Increasing the Robustness of a Preconditioned Filtered-X LMS Algorithm

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***Abstract*—This letter presents a robustification of the preconditioned Filtered-X LMS algorithm proposed by Elliott *et al.*. The method optimizes the average performance for probabilistic uncertainty in the secondary path and relaxes the SPR condition for global convergence. It also prevents large amplification in the preconditioning filters due to secondary path zeros on and/or close to the unit circle, which may yield overactuation in practical applications.**

***Index Terms*—Acoustic noise, adaptive control, adaptive signal processing, feedforward systems, robust filtering.**

I. Introduction

THE Filtered-X LMS (FxLMS) algorithm is a very popular algorithm for feedforward active noise and vibration control, because the implementation is simple and its recursions are well studied (e.g., see [1]–[5], just to name a few). In broadband applications the convergence rate of FxLMS may be poor due to correlation in the regression vector. To overcome this problem, [6], [7] proposes a preconditioning of the FxLMS (PFxLMS) algorithm, which removes all correlation in the regression vector. This can increase the convergence rate significantly as shown in [8] for a realistic active control problem.

However, in [6]–[8] it was also noted that regularization is necessary in case the system has zeros on and/or close to the unit circle to reliably calculate the prefilters and prevent large amplification of the preconditioning filters, which may yield oversteering of, for example, the DA converters. An even more important problem is, that undermodeling of and variations in the secondary path may yield instability of the filter update algorithm if a particular well known strictly positive real (SPR) condition is not satisfied [2], [9].

The main focus of this letter is to adjust the PFxLMS algorithm, without paying too much performance, such that the stability of the filter update algorithm is less sensitive to errors in the secondary path model. Stated otherwise, our objective is to increase the stability robustness of the PFxLMS update algorithm w.r.t. model errors.

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In the literature, two approaches are proposed to improve the robustness of the update algorithm: 1) online secondary path modeling; and 2) adjusting the adaptive algorithm to relax the SPR condition. Both approaches have their advantages and drawbacks. Online secondary path modeling (e.g., see [4]) may keep track of variations in the secondary path and may thus yield optimal performance even if the secondary path varies. However, the computational complexity is increased and injection of an auxiliary dither signal is usually necessary with the consequence of reduced performance.

An example of the second approach, is proposed in [10] where a model is derived which satisfies the SPR condition for *multiple* secondary plant systems by solving the so-called *robust SPR* problem. The method focuses on IIR filtering, but can also be applied to FIR filtering. However, the set of multiple secondary plant systems should satisfy a particular condition for solving the robust SPR problem [10]. Furthermore, for every secondary plant system a different precondition filter would be necessary.

An alternative method which relaxes the SPR condition is control effort weighting. In [5] and [6], this was done by tuning a scalar parameter which weights the trace of the control effort covariance matrix, and results in Leakage FxLMS/PFxLMS. Besides the necessity of tuning a scalar regularization parameter, the method may be too conservative.

The contribution of this letter, is the derivation of the robust versions of both FxLMS and PFxLMS in the framework of *probabilistic robust filtering* proposed in [11]. The robust method uses a model uncertainty model of the secondary path, which acts as a *frequency-dependent* control effort weighting. As such the method results in a generalization of standard control effort weighting and hence a generalization of Leakage FxLMS/PFxLMS (e.g., cf. [4] and [5]). It is shown that the SPR condition is relaxed in a well motivated manner, and hence the stability robustness of the update algorithm is increased. A simulation example shows that this method yields better performance than Leakage FxLMS/PFxLMS.

The letter is organized as follows. Section II derives the Robust FxLMS (RFxLMS) algorithm and its new SPR condition for global convergence. Section III derives the Robust PFxLMS (RPFxLMS) algorithm, shows that large amplification of the precondition filter is prevented and derives the SPR condition for RPFxLMS. Section IV illustrates the method by a simulation example.

