Will It Blow Up?

Existence of Global Solutions of Weighted Nonlocal Reaction-Diffusion Problems

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by

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Layman summary

In this thesis we study the existence of solutions for a particular type of partial differential equation. The equations considered in this thesis are useful in various parts of biology and are called weighted nonlocal reaction-diffusion equations. These equations allow us to simulate, for example, the way a population is spread throughout a territory and how this spread changes over time, but are also useful for models in chemistry, evolution and much more. The equation manages this through a reaction term and a diffusion term, the latter describes how the population naturally spreads out, while the former term describes the birth/death rate of the population. The terms 'weighted' and 'nonlocal' refer to certain properties our reaction term has. 'Nonlocal' means that the reaction term does not depend on the population at a single location in our territory but on the population of the entire territory. 'Weighted' means we put more importance on some parts of the territory than others.

With this interpretation in mind, we will attempt to give conditions for the reaction part of this equation that ensure that the population does not become infinitely large at some point in time. We accomplish this by showing that the equation has a solution and that, under certain conditions, this solution always remains finite.

Summary

In this thesis we study the existence of global classical solutions to a weighted nonlocal reaction-diffusion equation on the whole space $\mathbb{R}^m$. In this equation we consider the evolution as a result of diffusion, nonlinear local growth and a weighted nonlocal inhibition term that considers the current state of the entire domain. These equations are particularly useful for models in biology, chemistry and evolution theory.

The main objective is to extend an existing global existence result for an unweighted nonlocal reaction-diffusion equation to a weighted one for a positive bounded weight. Unlike the original result, showing only the boundedness of solutions, we show the entire process of proving the global existence of classical solutions. After introducing the necessary analytical tools and notations we go about proving the main result in five steps. Firstly we prove the existence of a unique mild solution using the well known Banach fixed-point method. Standard regularity results are then used to obtain a local classical solution. With a classical solution in hand we can study various properties of the solution. First, it is shown that non-negative initial data results in non-negative solutions by means of bounding the size of the negative part of the solution by 0. Next, uniform a priori estimates are derived in $L^k(\mathbb{R}^m)$ for $\beta \leq k \leq \infty$, where β comes from the nonlinearity inside the nonlocal inhibition term. These estimates hold under some restrictions concerning the exponents of the equation and the dimension of the x-domain. The most important aspect of these estimates is that they are independent of the local existence time. The proof for these estimates follows that of the unweighted nonlocal reaction-diffusion equation. Finally, a continuation argument is employed, using the a priori estimate and the fact it is independent of the local existence time, to show that every unique local classical solution of an equation satisfying the restrictions needed for the a priori estimate, can be extended to a unique global classical solution. This concludes the extension of the global existence of solutions to the weighted problem. As a last note, we also discuss the possibility of further study into the existence of global solution for less restricted weights.

1 Introduction

In this thesis, we investigate the existence of global solutions of a weighted nonlocal reaction-diffusion equation of the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = u^{\alpha}(x,t)\left(1 - \displaystyle\int_{\mathbb{R}^m} w(y)u^{\beta}(y,t)dy\right), & x \in \mathbb{R}^m, t > 0, \\ u(x,0) = \phi(x) \geq 0, & x \in \mathbb{R}^m, \end{cases} \tag{1.1}$$

where $\alpha, \beta \geq 1$ and the real function $w(y)$ is bounded above and below by

$$\sup_{y\in\mathbb{R}^m} \{w(y)\} = S < \infty \text{ and } \inf_{y\in\mathbb{R}^m} \{w(y)\} = I > 0 \tag{1.2}$$

respectively.

The equation models the evolution of a density $u(x,t)$, as the result of diffusion ($\Delta u(x,t)$), a local reaction term ($u^{\alpha}(x,t)$) and a weighted nonlocal reaction term ($u^{\alpha}(x,t)\int_{\mathbb{R}^m} w(y)u^{\beta}(y,t)dy$). The diffusion term models spatial spreading, while the reaction term models growth and decay. The nonlocal term adds feedback of the current state across the entire region to the equation, with the weight, $w(y)$, giving more or less importance to certain regions. The assumption that the weight function is bounded above and below by positive constants is needed to ensure the nonlocal term remains strictly inhibitory. If w is allowed to get arbitrarily close to 0 or even change sign, then the behavior of our equation changes completely and the techniques in this thesis no longer apply.

Nonlocal reaction-diffusion equations have many applications, particularly in biology. Examples include: modeling nonlocal consumption of resources in population dynamics models[1, 4]; breeding with different phenotypes[1, 5]; and modeling cancer[13]. Throughout this thesis we interpret the equation as a population model. In this setting $u(x,t)$ is the density of the population at the point $x \in \mathbb{R}^m$ and time t, where the x-space is the territory of the population. The simplest way to imagine the x-space would be as a two-dimensional surface, for instance $\mathbb{R}^2$. It is also possible to model a population in other domains, such as $\mathbb{R}^1$ or $\mathbb{R}^3$ if we consider, for instance, a river or the ocean respectively. Next the diffusion term, $\Delta u(x,t)$, would be the population spreading throughout the territory, the reaction term, $u^{\alpha}(x,t)\left(1 - \int_{\mathbb{R}^m} w(y)u^{\beta}(y,t)dy\right)$, is a combination of growth through reproduction given by $u^{\alpha}(x,t)$ and an inhibiting factor $u^{\alpha}(x,t)\int_{\mathbb{R}^m} w(y)u^{\beta}(y,t)dy$ representing the consumption of resources that are spread out according to $w(y)$. In this inhibiting term, the strength of using an integral is to show that the population does not only consume resources where they are but also throughout the entire territory.

Before moving on to the history section we note that certain writing conventions, simplifications, etc. are summarized at the start of Chapter 2.

1.1. History

There is a long history of people investigating the existence of local and global solutions to reaction-diffusion equations, with famous early contributions by H. Fujita[8] and F.B. Weissler[15] who focused on the "standard" reaction-diffusion model. Later a lot of research was done on various different nonlocal reaction-diffusion equations that better captured interactions found in biology and biomedical applications[1, 4, 5, 2, 13]. We provide an overview of important and related developments below.

1.1.1. Reaction-Diffusion equation and Fujita exponent

In 1966, Fujita [8] investigated the reaction-diffusion equation

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = u^{\alpha}(x,t), & x \in \mathbb{R}^m, t > 0, \\ u(x,0) = \phi(x) \geq 0, & x \in \mathbb{R}^m, \end{cases} \tag{1.3}$$

with $\alpha \geq 1$, $m = 1,2,\dots$ and discovered that there is a critical exponent for the reaction term of the equation, u^{α}, below which every solution of (1.3) will always blow up regardless of how small the initial data are. In this context blow-up means that

$$\|u(x,t)\|_{L^{\infty}(\mathbb{R}^m)} = \infty,$$

for some $t > 0$.

Fujita's paper contains two main results; the first result is that if $0 < \frac{m(\alpha-1)}{2} < 1$, then every non-negative solution of (1.3) blows up in finite time, regardless of the initial data (except for the trivial solution $u \equiv 0$); while the second result states that if $1 < \frac{m(\alpha-1)}{2}$, then there are non-negative initial data $\phi = \phi(x)$ which give global solutions. Both theorems require different methods to prove.

The first theorem is proven by contradiction using Lemma 2.1 of [8]. This lemma roughly states that if u is a solution of (1.3) then it must satisfy

$$J_0^{-(\alpha-1)} - u(t,0)^{-(\alpha-1)} \geq \alpha t, \quad t \in [0,T], \tag{1.4}$$

where, with $G(x,t)$ being the Green's function of the heat equation as in Definition 2.2.1, J_0 is defined as

$$J_0 = J_0(t) = \int_{\mathbb{R}^m} G(x,t)\phi(x)dx,$$

where $\phi(x)$ is the initial data. Then, it can be shown that in a neighborhood around the origin we must have

$$J_0 \geq Ct^{-\frac{m}{2}} \Rightarrow J_0^{-(\alpha-1)} \leq C^{-(\alpha-1)} t^{\frac{m(\alpha-1)}{2}}.$$

If we put this back into equation (1.4) and remove the term $u(t,0)$, which we assume to be positive, we find that

$$t^{\frac{m(\alpha-1)}{2}} \geq \alpha C^{\alpha-1} t.$$

It is easy to see that if $0 \leq \frac{m(\alpha-1)}{2} < 1$, this inequality will not hold for sufficiently large t giving us our contradiction. For this thesis we will not be using such methods as we only consider conditions for existence, not non-existence of local and global solutions.

The second theorem is proven in a similar way to what we will be doing later in this thesis. That is, constructing a solution using the integral equation associated with equation (1.3)

$$u(x,t) = e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta} u^{\alpha}(s)ds,$$

where the heat semi-group is as in Definition 2.2.1 and ϕ is the initial data. When using this integral equation as a mapping between function spaces, we can show that it is a contraction mapping when $\frac{m(\alpha-1)}{2} > 1$. This allows the use of the Banach fixed-point Theorem to show that we have a solution to the integral equation and, by extension, equation (1.3).

After this paper, research into sufficient conditions for the existence of global solutions of semi-linear heat equations received a lot of attention. The case $\frac{m(\alpha-1)}{2} = 1$, which was left open by Fujita, was proven to result in blow up in dimensions $m = 1,2$ by K. Hayakawa[9] followed by K. Kobayashi, T. Sirao and H. Tanaka's[11] proof for general $m \in \mathbb{N}$, even replacing $u^{\alpha}(x,t)$ with a nonlinear $f(u(x,t))$. Then in 1981 F.B. Weissler published his paper[15] on the same problem (1.3) but giving a considerably simplified proof for the case $\frac{m(\alpha-1)}{2} = 1$ by making use of L^p-norms.

1.1.2. Nonlocal reaction-diffusion equations

Studies on nonlocal reaction-diffusion equations began later, mostly after their usefulness in improving population dynamics models was proposed and studied by N.B. Britton[4], who was one of the first to study these problems, though their models were of the form

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = \mu u(x,t)\left(1 - \int_{\mathbb{R}^m} u(y,t)dy\right), \tag{1.5}$$

and did not feature a Fujita like exponents, instead only using exponent 1. However, it did show that these types of nonlocal reaction-diffusion equations can be very useful in biological models[4, 13]. If we follow the terminology of population dynamics, then, on the right hand side, $u(x,t)$ is a growth term that represents the reproduction rate, while $\int_{\mathbb{R}^m} u(y,t)dy$ is an inhibition term that represents competition. In equation (1.5), it is assumed that the reproduction rate scales linearly with population density u and the same for the term inside the integral, however this might not be the case in reality. A more realistic model might have the integral term like $\int_{\mathbb{R}^m} u^\beta(x,t)dx$ with $\beta \geq 1$ as a highly concentrated population over-exploits its resources. The reproduction term might also have superlinear growth like $u^\alpha(x,t)$ with $\alpha \geq 1$, where having a higher density gives the individuals of a species a higher chance of finding a mate.

This exact problem,

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = u^\alpha(x,t)\left(1 - \sigma \int_{\mathbb{R}^m} u^\beta(x,t)dx\right), & x \in \mathbb{R}^m, t > 0, \\ u(x,0) = \phi(x) \geq 0, & x \in \mathbb{R}^m, \end{cases} \tag{1.6}$$

was studied by Shen Bian and Li Chen in [2], where $\alpha, \beta \geq 1$ and $\sigma > 0$. Their paper focuses on getting an a priori estimate for the L^k-norm, for $\beta \leq k \leq \infty$, of the solution. They found conditions similar to those found by Fujita relating the exponents of (1.6) and the dimension of the x-domain to the existence of global solutions. However, the conditions they found were exactly the opposite of those for the regular reaction-diffusion equation (1.3). The main result of their paper states:

Assume ϕ is non-negative and $\phi \in L^\beta(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$, $m \geq 1$. If α satisfies

$$1 \leq \alpha < 1 + \left(1 - \frac{2}{p}\right)\beta,$$

where p is the exponent from the Sobolev embedding theorem

$$\begin{cases} p = \frac{2m}{m-2}, & m \geq 3, \\ 2 < p < \infty, & m = 2, \\ p = \infty, & m = 1, \end{cases} \tag{1.7}$$

then (1.6) has a bounded non-negative solution. Moreover, the following a priori estimate holds true. That is, for any $t > 0$ and $\beta \leq k \leq \infty$

$$\|u(t)\|_{L^k} \leq C\left(\|\phi\|_{L^\infty}, \|\phi\|_{L^\beta}\right).$$

If we consider the case $m \geq 3$ we see that the condition becomes $0 \leq \frac{m(\alpha-1)}{2\beta} < 1$, which is exactly the blow up condition of (1.3), save for the addition of β. This shows that the addition of the nonlocal effect has a distinct effect on the behavior of solutions.

We can now account for nonlinear growth and decay, but the nonlocal models we looked at so far all assume that the x-domain is perfectly uniform, meaning $w(y) = 1$ in (1.1). However, if we consider the population model again, where the x-domain represents the space in which a population lives and the nonlocal term, $\int_{\mathbb{R}^m} u(x,t)dx$, being the consumption of some resource like food, then the x-domain being perfectly uniform would imply the availability of this food is the same throughout the entire territory. A much more realistic scenario would be that this food is more concentrated in some regions and sparser in others. We can model this by adding a weight function to our nonlocal term, $\int_{\mathbb{R}^m} w(y)u(y,t)dy$.

