Explosion of Branching Processes

Finding sufficient conditions for infinitely large branching processes

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Layman's summary

In this research, we looked at a version of a branching process. A branching process is, similar to a family tree, a process consisting of individuals that reproduce. Every individual has a certain chance to get an amount of children. For instance, they might have a 6% chance to get one child, a 3% chance to get two children, and so on. Assuming that we know what those chances are, we are going to find how big every generation of individuals is. Once we know that, we can find a constraint on the birth times of the children, so that an explosion takes place. An explosion is an event where there are infinitely many individuals in a finite amount of time.

Summary

We provide sufficient criteria for explosion in an age-dependent branching process. For this, we assume the offspring distribution has a certain form. Given this form, we will construct a lower bound on the generation sizes. After we obtained this lower bound, we will start a process in which we will thin the tree. We will do this by pruning the children of an individual if that individual is born after its assigned time. Doing this for all generations gives rise to a thinned tree, with infinitely many non-empty generations in a finite time. Because of that, there are infinitely many individuals at a finite time, and thus the tree exploded. There is a constraint needed, dependent on the offspring distribution, for the thinned tree to survive. The goal of this thesis is to find that constraint.

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Chapter 1

Introduction

Branching processes have been studied for a long time in probability theory, with papers from before 1950 [11] [5]. Lots of research has been done since then and a lot of advancements have been made.

In this thesis we will study branching processes with birth times as described in [12] or [15]. This means that the distribution of birth times for individuals is independent and identically distributed (i.i.d.), so every individual will have the same distribution for when their children are born. There are no assumptions made on the dependency of birth times of consecutive children of the same person. In these processes, we look for explosion, i.e. we look for a criterion so that there are infinitely many individuals at a finite time.

If an individual is expected to get a finite number of children, then the population grows exponentially in time. That is, the population size at time t is of the order $\exp(\lambda \cdot t)$, with λ being the Malthusian parameter [14]. Under the assumption that the Malthusian parameter exists, the population is expected to grow exponentially [13], and thus explosion doesn't occur.

This thesis will focus on a branching process where explosion might occur. For that, the order of the population size can't be written as as $\exp(\lambda \cdot t)$, as that indicates a finite population. Therefore we will need that the expected number of children of an individual is infinite. This is however not sufficient for explosion, as the birth times can be quite high and thus only result in infinitely many individuals at an infinite time. In this thesis we will focus on finding a constraint on the birth times so that explosion occurs.

There are currently two kinds of branching processes with infinite mean. The first kind is the Galton-Watson branching process. This is a branching process where all children of an individual are born after one unit of time [9]. These types of branching processes do not have a lot of freedom, as their birth times are already set. Galton-Watson branching processes have been researched in the 70's by Schuh and Barbour [17], and Bingham and Doney [6].

The other kind of branching process is the age-dependent branching process. An age-dependent branching process is a process where each individual has an i.i.d. distributed lifetime, and they produce all their children when they die [19]. Around the 70's, Sevast'yanov and Grey gave a sufficient criterion for explosion[18, 10]. In 2013, [1] gave a sufficient condition for explosion using branching random walks.

This was done by starting with one individual at the origin and walking a random length in a direction. Upon arriving, the individual died and produced children, who underwent the same process and so on. This is very similar to a branching process, as the length and direction can be mapped as birth time in a branching process. The number of children that the individual produced is the same as in the branching process.

The branching process we will consider in this thesis is an age-dependent branching process, as there are more applications for age-dependent branching processes. We assume the lifetime is i.i.d. for all individuals and we do not assume anything about the dependency of lifetimes of consecutive births.

One application of these age-dependent branching processes, are random graph models. Random models are locally often similar to trees, i.e. there are no cycles in the process, and so they can locally be graphed as a branching process. These random graph models can be used to model a spreading epidemic. To fully understand how that works, random graph models will first need to be studied locally. Examples of this are the configuration model [16] and inhomogeneous random graphs [8].

A lot of real life networks can be modelled using power-law degree distributions with exponent in $(2,3)$. Some examples are the world wide web [2] and citation networks. If a model has an exponent between $(2,3)$ for the degrees in the graph, can be modelled by a branching process with exponent between $(1,2)$. That means that the expectation of the offspring is infinite, but the second moment is finite. For networks with these power-law degree distributions, it is very important that the branching processes are understood, and thus they can be researched.

So it is very important to research branching processes, as it helps understanding random graphs. More specifically, it helps understanding spreading events on random graphs. For example, very little is known about nondeterministic but non-explosive information diffusion on random graphs with infinite variance degrees [4].There is also not much known when we assume dependencies between the transmission times from a vertex to its neighbors[14]. This is very useful when researching an epidemic, where every individual has an incubation time. That is very crucial to know, as explosion in epidemics should be avoided, especially for deadly diseases.

In this thesis, we will find a sufficient criterion for explosion. There, we assume that we know the offspring distribution, and eventually gain a constraint on the birth times so that the branching process explodes. We will research two families of distributions, and we will arrive at a constraint for both for explosion. First, in Chapter 2 we will define the model we are working with. In Chapter 3 we will calculate a lower bound on the generation sizes for both offspring distributions. Finally in Chapter 4 we will define a thinned tree, which is a subset of the original tree. We will define this thinned tree in such a way, that existence of this tree immediately implies an explosion. Then we will arrive at a constraint for which the tree will survive and we can thus have explosion.

Chapter 2

Model definition

First, we use $\mathbb{N} := \{1, 2, \ldots\}$. *Individuals* are considered to be labeled as elements of the infinite *Ulam-Harris* tree $\mathscr{U}_\infty := \bigcup_{n \geq 0} \mathbb{N}^n$, where $\mathbb{N}^0 := \{\varnothing\}$ contains the *root*, which is the first element of the tree. Elements $u \in \mathscr{U}_\infty$ are denoted as a tuple, such that $u = (u_1, \ldots, u_k) \in \mathbb{N}^k$, $k \geq 1$. For simplicity, this is written as $u = u_1 \ldots u_k$. This is to be interpreted as the u_kth child of the individual $u_1 \ldots u_{k-1}$. For instance, 1 represents the first child of the root and 2 represents the second child of the root. We use $|\cdot|$ to measure the *length* of a tuple u, such that, if $u = \varnothing$, then $|u| = 0$, and if $u = u_1 \ldots u_k$, we have $|u| = k$. We say $p(u)$ is the parent of u, so if $u = u_1 \ldots u_k$, then we have $p(u) = u_1 \ldots u_{k-1}$. Conversely, u is the *child* of $p(u)$. We use *generations*, where generation k is defined as $\mathscr{G}_k := \{u : |u| = k\}$, and $|\mathscr{G}_k|$ is the *size of the generation*.

The *birth interval* $T(u)$ of an individual u is a random variable and is defined as the time between the birth of individual u and its parent $p(u)$. For instance, if $p(u)$ is born at time 1 and u is born at time 3, than $T(u) = 2$. Furthermore we define $T(\varnothing) := 0$. The birth time $\mathscr{B}(u)$ of an individual is defined as

$$\mathscr{B}(\varnothing) := 0 \quad \text{and for } u \in \mathscr{U}_\infty, i \in \mathbb{N}, \quad \mathscr{B}(ui) := \mathscr{B}(u) + T(ui).$$

We also define $F_T(x) := \mathbb{P}\big(T \leq x\big)$. The *population size* of the tree at time t is defined as

$$P(t) := |u \in \mathscr{U}_\infty : \mathscr{B}(u) < t|.$$

We speak of *explosion* if there is a time $t < \infty$ such that $P(t) = \infty$, so if there is a finite time where there are infinitely many individuals.

We say a tree *survives*, when there is no empty generation, so there are infinitely many non-empty generations.