The notation is standard. $(.)^T$ and $(.)^*$ denote the transpose and complex conjugate transpose, respectively. $E[.]$ the expectation operator, tr$(.)$ the trace and vec$(.)$ the column stacking operator. The estimated model of . is indicated by $\widehat{.}$.

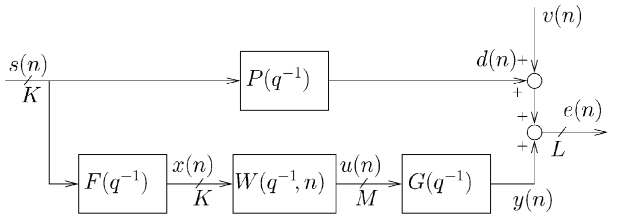

Fig. 1. Block scheme of the general multichannel feedforward active control system, with K reference, M control, and L residual signals.

II. Robust Filtered-X LMS

Consider Fig. 1, which illustrates the feedforward active control problem (acoustical feedback is neglected or assumed to be perfectly compensated by Internal Model Control). Here, $s(n) \in \mathbb{R}^K$ represents the signal from the disturbance source and is assumed to be a zero mean white noise stochastic process with $E[s(n)s^T(m)] = I_K\delta(n-m)\ \forall\ m,n$, where $\delta(0) := 1$, $\delta(n) := 0,\ n \neq 0$. Let $RH_\infty^{L\times M}$ the set of all stable proper rational $L\times M$ transfer functions matrices in the unit delay operator q^{-1} with real coefficients. Then the primary path, the detector path and the secondary path are denoted by $P(q^{-1}) \in RH_\infty^{L\times K}$, $F(q^{-1}) \in RH_\infty^{K\times K}$ and $G(q^{-1}) \in RH_\infty^{L\times M}$, respectively. The adaptive feedforward controller is an $M\times K$ matrix with FIR filters of length I and its m, kth element is given by

$$W_{mk}(q^{-1},n) = \sum_{i=0}^{I-1} w_{mk}^i(n)q^{-i}$$

with $w_{mk}^i(n) \in \mathbb{R}$. For ease of notation, we define $\mathbf{w}_{mk}(n) := [w_{mk}^0(n)\ \ldots\ w_{mk}^{I-1}(n)]^T \in \mathbb{R}^I$, $\mathbf{w}_m(n) := [\mathbf{w}_{m1}^T(n)\ \ldots\ \mathbf{w}_{mK}^T(n)]^T \in \mathbb{R}^{\mathrm{IK}}$ and $\mathbf{W}(n) := [\mathbf{w}_1(n)\ \cdots\ \mathbf{w}_M(n)] \in \mathbb{R}^{IK\times M}$ and the vector stacking of all controller coefficients $\theta(n) := \mathrm{vec}(\mathbf{W}(n)) \in \mathbb{R}^{\mathrm{MIK}}$. The input to the adaptive filter is the reference signal $x(n) \in \mathbb{R}^K$, let $\phi(n) := [x_1(n)\ \ldots\ x_1(n-I+1)\ \ldots\ x_K(n)\ \ldots\ x_K(n-I+1)]^T \in \mathbb{R}^{\mathrm{IK}}$. Then the control signal $u(n) \in \mathbb{R}^M$ is given by

$$u(n) := W(q^{-1},n)x(n) = \mathbf{W}^T(n)\phi(n).$$

The objective is to determine $u(n)$ such that $y(n) \in \mathbb{R}^L$, counteracts the disturbance signal $d(n) \in \mathbb{R}^L$. The measured residual signal is corrupted with a zero mean stochastic noise process $v(n) \in \mathbb{R}^L$, with intensity $\sigma_v^2 := \mathrm{tr}\ E[v(n)v^T(n)]$, which is independent of $s(n)$, i.e., $E[v(n)s^T(m)] = 0_{L\times K},\ \forall\ m,n$. The measured residual $e(n) \in \mathbb{R}^L$ is given by

$$e(n) = P(q^{-1})s(n) + G(q^{-1})u(n) + v(n).$$

Then, the FxLMS algorithm, which objective is to minimize $\xi = \mathrm{tr}E[e(n)e^T(n)]$ is given by