There are many studies that look into this exact kind of problem. For example the problems N.B. Britton studied in [4] were of the form

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = \mu u(x,t)\left(1 - \int_{\mathbb{R}^m} w(x-y)u(y,t)dy\right).$$

V. Volpert[14] also studied similar problems as well as many other nonlocal reaction-diffusion problems. However, these studies mostly consider properties of the solutions instead of their existence.

1.2. Weighted Nonlocal reaction-diffusion Equation

What we are ultimately interested in this thesis is the existence of solutions in the presence of a combination of the previously mentioned weighted nonlocal term and the nonlinear growth and decay. This is precisely what we have in (1.1), and throughout this thesis we will give a full proof of the following theorem for the existence of global solutions to these problems.

Theorem 1.2.1. *Let ϕ be non-negative, $\phi \in L^{\beta}(\mathbb{R}^m) \cap L^{\infty}(\mathbb{R}^m)$, $w(y)$ bounded above and below by*

$$\sup_{y\in\mathbb{R}^m}\{w(y)\} = S < \infty \text{ and } \inf_{y\in\mathbb{R}^m}\{w(y)\} = I > 0$$

respectively and $m \in \{1,2\}$. If $\alpha, \beta \geq 1$ satisfy

$$1 \leq \alpha < 1 + \left(1 - \frac{2}{p}\right)\beta,$$

where p is the exponent from the Sobolev embedding theorem

$$\begin{cases} 2 < p < \infty, & m = 2, \\ p = \infty, & m = 1, \end{cases} \tag{1.8}$$

then problem (1.1) has a unique global classical solution and the following a priori estimates hold true. For all $t > 0$ and $\beta \leq k \leq \infty$

$$\|u(t)\|_{L^k} \leq C\left(\|\phi\|_{L^\infty}, \|\phi\|_{L^\beta}, I\right).$$

In the rest of this thesis, we go about proving this theorem. To this end, we start with preliminaries in Chapter 2, where we give an overview of writing conventions and simplifications used in this thesis as well as the necessary definitions we use throughout thesis. In this chapter we also introduce various theorems, lemmas and propositions that we will need in the proof. Then, in Chapter 3, we go into the proof of Theorem 1.2.1. Said proof consists of 5 parts; showing that (1.1) has a unique local mild solution; showing that (1.1) has a local classical solution; proving that the solution is non-negative; giving an a priori estimate for the L^k-norm of the local classical solution; using this a priori estimate to extend to a global classical solution.

2

Preliminary

Before we present the proof of the main result and auxiliary results, we start with a summary of conventions that will be used in this thesis followed by a collection of definitions, theorems, lemmas and propositions that will be used throughout this thesis. The definitions will be for equations like (1.1) with the same restrictions on α, β and $w(y)$.

2.1. Conventions

Certain notational conventions are particularly useful in this thesis and are hence used throughout the document. They are outlined below and thereafter used without further explanation.

- As we study problems on $\mathbb{R}^m$, we will use m exclusively to refer to the dimension of the x-domain and when considering the L^k-norm, we specifically consider the norm of $L^k(\mathbb{R}^m)$ but will write $\|\cdot\|_{L^k}$ for simplicity.
- We will use $\phi(x)$ exclusively to refer to initial data of the various problems, both local and nonlocal, discussed in this thesis. The problem ϕ refers to will always be clear from the context.
- I and S will be used to refer to the infimum and supremum of the weight function $w(y)$ as described in (1.2).

2.2. Definitions

Definition 2.2.1 (Heat semi-group $e^{t\Delta}$)**.** The heat semi-group $\left\{e^{t\Delta}(t)\right\}_{t\in[0,T]}$ is defined by

$$e^{t\Delta}\psi = \int_{\mathbb{R}^m} G(x,t)\psi(x)\,dx, \quad \text{for } \psi(x) \in C_c^\infty(\mathbb{R}^m) \tag{2.1}$$

where

$$G(x,t) = \frac{1}{(4\pi t)^{\frac{m}{2}}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^m, t \in [0,T], \tag{2.2}$$

is the Green's function, or fundamental solution, of the regular heat equation

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t).$$

Definition 2.2.2 (Local Classical Solution)**.** For a $T > 0$ we call $u : \mathbb{R}^m \times [0,T] \to \mathbb{R}$ a **local classical solution** of (1.1) if $u \in C^{2,1}(\mathbb{R}^m \times (0,T))$, $u(x,0) = \phi(x)$ for all $x \in \mathbb{R}^m$ and u satisfies the differential equation (1.1) for all $t \in (0,T) \times \mathbb{R}^m$.

Definition 2.2.3 (Local Mild L^k-Solution)**.** For a $T > 0$ we call $u \in C\left([0,T]; L^k(\mathbb{R}^m)\right)$ a **local mild L^k-solution** with initial data $\phi \in L^k(\mathbb{R}^m)$ if $u(t) \in L^k(\mathbb{R}^m)$, $t \in [0,T]$, if

$$e^{t\Delta}\phi < \infty \text{ and } \int_0^t e^{(t-s)\Delta}u^\alpha(s)\left(1 - \int_{\mathbb{R}^m} w(y)u^\beta(y,s)\right)ds < \infty, \quad \text{for } t \in [0,T]. \tag{2.3}$$

and if

$$u(t) := e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta} u^\alpha(s)\left(1 - \int_{\mathbb{R}^m} w(y) u^\beta(y,s)\,dy\right) ds,\ t \in [0,T]. \tag{2.4}$$

Unlike the classical solution, we do not require the existence of continuous derivatives for mild solutions, this means classical solutions are stronger than mild solutions, and naturally any classical solution is also a mild solution but not every mild solution is a classical solution. It is usually easier to prove the existence of mild solutions, however, the lack of derivatives means that the function may not be very smooth. Still, mild solutions can be useful in proving the existence and uniqueness of solutions. Another solution concept often used is that of so-called weak solutions, which in some sense is even more general than a mild solution. However, we do not consider weak solutions as the methods we will be using naturally correspond to the mild solution concept.

Definition 2.2.4 (Global Mild L^k-Solution)**.** We call $u \in C\big([0,\infty); L^k(\mathbb{R}^m)\big)$ a **global mild L^k-solution** with initial data $\phi \in L^k(\mathbb{R}^m)$ if for all $T > 0$ we have (2.3) and the integral equation (2.4) holds for $t \in [0,T]$.

Definition 2.2.5 (Global Classical Solution)**.** We call $u : \mathbb{R}^m \times [0,\infty) \to \mathbb{R}$ a **global classical solution** of (1.1) if, for all $T > 0$, u is a local classical solution of (1.1) on $[0,T]$ with initial condition ϕ.

2.3. Fundamental Results

Lemma 2.3.1 (Hölder's inequality[7])**.** *Let $p,q \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all functions $f \in L^p(\mathbb{R}^m)$ and $g \in L^q(\mathbb{R}^m)$, we have $fg \in L^1(\mathbb{R}^m)$ and*

$$\|fg\|_{L^1} \le \|f\|_{L^p}\|g\|_{L^q}. \tag{2.5}$$

A direct result of this is the interpolation inequality. For

$$1 \le q \le r \le p \le \infty \text{ and } \theta = \frac{p(q-r)}{r(q-p)} \in (0,1)$$

we have

$$\|f\|_{L^r} \le \|f\|_{L^p}^{\theta}\|f\|_{L^q}^{1-\theta}. \tag{2.6}$$

Lemma 2.3.2 (Minkowski's inequality[7])**.** *If $1 \le p < \infty$ and $f,g \in L^p(\mathbb{R}^m)$, then*

$$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}. \tag{2.7}$$

Lemma 2.3.3 (Young's inequality)**.** *If $a,b \ge 0$ and if $p,q > 1$ such that*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.8}$$

A useful property of Young's inequality is that we can rewrite it to an inequality with a free parameter ε using

$$A = \varepsilon^{\frac{1}{p}} a,\ \ B = \varepsilon^{-\frac{1}{p}} b.$$

It is clear that $AB = ab$ but we can rewrite our equation into

$$ab = AB \le \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{-\frac{q}{p}} b^q}{q} \le \varepsilon a^p + \varepsilon^{-\frac{q}{p}} b^q. \tag{2.9}$$

Lemma 2.3.4. *[2] Let $m \in \{1,2\}$. Let p as in (1.8), $1 \le r < q < p$ and $\frac{q}{r} < \frac{2}{r} + 1 - \frac{2}{p}$, then for $v \in H^1(\mathbb{R}^m)$ and $v \in L^r(\mathbb{R}^m)$, it holds*

$$\|v\|_{L^q}^q \le C(m)\left(C_0^{-\frac{\lambda q}{2-\lambda q}} + C_1^{-\frac{\lambda q}{2-\lambda q}}\right)\|v\|_{L^r}^{\gamma} + C_0\|\nabla v\|_{L^2}^2 + C_1\|v\|_{L^2}^2. \tag{2.10}$$

Here $C(m)$ are constants depending on m. C_0, C_1 are arbitrary positive constants and

$$\lambda = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{r} - \frac{1}{p}},\ \ \gamma = \frac{2(1-\lambda)q}{2-\lambda q}.$$

Lemma 2.3.5 (Mean value theorem)**.** *Let $f(x):[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), where $a<b$. Then there exists $c\in(a,b)$ such that*

$$f'(c)=\frac{f(b)-f(a)}{b-a}. \tag{2.11}$$

If we enter a function into f instead of a point then we can use the mean value theorem pointwise to obtain the following: Let $F:\mathbb{R}\to\mathbb{R}$ be in $C^1(\mathbb{R})$ and $u(t),v(t):[0,T)\to\mathbb{R}$. Then for each $t\in[0,T)$ such that $u(t)\neq v(t)$ (without loss of generality assume $u(t)>v(t)$), there exists $\omega\in(u(t),v(t))$ such that

$$|F'(\omega)|=\frac{|F(u(t))-F(v(t))|}{|u(t)-v(t)|} \tag{2.12}$$

Specifically, if $F(u(t))=u^{\alpha}(t)$, then

$$\begin{aligned}|F(u(t))-F(v(t))| &\le \max\{|F'(u(t))|,|F'(v(t))|\}|u(t)-v(t)|\\ &\le \left(|F'(u(t))|+|F'(v(t))|\right)|u(t)-v(t)|\end{aligned} \tag{2.13}$$

Proof. The proof for functions follows directly from applying the mean value theorem pointwise. For $F(u(t))=u^{\alpha}(t)$ we use the fact that for $\omega\in(u(t),v(t))$ we have $F'(\omega)\le\max\{F'(u(t)),F'(v(t))\}$. □

Lemma 2.3.6 (Grönwall's inequality)**.** *Let $f,g:[0,T)\to\mathbb{R}$ be continuous functions, let f be differentiable on $(0,T)$ and satisfy*

$$f'(t)\le g(t)f(t), \textit{for } t\in(0,T).$$

Then f is bounded by the following

$$f(t)\le f(0)e^{\int_0^t g(s)ds},\quad t\in(0,T). \tag{2.14}$$

There is a specific case of Grönwall's inequality that will be very useful for us.

Proposition 1. *Let $f(t)\in C^1(0,T)$ and let a constant $K_1,K_2>0$. If f satisfies*

$$\frac{d}{dt}f(t)+K_1f(t)\le K_2,$$

then f is bounded by

$$f(t)\le f(0)e^{-K_1t}+\frac{K_2}{K_1},\quad t\in(0,T). \tag{2.15}$$

Proof. We multiply the inequality by e^{K_1s} to obtain

$$\frac{d}{ds}f(s)e^{K_1s}+K_1f(s)e^{K_1s}\le K_2e^{K_1s}.$$

Note that $\frac{d}{ds}f(s)e^{K_1s}+K_1f(s)e^{K_1s}=\frac{d}{ds}\left(f(s)e^{K_1s}\right)$ giving us

$$\frac{d}{ds}\left(f(s)e^{K_1s}\right)\le K_2e^{K_1s}.$$

If we now integrate over $(0,t)$ with $t\in(0,T)$, we get

$$\begin{aligned}f(t)e^{K_1t}-f(0)&\le\frac{K_2}{K_1}e^{K_1t}-\frac{K_2}{K_1},\\ f(t)e^{K_1t}&\le f(0)+\frac{K_2}{K_1}e^{K_1t}-\frac{K_2}{K_1},\\ f(t)&\le f(0)e^{-K_1t}+\frac{K_2}{K_1}-\frac{K_2}{K_1}e^{-K_1t}.\end{aligned}$$

Since $K_1,K_2>0$ we can remove $-\frac{K_2}{K_1}e^{-K_1t}$ from the right hand side giving us the desired result. □

Proposition 2. *[15] Let* $1 \le k \le \infty$ *and* $f : [0,T) \to \mathbb{R}$. *Then*

$$\|e^{t\Delta} f(t)\|_{L^k} \le \|f(t)\|_{L^k}, \quad \text{for } t \in [0,T). \tag{2.16}$$

Proposition 3. *Let* $f, g \in L^\infty(\mathbb{R}^m)$, $f(x) \ge 0$ *for all* $x \in \mathbb{R}^m$ *and* $\alpha \in \mathbb{R}$. *Then*

1. $\|f^\alpha\|_{L^\infty} = \|f\|_{L^\infty}^\alpha$,
2. $\|fg\|_{L^\infty} = \|f\|_{L^\infty} \cdot \|g\|_{L^\infty}$.