We look at random trees with the offspring distribution defined by the following distribution

$$\mathbb{P}(D \geq k) = \frac{l(k)}{k} \tag{2.1}$$

where D is the random variable that indicates the number of children of an individual. Every individual is assumed to have the distribution as stated in

Equation 2.1. The distribution of individual i is denoted as D_i, and it is assumed that $D, D_1, D_2, \ldots$ are i.i.d. Furthermore, $l(k)$ is any slowly varying function.

Definition 2.0.1. *Let l be a positive measurable function, defined on some neighbourhood $[X, \infty)$ of infinity, and satisfying*

$$l(\lambda x)/l(x) \to 1 \quad (x \to \infty) \quad \forall \lambda > 0$$

then l is said to be ***slowly varying****.*

In this paper we will look at two slowly varying function, namely $l_1(x) = c \cdot \log(x)^\alpha$ and $l_2(x) = c \cdot \exp\left(\log^\gamma(x)\right)$ where $\alpha > -1$ and $\gamma \in (0, 1)$.

First we will show that $l_1(x)$ is a slowly varying function.

$$\lim_{x \to \infty} \frac{l_1(\lambda x)}{l_1(x)} = \lim_{x \to \infty} \frac{\log(\lambda x)^\alpha}{\log(x)^\alpha} \tag{2.2}$$

$$= \lim_{x \to \infty} \frac{\left(\log(\lambda) + \log(x)\right)^\alpha}{\log(x)^\alpha} \tag{2.3}$$

$$= 1 \tag{2.4}$$

And $l_2(x)$ is a slowly varying function as well, as we have that

$$\lim_{x \to \infty} \frac{l_2(\lambda x)}{l_2(x)} = \lim_{x \to \infty} \frac{e^{\log(\lambda x)^\gamma}}{e^{\log(x)^\gamma}} \tag{2.5}$$

$$= \lim_{x \to \infty} \frac{e^{\left(\log(\lambda) + \log(x)\right)^\gamma}}{e^{\log(x)^\gamma}} \tag{2.6}$$

$$= \lim_{x \to \infty} e^{(\log(\lambda) + \log(x))^\gamma - \log(x)^\gamma} \tag{2.7}$$

As seen in Equation 2.3, dividing the differences in the power results in 1 in the limit, therefore they are equal and so their difference is equal to zero, which results in

$$= e^0 \tag{2.8}$$

$$= 1. \tag{2.9}$$

We will use both these functions later and for each of them we will look for a criterion for explosion.

Chapter 3

Lower bound for generation sizes

3.1 Creating recursive equation

We want to find explosion and for that we will look at the generation sizes. We will do that by finding a lower bound s_k for the generation sizes. Let $|\mathscr{G}_k| > s_k$. Then we know that $\mathbb{E}\big[|\mathscr{G}_k|\big] = \mathbb{E}\big[D\big] \cdot |\mathscr{G}_{k-1}|$, as every individual in $|\mathscr{G}_{k-1}|$ gets an expected number of $\mathbb{E}\big[D\big]$ children. We then let s_k grow slower than $\mathscr{G}_k$, so we set $s_k = (1-\delta)\mathbb{E}\big[D\big] \cdot s_{k-1}$ for some $\delta > 0$. We do this, as we expect the size of the k'th generation to be a little less than the size of the previous generation times the expected number of children of an individual. Later we will need that $\mathbb{E}\big[D\big] < \infty$, but $\mathbb{E}\big[D\big] = \infty$, so we decrease the expected value by setting some higher values to zero. We get $s_k = (1-\delta)\mathbb{E}\big[D\mathbb{1}_{(D \le d_k)}\big] \cdot s_{k-1}$, where d_k is some finite series.

We want to find a d_k for which the expectation is finite, so we can obtain a recursive formula for s_k. To calculate this, we want that it's very likely that $|\mathscr{G}_k| > s_k$, so we want that

$$\sum_{k=1}^{\infty} \mathbb{P}\big(|\mathscr{G}_k| \le s_k \big| |\mathscr{G}_{k-1}| \ge s_{k-1}\big) < \infty. \tag{3.1}$$

We can then bound this, as

$$\mathbb{P}\big(|\mathscr{G}_k| \le s_k \big| |\mathscr{G}_{k-1}| \ge s_{k-1}\big) \le \mathbb{P}\big(\sum_{i=1}^{s_{k-1}} D_i \ge s_k\big) \tag{3.2}$$

To further bound this, we need Bernstein's inequality [3], which states If X_i are i.i.d. with $\mathbb{E}[X] < \infty$ and $\mathbb{E}[X^2] < \infty$, then

$$\mathbb{P}\left(\sum_{i=1}^{s_{k-1}} X_i \ge (1-\delta) \cdot s_{k-1} \cdot \mathbb{E}[X]\right) \le \exp\Big(-\delta^2 \cdot \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]} \cdot s_{k-1}\Big). \tag{3.3}$$

As we chose $s_k = (1-\delta)\mathbb{E}\big[D\mathbb{1}_{(D\leq d_k)}\big]\cdot s_{k-1}$, we have finite expectation, so we can apply Bernstein's inequality. Setting $X_i := D_i\mathbb{1}_{(D_i\leq d_k)}$ and $m(d_k) = \mathbb{E}\big[D\mathbb{1}_{(D\leq d_k)}\big]$, we then have

$$\mathbb{P}\left(\sum_{i=1}^{s_{k-1}} D_i \geq (1-\delta)s_{k-1}\cdot m(d_k)\right) \leq \exp\left(-\delta^2\frac{\mathbb{E}[D\mathbb{1}_{(D_i\leq d_k)}]^2}{\mathbb{E}[D^2\mathbb{1}_{(D_i\leq d_k)}]}\cdot s_{k-1}\right) \tag{3.4}$$

We then want to calculate the expectations, so we can find a d_k for which the infinite sum converges. First we calculate $\mathbb{E}\big[D\mathbb{1}_{(D\leq x)}\big]$:

$$\mathbb{E}[D\mathbb{1}_{(D\leq x)}] = \sum_{k=1}^{\infty} k\cdot\mathbb{1}_{(D\leq x)}\cdot\mathbb{P}\big(D=k\big) \tag{3.5}$$

$$= \sum_{k=1}^{x} k\cdot\mathbb{P}\big(D=k\big) \tag{3.6}$$

$$\tag{3.7}$$

We can rewrite this as

$$\mathbb{E}[D\mathbb{1}_{(D\leq x)}] = \sum_{n=1}^{x}\sum_{k=n}^{x}\mathbb{P}\big(D=k\big) \tag{3.8}$$

$$= \sum_{k=1}^{x}\Big(\mathbb{P}\big(D\geq k\big) - \mathbb{P}\big(D>x\big)\Big) \tag{3.9}$$

Writing this in an integral form gives

$$\mathbb{E}[D\mathbb{1}_{(D\leq x)}] = \int_1^x \mathbb{P}\big(D\geq u\big)du - x\cdot\mathbb{P}\big(D>x\big) \tag{3.10}$$

$$= \int_1^x \frac{l(u)}{u}du - l(x) \tag{3.11}$$

$$=: L(x) - l(x) \leq L(x). \tag{3.12}$$

Then, calculating the second expectation:

$$\mathbb{E}[D^2\mathbb{1}_{(D\leq x)}] = \sum_{k=1}^{\infty} k^2\cdot\mathbb{1}_{(D\leq x)}\cdot\mathbb{P}\big(D=k\big) \tag{3.13}$$

$$= \sum_{k=1}^{x} k^2\cdot\mathbb{P}\big(D=k\big) \tag{3.14}$$

As the difference of two consecutive squares is $x^2-(x-1)^2 = x^2-x^2+2x-1 = 2x-1$, we can rewrite this as

$$= \sum_{n=1}^{x}(2n-1)\cdot\left(\sum_{k=n}^{x}\mathbb{P}\big(D=k\big)\right) \tag{3.15}$$

$$= \sum_{k=1}^{x}(2k-1)\Big(\mathbb{P}\big(D\geq k\big) - \mathbb{P}\big(D>x\big)\Big) \tag{3.16}$$

Further simplifying gives

$$\leq \sum_{k=1}^{x} 2k\Big(\mathbb{P}\big(D \geq k\big) - \mathbb{P}\big(D > x\big)\Big) \tag{3.17}$$

$$\leq \sum_{k=1}^{x} 2k\mathbb{P}\big(D \geq k\big) \tag{3.18}$$

$$\tag{3.19}$$

Writing this in an integral form gives

$$\mathbb{E}[D^2 \mathbb{1}_{(D \leq x)}] \leq 2\int_1^x u \cdot \mathbb{P}\big(D \geq u\big)du \tag{3.20}$$

$$= 2\int_1^x l(u)du \tag{3.21}$$

$$\leq Cxl(x), \tag{3.22}$$

for some constant C. The last equation holds because of Karamata theory [7].