$$\theta(n+1) = \theta(n) - \gamma(n)[\widehat{G}^T(q^{-1})\otimes\phi(n)]e(n)$$

with $\otimes$ denoting the Kronecker matrix product, $\gamma(n) \geq 0$ the step size. Using Ljung's [12]ordinary differential equation (ODE) method, [2] (see also [9]) shows that if $\gamma(n)$ suitably vanishes, $x(n)$ is persistently exciting and the following SPR condition is satisfied:

$$G^T(e^{j\omega})\widehat{G}(e^{-j\omega}) + \widehat{G}^T(e^{j\omega})G(e^{-j\omega}) > 0, \quad -\pi \leq \omega \leq \pi \tag{1}$$

then the associated ODE, which describes the asymptotic behavior of $\theta(n)$, is asymptotically stable. Hence, $\theta(n)$ converges, with probability one, to its unique global optimum [2]

$$\theta_{\mathrm{opt}} = -E\left[\left(\widehat{G}^T\otimes\phi(n)\right)\left(G^T\otimes\phi(n)\right)^T\right]^{-1} \cdot E\left[\left(\widehat{G}^T\otimes\phi(n)\right)d(n)\right].$$

To increase the robustness of the FxLMS algorithm w.r.t. uncertainty in $\widehat{G}$, we may want to have a (probabilistic) model of the uncertainty. Here, we will follow the idea of the probabilistic robust filtering approach proposed in [11]. We assume that G can be modeled as a stochastic variable, such that

$$G(q^{-1}) = \overline{G}(q^{-1}) + \Delta G(q^{-1})$$

with $\overline{E}[\Delta G(e^{-j\omega})\Delta G^T(e^{j\omega})] = \Phi_{\Delta G}(e^{-j\omega})$ and $\overline{E}[G(e^{-j\omega})] = \overline{G}(e^{-j\omega})$, for $-\pi \leq \omega \leq \pi$. $\overline{E}[.]$ denotes expectation over stochastic systems. Further, let ΔG be independent of $\overline{G}$, P, F, $s(n)$ and $v(n)\ \forall\ n$. The objective of the robust filtering approach is to minimize

$$\xi_{\mathrm{rob}} = \mathrm{tr}\ \overline{E}E\left[e(n)e^T(n)\right]. \tag{2}$$

By Parseval's equality and the independence between ΔG and the other factors,

$$\begin{aligned}\xi_{\mathrm{rob}} &= \frac{1}{2\pi}\mathrm{tr}\int_{-\pi}^{\pi}(.)^*(P+\overline{G}WF)\\ &\quad + F^*W^*\Phi_{\Delta G}WFd\omega + \sigma_v^2\\ &= \frac{1}{2\pi}\mathrm{tr}\int_{-\pi}^{\pi}(.)^*(P^{\mathrm{aug}}+G^{\mathrm{aug}}WF)d\omega + \sigma_v^2.\end{aligned}$$

Here $(.)^*$ indicates complex conjugate transpose of the following factor, $P^{\mathrm{aug}} = [P^T\ 0_{M\times J}^T]^T$, $G^{\mathrm{aug}} = [\overline{G}^T\ \widetilde{\Delta G}^T]^T$ and $\widetilde{\Delta G} \in RH_\infty^{L\times M}$ such that $\widetilde{\Delta G}^*\widetilde{\Delta G} = \Phi_{\Delta G}$. Now, let $\widehat{G} \in RH_\infty^{L\times M}$ and $\widehat{\Delta G} \in RH_\infty^{L\times M}$ be models of $\overline{G}$ and $\widetilde{\Delta G}$ respectively and $\widehat{G}^{\mathrm{aug}} = [\widehat{G}^T\ \widehat{\Delta G}^T]^T$. Then the Robust FxLMS (RFxLMS) algorithm is given by

$$\begin{aligned}\theta(n+1) &= \theta(n)\\ &\quad -\gamma(n)[\widehat{G}^{\mathrm{aug}\ T}(q^{-1})\otimes\phi(n)]\begin{bmatrix} e(n) \\ \widehat{\Delta}g(q^{-1})u(n)\end{bmatrix}.\end{aligned}$$

