potential conflicts have been solved by some predefined policy. Symmetrically, a single input transition implies that there is no competition in supplying tokens in the place. In the end, mostly synchronization phenomena (corresponding to the configuration in which a transition has several input places and/or several output places) can be considered, and this is the price to pay for linearity.

**Example 1.** Fig. 1 depicts a TEG, that is a Petri net in which each place (represented by a circle) has exactly one input transition (represented by a rectangle) and one output transition. The number next to a place indicates the sojourn time for a token, that is the number of units of time that must elapse before the token becomes available for the firing of the output transition. Let $u(k)$ denote the date of the $k^{\text{th}}$ firing of transition $u$ (same notation for $x_1$, $x_2$ and $y$). Considering the *earliest firing rule* (a transition is fired as soon as there is an available token in each input place), we have the following evolution equations

$$\begin{aligned} x_1(k) &= \max(2+u(k), 1+x_2(k-1)) \\ x_2(k) &= 3+x_1(k-1) \\ y(k) &= 1+x_2(k). \end{aligned}$$

Denoting $\oplus$ (resp. $\otimes$) the addition corresponding to max operation (resp. the multiplication corresponding to usual addition), we obtain linear equations in max-plus algebra, that is:

$$\begin{aligned} x_1(k) &= 2 \otimes u(k) \oplus 1 \otimes x_2(k-1) \\ x_2(k) &= 3 \otimes x_1(k-1) \\ y(k) &= 1 \otimes x_2(k) \end{aligned}$$

Rewriting the resulting equations in max-plus-algebraic matrix notation leads to a state-space representation:

$$\begin{cases} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} \varepsilon & 1 \\ 3 & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} x_1(k-1) \\ x_2(k-1) \end{bmatrix} \oplus \begin{bmatrix} 2 \\ \varepsilon \end{bmatrix} \otimes u(k) \\ y(k) = \begin{bmatrix} \varepsilon & 1 \end{bmatrix} \otimes \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \end{cases}$$

where $\varepsilon$ is equal to $-\infty$.

This new area of linear system theory has benefited from existing mathematical tools related to idempotent algebras such as lattice theory (Birkhoff, 1940), residuation theory (Blyth & Janowitz, 1972), graph theory (Gondran & Minoux, 1979), optimization (Zimmermann, 1981) and idempotent analysis (Kolokoltsov & Maslov, 1997), however it is worth mentioning that the progress has probably been impeded by the fact that some fundamental mathematical issues in this area are still open.

The overview of the contributions reveals that main concepts from linear system theory have been step by step specified into max-plus linear system theory. Without aiming to be exhaustive:

- several *possible representations* have been studied, namely state-space equations, transfer function in *event domain* (Cohen et al., 1983; 1985), *time domain* (Caspi & Halbwachs, 1986), and *two-dimensional domain* using series in two formal variables (Cohen, Moller, Quadrat, & Viot, 1986) (with more details in Cohen, Moller, Quadrat, & Viot, 1989);
- *performance analysis and stability* are mostly based on the interpretation of the *eigenvalue* of the state-matrix in terms of cycle-time, with its associated eigenspace and related cyclicity property (Baccelli, Cohen, Olsder, & Quadrat, 1992; Gaubert, 1997);
- a wide range of *control laws* have been adapted such as:
  - open-loop structures overcoming system output tracking (Baccelli et al., 1992, chap. 5.6), (Cofer & Garg, 1996; Menguy, Boimond, Hardouin, & Ferrier, 2000) or model reference tracking (Libeaut & Loiseau, 1996),
  - closed-loop structures taking into account disturbances and model-system mismatches (Cottenceau, Hardouin, Boimond, Ferrier et al., 1999; Lüders & Santos-Mendes, 2002), possibly including a state-observer (Hardouin, Maia, Cottenceau, & Lhommeau, 2010),
  - model predictive control scheme (De Schutter & van den Boom, 2001; van den Boom & De Schutter, 2002) with emphasis on stability in Necoara, De Schutter, van den Boom, and Hellendoorn (2007).

For a large survey on max-plus linear systems theory, we refer to books (Baccelli et al., 1992; Butkovič, 2010; Gunawardena, 1998; Heidergott, Olsder, & van der Woude, 2006), to manuscript (Gaubert, 1992) and surveys (Akian, Bapat, & Gaubert, 2003; Cohen, Gaubert, & Quadrat, 1999; Cohen et al., 1989; Gaubert, 1997).

## 2. Some extensions focused on synchronization in DES

There is an important connection between min-max-plus systems, in which time evolution depend on both max and min, but also addition operation and the game theory. It goes back to Olsder (1991), where spectral properties of such systems are studied. More recent references on this topic are Gunawardena (2003) and Akian, Gaubert, and Guterman (2012). The latter work establishes an equivalence with mean payoff games, an important open complexity problem in computer science, and it seems many verification problems for max-plus systems reduce to mean payoff games. We mention that many theoretical works on max-plus algebra and max-plus systems do not make use of the words “max-plus” but rather “tropical”. The adjective tropical was invented by French mathematicians, in honor of the Brazilian mathematician and computer scientist Imre Simon (1943–2009).