Now that we have bound for those expectations, we can substitute them into Equation 3.4, we get

$$\mathbb{P}\left(\sum_{i=1}^{s_{k-1}} D_i \geq (1-\delta)s_{k-1} \cdot m(d_k)\right) \leq \exp\left(-\delta^2 \cdot \frac{L(d_k)^2}{l(d_k) \cdot d_k} \cdot s_{k-1}\right) \tag{3.23}$$

Now that we have that bound, combining Equation 3.1, Equation 3.2 and Equation 3.23, we get

$$\sum_{k=1}^{\infty} \exp\left(-\delta^2 \cdot \frac{L(d_k)^2}{l(d_k) \cdot d_k} \cdot s_{k-1}\right) < \infty. \tag{3.24}$$

We have $l_1(x) = c \cdot \log^{\alpha}(x)$, and we defined $L(x) := \int_1^x \frac{l(u)}{u}du$, we get

$$L_1(x) = \int_1^x \frac{c \cdot \log(u)^{\alpha}}{u}du \tag{3.25}$$

Using the substitution $y = \log(u), dy = \frac{1}{u}du$, we get

$$= c \cdot \int_0^{\log(x)} y^{\alpha}dy \tag{3.26}$$

$$= c \cdot \Big[\frac{1}{\alpha+1}y^{\alpha+1}\Big]_0^{\log(x)} \tag{3.27}$$

$$= \frac{c}{\alpha+1}\log^{\alpha+1}(x) \tag{3.28}$$

Filling in the formulas for $l_1(x)$ and $L_1(x)$ into Equation 3.24 and summing them gives

$$\sum_{k=1}^{\infty} \exp\left(-\delta^2 \cdot A\frac{\log(d_k)^{2\alpha+2}}{\log(d_k)^{\alpha} \cdot d_k} \cdot s_{k-1}\right) = \sum_{k=1}^{\infty} \exp\left(-\delta^2 \cdot A\frac{\log(d_k)^{\alpha+2}}{d_k} \cdot s_{k-1}\right), \tag{3.29}$$

for constant $A = \frac{c}{(\alpha+1)^2}$. Our recursive equation is $s_k = (1-\delta)\mathbb{E}\big[D\mathbb{1}_{(D\leq d_k)}\big] \cdot s_{k-1} = (1-\delta)L(d_k)\cdot s_{k-1}$. Filling in the equation for $L(x)$, we get

$$s_k = (1-\delta)\cdot s_{k-1}\cdot \frac{c}{\alpha+1}\cdot \log^{\alpha+1}(d_k). \tag{3.30}$$

We want our lower bound to big as big as possible, so we choose d_k in such a way that Equation 3.30 grows as quickly as possible, but still satisfies Equation 3.29. Therefor we look specifically at the terms dependent on k in the exponent, so

$$\frac{\log^{\alpha+2}(d_k)}{d_k}\cdot s_{k-1}. \tag{3.31}$$

We will search for d_k of the form $d_k = s_{k-1}\cdot \log^{\beta}(s_{k-1})$. Equation 3.31 then becomes

$$\begin{aligned}\frac{\log^{\alpha+2}(s_{k-1}\cdot\log^{\beta}(s_{k-1}))}{s_{k-1}\cdot\log^{\beta}(s_{k-1})}\cdot s_{k-1} &= \frac{\log^{\alpha+2}(s_{k-1}\cdot\log^{\beta}(s_{k-1}))}{\log^{\beta}(s_{k-1})} &(3.32)\\ &= \frac{\Big(\log(s_{k-1})+\beta\log(\log(s_{k-1}))\Big)^{\alpha+2}}{\log^{\beta}(s_{k-1})} &(3.33)\end{aligned}$$

Rewriting this gives

$$\begin{aligned}\frac{\Big(\log(s_{k-1})+\log^{\beta}(s_{k-1})\Big)^{\alpha+2}}{\log^{\beta}(s_{k-1})} &= \frac{\log^{\alpha+2}(s_{k-1})\cdot(1+\frac{\beta\log(\log(s_{k-1}))}{\log(s_{k-1})})}{\log^{\beta}(s_{k-1})} &(3.34)\\ &= \log^{\alpha+2-\beta}(s_{k-1})\cdot\left(1+\frac{\beta\log(\log(s_{k-1}))}{\log(s_{k-1})}\right)^{\alpha+2} &(3.35)\\ &\leq \log^{\alpha+2-\beta}(s_{k-1})\cdot 2^{\alpha+2} &(3.36)\end{aligned}$$

To then obtain summability, we need $\alpha+2-\beta>0$, therefore we need $\beta<\alpha+2$. If we choose $\beta<\alpha+1$, we then have $1<\alpha+2-\beta$. We can write this as $1+\varepsilon\leq\alpha+2-\beta$ for some $\varepsilon>0$. In the power of the exponent we now have $-\log^{1+\varepsilon}(s_{k-1})$. Then we have

$$\begin{aligned}\exp\left(-\delta^2\cdot A\frac{\log(d_k)^{\alpha+2}}{d_k}\cdot s_{k-1}\right) &\sim \exp\left(-\log^{1+\varepsilon}(s_{k-1})\right) &(3.37)\\ &= \exp\left(-\log(s_{k-1})\cdot\log^{\varepsilon}(s_{k-1})\right) &(3.38)\\ &= \left(\frac{1}{s_{k-1}}\right)^{\log^{\varepsilon}(s_{k-1})}. &(3.39)\end{aligned}$$

As the exponent tends to infinity, the sum over all k converges, so

$$\sum_{k=1}^{\infty} \left(\frac{1}{s_{k-1}} \right)^{\log^{\varepsilon}(s_{k-1})} < \infty. \tag{3.40}$$

So now that we have found a d_k for which Equation 3.24 holds true, we can substitute that value of d_k into the recursive formula, which gives

$$\begin{aligned} s_k &= (1-\delta) \cdot s_{k-1} \cdot \frac{c}{\alpha+1} \cdot \log^{\alpha+1}(d_k) & (3.41) \\ &= (1-\delta) \cdot s_{k-1} \cdot \frac{c}{\alpha+1} \cdot \Big(\log^{\alpha+1}(s_{k-1}) + \beta \log(\log(s_{k-1})) \Big)^{\alpha+1} & (3.42) \\ &= (1-\delta) \cdot s_{k-1} \cdot \frac{c}{\alpha+1} \cdot \log^{\alpha+1}(s_{k-1}) \cdot \left(1 + \frac{\beta \log(\log(s_{k-1}))}{\log(s_{k-1})} \right)^{\alpha+1} & (3.43) \end{aligned}$$

As $\log(x)$ grows much faster than $\log(\log(x))$, therefore their fraction is $o(1)$. Filling that in and simplifying constants gives

$$\begin{aligned} &= (1-\delta) \cdot s_{k-1} \cdot \frac{c}{\alpha+1} \cdot \log^{\alpha+1}(s_{k-1}) \cdot \Big(1 + o(1) \Big)^{\alpha+1} & (3.44) \\ &= (1-\hat{\delta}) \cdot s_{k-1} \cdot \frac{c}{\alpha+1} \cdot \log^{\alpha+1}(s_{k-1}) & (3.45) \\ &= M \cdot s_{k-1} \cdot \log^{\alpha+1}(s_{k-1}), & (3.46) \end{aligned}$$

for some constant M. Now we have a recursive equation for s_k, which we want to solve.