Proof.

1. $\|f^\alpha\|_{L^\infty} = \text{ess sup}_{x\in\mathbb{R}^m} |f^\alpha(x)| = \text{ess sup}_{x\in\mathbb{R}^m} |f(x)|^\alpha = \|f\|_{L^\infty}^\alpha$,
2. $\|fg\|_{L^\infty} = \text{ess sup}_{x\in\mathbb{R}^m} |f(x)g(x)| = \text{ess sup}_{x\in\mathbb{R}^m} \big(|f(x)| \cdot |g(x)|\big) = \|f\|_{L^\infty} \cdot \|g\|_{L^\infty}$.

□

In Section 3.1 we use the spaces $L^k(\mathbb{R}^m)$ and $L^\infty(\mathbb{R}^m) \cap L^k(\mathbb{R}^m)$ and require them to have certain properties in order to use a theorem that gives us the existence and uniqueness of a solution. These properties and the relevant theorem are stated below.

Proposition 4. *For* $1 \le k \le \infty$, *both* $L^k(\mathbb{R}^m)$ *and* $L^\infty(\mathbb{R}^m) \cap L^k(\mathbb{R}^m)$, *with norms* $\|\cdot\|_{L^k}$ *and* $\|\cdot\|_{L^\infty} + \|\cdot\|_{L^k}$ *respectively, are Banach spaces and therefore also complete metric spaces[7, 10].*

Theorem 2.3.7 (Banach fixed-point Theorem, [12])**.** *Let* (X,d) *be a non-empty complete metric space with a contraction mapping* $\Psi : X \to X$. *Then* Ψ *has a unique fixed point* $x' \in X$, $\Psi(x') = x'$.

3

Weighted Nonlocal reaction-diffusion Equation

As stated in the introduction, we will be studying the weighted nonlocal reaction-diffusion equation (1.1) with $\alpha, \beta \geq 1$ and $w(y)$ bounded above and below as in (1.2). This may seem very restrictive on $w(y)$, but this is necessary to ensure the nonlocal term remains strictly greater than 0 since this term getting arbitrarily close to 0 breaks down many of the arguments in this thesis. Moreover, if it becomes 0 or negative the conditions for global existence change drastically.

For Sections 3.1-3.3 we will also use a more general form of equation (1.1) that allows u to be negative,

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = \operatorname{sign}(u(x,t))|u(x,t)|^{\alpha}\left(1 - \int_{\mathbb{R}^m} w(y)|u(y,t)|^{\beta} dy\right), & x \in \mathbb{R}^m, t \in [0,T], \\ u(x,0) = \phi(x) \geq 0, & x \in \mathbb{R}^m, \end{cases} \tag{3.1}$$

for some $T > 0$. This ensures that these terms make sense even when u is negative. In Section 3.3 we will prove that u is non-negative and that these definitions aren't needed.

To prove Theorem 1.2.1 we need to show five things, each of which we prove in its own section. In Section 3.1 we prove the existence of a local mild solution. We do this by showing that the formula obtained by Duhamel's principle (2.4), when considered as a mapping between appropriate function spaces is a contraction mapping. This then allows us to use the Banach fixed-point Theorem 2.3.7 to show that it has a unique fixed point that is a local mild solution of (3.1). Then in Section 3.2, we show that this local mild solution can be strengthened to a local classical solution which we outsource to the book [12]. We need this local classical solution in order to get an a priori estimate. But before this estimate we will prove that u is non-negative in Section 3.3. Then we give the a priori estimate in Section 3.4, where we show that under the conditions of Theorem 1.2.1, the solution is in $L^k(\mathbb{R}^m)$ for all $\beta \leq k \leq \infty$. Finally in Section 3.5 we use the local classical solution together with the a priori estimate to extend to a global solution.

3.1. Local mild solution

Consider equation (1.1) with initial data $\phi, \alpha, \beta \geq 0$ and $m \in \{1,2\}$. Let $X := L^{\infty}(\mathbb{R}^m) \cap L^{\beta}(\mathbb{R}^m)$ and $X_T := L^{\infty}([0,T]; X)$ with the norms $\|\cdot\|_X := \|\cdot\|_{L^\infty} + \|\cdot\|_{L^\beta}$ and $\|\cdot\|_{X_T} := \operatorname{ess\,sup}_{t\in[0,T]}\left(\|\cdot\|_{L^\infty} + \|\cdot\|_{L^\beta}\right)$. Consider the ball $B_{T,M} := \{u \in X_T : \|u\|_{X_T} \leq M\}$, where $M = \|\phi\|_X + 1$. Consider the mapping

$$\Psi(u) := \begin{cases} B_{T,M} & \to B_{T,M} \\ u & \mapsto e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta} \operatorname{sign}(u(s))|u(s)|^{\alpha}\left(1 - \int_{\mathbb{R}^m} w(y)|u(y,s)|^{\beta} dy\right) ds. \end{cases} \tag{3.2}$$

Since the X_T-norm is based on the essential suprema of the L^∞- and L^β-norms, we will show that for $u, v \in B_{T,M}$, there exists $T > 0$ such that, for all $t \in [0,T]$ the following four propositions, with the above definitions in mind, hold:

Proposition 5. *Fix* $0 \leq \varepsilon < \frac{1}{2}$ *and let*

$$T \cdot M^{\alpha}\left(1 + SM^{\beta}\right) < \varepsilon.$$

Then $\operatorname{ess\,sup}_{t\in[0,T]} \|\Psi(u(t))\|_{L^\infty} \leq \|\phi\|_{L^\infty} + \varepsilon$.

Proposition 6. *Fix $0 \le \varepsilon < \frac{1}{2}$ and let*

$$T \cdot \left(2\alpha M^{\alpha-1} + 2\alpha S M^{\beta+\alpha-1} + 2\beta S M^{\beta-1}\right) < \varepsilon.$$

Then $\operatorname{ess\,sup}_{t\in[0,T]} \|\Psi(u(t)) - \Psi(v(t))\|_{L^\infty} < \varepsilon \|u - v\|_{X_T}$.

Proposition 7. *Fix $0 \le \varepsilon < \frac{1}{2}$ and let*

$$T \cdot M^\alpha \left(1 + S M^\beta\right) < \varepsilon.$$

Then $\operatorname{ess\,sup}_{t\in[0,T]} \|\Psi(u(t))\|_{L^\beta} \le \|\phi\|_{L^\beta} + \varepsilon$.

Proposition 8. *Fix $0 \le \varepsilon < \frac{1}{2}$ and let*

$$T \cdot \left(2\alpha M^{\alpha-1} + 2\alpha S M^{\beta+\alpha-1} + 2\beta S M^{\beta-1}\right) < \varepsilon.$$

Then $\operatorname{ess\,sup}_{t\in[0,T]} \|\Psi(u(t)) - \Psi(v(t))\|_{L^\beta} < \varepsilon \|u - v\|_{X_T}$.

We will first prove these four propositions hold by giving bounds on the L^β- and L^∞-norms and then taking the essential supremum. If all four of the above propositions hold for the same ε, then we can use the following lemma to obtain a unique local mild solution.

Lemma 3.1.1. *Fix $0 \le \varepsilon < \frac{1}{2}$ and consider equation* (3.1) *with initial data $\phi \in X$. Let $M = \|\phi\|_X + 1$ and T satisfy the conditions set in propositions 5-8, that is,*

$$T \cdot \max\left\{M^\alpha \left(1 + S M^\beta\right), \left(2\alpha M^{\alpha-1} + 2\alpha S M^{\beta+\alpha-1} + 2\beta S M^{\beta-1}\right)\right\} < \varepsilon, \tag{3.3}$$

then

$$\Psi(B_{T,M}) \subset B_{T,M} \quad \text{and} \quad \|\Psi(u) - \Psi(v)\|_{X_T} < \varepsilon \|u - v\|_{X_T}, \quad u, v \in B_{T,M}.$$

Consequently, Ψ is a contraction mapping on the complete metric space $B_{T,M}$. Hence, by the Banach fixed-point Theorem 2.3.7, Ψ possesses a unique fixed point $u \in B_{T,M}$, that is the unique local mild solution of (3.1) *on* $[0, T]$.

We will first prove Propositions 5-8. Throughout these proofs we assume that $t \in [0, T]$ for T as in (3.3) and $s \in [0, t]$. We will not be referencing this at the end of equations containing t or s.

3.1.1. $\operatorname{ess\,sup}_{t\in[0,T]} \|\Psi(u(t))\|_{L^\infty} \le \|\phi\|_{L^\infty} + \varepsilon$

Proof of Proposition 5. Let $u \in X_T$, then we use Proposition 2 and the triangle inequality to remove the heat semi-group and split the terms, we also move the norm inside the integral (which allows us to remove $\operatorname{sign}(u(s))$) giving us

$$\begin{aligned}\|\Psi(u(t))\|_{L^\infty} &= \left\| e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta} \operatorname{sign}(u(s))|u(s)|^\alpha \left(1 - \int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy\right) ds \right\|_{L^\infty} \\ &\le \|\phi\|_{L^\infty} + \int_0^t \left\| |u(s)|^\alpha \left(1 - \int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy\right)\right\|_{L^\infty} ds.\end{aligned}$$

Now we use the second property of Proposition 3 to get

$$\|\Psi(u(t))\|_{L^\infty} \le \|\phi\|_{L^\infty} + \int_0^t \|u^\alpha(s)\|_{L^\infty} \left\|1 - \int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy\right\|_{L^\infty} ds.$$

Then we use the triangle inequality on $1 - \int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy$ together with the fact that $w(y)$ is bounded by $S = \sup_{y\in\mathbb{R}^m}\{w(y)\} < \infty$, giving us

$$\|\Psi(u(t))\|_{L^\infty} \le \|\phi\|_{L^\infty} + \int_0^t \|u^\alpha(s)\|_{L^\infty} \left\|1 + S\int_{\mathbb{R}^m} |u(y,s)|^\beta dy\right\|_{L^\infty} ds.$$

Lastly, we use the first property of Proposition 3 to get α outside the norm and the fact that $\|u(s)\|_{L^\beta}, \|u(s)\|_{L^\infty} \le M$ for all $s \in [0, T]$ to get

$$\|\Psi(u(t))\|_{L^\infty} \le \|\phi\|_{L^\infty} + tM^\alpha \left(1 + S M^\beta\right),$$

taking the essential supremum on $[0,T]$ then gives us

$$\operatorname{ess\,sup}_{t\in[0,T]} \|\Psi(u(t))\|_{L^\infty} \le \|\phi\|_{L^\infty} + TM^\alpha\left(1+SM^\beta\right).$$

This justifies the first part of the bound for T in (3.3), namely

$$T\cdot M^\alpha\left(1+SM^\beta\right) < \varepsilon. \tag{3.4}$$

□

3.1.2. $\operatorname{ess\,sup}_{t\in[0,T]} \|\Psi(u(t))-\Psi(v(t))\|_{L^\infty} < \varepsilon\|u-v\|_{X_T}$

Proof of Proposition 6. Let $u,v\in X_T$. Then we use Proposition 2 again to remove the heat semi-group and move the norm inside the integral. Next we use the triangle inequality to split the terms into those with an integral and those without

$$\begin{aligned}
\|\Psi(u(t))-\Psi(v(t))\|_{L^\infty} &= \Bigg\|\int_0^t e^{(t-s)\Delta}\operatorname{sign}(u(s))|u(s)|^\alpha\left(1-\int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy\right)\\
&\quad - e^{(t-s)\Delta}\operatorname{sign}(v(s))|v(s)|^\alpha\left(1-\int_{\mathbb{R}^m} w(y)|v(y,s)|^\beta dy\right)ds\Bigg\|_{L^\infty}\\
&\le \int_0^t \|\operatorname{sign}(u(s))|u(s)|^\alpha - \operatorname{sign}(v(s))|v(s)|^\alpha\|_{L^\infty}\\
&\quad + \Bigg\|\operatorname{sign}(u(s))|u(s)|^\alpha\int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy\\
&\quad - \operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m} w(y)|v(y,s)|^\beta dy\Bigg\|_{L^\infty} ds.
\end{aligned}$$

We consider these two terms separately, starting with $\|\operatorname{sign}(u(s))|u(s)|^\alpha - \operatorname{sign}(v(s))|v(s)|^\alpha\|_{L^\infty}$. Using Lemma 2.3.5 and the fact that $F(x)=\operatorname{sign}(x)|x|^\alpha \Rightarrow F'(x)=\alpha|x|^{\alpha-1}$ we find

$$\begin{aligned}
|\operatorname{sign}(u(s))|u(s)|^\alpha - \operatorname{sign}(v(s))|v(s)|^\alpha| &\le \alpha|u(s)-v(s)|\cdot\max\{|u(s)|^{\alpha-1},|v(s)|^{\alpha-1}\}\\
&\le \alpha|u(s)-v(s)|\cdot\left(|u(s)|^{\alpha-1}+|v(s)|^{\alpha-1}\right).
\end{aligned}$$

