Duality between Cooper-pair and vortex dynamics in two-dimensional Josephson-junction arrays

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The dynamics of Cooper pairs and vortices in a Josephson-junction array is investigated. For this purpose, a Hamiltonian is constructed in terms of vortex charges. Josephson-type equations for vortices are derived. A comparison with the Cooper-pair Hamiltonian shows that the roles of the magnetic field and induced charge density are reversed. The vortex and Cooper-pair Hamiltonians are approximately self-dual when $E_C/E_J=\pi^2/2$ ($E_C=e^2/2C$), which results in an array resistivity close to $h/4e^2$.

Two-dimensional arrays of superconducting islands coupled to their nearest neighbors by both Josephson junctions as well as capacitors are model systems, which are interesting from both experimental and theoretical points of view.[1] An interesting feature is that the two-dimensionality introduces the possibility of topological excitations. When the Josephson coupling dominates, these topological excitations are vortices.[2] In the opposite regime, when the Coulomb interaction dominates, the array can be described in terms of $2e$ charges (Cooper pairs).[3,4]

The duality between a description in terms of these Cooper pairs (CP's) or vortices has recently gained much attention,[5] in particular, in relation to the superconductor-insulator transition in thin films.[6–8] A two-dimensional (2D) array is a very suitable system for studying this duality, since its dynamics is determined by the junction parameters (capacitance, critical current), which are reasonably well known in experimental systems.[9]

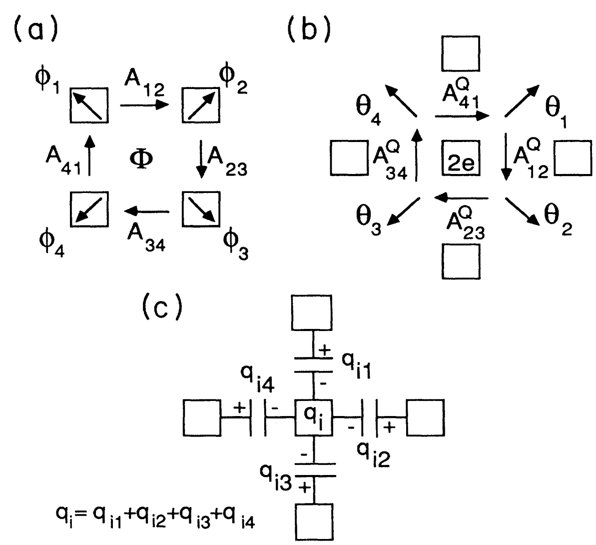

FIG. 1. (a) illustrates a vortex, which is defined by the phases ϕ_i of the superconducting islands. The vector potentials A_{ij} are associated with the flux Φ through the plaquette. (b) illustrates a CP, which is represented as a vortex in the ϑ phases which are defined on the plaquettes. (c) shows that the charge vector potentials A^Q_{ij} are related to the induced charges q_{in} on the capacitors between the plaquettes, e.g., $A^Q_{12}=q_{i2}$. Different choices of gauge correspond with different distributions of the charges over the capacitors.

I study a system which consists of a square array of $N\times N=M$ islands, each coupled to its four nearest neighbors with capacitors C, and Josephson junctions with a critical current I_c, and a coupling constant $E_J=\hbar I_c/2e$. The normal-state resistance of the junctions is assumed to be much larger than $h/4e^2$, so that dissipation can be neglected. Note that this model implies that the capacitances between non-nearest neighbors can be neglected[10] compared to C. Periodic boundary conditions are used. These boundary conditions, which correspond to a torus geometry, imply that all sites in the array are equivalent. Magnetic fluxes Φ_i can be applied through each of the plaquettes formed by four islands[11] (see Fig. 1). The islands are also coupled to voltage sources V_i by means of capacitors C_0. When $C_0 \ll C$, the screening length for the charge[3] $\lambda_c=\sqrt{C/C_0}$ is much larger than the array size. The induced charges on the islands are then fixed by the voltage sources V_i and given by $q_i=C_0V_i$. The Hamiltonian of the system now reads

$$H_c=4E_c\sum_{i,j}\left[N_i-\frac{q_i}{2e}\right](\underline{M}^{-1})_{ij}\left[N_j-\frac{q_j}{2e}\right]-\sum_{\langle i,j\rangle}E_J\cos\left[\phi_i-\phi_j+\frac{2e}{\hbar}A_{ij}\right]$$
$$\text{with } A_{ij}=\int_i^j \mathbf{A}\cdot d\mathbf{l}\ . \quad (1)$$