3.2 Solving for s_k

Lemma 1. *Let a_k be a series satisfying the differential equation*

$$a_k = N \cdot a_{k-1} \cdot \log^{\alpha+1}(a_{k-1})$$

for some constant $N > 0$ and some $\alpha + 1 > 0$. Then the solution of a_k is

$$\exp\Big(\alpha k \log(k)\Big) \leq a_k \leq \exp\Big(\alpha k \log(k) \log(\log(k))\Big)$$

Proof. Let a_k be a series satisfying the differential equation, with $N > 0$ and $\alpha + 1 > 0$ constants. Then, dividing by a_{k-1} gives

$$\frac{a_k}{a_{k-1}} = N \cdot \log^{\alpha+1}(a_{k-1}) \tag{3.47}$$

Taking the logarithm on both sides gives

$$\log(a_k) - \log(a_{k-1}) = \log(N) + (\alpha+1) \cdot \log(\log(a_{k-1})) \tag{3.48}$$

Substituting $y_k = \log(a_k)$ gives

$$y_k - y_{k-1} = \log(N) + (\alpha+1) \cdot \log(y_{k-1}) \tag{3.49}$$

This gives the differential equation

$$y'(k) = \log(N) + (\alpha+1) \cdot \log(y(k)) \tag{3.50}$$

Removing the constant gives

$$y'(k) \geq (\alpha+1) \cdot \log(y(k)) \tag{3.51}$$

We want to see if the solution $\hat{y}(k) = (\alpha+1)k \cdot \log(k)$ is a valid solution. Therefore, we have to calculate both sides of the equation and see if the inequality holds. We get that

$$\hat{y}'(k) = \alpha \cdot \big(1 + \log(k)\big) \tag{3.52}$$

and

$$\begin{aligned}
(\alpha+1) \cdot \log(\hat{y}(k)) &= (\alpha+1) \cdot \big(\log(\alpha) + \log(k) + \log(\log(k))\big) && (3.53)\\
&= (\alpha+1)\log(k) \cdot \left(1 + \frac{\log(\alpha)}{\log(k)} + \frac{\log(\log(k))}{\log(k)}\right) && (3.54)\\
&= (\alpha+1)\log(k) \cdot \big(1 + o(1)\big) && (3.55)
\end{aligned}$$

Simplifying Equation 3.55, we see

$$\begin{aligned}
(\alpha+1) \cdot \log(\hat{y}(k)) &= (\alpha+1)\log(k) \cdot \big(1 + o(1)\big) && (3.56)\\
&\geq (\alpha+1)\log(k). && (3.57)
\end{aligned}$$

Then, for $k > \exp\big(\alpha\big)$ we have

$$\begin{aligned}
(\alpha+1)\log(k) &> \alpha\big(\log(k) + 1\big) && (3.58)\\
&= \hat{y}(k) && (3.59)
\end{aligned}$$

So we do not satisfy Equation 3.51, so the function $y(k)$ that does satisfy the equation is greater than $\hat{y}(k) = \alpha k \log(k)$. Now that we have a lower bound, we want to find an upper bound that is as close as possible to this lower bound. We will try $\tilde{y}(k) = \alpha k \log(k) \log(\log(k))$. Computing both sides of Equation 3.51 we get

$$\tilde{y}'(k) = 2\alpha + \alpha \log(k) \cdot \log(\log(k)) \tag{3.60}$$

and

$$(\alpha+1) \cdot \log(\tilde{y}(k)) = (\alpha+1)\big(\log(\alpha) + \log(k) + \log(\log(k)) + \log(\log(\log(k)))\big). \tag{3.61}$$

If we set

$$K(k) := \left(\frac{\log(\alpha)}{\log(k)\log(\log(k))} + \frac{1}{\log(\log(k))} + \frac{1}{\log(k)} + \frac{\log(\log(\log(k)))}{\log(k)\log(\log(k))}\right),$$

we get that

$$(\alpha+1)\cdot\log(\tilde{y}(k)) = (\alpha+1)\log(k)\log(\log(k))\cdot K(k) \tag{3.62}$$

We can see that $K(k) = o\big(\frac{1}{\log(k)}\big)$, so substituting that in Equation 3.62 gives

$$\begin{aligned}(\alpha+1)\cdot\log(\tilde{y}(k)) &= (\alpha+1)\log(k)\log(\log(k))\cdot o\left(\frac{1}{\log(k)}\right) && (3.63)\\ &\leq (\alpha+1)\log(k)\log(\log(k))\cdot\frac{1}{\log(k)} && (3.64)\\ &= (\alpha+1)\log(\log(k)) && (3.65)\end{aligned}$$

We then get the inequality

$$\begin{aligned}(\alpha+1)\log(\log(k)) &\leq (\alpha+1)\log(k)\cdot\log(\log(k)) && (3.66)\\ &\leq (\alpha+2)\log(k)\cdot\log(\log(k)) && (3.67)\\ &= \tilde{y}'(k). && (3.68)\end{aligned}$$

So we have shown that for $\tilde{y}'(k) = \alpha k\log(k)\cdot\log(\log(k))$ it holds that $\tilde{y}'(k) \geq (\alpha+1)\cdot\log(\tilde{y}(k))$, and thus satisfies Equation 3.51. As $\tilde{y}(k)$ is only slightly greater than $\hat{y}(k)$, we see that $\hat{y}(k) = \alpha k\log(k)$ is a lower bound for $y(k)$ that cannot be improved much further. We now know that $y(k) \geq \alpha k\log(k)$, and as we took $y(k) = log(s_k)$, we have $\exp\big(\alpha k\log(k)\big) \leq a_k \leq \exp\big(\alpha k\log(k)\log(\log(k))\big)$. □

Using Lemma 1 on our differential equation, we get an upper and lower bound on our s_k. We now know that $|\mathscr{G}_k| \geq s_k \geq \exp\big(\alpha k\log(k)\big)$.

Now that we have a good lower bound on the generation sizes for this slowly varying function, we will now calculate a lower bound for the other slowly varying function.

3.3 The other slowly varying function

We already calculated s_k for $l_1(x)$, but now we will calculate it for $l_2(x)$. We have Equation 3.24, so we want to calculate $L_2(x)$. We get

$$L_2(x) = c\cdot\int_1^x \frac{\exp\big(\log^\gamma(u)\big)}{u}du. \tag{3.69}$$

Using the substitution $y = \log(u)$, $dy = \frac{1}{u}du$, we get

$$= c\cdot\int_0^{\log(x)} \exp(y^\gamma)dy. \tag{3.70}$$

We can bound this integral from above by $c \cdot \log(x) \cdot \exp\left(\log^\gamma(x)\right)$, as it is the maximum value of the integral times its width. The integral can be bounded from below by $c \cdot \exp\left(\log^\gamma(x)\right)$. So we have

$$c \cdot \exp\left(\log^\gamma(x)\right) \leq c \cdot \int_0^{\log(x)} \exp\left(y^\gamma\right) dy \leq c \cdot \log(x) \cdot \exp\left(\log^\gamma(x)\right). \tag{3.71}$$

Using the upper bound of $L_2(x)$, Equation 3.24 becomes

$$\sum_{k=1}^{\infty} \exp\left(-\delta^2 \cdot \frac{L(d_k)^2}{l(d_k) \cdot d_k} \cdot s_{k-1}\right) \tag{3.72}$$

$$\leq \sum_{k=1}^{\infty} \exp\left(-\delta^2 \cdot c \cdot \frac{\log^2(d_k)}{d_k} \exp\left(\log^\gamma(d_k)\right) \cdot s_{k-1}\right) < \infty \tag{3.73}$$

