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The Hilbert Transform and Orthogonal Martingales in Banach Spaces

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Let X be a given Banach space, and let M and N be two orthogonal X -valued local martingales such that N is weakly differentially subordinate to M . The paper contains the proof of the estimate $\mathbb{E}\Psi(N_t) \leq C_{\Phi, \Psi, X} \mathbb{E}\Phi(M_t)$, $t \geq 0$, where $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ are convex continuous functions and the least admissible constant $C_{\Phi, \Psi, X}$ coincides with the Φ, Ψ -norm of the periodic Hilbert transform. As a corollary, it is shown that the Φ, Ψ -norms of the periodic Hilbert transform, the Hilbert transform on the real line, and the discrete Hilbert transform are the same if Φ is symmetric. We also prove that under certain natural assumptions on Φ and Ψ , the condition $C_{\Phi, \Psi, X} < \infty$ yields the UMD property of the space X . As an application, we provide comparison of L^p -norms of the periodic Hilbert transform to Wiener and Paley–Walsh decoupling constants. We also study the norms of the periodic, nonperiodic, and discrete Hilbert transforms and present the corresponding estimates in the context of differentially subordinate harmonic functions and more general singular integral operators.

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1 Introduction

The purpose of this paper is to study a certain class of estimates for singular integral operators acting on Banach-space-valued functions. Let us start with a related classical problem that has served as a motivation for many mathematicians for almost a century. The question is “How does the size of a periodic function control the size of its conjugate?” Formally, assume that f is a trigonometric polynomial of the form

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^N (a_k \cos(k\theta) + b_k \sin(k\theta)), \quad \theta \in \mathbb{T} \simeq [-\pi, \pi),$$

with real coefficients $a_0, a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N$, and define the conjugate to f as

$$g(\theta) = \sum_{k=1}^N (a_k \sin(k\theta) - b_k \cos(k\theta)), \quad \theta \in [-\pi, \pi).$$

Alternatively, the conjugate function can be defined as $g = \mathcal{H}_{\mathbb{R}}^{\mathbb{T}} f$, where $\mathcal{H}_{\mathbb{R}}^{\mathbb{T}}$ is the periodic Hilbert transform given by

$$\mathcal{H}_{\mathbb{R}}^{\mathbb{T}} f(\theta) = \frac{1}{2\pi} \text{p.v.} \int_{-\pi}^{\pi} f(s) \cot \frac{\theta - s}{2} ds, \quad \theta \in [-\pi, \pi), \quad (1.1)$$

and the symbol \mathbb{R} in the lower index of $\mathcal{H}^{\mathbb{T}}$ indicates that the operator acts on real-valued functions. We can state the problem as follows. For a given $1 \leq p \leq \infty$, does there exist a universal constant C_p (i.e., not depending on the coefficients or the number N) such that

$$\left(\int_{[-\pi, \pi)} |g(\theta)|^p d\theta \right)^{1/p} \leq C_p \left(\int_{[-\pi, \pi)} |f(\theta)|^p d\theta \right)^{1/p} ?$$

Furthermore, if the answer is yes, what is the optimal value of C_p (i.e., what is the L^p norm of $\mathcal{H}_{\mathbb{R}}^{\mathbb{T}}$)? The 1st question was answered by M. Riesz in [45]: the inequality does hold if and only if $1 < p < \infty$. The best value of C_p was determined by Pichorides [42] and Cole (unpublished): the constant $\cot(\pi/(2p^*))$ is the best possible, where $p^* = \max\{p, p/(p-1)\}$. There is a natural further question concerning the version of the above result for Banach-space-valued functions (it is not difficult to see that the formula (1) makes perfect sense in the vector setting, at least for some special f , see Section 2 below). Few years after the results of Riesz, it was realized that not all spaces are well behaved: Bochner and Taylor [5] showed that $\|\mathcal{H}_{\ell_1}^{\mathbb{T}}\|_{L_p \rightarrow L_p} = \infty$ for all p . The problem of characterizing the “good” Banach spaces was solved over 40 years later: Burkholder

[8] and Bourgain [6] showed that the so-called UMD spaces form a natural environment to the study of the L^p boundedness ($1 < p < \infty$) of the periodic Hilbert transform, and more generally, for the L^p boundedness of a wider class of singular integral operators.

The above problems, though expressed in an analytic language, have a very strong connection with probability theory, especially with the theory of martingales (see, e.g., [1, 3, 4, 6, 9, 20, 25, 28, 40, 41]). Let us provide some necessary definitions. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, equipped with a continuous-time filtration $(\mathcal{F}_t)_{t \geq 0}$. Let $M = (M_t)_{t \geq 0}$, $N = (N_t)_{t \geq 0}$ be two adapted real-valued local martingales, whose trajectories are right continuous and have limits from the left. Let $[M]$ and $[N]$ stand for the associated quadratic variation (square brackets) of M and N , see [17] and (4) below. Furthermore, $M^* = \sup_{t \geq 0} |M_t|$ and $N^* = \sup_{t \geq 0} |N_t|$ denote the corresponding maximal functions. Following Bañuelos and Wang [3] and Wang [52], N is differentially subordinate to M (which we denote by $N \ll M$) if, with probability 1, the process $t \mapsto [M]_t - [N]_t$ is a nondecreasing and nonnegative function of $t \geq 0$. Furthermore, we say that M and N are orthogonal, if $[M, N] := \frac{[M+N] - [M-N]}{4} = 0$ almost surely.

One of the remarkable examples of the aforementioned connection between the theory of singular integral operators and martingale theory was provided by Bañuelos and Wang in [3]. They have shown that the L^p norm of $\mathcal{H}^\mathbb{T}$ acting on real-valued functions is equal to the sharp constant in the corresponding L^p inequality

$$(\mathbb{E}|N_t|^p)^{\frac{1}{p}} \leq C_p (\mathbb{E}|M_t|^p)^{\frac{1}{p}}, \quad t \geq 0, \quad (1.2)$$

where N is assumed to be differentially subordinate and orthogonal to M . The goal of the current article is to show that this interplay between the norm of $\mathcal{H}^\mathbb{T}$ and the martingale inequality (2) can be extended to (i) more general Φ, Ψ norms (see the beginning of Section 3 for the definition) and (ii) more general Banach spaces in which the functions and processes take values.

Let us say a few words about the structure of the paper. The next section is devoted to the introduction of the background that is needed for our further study. In particular, we recall there the notion of UMD spaces, define appropriate analogues of Banach-space-valued differential subordination, and, orthogonality, formulate the vector extensions of stochastic calculus and provide some basic information about plurisubharmonic functions, fundamental objects in the complex analysis of several variables. Section 3 contains the main result of the paper, connecting the best constants in certain Φ, Ψ estimates for the periodic Hilbert transform and their counterparts

in martingale theory. Though the rough idea of the proof can be tracked back to the classical works [3, 12, 26, 42] (the validity of a given estimate for the Hilbert transform/orthogonal differentially subordinate martingales is equivalent to the existence of a certain special plurisubharmonic function), there are several serious technical problems to be overcome, due to the fact that we work in the Banach-space-valued setting. Section 4 is devoted to some applications. The 1st and the most notable one connects together the Φ, Ψ norms of the periodic Hilbert transform $\mathcal{H}_X^{\mathbb{T}}$, the Hilbert transform $\mathcal{H}_X^{\mathbb{R}}$ defined on a real line, and the discrete Hilbert transform $\mathcal{H}_X^{\text{dis}}$ (for the definition of the latter object, consult Definition 4.1 and 4.2 below). It turns out that all these norms coincide for quite general class of Φ and Ψ . This in particular generalizes the recent result of Bañuelos and Kwaśnicki [1] on the discrete Hilbert transform $\mathcal{H}_{\mathbb{R}}^{\text{dis}}$, which asserts that

$$\|\mathcal{H}_{\mathbb{R}}^{\text{dis}}\|_{L^p(\mathbb{Z}) \rightarrow L^p(\mathbb{Z})} = \|\mathcal{H}_{\mathbb{R}}^{\mathbb{T}}\|_{L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T})} = \cot\left(\frac{\pi}{2p^*}\right), \quad 1 < p < \infty.$$

This used to be an open problem for 90 years (see [1, 35, 49]). Subsection 4.2 is devoted to the comparison of L^p norms of the periodic Hilbert transform to Wiener and Paley–Walsh decoupling constants. Application in Subsection 4.3 is concerned with UMD Banach spaces and can be regarded as an extension of Bourgain’s result [6]: we show that under some mild assumption on Φ and Ψ , the validity of the corresponding Φ, Ψ estimate (with some finite constant) implies the UMD property of X . In Subsection 4.4 we prove that the results obtained in this paper can be applied to obtain sharper estimates for weakly differentially subordinate martingales (not necessarily satisfying the orthogonality assumption). Subsection 4.5 contains the study of related estimates in the context of harmonic functions on Euclidean domains. In Subsection 4.6 we present the possibility of extending the estimates to the more general class of singular integral operators. Our final application, described in Subsection 4.7, discusses the vector-valued extension of the classical results of Hardy concerning Hilbert operators.

2 Preliminaries

This section contains the definitions of some basic notions and facts used later. Here and below, the scalar field is assumed to be \mathbb{R} , unless stated otherwise. In particular, all Banach spaces are real, unless stated otherwise.

2.1 Periodic Hilbert transform

In what follows, the symbol \mathbb{T} will stand for the torus $(\{z \in \mathbb{C} : |z| = 1\}, \cdot)$ equipped with the natural multiplication. Sometimes, passing to the argument of a complex number, we will identify \mathbb{T} with the interval $[-\pi, \pi)$. Let X be a Banach space. A function $f : \mathbb{T} \rightarrow X$ is called a step function, if it is of the form

$$f = \sum_{k=1}^N x_k \mathbf{1}_{A_k}(s), \quad -\pi \leq s < \pi,$$

where N is finite, $x_k \in X$ and A_k are intervals in \mathbb{T} . The periodic Hilbert transform $\mathcal{H}_X^{\mathbb{T}}$ of a step function $f : \mathbb{T} \rightarrow X$ is given by the singular integral

$$\mathcal{H}_X^{\mathbb{T}}f(t) = \frac{1}{2\pi} \text{p.v.} \int_{-\pi}^{\pi} f(s) \cot \frac{t-s}{2} ds, \quad -\pi \leq t < \pi. \tag{2.1}$$

2.2 UMD Banach spaces

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a nonatomic probability space. A Banach space X is called a UMD space if for some (or equivalently, for all) $p \in (1, \infty)$ there exists a finite constant β such that the following holds. If $(d_n)_{n=1}^{\infty}$ is any X -valued martingale difference sequence (relative to some discrete-time filtration) contained in $L^p(\Omega; X)$ and $(\varepsilon_n)_{n=1}^{\infty}$ is any deterministic sequence of signs, then

$$\left(\mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n d_n \right\|^p \right)^{\frac{1}{p}} \leq \beta \left(\mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p \right)^{\frac{1}{p}}.$$

The least admissible constant β above is denoted by $\beta_{p,X}$ and is called the UMD constant of X . It is well known that UMD spaces enjoy a large number of useful properties, such as being reflexive. Examples of UMD spaces include all finite dimensional spaces, Hilbert spaces (then $\beta_{p,X} = p^* - 1$ with $p^* = \max\{p, p/(p - 1)\}$), the reflexive range of L^q -spaces, Sobolev spaces, Schatten class spaces, and Orlicz spaces. On the other hand, all nonreflexive Banach spaces, for example, $L^1(0, 1)$ and $C([0, 1])$, are not UMD. We refer the reader to [14], [28], and [43] for further details.

Remark 2.1. As we have already mentioned in the introductory section, UMD Banach spaces form a natural environment for the L^p boundedness of the periodic Hilbert

transform. It follows from [9] and [6] that for every $1 < p < \infty$ we have

$$\sqrt{\beta_{p,X}} \leq \|\mathcal{H}_X^\mathbb{T}\|_{L^p(\mathbb{T},X) \rightarrow L^p(\mathbb{T},X)} \leq \beta_{p,X}^2. \quad (2.2)$$

It is not known whether the quadratic dependence can be improved on either of the sides (see, e.g., [14, 25, 28]). Notice that if $X = \mathbb{R}$ then the dependence becomes linear: indeed,

$$\frac{2}{\pi} \beta_{p,\mathbb{R}} = \frac{2}{\pi} (p^* - 1) \leq \cot\left(\frac{\pi}{2p^*}\right) = \|\mathcal{H}_X^\mathbb{T}\|_{\mathcal{L}(L^p(\mathbb{T},X))} \leq p^* - 1 = \beta_{p,\mathbb{R}},$$

where, as above, $p^* := \max\{p, p/(p-1)\}$.

We will see later that the context of UMD is also natural from the viewpoint of more general Φ, Ψ estimates for the periodic Hilbert transform (see Subsection 4.3).

2.3 Stochastic integration and Itô's formula

For given Banach spaces X and Y , the symbol $\mathcal{L}(X, Y)$ will denote the classes of all linear operators from X to Y . We will also use the notation $\mathcal{L}(X) = \mathcal{L}(X, X)$. Suppose that H is a Hilbert space. For each $h \in H$ and $x \in X$, we denote by $h \otimes x$ the associated linear operator given by $g \mapsto \langle g, h \rangle x$, $g \in H$. The process $\phi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(H, X)$ is called *elementary progressive* with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ if it is of the form

$$\phi(t, \omega) = \sum_{k=1}^K \sum_{m=1}^M \mathbf{1}_{(t_{k-1}, t_k] \times B_{mk}}(t, \omega) \sum_{n=1}^N h_n \otimes x_{kmn}, \quad t \geq 0, \omega \in \Omega.$$

Here $0 \leq t_0 < \dots < t_K < \infty$ is a finite increasing sequence of nonnegative numbers, the sets B_{1k}, \dots, B_{Mk} belong to $\mathcal{F}_{t_{k-1}}$ for each $k = 1, 2, \dots, K$, and the vectors h_1, \dots, h_N are assumed to be orthogonal. Suppose further that M is an adapted local martingale taking values in H . Then the stochastic integral $\phi \cdot M : \mathbb{R}_+ \times \Omega \rightarrow X$ of ϕ with respect to M is defined by the formula

$$(\phi \cdot M)_t = \sum_{k=1}^K \sum_{m=1}^M \mathbf{1}_{B_{mk}} \sum_{n=1}^N \langle (M(t_k \wedge t) - M(t_{k-1} \wedge t)), h_n \rangle x_{kmn}, \quad t \geq 0.$$

In what follows, we will also need a version of Itô formula, which is a variation of [32, Theorem 26.7] that does not use the Euclidean structure of a finite-dimensional Banach space. The proof can be found in [55, Theorem 5.5].

Lemma 2.2 (Itô formula). Let $d \geq 1$, X be a d -dimensional Banach space, $f \in C^2(X)$, $M : \mathbb{R}_+ \times \Omega \rightarrow X$ be a martingale. Let $(x_n)_{n=1}^d$ be a basis of X , $(x_n^*)_{n=1}^d$ be the corresponding dual basis. Then for each $t \geq 0$

$$\begin{aligned}
 f(M_t) &= f(M_0) + \int_0^t \langle \partial_x f(M_{s-}), dM_s \rangle \\
 &\quad + \frac{1}{2} \int_0^t \sum_{n,m=1}^d \frac{\partial^2 f(M_{s-})}{\partial x_n \partial x_m} d[\langle M, x_n^* \rangle, \langle M, x_m^* \rangle]_s^c \\
 &\quad + \sum_{s \leq t} (\Delta f(M_s) - \langle \partial_x f(M_{s-}), \Delta M_s \rangle).
 \end{aligned} \tag{2.3}$$

Here $\partial_x f(y) \in X^*$ is the Fréchet derivative of f in point $y \in X$. Recall that $(x_n^*)_{n=1}^d \subset X^*$ is called the corresponding dual basis of $(x_n)_{n=1}^d$ if $\langle x_n, x_m^* \rangle = \delta_{nm}$ for each $m, n = 1, \dots, d$.

2.4 Quadratic variation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ that satisfies the usual conditions. Let $M : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be a local martingale. We define a quadratic variation of M in the following way:

$$[M]_t := |M_0|^2 + \mathbb{P} - \lim_{mesh \rightarrow 0} \sum_{n=1}^N \|M(t_n) - M(t_{n-1})\|^2, \tag{2.4}$$

where the limit in probability is taken over partitions $0 = t_0 < \dots < t_N = t$. Note that $[M]$ exists and is nondecreasing a.s. The reader can find more on quadratic variations in [32], [44], and [17]. For any martingales $M, N : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ we can define a covariation $[M, N] : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ as $[M, N] := \frac{1}{4}([M+N] - [M-N])$. Since M and N have càdlàg versions, $[M, N]$ has a càdlàg version as well (see, e.g., [30, Theorem I.4.47]).

2.5 Weak differential subordination and orthogonal martingales

We have defined the notions of differential subordination and orthogonality of real-valued local martingales in the introductory section. We turn our attention to their vector analogues.

Definition 2.3. Let M and N be local martingales taking values in a given Banach space X . Then N is said to be weakly differentially subordinate to M (which will be denoted by $N \overset{w}{\ll} M$) if $\langle N, x^* \rangle \ll \langle M, x^* \rangle$ for any functional $x^* \in X^*$.

It is known (see [56]) that if N is weakly differentially subordinate to M then

$$(\mathbb{E}\|N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X}^2(\beta_{p,X} + 1)(\mathbb{E}\|M_t\|^p)^{\frac{1}{p}}, \quad t \geq 0. \quad (2.5)$$

This estimate can be improved under some additional assumptions on M and N (see [55, 56]). Here we will show such an improvement for M and N being orthogonal (see Section 3). Moreover, using this improvement we will strengthen (5) (see Remark 4.27).

Definition 2.4. Let M and N be local martingales taking values in a given Banach space X . Then M and N are said to be orthogonal, if $[\langle M, x^* \rangle, \langle N, x^* \rangle] = 0$ almost surely for all functionals $x^* \in X^*$.

Remark 2.5. Assume that M and N are local martingales taking values in some Banach space X . If M and N are orthogonal and N is weakly differentially subordinate to M , then $N_0 = 0$ almost surely (which follows immediately from the above definitions). Moreover, under these assumptions, N must have continuous trajectories with probability 1. Indeed, in such a case for any fixed $x^* \in X^*$ the real-valued local martingales $\langle M, x^* \rangle$ and $\langle N, x^* \rangle$ are orthogonal and we have $\langle N, x^* \rangle \ll \langle M, x^* \rangle$. Therefore, $\langle N, x^* \rangle$ has a continuous version for each $x^* \in X^*$ by [39, Lemma 3.1] (see also [4, Lemma 1]), which in turn implies that N is continuous: any X -valued local martingale has a càdlàg version (see [55] and [50, Proposition 2.2.2]).

Remark 2.6. The requirement $\langle M_0, x^* \rangle \cdot \langle N_0, x^* \rangle = 0$ for all $x^* \in X^*$ in Definition 2.4 is usually omitted (see, e.g., [3, 4, 30]). Nevertheless, we need this requirement in order to simplify all the statements in the sequel concerning orthogonal martingales.

Weakly differentially subordinate orthogonal martingales appear naturally while working with the periodic Hilbert transform, which can be seen by exploiting the classical argument of Doob (the composition of a harmonic function with a Brownian motion is a martingale). Indeed, suppose that X is a given Banach space. Suppose that f is a simple function and put $g = \mathcal{H}_X^{\mathbb{T}} f$. Let u_f and u_g denote the harmonic extensions of f and g to the unit disc, obtained by the convolution with the Poisson kernel. In particular, the equality $g = \mathcal{H}^{\mathbb{T}} f$ implies that $u_g(0, 0) = 0$ and for any functional $x^* \in X^*$, the function $\langle u_f, x^* \rangle + i\langle u_g, x^* \rangle$ is holomorphic on the disc.

Next, suppose that $W = (W^1, W^2)$ is a planar Brownian motion started from $(0, 0)$ and stopped upon leaving the unit disc. Then the processes $M = (M_t)_{t \geq 0} = (u_f(W_t))_{t \geq 0}$

and $N = (N_t)_{t \geq 0} = (u_g(W_t))_{t \geq 0}$ are X -valued martingales such that $N_0 = 0$. For any functional $x^* \in X^*$, we apply the standard, one-dimensional Itô's formula to obtain, for any $t \geq 0$,

$$\langle M_t, x^* \rangle = \langle M_0, x^* \rangle + \int_0^t \nabla \langle u_f(W_s), x^* \rangle dW_s$$

and

$$\langle N_t, x^* \rangle = \langle N_0, x^* \rangle + \int_0^t \nabla \langle u_g(W_s), x^* \rangle dW_s.$$

By the aforementioned connection to analytic functions, the gradients $\nabla \langle u_f, x^* \rangle$ and $\nabla \langle u_g, x^* \rangle$ are orthogonal and of equal length, so

$$[\langle M, x^* \rangle, \langle N, x^* \rangle]_t = \int_0^t \nabla \langle u_f(W_s), x^* \rangle \cdot \nabla \langle u_g(W_s), x^* \rangle ds = 0,$$

and

$$[\langle M, x^* \rangle]_t - [\langle N, x^* \rangle]_t = |\langle M_0, x^* \rangle|^2 + \int_0^t \nabla \langle u_f(W_s), x^* \rangle^2 - \nabla \langle u_g(W_s), x^* \rangle^2 ds = |\langle M_0, x^* \rangle|^2 \geq 0.$$

Hence M and N are orthogonal and satisfy the weak differential subordination $N \stackrel{w}{\ll} M$. Since the distribution of W_∞ is uniform on the unit circle \mathbb{T} , essentially any estimate of the form

$$\mathbb{E}V(M_t, N_t) \leq 0, \quad t \geq 0,$$

for weakly differentially subordinate orthogonal martingales leads to the analogous bound

$$\int_{\mathbb{T}} V(f, \mathcal{H}_{\mathbb{T}} f) dx \leq 0$$

for the periodic Hilbert transform, at least when restricted to the class of simple functions. (Later in Theorem 3.1 we will show that the reverse holds true).