The first term describes the Coulomb energy ($E_c=e^2/2C$), N_i is the number of Cooper pairs on the ith island, and $\underline{M}^{-1}=C\underline{C}^{-1}$, with $\underline{C}^{-1}$ the inverse capacitance matrix. The charges interact logarithmically[3] when their separation $r_{ij}<\lambda_c$ [this implies that $(\underline{M}^{-1})_{ij}\sim\ln(r_{ij})$ for $r_{ij}<\lambda_c$]. Note that, for fabricated arrays, λ_c was estimated to be about 20. The second term in (1) describes the Josephson coupling between nearest neighbors, which is a function of the gauge-invariant combination of the phases ϕ_i and a line integral of the magnetic vector potential, A_{ij}. The commutation relation between N_i and ϕ_i reads $[N_i,\phi_i]=i$. The eigenfunctions of (1) are determined by both sets of induced charges q_i and the fluxes Φ_i. The invariance of ϕ_i under the transformation $\phi_i \to \phi_i + 2\pi$ imposes the boundary conditions on the wave function

$$\Psi(\ldots,\phi_i,\ldots)=\Psi(\ldots,\phi_i+2\pi,\ldots)\ .$$

The eigenfunctions can then be written as [12]

$$\Psi(\phi_1,\phi_2,\ldots,\phi_M)=\sum_{\mathbf{k}} c_{\mathbf{k}} \exp(i\mathbf{k}\cdot\boldsymbol{\phi})$$

$$\text{with } \phi=(\phi_1,\phi_2,\ldots,\phi_M) \text{ and } \mathbf{k}=(k_1,k_2,\ldots,k_M)\ . \tag{2}$$

The coefficients $c_{\mathbf{k}}$ can now be obtained by diagonalizing the matrix with diagonal elements given by the Coulomb energy associated with a distribution $\mathbf{k}=(k_1,k_2,\ldots,k_M)$ of CP's, and off-diagonal elements $E_J/2$ between states which are coupled by the tunneling of a CP from one island to an adjacent one.

First I study the dynamics of a single CP, which can be introduced in the array by applying a uniform charge $q_i=-2e/M$ on the islands. When $E_J=0$, the ground state of the system is M-fold degenerate, since the Coulomb energy is minimized when one CP is present on one of the M islands. The excited states with the lowest energy have an additional pair of $2e$ or $-2e$ charges on nearest-neighbor islands and have an energy of about $2E_C$ above the ground state. One can therefore construct the eigenfunctions in the presence of weak Josephson coupling ($E_J \ll 2E_C$) from the states $|x,y\rangle$ which describe the presence of a single CP on the ith island, with coordinates x and y. The Hamiltonian (1) now reduces to a tight-binding Hamiltonian:

$$\begin{aligned} H_c=&\sum_{x,y} E_0|x,y\rangle\langle x,y| \\ &+\sum_{x,y}\frac{E_J}{2}\exp\left[i\frac{2e}{\hbar}A_{ij}\right]|x,y\rangle \\ &\quad\times(\langle x+1,y|+\langle x-1,y| \\ &\qquad+\langle x,y+1|+\langle x,y-1|)\ . \end{aligned} \tag{3}$$