For our recursive equation we will use the lower bound of $L_2(x)$ to get

$$s_k = (1-\delta) \cdot c \cdot s_{k-1} \cdot \exp\Big(\log^\gamma(d_k)\Big). \tag{3.74}$$

We want our s_k to be as big as possible, so we choose d_k in such a way that Equation 3.74 grows as quickly as possible, but still satisfies Equation 3.73. Therefor we look specifically at the terms dependent on k in the exponent, so

$$\frac{\log^2(d_k)}{d_k} \cdot \exp\Big(\log^\gamma(d_k)\Big) \cdot s_{k-1}. \tag{3.75}$$

We will search for d_k of the form $d_k = s_{k-1} \cdot \exp\Big(\log^\beta(s_{k-1})\Big)$. Equation 3.75 then becomes

$$\frac{\log^2(d_k)}{d_k} \cdot \exp\Big(\log^\gamma(d_k)\Big) \cdot s_{k-1} \tag{3.76}$$

$$= \exp\Big(N^\gamma(k)\Big) \cdot \frac{N^2(k)}{\exp\Big(\log^\beta(s_{k-1})\Big)} \tag{3.77}$$

$$= \exp\Big(N^\gamma(k) - \log^\beta(s_{k-1}) + 2\log\Big(N(k)\Big)\Big), \tag{3.78}$$

where $N(K) = \left(\log(s_{k-1}) + \log^\beta(s_{k-1})\right)$. If we choose $\beta \in (0, \gamma)$, we can simplify $N(k)$ as follows:

$$\left(\log(s_{k-1}) + \log^\beta(s_{k-1})\right) = \log(s_{k-1})\left(1 + \frac{\log^\beta(s_{k-1})}{\log(s_{k-1}}\right) \tag{3.79}$$

$$= \log(s_{k-1})\big(1 + o(1)\big) \tag{3.80}$$

$$\sim \log(s_{k-1}). \tag{3.81}$$

We can then simplify Equation 3.78 to

$$\sim \exp\left(\log^{\gamma}(s_{k-1}) - \log^{\beta}(s_{k-1}) + 2\log\Big(\log(s_{k-1})\Big)\right) \tag{3.82}$$

$$= \exp\left(\log^{\gamma}(s_{k-1})\Big(1 + \frac{\log^{\beta}(s_{k-1})}{\log^{\gamma}(s_{k-1})}\Big) + 2\log\Big(\log(s_{k-1})\Big)\right) \tag{3.83}$$

$$= \exp\left(\log^{\gamma}(s_{k-1})\big(1 + o(1)\big) + 2\log\Big(\log(s_{k-1})\Big)\right). \tag{3.84}$$

We can simplify this further to

$$\sim \exp\left(\log^{\gamma}(s_{k-1}) + 2\log\Big(\log(s_{k-1})\Big)\right) \tag{3.85}$$

$$= \exp\left(\log^{\gamma}(s_{k-1})\Big(1 + \frac{2\log(\log(s_{k-1}))}{\log^{\gamma}(s_{k-1})}\Big)\right) \tag{3.86}$$

$$\sim \exp\Big(\log^{\gamma}(s_{k-1})\Big) \tag{3.87}$$

So if we choose $\beta \in (0, \gamma)$, we have

$$\exp\left(-\delta^2 \cdot A\frac{\log(d_k)^{\alpha+2}}{d_k} \cdot s_{k-1}\right) \sim \exp\left(-\exp\Big(\log^{\gamma}(s_{k-1})\Big)\right). \tag{3.88}$$

As the exponent grows faster than a linear function, we have

$$\sum_{k=1}^{\infty} \exp\left(-\exp\Big(\log^{\gamma}(s_{k-1})\Big)\right) \leq \sum_{k=1}^{\infty} \exp\left(-k\right) < \infty \tag{3.89}$$

So now that we have found a d_k for which Equation 3.24 holds true, we cab substitute that value of d_k into the recursive formula, which gives

$$s_k = (1-\delta) \cdot c \cdot s_{k-1} \cdot \exp\left(\log^{\gamma}\Big(\exp\left(\log(s_{k-1})\right) \cdot s_{k-1}\Big)\right) \tag{3.90}$$

$$= (1-\delta) \cdot c \cdot s_{k-1} \cdot \exp\left(\Big(2\log(s_{k-1})\Big)^{\gamma}\right) \tag{3.91}$$

$$\geq M \cdot s_{k-1} \cdot \exp\left(\log^{\gamma}(s_{k-1})\right), \tag{3.92}$$

for $M = (1-\delta) \cdot c$. We now have a recursive equation for s_k, which we want to solve.

Lemma 2. *Let a_k be a series satisfying the differential equation*

$$a_k \geq N \cdot a_{k-1} \cdot \exp\Big(\log^{\gamma}(a_{k-1})\Big)$$

for some constant $N > 0$ and some $\gamma \in (0,1)$. Then the solution of a_k is

$$a_k \geq \exp\left(k^{\left(\frac{1}{1-\gamma}\right)}\right)$$

.

Proof. Let a_k be a series satisfying the differential equation, with $N > 0$ and $\gamma \in (0,1)$ constants. Then, dividing by a_{k-1} gives

$$\frac{a_k}{a_{k-1}} \geq N \cdot \exp\left(\log^{\gamma}(a_{k-1})\right). \tag{3.93}$$

Taking the logarithm on both sides gives

$$\log(a_k) - \log(a_{k-1}) \geq \log(N) + \log^{\gamma}(a_{k-1}). \tag{3.94}$$

Substituting $y_k = \log(a_k)$ gives

$$y_k - y_{k-1} \geq \log(N) + y_{k-1}^{\gamma} \tag{3.95}$$

Removing the constant and writing this as a differential equation gives

$$y'(k) \geq y(k)^{\gamma}. \tag{3.96}$$

We want to see if the solution $\hat{y}(k) = k^{\left(\frac{1}{1-\gamma}\right)}$ is a valid solution. Therefor, we have to calculate both sides of the equation and see if the inequality holds. We get that

$$\hat{y}'(k) = \frac{1}{1-\gamma} \cdot k^{\left(\frac{\gamma}{1-\gamma}\right)} \tag{3.97}$$

and

$$\hat{y}^{\gamma}(k) = k^{\left(\frac{\gamma}{1-\gamma}\right)}. \tag{3.98}$$

As we have $\gamma \in (0,1)$, we have that $\hat{y}'(k) \geq \hat{y}^{\gamma}(k)$, we have that $y(k) = k^{\left(\frac{1}{1-\gamma}\right)}$ is a valid solution to Equation 3.96. As we took $y(k) = \log(a_k)$, we have that $a_k = \exp\left(k^{\left(\frac{1}{1-\gamma}\right)}\right)$. □

Using Lemma 2 on out differential equation, we get an lower bound on out s_k. We now know that, for $l(x) = c \cdot \exp\left(\log^{\gamma}(x)\right)$, we have $|\mathscr{G}_k| \geq s_k \sim \exp\left(k^{\left(\frac{1}{1-\gamma}\right)}\right)$. Now that we have a good lower bound on the generation sizes, we will find a criteria for explosion in Section 4.

Chapter 4

Thinned Tree

4.1 New model

We will look at a specific thinned version of the model mentioned in Section 2, where we will thin the tree $\mathscr{U}_\infty$ by pruning some branches of the tree. The tree from this new model will be indicated by $\tilde{\mathscr{U}_\infty}$, and the generations of $\tilde{\mathscr{U}_\infty}$ will be written as $\tilde{\mathscr{G}_k}$, where $\tilde{\mathscr{G}_k} \subseteq \mathscr{G}_k$.