For more information and examples concerning the differential subordination, weak differential subordination, and orthogonal martingales, we refer the reader to [30, 28, 56, 44, 4, 52, 10], and [3].

2.6 Subharmonic and plurisubharmonic functions

An upper semicontinuous function $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty\}$ is called subharmonic if for any ball $B \subset \mathbb{R}^d$ and any harmonic function $g : B \rightarrow \mathbb{R}$ such that $f \leq g$ on ∂B one has the inequality $f \leq g$ on the whole B . The following lemma follows from [36, Proposition I.9].

Lemma 2.7. Let $d \geq 1$ and let $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty\}$ be a subharmonic function. Then either $f \equiv -\infty$, or f is locally integrable.

Let X be a Banach space. An upper semicontinuous function $F : X + iX \rightarrow \mathbb{R} \cup \{-\infty\}$ is called plurisubharmonic if for any $x, y \in X + iX$ the restriction $z \mapsto F(x + yz)$ is subharmonic in $z \in \mathbb{C}$.

Remark 2.8. Notice that $X + iX$ is a Banach space equipped with the norm

$$\|x + iy\|_{X+iX} := \sup_{x^* \in X^*, \|x^*\| \leq 1} (|\langle x, x^* \rangle|^2 + |\langle y, x^* \rangle|^2)^{\frac{1}{2}}, \quad x, y \in X$$

(see [28, Subsection B.4]).

Remark 2.9. Let X be finite dimensional. Then any plurisubharmonic function defined on $X + iX$ is subharmonic (see [36, Proposition I.9] and [23, Theorem 39]). Therefore, by Lemma 2.7, a plurisubharmonic function either identically equals $-\infty$, or is locally integrable.

Let $F : X + iX \rightarrow \mathbb{R}$ be k -times differentiable, $u_1, \dots, u_k \in X + iX$. Then we denote

$$\frac{\partial^k F(v)}{\partial u_1 \cdots \partial u_k} := \frac{\partial^k}{\partial t_1 \cdots \partial t_k} F(v + t_1 u_1 + \cdots + t_k u_k) \Big|_{t_1, \dots, t_k = 0}, \quad v \in X + iX.$$

In particular, for any $u \in X + iX$,

$$\frac{\partial^k F(v)}{\partial u^k} := \frac{\partial^k}{\partial t^k} F(v + tu) \Big|_{t=0}, \quad v \in X + iX.$$

Remark 2.10. Note that if X is finite dimensional, F is plurisubharmonic and twice differentiable, then for all $z_0 \in X + iX$ and $x \in X$ we have

$$\frac{\partial^2 F(z_0)}{\partial x^2} + \frac{\partial^2 F(z_0)}{\partial ix^2} = \left(\frac{\partial^2 F(z_0 + zx)}{\partial \Re z^2} + \frac{\partial^2 F(z_0 + zx)}{\partial \Im z^2} \right) \Big|_{z=0} = \Delta_z F(z_0 + zx) \Big|_{z=0} \geq 0.$$

Later on we will need the following result.

Proposition 2.11. Let X be a Banach space and let $F : X + iX \rightarrow \mathbb{R} \cup \{-\infty\}$ be plurisubharmonic. Assume further that $y \mapsto F(x + iy)$ is concave in $y \in X$ for any fixed $x \in X$. Then $x \mapsto F(x + iy)$ is convex in $x \in X$ for any $y \in X$, and F is continuous.

For the proof we will need the following lemma.

Lemma 2.12. Let X be a finite-dimensional Banach space and let $V : X + iX \rightarrow \mathbb{R}$ be a continuous twice differentiable plurisubharmonic function. Let $y \mapsto V(x + iy)$ be concave in $y \in X$ for all $x \in X$. Then $t \mapsto V(tx + z)$ is convex in $t \in \mathbb{R}$ for all $x \in X$ and $z \in X + iX$. In particular, since $t \mapsto V(tx + z)$ is differentiable, we have that

$$V(tx + z) \geq V(sx + z) + \partial_x V(sx + z)(t - s), \quad t, s \in \mathbb{R}. \tag{2.6}$$

Proof. The 1st part follows from the fact that V is plurisubharmonic and twice differentiable. Indeed, we have

$$\frac{\partial^2 V(tx + z)}{\partial t^2} = \left(\frac{\partial^2 V(tx + z + isx)}{\partial t^2} + \frac{\partial^2 V(tx + z + isx)}{\partial s^2} \right) \Big|_{s=0} - \frac{\partial^2 V(tx + z + isx)}{\partial s^2} \Big|_{s=0} \geq 0$$

since

$$\left(\frac{\partial^2 V(tx + z + isx)}{\partial t^2} + \frac{\partial^2 V(tx + z + isx)}{\partial s^2} \right) \Big|_{s=0} \geq 0$$

by plurisubharmonicity and $\frac{\partial^2 V(tx+z+isx)}{\partial s^2} \leq 0$ by concavity of $y \mapsto V(x + z + iy)$. The inequality (2.6) follows immediately from the convexity of $t \mapsto V(tx + iy)$ and twice differentiability of V . ■

To complete the proof of Proposition 2.11 we will need the following observation, which will allow us to integrate over a Banach space.

Remark 2.13. Let X be a finite-dimensional Banach space. Then due to [19, Theorem 2.20 and Proposition 2.21] there exists a unique translation-invariant measure λ_X on X such that $\lambda_X(\mathbb{B}_X) = 1$ for the unit ball \mathbb{B}_X of X . We will call λ_X the Lebesgue measure. In the sequel we will omit the Lebesgue measure notation while integrating over X (i.e., we will write $\int_X F(s) ds$ instead of $\int_X F(s)\lambda_X(ds)$).

Proof of Proposition 2.11. Without loss of generality we can assume that X is finite dimensional and that $f \not\equiv -\infty$. As X is finite dimensional, $X + iX \simeq \mathbb{C}^d$ for some d . Let $\phi : X + iX \rightarrow \mathbb{R}_+$ be a C^∞ function radial with respect to \mathbb{C}^d (i.e., depending only on $|z_1|^2 + \dots + |z_d|^2$ with z_1, \dots, z_d being the basis of \mathbb{C}^d) with bounded support such that

$$\int_{X+iX} \phi(s) ds = 1.$$

(This integral is well defined due to Remarks 2.8 and 2.13). For each $\varepsilon > 0$ we define $F_\varepsilon : X + iX \rightarrow \mathbb{R}$ in the following way:

$$F_\varepsilon(s) = \int_{X+iX} F(s - \varepsilon t)\phi(t) dt, \quad s \in X + iX. \quad (2.7)$$

Then F_ε is plurisubharmonic due to [27, Theorem 4.1.4]. Moreover, again by [27, Theorem 4.1.4], we have $F_\varepsilon \searrow F$ as $\varepsilon \searrow 0$. On the other hand, F_ε is well defined and of class C^∞ . Furthermore, the function $y \mapsto F_\varepsilon(x + iy)$ is concave in $y \in X$ for any $x \in X$ by (7): here we use the fact that F is locally integrable (see Remark 2.9) and the concavity of $y \mapsto F(x + iy)$ for any fixed $x \in X$. Therefore by Lemma 2.12, the function $x \mapsto F_\varepsilon(x + iy)$ is convex for any fixed $y \in X$; hence so is F , being the pointwise limit of $(F_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \rightarrow 0$.

Let us now show that $F > -\infty$. Assume that there exists $x_0, y_0 \in X$ such that $F(x_0 + iy_0) = -\infty$. Since the function $y \mapsto F(x_0 + iy)$ is concave, the set $A = \{y \in X : F(x_0 + iy) > \infty\} \subset X$ is convex and open; moreover, $y_0 \notin A$, so $X \setminus A$ is of positive measure. Now fix $(x, y) \in X \times (X \setminus A)$. Notice that $F(x_0 + iy) = -\infty$. On the other hand $x \mapsto F(x + iy)$ is convex, so $F(x + iy) = -\infty$ as well (if a convex function equals $-\infty$ in one point, it equals $-\infty$ on the whole X). Therefore, $F = -\infty$ in the set $X \times (X \setminus A)$ of positive measure; hence $F \equiv -\infty$ by Remark 2.9, which leads to a contradiction.

Finally, note that $F < \infty$: we have $F \leq F_1$ with F_1 defined in (7). Therefore, F is continuous as a finite concave–convex function (see [48, Proposition 3.3] and [31, Corollary 4.5]). ■

For further material on subharmonic and plurisubharmonic functions, we recommend the works [23, 27, 36, 46, 47].

2.7 Meyer–Yoeurp decomposition

Let X be a Banach space and let M be a local martingale with values in X . Then is called purely discontinuous if $[M]$ is a.s. a pure jump process (see [30, 32] for details). M is said to have the Meyer–Yoeurp decomposition if there exist an X -valued continuous local martingale M^c and an X -valued purely discontinuous local martingale M^d such that $M_0^c = 0$ and $M = M^c + M^d$ a.s. It was shown by Meyer in [38] and by Yoeurp in [57] that any real-valued martingale has the Meyer–Yoeurp decomposition. Later in [54] it was shown that any X -valued local martingale has the Meyer–Yoeurp decomposition if and only if X has the UMD property. See [32], [30], and [56] for further details.

The following result shows the connection between the Meyer–Yoeurp decomposition and the weak differential subordination.

Proposition 2.14. Let X be a Banach space and let M and N be local X -valued martingales possessing the Meyer-Yoeurp decompositions $M = M^c + M^d$, $N = N^c + N^d$. Then $N \overset{w}{\ll} M$ if and only if $N^c \overset{w}{\ll} M^c$ and $N^d \overset{w}{\ll} M^d$. Moreover, if M and N are orthogonal, then M^c and N^c , M^d and N^d are pairwise orthogonal.

Proof. The 1st part can be found in [56] (see also [52, Lemma 1]). Due to Remark 2.5 we know that $N^d = 0$, so it is sufficient to show that M^c and N^c are orthogonal. The latter is equivalent to the fact that $\langle M^c, x^* \rangle$ and $\langle N^c, x^* \rangle$ are orthogonal for any $x^* \in X^*$, which holds true by [4, Lemma 1]. ■

3 Main theorem

Having introduced all the necessary notions, we turn to the study of our new results. For given two nonnegative and continuous functions $\Phi, \Psi : X \rightarrow \mathbb{R}_+$, we define the associated “ Φ, Ψ -norm” of $\mathcal{H}_X^{\mathbb{T}}$ by the formula

$$|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} := \inf \left\{ c \in [0, \infty] : \int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}} f(s)) \, ds \leq c \int_{\mathbb{T}} \Phi(f(s)) \, ds \text{ for all step functions } f : \mathbb{T} \rightarrow X \right\}.$$

Notice that if $\Psi \equiv 0$, then $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} = 0$, and if $\Phi \equiv 0$, then $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} \in \{0, +\infty\}$. Throughout the paper we exclude these trivial cases: we will assume that both Φ and Ψ are not identically zero. Furthermore, for any $1 < p < \infty$, we will denote the L^p norm of $\mathcal{H}_X^{\mathbb{T}}$ by $\mathfrak{h}_{p, X}$ (in the language of Φ, Ψ -norms, we have $\mathfrak{h}_{p, X}^p = |\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi}$ with $\Phi(x) = \Psi(x) = \|x\|^p$).

The following theorem is the main result of this section.

Theorem 3.1. Let X be a separable Banach space and let $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ be continuous convex functions such that $\Psi(0) = 0$ and $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} < \infty$. Let M and N be two orthogonal X -valued local martingales such that $N \overset{w}{\ll} M$. Then

$$\mathbb{E}\Psi(N_t) \leq C_{\Phi, \Psi, X} \mathbb{E}\Phi(M_t), \quad t \geq 0, \tag{3.1}$$

and the least admissible $C_{\Phi, \Psi, X}$ equals $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi}$.

The idea behind the proof of (3.1) can be roughly described as follows. First, we will show that the condition $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} < \infty$ (i.e., the validity of a Φ, Ψ -estimate for the periodic Hilbert transform) implies the existence of a certain special function on

$X + iX$, enjoying appropriate size conditions and concavity. Next, we will compose this function with $M + iN$ and prove, using the concavity and Itô's formula from the previous section, that the resulting process has nonnegative expectation. This in turn will give the desired bound, in the light of the size condition of the special function. Though this reasoning is typical for this kind of martingale inequalities, there are two essential differences. First, we will see that the special function will not have any explicit form: in particular, this makes the exploitation of its properties much harder, as one can get them only from some abstract (and restricted) reasoning. The 2nd difference is related to the fact that we work with Banach-space-valued processes: this enforces us to study some additional, structural properties of the local martingales involved. Moreover, since we will work in infinite-dimensional Banach spaces, the approximation to finite dimensions exploited in the proof should be especially delicate because we do not want to ruin weak differential subordination and orthogonality of the corresponding martingales.

Having described our plan, we turn to its realization. We will need several intermediate facts. The following theorem links the quantity $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi}$ with a certain special plurisubharmonic function.

Theorem 3.2. Let X be a separable Banach space and let $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ be continuous functions such that $\Psi(0) = 0$ and $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} < \infty$. Then there exists a plurisubharmonic function $U_{\Phi, \Psi} : X + iX \rightarrow \mathbb{R}$ such that $U_{\Phi, \Psi}(x) \geq 0$ for all $x \in X$ and

$$U_{\Phi, \Psi}(x + iy) \leq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} \Phi(x) - \Psi(y), \quad x, y \in X.$$

Moreover, if Ψ is convex, then $y \mapsto U_{\Phi, \Psi}(x + iy)$ is concave in $y \in X$ for all $x \in X$.

Proof. (sketch) We repeat the reasoning presented in [26, Theorem 2.3] (the separability of X is a key part of the construction $U_{\Phi, \Psi}$). The last property follows from the construction of $U_{\Phi, \Psi}$, the fact that $y \mapsto |\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} \Phi(x) - \Psi(y)$ is a concave function in $y \in X$, and the fact that a minimum of concave functions is a concave function as well. ■

Corollary 3.3. Let X be a Banach space, $1 < p < \infty$. Then X is a UMD Banach space if and only if there exists a plurisubharmonic function $U_{p, X} : X + iX \rightarrow \mathbb{R}$ such that $U_{p, X}(x) \geq 0$ for all $x \in X$ and

$$U_{p, X}(x + iy) \leq h_{p, X}^p \|x\|^p - \|y\|^p, \quad x, y \in X.$$

Moreover, if this is the case, then $y \mapsto U_{p,X}(x + iy)$ is concave in $y \in X$ for all $x \in X$.

Proof. It is sufficient to take $\Phi(x) = \Psi(x) = \|x\|^p$, $x \in X$, and apply Theorem 3.2 and the fact that $h_{p,X} < \infty$ if and only if X is a UMD Banach space (see [6, 8]). ■

Lemma 3.4. Let X be a Banach space, let M be an X -valued local martingale, and let $(\tau_n)_{n \geq 1}$ be a sequence of stopping times increasing to infinity almost surely. Let $\Phi : X \rightarrow \mathbb{R}_+$ be a convex function such that $\mathbb{E}\Phi(M_t) < \infty$ for some $t \geq 0$. Then $\mathbb{E}\Phi(M_{t \wedge \tau_n}) \nearrow \mathbb{E}\Phi(M_t)$ as $n \rightarrow \infty$.

Proof. Notice that $(\mathbb{E}\Phi(M_{t \wedge \tau_n}))_{n \geq 1}$ is an increasing sequence that is less than $\mathbb{E}\Phi(M_t)$ by the conditional Jensen’s inequality, [32, Theorem 7.12], and [32, Lemma 7.1(iii)]. On the other other hand $\Phi(M_{t \wedge \tau_n}) \rightarrow \Phi(M_t)$ a.s. since $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$. It suffices to apply Fatou’s lemma to get the assertion. ■

The next statement contains the proof of a structural property of orthogonal martingales. We need an additional notion. A linear operator T acting on a Hilbert space H is called skew-symmetric (or antisymmetric) if $\langle Th, h \rangle = 0$ for all $h \in H$.

Proposition 3.5. Let $d \geq 1$, W be a d -dimensional standard Brownian motion, let X be a finite-dimensional Banach space, and let $\phi, \psi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(\mathbb{R}^d, X)$ be progressively measurable processes such that $M := \phi \cdot W$ and $N := \psi \cdot W$ are well-defined orthogonal martingales. Assume further that $N \stackrel{w}{\ll} M$. Then there exists a operator-valued progressively measurable process $A : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(\mathbb{R}^d)$ such that $\|A\| \leq 1$, $\psi^* = A\phi^*$ a.s. on $\mathbb{R}_+ \times \Omega$, and $P_{\text{Ran}(\phi^*)}(s, \omega)A(s, \omega)$ is skew-symmetric for all $s \geq 0$ and $\omega \in \Omega$, where $P_{\text{Ran}(\phi^*)} \in \mathcal{L}(\mathbb{R}^d)$ is the orthoprojection on $\text{Ran}(\phi^*)$.

Proof. Let $(x_n^*)_{n \geq 1}$ be a dense sequence in X^* . Then by the orthogonality of M and N and the condition $N \stackrel{w}{\ll} M$, we have

$$\begin{aligned} \|\psi^*(t, \omega)x_n^*\| &\leq \|\phi^*(t, \omega)x_n^*\|, \\ \langle \psi^*(t, \omega)x_n^*, \phi^*(t, \omega)x_n^* \rangle &= 0 \end{aligned}$$

for almost all $\omega \in \Omega$, all $t \in \mathbb{R}_+$ and all $n \geq 1$. Hence by a density argument, for any $x^* \in X^*$, almost all $\omega \in \Omega$ and all $t \in \mathbb{R}_+$,

$$\|\psi^*(t, \omega)x^*\| \leq \|\phi^*(t, \omega)x^*\|, \tag{3.2}$$

$$\langle \psi^*(t, \omega)x^*, \phi^*(t, \omega)x^* \rangle = 0. \tag{3.3}$$

Fix $t \in \mathbb{R}_+$ and $\omega \in \Omega$ such that (3.2) and (3.3) hold for any $x^* \in X^*$. Define $A(t, \omega) : H \rightarrow H$ in the following way (we omit (t, ω) for the convenience of the reader):

$$Ah := \begin{cases} \psi^* x^*, & \text{if } \exists x^* \in X^* \text{ such that } h = \phi^* x^*; \\ 0, & \text{if } h \perp \text{Ran}(\phi^*). \end{cases} \quad (3.4)$$

Then A is well defined since if $h = \phi^*(t, \omega)x_1^* = \phi^*(t, \omega)x_2^*$ for some different $x_1^*, x_2^* \in X^*$, then by (3.2),

$$\begin{aligned} \|\psi^*(t, \omega)x_1^* - \psi^*(t, \omega)x_2^*\| &= \|\psi^*(t, \omega)(x_1^* - x_2^*)\| \leq \|\phi^*(t, \omega)(x_1^* - x_2^*)\| \\ &= \|\phi^*(t, \omega)x_1^* - \phi^*(t, \omega)x_2^*\| = \|h - h\| = 0. \end{aligned}$$

Moreover, A is linear on both $\text{Ran}(\phi^*)$ and $(\text{Ran}(\phi^*))^\perp$, so it can be extended to a linear operator $A \in \mathcal{L}(H)$. Notice that then we have $\psi^* = A\phi^*$. Furthermore, the conditions (3.2) and (3.4) imply that $\|A\| \leq 1$, while (3.3) and (3.4) give that $P_{\text{Ran}(\phi^*)}A$ is skew-symmetric ($P_{\text{Ran}(\phi^*)}$ being the orthoprojection on $\text{Ran}(\phi^*)$). ■

In our later considerations, we will also need the following technical result.