In the absence of a magnetic field the solutions are given by[13]

$$\Psi(k_x,k_y)=\sum_{x,y}\exp(ik_x x+ik_y y)|x,y\rangle \tag{4}$$

with eigenvalues $E(k_x,k_y)=E_J[2-\cos(k_x x)-\cos(k_y y)]$. Note that due to the periodic boundary conditions, the values of k_x and k_y are restricted to multiples of *$2\pi/N$. In a large array the discreteness of k_x and k_y can be neglected, and $E(\mathbf{k})=\hbar^2\mathbf{k}^2/(2m_c)$ for $E \ll E_J$, with $m_c=\hbar^2/E_J$. This shows that the dynamics of a CP can be described by that of a particle with a mass m_c, determined by the Josephson coupling E_J. An alternative way to understand this kinetic energy is that a (Josephson) current $I=2ev$ flows through the junction which is being crossed by a CP which moves with a velocity v. The energy can then be written as

$$E_J[1-\cos(\Delta\phi)]\approx E_J[1-(1-I^2/I_c^2)^{1/2}]\ ,$$

which reduces to $\frac{1}{2}m_c v^2$ for $I \ll I_c$. When the Josephson coupling between the islands is weak and m_c is large, the CP motion can be described classically. When E_J is increased, quantum effects will become prominent (such as tunneling through electrostatic potential barriers). The motion of the CP in a uniform magnetic field can be obtained by the substitution $\hbar\mathbf{k}\to\mathbf{p}+2e\,\mathbf{A}$, which yields

$$H_c=\frac{(\mathbf{p}+2e\,\mathbf{A})^2}{2m_c} \quad \text{with } \nabla\times\mathbf{A}=\mathbf{B} \tag{5}$$

with $\mathbf{p}$ the canonical momentum of the CP. The validity of (5) was checked by a calculation of the energy spectrum of (3) for a 6×6 array. For low energies, $E \ll E_J$ and fluxes $\Phi_i \ll \Phi_0$, the obtained eigenvalues correspond closely[14] with Landau level energies $E_n=(n+\frac{1}{2})\hbar\omega_c$, with $\omega_c=2eB/m_c$. Equation (5) shows that, in the classical limit, the CP experiences a Lorentz force $\mathbf{F}=2e\mathbf{v}\times\mathbf{B}$, even though no magnetic field acts on the CP (the magnetic field is screened from the superconductor).[15]

When E_J is increased and becomes of the order of $2E_C$, the description of the array dynamics in terms of single CP's fails. However, when the opposite limit $E_J \gg E_C$ is reached, it is known that the array dynamics can again be described in terms of particles, which, in this case, are vortices.[16] In this limit one can also start from (1) and (2) to calculate the dynamics of the array. However, this requires large computational effort since many charge states $|k_1,k_2,\ldots,k_M\rangle$ have to be included.[17] Therefore, it is worthwhile constructing a Hamiltonian which is defined in terms of vortices instead of CP charges. Such a Hamiltonian will be particularly useful to describe the quantum-mechanical behavior of vortices.

First the wave function $\Psi(\phi_1,\phi_2,\ldots,\phi_M)$ is expressed as a wave function $\Psi(N_1^v,N_2^v,\ldots,N_M^v)$ of the vortex charges N_i^v on each of the plaquettes. Note that the replacement of the phase variables by discrete variables N_i^v is not exact, since, in general, a phase configuration cannot be exactly represented by a set of vortex charges.[18] Because both Josephson and Coulomb interactions take place exclusively between nearest neighbors, it is possible to express the Josephson energy which is associated with a distribution of vortices in a similar way as the Coulomb energy of a distribution of CP charges [this is the first term in (6)]. In the absence of capacitive coupling, the eigenstates of (6) are given by distributions $(N_1^v,N_2^v,\ldots,N_M^v)$ of vortex charges. The presence of capacitors will couple states by the transfer of a vortex from a plaquette to one of its four nearest-neighbor plaquettes, in a similar way as Josephson coupling couples states by a transfer of a CP to a nearest-neighbor island. When one describes the vortices as bosons, the conjugated variables of the vortex charges N_i^v are phases ϑ_i with the commutation relation[19] $[N_i^v,\vartheta_i]=i$. The phase ϑ_i can be regarded as the phase of the macroscopic wave function of the vortices, in a similar way as the phase ϕ_i of the superconducting islands describes the macroscopic wave function of the CP's.