Definition 4.1.1 (Thinned tree)**.** *Given a series $t_k > 0$, the thinned tree is defined as*

$$\tilde{\mathscr{U}_\infty} := \bigcup_{k \geq 2} \{u \in \mathscr{G}_k \big| T(p(u)) < t_{k-1}\} \cup \mathscr{G}_1 \cup \varnothing$$

In generation 1, nothing gets pruned, so $\tilde{\mathscr{G}_1} = \mathscr{G}_1$.In generation k, $k > 1$, an individual u will get pruned if its parent $p(u)$ has a greater birth interval than t_{k-1}. So every element of the series t_k is assigned to generation $\mathscr{G}_k$, and if an individual u from generation $\mathscr{G}_k$ has a greater birth interval than t_k, all its children and their children and so on will be pruned.

As $\tilde{\mathscr{U}_\infty} \subseteq \mathscr{U}_\infty$, if explosion happens in $\tilde{\mathscr{U}_\infty}$, it also happens in $\mathscr{U}_\infty$. Therefore we will search for explosion in $\tilde{\mathscr{U}_\infty}$. If the tree $\tilde{\mathscr{U}_\infty}$ survives, by definition that means that there are infinitely many non-empty generations. As individuals in generation k are thinned in a way so that they only exist if their parents in the previous generation $k-1$ are all born before time $\sum_{i=1}^{k-1} t_{k-1}$. Therefore, if there are infinitely many generations, they all exist before time $\sum_{k=1}^{\infty} t_k$. Therefore, we want to find a series t_k for which $\sum_{k=1}^{\infty} t_k < \infty$. Then there are infinitely many non-empty generations in a finite time, and therefore infinitely many individuals in a finite time. In short, if $\tilde{\mathscr{U}_\infty}$ survives, explosion occurs.

4.2 Explosion in thinned tree

First we have two conditions for survivability:

Theorem 3 (Condition for explosion for $l_1(x)$)**.** *Let $\tilde{\mathscr{U}}_\infty$ be a thinned tree by t_k with $\sum_{k=1}^{\infty} t_k < \infty$ and offspring distribution $\mathbb{P}\big(D \geq k\big) = \frac{c \cdot \log^{\alpha}(k)}{k}$. Then the tree survives if*

$$F_T(x) >> x^{\alpha+1-\varepsilon}$$

for x close to zero and for some $\varepsilon > 0$.

Theorem 4 (Condition for explosion for $l_2(x)$)**.** *Let $\tilde{\mathscr{U}}_\infty$ be a thinned tree by t_k with $\sum_{k=1}^{\infty} t_k < \infty$ and offspring distribution $\mathbb{P}\big(D \geq k\big) = \frac{c \cdot \exp\left(\log^{\gamma}(k)\right)}{k}$. Then the tree survives if*

$$F_T(x) >> \exp\left((\varepsilon - \alpha)x^{\left(\frac{1}{1-\gamma}\right)}\right)$$

for some $\varepsilon > 0$.

Before we prove these theorems, we want to find a lower bound $\tilde{s}_k$ on the generation size of the thinned tree. To find this lower bound, we use the same equation as in Section 3, but we introduce the term $p_k := \mathbb{P}\big(T \leq t_k\big)$. So p_k is the chance that an individuals doesn't get pruned. We get

$$\tilde{s}_k = (1-\delta)p_{k-1}\mathbb{E}\big[D\mathbb{1}_{(D \leq \tilde{d}_k)}\big]\tilde{s}_{k-1} \tag{4.1}$$

Repeating the calculations done in Equation 3.2 until Equation 3.23 in Section 3, we get that

$$\mathbb{P}\big(|\tilde{\mathscr{G}}_k| \leq \tilde{s}_k \big| |\tilde{\mathscr{G}}_{k-1}| \geq \tilde{s}_{k-1}\big) \leq \exp\left(-\delta^2 \cdot \frac{L(\tilde{d}_k)^2}{l(\tilde{d}_k) \cdot \tilde{d}_k} \cdot p_{k-1} \cdot \tilde{s}_{k-1}\right) \tag{4.2}$$

$$=: e_k. \tag{4.3}$$

We want all generation sizes to be greater than the lower bound. Therefore we want

$$\mathbb{P}\left(\forall k : |\tilde{\mathscr{G}}_k| \geq \tilde{s}_k\right) > 0. \tag{4.4}$$

We also have

$$\mathbb{P}\left(\forall k : |\tilde{\mathscr{G}}_k| \geq \tilde{s}_k\right) = \prod_{k=1}^{\infty} \mathbb{P}\left(|\tilde{\mathscr{G}}_k| \geq \tilde{s}_k \big| |\tilde{\mathscr{G}}_{k-1}| \geq \tilde{s}_{k-1}\right) \tag{4.5}$$

$$\geq \prod_{k=1}^{\infty}\Big(1 - e_k\Big). \tag{4.6}$$

So, combining Equation 4.4 and Equation 4.6, we have

$$\prod_{k=1}^{\infty}\Big(1 - e_k\Big) > 0. \tag{4.7}$$

Corollary 4.1. *Let a_i be a series with $a_i \in [0,1) \quad \forall i \in \mathbb{N}$. Then*

$$\sum_{i=1}^{\infty} a_i < \infty \implies \prod_{i=1}^{\infty}(1-a_i) > 0$$

holds true.

Proof. Suppose $\sum_{i=1}^{\infty} a_i < \infty$. We want to show that $\prod_{i=1}^{\infty}(1-a_i) > 0$. By taking the natural logarithm it suffices to show that $\sum_{i=1}^{\infty} \log(1-a_i) > -\infty$. Then we can use the Taylor expansion of the natural logarithm.

$$\log(1-x) = -x - \frac{x^2}{2} - O(x^3)$$

Therefore we have

$$\sum_{i=1}^{\infty} \log(1-a_i) = \sum_{i=1}^{\infty}\Big(-a_i - \frac{a_i^2}{2} - O(a_i^3)\Big) \tag{4.8}$$

$$= -\sum_{i=1}^{\infty} a_i - \sum_{i=1}^{\infty} O(a_i^2) \tag{4.9}$$

$$> -2\sum_{i=1}^{\infty} a_i \tag{4.10}$$

$$> -\infty \tag{4.11}$$

Since the infinite sum only contains numbers in $[0,1)$, the second sum is smaller than the first sum Which is the desired inequality. □

Proof of Theorem 3. Using Corollary 4.1 and Equation 4.7, we see that it is sufficient to show

$$\sum_{k=1}^{\infty} e_k < \infty. \tag{4.12}$$

if we want the tree to survive. Repeating the calculations done in Equation 3.24 until Equation 3.46 in Section 3, we get the recursive equation

$$\tilde{s}_k = M \cdot p_{k-1}\tilde{s}_{k-1} \log^{\alpha+1}(p_{k-1}\tilde{s}_{k-1}). \tag{4.13}$$

Now we want to find a p_k, such that we can get a constraint on T to find explosion. If we want explosion, we want $p_{k-1}\tilde{s}_{k-1} \to \infty$, so that $\tilde{s}_k$ grows as quickly as possible. Therefore we need that $p_{k-1} >> \tilde{s}_{k-1}$. We can construct another constraint for p_{k-1} from Equation 4.13, as we want $\tilde{s}_k$ to be an increasing function, so $\frac{\tilde{s}_k}{\tilde{s}_{k-1}} \geq 1$. From Equation 4.13 we can then see that

$$M \cdot p_{k-1} \log^{\alpha+1}(p_{k-1}\tilde{s}_{k-1}) \geq 1. \tag{4.14}$$

We just want the order of magnitude of p_k, so the term M can be left out of the equation. Rewriting this equation gives

$$p_{k-1} \sim \frac{1}{\log^{\alpha+1}(p_{k-1}\tilde{s}_{k-1})}. \tag{4.15}$$

This is a difficult recursive equation so solve with the p_{k-1} term on both sides of the equation. Therefore we will remove that term on the right-hand side of the equation in the following way:

$$\log^{\alpha+1}(p_{k-1}\tilde{s}_{k-1}) = \Big(\log(p_{k-1}) + \log(\tilde{s}_{k-1})\Big)^{\alpha+1}. \tag{4.16}$$