A natural generalization of deterministic max-plus-linear systems are stochastic max-plus-linear systems, which have been studied for more than two decades. Ergodic theory of stochastic timed event graphs is developed in Baccelli et al. (1992), where most of the theory is covered. In particular, asymptotic properties of stochastic max-plus-linear systems are studied therein in terms of the so-called Lyapunov exponents that correspond to the

[1] Petri nets in which all arc weights are 1.

[2] Stationarity is defined conventionally but over operators of max-plus algebra.

[3] Max Plus is a collective name for a working group on max-plus algebra, at INRIA Rocquencourt, comprising: Marianne Akian, Guy Cohen, Stéphane Gaubert, Jean-Pierre Quadrat and Michel Viot.

asymptotic mean value of the norm of the state variables. In the case the underlying event graph is strongly connected the Lyapunov exponent is the unique value to which the mean value almost surely converges. For general event graphs there is a maximal Lyapunov exponent.

Uncertainty can also be considered through intervals defining the possible values for parameters of the system. In Lhommeau, Hardouin, Cottenceau, and Jaulin (2004) TEGs, in which the number of initial tokens and the time delays are only known to belong to intervals, are represented over a semiring of intervals and robust controllers are designed.

Another way of extending techniques for linear systems is to consider that parameters of the models may vary, that is study non-stationary linear systems. This possibility has been examined within the max-plus linear setting (Brat & Garg, 1998; Lahaye, Boimond, & Hardouin, 1999; 2004) with contributions mainly focused on representation, control and performance analysis.

Continuous TEGs in which fluids hold rather than discrete tokens and the fluid flow through transitions can be limited to a maximum value. Moreover, an initial volume of fluid can be defined in places and times can be associated with places to model fluid transportation times (MaxPlus, 1991). Such graphs are relevant for example to approximate the behavior of high throughput manufacturing systems in which the number of processed parts is very large. In parallel, a similar approach called network calculus has been developed by considering computer network traffic as a flow (based on the use of 'leaky buckets') to approximate the high number of conveyed packets (Cruz, 1991; Le Boudec, 2001). Extension on fluid timed event graphs with multipliers in a new algebra, analogous to the min-plus algebra, has been proposed in Cohen, Gaubert, and Quadrat (1995, 1998).

Switching Max-Plus-Linear (SMPL) systems are discrete-event systems that can switch between different modes of operation (van den Boom & De Schutter, 2006). The switching allows to change the structure of the system, to break synchronization, or to change the order of events. In each mode the system is described by a max-plus-linear state equation and a max-plus-linear output equation. Note that regular max-plus-linear systems are a subclass of SMPL system, namely with only one mode. In van den Boom and De Schutter (2011), authors describe the commutation between different max-plus-linear modes and shows that an SMPL system can be written as a piecewise affine system which allows for using similar techniques in such seemingly different classes of systems. In cyclic DES the operations appear in a cyclic way. After all operations in a system have been completed, the cycle is closed and a new cycle begins. In the case of changes in operations and resources per cycle the system is called semi-cyclic. SMPL models can be used to describe the dynamics of various semi-cyclic DES.

Another extension of the class of systems that can be modeled in max-plus algebras consists in considering hybrid Petri nets, and more particularly, hybrid TEGs that consist of a discrete part (a TEG) and a continuous part (a continuous TEG). It has been shown in Komenda, El Moudni, and Zerhouni (2001) that a linear model can be obtained based on counter function if only one type of the interface between continuous and discrete part is present. However, for application to just in time control this constraint can be relaxed as it has been shown in Hamaci, Boimond, and Lahaye (2006).

The weighted[4] TEGs make it possible to describe batching and duplication (unbatching) phenomena. In Cottenceau, Hardouin, and Trunk (2017), authors show that such graphs can be also linearly modeled by transfer series in a particular max-plus algebra. In addition to event and time shifts, two additional operators are used to describe the batching/unbatching operations.

P-time Petri nets form an important extension of Petri nets, where the timing of places/transitions is nondeterministic. P-time Event Graphs have been studied in max-plus algebras in Declerck and Alaoui (2004, 2005). They find their applications e.g. in modeling of electroplating lines or chemical processes, where both upper and lower bound constraints processing time are required, see e.g. Spacek, Manier, and Moudni (1999).

## 3. Timed DES with shared resources

A major issue with application of max-plus linear systems to modeling of timed DES (important among others in manufacturing systems or in computer and communication networks) is that it appears difficult to model resource sharing within TEGs that correspond to stationary max-plus linear systems. In real manufacturing systems, however, there are typically several processes (tasks) that share (and compete for) given resources such as robots in manufacturing systems or memory in computer systems. In the max-plus systems literature various resource allocations policies have been proposed to integrate conflict resolution with other typical phenomena of timed DES, namely synchronization and parallelism. For instance, within timed Petri nets resource allocations policies has been studied based on the dual counter function description in the idempotent semiring min-plus (Cohen, Gaubert, & Quadrat, 1997). General Petri net models have been addressed by preselection rules in Cohen et al. (1995), which enables to describe their evolution using max-plus dynamics. More general monotone homogeneous dynamics, relevant to free choice Petri nets, and their optimal routing is studied in Gaujal and Giua (2004).