In Ref. 20 the dynamics of a single vortex in a ring-shaped (Corbino) array was studied. For the description of the response of the vortex to the induced charge Q_0 on the center island, a charge vector potential $\mathbf{A}^Q$ was introduced, which was defined by the line integral $\oint \mathbf{A}^Q \cdot dl = Q_0$. In a discrete lattice, the vector potentials, A_{ij}^Q between plaquettes are defined by the sum around an island $\oint\!\!\!\sum A_{ij}^Q = q_i$ (see Fig. 1). The coupling between adjacent plaquettes i and j is now expressed as a function of the gauge-invariant combination $\vartheta_i - \vartheta_j + 2\pi/(2e) A_{ij}^Q$, which yields the following Hamiltonian:

$$H_v = 2\pi^2 E_J \sum_{i,j} \left[N_i^v - \frac{2e}{h}\Phi_i \right] (\underline{M}^{-1})_{ij} \left[N_j^v - \frac{2e}{h}\Phi_j \right] - \frac{2}{\pi^2} E_C \sum_{\langle i,j \rangle} \cos\left[\vartheta_i - \vartheta_j + \frac{2\pi}{2e} A_{ij}^Q \right] \quad (6)$$

The cosine in the second term of (6) implies that the coupling between adjacent plaquettes can be described by the transfer of single vortices. The prefactor of this term is obtained from a comparison with a single low capacity junction which is driven by an external induced charge Q_0, which acts as a vector potential $\mathbf{A}^Q = \mathbf{Q}_0$. For $A_{ij}^Q \ll 2e$, the second term in (6) can then be written as $\frac{1}{2} Q_0^2 / C$, which is equal to the Coulomb energy of the junction. I emphasize that (6) is not exact.[21,22] Probably the most important disparity between (1) and (6) is due to the fact that the vortex charge on a plaquette is limited to values -1, 0, and 1, whereas (1) correctly describes the Coulomb interaction for an arbitrary number of CP's on the islands. This may mean that a description of the array dynamics with (6) fails when $E_C \gg E_J$.

A comparison of (1) and (6) shows that the magnetic field, which enters (1) as a vector potential, is included in (6) in the potential-energy term of the vortices. On the other hand, the induced charge density q_i is included in the CP Hamiltonian in the potential-energy term, but is present in (6) as a vector potential. A comparison between the properties of Cooper pairs and vortices is given in Table I.

From (6) one can derive two Josephson-type equations for vortices:

$$\frac{d\vartheta_i}{dt} = \frac{1}{\hbar}\frac{\partial H}{\partial N_i^v} = \frac{2\pi^2}{\hbar} E_J \sum_j (\underline{M}^{-1})_{ij} N_j^v = \frac{2\pi}{2e} V_i^v , \quad (7)$$

where V_i^v is the vortex potential of the ith plaquette. Similarly, one obtains

$$\frac{dN_i^v}{dt} = -\frac{1}{\hbar}\frac{\partial H}{\partial \vartheta_i} = \frac{2E_C}{\pi^2 \hbar} \sum_{\langle i,j \rangle} \sin\left[\vartheta_i - \vartheta_j + \frac{2\pi}{2e} A_{ij}^Q \right] . \quad (8)$$

Alternatively, the vortex current I_{ij}^v which flows between the plaquettes i and j can be expressed as

TABLE I. Comparison between the properties of vortices and Cooper pairs. F_v denotes the force on vortex, F_c denotes the force on CP, j_c denotes the CP current density, j_v denotes the vortex current density, v denotes the velocity, B denotes the magnetic field, and q denotes the induced charge density.