Taking the L^∞-norm and using Proposition 3, we get

$$\begin{aligned}
\|\operatorname{sign}(u(s))|u(s)|^\alpha - \operatorname{sign}(v(s))|v(s)|^\alpha\|_{L^\infty} &\le \alpha\|u(s)-v(s)\|_{L^\infty}\left(\||u(s)|^{\alpha-1}\|_{L^\infty}+\||v(s)|^{\alpha-1}\|_{L^\infty}\right)\\
&\le \alpha\|u(s)-v(s)\|_{L^\infty}\left(\|u(s)\|_{L^\infty}^{\alpha-1}+\|v(s)\|_{L^\infty}^{\alpha-1}\right) \qquad (3.5)\\
&\le 2\alpha M^{\alpha-1}\|u(s)-v(s)\|_{L^\infty}.
\end{aligned}$$

For the term $\left\|\operatorname{sign}(u(s))|u(s)|^\alpha\int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy - \operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m} w(y)|v(y,s)|^\beta dy\right\|_{L^\infty}$, we will add 0 such that we can use the triangle inequality in a convenient way

$$\begin{aligned}
&\left\|\operatorname{sign}(u(s))|u(s)|^\alpha\int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy - \operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m} w(y)|v(y,s)|^\beta dy\right\|_{L^\infty}\\
&= \Bigg\|\operatorname{sign}(u(s))|u(s)|^\alpha\int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy + \operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy\\
&\quad - \operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy - \operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m} w(y)|v(y,s)|^\beta dy\Bigg\|_{L^\infty}\\
&\le \left\|\left(\operatorname{sign}(u(s))|u|^\alpha - \operatorname{sign}(v(s))|v|^\alpha\right)\int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy\right\|_{L^\infty}\\
&\quad + \left\|\operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m} w(y)\left(|u(y,s)|^\beta-|v(y,s)|^\beta\right)dy\right\|_{L^\infty}.
\end{aligned}$$

The first term is the same as equation (3.5) but multiplied by

$$\int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy \le S\|u(s)\|_{L^\beta}^\beta \le SM^\beta,$$

so

$$\left\| \left(\operatorname{sign}(u(s))|u(s)|^{\alpha} - \operatorname{sign}(v(s))|v(s)|^{\alpha}\right) \int_{\mathbb{R}^m} w(y)|u(y,s)|^{\beta} dy \right\|_{L^\infty} \leq 2\alpha S M^{\beta+\alpha-1} \|u(s) - v(s)\|_{L^\infty}. \tag{3.6}$$

For the right term we can take the constant $\|\operatorname{sign}(v(s))|v(s)|^{\alpha}\|_{L^\infty}$ out and remove the sign function, use the supremum of $w(y)$ again and use Lemma 2.3.5 inside the integral

$$\begin{aligned}
&\left\| \operatorname{sign}(v(s))|v(s)|^{\alpha} \int_{\mathbb{R}^m} w(y)\left(|u(y,s)|^{\beta} - |v(y,s)|^{\beta}\right) dy \right\|_{L^\infty} \\
&\leq \|v(s)\|_{L^\infty}^{\alpha} \cdot S \int_{\mathbb{R}^m} \beta |u(y,s) - v(y,s)| \max\left\{\operatorname{sign}(u(y,s))|u(y,s)|^{\beta-1}, \operatorname{sign}(v(y,s))|v(y,s)|^{\beta-1}\right\} dy \\
&\leq \|v(s)\|_{L^\infty}^{\alpha} \cdot S \int_{\mathbb{R}^m} \beta |u(y,s) - v(y,s)| \left(|u(y,s)|^{\beta-1} + |v(y,s)|^{\beta-1}\right) dy.
\end{aligned}$$

Now we can use Hölder's inequality, Lemma 2.3.1, with $f = |u(x,s) - v(x,s)|$, $|g = (|u(x,s)|^{\beta-1} + |v(x,s)|^{\beta-1})|$, $p = \beta$ and $q = \frac{\beta}{\beta-1}$. This gives us

$$\begin{aligned}
\int_{\mathbb{R}^m} \beta |u(y,s) - v(y,s)| \left(|u(y,s)|^{\beta-1} + |v(y,s)|^{\beta-1}\right) dy &\leq \beta \|u(s) - v(s)\|_{L^\beta} \left(\|u^{\beta-1}(s)\|_{L^{\frac{\beta}{\beta-1}}} + \|v^{\beta-1}(s)\|_{L^{\frac{\beta}{\beta-1}}} \right) \\
&\leq \beta \|u(s) - v(s)\|_{L^\beta} \left(\|u(s)\|_{L^\beta}^{\beta-1} + \|v(s)\|_{L^\beta}^{\beta-1} \right) \\
&\leq 2\beta M^{\beta-1} \|u(s) - v(s)\|_{L^\beta}.
\end{aligned} \tag{3.7}$$

Combining the terms from (3.5),(3.6) and (3.7) we get

$$\begin{aligned}
\|\Psi(u(t)) - \Psi(v(t))\|_{L^\infty} &= \left\| \int_0^t e^{(t-s)\Delta} \operatorname{sign}(u(s))|u(s)|^{\alpha} \left(1 - \int_{\mathbb{R}^m} w(y)|u(y,s)|^{\beta} dy\right) \right. \\
&\qquad \left. - e^{(t-s)\Delta} \operatorname{sign}(v(s))|v(s)|^{\alpha} \left(1 - \int_{\mathbb{R}^m} w(y)|v(y,s)|^{\beta} dy\right) ds \right\|_{L^\infty} \\
&\leq t \left(\|u(t) - v(t)\|_{L^\infty} \left(2\alpha M^{\alpha-1} + 2\alpha S M^{\beta+\alpha-1}\right) + 2\beta S M^{\beta-1} \|u(t) - v(t)\|_{L^\beta} \right) \\
&\leq t \left(\|u(t) - v(t)\|_{L^\beta} + \|u(t) - v(t)\|_{L^\infty} \right) \left(2\alpha M^{\alpha-1} + 2\alpha S M^{\beta+\alpha-1} + 2\beta S M^{\beta-1}\right).
\end{aligned} \tag{3.8}$$

Taking the essential supremum then gives us

$$\operatorname{ess\,sup}_{t\in[0,T]} \|\Psi(u(t)) - \Psi(v(t))\|_{L^\infty} \leq T \|u - v\|_{X_T} \left(2\alpha M^{\alpha-1} + 2\alpha S M^{\beta+\alpha-1} + 2\beta S M^{\beta-1}\right).$$

We see that both $\|u(t) - v(t)\|_{L^\infty}$ and $\|u(t) - v(t)\|_{L^\beta}$ appear in the second step of (3.8). This is why our X_T-norm is defined as the essential supremum of the sum of the L^∞- and L^β-norm. We also see that the second term of the bound on T in (3.3), is justified since we need

$$T\left(2\alpha M^{\alpha-1} + 2\alpha S M^{\beta+\alpha-1} + 2\beta S M^{\beta-1}\right) < \varepsilon < 1, \tag{3.9}$$

for the proposition to be true. □

Next we look at the bounds of the L^β-norm, which are very similar but require the use of Lemma 2.3.1 to get the powers out of the norm.

3.1.3. $\operatorname{ess\,sup}_{t\in[0,T]} \|\Psi(u(t))\|_{L^\beta} \leq \|\phi\|_{L^\beta} + \varepsilon$

Proof. We start the same as in Subsection 3.1.1 using Proposition 2 and the triangle inequality to remove the heat semi-group and split the terms, then move the L^β-norm inside the integral

$$\begin{aligned}
\|\Psi(u(t))\|_{L^\beta} &= \left\| e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta} \operatorname{sign}(u(s))|u(s)|^{\alpha} \left(1 - \int_{\mathbb{R}^m} w(y)|u(y,s)|^{\beta} dy\right) ds \right\|_{L^\beta} \\
&\leq \|\phi\|_{L^\beta} + \int_0^t \left\| \operatorname{sign}(u(s)))|u(s)|^{\alpha} \left(1 - \int_{\mathbb{R}^m} w(y)|u(y,s)|^{\beta} dy\right) \right\|_{L^\beta} ds.
\end{aligned}$$

The term $\left(1-\int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy\right)$ is just a constant here allowing us to take it out of the norm. It is also simple to see that

$$1-\int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy \le 1+\int_{\mathbb{R}^m} S|u(y,s)|^\beta dy \le 1+S\|u(s)\|_{L^\beta}^\beta,$$

since $|u(y,s)|^\beta$ is non-negative. This combined with the bound $\|u(s)\|_{L^\beta} \le M$ gives us

$$\|\Psi(u(t))\|_{L^\beta} \le \|\phi\|_{L^\beta} + \int_0^t \||u(s)|^\alpha\|_{L^\beta}\left(1+SM^\beta\right) ds$$

Next we note

$$\||u(s)|^\alpha\|_{L^\beta} \le \|u(s)\|_{L^\beta}\|u(s)\|_{L^\infty}^{\alpha-1} \le M^\alpha. \tag{3.10}$$

Substitution back into our equation gives us

$$\begin{aligned}\|\Psi(u(t))\|_{L^\beta} &\le \|\phi\|_{L^\beta} + \int_0^t \|\operatorname{sign}(u(s))|u(s)|^\alpha\|_{L^\beta}\left(1+SM^\beta\right) ds\\ &\le \|\phi\|_{L^\beta} + tM^\alpha\left(1+SM^\beta\right),\end{aligned}$$

and taking the essential supremum we find,

$$\operatorname{ess\,sup}_{t\in[0,T]}\|\Psi(u(t))\|_{L^\beta} \le \|\phi\|_{X_T} + TM^\alpha\left(1+SM^\beta\right),$$

giving us the same bound for T as in Subsection 3.1.1. □

3.1.4. $\operatorname{ess\,sup}_{t\in[0,T]}\|\Psi(u(t))-\Psi(v(t))\|_{L^\beta} < \varepsilon\|u-v\|_{X_T}$

Proof. We start the same as with the L^∞-norm, removing the heat semi-group, splitting the terms, using the supremum of $w(y)$ and moving the norm inside the integral.

$$\begin{aligned}\|\Psi(u(t))-\Psi(v(t))\|_{L^\beta} &= \Bigg\|\int_0^t e^{(t-s)\Delta}\operatorname{sign}(u(s))|u(s)|^\alpha\left(1-\int_{\mathbb{R}^m} w(y)|u(y,s)|^\beta dy\right)\\ &\qquad - e^{(t-s)\Delta}\operatorname{sign}(v(s))|v(s)|^\alpha\left(1-\int_{\mathbb{R}^m} w(y)|v(y,s)|^\beta dy\right) ds\Bigg\|_{L^\beta}\\ &\le \int_0^t \|\operatorname{sign}(u(s))|u(s)|^\alpha-\operatorname{sign}(v(s))|v(s)|^\alpha\|_{L^\beta}\\ &\qquad + S\left\|\operatorname{sign}(u(s))|u(s)|^\alpha\int_{\mathbb{R}^m}|u(y,s)|^\beta dy-\operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m}|v(y,s)|^\beta dy\right\|_{L^\beta} ds\end{aligned}$$

We consider the two terms separately. Using the same method as in equation (3.5) for $\|\operatorname{sign}(u(s))|u(s)|^\alpha - \operatorname{sign}(v(s))|v(s)|^\alpha\|_{L^\beta}$ gives us

$$\|\operatorname{sign}(u(s))|u(s)|^\alpha-\operatorname{sign}(v(s))|v(s)|^\alpha\|_{L^\beta} \le \|\alpha|u(s)-v(s)|\cdot\left(|u(s)|^{\alpha-1}+|v(s)|^{\alpha-1}\right)\|_{L^\beta}.$$

We want to have just $\|u(s)-v(s)\|_{L^\beta}$ multiplied by some constant. In order to do this, we note that $|u(s)|^{\alpha-1} \le \||u(s)|^{\alpha-1}\|_{L^\infty} \le \|u(s)\|_{L^\infty}^{\alpha-1} \le M^{\alpha-1}$, which is just a constant and can thus be moved outside the L^β-norm giving us

$$\begin{aligned}\|\operatorname{sign}(u(s))|u(s)|^\alpha-\operatorname{sign}(v(s))|v(s)|^\alpha\|_{L^\beta} &\le \|\alpha|u(s)-v(s)|\cdot\left(|u(s)|^{\alpha-1}+|v(s)|^{\alpha-1}\right)\|_{L^\beta}\\ &\le 2\alpha M^{\alpha-1}\|u(s)-v(s)\|_{L^\beta}.\end{aligned} \tag{3.11}$$

For the other term, like with the L^∞-norm, we will add 0 so we can split it

$$\begin{aligned}&\left\|\operatorname{sign}(u(s))|u(s)|^\alpha\int_{\mathbb{R}^m}|u(y,s)|^\beta dy-\operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m}|v(y,s)|^\beta dy\right\|_{L^\beta}\\ &= \Bigg\|\operatorname{sign}(u(s))|u(s)|^\alpha\int_{\mathbb{R}^m}|u(y,s)|^\beta dy+\operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m}|u(y,s)|^\beta dy\\ &\quad -\operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m}|u(y,s)|^\beta dy-\operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m}|v(y,s)|^\beta dy\Bigg\|_{L^\beta}\\ &\le \left\|\left(\operatorname{sign}(u(s))|u(s)|^\alpha-\operatorname{sign}(v(s))|v(s)|^\alpha\right)\int_{\mathbb{R}^m}|u(y,s)|^\beta dy\right\|_{L^\beta}\\ &\quad + \left\|\operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m}|u(y,s)|^\beta-|v(y,s)|^\beta dy\right\|_{L^\beta}.\end{aligned}$$