More recently, conflicts among several TEGs have been studied in Addad, Amari, and Lesage (2012), where conflicting TEGs (CTEG) have been proposed with some fairly restrictive assumptions. It should be stated that resource allocation policies studied in Addad et al. (2012) are either FIFO or cyclic (periodic) policies. On one hand the performance analysis (computation of an upper bound on the cycle time of CTEG) has been proposed and it is dependent on the cycle time of individual TEGs and on timing of the conflict places. On the other hand, the approach has not yet been applied to control problems.

Unlike the approach based on TEGs, where different resource allocation policies are handled one by one, there exists a max-plus automata based approach that allows simultaneous modeling of different resource allocation policies within a single model as long as these policies can be represented by a regular language. An automaton-based model it can handle several such policies at the same time within a single model without having to rebuild the model each time the policy is changed.

The framework of max-plus automata has enabled a deep investigation of performance evaluation (Gaubert, 1995) of DES with shared resources. Max-plus automata can be viewed as a rather special class of automata models enriched with time, because unlike timed automata they do not time non determinism, where both lower and upper bounds on timing of events can be defined. However, they have strong expressive power in terms of timed Petri nets as shown in Gaubert and Mairesse (1999) and Lahaye, Komenda, and Boimond (2015a). In particular, every safe timed Petri net can be represented by special max-plus automaton, called heap model.

**Example 2.** A safe timed Petri net is depicted on Fig. 2. The number next to the label of a transition specifies its firing duration, that is the minimal time that must elapse, starting from the time at which it is enabled, until the transition can fire. All the places are assumed to have a sojourn duration equal to 0 unit of time. Its

[4] Arcs weights can be any positive integers.

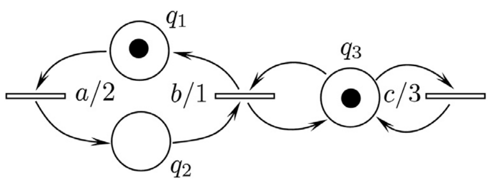


**Fig. 2.** A timed Petri net.

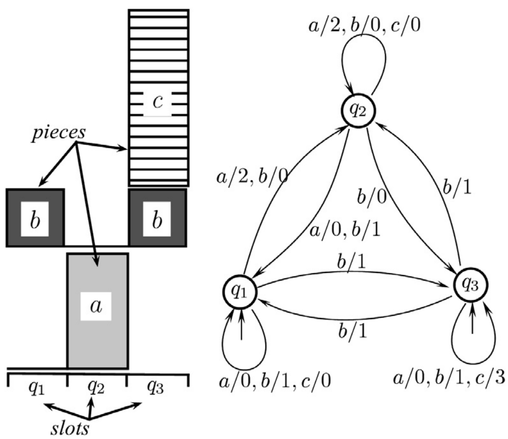


**Fig. 3.** Heap model and associated (max,+) automaton to represent the timed Petri net of Fig. 2.

behavior can be described by the heap model on the left-hand side of Fig. 3. In few words, a heap model is composed of

- slots which correspond to resources (tokens in the Petri net) and one slot is associated to each place (possibly containing a token),
- pieces which represent activities (firings of transitions in the Petri net) and one piece is associated to each transition.

The activities require resources (transitions consume token(s) to be fired in the Petri net and pieces occupy slots in the heap model) during predefined durations (transitions firing durations rendered by specific heights of pieces in slots). The dynamics is then modeled by the sequence of pieces (transitions firing sequence in the Petri net) pilling up according to the Tetris game mechanism. It can be shown that the height of heaps of pieces is recognized by a particular (max,+) automaton (especially useful for algebraic computations). The (max,+) automaton derived from the example of heap model is depicted[5] on the right-hand side of Fig. 3.

S. Gaubert has presented several important analysis results in Gaubert (1995), where the worst case, the optimal case, and the mean case performance of max-plus automata are examined in detail. Better results are naturally obtained for deterministic max-plus automata, but most of the paper is focused on general nondeterministic max-plus automata. We emphasize that not all max-plus automata can be determinized and their determinization, i.e. existence of a deterministic max-plus automaton having the same behavior (recognizing the same formal power series), is still an open problem and it is not even known if determinization of a given nondeterministic max-plus automaton is decidable. In Gaubert (1995) a sufficient condition in terms of projectively finite semigroups for determinization of max-plus automata is provided, which can be used as a semi-algorithm for determinization (with no guarantee of success). Several other works on determinization of max-plus automata have appeared later. Mohri developed a semi-algorithm for determinization of max-plus automata in Mohri (1997) with a very successful application in speech recognition. It is known that the twins property, generalized to clones property for polynomially ambiguous in Kirsten (2008), is a sufficient condition for the termination of this algorithm. In Kirsten and Lombardy (2009) the authors propose an algorithm for deciding unambiguity and sequentiality of polynomially ambiguous min-plus automata, which leaves the unambiguity and sequentiality problem open only for non polynomially ambiguous class of both max-plus and min-plus automata.