We observe, that the RFxLMS algorithm is identical to the FxLMS algorithm with the secondary path model augmented by $\widehat{\Delta G}$ and the performance channels $e(n)$ augmented by $\widehat{\Delta G}u(n)$. This additional term reduces the energy of the control signal at frequencies where the uncertainty, i.e., $\Phi_{\Delta G}(e^{-j\omega})$, is large. The uncertainty model $\widehat{\Delta G}$ can be obtained from e.g., identification of $\widehat{G}$; see also [11], [13], and [14]. An other approach is by performing a series of identification experiments under different secondary path conditions which yields $\widehat{G}$ as the average model and $\widehat{\Delta G}$ as a stable spectral factor of the estimated covariance $\widehat{\Phi}_{\Delta G}$. In case $\widehat{\Delta G} = \beta I_M$ with $\beta > 0$ a constant real scalar, the RFxLMS algorithm can be reduced to the Leakage FxLMS algorithm (e.g., cf. [4] and [5]).

To derive the SPR condition for the RFxLMS algorithm, we have to rewrite the FxLMS SPR condition (1) for the augmented system, which yields

$$G^T(e^{j\omega})\widehat{G}(e^{-j\omega})+\widehat{G}^T(e^{j\omega})G(e^{-j\omega})\\ +2\widehat{\Delta G}^T(e^{j\omega})\widehat{\Delta G}(e^{-j\omega})>0,\quad -\pi\le\omega\le\pi. \tag{3}$$

Because, $\widehat{\Delta G}^T(e^{j\omega})\widehat{\Delta G}(e^{-j\omega})=\widehat{\Phi}_{\Delta G}(e^{-j\omega})\ge 0$ for $-\pi\le\omega\le\pi$, the SPR condition is relaxed, especially at frequencies where the magnitude of the uncertainty model is large.

III. Robust Preconditioned Filtered-X LMS

The robustness of the PFxLMS algorithm can be increased too by minimizing the *robust* cost function (2). Like the preconditioning filters for the FxLMS algorithm are factors of the Causal Wiener filter (see [6] and [7]), the robust preconditioning filters are factors of the robust Wiener filter -called the *Cautious* Wiener filter in [11]—which minimizes (2) and is given by

$$W_{\text{rob,opt}}=-(G_o^{\text{aug}})^{-1}[G_i^{\text{aug}*}P^{\text{aug}}F_i^*]_+F_o^{-1}$$

with $[.]_+$ the causality operator, $F=F_oF_i$ is the outer-inner factorization of F and $G^{\text{aug}}=G_i^{\text{aug}}G_o^{\text{aug}}$ is the inner-outer factorization of G^{aug}. Note, that F_o^{-1} is a whitening filter for the reference signal and $(G_o^{\text{aug}})^{-1}$ inverts the minimum phase part of the *augmented* secondary path (if G_o^{aug} is nonsquare $(G_o^{\text{aug}})^{-1}$ denotes a right inverse).

Models of F_o^{-1} and $(G_o^{\text{aug}})^{-1}$ can be used to precondition the RFxLMS problem by removing the correlation in the regression vector, which yields the RPFxLMS algorithm

RPFxLMS algorithm:

The control law is given by

$$u(n)=(\widehat{G}_o^{\text{aug}}(q^{-1}))^{-1}\tilde{u}(n)$$
$$\tilde{u}(n)=W(q^{-1},n)\tilde{x}(n)$$
$$\tilde{x}(n)=(\widehat{F}_o(q^{-1}))^{-1}x(n)$$

and the update algorithm by

$$\theta(n+1)=\theta(n)\\ -\gamma(n)\left[(\widehat{G}_i^{\text{aug}}(q^{-1}))^T\otimes\tilde{\phi}(n)\right]\begin{bmatrix}e(n)\\ \widehat{G}_{i2}^{\text{aug}}(q^{-1})\tilde{u}(n)\end{bmatrix} \tag{4}$$

with $\tilde{\phi}(n)$ is defined similar to $\phi(n)$ but $x(n)$ is replaced by $\tilde{x}(n)$, and $\widehat{\Delta G}(\widehat{G}_o^{\text{aug}})^{-1}=\widehat{G}_{i2}^{\text{aug}}$ equals the last L rows of $\widehat{G}_i^{\text{aug}}$.