The first term is just equation (3.11) multiplied by $\|u(s)\|_{L^\beta}^\beta$, so

$$\left\|\left(\operatorname{sign}(u(s))|u(s)|^\alpha - \operatorname{sign}(v(s))|v(s)|^\alpha\right)\int_{\mathbb{R}^m}|u(s)|^\beta dx\right\|_{L^\beta} \begin{aligned}&\le 2\alpha M^{\alpha-1}\|u(s)-v(s)\|_{L^\beta}\cdot\|u(s)\|_{L^\beta}^\beta \\ &\le 2\alpha M^{\beta+\alpha-1}\|u(s)-v(s)\|_{L^\beta}.\end{aligned} \tag{3.12}$$

The right term requires slightly more effort. The term $\int_{\mathbb{R}^m}|u(y,s)|^\beta - |v(y,s)|^\beta dy$ is just a constant and can be moved out of the norm

$$\left\|\operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m}|u(y,s)|^\beta - |v(y,s)|^\beta dy\right\|_{L^\beta} \le \||v(s)|^\alpha\|_{L^\beta}\left|\int_{\mathbb{R}^m}|u(y,s)|^\beta - |v(y,s)|^\beta dy\right|.$$

For $\||v(s)|^\alpha\|_{L^\beta}$ we recall equation (3.10) while we can get $\|u(s)-v(s)\|_{L^\beta}$ from $\int_{\mathbb{R}^m}|u(y,s)|^\beta - |v(y,s)|^\beta dy$ using a combination of Lemma 2.3.5 and Hölder's inequality with parameters

$$f = |u(y,s)-v(y,s)|, \quad g = \left(|u(y,s)|^{\beta-1}+|v(y,s)|^{\beta-1}\right), \quad p=\beta, \quad q=\frac{\beta}{\beta-1},$$

giving us

$$\begin{aligned}\left|\int_{\mathbb{R}^m}|u(y,s)|^\beta - |v(y,s)|^\beta dy\right| &\le \int_{\mathbb{R}^m}\beta|u(y,s)-v(y,s)|\left(|u(y,s)|^{\beta-1}+|v(y,s)|^{\beta-1}\right)dy \\ &\le \beta\|u(s)-v(s)\|_{L^\beta}\left\||u|^{\beta-1}+|v|^{\beta-1}\right\|_{L^{\frac{\beta}{\beta-1}}}.\end{aligned}$$

Then we use Minkowski's inequality, Lemma 2.3.2, to get

$$\left\||u|^{\beta-1}+|v|^{\beta-1}\right\|_{L^{\frac{\beta}{\beta-1}}} \le \||u|^{\beta-1}\|_{L^{\frac{\beta}{\beta-1}}} + \||v|^{\beta-1}\|_{L^{\frac{\beta}{\beta-1}}} \le \|u\|_{L^\beta}^{\beta-1}+\|v\|_{L^\beta}^{\beta-1} \le 2M^{\beta-1}$$

Putting them back together gives us

$$\left\|\operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m}|u(y,s)|^\beta - |v(y,s)|^\beta dy\right\|_{L^\beta} \le 2\beta M^{\beta+\alpha-1}\|u(s)-v(s)\|_{L^\beta}. \tag{3.13}$$

Now combining equations (3.11),(3.12) and (3.13) back together we find

$$\begin{aligned}\|\Psi(u(t))-\Psi(v(t))\|_{L^\beta} \le \int_0^t &\|\operatorname{sign}(u(s))|u(s)|^\alpha - \operatorname{sign}(v(s))|v(s)|^\alpha\|_{L^\beta} \\ &+ S\left\|\operatorname{sign}(u(s))|u(s)|^\alpha\int_{\mathbb{R}^m}|u(y,s)|^\beta dy - \operatorname{sign}(v(s))|v(s)|^\alpha\int_{\mathbb{R}^m}|v(y,s)|^\beta dy\right\|_{L^\beta} ds \\ &\le t\|u(t)-v(t)\|_{L^\beta}\left(2\alpha M^{\alpha-1}+2\alpha SM^{\beta+\alpha-1}+2\beta SM^{\beta-1}\right),\end{aligned} \tag{3.14}$$

and taking the essential supremum gives us,

$$\operatorname{ess\,sup}_{t\in[0,T]}\|\Psi(u(t))-\Psi(v(t))\|_{L^\beta} \le T\|u-v\|_{X_T}\left(2\alpha M^{\alpha-1}+2\alpha SM^{\beta+\alpha-1}+2\beta SM^{\beta-1}\right),$$

once again justifying the bound on T given by (3.3). □

Now we know that, using the bound $T\cdot\max\{M^\alpha\left(1+SM^\beta\right),\left(2\alpha M^{\alpha-1}+2\alpha SM^{\beta+\alpha-1}+2\beta SM^{\beta-1}\right)\}<\varepsilon$, we have that $\Psi: B_{T,M}\to B_{T,M}$ is indeed a contraction mapping.

3.1.5. Banach fixed-point Theorem

Proof of Lemma 3.1.1. We defined $\|u\|_{X_T} := \operatorname{ess\,sup}_{t\in[0,T]}\left(\|u(t)\|_{L^\infty}+\|u(t)\|_{L^\beta}\right)$ and showed that if

$$T\cdot\max\left\{M^\alpha\left(1+SM^\beta\right),(2\alpha M^{\alpha-1}+2\alpha SM^{\beta+\alpha-1}+2\beta SM^{\beta-1}\right\}<\varepsilon,$$

for $\varepsilon<\frac{1}{2}$, we satisfy Propositions 5-8. Thus, for $u,v\in B_{T,M}$, we have

$$\|\Psi(u)\|_{X_T} \le \|\phi\|_{L^\infty}+\|\phi\|_{L^\beta}+2\varepsilon < \|\phi\|_X+1 \Rightarrow \Psi(u)\in B_{T,M},$$

and

$$\|\Psi(u)-\Psi(v)\|_{X_T} < \varepsilon\|u-v\|_{X_T}.$$

We conclude that $\Psi: B_{T,M}\to B_{T,M}$ is a contraction mapping. We know from Proposition 4 that $B_{T,M}$ is a complete metric space. Therefore, the Banach fixed-point Theorem 2.3.7 tells us that Ψ has a unique fixed point $u(t)\in B_{T,M}$ which is a local mild solution to (3.1). □

3.2. Local classical solution

Improving the local mild solution to a local classical solution is a non-trivial matter and the details are beyond the scope of this thesis. As such we will assume that we can improve our local mild solution to a local classical solution and direct the reader to Section 44 of the book Superlinear Parabolic Problems by P. Quittner and P. Souplet[12] where the setting is very similar and the reader may check that the proof and statement are also valid here.

Lemma 3.2.1. *Let $T > 0$, $m \in \{1,2\}$ and consider equation* (3.1) *with $\alpha, \beta \geq 1$ and $\phi \in L^{\infty}(\mathbb{R}^m) \cap L^{\beta}(\mathbb{R}^m)$. Then, u, the unique mild solution of* (3.1) *on $[0,T]$ given by Lemma 3.1.1, is also a local classical solution, $u \in C(\mathbb{R}^m \times [0,T]) \cap C^{2,1}(\mathbb{R}^m \times (0,T))$.*

3.3. Non-negativity

Now we will prove that the solution u is non-negative. We start by splitting u into its positive and negative parts

$$\begin{aligned} u^+(x,t) &:= \max\{u(x,t), 0\}, \\ u^-(x,t) &:= \max\{-u(x,t), 0\}, \end{aligned}$$

such that $u(x,t) = u^+(x,t) - u^-(x,t)$. We will also need $u^{\pm}$ to be differentiable in the weak sense. We will not prove this is the case but instead refer to the book [6], specifically Sections 5.9-5.10, from which we derive the following lemma

Lemma 3.3.1 (Weak differentiability of $u^{\pm}$, [6])**.** *Let u be the unique local classical solution of* (3.1) *from Lemma 3.2.1. Then $u^{\pm}$ are weakly differentiable and*

$$\nabla u^+ = \nabla u \mathbb{1}_{\{u>0\}} \quad \textit{and,} \quad \nabla u^- = -\nabla u \mathbb{1}_{\{u<0\}}$$

almost everywhere, and

$$\frac{\partial u^+}{\partial t} = \frac{\partial u}{\partial t} \mathbb{1}_{\{u>0\}} \quad \textit{and,} \quad \frac{\partial u^-}{\partial t} = -\frac{\partial u}{\partial t} \mathbb{1}_{\{u<0\}}$$

almost everywhere.

Then the following lemma gives us non-negativity when the initial data is non-negative.

Lemma 3.3.2. *Let $u(x,t)$ be a local classical solution of (1.1) on $[0,T)$ for some $T > 0$. If the initial data $\phi \in L^{\beta}(\mathbb{R}^m) \cap L^{\infty}(\mathbb{R}^m)$, for $\beta \geq 1$, satisfies*

$$\phi(x) \geq 0, \forall x \in \mathbb{R}^m,$$

then

$$\|u^-(t)\|_{L^{\beta}} = 0, \quad t \in [0,T).$$

Hence $u(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^m \times [0,T)$.

We will only prove this for the case $\beta = 1$, however the proof for $\beta \geq 1$ is analogous. Throughout the proof we once again assume that $t \in [0,T)$ for some $T > 0$ and will not include this notion at the end of every equation.

Proof. To start, note that $u^-(x,t) \cdot u(x,t) = u^-(x,t) \cdot (-u^-(x,t)) = -|u^-(x,t)|^2$ and that the necessary weak derivatives of u^- exist, therefore

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\|u^-(t)\|_{L^2}^2 &= \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^m} |u^-(y,t)|^2 dy \\ &= \int_{\mathbb{R}^m} u^-(y,t)\frac{\partial u^-}{\partial t}(y,t) dy \\ &= -\int_{\mathbb{R}^m} u^-(y,t)\frac{\partial u}{\partial t}(y,t)\, dy. \end{aligned} \tag{3.15}$$

This means we can multiply equation (1.1) by u^- to obtain $\frac{\partial}{\partial t}\|u^-(t)\|_{L^2}$, which we could use to bound $\|u^-(t)\|_{L^2}$ if we can get it in the form of Grönwall's inequality, Lemma 2.3.6. Multiplying equation (1.1) by u^- and integrating over $\mathbb{R}^m$ gives us

$$\begin{aligned}
&-\int_{\mathbb{R}^m} u^-(x,t)\frac{\partial u}{\partial t}(x,t)\,dx + \int_{\mathbb{R}^m} u^-(x,t)\Delta u(x,t)\,dx \\
&\quad = -\int_{\mathbb{R}^m} u^-(x,t)\,\mathrm{sign}(u(x,t)|u(x,t)|^\alpha \left(1-\int_{\mathbb{R}^m} w(y)|u(y,t)|^\beta dy\right) dx.
\end{aligned} \tag{3.16}$$

Clearly the left most term of (3.16) corresponds to the term in (3.15). As for the term $\int_{\mathbb{R}^m} u^-\Delta u(x,t)\,dx$, integration by parts gives us

$$\int_{\mathbb{R}^m} u^-(x,t)\Delta u(x,t)\,dx = -\int_{\mathbb{R}^m} u^-(x,t)\Delta u(x,t)^-\,dx = \int_{\mathbb{R}^m} |\nabla u^-(x,t)|^2\,dx = \|\nabla u^-(t)\|_{L^2}.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|u^-(t)\|_{L^2}^2 + \|\nabla u^-(t)\|_{L^2}^2 = \\
&\quad -\int_{\mathbb{R}^m} u^-(x,t)\,\mathrm{sign}(u(x,t))|u(x,t)|^\alpha \left(1-\int_{\mathbb{R}^m} w(y)|u(y,t)|^\beta dy\right) dx.
\end{aligned}$$

We know that $\|\nabla u^-(t)\|_{L^2}^2$ is positive, so we can remove it from the left hand side to get an inequality. Then, to use Grönwall's inequality, we need to get a term involving $\|u^-\|_{L^2}^2$ on the right hand side. We start by simplifying the inequality

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\|u^-\|_{L^2}^2 &\leq -\int_{\mathbb{R}^m} u^-(x,t)\,\mathrm{sign}(u(x,t))|u(x,t)|^\alpha \left(1-\int_{\mathbb{R}^m} w(y)|u(y,t)|^\beta dy\right) dx \\
&\leq \int_{\mathbb{R}^m} |u^-(x,t)|^{\alpha+1}\left(1-\int_{\mathbb{R}^m} w(y)|u^-(y,t)|^\beta dy\right) dx \\
&\leq \int_{\mathbb{R}^m} |u^-(x,t)|^{\alpha+1} dx,
\end{aligned}$$

where we remove $\int_{\mathbb{R}^m} w(y)|u^-(y,t)|^\beta dy$ since it is non-negative and $u^-(x,t)\,\mathrm{sign}(u(x,t)) = -u^-(x,t)$. We note that

$$\|u^-(t)\|_{L^{\alpha+1}}^{\alpha+1} \leq \|u^-(t)\|_{L^2}^2 \|u^-(t)\|_{L^\infty}^{\alpha-1},$$

giving us the exact form we need for Grönwall's inequality, with $f(t) = \|u^-(t)\|_{L^2}^2$ and $g(t) = 2\|u^-(t)\|_{L^\infty}^{\alpha-1}$ giving us

$$\|u^-\|_{L^2}^2 \leq \|u^-(0)\|_{L^2}^2 e^{\int_0^t 2\|u^-(s)\|_{L^\infty}^{\alpha-1} ds} = 0, \tag{3.17}$$

when the initial condition $\phi(x) \geq 0$ for all $x \in \mathbb{R}^m$, where the integral is well-defined since our solution u is bounded for $t < T$ (see Section 3.1). □

We can now conclude that the solutions u of equation (1.1) are non-negative when $\phi(x) \geq 0$ for all $x \in \mathbb{R}^m$. Consequently equation (3.1) reduces to equation (1.1).