## 4. Max-plus-algebra and theoretical computer science

In this section we will address algorithms in max-plus-algebra from a computer science perspective, where a special emphasis is put on complexity issues. Time and space complexities of algorithms are important in the whole theory of discrete-event systems, which includes among others supervisory control, stochastic discrete systems (Markov Chains), and timed discrete event systems. Max-plus algebra finds its applications mainly in timed discrete event systems. Although the underlying system models vary from deterministic-time models such as timed event graphs to non deterministic-time models such as time event graphs, the main operations used in equations describing the evolution of these systems are the rational matrix operations: sum, product, and the Kleene star. Matrix multiplication, including the one in the max-plus-algebra is well known to have the worst case complexity $O(n^3)$ for square matrices with $n$ lines and columns. This rather naive bound can be improved, which is a major topic of research in algebraic complexity theory, where algorithms for matrix multiplication and inversion in $n^\omega$ with $\omega \leq 2.373$ are known for matrices over field. The complexity of the matrix product and Kleene star in the tropical setting is a major open problem, see Williams (2014). The important aspect is, however, that all rational operations on matrices (including the Kleene star, which reduces to the finite sum of the first $n+1$ max-plus-powers) are of polynomial worst case complexity. The same complexity result obviously holds for matrix residuation used in solutions to various control problems, because matrix residuation can be viewed as a dual multiplication with the so called conjugate matrix (Cuninghame-Green & Butkovic, 2008). This is a very good news, because control problems for max-plus-linear systems listed in Section 1 are based on matrix multiplication and residuation, i.e. can be solved in polynomial time.

Similarly, the max-plus and min-plus operations on vectors are of polynomial time complexity. We recall that min-plus convolution of two vectors plays a central role in the dynamic programming. Since the early 1960s, it is known that it can be computed in $O(n^2)$ time. This bound can be further improved in some special cases, e.g. for two convex sequences it can be computed in $O(n)$ time (Brenier, 1989; Lucet, 1996) by a simple merge (the Minkowski sum) of two convex polygons (Rockafellar, 1970). This special case is already used in image processing and computer vision.

However, the above discussion mainly applies to approaches covered in Sections 1 and 2. One should bear in mind that choice phenomena (i.e. resource sharings) are then excluded and the models correspond to recurrent equations on natural numbers that count events, without making distinction between them. As mentioned in Section 3, some DES including choice phenomena can be seen as max-plus linear systems, e.g. max-plus automata, if one uses recurrent equation on words reflecting sequences of differentiated events. The complexity picture is then very different,

[5] The graphical representation of a (max,+) automaton is such that: nodes correspond to states, an arrow from a state to another with label $a/n$ denotes a state transition requiring $n$ units of time before event $a$ can occur, an input arrow symbolizes an initial state.

because we will recall below that many fundamental verification problems are already undecidable.

### 4.1. Max-plus automata and properties of their series

Max-plus automata have been introduced by S. Gaubert in Gaubert (1995) as a generalization of both max-plus-linear systems and standard Boolean automata. Max-plus automata as weighted automata with weights (sometimes called multiplicities) in the max-plus semiring have also been studied by the computer science community. The basic reference on the theory of automata of multiplicities developed by Eilenberg and Schutzenberger is Eilenberg (1974). It has been understood in 1990s that several fundamental problems undecidable for general timed systems, such as timed automata, are already undecidable for max-plus automata. In particular, it has been shown in Krob (1992) that equalities and inequalities of rational max-plus formal power series are undecidable. Since it is well known that rational max-plus formal power series are behaviors (weighted languages) of finite (state) max-plus automata, the result of D. Krob means that, in general, it is not algorithmically possible to compare the behaviors of finite max-plus automata. Some other verification problems are decidable for max-plus automata. For instance, it can be decided in polynomial time (namely $O(n^3)$ with n the size of the state set of the recognizer, see Lombardy & Mairesse, 2006) if a rational max-plus power series has all coefficients non positive, i.e. $\langle S, w\rangle \leq 0$ for all $w \in A^*$. This means that existence of $w \in A^*$ with $\langle S, w\rangle > 0$ can be decided in polynomial time as well. On the other hand, the problem $\langle S, w\rangle \geq 0$ for all $w \in A^*$ is undecidable, cf. (Krob, 1992). It is also known that equality to a constant is decidable, i.e. for a rational max-plus power series it can be decided if for all $w \in A^*$ it holds that $\langle S, w\rangle = c$ for some real constant $c$, see Lombardy and Mairesse (2006). In the literature one encounters also min-plus automata, which are weighted automata with weight in the (dual) min-plus semiring. They are also nondeterministic: minimum, instead of maximum, of weights of paths that shared a label is taken for computation of the corresponding min-plus formal series. For min-plus automata dual decidability results hold meaning that it is decidable in $O(n^3)$ to check if $\langle S, w\rangle \geq 0$ for all $w \in A^*$, while it is undecidable to check if $\langle S, w\rangle \leq 0$ for all $w \in A^*$. Very interesting are complexity results concerning the comparisons of max-plus and min-plus series. It is known that inequality $S(w) \leq S'(w)$ for all $w \in A^*$ can be decided if $S$ is a max-plus rational formal power series and $S'$ is a min-plus rational series, but the opposite inequality is then undecidable! In Lombardy and Mairesse (2006) the series which are recognized both by a finite max-plus and a finite min-plus automaton, i.e. series at the same time max-plus and min-plus rational, have been characterized. It has been shown that these series are precisely the unambiguous max-plus (equivalently, unambiguous min-plus) series. We recall that unambiguous max-plus series are those recognized by unambiguous max-plus automata: for every word $w$, there is at most one successful path labeled by $w$. Note that inverting the coefficient of a rational max-plus series, i.e. multiplying all its coefficients by $-1$ does not yield a rational max-plus series, but rather a rational min-plus series. This helps understanding the above discussed asymmetries in the fundamental decision problems discussed above.