As we already found that $p_{k-1} >> \tilde{s}_{k-1}$, we know that p_k is of a greater order than $\tilde{s}_k$, so we can bound the right-hand side of Equation 4.16 by

$$\Big(\log(p_{k-1}) + \log(\tilde{s}_{k-1})\Big)^{\alpha+1} \leq \log^{\alpha+1-\varepsilon}(\tilde{s}_{k-1}), \tag{4.17}$$

for some $\varepsilon > 0$. Substituting Equation 4.16 and Equation 4.17 into Equation 4.15, we get

$$p_{k-1} \sim \frac{1}{\log^{\alpha+1-\varepsilon}(\tilde{s}_{k-1})}. \tag{4.18}$$

To further simplify Equation 4.13, we substitute Equation 4.18 to obtain

$$\tilde{s}_k \geq M \cdot \frac{1}{\log^{\alpha+1-\varepsilon}(\tilde{s}_{k-1})}\tilde{s}_{k-1}\log^{\alpha+1}\left(\frac{1}{\log^{\alpha+1-\varepsilon}(\tilde{s}_{k-1})}\tilde{s}_{k-1}\right) \tag{4.19}$$

Further simplifying this equation gives

$$\tilde{s}_k \geq M \cdot \log^{\varepsilon-\alpha-1}(\tilde{s}_{k-1})\tilde{s}_{k-1}\left(\log(\tilde{s}_{k-1}) - (\alpha+1-\varepsilon)\frac{\log(\log(\tilde{s}_{k-1}))}{\log(\tilde{s}_{k-1})}\right)^{\alpha+1} \tag{4.20}$$

$$\geq M \cdot \log^{\varepsilon}(\tilde{s}_{k-1})\tilde{s}_{k-1}\left(1 - \frac{(\alpha+1-\varepsilon)\log(\log(\tilde{s}_{k-1}))}{\log(\tilde{s}_{k-1})}\right)^{\alpha+1}. \tag{4.21}$$

For $\varepsilon \in (0, \alpha+1)$, we can further rewrite this as

$$\tilde{s}_k \geq M \cdot \log^{\varepsilon}(\tilde{s}_{k-1})\tilde{s}_{k-1}\big(1 + o(1)\big)^{\alpha+1} \tag{4.22}$$

$$\geq \tilde{s}_{k-1} \cdot \log^{\varepsilon}(\tilde{s}_{k-1}). \tag{4.23}$$

Using Lemma 1, we get the solution $\tilde{s}_k \geq \exp\Big(\varepsilon k \log(k)\Big)$. As we chose p_k in Equation 4.18, we now have

$$p_k \sim \frac{1}{\log^{\alpha+1-\varepsilon}\Big(\exp\big(\varepsilon k \log(k)\big)\Big)} \tag{4.24}$$

$$= \frac{1}{\big(\varepsilon k \log(k)\big)^{\alpha+1-\varepsilon}} \tag{4.25}$$

As we chose $p_k = \mathbb{P}\Big(T \leq t_k\Big)$, we now have

$$\mathbb{P}\Big(T \leq t_k\Big) = \frac{1}{\big(\varepsilon k \log(k)\big)^{\alpha+1-\varepsilon}} \tag{4.26}$$

From here, we want to find a series t_k. We can write $F_T(x) = \mathbb{P}\Big(T \leq x\Big)$. Using this notation in Equation 4.26, we get

$$F_T(t_k) = \frac{1}{\big(\varepsilon k \log(k)\big)^{\alpha+1-\varepsilon}}. \tag{4.27}$$

Now we can take the inverse of $F_T(x)$ to find an expression for t_k.

$$t_k = F_T^{-1}\left(\frac{1}{\big(\varepsilon k \log(k)\big)^{\alpha+1-\varepsilon}}\right) \tag{4.28}$$

To obtain survival, the sum of time should be finite, so we must have

$$\sum_{k=1}^{\infty} F_T^{-1}\left(\frac{1}{\big(\varepsilon k \log(k)\big)^{\alpha+1-\varepsilon}}\right) < \infty \tag{4.29}$$

The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to infinity, so therefore we know that

$$F_T^{-1}\left(\frac{1}{\big(\varepsilon k \log(k)\big)^{\alpha+1-\varepsilon}}\right) = o\Big(\frac{1}{k}\Big) \tag{4.30}$$

To simplify this, we will substitute $x := \frac{1}{k}$, so $k = \frac{1}{x}$ into Equation 4.30:

$$F_T^{-1}\left(\frac{x^{\alpha+1-\varepsilon}}{\big(\varepsilon \log\big(\frac{1}{x}\big)\big)^{\alpha+1-\varepsilon}}\right) = o\big(x\big) \tag{4.31}$$

Then we can take the cumulative distribution function $F_T(x)$ on both sides of Equation 4.31, which gives

$$\frac{x^{\alpha+1-\varepsilon}}{\big(\varepsilon \log\big(\frac{1}{x}\big)\big)^{\alpha+1-\varepsilon}} = F_T\Big(o\big(x\big)\Big) << F_T(x), \tag{4.32}$$

for $\varepsilon \in (0, \alpha+1)$. Then, if we choose $\varepsilon > 0$ close to zero, the left-hand side of Equation 4.32 is close to zero around $x = 0$, and thus, if there exists an $\varepsilon > 0$ for which we have that $F_T(x) >> x^{\alpha+1-\varepsilon}$, the tree survives. □

Then we have that, if $F_T(x) >> x^{\alpha+1-\varepsilon}$ and $l(x) = c \cdot \log^{\alpha}(x)$ for some $\varepsilon > 0$, the tree survives and there are infinitely many non-empty generations. As all generations only exist before time $\sum_{k=1}^{\infty} t_k < \infty$, we have an infinite number of individuals at a finite time in the thinned tree, and thus explosion. As $\tilde{\mathscr{G}}_k \subseteq \mathscr{G}_k$, explosion also happens in the original tree, and we have found a condition for explosion.

Proof of Theorem 4. Using Corollary 4.1 and Equation 4.7, we see that it is sufficient to show

$$\sum_{k=1}^{\infty} e_k < \infty. \tag{4.33}$$

if we want the tree to survive. Repeating the calculations done in Equation 3.24 until Equation 3.46 in Section 3, we get the recursive equation

$$\tilde{s}_k = M \cdot p_{k-1}\tilde{s}_{k-1} \cdot \exp\Big(\log^{\gamma}(p_{k-1}\tilde{s}_{k-1})\Big). \tag{4.34}$$

Now we want to find a p_k, such that we can get a constraint on T to find explosion. If we want explosion, we want $p_{k-1}\tilde{s}_{k-1} \to \infty$, so that $\tilde{s}_k$ grows as quickly as possible. Therefore we need that $p_{k-1} >> \tilde{s}_{k-1}$. We can construct another constraint for p_{k-1} from Equation 4.34, as we want $\tilde{s}_k$ to be an increasing function, so $\frac{\tilde{s}_k}{\tilde{s}_{k-1}} \geq 1$. From Equation 4.34 we can then see that

$$M \cdot p_{k-1} \exp\Big(\log^{\gamma}(p_{k-1}\tilde{s}_{k-1})\Big) \geq 1. \tag{4.35}$$

We just want the order of magnitude of p_k, so the term M can be left out of the equation. Rewriting this equation gives

$$p_{k-1} \sim \frac{1}{\exp\Big(\log^{\gamma}(p_{k-1}\tilde{s}_{k-1})\Big)}. \tag{4.36}$$