3.4. A priori estimate

Now that we have a local classical solution in hand we will give an a priori estimate on the L^k-norm for $\beta \leq k \leq \infty$. This proof will follow that of [2] very closely but adjusted for our case and with some additional clarifications and justifications. We will consider only the case where $m = 1,2$, the proof for $m \geq 3$ follows the same structure but is slightly simpler. The ultimate goal of this section is to prove the following theorem.

Theorem 3.4.1. *Assume ϕ is non-negative and $\phi \in L^\beta(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$, $m = 1,2$. If α satisfies*

$$1 \leq \alpha < 1 + \left(1 - \frac{2}{p}\right)\beta, \tag{3.18}$$

where p is as in (1.8)*. Then equation (1.1) has a bounded non-negative local classical solution. Moreover, the following a priori estimates hold true. That is, for any $t \in [0,T)$ and $\beta \leq k \leq \infty$*

$$\|u(t)\|_{L^k} \leq C\left(\|\phi\|_{L^\infty}, \|\phi\|_{L^\beta}, S\right). \tag{3.19}$$

The proof of this theorem mainly consists of using inequalities to obtain

$$\frac{d}{dt}\|u(t)\|_{L^k}^k + \|u(t)\|_{L^k}^k \le C, \quad t \in [0,T)$$

for $\beta \le k \le \infty$. After this, Proposition 1 gives us boundedness. Throughout the proof we once again assume $t \in [0,T)$ for some $T > 0$ and will specify this at the end of every equation. We will also be collecting all constants that appear into a single constant $C(\dots)$, where any new parameters defining the constant will be added in the brackets.

Proof. We start by multiplying (1.1) by ku^{k-1} and integrating over $\mathbb{R}^m$ giving us

$$\int_{\mathbb{R}^m} \frac{\partial u}{\partial t}(x,t)\cdot ku^{k-1}(x,t)dx - \int_{\mathbb{R}^m} \Delta u(x,t)\cdot ku^{k-1}(x,t)dx = k\int_{\mathbb{R}^m} u^{k+\alpha-1}(x,t)\left(1-\int_{\mathbb{R}^m} w(y)u^{\beta}(y,t)dy\right)dx$$

On the left hand side we can simplify both terms, starting with the leftmost term,

$$\int_{\mathbb{R}^m} \frac{\partial u}{\partial t}(x,t)\cdot ku^{k-1}(x,t)dx = \frac{d}{dt}\int_{\mathbb{R}^m} u^k(x,t)dx. \tag{3.20}$$

For the Laplacian term we use integration by parts

$$\int_{\mathbb{R}^m} \Delta u(x,t)\cdot ku^{k-1}(x,t)dx = -k\int_{\mathbb{R}^m} \nabla\left(u^{k-1}(x,t)\right)\cdot\nabla u(x,t)\,dx = -k(k-1)\int_{\mathbb{R}^m} u^{k-2}(x,t)|\nabla u(x,t)|^2dx,$$

followed by substituting in

$$\frac{k^2}{4}u^{k-2}(x,t)|\nabla u(x,t)|^2 = |\nabla u^{\frac{k}{2}}(x,t)|^2,$$

to get

$$\int_{\mathbb{R}^m} \Delta u(x,t)\cdot ku^{k-1}(x,t)dx = -\frac{4(k-1)}{k}\int_{\mathbb{R}^m} |\nabla u^{\frac{k}{2}}(x,t)|^2dx = -\frac{4(k-1)}{k}\|\nabla u^{\frac{k}{2}}(x,t)\|_{L^2}^2. \tag{3.21}$$

Note also that the nonlocal term can be moved to the left hand side. With these simplifications we get

$$\frac{d}{dt}\int_{\mathbb{R}^m} u^k(x,t)dx + \frac{4(k-1)}{k}\|\nabla u^{\frac{k}{2}}(t)\|_{L^2}^2 + k\int_{\mathbb{R}^m} w(x)u^{\beta}(x,t)dx\|u(t)\|_{L^{k+\alpha-1}}^{k+\alpha-1} = k\|u(t)\|_{L^{k+\alpha-1}}^{k+\alpha-1}.$$

Now we can use the fact that $\inf_{x\in\mathbb{R}^m}\{w(x)\} = I > 0$ to get the following inequality

$$\frac{d}{dt}\int_{\mathbb{R}^m} u^k(x,t)dx + \frac{4(k-1)}{k}\|\nabla u(t)^{\frac{k}{2}}\|_{L^2}^2 + kI\|u(t)\|_{L^\beta}^{\beta}\|u(t)\|_{L^{k+\alpha-1}}^{k+\alpha-1} \le k\|u(t)\|_{L^{k+\alpha-1}}^{k+\alpha-1}. \tag{3.22}$$

From here on our goal is to get $\|u(t)\|_{L^k}$ as a positive term on the left hand side. We can introduce it using Lemma 2.3.4 with the parameters

$$v = u^{\frac{k}{2}},\ q = \frac{2(k+\alpha-1)}{k},\ r = \frac{2k'}{k} = \frac{k+\alpha-1+\beta}{k},\ C_0 = \frac{k-1}{k^2},\ C_1 = \frac{1}{2k},$$

where we satisfy all the necessary inequalities when $\alpha < 1+\left(1-\frac{2}{p}\right)\beta$, $k > \max\left\{\frac{2(\alpha-1)}{p-2}, 1\right\}$ and $k' = \frac{k+\alpha-1+\beta}{2}$. These parameters give us the following bound on $\|u(t)\|_{L^{k+\alpha-1}}^{k+\alpha-1}$

$$\int_{\mathbb{R}^m} u^{k+\alpha-1}(x,t)dx \le \frac{k-1}{k^2}\|\nabla u^{\frac{k}{2}}(t)\|_{L^2}^2 + \frac{1}{2k}\|u(t)\|_{L^k}^k + C(k)\|u(t)\|_{L^{k'}}^b,$$

where

$$b = \frac{(1-\lambda)(k+\alpha-1)}{1-\lambda\frac{k+\alpha-1}{k}},$$

and λ as described in Lemma 2.3.4. If we substitute back into equation (3.22) we get

$$\frac{d}{dt}\int_{\mathbb{R}^m} u^k(x,t)dx + \frac{3(k-1)}{k}\|\nabla u^{\frac{k}{2}}(t)\|_{L^2}^2 + kI\|u(t)\|_{L^\beta}^{\beta}\|u(t)\|_{L^{k+\alpha-1}}^{k+\alpha-1} \le \frac{1}{2}\|u(t)\|_{L^k}^k + C(k)\|u(t)\|_{L^{k'}}^b. \tag{3.23}$$

Now we need to eliminate the $\|u\|^b_{L^{k'}}$ term from the right hand side, which we can do by using the interpolation inequality from Lemma 2.3.1 to turn it into $\|u\|^{\beta}_{L^\beta}\|u\|^{k+\alpha-1}_{L^{k+\alpha-1}}$. We find

$$\|u(t)\|^b_{L^{k'}} \le \|u(t)\|^{b\theta}_{L^{k+\alpha-1}}\|u(t)\|^{(1-\theta)b}_{L^\beta} = \left(\|u(t)\|^{k+\alpha-1}_{L^{k+\alpha-1}}\|u(t)\|^{\beta}_{L^\beta}\right)^{\frac{b\theta}{k+\alpha-1}}\|u(t)\|^{b\left(1-\theta-\frac{\theta\beta}{k+\alpha-1}\right)}_{L^\beta},$$

where

$$\theta = \frac{\frac{1}{\beta}-\frac{1}{k'}}{\frac{1}{\beta}-\frac{1}{k+\alpha-1}} = \frac{k+\alpha-1}{k+\alpha-1+\beta}.$$

We can easily derive that the power $1-\theta-\frac{\theta\beta}{k+\alpha-1}=0$ thanks to our choice of k'. We also note that $\frac{b\theta}{k+\alpha-1}<1$ allowing us to get rid of this power using Young's inequality 2.3.3 with the free parameter $\varepsilon=\frac{kI}{4}$

$$\begin{aligned} C(k)\|u(t)\|^b_{L^{k'}} &\le C(k)\left(\|u(t)\|^{k+\alpha-1}_{L^{k+\alpha-1}}\|u(t)\|^{\beta}_{L^\beta}\right)^{\frac{b\theta}{k+\alpha-1}} \\ &\le \frac{kI}{4}\left(\|u(t)\|^{k+\alpha-1}_{L^{k+\alpha-1}}\|u(t)\|^{\beta}_{L^\beta}\right)+C(k,I), \end{aligned}$$

where we simply absorb the multiplication and power into $C(k)$. Substitution back into equation (3.23) gives us

$$\frac{d}{dt}\int_{\mathbb{R}^m}u^k(x,t)dx+\frac{3(k-1)}{k}\|\nabla u^{\frac{k}{2}}(t)\|^2_{L^2}+\frac{3kI}{4}\|u(t)\|^{\beta}_{L^\beta}\|u(t)\|^{k+\alpha-1}_{L^{k+\alpha-1}} \le C(k,I)+\frac{1}{2}\|u(t)\|^k_{L^k}. \tag{3.24}$$

Now we must consider three separate cases: $\beta\le k\le\beta+\alpha-1$, $\beta+\alpha-1<k<\infty$ and $k=\infty$. The reason these cases must be considered separately is that the case $\beta\le k\le\beta+\alpha-1$ uses the interpolation inequality to show that all L^k-norms for these k are bounded. However this requires showing that the L^β-norm is bounded for which we need a bound on the $L^{\beta+\alpha-1}$-norm. Then for $\beta+\alpha-1<k<\infty$ we are able to satisfy the inequalities need by Lemma 2.3.4 a second time which we can use to move $\|u\|_{L^k}$ to the left hand side. Lastly, the L^∞-norm is a special case which we solve using Moser's iteration.

3.4.1. $\boldsymbol{\beta\le k\le\beta+\alpha-1}$

We will show that both $\|u(t)\|_\beta$ and $\|u(t)\|_{\beta+\alpha-1}$ are bounded, starting with the latter. Then, by the interpolation inequality from Lemma 2.3.1, we can bound $\|u(t)\|_k$ for all $\beta\le k\le\beta+\alpha-1$.

Setting $\boldsymbol{k=\beta+\alpha-1}$ in equation (3.24), giving us

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^m}u^{\beta+\alpha-1}(x,t)dx+\frac{3(\beta+\alpha-2)}{\beta+\alpha-1}\|\nabla u^{\frac{\beta+\alpha-1}{2}}(t)\|^2_{L^2}+\frac{3I(\beta+\alpha-1)}{4}\|u(t)\|^{\beta}_{L^\beta}\|u(t)\|^{\beta+2(\alpha-1)}_{L^{\beta+2(\alpha-1)}} \\ \le C(\alpha,\beta,I)+\frac{1}{2}\|u(t)\|^{\beta+\alpha-1}_{L^{\beta+\alpha-1}}. \end{aligned} \tag{3.25}$$

We use the interpolation inequality with

$$\beta\le\beta+\alpha-1\le\beta+2(\alpha-1),\quad \theta=\frac{\beta}{2(\beta+\alpha-1)},$$

to find

$$\|u(t)\|^{\beta+\alpha-1}_{L^{\beta+\alpha-1}} \le \left(\|u(t)\|^{\theta}_{L^\beta}\|u(t)\|^{1-\theta}_{L^{\beta+2(\alpha-1)}}\right)^{\beta+\alpha-1} = \left(\|u(t)\|^{\beta}_{L^\beta}\|u(t)\|^{\beta+2(\alpha-1)}_{L^{\beta+2(\alpha-1)}}\right)^{\frac{1}{2}}.$$

Then by Young's inequality with

$$\varepsilon=I(\beta+\alpha-1),\quad p,q=2,$$

we obtain

$$\|u(t)\|^{\beta+\alpha-1}_{L^{\beta+\alpha-1}} \le \frac{I(\beta+\alpha-1)}{2}\|u(t)\|^{\beta}_{L^\beta}\|u(t)\|^{\beta+2(\alpha-1)}_{L^{\beta+2(\alpha-1)}}+\frac{1}{2I(\beta+\alpha-1)}. \tag{3.26}$$

Now we substitute (3.26) into equation (3.25), removing $\|\nabla u^{\frac{\beta+\alpha-1}{2}}(t)\|_{L^2}$ from the left hand side since it is positive, to get

$$\frac{d}{dt}\int_{\mathbb{R}^m}u^{\beta+\alpha-1}(x,t)dx+\frac{I(\beta+\alpha-1)}{2}\|u(t)\|^{\beta}_{L^\beta}\|u(t)\|^{\beta+2(\alpha-1)}_{L^{\beta+2(\alpha-1)}} \le C(\alpha,\beta,I)+\frac{1}{4I(\beta+\alpha-1)}.$$