Proposition 3.6. Let X be a finite-dimensional Banach space and let $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ be continuous functions such that Ψ is convex, $\Psi(0) = 0$ and $|\mathcal{H}_X^\mathbb{T}|_{\Phi, \Psi} < \infty$. Let $U_{\Phi, \Psi} : X + iX \rightarrow \mathbb{R}$ be the special function from Theorem 3.2. Assume additionally that $U_{\Phi, \Psi}$ is twice differentiable. Then for any $x, y \in X$, $z_0 \in X + iX$ and any $\lambda \in [-1, 1]$ we have

$$\begin{aligned} &\frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial x^2} \\ &+ \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial y^2} + 2\lambda \left(\frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial x \partial iy} - \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial y \partial ix} \right) + \lambda^2 \left(\frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial ix^2} + \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial iy^2} \right) \geq 0. \end{aligned} \quad (3.5)$$

Proof. Notice that the function

$$\begin{aligned} \lambda \mapsto &\frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial x^2} \\ &+ \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial y^2} + 2\lambda \left(\frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial x \partial iy} - \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial y \partial ix} \right) + \lambda^2 \left(\frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial ix^2} + \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial iy^2} \right) \end{aligned}$$

is concave due to the fact that $\frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial ix^2}, \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial iy^2} \leq 0$ by the last part of Theorem 3.2. Therefore it is sufficient to show (3.5) for $\lambda = 1$ and $\lambda = -1$. We will consider the 1st possibility only; the 2nd can be handled analogously. We have

$$\begin{aligned} & \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial x^2} + \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial y^2} + 2\left(\frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial x \partial iy} - \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial y \partial ix}\right) + \left(\frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial ix^2} + \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial iy^2}\right) \\ &= \frac{\partial^2 U_{\Phi, \Psi}(z_0 + t(x + iy))}{\partial t^2} + \frac{\partial^2 U_{\Phi, \Psi}(z_0 + t(y - ix))}{\partial t^2} \\ &= \Delta_z U_{\Phi, \Psi}(z_0 + z(y - ix)) \geq 0, \end{aligned}$$

since $U_{\Phi, \Psi}$ is plurisubharmonic (here Δ_z is the Laplace operator acting with respect to the z variable). ■

Corollary 3.7. Under the assumptions of the previous proposition, for any $x, y \in X$, $z_0 \in X + iX$, $\lambda \in [-1, 1]$ and any $\mu \in [-|\lambda|, |\lambda|]$ we have

$$\begin{aligned} & \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial x^2} + \\ & \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial y^2} + 2\mu\left(\frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial x \partial iy} - \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial y \partial ix}\right) + \lambda^2\left(\frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial ix^2} + \frac{\partial^2 U_{\Phi, \Psi}(z_0)}{\partial iy^2}\right) \geq 0. \end{aligned} \tag{3.6}$$

Proof. The left-hand side of (3.6) is linear in μ , so it is sufficient to check the estimate for $\mu = \pm\lambda$. ■

The following lemma was proven in the supplement to [56, Lemma 3.7].

Lemma 3.8. Let d be a natural number and E be a d -dimensional linear space. Let $V : E \times E \rightarrow \mathbb{R}$ and $W : E^* \times E^* \rightarrow \mathbb{R}$ be two bilinear functions. Then the expression

$$\sum_{n,m=1}^d V(e_n, e_m) W(e_n^*, e_m^*)$$

does not depend on the choice of basis $(e_n)_{n=1}^d$ of E (here $(e_n^*)_{n=1}^d$ is the corresponding dual basis of $(e_n)_{n=1}^d$).

Corollary 3.9. Let d be a natural number and E be a d -dimensional linear space. Let $V : E \times E \rightarrow \mathbb{R}$ and $W_1, W_2 : E^* \times E^* \rightarrow \mathbb{R}$ be bilinear functions. Assume additionally that V

is symmetric nonnegative (i.e., $V(x, x) \geq 0$ for all $x \in E$) and that $W_1(x^*, x^*) \leq W_2(x^*, x^*)$ for all $x^* \in X^*$. Then

$$\sum_{n,m=1}^d V(e_n, e_m) W_1(e_n^*, e_m^*) \leq \sum_{n,m=1}^d V(e_n, e_m) W_2(e_n^*, e_m^*)$$

for any basis $(e_n)_{n=1}^d$ of E (here $(e_n^*)_{n=1}^d$ is the corresponding dual basis of $(e_n)_{n=1}^d$).

Proof. Since V is symmetric and nonnegative it defines an inner product on $E \times E$. Let $(\tilde{e}_n)_{n=1}^d$ be an orthogonal basis of E under the inner product V (i.e., $V(\tilde{e}_n, \tilde{e}_m) = 0$ for all $n \neq m$, and $V(\tilde{e}_n, \tilde{e}_n) \geq 0$ for all $n = 1, \dots, d$). Then we have that

$$\begin{aligned} \sum_{n,m=1}^d V(\tilde{e}_n, \tilde{e}_m) W_1(\tilde{e}_n^*, \tilde{e}_m^*) &= \sum_{n=1}^d V(\tilde{e}_n, \tilde{e}_n) W_1(\tilde{e}_n^*, \tilde{e}_n^*) \\ &\leq \sum_{n=1}^d V(\tilde{e}_n, \tilde{e}_n) W_2(\tilde{e}_n^*, \tilde{e}_n^*) = \sum_{n,m=1}^d V(\tilde{e}_n, \tilde{e}_m) W_2(\tilde{e}_n^*, \tilde{e}_m^*), \end{aligned} \tag{3.7}$$

where $(\tilde{e}_n^*)_{n=1}^d$ is the corresponding dual basis of $(\tilde{e}_n)_{n=1}^d$. Consequently, the desired follows from (3.7) and Lemma 3.8. \blacksquare

The next few statements aim at establishing an appropriate ‘‘localization’’ procedure: we will prove how to deduce the general, possibly infinite-dimensional context from its finite-dimensional counterpart. We need some additional notation. Let X be a Banach space with a dual X^* , $Y \subset X^*$ be a linear subspace. Let $P : Y \hookrightarrow X^*$ be the continuous embedding operator. Then P^* is a well-defined bounded linear operator from X^{**} to $X_Y := Y^*$ such that $\text{Ran}(P^*) = X_Y$. Moreover, if Y is finite dimensional, then $\text{Ran}(P^*|_X) = X_Y$, where $P^*|_X : X \rightarrow X_Y$ is a well-defined restriction of P^* on X due to the natural embedding $X \hookrightarrow X^{**}$. For any function $\phi : X \rightarrow \mathbb{R}_+$, we can define $\phi_Y : X_Y \rightarrow \mathbb{R}_+$ by the formula

$$\phi_Y(\tilde{x}) = \inf\{\phi(x) : x \in X, P^*x = \tilde{x}\}, \quad \tilde{x} \in X_Y. \tag{3.8}$$

In other words, $\phi_Y(\tilde{x})$ denotes the infimum of $\phi(x)$ over all $x \in X$ satisfying $x|_Y = \tilde{x}$ where we consider x as an element of Y^* .

Lemma 3.10. Let X be a Banach space with a dual X^* and let $Y \subset X^*$ be a closed linear subspace. Let $\phi : X \rightarrow \mathbb{R}_+$ be a convex function. Then $\phi_Y : X_Y \rightarrow \mathbb{R}_+$ defined by (3.8) is convex and we have $\phi_Y(P^*x) \leq \phi(x)$ for all $x \in X$.

Proof. Fix $\tilde{x}_1, \tilde{x}_2 \in X_Y$, $\lambda \in [0, 1]$ and set $\tilde{x} = \lambda\tilde{x}_1 + (1 - \lambda)\tilde{x}_2$. Then

$$\begin{aligned} \phi_Y(\tilde{x}) &= \inf_{\substack{x \in X \\ P^*x = \tilde{x}}} \phi(x) = \inf_{\substack{x_1 \in X, P^*x_1 = \tilde{x}_1 \\ x_2 \in X, P^*x_2 = \tilde{x}_2}} \phi(\lambda x_1 + (1 - \lambda)x_2) \leq \inf_{\substack{x_1 \in X, P^*x_1 = \tilde{x}_1 \\ x_2 \in X, P^*x_2 = \tilde{x}_2}} \lambda\phi(x_1) + (1 - \lambda)\phi(x_2) \\ &= \lambda \inf_{x_1 \in X, P^*x_1 = \tilde{x}_1} \phi(x_1) + (1 - \lambda) \inf_{x_2 \in X, P^*x_2 = \tilde{x}_2} \phi(x_2) = \lambda\phi_Y(\tilde{x}_1) + (1 - \lambda)\phi_Y(\tilde{x}_2), \end{aligned}$$

so ϕ_Y is convex. The last part of the lemma follows from the definition of ϕ_Y . ■

Lemma 3.11. Let X be a separable Banach space, $\phi : X \rightarrow \mathbb{R}_+$ be convex lower semi-continuous. Then there exists an increasing sequence of finite-dimensional subspaces $(Y_n)_{n \geq 1}$ of X^* such that the following holds. If $P_n : Y_n \hookrightarrow X^*$ is the corresponding embedding for each $n \geq 1$ and $\phi_n : Y_n^* \rightarrow \mathbb{R}_+$ satisfies

$$\phi_n(\tilde{x}) = \inf\{\phi(x) : x \in X, P_n^*x = \tilde{x}\}, \quad \tilde{x} \in Y_n^*, \tag{3.9}$$

then for each $x \in X$ the sequence $(\phi_n(P_n^*x))_{n \geq 1}$ increases to $\phi(x)$ as $n \rightarrow \infty$.

Proof. By [28, Lemma 1.2.10] there exist a sequence $(x_n^*)_{n \geq 1}$ in X^* and a sequence $(a_n)_{n \geq 1}$ of real numbers such that

$$\phi(x) = \sup_n \langle x, x_n^* \rangle + a_n, \quad x \in X. \tag{3.10}$$

Let $Y_n := \text{span}(x_1^*, \dots, x_n^*)$ for each $n \geq 1$. Fix $x \in X$. First notice that $\phi_n(P_n^*x) \leq \phi(x)$ by Lemma 3.10. Moreover, $\phi_n(P_n^*x) \leq \phi_{n+1}(P_{n+1}^*x)$ for each $n \geq 1$ since $Y_n \subset Y_{n+1}$ (see (3.9)). Fix $n \geq 1$. Then for any $y \in X$ such that $P_n^*x = P_n^*y$ we have $\langle x, x_k^* \rangle = \langle y, x_k^* \rangle$ for any $k = 1, \dots, n$, so by (3.10),

$$\begin{aligned} \phi_n(P_n^*x) &= \inf\{\phi(y) : y \in X, P_n^*y = P_n^*x\} \geq \inf\{\sup_{1 \leq k \leq n} \langle y, x_k^* \rangle + a_k : y \in X, P_n^*y = P_n^*x\} \\ &= \inf\{\sup_{1 \leq k \leq n} \langle x, x_k^* \rangle + a_k : y \in X, P_n^*y = P_n^*x\} = \sup_{1 \leq k \leq n} \langle x, x_k^* \rangle + a_k. \end{aligned}$$

Since the latter expression tends to $\phi(x)$ as $n \rightarrow \infty$, we obtain the desired monotone convergence $\phi_n(P_n^*x) \nearrow \phi(x)$. ■

Proposition 3.12. Let X be a Banach space with a dual X^* and let $Y \subset X^*$ be a finite-dimensional linear subspace. Assume further that $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ are convex continuous

functions and let $\Phi_Y, \Psi_Y : X_Y \rightarrow \mathbb{R}_+$ be defined by (3.8). Then

$$|\mathcal{H}_{X_Y}^{\mathbb{T}}|_{\Phi_Y, \Psi_Y} \leq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi}.$$

Proof. Recall that

$$|\mathcal{H}_{X_Y}^{\mathbb{T}}|_{\Phi_Y, \Psi_Y} = \sup_{f \in F_{X_Y}^{step}} \frac{\int_{\mathbb{T}} \Psi_Y(\mathcal{H}_{X_Y}^{\mathbb{T}} f) \, d\mu}{\int_{\mathbb{T}} \Phi_Y(f) \, d\mu},$$

where μ is the Lebesgue measure on \mathbb{T} . Fix $f \in F_{X_Y}^{step}$ and $\varepsilon > 0$. Let $(\tilde{x}_n)_{n=1}^N \subset X_Y$ be the range of f . For each $n = 1, \dots, N$ we define $x_n \in X$ to be such that $P^*x_n = \tilde{x}_n$ and $\Phi(x_n) \leq (1 + \varepsilon)\Phi_Y(\tilde{x}_n)$ (existence of such x_n follows from the fact that $\text{Ran}(P^*) = X_Y$); we define $g : \mathbb{T} \rightarrow X$ to be such that $f(s) = \tilde{x}_n$ if and only if $g(s) = x_n$, $s \in \mathbb{T}$. Then $\Phi_Y(f) = \Phi_Y(P^*g)$ and $\Psi_Y(\mathcal{H}_{X_Y}^{\mathbb{T}} f) = \Psi_Y(\mathcal{H}_{X_Y}^{\mathbb{T}} P^*g) = \Psi_Y(P^* \mathcal{H}_X^{\mathbb{T}} g)$ for any $s \in \mathbb{T}$ by the definition of the Hilbert transform on the torus. Therefore,

$$\frac{\int_{\mathbb{T}} \Psi_Y(\mathcal{H}_{X_Y}^{\mathbb{T}} f) \, d\mu}{\int_{\mathbb{T}} \Phi_Y(f) \, d\mu} = \frac{\int_{\mathbb{T}} \Psi_Y(P^* \mathcal{H}_X^{\mathbb{T}} g) \, d\mu}{\int_{\mathbb{T}} \Phi_Y(P^*g) \, d\mu} \stackrel{(*)}{\leq} (1 + \varepsilon) \frac{\int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}} g) \, d\mu}{\int_{\mathbb{T}} \Phi(g) \, d\mu} \stackrel{(**)}{\leq} (1 + \varepsilon) |\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi},$$

where (*) follows from the fact that $\Phi(g(s)) \leq (1 + \varepsilon)\Phi_Y(f(s))$ for any $s \in \mathbb{T}$ and from the fact that $\Psi_Y(P^*\cdot) \leq \Psi(\cdot)$ on X , while (**) follows from the definition of $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi}$. Since $f \in F_{X_Y}^{step}$ and $\varepsilon > 0$ were arbitrary, the claim follows. \blacksquare

The final ingredient is the following well-known statement from the theory of stochastic integration.

Lemma 3.13. Let $d \geq 1$ and let M be a martingale with values in \mathbb{R}^d satisfying the condition $\mathbb{E}M_{\infty}^* < \infty$. Let $V : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$ be a predictable and bounded process. Then $V \cdot M := \int \langle V, dM \rangle$ is a well-defined martingale and $\mathbb{E}(V \cdot M)_{\infty}^* < \infty$.

Equipped with the above statements, we are ready for the study of our main result. We should point out that the main difficulty lies in proving the inequality (3.1) for finite-dimensional Banach spaces. The novelty in comparison to other results from the literature is that we work under slightly different condition of weak differential subordination and orthogonality; therefore, though at some places the arguments might look similar to, for instance, those appearing in [3], there is no apparent connection between them.

Proof of (3.1) for finite-dimensional X . We split the reasoning into several intermediate parts.

Step 1. Some reductions. First assume that the function $U_{\Phi,\Psi}$ (defined in Theorem 3.2) is continuous and twice differentiable. Since N has continuous paths almost surely, we may assume that N is a bounded martingale: this is due to a simple stopping time argument combined with Lemma 3.4. Moreover, we may assume that $\mathbb{E}\Phi(M_t) < \infty$, since otherwise there is nothing to prove. Let d be the dimension of X . Then analogously to [56, Section 4] we can find a continuous time-change $\tau = (\tau_s)_{s \geq 0}$ and redefine $M := M \circ \tau$ and $N := N \circ \tau$, so that the following holds. For some $2d$ -dimensional standard Brownian motion W on an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ equipped with an extended filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0}$, there exist progressively measurable processes $\phi, \psi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(\mathbb{R}^{2d}, X)$ such that $M^c = \phi \cdot W$ and $N = \psi \cdot W$, where $M = M^c + M^d$ is the Meyer–Yœurp decomposition of M (see [32, 54, 56]). In addition, the arguments in [56, Section 4] also yield the identities $[M \circ \tau] = [M] \circ \tau$, $[N \circ \tau] = [N] \circ \tau$ and $[M \circ \tau, N \circ \tau] = [M, N] \circ \tau$, so the weak differential subordination and orthogonality are not ruined under the time-change.

Now, for each $n \geq 1$, introduce the stopping time

$$\sigma_n := \inf\{t \geq 0 : M_t > n\}. \tag{3.11}$$

By Lemma 3.4 it is sufficient to show that

$$\mathbb{E}\Psi(N_{t \wedge \sigma_n}) \leq |\mathcal{H}_X^\mathbb{T}|_{\Phi,\Psi} \mathbb{E}\Phi(M_{t \wedge \sigma_n}) \tag{3.12}$$

for any $n \geq 1$. Actually, passing to M/n and N/n (and by modifying Φ and Ψ accordingly), we see that it is enough to show the above estimate for $n = 1$. For the sake of notational convenience, we redefine $M := M^{\sigma_1}$ and $N := N^{\sigma_1}$ and observe that it suffices to show $\mathbb{E}U_{\Phi,\Psi}(M_t + iN_t) \geq 0$, since then (3.12) follows at once from the majorization property of $U_{\Phi,\Psi}$.

Step 2. Application of Itô’s formula. Let $(e_n)_{n=1}^d$ be a basis of X , and $(e_n^*)_{n=1}^d$ be the corresponding dual basis. Then by the Itô formula (2.3), we get

$$\begin{aligned} \mathbb{E}U_{\Phi,\Psi}(M_t + iN_t) &= \mathbb{E}U_{\Phi,\Psi}(M_0 + iN_0) + \mathbb{E} \int_0^t \langle \partial_x U_{\Phi,\Psi}(M_{s-} + iN_{s-}), dM_s \rangle \\ &\quad + \mathbb{E} \int_0^t \langle \partial_{ix} U_{\Phi,\Psi}(M_{s-} + iN_{s-}), dN_s \rangle + \mathbb{E}I_1 + \mathbb{E}I_2, \end{aligned} \tag{3.13}$$

where $\partial_x U_{\Phi, \Psi}(\cdot), \partial_{ix} U_{\Phi, \Psi}(\cdot) \in X^*$ are the corresponding Fréchet derivatives of $U_{\Phi, \Psi}$ in the real and the imaginary subspaces of $X + iX$ respectively,

$$I_1 = \sum_{0 \leq s \leq t} (\Delta U_{\Phi, \Psi}(M_s + iN_s) - \langle \partial_x U_{\Phi, \Psi}(M_{s-} + iN_{s-}), \Delta M_s \rangle),$$

and

$$\begin{aligned} I_2 &= \frac{1}{2} \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi, \Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial e_j} d[\langle M^c, e_i^* \rangle, \langle M^c, e_j^* \rangle]_s \\ &\quad + \frac{1}{2} \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi, \Psi}(M_{s-} + iN_{s-})}{\partial i e_i \partial i e_j} d[\langle N, e_i^* \rangle, \langle N, e_j^* \rangle]_s \\ &\quad + \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi, \Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial i e_j} d[\langle M^c, e_i^* \rangle, \langle N, e_j^* \rangle]_s \\ &= \frac{1}{2} \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi, \Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial e_j} \langle \phi^*(s) e_i^*, \phi^*(s) e_j^* \rangle ds \\ &\quad + \frac{1}{2} \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi, \Psi}(M_{s-} + iN_{s-})}{\partial i e_i \partial i e_j} \langle \psi^*(s) e_i^*, \psi^*(s) e_j^* \rangle ds \\ &\quad + \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi, \Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial i e_j} \langle \phi^*(s) e_i^*, \psi^*(s) e_j^* \rangle ds. \end{aligned}$$

Step 3. Analysis of the terms on the right of (3.13). Let us first show that

$$\mathbb{E} \int_0^t \langle \partial_x U_{\Phi, \Psi}(M_{s-} + iN_{s-}), dM_s \rangle + \mathbb{E} \int_0^t \langle \partial_{ix} U_{\Phi, \Psi}(M_{s-} + iN_{s-}), dN_s \rangle$$

exists and equals zero. First notice that since $M = M^{\sigma_1}$, the variable M_{s-} is bounded by 1 for any $0 \leq s \leq \sigma_1$. Furthermore, as we have assumed above, the process N is also bounded. Since $U_{\Phi, \Psi}$ is twice differentiable, both $\partial_x U_{\Phi, \Psi}(\cdot)$ and $\partial_{ix} U_{\Phi, \Psi}(\cdot)$ are continuous functions, so $s \mapsto \partial_x U_{\Phi, \Psi}(M_{s-} + iN_{s-})$ and $s \mapsto \partial_{ix} U_{\Phi, \Psi}(M_{s-} + iN_{s-})$ define bounded processes on $0 \leq s \leq \sigma_1$. Furthermore, it is easy to see that

$$\mathbb{E} M_t^* = \mathbb{E} M_{t \wedge \sigma_1}^* \leq \mathbb{E} \|M_{t \wedge \sigma_1}\| + 1 \leq \mathbb{E} \|M_t\| + 1 < \infty,$$

and hence by Lemma 3.13,

$$\begin{aligned}
 t &\mapsto \int_0^t \langle \partial_x U_{\Phi, \Psi}(M_{s-} + iN_{s-}) \mathbf{1}_{s \in [0, \sigma_1]}, dM_s \rangle, \quad t \geq 0, \\
 t &\mapsto \int_0^t \langle \partial_{ix} U_{\Phi, \Psi}(M_{s-} + iN_{s-}) \mathbf{1}_{s \in [0, \sigma_1]}, dN_s \rangle, \quad t \geq 0,
 \end{aligned}
 \tag{3.14}$$

define martingales. Moreover, with Probability 1,

$$\begin{aligned}
 \int_0^t \langle \partial_x U_{\Phi, \Psi}(M_{s-} + iN_{s-}) \mathbf{1}_{s \in [0, \sigma_1]}, dM_s \rangle &= \int_0^t \langle \partial_x U_{\Phi, \Psi}(M_{s-} + iN_{s-}), dM_s \rangle, \\
 \int_0^t \langle \partial_{ix} U_{\Phi, \Psi}(M_{s-} + iN_{s-}) \mathbf{1}_{s \in [0, \sigma_1]}, dN_s \rangle &= \int_0^t \langle \partial_{ix} U_{\Phi, \Psi}(M_{s-} + iN_{s-}), dN_s \rangle,
 \end{aligned}$$

since $M = M^{\sigma_1}$ and $N = N^{\sigma_1}$, and consequently the expectations of the above integrals vanish. Let us now show that $I_1, I_2 \geq 0$ almost surely. For the 1st term, the argument is simple: by (2.6), each summand in I_1 is nonnegative. The analysis of I_2 is slightly more complex. By Proposition 2.14, we get that $N \ll^w M^c$ and M^c, N are orthogonal, so Proposition 3.5 implies the existence of a progressively measurable operator-valued process $A : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(\mathbb{R}^d)$ such that $\|A\| \leq 1$, $\psi^* = A\phi^*$, and $P_{\text{Ran}(\phi^*)}A$ is skew-symmetric on $\mathbb{R}_+ \times \Omega$ (here $P_{\text{Ran}(\phi^*)}$ is an orthoprojection on $\text{Ran}(\phi^*)$). Thus it is enough to show that

$$\begin{aligned}
 &\sum_{i,j=1}^d \frac{\partial^2 U_{\Phi, \Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial e_j} \langle \phi^*(s)e_i^*, \phi^*(s)e_j^* \rangle ds \\
 &+ \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi, \Psi}(M_{s-} + iN_{s-})}{\partial ie_i \partial ie_j} \langle \psi^*(s)e_i^*, \psi^*(s)e_j^* \rangle ds \\
 &+ 2 \sum_{i,j=1}^d \frac{\partial^2 U_{\Phi, \Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial ie_j} \langle \phi^*(s)e_i^*, P_{\text{Ran}(\phi^*)}A\phi^*(s)e_j^* \rangle ds \geq 0.
 \end{aligned}
 \tag{3.15}$$