### 4.2. Bisimulation properties

It is quite disappointing that several fundamental problems are undecidable for max-plus automata. We point out that recently there are also more optimistic complexity results about max-plus automata. The well known concept of bisimulation, which captures behavioral equivalence of nondeterministic transition systems, has been introduced for max-plus automata in Buchholz and Kemper (2003). It is a stronger property that equality of formal power series, but may serve as a partial remedy to undecidability of inequalities and equalities between formal power series. Bisimulation between two max-plus automata means that the related (equivalent) states match each other's transitions (not only from the logical viewpoint: existence of transitions, but also from quantitative view point: the weights of two matching transitions should be identical). Algebraic approach to the investigation of bisimulation relations encoded as Boolean matrices has been adopted in Damljanović, Ćirić, and Ignjatović (2014), where bisimulation is characterized by max-plus-linear matrix inequalities (to be distinguished from MLI's in classical control theory) and a fix-point algorithm with a polynomial complexity for algebraic computation of largest bisimulations has been proposed.

In concurrency theory there is a concept of weak bisimulation, which weakens the bisimulation by not requiring internal (externally invisible) transitions to be preserved. In Buchholz and Kemper (2003) the concept of projected max-plus automaton has appeared first (although it is not explicitly named so). The authors of that paper define weak bisimulations as (strong) bisimulations between projected automata, which enables to use their algebraic characterization. Since bisimulations are stronger than language (formal power series) equalities, it immediately follows that existence of a weak bisimulation between two max-plus automata implies the equality of their projected behaviors. This particular definition of bisimulation is very much influenced by the concurrency theory community, where instead of unobservable events (as a subset of the event set) as customary in DES community the notion of an internal action denoted by $\tau$ is used. We believe that concepts like weak bisimulation will be proven very useful for partially observed max-plus automata in a near future, because firstly they admit nice algebraic characterization and secondly they can be computed in a polynomial time.

### 4.3. Supervisory control

Supervisory control theory can be viewed as a generalization of verification. The idea is that if a property to be verified fails to be satisfied it can still be imposed by a supervisor. Supervisory control is a formal approach introduced first for control of logical automata with partial transition functions that aims to solve the safety issue (avoidance of forbidden states given by control specifications) and nonblockingness (avoidance of deadlocks and livelocks). If a control specification (property) is given informally, software engineers must translate them into control software manually. The ultimate goal of supervisory control is to develop formal theory that enables an automated synthesis of controllers that are correct by construction so that further verification is not needed. Given a control specification describing required behavior of the system, one has to construct a supervisor that observes a subset of events (yielding a possibly partial information about the state of the plant) and selects actuators, that can control the execution of some controllable events in order to meet the prescribed specification language, which specifies a property of the system such as certain states must be forbidden. An interesting control approach to max-plus automata is presented in Klimann (2003), where the problem of $(A, B)$-invariance for formal power series is solved.

Supervisory control theory of max-plus automata with complete observations has been proposed in Komenda, Lahaye, and Boimond (2009), where the basic elements of supervisory control, such as supervisor, closed-loop system, and controllability are extended from logical to max-plus automata. However, it follows from results presented therein that rational (i.e. finite state) controllers can only be obtained for systems (plants) which have behaviors at the same time max-plus and min-plus rational. The problem is that the controller series is based on residuation of the

Hadamard product of series, which can be seen as a Hadamard product with a series having all its coefficients inverted. This operation has already been discussed in Section 4.1 and it outputs a min-plus rational series for a given max-plus rational series. We then need to work with the class of series that are at the same time max-plus and min-plus rational in order to have rational (finite-state) controllers. Unfortunately, it has been shown in Lombardy and Mairesse (2006) that this class of series coincides with the class of unambiguous series (series recognized by an unambiguous automaton). Although unambiguous series is less restrictive property than deterministic series (series recognized by a deterministic max-plus automaton), a typical approach for imposing unambiguity is to determinize a max-plus automaton.