Now we can use the Young's inequality result (3.26) again to regain $\|u(t)\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1}$

$$\frac{d}{dt}\int_{\mathbb{R}^m} u^{\beta+\alpha-1}(x,t)dx + \|u(t)\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1} \le C(\alpha,\beta,I) + \frac{3}{4I(\beta+\alpha-1)} = C(\alpha,\beta,I). \tag{3.27}$$

Equation (3.27) is now precisely in the form required by Proposition 1, therefore

$$\|u(t)\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1} \le \|\phi\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1} e^{-t} + C(\alpha,\beta,I). \tag{3.28}$$

Now we know that $\|u(t)\|_{L^{\beta+\alpha-1}}$ is bounded, we can use this to prove that $\|u(t)\|_{L^\beta}$ is also bounded,

Setting $k=\beta$ in equation (3.22) gives us

$$\frac{d}{dt}\int_{\mathbb{R}^m} u^{\beta}(x,t)dx + \frac{4(\beta-1)}{\beta}\|\nabla u^{\frac{\beta}{2}}(t)\|_{L^2}^2 + \beta I\|u(t)\|_{L^\beta}^{\beta}\|u(t)\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1} \le \beta\|u(t)\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1}.$$

We can remove $\|\nabla u^{\frac{\beta}{2}}(t)\|_{L^2}^2$ since it is positive and we know that $\|u(t)\|_{L^{\beta+\alpha-1}}$ is bounded, therefore

$$\frac{d}{dt}\int_{\mathbb{R}^m} u^{\beta}(x,t)dx + \beta I\|u(t)\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1}\|u(t)\|_{L^\beta}^{\beta} \le \beta\|u(t)\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1}$$

$$\frac{d}{dt}\int_{\mathbb{R}^m} u^{\beta}(x,t)dx + C(\|\phi\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1},\alpha,\beta,I)\|u(t)\|_{L^\beta}^{\beta} \le C(\|\phi\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1},\alpha,\beta,I).$$

We are in the form required by Proposition 1 again. Therefore,

$$\|u(t)\|_{L^\beta}^{\beta} \le \|\phi\|_{L^\beta}^{\beta} e^{-tC(\|\phi\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1},\alpha,\beta,I)} + C(\|\phi\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1},\alpha,\beta,I). \tag{3.29}$$

Now that we have shown both $\|u(t)\|_{L^\beta}^{\beta}$ and $\|u(t)\|_{L^{\beta+\alpha-1}}^{\beta+\alpha-1}$ are bounded we can use the interpolation inequality to show that $\|u(t)\|_{L^k}$ is bounded for any $\beta \le k \le \beta+\alpha-1$

$$\|u(t)\|_{L^k} \le C(\|\phi\|_{L^\beta},\|\phi\|_{L^{\beta+\alpha-1}},\alpha,\beta,I). \tag{3.30}$$

3.4.2. $\boldsymbol{\beta+\alpha-1<k<\infty}$

For this case, we will use Lemma 2.3.4 followed by a combination of the interpolation inequality and Young's inequality to get the term $\|u(t)\|_{L^k}$, in equation (3.24) on the left hand side and only a constant on the right hand side.

Letting $k>\beta+\alpha-1$ and $k'=\frac{\beta+\alpha-1+k}{2}$, the parameters

$$v=u^{\frac{k}{2}},\quad q=2,\quad r=\frac{2k'}{k},\quad C_0=\frac{k-1}{k},\quad C_1=\frac{1}{2},$$

satisfy the inequalities needed for Lemma 2.3.4. Therefore

$$\begin{aligned}\|u^{\frac{k}{2}}(t)\|_{L^2}^2 = \|u(t)\|_{L^k}^k &\le C(k,I)\|u^{\frac{k}{2}}(t)\|_{L^{\frac{2k'}{k}}}^2 + \frac{k-1}{k}\|\nabla u^{\frac{k}{2}}(t)\|_{L^2}^2 + \frac{1}{2}\|u^{\frac{k}{2}}(t)\|_{L^2}^2\\ &\le C(k,I)\|u(t)\|_{L^{k'}}^k + \frac{k-1}{k}\|\nabla u^{\frac{k}{2}}(t)\|_{L^2}^2 + \frac{1}{2}\|u(t)\|_{L^k}^k,\end{aligned}$$

we can move $\frac{1}{2}\|u(t)\|_{L^k}^k$ to the left hand side and multiply both sides by 2 to get,

$$\|u(t)\|_{L^k}^k \le C(k,I)\|u(t)\|_{L^{k'}}^k + \frac{2(k-1)}{k}\|\nabla u(t)^{\frac{k}{2}}\|_{L^2}^2. \tag{3.31}$$

Applying the interpolation inequality and Young's inequality, with

$$\beta<k'<k+\alpha-1,\quad \theta=\frac{\beta}{\beta+k+\alpha-1},$$

gives us

$$C(k,I)\|u(t)\|_{L^{k'}}^k \le C(k,I)\left(\|u(t)\|_{L^\beta}^{\frac{\beta}{\beta+k+\alpha-1}}\|u(t)\|_{L^{k+\alpha-1}}^{1-\frac{\beta}{\beta+k+\alpha-1}}\right)^k = C(k,I)\left(\|u(t)\|_{L^\beta}^{\beta}\|u(t)\|_{L^{k+\alpha-1}}^{k+\alpha-1}\right)^{\frac{k}{\beta+k+\alpha-1}}.$$

Then using Young's inequality with

$$p=\frac{\beta+k+\alpha-1}{\beta},\quad q=\frac{\beta+\alpha-1}{\beta+k+\alpha-1},\quad \varepsilon=\frac{kI}{2},$$

we obtain

$$C(k,S)\|u(t)\|_{L^{k'}}^{k}\le\frac{kI}{2}\|u(t)\|_{L^{\beta}}^{\beta}\|u(t)\|_{L^{k+\alpha-1}}^{k+\alpha-1}+C(k,I). \tag{3.32}$$

Before substituting equations (3.31) and (3.32) into equation (3.24), we rewrite $\frac{1}{2}\|u(t)\|_{L^k}^k$ as $\left(\frac{3}{2}-1\right)\|u(t)\|_{L^k}^k$. Then

$$\frac{3}{2}\|u(t)\|_{L^k}^k\le\frac{3(k-1)}{k}\|\nabla u^{\frac{k}{2}}(t)\|_{L^2}^2+\frac{3kS}{4}\|u(t)\|_{L^{\beta}}^{\beta}\|u(t)\|_{L^{k+\alpha-1}}^{k+\alpha-1}+C(k,S)$$

will perfectly cancel out both the terms $\|\nabla u^{\frac{k}{2}}(t)\|_{L^2}^2$ and $\|u(t)\|_{L^{\beta}}^{\beta}\|u(t)\|_{L^{k+\alpha-1}}^{k+\alpha-1}$ on the left hand side while we can move the remaining $\|u(t)\|_{L^k}^k$ to the left hand side giving us the form needed by Proposition 1

$$\frac{d}{dt}\int_{\mathbb{R}^m}u^k(x,t)\,dx+\|u(t)\|_{L^k}^k\le C(k,I),$$

along with the following bound on the L^k-norm for $\beta+\alpha-1<k<\infty$,

$$\|u(t)\|_{L^k}^k\le\|\phi\|_{L^k}^k e^{-t}+C(k,I). \tag{3.33}$$

3.4.3. $k=\infty$

Like with the rest of the a priori estimate we will follow the proof from [2], given how technical this proof is we will not be adding much to the explanation except for a brief overview of the steps of the proof and the idea behind Moser's iteration. In this step we focus on the case $m=2$ and $2<p<\infty$, as the other case, $m=1$ and $p=\infty$, follows more easily.

Let $q_n=2^n+\beta+\alpha-1$ and take $k=q_n$ in equation (3.22), then

$$\frac{d}{dt}\int_{\mathbb{R}^m}u^{q_n}(x,t)\,dx+\frac{4(q_n-1)}{q_n}\|\nabla u^{\frac{q_n}{2}}(t)\|_{L^2}^2+q_nI\|u(t)\|_{L^{\beta}}^{\beta}\|u(t)\|_{L^{q_n+\alpha-1}}^{q_n+\alpha-1}\le q_n\|u(t)\|_{L^{q_n+\alpha-1}}^{q_n+\alpha-1}. \tag{3.34}$$

The goal is to rewrite this inequality into the form

$$\frac{d}{dt}\int_{\mathbb{R}^m}u^{q_n}(x,t)\,dx+\int_{\mathbb{R}^m}u^{q_n}(x,t)\,dx\le Cq_n^{d_0}\left(\int_{\mathbb{R}^m}u^{q_{n-1}}(x,t)\right)^2,$$

where $d_0>1$. This then gives us

$$\int_{\mathbb{R}^m}u^{q_n}(x,t)\,dx\le Cq_n^{d_0}\max\left\{\left(\int_{\mathbb{R}^m}u^{q_{n-1}}(x,t)\,dx\right)^2,K_0^{q_n}\right\},$$

where $K_0=\max\{1,\|\phi\|_{L^{\beta}},\|\phi\|_{L^{\infty}}\}$. We can see that this is a recursive estimate for $\|u(t)\|_{L^{q_n}}^{q_n}$, depending on the previous iteration, $u^{q_{n-1}}$.

However, note that we are not interested in $\int_{\mathbb{R}^m}u^{q_n}(x,t)\,dx$ but in $\|u(t)\|_{L^{q_n}}$, so we take the q_n-th root. Then

$$\left(Cq_n^{d_0}\right)^{\frac{1}{q_n}}=\exp\frac{\log(C)+d_0\log(q_n)}{q_n}\to 1,$$

since $\frac{\log(q_n)}{q_n}\to 0$ as $n\to\infty$. Next note that

$$\left(\int_{\mathbb{R}^m}u^{q_{n-1}}(x,t)\,dx\right)^{\frac{2}{q_n}}=\|u(t)\|_{L^{q_{n-1}}}^{\frac{2q_{n-1}}{q_n}},$$

where $\frac{2q_{n-1}}{q_n}\to 1$ as $n\to\infty$ since $q_n\approx 2^n$. These two factors combined are what allows us to use Moser's iteration. We then use an external lemma to introduce a supremum over time in the right hand side of our estimate for $\|u(t)^{q_n}\|_{L^{q_n}}$, which allows us to bound each iteration by this same constant. From this we conclude that the L^{∞}-norm is bounded after letting $n\to\infty$.

More precisely, using Lemma 2.3.4 with parameters

$$v = u^{\frac{q_n}{2}},\ q = \frac{2(q_n+\alpha-1)}{q_n},\ r = \frac{2q_{n-1}}{q_n},\ C_0 = C_1 = \frac{1}{2q_n},$$

we find

$$\begin{aligned} \|u(t)\|_{L^{q_n+\alpha-1}}^{q_n+\alpha-1} &\le C(m)\left(\frac{1}{2q_n}\right)^{-\frac{1}{\delta_1-1}}\left(\int_{\mathbb{R}^m} u^{q_{n-1}}(x,t)dx\right)^{\gamma_1} + \frac{1}{2q_n}\|\nabla u^{\frac{q_n}{2}}(t)\|_{L^2}^2 + \frac{1}{2q_n}\|u(t)\|_{L^{q_n}}^{q_n} \\ &\le C(m,\delta_1)q_n^{\frac{1}{\delta_1-1}}\left(\int_{\mathbb{R}^m} u^{q_{n-1}}(x,t)dx\right)^{\gamma_1} + \frac{1}{2q_n}\|\nabla u^{\frac{q_n}{2}}(t)\|_{L^2}^2 + \frac{1}{2q_n}\|u(t)\|_{L^{q_n}}^{q_n}, \end{aligned} \tag{3.35}$$

where

$$\gamma_1 = 1 + \frac{q_n+\alpha-1-q_{n-1}}{q_{n-1}-\frac{p(\alpha-1)}{p-2}} \le 2, \quad \delta_1 = \frac{q_n - \frac{2q_{n-1}}{p}}{q_n+\alpha-1-q_{n-1}}.$$

Substitution back into (3.34) together with the fact that $\frac{4(q_n-1)}{q_n} \ge 2$, gives us

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^m} u^{q_n}(x,t)dx + \frac{3}{2}\|\nabla u(t)^{\frac{q_n}{2}}\|_{L^2}^2 + q_n S\|u(t)\|_{L^\beta}^\beta \|u(t)\|_{L^{q_n+\alpha-1}}^{q_n+\alpha-1} \\ &\le C(m,\delta_1)q_n^{\frac{\delta_1}{\delta_1-1}}\left(\int_{\mathbb{R}^m} u^{q_{n-1}}(x,t)dx\right)^{\gamma_1} + \frac{1}{2}\|u(t)\|_{L^{q_n}}^{q_n}. \end{aligned} \tag{3.36}$$

Now we use Lemma 2.3.4 again, but with the parameters

$$v = u^{\frac{q_n}{2}}, \quad q = 2, \quad r = \frac{2q_{n-1}}{q_n}, \quad C_0 = C_1 = \frac{1}{2},$$

to obtain

$$\frac{3}{2}\|u(t)\|_{L^{q_n}}^{q_n} \le C(m)\left(\int_{\mathbb{R}^m} u^{q_{n-1}}(x,t)dx\right)^{\gamma_2} + \frac{3}{2}\|\nabla u^{\frac{q_n}{2}}(t)\|_{L^2}^2 \tag{3.37}$$

where

$$\gamma_2 = 1 + \frac{q_n - q_{n-1}}{q_{n-1}} < 2.$$

Adding the left and right hand side terms of equation (3.37) to the left and right hand sides of equation (3.36), canceling the terms and using the fact $\gamma_1 \le 2, \gamma_2 < 2$ we find

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^m} u^{q_n}(x,t)dx + \|u(t)\|_{L^{q_n}}^{q_n} &\le C(m,\delta_1)q_n^{\frac{\delta_1}{\delta_1-1}}\left(\int_{\mathbb{R}^m} u^{q_{n-1}}(x,t)dx\right)^{\gamma_1} + C(m)\left(\int_{\mathbb{R}^m} u^{q_{n-1}}(x,t)dx\right)^{\gamma_2} \\ &\le \max\left[C(m,\delta_1),C(m)\right] q_n^{\frac{\delta_1}{\delta_1-1}}\left[\left(\int_{\mathbb{R}^m} u^{q_{n-1}}(x,t)dx\right)^{\gamma_1} + \left(\int_{\mathbb{R}^m} u^{q_{n-1}}(x,t)dx\right)^{\gamma_2}\right] \\ &\le 2\max\left[C(m,\delta_1),C(m)\right] q_n^{\frac{\delta_1}{\delta_1-1}} \max\left[\left(\int_{\mathbb{R}^m} u^{q_{n-1}}(x,t)dx\right)^2, 1\right]. \end{aligned}$$

This is the form mentioned in the introduction of this subsection that we require.