By the spectral theory of skew-symmetric matrices (see, e.g., [58, Corollary 2]) there exist $L \geq 0$, positive numbers $(\lambda_n)_{n=1}^L$, and an orthonormal basis $(h_n)_{n=1}^{2d}$ of \mathbb{R}^{2d} such that $P_{\text{Ran}(\phi^*)}Ah_{2n-1} = \lambda_n h_{2n}$ and $P_{\text{Ran}(\phi^*)}Ah_{2n} = -\lambda_n h_{2n-1}$ for all $n = 1, \dots, L$, and $P_{\text{Ran}(\phi^*)}Ah_n = 0$ for all $2L < n \leq d$. Moreover, the condition $\|A\| \leq 1$ implies that $|\lambda_1|, \dots, |\lambda_L| \leq 1$, and since $(\text{Ran}(\phi^*))^\perp$ is a zero eigenspace of $P_{\text{Ran}(\phi^*)}A$ (see the construction of A in the proof of Proposition 3.5), we conclude that $h_n \in \text{Ran}(\phi^*)$ for $n = 1, 2, \dots, 2L$. By a usual orthogonalization procedure, we may assume that there

exists $K \geq 2L$ such that $h_n \in \text{Ran}(\phi^*)$ for $2L < n \leq K$ and $h_n \perp \text{Ran}(\phi^*)$ for $K < n \leq 2d$ (then K is the dimension of $\text{Ran}(\phi^*)$). Notice that X^* is d dimensional, so $\text{Ran}(\phi^*)$ is at most d dimensional and hence obviously $K \leq d$. Due to Lemma 3.8, the expression (3.15) does not depend on the basis $(e_n)_{n=1}^d$ (and the corresponding dual basis $(e_n^*)_{n=1}^d$), so we can choose a basis $(e_n)_{n=1}^d$ such that $\phi^* e_n = h_n$ for all $n = 1, \dots, K$ and $\phi^* e_n = 0$ for all $K < n \leq d$ (such a basis exists since $\text{span}\{h_1, \dots, h_K\} = \text{Ran}(\phi^*)$). Then (3.15) becomes

$$\begin{aligned} & \sum_{i,j=1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s^-} + iN_{s^-})}{\partial e_i \partial e_j} \langle h_i, h_j \rangle \\ & + \sum_{i,j=1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s^-} + iN_{s^-})}{\partial i e_i \partial i e_j} \langle \psi^* e_i, \psi^* e_j \rangle \\ & + 2 \sum_{i,j=1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s^-} + iN_{s^-})}{\partial e_i \partial i e_j} \langle h_i, P_{\text{Ran}(\phi^*)} A h_j \rangle \geq 0 \end{aligned} \quad (3.16)$$

(The 2nd sum is up to K due to the fact that $\phi^* x^* = 0$ implies $\psi^* x^* = 0$ for any $x^* \in X^*$, see (3.2)). Notice that the bilinear form $V : X \times X \rightarrow \mathbb{R}$ defined by

$$V(x, y) := -\frac{\partial^2 U_{\Phi,\Psi}(M_{s^-} + iN_{s^-})}{\partial i x \partial i y}, \quad x, y \in X,$$

is nonnegative by Theorem 3.2 and symmetric by the definition. Moreover, by (3.2),

$$\langle \psi^* x^*, \psi^* x^* \rangle = \|\psi^* x^*\|^2 \leq \|\phi^* x^*\|^2 = \langle \phi^* x^*, \phi^* x^* \rangle, \quad \text{for } x^* \in X^*.$$

Therefore, Corollary 3.9 yields

$$\begin{aligned} \sum_{i,j=1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s^-} + iN_{s^-})}{\partial i e_i \partial i e_j} \langle \psi^* e_i, \psi^* e_j \rangle & \geq \sum_{i,j=1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s^-} + iN_{s^-})}{\partial i e_i \partial i e_j} \langle \phi^* e_i, \phi^* e_j \rangle \\ & = \sum_{i,j=1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s^-} + iN_{s^-})}{\partial i e_i \partial i e_j} \langle h_i, h_j \rangle, \end{aligned}$$

so (3.16) is not less than

$$\begin{aligned} & \sum_{i,j=1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial e_j} \langle h_i, h_j \rangle + \sum_{i,j=1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial i e_i \partial i e_j} \langle h_i, h_j \rangle \\ & \quad + 2 \sum_{i,j=1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial i e_j} \langle h_i, P_{\text{Ran}(\phi^*)} A h_j \rangle \\ = & \sum_{i=1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial e_i} \langle h_i, h_i \rangle + \sum_{i=1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial i e_i \partial i e_i} \langle h_i, h_i \rangle \\ & \quad + 2 \sum_{i,j=1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial i e_j} \langle h_i, P_{\text{Ran}(\phi^*)} A h_j \rangle. \end{aligned}$$

The latter expression consists of two parts:

$$\begin{aligned} & \sum_{i=1}^{2L} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial e_i} + \sum_{i=1}^{2L} \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial i e_i \partial i e_i} \\ & \quad + 2 \sum_{n=1}^L \lambda_n \left(\frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_{2n-1} \partial i e_{2n}} - \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_{2n} \partial i e_{2n-1}} \right) \\ = & \sum_{n=1}^L \left\{ \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_{2n-1} \partial e_{2n-1}} + \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_{2n} \partial e_{2n}} \right. \\ & \quad + 2\lambda_n \left(\frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_{2n-1} \partial i e_{2n}} - \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_{2n} \partial i e_{2n-1}} \right) \\ & \quad \left. + \left(\frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial i e_{2n-1} \partial i e_{2n-1}} + \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial i e_{2n} \partial i e_{2n}} \right) \right\} \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} & \sum_{i=2L+1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial e_i} + \sum_{i=2L+1}^K \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial i e_i \partial i e_i} \\ = & \sum_{i=2L+1}^K \left(\frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial e_i \partial e_i} + \frac{\partial^2 U_{\Phi,\Psi}(M_{s-} + iN_{s-})}{\partial i e_i \partial i e_i} \right). \end{aligned} \tag{3.18}$$

Now, the expression (3.17) is nonnegative by Corollary 3.7 and (3.18) is nonnegative by Remark 2.10. This gives $I_2 \geq 0$. Putting all the above facts together, we obtain

$$\mathbb{E}U_{\Phi,\Psi}(M_t + iN_t) \geq \mathbb{E}U_{\Phi,\Psi}(M_0 + iN_0).$$

However, by Remark 2.5, we have $N_0 = 0$ almost surely, so Theorem 3.2 implies

$$\mathbb{E}U_{\Phi,\Psi}(M_0 + iN_0) = \mathbb{E}U_{\Phi,\Psi}(M_0) \geq 0,$$

which completes the proof.

Step 4. Now we assume that $U_{\Phi,\Psi}$ is general (i.e., not necessarily twice integrable). We will use a standard mollification argument. Let $\phi : X + iX \rightarrow \mathbb{R}_+$ be a C^∞ radial function with compact support such that $\int_{X+iX} \phi(s) ds = 1$. For each $\varepsilon > 0$, define $U_{\Phi,\Psi}^\varepsilon : X + iX \rightarrow \mathbb{R}$ via the convolution

$$U_{\Phi,\Psi}^\varepsilon(x + iy) := \int_{X+iX} U_{\Phi,\Psi}(x + iy - \varepsilon s) \phi(s) ds, \quad x, y \in X.$$

Then $U_{\Phi,\Psi}^\varepsilon$ is of class C^∞ and for any $x \in X$ we have

$$U_{\Phi,\Psi}^\varepsilon(x) = \int_{X+iX} U_{\Phi,\Psi}(x - \varepsilon s) \phi(s) ds \geq U_{\Phi,\Psi}(x) \geq 0, \quad (3.19)$$

since $U_{\Phi,\Psi}$ is subharmonic (see Remark 2.9). Therefore, repeating the arguments from the above steps, we get

$$\begin{aligned} & \mathbb{E} \int_{X+iX} \left[|\mathcal{H}_X^\mathbb{T}|_{\Phi,\Psi} \Phi(M_t - \varepsilon r) - \Psi(N_t - \varepsilon u) \right] \phi(r + iu) dr + iu \\ & \geq \mathbb{E}U_{\Phi,\Psi}^\varepsilon(M_t + iN_t) \geq \mathbb{E}U_{\Phi,\Psi}^\varepsilon(M_0) \geq 0, \end{aligned} \quad (3.20)$$

where the latter bound follows from (3.19). Note that $\Psi(N_t + \varepsilon u)$ is uniformly bounded (when $r + iu$ runs over the support of ϕ) and notice that for any x , $\varepsilon \mapsto \frac{\Phi(x-\varepsilon) + \Phi(x+\varepsilon)}{2}$ is an increasing function of $\varepsilon > 0$. Furthermore, we have $\phi(r + iu) = \phi(-r + iu) \geq 0$ and hence

$$\varepsilon \mapsto \int_{X+iX} \Phi(M_t - \varepsilon r) \phi(r + iu) d(r + iu) = \int_{X+iX} \frac{\Phi(M_t - \varepsilon r) + \Phi(M_t + \varepsilon r)}{2} \phi(r + iu) d(r + iu), \quad (3.21)$$

decreases as $\varepsilon \downarrow 0$. Combining these observations with standard limiting theorems, we deduce the desired claim. \blacksquare

Now we prove our main result in full generality. Of course, we will exploit an appropriate limiting procedure, which enables us to deduce the claim from its finite-dimensional version just established above.

Proof of (3.1) for infinite-dimensional X . We may assume that $\mathbb{E}\Phi(M_t) < \infty$, since otherwise the claim is obvious. Suppose that $(Y_n)_{n \geq 1}$ is a sequence of finite-dimensional subspaces of X^* such that $Y_n \subset Y_{n+1}$ for any $n \geq 1$ and $\overline{\cup_{n \geq 1} Y_n} = X^*$. For each $n \geq 1$ define $X_n := Y_n^*$, let $P_n : Y_n \hookrightarrow X^*$ be the corresponding embedding operator, and let $P_n^* : X \rightarrow X_n$ be its adjoint (recall that X is reflexive). Finally, define $\Phi_n, \Psi_n : X_n \rightarrow \mathbb{R}_+$ by the formulas

$$\Phi_n(\tilde{x}) = \inf\{\Phi(x) : x \in X, P_n^*x = \tilde{x}\}, \quad \Psi_n(\tilde{x}) = \inf\{\Psi(x) : x \in X, P_n^*x = \tilde{x}\},$$

for $\tilde{x} \in X_n$. In the light of Lemma 3.10, both Φ_n and Ψ_n are convex functions. Moreover, by Proposition 3.12,

$$|\mathcal{H}_{X_n}^{\mathbb{T}}|_{\Phi_n, \Psi_n} \leq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi}. \tag{3.22}$$

Let us show that the processes P_n^*M and P_n^*N are orthogonal for each $n \geq 1$. By the very definition, we must prove that for a fixed functional $x^* \in X_n^*$, the local martingales $\langle P_n^*M, x^* \rangle$ and $\langle P_n^*N, x^* \rangle$ are orthogonal. This follows at once from orthogonality of M, N and the identities

$$\langle P_n^*M, x^* \rangle = \langle M, P_n x^* \rangle, \quad \langle P_n^*N, x^* \rangle = \langle N, P_n x^* \rangle. \tag{3.23}$$

These identities also immediately give the weak differential subordination $P_n^*N \overset{w}{\ll} P_n^*M$, since M and N enjoy this condition. Finally, observe that by Lemma 3.10, we have $\mathbb{E}\Phi_n(P_n^*M_t) \leq \mathbb{E}\Phi(M_t) < \infty$. Therefore, applying the finite-dimensional version of (3.1), we see that for each $n \geq 1$,

$$\mathbb{E}\Psi_n(P_n^*N_t) \leq |\mathcal{H}_{X_n}^{\mathbb{T}}|_{\Phi_n, \Psi_n} \mathbb{E}\Phi_n(P_n^*M_t) \leq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} \mathbb{E}\Phi_n(P_n^*M_t), \tag{3.24}$$

where the 2nd passage is due to (3.22). Note that with probability 1 we have $\Phi_n(P_n^*M_t) \nearrow \Phi(M_t)$ and $\Psi_n(P_n^*N_t) \nearrow \Psi(N_t)$ monotonically as $n \rightarrow \infty$ by Lemma 3.11. This establishes the desired estimate, by Lebesgue’s monotone convergence theorem. ■

It remains to handle the sharpness of (3.1).

Proof of the estimate $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} \leq C_{\Phi, \Psi, X}$. This follows immediately from the reasoning presented in Section 2.5: indeed, (1) implies the corresponding bound

$$\int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}} f) dx \leq C_{\Phi, \Psi, X} \int_{\mathbb{T}} \Phi(f) dx$$

for any step function $f : \mathbb{T} \rightarrow X$. ■

Remark 3.14. It is easy to see that if X is finite dimensional, then there is no need for Φ to be convex. The limiting argument presented in the above proof does not need this requirement. (The only place where the convexity of Φ is used is (3.21); we leave to the reader the question how to avoid this issue).

4 Applications

4.1 Hilbert transforms on \mathbb{T} , \mathbb{R} , and \mathbb{Z}

Let X be a Banach space and let $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ be continuous functions. Let (S, Σ, μ) be a measure space, with S equal to \mathbb{T} , \mathbb{R} , or \mathbb{Z} . A function $f : S \rightarrow X$ is called a step function, if it is of the form

$$f(t) = \sum_{k=1}^N x_k \mathbf{1}_{A_k}(t), \quad t \in S,$$

where N is finite, $x_k \in X$, and A_k are intervals in S of a finite measure.

Definition 4.1. The Hilbert transform $\mathcal{H}_X^{\mathbb{R}}$ is a linear operator that maps a step function $f : \mathbb{R} \rightarrow X$ to the function

$$(\mathcal{H}_X^{\mathbb{R}}f)(t) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(s)}{t-s} ds, \quad t \in \mathbb{R}. \quad (4.1)$$

The associated Φ, Ψ -norms $|\mathcal{H}_X^{\mathbb{R}}|_{\Phi, \Psi}$ are given by a formula similar to that used previously:

$$|\mathcal{H}_X^{\mathbb{R}}|_{\Phi, \Psi} := \inf \left\{ c \in [0, \infty] : \int_{\mathbb{R}} \Psi(\mathcal{H}_X^{\mathbb{R}}f(s)) ds \leq c \int_{\mathbb{R}} \Phi(f(s)) ds \text{ for all step functions } f : \mathbb{R} \rightarrow X \right\}.$$

Definition 4.2. The discrete Hilbert transform $\mathcal{H}_X^{\text{dis}}$ is a linear operator that maps a step function $f : \mathbb{Z} \rightarrow X$ to the function

$$(\mathcal{H}_X^{\text{dis}}f)(t) := \frac{1}{\pi} \sum_{s \in \mathbb{Z} \setminus \{t\}} \frac{f(s)}{t-s}, \quad t \in \mathbb{Z}.$$

The associated Φ, Ψ -norms $|\mathcal{H}_X^{dis}|_{\Phi, \Psi}$ are given by

$$|\mathcal{H}_X^{dis}|_{\Phi, \Psi} := \inf \left\{ c \in [0, \infty] : \sum_{s \in \mathbb{Z}} \Psi(\mathcal{H}_X^{dis} f(s)) \leq c \sum_{s \in \mathbb{Z}} \Phi(f(s)) \text{ for all step functions } f : \mathbb{Z} \rightarrow X \right\}.$$

We will also need a certain variant of Φ, Ψ -norm in the periodic setting. Namely, define $|\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi}$ by

$$|\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} := \inf \left\{ c \in [0, \infty] : \int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}} f(s)) ds \leq c \int_{\mathbb{T}} \Phi(f(s)) ds \right. \\ \left. \text{for all step functions } f : \mathbb{T} \rightarrow X \text{ with } \int_{\mathbb{T}} f(s) ds = 0 \right\}.$$

The following theorem demonstrates that the norm of the Hilbert transform does not depend whether it is defined on \mathbb{T}, \mathbb{R} , or \mathbb{Z} .

Theorem 4.3. Let X be a Banach space and let $\Phi, \Psi : X \rightarrow \mathbb{R}$ be continuous convex functions such that $\Phi(0) = 0$. Then

$$|\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} = |\mathcal{H}_X^{\mathbb{R}}|_{\Phi, \Psi} \leq |\mathcal{H}_X^{dis}|_{\Phi, \Psi} \leq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi}.$$

Moreover, if Φ is symmetric, then

$$|\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} = |\mathcal{H}_X^{\mathbb{R}}|_{\Phi, \Psi} = |\mathcal{H}_X^{dis}|_{\Phi, \Psi} = |\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi}.$$

The proof will consist of several steps.

Proposition 4.4. Let X be a Banach space and let $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ be convex functions. Then we have

$$|\mathcal{H}_X^{\mathbb{R}}|_{\Phi, \Psi} \leq |\mathcal{H}_X^{dis}|_{\Phi, \Psi} \leq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi}.$$

Proof. Introduce yet another Hilbert-type operator acting on step functions $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(\mathcal{H}_X^{\mathbb{R}, dis} f)(t) := \frac{1}{\pi} \sum_{s \in \mathbb{Z} \setminus \{0\}} \frac{f(t-s)}{s}, \quad t \in \mathbb{R},$$

and define its Φ, Ψ -norm analogously. We will first prove that $|\mathcal{H}_X^{\mathbb{R}}|_{\Phi, \Psi} \leq |\mathcal{H}_X^{\mathbb{R}, \text{dis}}|_{\Phi, \Psi}$. To this end, fix a step function f on \mathbb{R} and define its ε -dilation by $f_\varepsilon(\cdot) := f(\varepsilon \cdot)$. Then similarly to [35, Theorem 4.3], we have

$$\begin{aligned} \frac{\int_{\mathbb{R}} \Psi((\mathcal{H}_X^{\mathbb{R}, \text{dis}} f_\varepsilon)(s)) \, ds}{\int_{\mathbb{R}} \Phi(f_\varepsilon(s)) \, ds} &= \frac{\int_{\mathbb{R}} \Psi(\pi^{-1} \sum_{k \in \mathbb{Z} \setminus \{0\}} f_\varepsilon(s - k)/k) \, ds}{\int_{\mathbb{R}} \Phi(f_\varepsilon(s)) \, ds} \\ &= \frac{\int_{\mathbb{R}} \Psi(\pi^{-1} \sum_{k \in \mathbb{Z} \setminus \{0\}} \varepsilon f(\varepsilon s - \varepsilon k)/(\varepsilon k)) \, d(\varepsilon s)}{\int_{\mathbb{R}} \Phi(f(\varepsilon s)) \, d(\varepsilon s)} \\ &= \frac{\int_{\mathbb{R}} \Psi(\pi^{-1} \sum_{k \in \mathbb{Z} \setminus \{0\}} \varepsilon f(s - \varepsilon k)/(\varepsilon k)) \, ds}{\int_{\mathbb{R}} \Phi(f(s)) \, ds}. \end{aligned}$$

Since $\frac{1}{\pi} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{f(s - \varepsilon k)}{\varepsilon k} \varepsilon \rightarrow \mathcal{H}_X^{\mathbb{R}} f(s)$ for a.e. $s \in \mathbb{R}$, Fatou's lemma yields

$$\begin{aligned} |\mathcal{H}_X^{\mathbb{R}}|_{\Phi, \Psi} &= \sup_{f \in F_X^{\text{step}}} \frac{\int_{\mathbb{R}} \Psi(\mathcal{H}_X^{\mathbb{R}} f(s)) \, ds}{\int_{\mathbb{R}} \Phi(f(s)) \, ds} \\ &\leq \sup_{f \in F_X^{\text{step}}} \liminf_{\varepsilon \rightarrow 0} \frac{\int_{\mathbb{R}} \Psi((\mathcal{H}_X^{\mathbb{R}, \text{dis}} f_\varepsilon)(s)) \, ds}{\int_{\mathbb{R}} \Phi(f_\varepsilon(s)) \, ds} \leq |\mathcal{H}_X^{\mathbb{R}, \text{dis}}|_{\Phi, \Psi} = |\mathcal{H}_X^{\text{dis}}|_{\Phi, \Psi}, \end{aligned}$$

where the latter equality follows from the direct repetition of the arguments from [35, Theorem 4.2]. This gives us the 1st inequality of the assertion. The proof of the fact that $|\mathcal{H}_X^{\text{dis}}|_{\Phi, \Psi} \leq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi}$ follows word-by-word from the infinite-dimensional analogue of the recent approach of Bañuelos and Kwaśnicki [1] combined with the estimate (3.1). ■

Theorem 4.5. Let X be a Banach space and let $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ be continuous functions. Then $|\mathcal{H}_X^{\mathbb{R}}|_{\Phi, \Psi} \leq |\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi}$.