More complete picture about rationality issues extended to more general setting is presented in Lahaye, Komenda, and Boimond (2015b). More specifically, minimally permissive and just-after-time supervisors are studied in order to guarantee a minimal required behavior and to delay the system as little as possible so that sequences of events occur later than prescribed dates, which is important for applications in transportation networks (e.g. improving train connections in railway systems), but also in manufacturing systems and communication networks. It has been shown that finite state controllers exist if the system-series and the specification (reference-series) are both unambiguous. This assumption is met for several classes of practically relevant max-plus automata, e.g. those modeling a type of manufacturing systems such as safe Flow-shops and Job-shops. Another class of timed system called timed weighted systems has been studied in Su, van Schuppen, and Rooda (2012). Timed weighted systems are simply modular automata (collection of local automata) endowed with the so called mutual exclusion function as well as a time-weighted function. Timed weighted systems can be understood as a synchronous product of max-plus automata, which is not made explicit and the durations of events are described by time-weighted function.

In our opinion max-plus automata form a gateway to the general timed automata, because systems modeled by max-plus automata exhibit most of decidability and determinization issues that are present for general timed automata, while they are conceptually simpler, which allows for better grasping the core of these fundamental problems. Fortunately, there exist several ways how to deal with these issues. For instance, further progress in determinization of max-plus automata is possible as it is shown in Lahaye, Lai, and Komenda (2017). There exist approaches to approximate determinization of weighted automata (Filiot, Jecker, Lhote, A. Pérez, & Raskin, 2017). Finally, one may replace the control specification (requirement) in terms of inequality of formal power series by a simulation-based specification and introduce the supervisory control theory for imposition of simulation properties.

## 5. Max-plus planning and model predictive control

The Model Predictive Control (MPC) design method can be applied to (switching) max-plus linear systems (van den Boom & De Schutter, 2006). MPC for conventional (non-DES) systems is very popular in the process industry (Maciejowski, 2002) and a key advantage of MPC is that it can accommodate constraints on the inputs and outputs. For every cycle the future control actions are optimized by minimizing a cost function over a prediction window subject to constraints. If the cost function and the constraints are piecewise affine functions in the input, output, and state variables, the resulting optimization problem will be a mixed-integer linear programming (MILP) problem, for which fast and reliable algorithms exist. An alternative approach is to use optimistic optimization (Xu, De Schutter, & van den Boom, 2014). In De Schutter and van den Boom (2001) MPC for regular max-plus-linear systems was studied using the just-in-time cost function with constraints that were monotonically nondecreasing in the output. In that case the problem turns out to be a linear programming problem.

A natural generalization of deterministic max-plus-linear systems are max-plus-linear systems with uncertainty. This uncertainty can either have a bounded nature or a stochastic nature. The uncertainty will appear in a max-plus-multiplicative way as perturbations of the system parameters (Olsder, Resing, Vries, Keane, & Hooghiemstra, 1990).

In the bounded uncertainty approach the parameters of the models may vary, which leads to the study of non-stationary linear systems approach. This possibility has been examined within the max-plus linear setting (Brat & Garg, 1998; Lahaye et al., 1999; 2004) with contributions mainly focused on representation, control and performance analysis. Bounded uncertainty can also be considered through intervals defining the possible values for parameters of the system. In Lhommeau et al. (2004) TEGs, in which the number of initial tokens and the time delays are only known to belong to intervals, are represented over a semiring of intervals and robust controllers are designed. A dynamic programming approach to robust state-feedback control of max-plus-linear systems with interval bounded matrices is given in Necoara, De Schutter, van den Boom, and Hellendoorn (2009) in which it is shown that the min-max control problem can be recast as a deterministic optimal control problem by employing results from dynamic programming.

Stochastic Max Plus Linear systems, defined as MPL systems where the matrices entries are characterized by stochastic variables (Heidergott et al., 2006; Heidergott, 2007), have been studied for more than two decades. As noticed in Section 2, most of the ergodic theory of stochastic timed event graphs is covered in Baccelli et al. (1992). Results for MPC of stochastic max-plus linear systems are given in van den Boom and De Schutter (2004), where the authors show that under quite general conditions the resulting optimization problems turn out to be convex. The main problem with this method is that the computation of the expected value can be highly complex and expensive, which also results in a high computation time to solve the optimization problem. To this end Farahani, van den Boom, van der Weide, and De Schutter (2016) use an approximation method based on the moments of a random variable to obtain a much lower computation time while still guaranteeing a comparable performance.

Switching max-plus linear systems with both stochastic and deterministic switching are discussed in van den Boom and De Schutter (2012). In general, the optimization in the model predictive control approach boils down to a nonlinear nonconvex optimization problem, where the cost criterion is piecewise polynomial on polyhedral sets and the inequality constraints are linear. However, in the case of stochastic switching that depends on the previous mode only, the resulting optimization problem can be solved using linear programming algorithms.