Now let $K_0 = \max\{1, \|\phi\|_{L^\beta}, \|\phi\|_{L^\infty}\}$, then

$$\int_{\mathbb{R}^m} \phi^{q_n} dx \le \left(\max\{\|\phi\|_{L^\beta}, \|\phi\|_{L^\infty}\}\right)^{q_n} \le K_0^{q_n}.$$

Let $d_0 = \frac{\delta_1}{\delta_1-1}$ and note that $q_n^{d_0} = [2^n+\beta+\alpha-1]^{d_0} \le [2^n+2^n(\beta+\alpha-1)]^{d_0}$. Setting $\overline{a} = 2\max\{C(m,\delta_1), C(m)\}(\beta+\alpha)^{d_0}$ in Lemma 4.1 of [3], gives us

$$\int_{\mathbb{R}^m} u^{q_n}(x,t)dx \le (2\overline{a})^{2^n-1} 2^{d_0(2^{n+1}-n-2)} \max\left\{\sup_{t\in(0,T)}\left(\int_{\mathbb{R}^m} u^{q_0}(x,t)dx\right)^{2^n}, K_0^{q_n}\right\}. \tag{3.38}$$

Since $q_n = 2^n + \beta + \alpha - 1$ and taking the $\frac{1}{q_n}$-th power of both sides of (3.38) allows us to bound all $\|u^{q_n}(t)\|_{L^{q_n}}$ by the same constant allowing us to let $n \to \infty$ to give us

$$\|u(t)\|_{L^\infty} \leq 2\overline{a}2^{2d_0} \max\left\{ \sup_{t\in(0,T)} \int_{\mathbb{R}^m} u^{q_0}(x,t)\,dx, K_0 \right\}. \tag{3.39}$$

From Subsection 3.4.2 we know that for $q_0 = \beta + \alpha > \beta + \alpha - 1$, we have

$$\int_{\mathbb{R}^m} u^{q_0}(x,t)\,dx = \int_{\mathbb{R}^m} u^{\beta+\alpha}(x,t)\,dx \leq \|\phi\|^{\beta+\alpha}_{L^{\beta+\alpha}} + C(\alpha,\beta,I) \leq K_0^{\beta+\alpha} + C(\alpha,\beta,I)$$

which ultimately gives us

$$\|u(t)\|_{L^\infty} \leq C(\alpha,\beta,K_0,I). \tag{3.40}$$

□

3.5. Global classical solution

Now that we have both a non-negative local classical solution of (1.1) for some T and conditions under which it remains bounded, we can extend it to a global solution.

Theorem 3.5.1. *Let $m \in \{1,2\}$ and $\alpha, \beta \geq 1$, where α satisfies*

$$1 \leq \alpha < 1 + \left(1 - \frac{2}{p}\right)\beta,$$

with p as in (1.8)*. Consider equation* (1.1) *with initial data $\phi \in L^\infty(\mathbb{R}^m) \cap L^\beta(\mathbb{R}^m)$. Then there exists a non-negative global classical solution u on* $[0,\infty)$*, such that for $\beta \leq k \leq \infty$*

$$\|u(t)\|_{L^k} \leq C\left(\|\phi\|_{L^\infty}, \|\phi\|_{L^\beta}, I\right), \quad t \in [0,\infty). \tag{3.41}$$

Proof. Let $\phi \in X := L^\infty(\mathbb{R}^m) \cap L^\beta(\mathbb{R}^m)$ such that $\|\phi\|_X := \|\phi\|_{L^\infty} + \|\phi\|_{L^\beta} \leq M$ for some $M > 0$, then we know from Lemma 3.1.1, choosing $\varepsilon = \frac{1}{4}$, that there is a $T := T(M)$ such that u is a local mild solution to (1.1) on $[0,T]$. Specifically, we may take

$$T := T(M) := \frac{1}{8}\left(\max\{M^\alpha\left(1 + SM^\beta\right), (2\alpha M^{\alpha-1} + 2\alpha SM^{\beta+\alpha-1} + 2\beta SM^{\beta-1}\}\right)^{-1}.$$

By Lemma 3.2.1 we further know that u is also a classical solution on $[0,T)$, which by Lemma 3.3.2 is non-negative. From our a priori estimate, Theorem 3.4.1, we know that

$$\sup_{t\in[0,T]} \|u(t)\|_X \leq C,$$

for a constant $C = C(M)$ depending only on M.

Consider the equation

$$\begin{cases} \dfrac{\partial v_1}{\partial t}(x,t) - \Delta v_1(x,t) = v^\alpha(x,t)\left(1 - \displaystyle\int_{\mathbb{R}^m} w(y) v_1^\beta(y,t)\,dy\right), & x \in \mathbb{R}^m, t \in [T(M), T(M)+T(C)], \\ v_1(x,T(M)) = u(x,T(M)), & x \in \mathbb{R}^m, \end{cases} \tag{3.42}$$

where

$$T(C) := \frac{1}{8}\left(\max\{C^\alpha\left(1 + SC^\beta\right), (2\alpha C^{\alpha-1} + 2\alpha SC^{\beta+\alpha-1} + 2\beta SC^{\beta-1}\}\right)^{-1}.$$

By applying Lemma 3.1.1 to the above equation (3.42), we get a unique local mild solution v_1 on $[T(M), T(M)+T(C)]$. Due to the integral representation of the mild solution, equation (2.4), we may define

$$u_1(t) := u(t)\mathbb{1}_{[0,T(M))} + v_1(t)\mathbb{1}_{[T(M),T(M)+T(C)]}, \quad t \in [0, T(M)+T(C)], \tag{3.43}$$

where

$$\mathbb{1}_{[x,y]} := \begin{cases} 1, & \text{on } [x,y], \\ 0, & \text{otherwise.} \end{cases}$$

We compute

$$\begin{aligned}
u_1(t) &= e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta} u_1^\alpha(s)\left(1 - \int_{\mathbb{R}^m} w(x) u_1^\beta(x,s)\,dx\right) ds \\
&= e^{t\Delta}\phi + \int_0^{\min\{t,T(M)\}} e^{(t-s)\Delta} u_1^\alpha(s)\left(1 - \int_{\mathbb{R}^m} w(x) u_1^\beta(x,s)\,dx\right) ds \\
&\quad + \int_{\min\{t,T(M)\}}^{\min\{t,T(M)+T(C)\}} e^{(t-s)\Delta} u_1^\alpha(s)\left(1 - \int_{\mathbb{R}^m} w(x) u_1^\beta(x,s)\,dx\right) ds \\
&= e^{t\Delta}\phi + \int_0^{\min\{t,T(M)\}} e^{(t-s)\Delta} u^\alpha(s)\left(1 - \int_{\mathbb{R}^m} w(x) u^\beta(x,s)\,dx\right) ds \\
&\quad + \int_{\min\{t,T(M)\}}^{\min\{t,T(M)+T(C)\}} e^{(t-s)\Delta} v_1^\alpha(s)\left(1 - \int_{\mathbb{R}^m} w(x) v_1^\beta(x,s)\,dx\right) ds.
\end{aligned}$$

From Section 3.2 we further know that u_1 is a classical solution on $[0, T(M)+T(C)]$ and since $u(T(M)) = v_1(T(M))$ we know that u_1 is continuous.

Moreover, since the a priori estimate from Theorem 3.4.1 depends only on the size of the initial data, we can bound u_1 by the same constant $C := C(M)$, giving us

$$\sup_{t\in[0,T(M)+T(C)]} \|u_1(t)\|_X \le C.$$

Therefore, we can repeat the process of the previous paragraph to obtain local classical solutions to

$$\begin{cases} \dfrac{\partial v_i}{\partial t}(t) - \Delta v_i(t) = v_i^\alpha(t)\left(1 - \displaystyle\int_{\mathbb{R}^m} w(y) v_i^\beta(y,t)\,dy\right), & t \in [T(M)+(i-1)T(C), T(M)+iT(C)], \\ v_i(x, T(M)+(i-1)T(C)) = v_{i-1}(x, T(M)+(i-1)T(C)), & x \in \mathbb{R}^m, \end{cases} \tag{3.44}$$

for $i = 2,3,\dots$, giving us the following local classical solution on $[0, T(M)+nT(C)]$

$$\begin{aligned} u_n(t) := u(t)\mathbb{1}_{[0,T(M))} &+ \sum_{i=1}^{n-1} v_i(t)\mathbb{1}_{[T(M)+(i-1)T(C),T(M)+iT(C))} \\ &+ v_n(t)\mathbb{1}_{[T(M)+(n-1)T(C),T(M)+nT(C)]}, \quad t \in [0, T(M)+nT(C)]. \end{aligned} \tag{3.45}$$

Since $T(C) > 0$ we get $T(M)+nT(C) \to \infty$ as $n \to \infty$, giving us the global classical solution

$$\tilde{u}(t) := u(t)\mathbb{1}_{[0,T(M))} + \sum_{i=1}^{\infty} v_i(t)\mathbb{1}_{[T(M)+(i-1)T(C),T(M)+iT(C))}, \quad t \in [0,\infty), \tag{3.46}$$

bounded by the a priori estimate

$$\|\tilde{u}(t)\|_{L^k} \le C\left(\|\phi\|_{L^\infty}, \|\phi\|_{L^\beta}, S\right), \quad t \in [0,\infty).$$

$\square$

4

Discussion

With each of the five parts of Chapter 3 now proven we can conclude that

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = u^{\alpha}(x,t)\left(1 - \displaystyle\int_{\mathbb{R}^m} w(y)u^{\beta}(y,t)\,dy\right), & x \in \mathbb{R}^m, t > 0, \\ u(x,0) = \phi(x) \geq 0, & x \in \mathbb{R}^m, \end{cases} \tag{1.1}$$

has a non-negative, global classical solution when

$$1 \leq \alpha < 1 + \left(1 - \frac{2}{p}\right)\beta,$$

and $w(x)$ is bounded above and below by

$$\sup_{x\in\mathbb{R}^m} \{w(x)\} = S < \infty \text{ and } \inf_{x\in\mathbb{R}^m} \{w(x)\} = I > 0$$

respectively. We also obtained the a priori estimate

$$\|u(t)\|_{L^k} \leq C\left(\|\phi\|_{L^\infty}, \|\phi\|_{L^\beta}, I\right), \quad t \in [0,\infty).$$

for $\beta \leq k \leq \infty$.

However, our problem is very restrictive on what the weight, $w(x)$, is allowed to be. As stated in the introduction, allowing the weight function to get arbitrarily close to 0 or even become negative drastically changes the conditions for global existence. Ensuring that $w(x)$ remains strictly positive prevents this from happening. In this chapter we will discuss what might happen when the weight function is less restricted.

We start by considering when $w(x) \geq 0$. In this thesis we never allowed the weight to become 0 because if it did we would completely lose the inhibiting effect of the nonlocal term and simply be back in the Fujita case (1.3). If $w(x) = 0$ at finitely many points this may not be a problem. However if $w(x) = 0$ on an infinite subset of the domain this would be problematic as the conditions for global existence of the Fujita case are the opposite of those for the nonlocal problem.

If we let $w(x)$ take only strictly negative values, it would be equivalent to changing the sign in front of the nonlocal term. This would turn the inhibiting effect of the nonlocal term into an increasing term making the problem blow up faster. In this case the conditions for existence would likely still exist but be a stricter version of those found in Fujita's paper with the term β also playing a role.

Now for the most difficult yet interesting scenario, the case where $w(x)$ takes both positive and negative values. Like with the case of the weight taking 0 value, this would divide the domain into regions with opposite conditions for global existence. We do not know if it is possible for there to be nontrivial global solutions in this case as it is beyond the scope of this thesis; however we encourage the reader to consider this problem for further study.

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