Proof. Fix a step function $f : \mathbb{R} \rightarrow X$. It takes only a finite number of values, so we may assume that X is finite dimensional (which will guarantee the validity of the reasoning below). For any $n \geq 1$, introduce the function $g_n : \mathbb{R} \rightarrow X$ by

$$g_n(x) = \frac{1}{2\pi n} \int_{-\pi n}^{\pi n} f(t) \cot \frac{x-t}{2n} dt, \quad x \in \mathbb{R}.$$

It follows from the observation of Zygmund [59, p. 256] that $g_n \rightarrow \mathcal{H}_X^{\mathbb{R}} f$ a.e. as $n \rightarrow \infty$. On the other hand, the function $x \mapsto g_n(nx)$, $|x| \leq \pi$, is precisely the periodic Hilbert transform of the function $x \mapsto f(nx)$, $|x| \leq \pi$ (see (2.1)). Therefore, it is also the periodic

Hilbert transform of the centered function

$$x \mapsto f(nx) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(ns)ds, \quad |x| \leq \pi.$$

Clearly, the latter is a step function. Consequently, by Fatou’s lemma and the definition of $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}$,

$$\begin{aligned} \int_{\mathbb{R}} \Psi(\mathcal{H}_X^{\mathbb{R}}f)dx &\leq \liminf_{n \rightarrow \infty} \int_{-\pi n}^{\pi n} \Psi(g_n)dx = \liminf_{n \rightarrow \infty} \int_{-\pi}^{\pi} \Psi(g_n(nx))ndx \\ &\leq |\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \liminf_{n \rightarrow \infty} \int_{-\pi}^{\pi} \Phi \left(f(nx) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(ns)ds \right) ndx \\ &= |\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \liminf_{n \rightarrow \infty} \int_{-\pi n}^{\pi n} \Phi \left(f(x) - \frac{1}{2\pi n} \int_{-\pi n}^{\pi n} f(s)ds \right) dx. \end{aligned}$$

However, $\frac{1}{2\pi n} \int_{-\pi n}^{\pi n} f(s)ds \rightarrow 0$ by the fact that f is a step function. Therefore, again using this property of f and the continuity of Φ , the last expression of the above chain equals $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \int_{\mathbb{R}} \Phi(f)dx$. Since f was arbitrary, the result follows. ■

Now we turn our attention to the estimate in the reverse direction. We start from the observation that it does not hold true if $\Phi(0) > 0$ and $\Psi \neq 0$. Indeed, if $\Phi(0) > 0$, then $\int_{\mathbb{R}} \Phi(f)dx = \infty$ for any step function and hence $|\mathcal{H}_X^{\mathbb{R}}|_{\Phi,\Psi} = 0$. On the other hand, the condition $\Psi \neq 0$ implies that $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} > 0$: it is easy to construct a step function $f : \mathbb{T} \rightarrow X$ of mean zero for which $\int_{\mathbb{R}} \Psi(\mathcal{H}^{\mathbb{T}}f)dx > 0$.

In other words, the inequality $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \leq |\mathcal{H}_X^{\mathbb{R}}|_{\Phi,\Psi}$ fails, because of obvious reasons, if $\Phi(0) > 0$ and $\Psi \neq 0$. If Ψ is identically 0, then the estimate holds true: the reason is even more trivial – both sides are zero. It remains to study the key possibility when $\Phi(0) = 0$ and $\Psi \neq 0$.

Theorem 4.6. Let X be a Banach space and let $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ be arbitrary continuous functions such that $\Phi(0) = 0$ and $\Psi \neq 0$. Then $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \leq |\mathcal{H}_X^{\mathbb{R}}|_{\Phi,\Psi}$.

Proof. As was mentioned above, the assumption $\Psi \neq 0$ implies $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} > 0$. For the sake of clarity, we split the reasoning into a few separate parts.

Step 1. Auxiliary analytic maps. Let D denote the open unit disc of \mathbb{C} and let $H = \mathbb{R} \times (0, \infty)$ be the upper halfplane. Define $K : D \cap H \rightarrow H$ by the formula $K(z) = -(1 - z)^2/(4z)$. It is not difficult to verify that K is conformal and hence so is its inverse L . Let us extend L to the continuous function on \bar{H} . It is easy to see that $L(z) \rightarrow 0$ as

$z \rightarrow \infty$. Furthermore, L maps the interval $[0, 1]$ onto $\{e^{i\theta} : 0 \leq \theta \leq \pi\}$. More precisely, we have the following formula: if $x \in [0, 1]$, then

$$L(x) = e^{i\theta}, \text{ where } \theta \in [0, \pi] \text{ is uniquely determined by } x = \sin^2(\theta/2). \quad (4.2)$$

In addition, L maps the set $\mathbb{R} \setminus [0, 1]$ onto the open interval $(-1, 1)$; precisely, we have the identity

$$L(x) = \begin{cases} 1 - 2x - 2\sqrt{x^2 - x} & \text{if } x < 0, \\ 1 - 2x + 2\sqrt{x^2 - x} & \text{if } x > 1. \end{cases} \quad (4.3)$$

In particular, we easily check that for any $\delta > 0$, the function L is bounded away from 1 outside any interval of the form $[-\delta, 1 + \delta]$ and $|L(x)| = O(|x|^{-1})$ as $x \rightarrow \pm\infty$.

Step 2. A function on \mathbb{T} and its extension to a disc. Fix a positive number ε and pick a step function $f : \mathbb{T} \rightarrow X$ of integral 0 such that

$$\int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}} f) dx > (|\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} - \varepsilon) \cdot \int_{\mathbb{T}} \Phi(f) dx.$$

We may assume that X is finite dimensional, restricting to the range of f if necessary. Given a big number $R > 0$, consider a continuous function $\kappa^R : X \rightarrow [0, 1]$ equal to 1 on $B(0, R)$ and equal to 0 outside $B(0, 2R)$. Set $\Psi^R(x) = \Psi(x) \cdot \kappa^R(x)$ for $x \in X$. Note that Ψ^R is uniformly continuous, since it is continuous and supported on the compact ball $B(0, 2R)$ (recall that X is finite dimensional). By Lebesgue's monotone convergence theorem, if R is sufficiently big, we also have

$$\int_{\mathbb{T}} \Psi^R(\mathcal{H}_X^{\mathbb{T}} f) dx > (|\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} - \varepsilon) \cdot \int_{\mathbb{T}} \Phi(f) dx. \quad (4.4)$$

There is an analytic function $F : D \rightarrow X + iX$ with the property that the radial limit $\lim_{r \rightarrow 1^-} F(re^{i\theta})$ is equal to $f(e^{i\theta}) + i\mathcal{H}_X^{\mathbb{T}} f(e^{i\theta})$ for almost all $|\theta| \leq \pi$. Note that we have

$$F(0) = \frac{1}{2\pi} \int_{\mathbb{T}} f dx + i \cdot 0 = 0 \quad (4.5)$$

and that the "real part" of F is bounded (by the supremum norm of f). Consider the analytic function $M_n : H \rightarrow X + iX$ given by the composition

$$M_n(z) = F(L^{2n}(z))$$

and decompose it as $M_n(z) = \Re M_n(z) + i\Im M_n(z)$, with $\Re M_n$ and $\Im M_n$ taking values in X . Observe that for each n the function $\Re M_n$ is bounded by the supremum norm of f (which is directly inherited from the “real part” of the function F). In addition, $h = \mathbf{1}_{[0,1]} \Re M_n$, considered as a function on \mathbb{R} , is a step function (with the number of steps depending on n and going to infinity). Since $\lim_{z \rightarrow \infty} L(z) = 0$, we have $\lim_{z \rightarrow \infty} M_n(z) = 0$ and therefore $\mathcal{H}_X^\mathbb{T} \Re M_n(x) = \Im M_n(x)$ for $x \in \mathbb{R}$.

Step 3. Calculations. We compute that

$$\begin{aligned} \int_{\mathbb{R}} \Phi(h(x)) \, dx &= \int_0^1 \Phi(\Re M_n(x)) \, dx = \int_0^1 \Psi(f(L^{2n}(x))) \, dx = \frac{1}{2} \int_0^\pi \Phi(f(e^{2in\theta})) \sin \theta \, d\theta \\ &= \frac{1}{2} \int_0^{2n\pi} \Phi(f(e^{i\theta})) \sin\left(\frac{\theta}{2n}\right) \frac{d\theta}{2n} = \frac{1}{2} \int_0^{2\pi} \Phi(f(e^{i\theta})) \sum_{k=0}^{n-1} \sin\left(\frac{k\pi}{n} + \frac{\theta}{2n}\right) \frac{d\theta}{2n} \\ &= \frac{1}{2} \int_0^{2\pi} \Phi(f(e^{i\theta})) \frac{\cos\left(\frac{\theta-\pi}{2n}\right)}{2n \sin\left(\frac{\pi}{2n}\right)} \, d\theta \xrightarrow{n \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \Phi(f(e^{i\theta})) \, d\theta. \end{aligned} \tag{4.6}$$

Now, let us similarly handle the integral $\int_{\mathbb{R}} \Psi^K(\mathcal{H}^\mathbb{R} h) \, dx$. We have

$$\begin{aligned} \int_{\mathbb{R}} \Psi^R(\mathcal{H}_X^\mathbb{R} h(x)) \, dx &\geq \int_0^1 \Psi^R(\mathcal{H}_X^\mathbb{R} h(x)) \, dx = \int_0^1 \Psi^R(\mathcal{H}_X^\mathbb{R} \Re M_n - \mathcal{H}_X^\mathbb{R}(\mathbf{1}_{\mathbb{R} \setminus [0,1]} \Re M_n)) \, dx \\ &= \int_0^1 \Psi^R(\mathcal{H}_X^\mathbb{R} \Re M_n) \, dx + \int_0^1 \left[\Psi^R(\mathcal{H}_X^\mathbb{R} \Re M_n - \mathcal{H}_X^\mathbb{R}(\mathbf{1}_{\mathbb{R} \setminus [0,1]} \Re M_n)) - \Psi^R(\mathcal{H}_X^\mathbb{R} \Re M_n) \right] \, dx. \end{aligned} \tag{4.7}$$

Now, we have $\mathcal{H}_X^\mathbb{R} \Re M_n(x) = \Im M_n(x) = \mathcal{H}_X^\mathbb{T} f(L^{2n}(x))$, so a calculation similar to that in (4.6) gives

$$\int_0^1 \Psi^R(\mathcal{H}_X^\mathbb{R} \Re M_n) \, dx \xrightarrow{n \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \Psi^R(\mathcal{H}_X^\mathbb{T} f(e^{i\theta})) \, d\theta.$$

To deal with the last integral in (4.7) we will first show that $\mathcal{H}_X^\mathbb{R}(\mathbf{1}_{\mathbb{R} \setminus [0,1]} \Re M_n)$ converges to 0 in L^2 , as $n \rightarrow \infty$. To this end, recall that X is finite dimensional, and hence it has the UMD property. Consequently, by [28, Corollary 5.2.11]

$$\int_{\mathbb{R}} |\mathcal{H}_X^\mathbb{R}(\mathbf{1}_{\mathbb{R} \setminus [0,1]} \Re M_n)|^2 \, dx \leq C_X \int_{\mathbb{R} \setminus [0,1]} |\Re M_n|^2 \, dx \tag{4.8}$$

for some constant C_X depending only on X . Fix an arbitrary $\eta > 0$. As we have already noted above, $\Re M_n$ is bounded by the supremum norm of f . Setting $\delta = \eta / (C_X \sup_X \|f\|^2)$,

we see that

$$\int_{(-\delta,0)} |\Re M_n(x)|^2 dx + \int_{(1,1+\delta)} |\Re M_n(x)|^2 dx \leq 2\eta C_X^{-1}. \quad (4.9)$$

Furthermore, recall that L maps $\mathbb{R} \setminus [0, 1]$ onto $(-1, 1)$, it is bounded away from 1 outside $[-\delta, 1 + \delta]$ and $|L(x)| = O(|x|^{-1})$ as $x \rightarrow \pm\infty$. Since F is analytic and vanishes at 0, we conclude that $M_n(x) = F(L^{2n}(x)) = O(|x|^{-2n})$ and hence

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R} \setminus [-\delta, 1+\delta]} |\Re M_n(x)|^2 dx = 0. \quad (4.10)$$

Putting (4.8), (4.9), and (4.10) together, we see that if n is sufficiently large, then $\int_{\mathbb{R}} |\mathcal{H}_X^{\mathbb{R}}(\mathbf{1}_{\mathbb{R} \setminus [0,1]} \Re M_n)|^2 dx \leq 3\eta$ and the aforementioned convergence in L^2 holds. In particular, passing to a subsequence if necessary, we see that $\mathcal{H}_X^{\mathbb{R}}(\mathbf{1}_{\mathbb{R} \setminus [0,1]} \Re M_n) \rightarrow 0$ almost everywhere. However, as we have already mentioned above, the function Ψ^R is uniformly continuous, so the expression in the square brackets in the last term in (4.7) converges to zero almost everywhere. In addition, this expression is bounded in absolute value by $\sup \Psi^R$. Consequently, by Lebesgue's dominated convergence theorem, the last integral in (4.7) converges to 0 as $n \rightarrow \infty$. Putting all the above facts together, we see that if n is sufficiently large, then

$$\int_{\mathbb{R}} \Psi^R(\mathcal{H}_X^{\mathbb{R}} h(x)) dx \geq (1 - \varepsilon) \cdot \frac{1}{2\pi} \int_0^{2\pi} \Psi^R(\mathcal{H}_X^{\mathbb{T}} f(e^{i\theta})) d\theta.$$

Combining this with (4.4) and (4.6), we obtain that for n large enough we have

$$\int_{\mathbb{R}} \Psi(\mathcal{H}_X^{\mathbb{R}} h(x)) dx \geq \int_{\mathbb{R}} \Psi^R(\mathcal{H}_X^{\mathbb{R}} h(x)) dx \geq (1 - \varepsilon)(|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi, \Psi} - \varepsilon) \int_{\mathbb{R}} \Phi(h) dx.$$

Since h is a step function and ε was arbitrary, the claim follows. \blacksquare

Remark 4.7. Note that if $\Psi(0) \neq 0$ then Theorems 4.5 and 4.6 do not make any sense. Indeed, if this is the case, then there exists $\varepsilon > 0$ and R such that $\Psi(x) \geq \varepsilon$ for any $x \in X$ with $\|x\| \leq R$. Since for any step function $f : \mathbb{R} \rightarrow X$ the function $\mathcal{H}_X^{\mathbb{R}} f$ is in $L^2(\mathbb{R}; X)$, the set $\{\|\mathcal{H}_X^{\mathbb{R}} f\| \leq R\} \subset \mathbb{R}$ is of infinite measure, so

$$\int_{\mathbb{R}} \Psi(\mathcal{H}_X^{\mathbb{R}} f(s)) ds \geq \int_{\mathbb{R}} \mathbf{1}_{\|\mathcal{H}_X^{\mathbb{R}} f\| \leq R}(s) \varepsilon ds = \infty,$$

so $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} \geq |\mathcal{H}_X^{\mathbb{T},0}|_{\Phi, \Psi} = |\mathcal{H}_X^{\mathbb{R}}|_{\Phi, \Psi} = \infty$.

Remark 4.8. The finiteness of $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}$ implies the existence of a plurisubharmonic function $U_{\Phi,\Psi} : X+iX \rightarrow \mathbb{R}$ such that $U_{\Phi,\Psi}(0) \geq 0$. Hence, modifying the proof of Theorem 3.1, we see that the inequality (3.1) holds, with $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$ replaced with $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}$, if the dominating martingale M is additionally assumed to start from 0.

Theorem 4.9. Let $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ be continuous such that Φ is symmetric (i.e., $\Phi(x) = \Phi(-x)$ for all $x \in X$) and Ψ is convex. Then $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} = |\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$.

Proof. It suffices to show the estimate $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \geq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$. Fix $\varepsilon > 0$. By the definition of $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi}$, there is a step function $f : \mathbb{T} \rightarrow X$ such that

$$\int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}} f) dx > (|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} - \varepsilon) \int_{\mathbb{T}} \Phi(f) dx. \tag{4.11}$$

Let $F = F_1 + iF_2$ be the analytic extension of $f + i\mathcal{H}_X^{\mathbb{T}} f : \mathbb{T} \rightarrow X + iX$ to the unit disc and suppose that $B = (B^1, B^2)$ is the planar Brownian motion started at 0 and stopped upon hitting \mathbb{T} . Let $\tau = \inf\{t \geq 0 : |B_t| = 1\}$ be the lifetime of B . The processes $M_t = F_1(B_t), N_t = F_2(B_t)$ are orthogonal martingales such that N is weakly differentially subordinate to M . By Fatou’s lemma and Lebesgue’s monotone convergence theorem (observe that f , being a step function, is bounded) we see that if t is sufficiently large, then

$$\mathbb{E}\Psi(N_t) > (|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} - \varepsilon)\mathbb{E}\Phi(M_t).$$

If the expectation of M is zero, then by Remark 4.8 we know that

$$\mathbb{E}\Psi(N_t) \leq |\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}\mathbb{E}\Phi(M_t)$$

and hence we obtain that

$$|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \geq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} - \varepsilon. \tag{4.12}$$

We will show that this is also true if the expectation $x = \mathbb{E}M_t$ does not vanish. To this end, consider another Brownian motion $W = (W^1, W^2)$ in \mathbb{R}^2 started at 0 and stopped upon reaching the boundary of the strip $S = \{(x, y) : |x| \leq 1\}$. Let $\sigma = \inf\{t : |W_t^1| = 1\}$

denote its lifetime. We may assume that W is constructed on the same probability space as B and that both processes are independent. We splice these processes as follows: set

$$\tilde{M}_s = \begin{cases} xW_s^1 & \text{if } s \leq \sigma, \\ \text{sgn}(W_\sigma^1)M_{s-\sigma} & \text{if } s > \sigma \end{cases}$$

and

$$\tilde{N}_s = \begin{cases} xW_s^2 & \text{if } s \leq \sigma, \\ xW_\sigma^2 + N_{s-\sigma} & \text{if } s > \sigma. \end{cases}$$

In other words, the pair (\tilde{M}, \tilde{N}) behaves like a Brownian motion evolving in the strip Sx until its 1st coordinate reaches x or $-x$, and then it starts behaving like the pair $(M, \tilde{N}_\sigma + N)$ or $(-M, \tilde{N}_\sigma + N)$, depending on which the side of the boundary of Sx the process \tilde{M} reaches. Note that \tilde{M} and \tilde{N} are orthogonal martingales such that \tilde{N} is weakly differentially subordinate to \tilde{M} and $\tilde{M}_0 = 0$. Consequently, by Remark 4.8 for any t ,

$$\mathbb{E}\Psi(\tilde{N}_t) \leq |\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} \mathbb{E}\Phi(\tilde{M}_t). \quad (4.13)$$

Now,

$$\mathbb{E}\Psi(\tilde{N}_t) \geq \mathbb{E}\Psi(\tilde{N}_t)\mathbf{1}_{\{t \geq \sigma\}} = \mathbb{E}\Psi(xW_\sigma^2 + N_{t-\sigma})\mathbf{1}_{\{t \geq \sigma\}}.$$

However, W and B are independent, and the random variable xW_σ^2 is symmetric. Therefore, using the fact that Ψ is convex, we see that

$$\mathbb{E}\Psi(\tilde{N}_t) \geq \mathbb{E}\Psi(N_{t-\sigma})\mathbf{1}_{\{t \geq \sigma\}}.$$

Furthermore, using the symmetry of Φ , we have

$$\mathbb{E}\Phi(\tilde{M}_t)\mathbf{1}_{\{t \geq \sigma\}} = \mathbb{E}\Phi(\text{sgn}(W_\sigma^1)M_{t-\sigma})\mathbf{1}_{\{t \geq \sigma\}} = \mathbb{E}\Phi(M_{t-\sigma})\mathbf{1}_{\{t \geq \sigma\}}.$$

As previously, combining (4.11) with Fatou's lemma and Lebesgue's dominated convergence theorem, if t is sufficiently large, then

$$\mathbb{E}\Psi(N_{t-\sigma})\mathbf{1}_{\{t \geq \sigma\}} > (|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} - \varepsilon)\mathbb{E}\Phi(M_{t-\sigma})\mathbf{1}_{\{t \geq \sigma\}}$$

and hence also

$$\mathbb{E}\Psi(\tilde{N}_t) > (|\mathcal{H}_X^{\mathbb{T}}|_{\Phi,\Psi} - \varepsilon)\mathbb{E}\Phi(\tilde{M}_t)\mathbf{1}_{\{t \geq \sigma\}}.$$

But $\lim_{t \rightarrow \infty} \mathbb{E}\Phi(\tilde{M}_t)1_{\{t < \sigma\}} = 0$, by Lebesgue’s dominated convergence theorem (we have $1_{\{t < \sigma\}} \rightarrow 0$ and the norm of \tilde{M}_t is bounded by $\|x\|$ for $t \in [0, \sigma]$). Therefore, the preceding estimate gives

$$\mathbb{E}\Psi(\tilde{N}_t) > (|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} - \varepsilon)\mathbb{E}\Phi(\tilde{M}_t)$$

if t is sufficiently big. By (4.13), this gives (4.12) and completes the proof of the theorem, since ε was arbitrary. ■

Remark 4.10. Assume that $|\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi} = |\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi}$ (this holds true under some additional assumptions on Φ and Ψ , see Theorem 4.9). Then the plurisubharmonic function $U_{\Phi, \Psi}$ considered in Remark 4.8 coincides with the one considered in Theorem 3.2, and hence we automatically have that $U_{\Phi, \Psi}(x) \geq 0$ for all $x \in X$.