Note, that $\widehat{G}_o^{\text{aug}*}\widehat{G}_o^{\text{aug}}=\widehat{G}^*\widehat{G}+\widehat{\Delta G}^*\widehat{\Delta G}$ and thus the gain of $(\widehat{G}_o^{\text{aug}})^{-1}$ will be reduced where $\widehat{\Delta G}^*\widehat{\Delta G}>0$, which may prevent oversteering of, for example, the DA converters.

Assuming $\widehat{F}_o^{-1}=F_o^{-1}$, which is such that $E[\tilde{x}(m)\tilde{x}^T(n)]=I_K\delta(m-n)$, it can be proven that the autocorrelation matrix of the regression vector

$$R=E\left[\left(\widehat{G}_i^{\text{aug}\,T}\otimes\tilde{\phi}(n)\right)\left(\widehat{G}_i^{\text{aug}\,T}\otimes\tilde{\phi}(n)\right)^T\right]$$

equals the identity matrix I_{MIK}. Therefore, under this condition all modes converge at the same rate, which is determined by the step size $\gamma(n)$.

Using the ordinary differential equation (ODE) method as in [2] the following theorem on the convergence of RPFxLMS is obtained.

Theorem 1 (Convergence RPFxLMS): If $\gamma(n)$ suitably vanishes, $x(n)$ is persistently exciting, the regularity conditions of the ODE theorem [12] are satisfied and the following SPR condition holds:

$$(\widehat{G}_o^{\text{aug}}(e^{j\omega}))^{-T}\left(G^T(e^{j\omega})\widehat{G}(e^{-j\omega})+\widehat{G}^T(e^{j\omega})+G(e^{-j\omega})\\ +2\widehat{\Delta G}^T(e^{j\omega})\widehat{\Delta G}(e^{-j\omega})\right)\\ \cdot(\widehat{G}_o^{\text{aug}}(e^{-j\omega}))^{-1}>0,\qquad \text{for }-\pi\le\omega\le\pi. \tag{5}$$

Then the associated ODE, which describes the asymptotic behavior of $\theta(n)$, is asymptotically stable. Furthermore, $\theta(n)$ converges, with probability one, to its unique global optimum

$$\theta_{\text{opt}}=-E\left[\left(\widehat{G}_{i1}^{\text{aug}T}\otimes\widetilde{\phi}(n)\right)\left((G\widehat{G}_o^{\text{aug}-1})^T\otimes\widetilde{\phi}(n)\right)^T\\ +\left(\widehat{G}_{i2}^{\text{aug}T}\otimes\widetilde{\phi}(n)\right)\left(\widehat{G}_{i2}^{\text{aug}T}\otimes\widetilde{\phi}(n)\right)^T\right]^{-1}\\ \cdot E\left[\left(\widehat{G}_{i1}^{\text{aug}T}\otimes\widetilde{\phi}(n)\right)d(n)\right].$$

Proof: The proof is along the same lines as in [2], but with augmented secondary path and precondition filters. ■

The SPR condition (5) for RPFxLMS is a weighted version of the SPR condition (3) for RFxLMS, with weighting function $(\widehat{G}_o^{\text{aug}})^{-1}$. If $(\widehat{G}_o^{\text{aug}})^{-1}$ is square and full rank (which is usually the case), then the SPR condition for RPFxLMS (5) can be simplified further to the SPR condition of RFxLMS (3). In the case $(\widehat{G}_o^{\text{aug}})^{-1}$ is tall (i.e., if $\widehat{G}^{\text{aug}}$ has more columns than rows), (5) is less strict than (3). Hence, if RFxLMS converges then RPFxLMS converges, provided the step size is small enough.