In van den Boom, Lopes, and Schutter (2013) a general framework has been set up for model predictive scheduling of semi-cyclic discrete event systems. In a systematic way the main scheduling steps, i.e. routing, ordering, and synchronization, can be modeled. A switching max-plus linear model has been derived with scheduling parameters for each scheduling step. The system matrix is max-plus affine in the max-plus binary scheduling parameters and a model predictive scheduling problem has been formulated. This model predictive scheduling problem can be recast into a mixed integer linear programming problem. This scheduling technique has been applied in Kersbergen, Rudan, van den Boom, and De Schutter (2016), where a railway traffic management algorithm has been derived that can determine new conflict-free schedules and routes for a railway traffic network when delays oc-

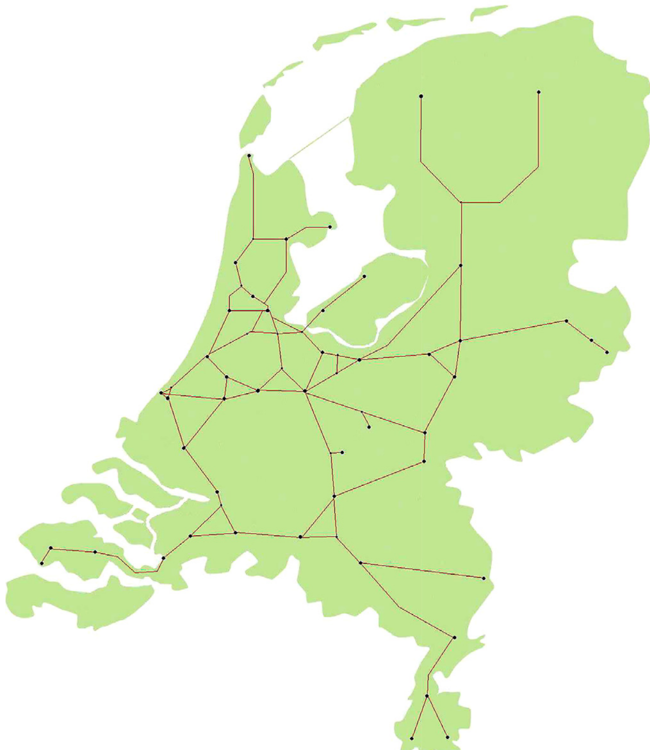

**Fig. 4.** Dutch railway network.

cur. See Fig. 4 for the scheme of the railway network in the Netherlands.

Scheduling using switching max-plus linear models has also been described in Lopes, Kersbergen, van den Boom, De Schutter, and Babuška (2014), where the transition between different gait transition schemes in legged robots has been discussed and optimal transitions are derived such that the stance time variation is minimized, allowing for constant acceleration and deceleration. In Alirezaei, van den Boom, and Babuška (2012) an optimal scheduler for paper-handling in a duplex printer is presented. The scheduling is based on the max-plus modeling framework. It is shown that the proposed method successfully finds the globally optimal schedule for different types of the sheets.

## 6. Max-plus and min-plus geometry

It is well understood that linear algebra is closely connected to geometry and that geometric concepts play an important role in control of linear systems.

As we have argued in previous sections, control theory for max-plus-linear systems has been inspired mainly by the theory of linear systems. A fundamental concept in both linear algebra and geometry is that of vector spaces or more generally modules. Their max-plus counterparts are known as idempotent semimodules, which are module-like structures, but over an idempotent semiring (such as max-plus semiring) rather than over a ring. The basic properties of idempotent semimodules including the concepts of independence and dimension have been studied since late 1980's by Wagneur (1991) or Russian school (Litvinov, Maslov, & Shpiz, 2001). A fundamental control theoretic concept is $(A, B)$ invariant space, which is a controlled invariance of a semimodule. Namely, it requires that any trajectory starting in this semimodule can be controlled such that it remains forever within this semimodule. It has been investigated in Katz (2007), where a classical algorithm for the computation of the maximal (A, B)-invariant subspace contained in a given space is generalized to the max-plus linear systems. Although the algorithm needs not converge in a finite number of steps, the sufficient conditions (of demonstrated practical interest for a class of semimodules) have been proposed for the convergence in a finite number of steps.

The study of invariance properties for max-plus linear systems are inspired by the Wonham's geometric theory of linear systems (Wonham, 1974). We emphasize that this theory has been at the very origin of the supervisory control theory developed in early 1980's in parallel with the theory of max-plus-linear systems. The geometric framework has enabled among others to solve disturbance decoupling problem for linear systems. Following similar ideas, modified disturbance decoupling problem for max-plus linear systems has been studied in Shang, Hardouin, Lhommeau, and Maia (2016).

Another interesting work is Cohen, Gaubert, and Quadrat (1996), where projection onto images of operators and parallel to the kernels of operators have been studied. It should be noted that these operators are useful not only in control of max-plus linear systems, but admit also specific application to aggregation and other problems for Markov chains.

Very important concept, convexity, a powerful tool in optimization and operational research, has been extended to the max-plus framework. Well known Minkowski theorem from linear algebra states that a non-empty compact convex subset of a finite dimensional space is the convex hull of its set of extreme points. The max-plus counterpart of Minkowski theorem presented in Gaubert and Katz (2007) extends this result to max-plus convex sets. This result is very important, because max-plus convex sets arise in many different domains, ranging from max-plus-linear systems, abstractions of timed automata to solutions of Hamilton–Jacobi equations associated with a deterministic optimal control problem, see e.g. Litvinov et al. (2001).