Similar to Equation 4.16 and Equation 4.17 we get

$$p_{k-1} \sim \frac{1}{\exp\Big(\log^{\gamma-\varepsilon}(\tilde{s}_{k-1})\Big)}, \tag{4.37}$$

for some $\varepsilon \in (0, \gamma)$. To further simplify Equation 4.34, we substitute Equation 4.37 to obtain

$$\tilde{s}_{k-1} \geq M \frac{1}{\exp\Big(\log^{\gamma-\varepsilon}(\tilde{s}_{k-1})\Big)} \tilde{s}_{k-1} \cdot \exp\left(\log^{\gamma}\left(\frac{1}{\exp\Big(\log^{\gamma-\varepsilon}(\tilde{s}_{k-1})\Big)} \cdot \tilde{s}_{k-1}\right)\right) \tag{4.38}$$

$$\geq M \cdot \exp\Big(-\log^{\gamma-\varepsilon}(\tilde{s}_{k-1})\Big) \cdot \tilde{s}_{k-1} \cdot \exp\left(\Big(\log(\tilde{s}_{k-1}) - \log^{\gamma-\varepsilon}(\tilde{s}_{k-1})\Big)^{\gamma}\right) \tag{4.39}$$

$$= M \cdot s_{k-1} \cdot \exp\left(\Big(\log(\tilde{s}_{k-1}) - \log^{\gamma-\varepsilon}(\tilde{s}_{k-1})\Big)^{\gamma} - \log^{\gamma-\varepsilon}(\tilde{s}_{k-1})\right). \tag{4.40}$$

We can simplify the exponent, so in the exponent we have

$$\Big(\log(\tilde{s}_{k-1}) - \log^{\gamma-\varepsilon}(\tilde{s}_{k-1})\Big)^{\gamma} - \log^{\gamma-\varepsilon}(\tilde{s}_{k-1}) \tag{4.41}$$

$$= \log^{\gamma}(\tilde{s}_{k-1})\Big(1 + \frac{\log^{\gamma-\varepsilon}(\tilde{s}_{k-1})}{\log(\tilde{s}_{k-1})}\Big)^{\gamma} - \log^{\gamma-\varepsilon}(\tilde{s}_{k-1}) \tag{4.42}$$

$$= \log^{\gamma}(\tilde{s}_{k-1})\Big(1 + o(1)\Big)^{\gamma} - \log^{\gamma-\varepsilon}(\tilde{s}_{k-1}). \tag{4.43}$$

Further simplifying gives

$$\geq \log^{\gamma}(\tilde{s}_{k-1}) - \log^{\gamma-\varepsilon}(\tilde{s}_{k-1}) \tag{4.44}$$

$$= \log^{\gamma}(\tilde{s}_{k-1}) \cdot \left(1 - \frac{\log^{\gamma-\varepsilon}(\tilde{s}_{k-1}}{\log^{\gamma}(\tilde{s}_{k-1}}\right) \tag{4.45}$$

$$\geq \log^{\gamma}(\tilde{s}_{k-1}). \tag{4.46}$$

This gives the differential equation

$$\tilde{s}_k \geq M \cdot \tilde{s}_{k-1} \cdot \exp\Big(\log^{\gamma}(\tilde{s}_{k-1})\Big). \tag{4.47}$$

Using Lemma 2, we get the solution as $\tilde{s}_k \geq \exp\Big(k^{\left(\frac{1}{1-\gamma}\right)}\Big)$. As we chose p_k in Equation 4.36, we now have

$$p_{k-1} \sim \frac{1}{\exp\Big(\big(k^{\left(\frac{1}{1-\gamma}\right)}\big)^{\gamma-\varepsilon}\Big)} \tag{4.48}$$

$$= \frac{1}{\exp\Big((\gamma-\varepsilon) \cdot k^{\left(\frac{1}{1-\gamma}\right)}\Big)}. \tag{4.49}$$

Fully writing p_k gives

$$\mathbb{P}\Big(T \leq t_k\Big) = \frac{1}{\exp\Big((\gamma-\varepsilon) \cdot k^{\left(\frac{1}{1-\gamma}\right)}\Big)}. \tag{4.50}$$

From here, we want to find a series t_k. Again, we will write $F_T(x) = \mathbb{P}\Big(T \leq t_k\Big)$. We get

$$F_T(t_k) = \frac{1}{\exp\Big((\gamma-\varepsilon) \cdot k^{\left(\frac{1}{1-\gamma}\right)}\Big)}. \tag{4.51}$$

Taking the inverse of $F_T(x)$ we get

$$t_k = F_T^{-1}\left(\frac{1}{\exp\Big((\gamma-\varepsilon) \cdot k^{\left(\frac{1}{1-\gamma}\right)}\Big)}\right). \tag{4.52}$$

To obtain survival, we must have $\sum_{k=1}^{\infty} t_k < \infty$, and as the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to infinity, we have

$$F_T^{-1}\left(\frac{1}{\exp\left((\gamma-\varepsilon)\cdot k^{\left(\frac{1}{1-\gamma}\right)}\right)}\right) = o\left(\frac{1}{k}\right) \tag{4.53}$$

To simplify, we substitute $x := \frac{1}{k}$, so $k = \frac{1}{x}$ into Equation 4.53 to get

$$F_T^{-1}\left(\frac{1}{\exp\left((\gamma-\varepsilon)\cdot x^{\left(\frac{1}{\gamma-1}\right)}\right)}\right) = o\big(x\big). \tag{4.54}$$

Then taking $F_T(x)$ on both sides of Equation 4.54, which gives

$$\frac{1}{\exp\left((\gamma-\varepsilon)\cdot x^{\left(\frac{1}{\gamma-1}\right)}\right)} = F_T\big(o(x)\big) << F_T(x), \tag{4.55}$$

for $\varepsilon \in (0, \gamma)$. Therefore, if $\frac{1}{\exp\left((\gamma-\varepsilon)\cdot x^{\left(\frac{1}{\gamma-1}\right)}\right)} = F_T\big(o(x)\big) << F_T(x)$ the tree survives. $\square$

Then by Theorem 3 and Theorem 4, we have two constraints where the thinned tree $\tilde{\mathscr{U}}_\infty$ survives in a finite time, and thus explodes.

Chapter 5

Conclusion

In this thesis we wanted to find an criterion for explosion in a branching process, where the offspring distribution is defined by $\mathbb{P}\big(D \geq k\big) = \frac{l(k)}{k}$, where $l(x)$ is a slowly varying function. We looked at two functions, namely $l_1(x) = c \cdot \log^{\alpha+1}(x)$ and $l_2(x) = c \cdot \exp\Big(\log^{\gamma}(x)\Big)$ for $\alpha > -1$ and $\gamma \in (0,1)$. We did this by constructing a lower bound s_k for the generation sizes. Using Bernstein's inequality, we were able to find a recursive equation for s_k. By writing this recursive equation as a differential equation, we were able to find a solution for the lower bound of the generation sizes.

After this, we applied a thinning method, which resulted in a new tree $\tilde{\mathscr{U}}_\infty \subseteq \mathscr{U}_\infty$. In this thinned tree, we removed individuals from the original tree if their parents were born too late. After that, we found that for $l_1(x)$, if $F_T(x) >> x^{\alpha+1-\varepsilon}$ for some $\varepsilon > 0$, the thinned tree survives. As we thinned the tree by choosing $\sum_{k=1}^{\infty} t_k < \infty$, all generations existed in a finite time, and thus explosion occurred in $\tilde{\mathscr{U}}_\infty$. As it is a subset of $\mathscr{U}_\infty$, explosion also occurred in $\mathscr{U}_\infty$.

For $l_2(x)$, if $F_T(x) >> \exp\Big((\varepsilon - \gamma) \cdot x^{\left(\frac{1}{\gamma-1}\right)}\Big)$ for some $\varepsilon > 0$, the thinned tree $\tilde{\mathscr{U}}_\infty$ survives, and thus the whole tree $\mathscr{U}_\infty$ explodes.

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