Uncertainty in the detector path F can be taken into account similar. But, instead of augmenting the performance channels to deal with uncertainty in S, the *reference* channels has to be augmented with an additional noise signal uncorrelated with $s(n)$ and $v(n)$ (cf. [15]). Furthermore, the same robustification method can be used in the Adjoint FxLMS algorithms as in [6] and [7].

IV. Simulation Example

The RPFxLMS algorithm is tested on a one-dimensional acoustical duct simulation model, discretized using a sampling rate of $1\cdot10^3$ Hz. The delay in the secondary path *model* has been varied, from 0 to $4\cdot10^{-3}$ s additional delay. Depending on the amount of additional delay the SPR condition (1) does not hold anymore, especially for high frequencies. The RPFxLMS algorithm has been applied for various choices of $\widehat{\Delta G}$: 1) $\widehat{\Delta G}=0$, i.e., the nominal case; 2) $\widehat{\Delta G}=1.87\cdot10^{-2}$, which is such that (5) just holds for a delay of $1\cdot10^{-2}$ s; 3) $\widehat{\Delta G}=4.27\cdot10^{-2}$, which is such that (5) just holds for a delay of $2\cdot10^{-2}$ s; and finally 4) $\widehat{\Delta G}$ estimated via the covariance of the model error $\Phi_{\Delta G}$ due to a delay uniformly distributed from

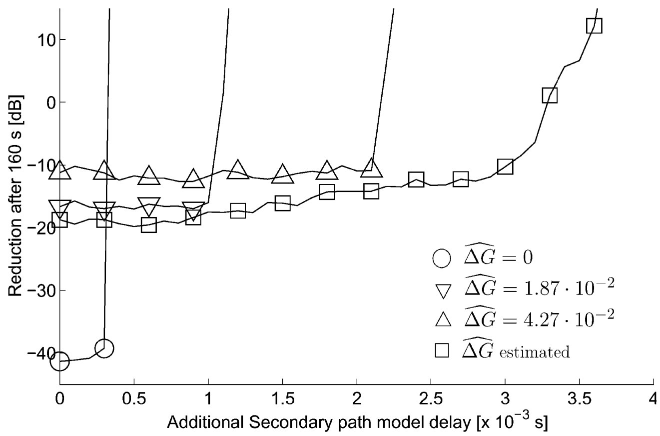

Fig. 2. Reduction of RPFxLMS after 160 s obtained for various choices of $\widehat{\Delta G}$, versus additional delay in $\widehat{G}$.

0 to $2 \cdot 10^{-2}$ s. In all experiments, the normalized step size is chosen to be 0.1, the number of filter coefficients $I = 400$ and the measurement noise is absent ($\sigma_v^2 = 0$).

Fig. 2 shows the reduction after 160 s (if the algorithm converges it is usually converged after $\approx$ 30 s, but 160 s has been chosen to fully guarantee the algorithm is converged). The nominal case (marked with $\bigcirc$) yields best performance between 0–$0.3 \cdot 10^{-3}$ s, however the adaptive algorithm diverges for larger delays. Using scalar regularization (marked with $\bigtriangledown$ and $\bigtriangleup$), the robustness can be improved, but at the expense of significant performance. By estimating the uncertainty model $\widehat{\Delta G}$ via the covariance $\Phi_{\Delta G}$ with delay uniformly distributed between 0 and $2 \cdot 10^{-3}$ s (marked with $\square$), the robustness of the update algorithm is increased significantly without paying too much performance.

V. Conclusion

The robustness of the preconditioned FxLMS algorithm, proposed by Elliott *et al.* is increased by following a probabilistic robust filtering method. The SPR condition is relaxed by taking the model uncertainty in the secondary path model explicitly into account. Furthermore, the gain of the precondition filter is reduced, which may prevent oversteering problems.

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