Proof of Theorem 4.3. The theorem follows from Proposition 4.4, Theorems 4.5, and 4.6, 4.9, and the fact that $|\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} \leq |\mathcal{H}_X^{\mathbb{T}}|_{\Phi, \Psi}$. ■

Remark 4.11. Notice that Theorem 4.3 can not be applied to more general norms. For example, if X is a UMD Banach space, $1 < q < p < \infty$, then

$$\|\mathcal{H}_X^{\mathbb{T}}\|_{\mathcal{L}(L^p(\mathbb{T}; X), L^q(\mathbb{T}; X))} < \infty \text{ and } \|\mathcal{H}_X^{\mathbb{R}}\|_{\mathcal{L}(L^p(\mathbb{R}; X), L^q(\mathbb{R}; X))} = \infty.$$

4.2 Decoupling constants

We turn our attention to the next important application. We need some additional notation. Consider the probability space $([0, 1], \mathcal{B}(0, 1), |\cdot|)$, equipped with the dyadic filtration $(\mathcal{F}_n)_{n \geq 0}$ (i.e., generated by the Haar system $(h_n)_{n=0}^\infty$, see, e.g., [28]). A martingale f adapted to this filtration is called a Paley–Walsh martingale.

Definition 4.12. Let X be a Banach space and let $1 < p < \infty$ be a fixed parameter. Then we define $\beta_{p, X}^{\Delta, +}$ and $\beta_{p, X}^{\Delta, -}$ to be the smallest β^+ and β^- such that

$$\frac{1}{(\beta^-)^p} \mathbb{E} \left\| \sum_{n=0}^\infty df_n \right\|^p \leq \mathbb{E} \left\| \sum_{n=0}^\infty r_n df_n \right\|^p \leq (\beta^+)^p \mathbb{E} \left\| \sum_{n=0}^\infty df_n \right\|^p$$

for any finite Paley–Walsh martingale $(f_n)_{n \geq 0}$ and any independent Rademacher sequence $(r_n)_{n \geq 0}$. Furthermore, we define $\beta_{p, X}^{\gamma, +}$ and $\beta_{p, X}^{\gamma, -}$ to be the least possible values

of β^+ and β^- for which

$$\frac{1}{(\beta^-)^p} \mathbb{E} \left\| \int_0^\infty \phi \, dW \right\|^p \leq \mathbb{E} \left\| \int_0^\infty \phi \, d\tilde{W} \right\|^p \leq (\beta^+)^p \mathbb{E} \left\| \int_0^\infty \phi \, dW \right\|^p,$$

where W is a standard Brownian motion, $\phi : \mathbb{R}_+ \times \Omega \rightarrow X$ is an elementary progressive process, and \tilde{W} is another Brownian motion independent of ϕ and W .

Decoupling constants appear naturally while working with UMD Banach spaces (see, e.g., [15, 16, 20, 24, 28, 37, 51]). The following result, a natural corollary of Theorem 3.1 for $\Phi(x) = \Psi(x) = \|x\|^p$, exhibits the direct connection between decoupling constants and $h_{p,X} := \|\mathcal{H}_X^\mathbb{T}\|_{\mathcal{L}(L^p(\mathbb{T};X))}$ (see Corollary 3.3).

Corollary 4.13. Let X be a Banach space and let $1 < p < \infty$ be a fixed parameter. Then we have

$$h_{p,X} \geq \max\{\beta_{p,X}^{\gamma,+}, \beta_{p,X}^{\gamma,-}\} \quad (4.14)$$

and hence

$$h_{p,X} \geq C \max\{\beta_{p,X}^{\Delta,+}, \beta_{p,X}^{\Delta,-}\}. \quad (4.15)$$

Here $C = \mathbb{E}|\gamma| \mathbb{E}\sqrt{\tau}$, where γ is a standard normal random variable and $\tau = \inf\{t \geq 0 : |W_t| = 1\}$ for a standard Brownian motion W .

Note that $\mathbb{E}\tau \leq (\mathbb{E}\sqrt{\tau})^{\frac{2}{3}} (\mathbb{E}\tau^2)^{\frac{1}{3}}$ by Hölder's inequality, so C in (15) is bounded from below by $\frac{(\mathbb{E}\tau)^{\frac{3}{2}}}{(\mathbb{E}\tau^2)^{\frac{1}{2}}} \mathbb{E}|\gamma| = \frac{\sqrt{6}}{\sqrt{5\pi}} \approx 0.618$ (since $\mathbb{E}\tau = 1$ and $\mathbb{E}\tau^2 = \frac{5}{3}$).

Proof. The inequality (4.14) follows directly from the definition of $\beta_{p,X}^{\gamma,+}$ and $\beta_{p,X}^{\gamma,-}$. Indeed, for any Brownian motion W , elementary progressive process ϕ , and a Brownian motion \tilde{W} independent of ϕ and W we have, for any $x^* \in X^*$,

$$\begin{aligned} \left[\left\langle \int_0^\cdot \phi \, dW, x^* \right\rangle \right]_t &= \left[\int_0^\cdot \langle \phi, x^* \rangle \, dW \right]_t = \int_0^t |\langle \phi(s), x^* \rangle|^2 \, ds, \\ \left[\left\langle \int_0^\cdot \phi \, d\tilde{W}, x^* \right\rangle \right]_t &= \left[\int_0^\cdot \langle \phi, x^* \rangle \, d\tilde{W} \right]_t = \int_0^t |\langle \phi(s), x^* \rangle|^2 \, ds, \end{aligned}$$

so $\int \phi \, dW \stackrel{w}{\ll} \int \phi \, d\tilde{W} \stackrel{w}{\ll} \int \phi \, dW$. Moreover, by [32, Lemma 17.10],

$$\begin{aligned} &\left[\left\langle \int_0^\cdot \phi \, dW, x^* \right\rangle, \left\langle \int_0^\cdot \phi \, d\tilde{W}, x^* \right\rangle \right]_t \\ &= \left[\int_0^\cdot \langle \phi, x^* \rangle \, dW, \int_0^\cdot \langle \phi, x^* \rangle \, d\tilde{W} \right]_t = \int_0^t |\langle \phi(s), x^* \rangle|^2 \, d[W, \tilde{W}]_s = 0, \end{aligned}$$

where the latter holds since W and \tilde{W} are independent. Therefore $\int \phi dW$ and $\int \phi d\tilde{W}$ are orthogonal local martingales satisfying the differential subordination (“in both directions”), so by Theorem 3.1,

$$\frac{1}{(\mathfrak{h}_{p,X})^p} \mathbb{E} \left\| \int_{\mathbb{R}_+} \phi dW \right\|^p \leq \mathbb{E} \left\| \int_{\mathbb{R}_+} \phi d\tilde{W} \right\|^p \leq (\mathfrak{h}_{p,X})^p \mathbb{E} \left\| \int_{\mathbb{R}_+} \phi dW \right\|^p.$$

Let us now turn to the 2nd part. First notice that $\beta_{p,X}^{\gamma,+} \geq C\beta_{p,X}^{\Delta,+}$ (see [51, (2.5)] and the discussion thereafter), so $\mathfrak{h}_{p,X} \geq \beta_{p,X}^{\gamma,+} \geq C\beta_{p,X}^{\Delta,+}$. On the other hand, X can be assumed UMD (and hence reflexive), so by the discussion above we have $\mathfrak{h}_{p',X^*} \geq C\beta_{p',X^*}^{\Delta,+}$. But $\mathfrak{h}_{p',X^*} = \mathfrak{h}_{p,X}$ (since $(\mathcal{H}_X^{\mathbb{T}})^* = \mathcal{H}_{X^*}^{\mathbb{T}}$), and $\beta_{p',X^*}^{\Delta,+} \geq \beta_{p,X}^{\Delta,-}$ analogously to [21, Theorem 1], so $\mathfrak{h}_{p,X} \geq C\beta_{p,X}^{\Delta,-}$. ■

Remark 4.14. Notice that (4.14) together with [20, Theorem 3] yields the related estimate $\max\{\beta_{p,X}^{\gamma,+}, \beta_{p,X}^{\gamma,-}\} \leq \mathfrak{h}_{p,X} \leq \beta_{p,X}^{\gamma,+} \beta_{p,X}^{\gamma,-}$.

Remark 4.15. Let X be a UMD Banach function space. Then inequality (4.15) together with [33] provide the lower bound for $\mathfrak{h}_{p,X}$ in terms of $\beta_{p,X}$ of the same order as (2.2). Indeed, by [33] thanks to Banach function space techniques one can show that

$$\beta_{p,X} \lesssim_p q(c_{q,X} \beta_{p,X}^{\Delta,+})^2,$$

where q is the cotype of X and $c_{q,X}$ is the corresponding cotype constant. Therefore by applying (4.15) we get the following square root dependence:

$$\sqrt{\beta_{p,X}} \lesssim_p \sqrt{q} c_{q,X} \mathfrak{h}_{p,X}.$$

4.3 Necessity of the UMD property

Our next result answers a very natural question about the link of the number $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi}$ to the UMD property.

Theorem 4.16. Let $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ be continuous convex functions such that $\Psi(0) = 0$. Assume in addition that there is a positive number C such that the sets $\{x \in X : \Psi(x) < C\}$ and $\Phi(B(0, C))$ are bounded. If $|\mathcal{H}_X^{\mathbb{T},0}|_{\Phi,\Psi} < \infty$, then X is UMD.

Remark 4.17. It is easy to see that the assumption $\Psi(0) = 0$ combined with the boundedness of $\{\Psi < C\}$ enforces the function Ψ to explode “uniformly” in the whole

space. That is, if $B(0, R)$ is the ball containing $\{\Psi < C\}$, then the convexity of Ψ implies $\Psi(x) \geq C\|x\|/R$ for all $x \notin B(0, R)$. Some condition of this type is necessary, as the following simple example indicates. Take $X = \ell_\infty$ and set $\Phi(x) = |x_1|^2 = \Psi(x)$. Then $|\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} = 1 < \infty$, while X is not UMD. The reason is that the function Ψ controls only the subspace generated by the 1st coordinate.

Remark 4.18. Note that X being UMD does not imply $|\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} < \infty$. Indeed, if Φ and Ψ are of different homogeneity (i.e., $\Phi(ax) = a^\alpha \Phi(x)$, $\Psi(ax) = a^\beta \Psi(x)$ for any $x \in X$, $a \geq 0$, and for some fixed positive $\alpha \neq \beta$), then for any nonzero step function $f : \mathbb{T} \rightarrow X$ such that $\int_{\mathbb{T}} f(s) ds = 0$ and for any $a \geq 0$ we have that

$$\begin{aligned} & \int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}} f(s)) ds \\ &= \frac{1}{a^\beta} \int_{\mathbb{T}} \Psi(\mathcal{H}_X^{\mathbb{T}}(af)(s)) ds \leq \frac{1}{a^\beta} |\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} \int_{\mathbb{T}} \Phi(af(s)) ds = a^{\alpha-\beta} |\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} \int_{\mathbb{T}} \Phi(f(s)) ds, \end{aligned}$$

so $a^{\alpha-\beta} |\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} \leq |\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi}$ for any $a > 0$, and since $\alpha \neq \beta$, $|\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} = \infty$. The classical examples of such Φ and Ψ are $\Phi(x) = \|x\|^p$, $\Psi(x) = \|x\|^q$, $x \in X$ for different p and q .

The proof of Theorem 4.16 will exploit the following four lemmas. In what follows, $N^* = \sup_{t \geq 0} \|N_t\|$ is the maximal function of N .

Lemma 4.19. Under the assumptions of Theorem 4.16, there exists a constant c_1 depending on Φ , Ψ , and X , such that if M and N are orthogonal martingales such that N is weakly differentially subordinate to M , $M_0 = 0$ and $\|M\|_\infty \leq c_1$, then $\mathbb{P}(N^* \geq 1) < 1$.

Proof. Let R be as in Remark 4.17 and suppose that $\Phi(B(0, C)) \subseteq [-R', R']$. Then for any $\lambda \geq 1$ we have, in the light of Remark 4.8,

$$\mathbb{P}(\|N_t\| \geq 1) = \mathbb{P}(R\lambda\|N_t\| \geq R\lambda) \leq \frac{\mathbb{E}\Psi(R\lambda N_t)}{C\lambda} \leq \frac{|\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi} \mathbb{E}\Phi(R\lambda M_t)}{C\lambda}.$$

It suffices to take $\lambda = \frac{2R'|\mathcal{H}_X^{\mathbb{T}, 0}|_{\Phi, \Psi}}{C}$ and $c_1 = C/(R\lambda)$. ■

Lemma 4.20. Suppose that the assumptions of Theorem 4.16 are satisfied. Let M and N be continuous-path orthogonal martingales such that N is weakly differentially subordinate to M , $M_0 = 0$ and $\mathbb{P}(N^* > 1) = 1$. Then there exist continuous-path martingales \tilde{M} and \tilde{N} such that \tilde{N} is weakly differentially subordinate to \tilde{M} , $\tilde{M}_0 = 0$, $\mathbb{P}(\tilde{N}^* > 1) \geq 1/2$ and $\|\tilde{M}\|_\infty \leq 2\|M\|_1$.

Proof. Define $\tau = \inf\{t \geq 0 : \|M_t\| \geq 2\|M\|_1\}$ (as usual, $\inf \emptyset = +\infty$) and put $\tilde{M} = M^\tau$, $\tilde{N} = N^\tau$. Since M has continuous paths and starts from 0, we have $\|\tilde{M}\|_\infty \leq 2\|M\|_1$. Furthermore, $\mathbb{P}(\tilde{N}^* > 1) \geq \mathbb{P}(\tilde{N} = N) \geq 1/2$, since

$$\mathbb{P}(\tilde{N} \neq N) = \mathbb{P}(\tau < \infty) = \mathbb{P}(M^* \geq 2\|M\|_1) \leq 1/2$$

by [34, Theorem 1.3.8(i)]. ■

Lemma 4.21. Suppose that the assumptions of Theorem 4.16 are satisfied. Then there exists a constant $c > 0$ such that if M and N are continuous-path orthogonal martingales such that N is weakly differentially subordinated to M , $M_0 = 0$ and $N^* > 1$ almost surely, then $\|M\|_1 \geq c$.

Proof. Let c_1 be the number guaranteed by Lemma 4.19. Suppose that such a c does not exist. Then for any positive integer j there exist a pair (M^j, N^j) of orthogonal martingales such that N^j is weakly differentially subordinate to M^j , $M_0^j = 0$, $\mathbb{P}((N^j)^* > 2) = 1$ and $\|M^j\|_1 \leq 2^{-j-1}c_1$. By Lemma 4.20, for each j there is a pair $(\tilde{M}^j, \tilde{N}^j)$ of orthogonal, weakly differentially subordinate martingales satisfying $\tilde{M}_0^j = 0$, $\mathbb{P}((\tilde{N}^j)^* > 2) \geq 1/2$ and $\|\tilde{M}^j\|_\infty \leq 2^{-j}c_1$. We may assume that the underlying probability space is the same for all pairs and that all the pairs are independent. For each j there is a positive number t_j such that the event

$$A_j = \{\|\tilde{N}_t^j\| > 2 \text{ for some } t \leq t_j\}$$

has probability greater than $1/3$. Set $t_0 = 0$ and consider the martingale pair (M, N) defined as follows: if $t \in [t_0 + t_1 + \dots + t_n, t_0 + t_1 + \dots + t_{n+1})$ for some n , then

$$M_t = \tilde{M}_{t_1}^1 + \tilde{M}_{t_2}^2 + \dots + \tilde{M}_{t_n}^n + \tilde{M}_{t-t_1-t_2-\dots-t_n}^{n+1} \tag{4.16}$$

and analogously for N . Then M and N are orthogonal, N is weakly differentially subordinate to M , $M_0 = 0$ and

$$\|M\|_\infty \leq \sum_{j=1}^{\infty} \|\tilde{M}^j\|_\infty \leq \sum_{j=1}^{\infty} 2^{-j}c_1 = c_1.$$

Furthermore, by Borel–Cantelli lemma,

$$\mathbb{P}(N^* \geq 1) \geq \mathbb{P}\left(\limsup_{j \rightarrow \infty} A_j\right) = 1,$$

since the events A_j are independent and $\sum_{j=1}^{\infty} \mathbb{P}(A_j) = \infty$. Therefore we have that $\|M\|_{\infty} \leq c_1$, $\mathbb{P}(N^* \geq 1) = 1$, $N \overset{w}{\ll} M$, and M and N are orthogonal, which contradicts the assertion of Lemma 4.19. ■

Lemma 4.22. Suppose that the assumptions of Theorem 4.16 are satisfied. Then there exists a positive constant C such that if M, N are continuous-path orthogonal martingales such that N is weakly differentially subordinate to M and $M_0 = 0$, then

$$\mathbb{P}(N^* > 1) \leq C \|M\|_1. \quad (4.17)$$

Proof. Let c be the constant guaranteed by the previous lemma. Suppose that the assertion is not true. Then for any positive integer j there is a martingale pair (M^j, N^j) satisfying the usual structural properties such that

$$\mathbb{P}((N^j)^* > 2) > 2^{j+1} c^{-1} \|M^j\|_1. \quad (4.18)$$

We splice these martingale pairs into one pair (M, N) as previously; however, this time we allow pairs to appear several times. More precisely, denote $a_j = \mathbb{P}((N^j)^* > 2)$. Consider $\lceil 1/a_1 \rceil$ copies of (M^1, N^1) , $\lceil 1/a_2 \rceil$ copies of (M^2, N^2) , and so on (all the pairs are assumed to be independent). Let t_j be positive numbers such that the events $A_j = \{\|N^j_t\| > 2 \text{ for some } t \leq t_j\}$ have probability greater than $a_j/2$. Splice the aforementioned independent martingale pairs (with multiplicities) into one pair (M, N) using a formula analogous to (16). Then, by (18),

$$\|M\|_1 \leq \sum \|M^j\|_1 \leq \sum_{j=1}^{\infty} \left\lceil \frac{1}{a_j} \right\rceil \|M^j\|_1 \leq \sum_{j=1}^{\infty} \frac{2}{a_j} \cdot a_j c 2^{-j-1} = c$$

and, again by Borel–Cantelli lemma, $\mathbb{P}(N^* > 1) = 1$. Here we use the independence of the events A_j and

$$\sum \mathbb{P}(A_j) \geq \sum_{j=1}^{\infty} \frac{1}{a_j} \cdot \frac{a_j}{2} = \infty.$$

This contradicts Lemma 4.21. ■

Proof of Theorem 4.16. We will prove that theorem using the well-known extrapolation technique (good- λ inequalities) of Burkholder [7].

Step 1. First we show that for any fixed $0 < \delta < 1$ and $\beta > 1$ there exists $\varepsilon > 0$ depending only on δ, β , and X such that for any orthogonal continuous-path martingales $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ with $M_0 = N_0 = 0$ and $N \overset{w}{\ll} M$,

$$\mathbb{P}(N^* > \beta\lambda, M^* \leq \delta\lambda) \leq \varepsilon \mathbb{P}(N^* > \lambda) \tag{4.19}$$

for any $\lambda > 0$. Without loss of generality assume that both martingales take their values in a finite-dimensional subspace of X . Define three stopping times

$$\begin{aligned} \mu &:= \inf\{t \geq 0 : \|N_t\| > \lambda\}, \\ \nu &:= \inf\{t \geq 0 : \|M_t\| > \delta\lambda\}, \\ \sigma &:= \inf\{t \geq 0 : \|N_t\| > \beta\lambda\}. \end{aligned} \tag{4.20}$$

All the stopping times are predictable since M and N are continuous. Therefore, the equation $U(t) = \mathbf{1}_{[\mu, \nu \wedge \sigma]}(t)$ defines a predictable process (where $U = 0$ on \mathbb{R}_+ if $\nu \wedge \sigma < \mu$), which in turn gives rise to the martingales

$$\begin{aligned} \tilde{M} &:= \int U \, dM = M^{\nu \wedge \delta} - M^{\mu \wedge \nu \wedge \delta}, \\ \tilde{N} &:= \int U \, dN = N^{\nu \wedge \delta} - N^{\mu \wedge \nu \wedge \delta}. \end{aligned} \tag{4.21}$$

Notice that by (4.20) and (4.21), $\tilde{M}^* \leq 2\delta\lambda$ on $\{\mu < \infty\}$ and $\tilde{M}^* = 0$ on $\{\mu = \infty\}$, so

$$\|\tilde{M}\|_1 \leq 2\delta\lambda \mathbb{P}(N^* > \lambda). \tag{4.22}$$

Since $\tilde{N} \overset{w}{\ll} \tilde{M}$, $\tilde{M}_0 = \tilde{N}_0 = 0$ and \tilde{M} and \tilde{N} are orthogonal,

$$\mathbb{P}(N^* > \beta\lambda, M^* \leq \delta\lambda) \leq \mathbb{P}(\tilde{N}^* > (\beta - 1)\lambda) \stackrel{(i)}{\leq} \frac{C}{(\beta - 1)\lambda} \|\tilde{M}\|_1 \stackrel{(ii)}{\leq} \frac{2\delta C}{(\beta - 1)} \mathbb{P}(N^* > \lambda),$$

where (i) follows from (4.17) with the same constant C depending only on X , and (ii) follows from (4.22). Therefore (4.19) holds with $\varepsilon = 2\delta C/(\beta - 1)$.