More recently, an interesting relation has been discovered between geometric approach to max-plus-linear systems proposed in Gaubert and Katz (2007) and reachability analysis of timed automata, cf. Allamigeon, Fahrenberg, Gaubert, Katz, and Legay (2014). Timed automata are very general models of timed DES that involve several parallel clocks variables that measure time elapsed since their last reset and define time constraints (known as guards) for enabling logical transitions in timed automata.

Interestingly, max-plus geometry can be applied in reachability analysis of timed automata. Timed automata with infinite (but finite dimensional) clock spaces are abstracted into finite automata called region or zone automata, where the infinite clock space is abstracted by a finite number of regions or geometric zones. This abstraction is shown to be a timed bisimulations and this enables to solve several fundamental problems for timed automata such as non emptiness. The zones are represented by efficient data structures called difference bound matrices (DBM) that represents the bounds on differences between state variables. The reachability of different zones can be studied using max-plus-cones from geometric theory of max-plus-linear systems.

It has been shown in Lu et al. (2012) that every max-plus cone (also called max-plus polyhedron) can actually be described as a union of finitely many DBM's as shown in Adzkiya, De Schutter, and Abate (2013). These geometric objects have proven to be extremely useful for both forward and backward reachability analysis, see e.g. Allamigeon et al. (2014). Forward reachability analysis aims at computing the set of possible states that can be reached under the model dynamics, over a set of consecutive events from a set of initial conditions and possibly by choosing control actions (Adzkiya, De Schutter, & Abate, 2015). Backward reachability analysis consists in computing the set of states that enter a given set of final states, possibly by choosing control actions. This is of practical importance in safety control problems consisting in the determination of the set of initial conditions leading to unsafe states.

However, for backward reachability analysis the system matrix has to be max-plus invertible, i.e. in each row and in each column there should be a single finite element (not equal to $-\infty$), which is restrictive. The main advantage of using max-plus polyhedra is in saving computational complexity, because time complexity of these approaches is polynomial as all standard DBM based algorithms.

From a practical perspective, there are two basic ways of describing max-plus polyhedra. The first one, internal, gives the extreme points and rays, the second one, external, gives linear inequalities over max-plus semiring. It has been shown in Allamigeon, Gaubert, and Goubault (2010), see Allamigeon, Gaubert, and Goubault (2013) for more recent work, how to pass from the external description of a polyhedron to the internal description. Namely, the extremal points are computed in a recursive way, where the problem of checking the extremality of a point reduces to checking whether there is only one minimal strongly connected component in an hyper-graph. For the latter problem there exists a fast (almost linear time) algorithm, which allows quick elimination of redundant generators, but the number of generators can be exponential in general.

## 7. Applications

It can appear somewhat surprising that methods based on very particular structure of max-plus algebra can find a large number of applications. But it turns out that max-plus system theory has been indeed applied to a large variety of domains, such as:

- capacity assessment, evaluation and control of delays in *transportation systems* (Braker, 1991; Heidergott et al., 2006; Houssin, Lahaye, & Boimond, 2007; Kersbergen et al., 2016) and *car traffic* (Farhi, Goursat, & Quadrat, 2011),
- sizing, optimization and production management in *manufacturing systems* (Cohen et al., 1985; Cottenceau, Hardouin, & Ouerghi, 2008; Imaev & Judd, 2008; Martinez & Castagna, 2003),
- performance guarantees in *communication networks* through so-called *network calculus* (Cruz, 1991; Le Boudec, 2001),
- *high throughput screening* in biology and chemistry (Brunsch, Raisch, & Hardouin, 2012),
- modeling, analysis and control of *legged locomotion* (Lopes et al., 2014; Lopes, Kersbergen, De Schutter, van den Boom, & Babuška, 2016),
- *speech recognition* (Mohri, Pereira, & Riley, 2002) or *image processing* (Culik & Kari, 1997) through weighted automata such as max-plus automata,
- *optimization of crop rotation in agriculture* (Bacaër, 2003),
- *scheduling of energy flows for parallel batch processes* (Mutsaers, Özkan, & Backx, 2012),
- *max-plus model of ribosome dynamics during mRNA translation* (Brackley, Broomhead, Romano, & Thiel, 2012),
- *paper handling in printers* (Alirezaei et al., 2012),
- *performance evaluation of the emergency call center 17-18-112 in the Paris area* (Allamigeon, Bœuf, & Gaubert, 2015),
- *control of cluster tools in semiconductor manufacturing* (Kim & Lee, 2015),
- *biological sequence comparisons* (Comet, 2003).

This diversity is to be emphasized all the more since these applications have sometimes suggested new theoretical questions.

We cannot finish this brief overview of the role of max-plus algebra in the history of DES without mentioning the important connections with other fields of research: *dynamic programming and optimal control* with solutions to Hamilton-Jacobi-Bellman (partial) differential equations (Maslov & Kolokoltsov, 1994; Quadrat & Max-Plus, 1994), *statistical mechanics* (Quadrat & Max-Plus, 1997), *operations research* (Zimmermann, 2003).

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