Step 2. Now a straightforward integration argument (cf. [7, Lemma 7.1]), together with Doob’s maximal inequality, yield the L^p estimate

$$\sup_{t \geq 0} \|N_t\|_p \leq \|N^*\|_p \leq C_{p,X} \|M^*\|_p \leq \frac{pC_{p,X}}{p-1} \sup_{t \geq 0} \|M_t\|_p, \quad 1 < p < \infty,$$

for any pair of continuous, orthogonal, differentially subordinated martingales such that $M_0 = 0$. Here

$$C_{p,X}^p = \frac{\delta^{-p} \beta^p}{1 - \beta^p \cdot 2\delta C / (\beta - 1)}, \quad (4.23)$$

which, if we let $\beta = 1 + p^{-1}$ and $\delta = (10Cp)^{-1}$, depends only on p and the constant in (4.17). This in turn yields the corresponding L^p inequality for the periodic Hilbert transform for functions of integral 0. By Theorem 4.3 the assumption on the zero-average can be omitted, and hence X is UMD by [28, Corollary 5.2.11]. ■

Now we will take a closer look at the classical “LlogL” estimates of Zygmund [59]. For a Banach space X and a step function $f : \mathbb{T} \rightarrow X$, we define

$$\|f\|_{L \log L(\mathbb{T}; X)} := \int_{\mathbb{T}} (\|f(s)\| + 1) \log(\|f(s)\| + 1) \, ds$$

and denote

$$\mathfrak{h}_{L \log L, X} = |\mathcal{H}_X^{\mathbb{T}}|_{L \log L(\mathbb{T}; X) \rightarrow L^1(\mathbb{T}; X)} := \sup_{f: \mathbb{T} \rightarrow X \text{ step}} \frac{\|\mathcal{H}_X^{\mathbb{T}} f\|_{L^1(\mathbb{T}; X)}}{\|f\|_{L \log L(\mathbb{T}; X)}}.$$

Remark 4.23. In the light of Theorem 4.3, we have

$$\mathfrak{h}_{L \log L, X} = |\mathcal{H}_X^{\mathbb{T}, 0}|_{L \log L(\mathbb{T}; X) \rightarrow L^1(\mathbb{T}; X)} = |\mathcal{H}_X^{\mathbb{R}}|_{L \log L(\mathbb{R}; X) \rightarrow L^1(\mathbb{R}; X)} = |\mathcal{H}_X^{\text{dis}}|_{L \log L(\mathbb{Z}; X) \rightarrow L^1(\mathbb{Z}; X)}$$

for any Banach space X .

We will establish the following statement.

Theorem 4.24. Let X be a Banach space. Then X has the UMD property if and only if $\mathfrak{h}_{L \log L, X} < \infty$.

For the proof we will need the following lemma.

Lemma 4.25. Let X be a UMD Banach space. Then there exists a constant C_X depending only on X such that $\mathfrak{h}_{p,X} \leq C_X \frac{p}{p-1}$ for all $1 < p < 2$.

Proof. Let $M, N : \mathbb{R}_+ \times \Omega \rightarrow X$ be continuous orthogonal martingales such that $N \overset{w}{\ll} M$ and $N_0 = 0$. As we have already seen above,

$$\sup_{t \geq 0} (\mathbb{E} \|N_t\|^p)^{\frac{1}{p}} \leq \frac{p}{p-1} C_{p,X} \sup_{t \geq 0} (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}},$$

where $C_{p,X} \leq 10Cpe(1 - e/5)^{-1/p}$ (see (4.23) and the discussion following it). Therefore, if $1 < p < 2$, we may assume that this constant depends only on C (which essentially depends only on X). The claim follows from the sharpness part of Theorem 3.1. ■

Proof of Theorem 4.24. The inequality $h_{L \log L, X} < \infty$ implies UMD by Theorem 4.16 applied to $\Phi(x) = (\|x\| + 1) \log(\|x\| + 1)$ and $\Psi(x) = \|x\|$, $x \in X$. The converse holds true by Lemma 4.25 and Yano’s extrapolation argument (see, e.g., [18, 53]). ■

4.4 Weak differential subordination of martingales: sharper L^p inequalities

As it was noticed in (2.5), for a UMD Banach space X , any $1 < p < \infty$ and any X -valued local martingales M and N such that $N \overset{w}{\ll} M$, we have

$$\mathbb{E} \|N_t\|^p \leq c_{p,X}^p \mathbb{E} \|M_t\|^p, \quad t \geq 0,$$

with $c_{p,X} \leq \beta_{p,X}^2 (\beta_{p,X} + 1)$. The purpose of this subsection is to show that the upper bound can be substantially improved.

Theorem 4.26. Let X be a Banach space, let $1 < p < \infty$, and assume that M and N are local martingales satisfying $N \overset{w}{\ll} M$. Then

$$\mathbb{E} \|N_t\|^p \leq (\beta_{p,X} + h_{p,X})^p \mathbb{E} \|M_t\|^p \quad \text{for any } t \geq 0. \tag{4.24}$$

Remark 4.27. Note that $h_{p,X} \leq \beta_{p,X}^2$ (see (2.2)), so (4.24) gives

$$(\mathbb{E} \|N_t\|^p)^{\frac{1}{p}} \leq \beta_{p,X} (\beta_{p,X} + 1) (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}} \quad t \geq 0,$$

which is better than (2.5).

For the proof of Theorem 4.26 we will need the notion of the Burkholder function.

Definition 4.28. Let E be a linear space. A function $f : E \rightarrow \mathbb{R}$ is called concave if for any $x, y \in E$ and any $\lambda \in [0, 1]$ we have $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$. A function $f : E \times E \rightarrow \mathbb{R}$ is called zigzag–concave if for each $x, y \in E$ and $\varepsilon \in [-1, 1]$ the function $z \mapsto f(x + z, y + \varepsilon z)$ is concave.

The following theorem can be found in [11, 28, 55].

Theorem 4.29 (Burkholder). For a Banach space X the following conditions are equivalent:

1. X is a UMD Banach space;
2. for each $p \in (1, \infty)$ there exists a constant β and a zigzag–concave function $U : X \times X \rightarrow \mathbb{R}$, convex in the 2nd variable, such that

$$U(x, y) \geq \|y\|^p - \beta^p \|x\|^p, \quad x, y \in X. \quad (4.25)$$

The smallest admissible β , for which such U exists, is equal to $\beta_{p,X}$.

Any function U as in the above theorem will be called a Burkholder function.

Remark 4.30. Suppose that the Banach space X is finite dimensional and let $U : X \times X \rightarrow \mathbb{R}$ be a zigzag–concave function. Let $\rho : X \times X \rightarrow \mathbb{R}_+$ be a compactly supported nonnegative function of class C^∞ . Then the convolution $U_\rho := U * \rho : X \times X \rightarrow \mathbb{R}$ is zigzag–concave and of class C^∞ (see, e.g., [2]).

While working with the Burkholder function $U : X \times X \rightarrow \mathbb{R}$ we will use the following notation: for given vectors $x, y \in X$ instead of writing

$$\frac{\partial^2 U}{\partial(x, 0)^2}, \quad \frac{\partial^2 U}{\partial(0, y)^2}, \quad \frac{\partial^2 U}{\partial(x, 0)\partial(0, y)}$$

we will write

$$\frac{\partial^2 U}{\partial x^2}, \quad \frac{\partial^2 U}{\partial y^2}, \quad \frac{\partial^2 U}{\partial x \partial y}.$$

Therefore for the convenience of the reader throughout this subsection we always assume that the 1st coordinate of any vector in $X \times X$ is x (perhaps with a subscript), while the 2nd coordinate is y (perhaps with a subscript). The same holds for partial derivatives.

We also will need the following lemma.

Lemma 4.31. Let X be a finite-dimensional Banach space, let $F : X \times X \rightarrow \mathbb{R}$ be a zigzag-concave function and let $(x_0, y_0) \in X \times X$ be such that F is twice Fréchet differentiable at (x_0, y_0) . Let $(x, y) \in X \times X$ be such that $y = x$. Then for each $\lambda \in [-1, 1]$,

$$\frac{\partial^2 F(x_0, y_0)}{\partial x^2} + 2\lambda \frac{\partial^2 F(x_0, y_0)}{\partial x \partial y} + \frac{\partial^2 F(x_0, y_0)}{\partial y^2} \leq 0.$$

Proof. Since the function

$$\lambda \mapsto \frac{\partial^2 F(x_0, y_0)}{\partial x^2} + 2\lambda \frac{\partial^2 F(x_0, y_0)}{\partial x \partial y} + \frac{\partial^2 F(x_0, y_0)}{\partial y^2}$$

is linear in $\lambda \in [-1, 1]$, it is sufficient to check the cases $\lambda = \pm 1$. To this end notice that

$$\frac{\partial^2 F(x_0, y_0)}{\partial x^2} \pm 2 \frac{\partial^2 F(x_0, y_0)}{\partial x \partial y} + \frac{\partial^2 F(x_0, y_0)}{\partial y^2} = \frac{\partial^2}{\partial t^2} F(x_0 + tx, y_0 \pm tx) \Big|_{t=0} \leq 0,$$

where the latter follows from Definition 4.28. ■

Proof. of Theorem 4.26 We begin with similar reductions as in the proof of Theorem 3.1. First, we may assume that X is a finite-dimensional Banach space. Let $d \geq 1$ be the dimension of X . Let $M = M^c + M^d$ and $N = N^c + N^d$ be the Meyer–Yoeurp decompositions (see Subsection 2.7). Then by Proposition 2.14 $N^c \ll^w M^c$ and $N^d \ll^w M^d$. Let $\tau = (\tau_s)_{s \geq 0}$ be the time-change constructed in Step 1 of the proof of Theorem 3.1 (see [56, Section 4]). So, there exists a $2d$ -dimensional standard Brownian motion W on an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ equipped with an extended filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0}$, and there exist two progressively measurable processes $\phi, \psi : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(\mathbb{R}^{2d}, X)$ such that $M^c \circ \tau = \phi \cdot W$ and $N^c \circ \tau = \psi \cdot W$. Let us redefine $M := M \circ \tau$ and $N := N \circ \tau$ (hence $M^c := M^c \circ \tau$, $M^d := M^d \circ \tau$, $N^c := N^c \circ \tau$, and $N^d := N^d \circ \tau$, see [56, Subsection 2.6]). Without loss of generality we may further assume that M and N terminate after some deterministic time: $M_t = M_{t \wedge T}$ and $N_t = N_{t \wedge T}$ for some fixed parameter $T \geq 0$. Analogously to Proposition 3.5 there exists a progressively measurable $A : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{L}(\mathbb{R}^{2d})$ that satisfies $\|A\| \leq 1$ on $\mathbb{R}_+ \times \Omega$ and $\psi = \phi A$. Let us define $A^{\text{sym}} := \frac{A+A^T}{2}$, $A^{\text{asym}} := \frac{A-A^T}{2}$. If we set

$$N^{\text{sym}} := N^d + (\phi A^{\text{sym}}) \cdot W, \quad N^{\text{asym}} := (\phi A^{\text{asym}}) \cdot W,$$

then $N^{\text{sym}} \ll^w M$ and $N^{\text{asym}} \ll^w M$. Indeed, if $N^{\text{sym}} = N^{\text{sym},c} + N^{\text{sym},d}$ and $N^{\text{asym}} = N^{\text{asym},c} + N^{\text{asym},d}$ are the corresponding Meyer–Yoeurp decompositions, then $N^{\text{sym},d} = N^d \ll^w M^d$,

$N^{\text{asym},d} = 0 \ll M^d$, and for any $x^* \in X^*$ and $t \geq 0$, we have

$$\begin{aligned} [\langle N^{\text{sym},c}, x^* \rangle]_t &= \int_0^t \left\| \frac{A(s)+A^T(s)}{2} \phi^*(s)x^* \right\|^2 ds \\ &\leq \int_0^t \left\| \frac{A(s)+A^T(s)}{2} \right\|^2 \|\phi^*(s)x^*\|^2 ds \leq \int_0^t \|\phi^*(s)x^*\|^2 ds = [\langle M^c, x^* \rangle]_t. \end{aligned}$$

Here $\left\| \frac{A(s)+A^T(s)}{2} \right\| \leq 1$ by the triangle inequality. Therefore $N^{\text{sym},c} \ll M^c$ and, analogously, $N^{\text{asym},c} \ll M^c$, so the weak differential subordination holds by virtue of Proposition 2.14.

Let us now show that

$$\mathbb{E} \|N_t^{\text{asym}}\|^p \leq h_{p,X}^p \mathbb{E} \|M_t\|^p \quad \text{for } t \geq 0. \quad (4.26)$$

We have $N_0^{\text{asym}} = 0$ and $N^{\text{asym}} \ll M$; we will prove in addition that M and N^{asym} are orthogonal. For fixed $x^* \in X^*$ and $t \geq 0$ we may write

$$\begin{aligned} [\langle M, x^* \rangle, \langle N^{\text{asym}}, x^* \rangle]_t &= [\langle M^c, x^* \rangle, \langle N^{\text{asym}}, x^* \rangle]_t + [\langle M^d, x^* \rangle, \langle N^{\text{asym}}, x^* \rangle]_t \\ &= [\langle M^c, x^* \rangle, \langle N^{\text{asym}}, x^* \rangle]_t = [\langle \phi \cdot W, x^* \rangle, \langle (\phi A^{\text{asym}}) \cdot W, x^* \rangle]_t \\ &= [\langle \phi, x^* \rangle \cdot W, \langle (\phi A^{\text{asym}}), x^* \rangle \cdot W]_t = \int_0^t \langle \phi^*(s)x^*, A^{\text{asym}*}(s)\phi^*(s)x^* \rangle ds = 0, \end{aligned}$$

where the 2nd equality is a consequence of pure discontinuity of M^d and continuity of N^{asym} , while the last equality follows from the fact that A^{asym} is antisymmetric. This gives the orthogonality of the processes and (4.26) follows from (3.1).

The next step is to show that

$$\mathbb{E} \|N_t^{\text{sym}}\|^p \leq \beta_{p,X}^p \mathbb{E} \|M_t\|^p \quad \text{for } t \geq 0. \quad (4.27)$$

Let $U : X \times X \rightarrow \mathbb{R}$ be the Burkholder function guaranteed by Theorem 4.29. Using the same argument as in [2], we may assume that U is of class C^∞ (see also Remark 4.30). Applying Itô's formula (2.3) for a fixed basis $(x_i)_{i=1}^d$ of X with the dual basis $(x_i^*)_{i=1}^d$ of X^* , we get

$$\mathbb{E} U(M_t, N_t^{\text{sym}}) = \mathbb{E} U(M_0, N_0^{\text{sym}}) + \frac{1}{2} \mathbb{E} I_1 + \mathbb{E} I_2,$$

where

$$\begin{aligned}
 I_1 &:= \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial x_i \partial x_j} d[\langle M, x_i^* \rangle, \langle M, x_j^* \rangle]_s^c \\
 &\quad + \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial Y_i \partial Y_j} d[\langle N^{\text{sym}}, x_i^* \rangle, \langle N^{\text{sym}}, x_j^* \rangle]_s^c \\
 &\quad + 2 \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial x_i \partial Y_j} d[\langle M, x_i^* \rangle, \langle N^{\text{sym}}, x_j^* \rangle]_s^c
 \end{aligned} \tag{4.28}$$

and

$$I_2 := \sum_{0 \leq s \leq t} (\Delta U(M_s, N_s^{\text{sym}}) - \langle \partial_x U(M_{s-}, N_{s-}^{\text{sym}}), \Delta M_s \rangle - \langle \partial_Y U(M_{s-}, N_{s-}^{\text{sym}}), \Delta N_s^{\text{sym}} \rangle).$$

Here $\partial_x U(\cdot), \partial_Y U(\cdot) \in X^*$ are the corresponding Fréchet derivatives of U in the 1st and the 2nd X -subspace of the product space $X \times X$. Let us first show that $\mathbb{E}I_1 \leq 0$. Indeed, note that

$$\begin{aligned}
 &\sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial x_i \partial x_j} \langle \phi^* x_i^*, \phi^* x_j^* \rangle \\
 &\quad + \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial Y_i \partial Y_j} \langle A^{\text{sym}*} \phi^* x_i^*, A^{\text{sym}*} \phi^* x_j^* \rangle \\
 &\quad + 2 \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial x_i \partial Y_j} \langle \phi^* x_i^*, A^{\text{sym}*} \phi^* x_j^* \rangle \leq 0.
 \end{aligned} \tag{4.29}$$

Note also that by Corollary 3.9 and convexity of U in the 2nd variable,

$$\sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial Y_i \partial Y_j} \langle A^{\text{sym}*} \phi^* x_i^*, A^{\text{sym}*} \phi^* x_j^* \rangle \leq \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial Y_i \partial Y_j} \langle \phi^* x_i^*, \phi^* x_j^* \rangle. \tag{4.30}$$

The operator $P_{\text{Ran}(\phi^*)} A^{\text{sym}*} P_{\text{Ran}(\phi^*)}$ is symmetric and

$$\|P_{\text{Ran}(\phi^*)} A^{\text{sym}*} P_{\text{Ran}(\phi^*)}\| \leq 1.$$

Therefore by the spectral theorem there exist a $[-1, 1]$ -valued sequence $(\lambda_i)_{i=1}^{2d}$ and an orthonormal basis $(\tilde{h}_i)_{i=1}^{2d}$ of $(\mathbb{R}^{2d})^*$ such that $P_{\text{Ran}(\phi^*)} A^{\text{sym}*} P_{\text{Ran}(\phi^*)} \tilde{h}_i = \lambda_i \tilde{h}_i$. Moreover,

since $\text{Ran}(P_{\text{Ran}(\phi^*)} A^{\text{sym}*} P_{\text{Ran}(\phi^*)}) \subset \text{Ran}(\phi^*)$, $\tilde{h}_i \in \text{Ran}(\phi^*)$ if $\lambda_i \neq 0$, so we may assume that there exists a basis $(\tilde{x}_i)_{i=1}^d$ of X with the dual basis $(\tilde{x}_i^*)_{i=1}^d$ such that $\phi^* \tilde{x}_i^* = \tilde{h}_i$ for $1 \leq i \leq m$ and $\phi^* \tilde{x}_i^* = 0$ for $m < i \leq d$, where $m \in \{0, \dots, d\}$ is the dimension of ϕ^* . By Lemma 3.8 the expression on the left-hand side of (4.29) does not depend on the choice of the basis of X and the corresponding dual basis. Therefore, using (4.30), it is not bigger than

$$\sum_{i=1}^m \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial x_i \partial x_i} + \sum_{i=1}^m \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial y_i \partial y_i} + 2 \sum_{i=1}^m \lambda_i \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial x_i \partial y_i},$$

which is bounded from above by 0 (see Lemma 4.31). Thus, (4.29) follows. Therefore by (4.28) and (4.29), we see that

$$\begin{aligned} I_1 &= \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial x_i \partial x_j} \langle \phi^* x_i^*, \phi^* x_j^* \rangle \, ds \\ &\quad + \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial y_i \partial y_j} \langle A^{\text{sym}*} \phi^* x_i^*, A^{\text{sym}*} \phi^* x_j^* \rangle \, ds \\ &\quad + 2 \int_0^t \sum_{i,j=1}^d \frac{\partial^2 U(M_{s-}, N_{s-}^{\text{sym}})}{\partial x_i \partial y_j} \langle \phi^* x_i^*, A^{\text{sym}*} \phi^* x_j^* \rangle \, ds \leq 0, \end{aligned}$$

and hence the expectation of I_1 is nonpositive. The inequality $I_2 \leq 0$ can be proved by repeating the arguments from [55, Proof of Theorem 3.18], while for the estimate $U(M_0, N_0^{\text{sym}}) \leq 0$, consult [55, Remark 3.10]. Therefore, we have

$$\mathbb{E} \|N_t^{\text{sym}}\|^p - \beta_{p,X}^p \mathbb{E} \|M_t\|^p \leq \mathbb{E} U(M_t, N_t^{\text{sym}}) \leq \mathbb{E} U(M_0, N_0^{\text{sym}}) \leq 0,$$

so (4.27) holds. The general inequality (4.24) follows from (4.26), (4.27), and the triangle inequality. \blacksquare

Remark 4.32. It is an open problem whether there exists a Burkholder function U such that $-U$ is plurisubharmonic (note that $X \times X \simeq X + iX$, so the plurisubharmonicity condition is well defined). If it exists, then $\mathfrak{h}_{p,X} \leq \beta_{p,X}$ by Theorem 3.2, and so the open problem outlined in Remark 2.1 is solved. Unfortunately, plurisubharmonicity of $-U$ is discovered only in the Hilbert space case (see [52] and [55, Remark 5.6]).

4.5 Weak differential subordination of harmonic functions

Let X be a Banach space, let $d \geq 1$ be a fixed dimension, and let \mathcal{O} be an open subset of \mathbb{R}^d . A function $f : \mathcal{O} \rightarrow X$ is called harmonic if it takes its values in a finite-dimensional subspace of X , is twice-differentiable, and

$$\Delta f(s) := \sum_{i=1}^d \partial_i^2 f(s) = 0, \quad s \in \mathcal{O}.$$

For each $s \in \mathcal{O}$, we define $\nabla f(s) \in \mathcal{L}(\mathbb{R}^d, X)$ by

$$\nabla f(s)(a_1 e_1 + \cdots + a_d e_d) = \sum_{i=1}^d a_i \partial_i f(s), \quad a_1, \dots, a_d \in \mathbb{R},$$

where $(e_i)_{i=1}^d$ is the basis of \mathbb{R}^d .

Definition 4.33. Let X, d, \mathcal{O} be as above and assume that $f, g : \mathcal{O} \rightarrow X$ are harmonic functions. Then

1. g is said to be weakly differentially subordinate to f (which will be denoted by $g \overset{w}{\ll} f$) if

$$|\langle \nabla g(s), x^* \rangle| \leq |\langle \nabla f(s), x^* \rangle|, \quad s \in \mathcal{O}, x^* \in X^*; \tag{4.31}$$

2. f and g are said to be orthogonal if

$$\langle \langle \nabla f(s), x^* \rangle, \langle \nabla g(s), x^* \rangle \rangle = 0, \quad s \in \mathcal{O}, x^* \in X^*. \tag{4.32}$$

Here $|\cdot|$ in (4.31) is assumed to be the usual Euclidean norm in $(\mathbb{R}^d)^* \simeq \mathbb{R}^d$, and $\langle \cdot, \cdot \rangle$ in (4.32) is the usual scalar product in $(\mathbb{R}^d)^* \simeq \mathbb{R}^d$.

The notion of weak differential subordination of vector-valued harmonic functions extends the concept originally formulated in the one-dimensional case by Burkholder [13]. As shown in that paper, the differential subordination of harmonic functions lead to the corresponding L^p inequalities for $1 < p < \infty$. The aim of this subsection is to show the extension of that result to general weakly differentially subordinated harmonic functions and to show more general Φ, Ψ -type estimates under the orthogonality assumption. We start with recalling the definition of a harmonic measure.

Definition 4.34. Let $\mathcal{O} \subset \mathbb{R}^d$ be an open set containing the origin and let $\partial\mathcal{O}$ be the boundary of \mathcal{O} . The probability measure μ on $\partial\mathcal{O}$ is called a harmonic measure with respect to the origin, if for any Borel subset $A \subset \partial\mathcal{O}$ we have

$$\mu(A) := \mathbb{P}\{W_\tau \in A\}.$$

Here $W : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$ is a standard Brownian motion starting from 0 and τ is the exit time of W from \mathcal{O} .

Theorem 4.35. Let X be a Banach space, let $d \geq 1$ be a fixed dimension, and let \mathcal{O} be an open, bounded subset of \mathbb{R}^d containing the origin. Assume further that $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ are continuous functions such that Ψ is convex and $\Psi(0) = 0$. Then for any continuous functions $f, g : \overline{\mathcal{O}} \rightarrow X$ harmonic and orthogonal on \mathcal{O} satisfying $g \overset{w}{\ll} f$ and $g(0) = 0$ we have

$$\int_{\partial\mathcal{O}} \Psi(g(s)) \, d\mu(s) \leq C_{\Phi, \Psi, X} \int_{\partial\mathcal{O}} \Phi(f(s)) \, d\mu(s).$$

Here μ is the harmonic measure on $\partial\mathcal{O}$ with respect to the origin and the least admissible $C_{\Phi, \Psi, X}$ equals $|\mathcal{H}_X^\mathbb{T}|_{\Phi, \Psi}$.

Remark 4.36. We do not assume that Φ is convex because both f and g take their values in a finite-dimensional subspace of X , see Remark 3.14.

Proof. of Theorem 4.35 Let $W : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^d$ be a standard Brownian motion and let $\tau := \inf\{t \geq 0 : W_t \notin \mathcal{O}\}$. Then both $M := f(W^\tau)$ and $N := g(W^\tau)$ are martingales since both f and g are harmonic on \mathcal{O} (see, e.g., [32, Theorem 18.5]). By Itô's formula and the fact that both f and g are harmonic we have

$$\begin{aligned} M_t &= f(W_t^\tau) = f(0) + \int_0^t \nabla f(W_s^\tau) \, dW_s^\tau, \quad t \geq 0, \\ N_t &= g(W_t^\tau) = \int_0^t \nabla g(W_s^\tau) \, dW_s^\tau, \quad t \geq 0, \end{aligned}$$

where in the 2nd line we have used the equality $g(0) = 0$. Therefore for any $x^* \in X^*$ and any $0 \leq u \leq t$ we have

$$\begin{aligned} [\langle N, x^* \rangle]_t - [\langle N, x^* \rangle]_u &= \int_u^t \|\langle \nabla g(W_s^\tau), x^* \rangle\|^2 \, ds \\ &\leq \int_u^t \|\langle \nabla f(W_s^\tau), x^* \rangle\|^2 \, ds = [\langle M, x^* \rangle]_t - [\langle M, x^* \rangle]_u, \end{aligned}$$

and

$$[\langle M, x^* \rangle, \langle N, x^* \rangle]_t = \int_0^t \langle \nabla g(W_s^t), x^* \rangle, \langle \nabla f(W_s^t), x^* \rangle \rangle ds = 0.$$

Consequently, M and N are orthogonal and $N \overset{w}{\ll} M$, so

$$\int_{\partial\mathcal{O}} \Psi(g(s)) d\mu(s) = \lim_{t \rightarrow \infty} \mathbb{E} \Psi(g(W_t^t)) \leq \lim_{t \rightarrow \infty} |\mathcal{H}_X^\mathbb{T}|_{\Phi, \Psi} \mathbb{E} \Phi(f(W_t^t)) = |\mathcal{H}_X^\mathbb{T}|_{\Phi, \Psi} \int_{\partial\mathcal{O}} \Phi(f(s)) d\mu(s).$$

Here the 1st and the last equality follow from the dominated convergence theorem and the definition of μ , while the middle one is due to Theorem 3.1.

The sharpness of the constant $C_{\Phi, \Psi, X} = |\mathcal{H}_X^\mathbb{T}|_{\Phi, \Psi}$ follows from the case $d = 2$, $\mathcal{O} \subset \mathbb{R}^2$ being the unit disc, f and g being such that $g|_{\partial\mathcal{O}} = \mathcal{H}_X^\mathbb{T}(f|_{\partial\mathcal{O}})$ (in this case μ becomes the probability Lebesgue measure on the unit circle $\partial\mathcal{O}$). ■

Remark 4.37. Sharpness of the estimate

$$\int_{\partial\mathcal{O}} \Psi(g(s)) d\mu(s) \leq |\mathcal{H}_X^\mathbb{T}|_{\Phi, \Psi} \int_{\partial\mathcal{O}} \Phi(f(s)) d\mu(s)$$

for a fixed domain \mathcal{O} remains open. Nevertheless, in the case $d = 2$ and \mathcal{O} being bounded with a Jordan boundary (e.g., polygon-shaped) the sharpness follows immediately from the Carathéodory’s theorem (see, e.g., [22, Subsection I.3 and Appendix F]).

Let us turn to the corresponding result for L^p estimates for differentially subordinate harmonic functions (i.e., not necessarily orthogonal).

Theorem 4.38. Let X , d , and \mathcal{O} be as in the previous statement. Assume further that $f, g : \overline{\mathcal{O}} \rightarrow X$ are continuous functions harmonic on \mathcal{O} satisfying $g \overset{w}{\ll} f$ and $g(0) = a_0 f(0)$ for some $a_0 \in [-1, 1]$. Then for any $1 < p < \infty$ we have

$$\left(\int_{\partial\mathcal{O}} \|g(s)\|^p d\mu(s) \right)^{\frac{1}{p}} \leq C_{p,X} \left(\int_{\partial\mathcal{O}} \|f(s)\|^p d\mu(s) \right)^{\frac{1}{p}}, \tag{4.33}$$

where μ is the harmonic measure of $\partial\mathcal{O}$, and the least admissible constant $C_{p,X}$ is within the segment $[\hbar_{p,X}, \beta_{p,X} + \hbar_{p,X}]$.

Remark 4.39. In the scalar-valued setting it is known that the optimal $C_{p,\mathbb{R}}$ is within the range $[\cot(\frac{\pi}{2p^*}), p^* - 1]$. The precise identification of $C_{p,\mathbb{R}}$ is an open problem formulated by Burkholder in [13].

Proof. of Theorem 4.38 This is quite similar to the proof of the latter statement, so we will be brief and only indicate the necessary changes that need to be implemented. For the lower bound $C_{p,X} \geq \mathfrak{h}_{p,X}$, modify appropriately the last sentence of the proof of Theorem 4.35. To show the upper bound for $C_{p,X}$, consider the martingales $M := f(W^\tau)$ and $N := g(W^\tau)$, where W and τ are as previously. Arguing as in the proof of Theorem 4.35, we show that $N \stackrel{w}{\ll} M$ and hence by Theorem 4.26,

$$\begin{aligned} \left(\int_{\partial\mathcal{O}} \|g(s)\|^p d\mu(s) \right)^{\frac{1}{p}} &= \lim_{t \rightarrow \infty} (\mathbb{E} \|N_t\|^p)^{\frac{1}{p}} \leq \limsup_{t \rightarrow \infty} (\beta_{p,X} + \mathfrak{h}_{p,X}) (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}} \\ &\leq \lim_{t \rightarrow \infty} (\beta_{p,X} + \mathfrak{h}_{p,X}) (\mathbb{E} \|M_t\|^p)^{\frac{1}{p}} = (\beta_{p,X} + \mathfrak{h}_{p,X}) \left(\int_{\partial\mathcal{O}} \|f(s)\|^p d\mu(s) \right)^{\frac{1}{p}}. \end{aligned}$$

This completes the proof. ■

Remark 4.40. Note that any significant improvement for the upper bound of $C_{p,X}$ in (33) could automatically solve an open problem. Let us outline two remarkable examples. If one could show that $C_{p,X} \leq C\beta_{p,X}$ for some universal constant $C > 0$, then the open problem outlined in Remark 2.1 will be solved. On the other hand, if one could show that $C_{p,X} = \mathfrak{h}_{p,X}$, then the question of Burkholder concerning the optimal constant $C_{p,\mathbb{R}}$ in the real-valued case would be answered (see Remark 4.39).

4.6 Inequalities for singular integral operators

Our final application concerns the extension of Φ, Ψ -estimates from the setting of nonperiodic Hilbert transform to the case of odd-kernel singular integral operators on \mathbb{R}^d . We start with the notion of a directional Hilbert transform: given a unit vector $\theta \in \mathbb{R}^d$, we define the operator \mathcal{H}_θ by

$$\mathcal{H}_\theta f(x) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} f(x - t\theta) \frac{dt}{t}, \quad x \in \mathbb{R}^d,$$

where f is a sufficiently regular real-valued function on \mathbb{R}^d , and call it the Hilbert transform of f in the direction θ . For example, if e_1 stands for the unit vector $(1, 0, 0, \dots, 0) \in \mathbb{R}^d$, then \mathcal{H}_{e_1} is obtained by applying the Hilbert transform in the 1st

variable followed by the identity operator in the remaining variables. Consequently, by Fubini's theorem, we see that for any functions $\Phi, \Psi : X \rightarrow [0, \infty)$ and any step function $f : \mathbb{R}^d \rightarrow X$ (finite linear combination of characteristic functions of rectangles) we have

$$\int_{\mathbb{R}^d} \Psi(\mathcal{H}_{e_1} f) dx \leq |\mathcal{H}_X^{\mathbb{R}}|_{\Phi, \Psi} \int_{\mathbb{R}^d} \Phi(f) dx.$$

Now, if A is an arbitrary orthogonal matrix, we have

$$\mathcal{H}_{Ae_1}(f)(x) = \mathcal{H}_{e_1}(f \circ A)(A^{-1}x), \quad x \in \mathbb{R}^d,$$

so the above inequality holds true for any directional Hilbert transform \mathcal{H}_θ .

Suppose that $\Omega : S^{d-1} \rightarrow \mathbb{R}$ is an odd function satisfying $\|\Omega\|_{L^1(S^{d-1})} = 1$ and define the associated operator

$$T_\Omega f(x) = \frac{2}{\pi} \text{p.v.} \int_{\mathbb{R}^d} \frac{\Omega(y/|y|)}{|y|^d} f(x-y) dy, \quad x \in \mathbb{R}^d.$$

Then T_Ω can be expressed as an average of directional Hilbert transforms:

$$T_\Omega f(x) = \int_{S^{d-1}} \Omega(\theta) \mathcal{H}_\theta f(x) d\theta, \quad x \in \mathbb{R}^d.$$

(Sometimes this identity is referred to as the method of rotations.) Consequently, if Ψ is convex and even, we get

$$\begin{aligned} \int_{\mathbb{R}^d} \Psi(T_\Omega f) dx &= \int_{\mathbb{R}^d} \Psi \left(\int_{S^{d-1}} \Omega(\theta) \mathcal{H}_\theta f(x) d\theta \right) dx \\ &\leq \int_{S^{d-1}} |\Omega(\theta)| \int_{\mathbb{R}^d} \Psi(\mathcal{H}_\theta f(x)) dx d\theta \leq |\mathcal{H}_{\Phi, \Psi}^{\mathbb{R}}| \int_{\mathbb{R}^d} \Phi(f) dx. \end{aligned}$$

In particular, if we fix d and $j \in \{1, 2, \dots, d\}$, then the kernel

$$\Omega_{j,d}(\theta) = \frac{\pi \Gamma\left(\frac{d+1}{2}\right)}{2\pi^{(d+1)/2}} \theta_j, \quad \theta \in S^{d-1},$$

gives rise to the Riesz transform R_j . Therefore, we see that any Φ, Ψ -estimate for the nonperiodic Hilbert transform (where Ψ is assumed to be a convex and even function on X) holds true, with an unchanged constant, also in the context of Riesz transforms. In particular, the L^p norms of Riesz transforms are dominated by the L^p norms of Hilbert transform.

The following theorem connects the Φ, Ψ -norm of an odd power of a Riesz transform with the Φ, Ψ -norm of the Hilbert transform.

Theorem 4.41. Let X be a Banach space, $d \geq 1$, $j \in \{1, \dots, d\}$, $m \geq 1$ be odd. Let $R_{j,X}$ be the corresponding Riesz transform acting on X -valued step functions, $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ be convex continuous such that Ψ is even. Then

$$|R_{j,X}^m|_{\Phi, \Psi} \leq \left| \frac{2\Gamma(\frac{m+d}{2})}{\Gamma(\frac{d}{2})\Gamma(\frac{m}{2})} \mathcal{H}_X^{\mathbb{R}} \right|_{\Phi, \Psi}.$$

Proof. The proof follows from the discussion above, the fact that $R_{j,X}^m$ is a singular integral of the following form (see, e.g., [29, p. 33]):

$$R_{j,X}^m f(x) = \frac{\Gamma(\frac{m+d}{2})}{\pi^{\frac{d}{2}} \Gamma(\frac{m}{2})} \int_{\mathbb{R}^d} \frac{f(x-y)y_j^m}{|y|^{m+d}} dy, \quad x \in \mathbb{R}^d,$$

where $f : \mathbb{R}^d \rightarrow X$ is a step function, and the fact that the volume of S^{d-1} equals $2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$. ■

Notice that if d is fixed, then $\frac{2\Gamma(\frac{m+d}{2})}{\Gamma(\frac{d}{2})\Gamma(\frac{m}{2})}$ is of the order $m^{d/2}$, so in particular we have that for all $1 < p < \infty$,

$$\|R_{j,X}^m\|_{L^p(\mathbb{R}^d; X) \rightarrow L^p(\mathbb{R}^d; X)} \lesssim_d m^{d/2} \|\mathcal{H}_X^{\mathbb{R}}\|_{L^p(\mathbb{R}; X) \rightarrow L^p(\mathbb{R}; X)}.$$

4.7 Hilbert operators

Let X be a Banach space, let d be a positive integer, and pick $j \in \{1, \dots, d\}$. Let $f : \mathbb{R}_{j+}^d \rightarrow X$ be locally integrable function, where $\mathbb{R}_{j+}^d = \{x \in \mathbb{R}^d : x_j > 0\}$. We define $T_j f : \mathbb{R}_{j+}^d \rightarrow X$ by the formula

$$T_j f(x) := \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}} \int_{\mathbb{R}_{j+}^d} \frac{f(y)(x_j + y_j)}{|x + y|^{d+1}} dy, \quad x \in \mathbb{R}_{j+}^d.$$

This type of operators resembles Riesz transforms, but due to the domain restrictions the use of principal value is not necessary. Note that if $d = 1$, then T_j is the Hilbert operator T given by

$$Tf(x) := \frac{1}{\pi} \int_{\mathbb{R}_+} \frac{f(y)}{x+y} dy, \quad x \in \mathbb{R}_+.$$

We have the following statement.

Theorem 4.42. Let X be a Banach space, $\Phi, \Psi : X \rightarrow \mathbb{R}_+$ be convex continuous such that Ψ is even, $d \geq 1, j \in \{1, \dots, d\}, 1 < p < \infty$. Then

$$|T_j|_{\Phi, \Psi} \leq |\mathcal{H}_X^{\mathbb{R}}|_{\Phi, \Psi}. \tag{4.34}$$

Proof. By the discussion in Subsection 4.6 it is sufficient to show that

$$|T_j|_{\Phi, \Psi} \leq |R_{j, X}|_{\Phi, \Psi}.$$

Fix a step function $f : \mathbb{R}_{j+}^d \rightarrow X$. Let $\tilde{f} : \mathbb{R}^d \rightarrow X$ be such that $\tilde{f}(x_1, \dots, x_d) = 0$ if $x_j < 0$ and $\tilde{f}|_{\mathbb{R}_{j+}^d} = f$. Then $T_j f(x) = R_{j, X} \tilde{f}(-x)$ for any $x \in \mathbb{R}_{j+}^d$, and therefore

$$\begin{aligned} \int_{\mathbb{R}_{j+}^d} \Psi(T_j f(x)) \, dx &= \int_{\mathbb{R}^d} \Psi(R_{j, X} \tilde{f}(-x)) \mathbf{1}_{x_j > 0} \, dx \leq \int_{\mathbb{R}^d} \Psi(R_{j, X} \tilde{f}(-x)) \, dx \\ &= \int_{\mathbb{R}^d} \Psi(R_{j, X} \tilde{f}(x)) \, dx \leq |R_{j, X}|_{\Phi, \Psi} \int_{\mathbb{R}^d} \Phi(\tilde{f}(x)) \, dx \\ &= |R_{j, X}|_{\Phi, \Psi} \int_{\mathbb{R}_{j+}^d} \Phi(f(x)) \, dx. \end{aligned}$$

■

Remark 4.43. Notice that if Φ and Ψ are of the form $\Phi(x) = \phi(\|x\|), \Psi(x) = \psi(\|x\|)$ for some convex symmetric functions $\phi, \psi : \mathbb{R} \rightarrow \mathbb{R}_+$, then one can improve (4.34). Indeed, one can show that $|T_j|_{\Phi, \Psi} = |T_j|_{\phi, \psi}$, which does not depend on the Banach space X : for any step function $f : \mathbb{R}_{j+}^d \rightarrow X$ one has that

$$\begin{aligned} \int_{\mathbb{R}_{j+}^d} \Psi(T_j f(x)) \, dx &= \int_{\mathbb{R}_{j+}^d} \psi(\|T_j f(x)\|) \, dx \\ &= \int_{\mathbb{R}_{j+}^d} \psi\left(\left\| \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}} \int_{\mathbb{R}_{j+}^d} \frac{f(y)(x_j + y_j)}{|x + y|^{d+1}} \, dy \right\|\right) \, dx \\ &\leq \int_{\mathbb{R}_{j+}^d} \psi\left(\frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}} \int_{\mathbb{R}_{j+}^d} \frac{g(y)(x_j + y_j)}{|x + y|^{d+1}} \, dy\right) \, dx \\ &= \int_{\mathbb{R}_{j+}^d} \psi(T_j g(x)) \, dx \leq |T_j|_{\phi, \psi} \int_{\mathbb{R}_{j+}^d} \phi(g(x)) \, dx \\ &= |T_j|_{\phi, \psi} \int_{\mathbb{R}_{j+}^d} \Phi(f(x)) \, dx, \end{aligned}$$

where $g : \mathbb{R}_{j+}^d \rightarrow \mathbb{R}_+$ is a step function such that $g(\cdot) = \|f(\cdot)\|$. In particular, if $\Phi(x) = \Psi(x) = \|x\|^p$ for some $1 < p < \infty$, then by [41, Theorem 1.1]

$$\|T_j\|_{L^p(\mathbb{R}_{j+}^d; X) \rightarrow L^p(\mathbb{R}_{j+}^d; X)} = \sin^{-1}(\pi/p).$$

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