	Vortex	CP
Charge	$\frac{h}{2e}$	$2e$
Mass	$\frac{\pi^2}{2}\frac{\hbar^2}{E_c}$	$\frac{\hbar^2}{E_J}$
Force	$\mathbf{F}_v = \frac{h}{2e}\mathbf{j}_c \times \mathbf{n}$	$\mathbf{F}_c = 2e\mathbf{j}_v \times \mathbf{n}$
"Lorentz" force	$\mathbf{F}_L = (h/2e) q \mathbf{v} \times \mathbf{n}$	$\mathbf{F}_L = 2e\mathbf{v} \times \mathbf{B}$

$$I_{ij}^v = \frac{\partial H}{\partial A_{ij}^Q} = \frac{2}{\pi e} E_C \sin\left[\vartheta_i - \vartheta_j + \frac{2\pi}{2e} A_{ij}^Q \right] . \quad (9)$$

One can now use (6) as a starting point for the study of the dynamics of a single vortex, which can be introduced in the array by applying a magnetic flux $\Phi_i = \Phi_0 / M$ per plaquette. By using the duality between vortices and CP's, the results for the vortex can now be obtained from the CP results by the appropriate substitutions. In particular, when $E_C \ll E_J$ the continuum Hamiltonian of a single vortex reads[20]

$$H_v = \frac{(\mathbf{p} + \Phi_0 \mathbf{A}^Q)^2}{2m_v} \quad \text{with } \nabla \times \mathbf{A}^Q = q\mathbf{n} \quad (10)$$

with $\mathbf{p}$ the canonical momentum of the vortex, $\mathbf{n}$ the unit vector normal to the array, q the induced charge density, and the vortex mass $m_v = \pi^2\hbar^2/(2E_C)$. A vortex which moves with a velocity v generates a voltage across the junctions which it traverses given by

$$V = \hbar/2e d(\Delta\phi)/dt = h/2ev .$$

This results in a Coulomb energy

$$E = \tfrac{1}{2} C (h/2e)^2 v^2 = \tfrac{1}{2} m_v v^2 .$$

A striking implication of (10) is that the motion of a vortex in a uniform induced charge density q is similar to that of a CP in a uniform magnetic field B. Therefore, (10) predicts that the vortex experiences a Lorentz-type force, perpendicular to its direction of motion, which is given by [23] $\mathbf{F}_L = (h/2e) q \mathbf{v} \times \mathbf{n}$.

The duality between vortices and CP's is important for the phase transitions in the array. When $E_J \gg E_C$, a Kosterlitz-Thouless phase transition takes place at $T_v = \pi/(2\varepsilon_c) E_J$ (with ε_c slightly larger than 1), when pairs of vortices and antivortices unbind.[2] The array is superconducting for $T < T_v$. It was also proposed [3,4] that a similar phase transition should occur in the opposite regime $E_C \gg E_J$ at a temperature $T_c = 1/(\pi\varepsilon_c) E_C$, which, in this case, is driven by the unbinding of charge $2e/-2e$ pairs. Below this temperature the system is an insulator. A comparison of (1) and (6) shows that vortices and CP's play equal roles, and the system is self-dual, when

$E_C = (\pi^2/2)E_J$. Therefore one expects that, for this ratio of E_C and E_J, the system will neither become an insulator nor a superconductor at low temperatures.

The vortex and CP currents can be written as $j_v = \partial H/\partial A^Q$ and $j_c = \partial H/\partial A$, respectively. The self-duality of the array implies that these expressions should give identical results when they are expressed in dimensionless units. Therefore, the array resistivity

$$\rho_{xx} = (h/4e^2) j_v/j_c = h/4e^2 \ .$$

A similar result can be obtained when uniform fluxes $\Phi_i = c\Phi_0$ are applied. Note that, in this case, the system is self-dual only when charges $q_i = c2e$ are induced simultaneously. In this case the relation

$$\rho_{xx}^2 + \rho_{xy}^2 = h^2/(4e^2)^2$$

is obtained.[7]

It was already mentioned that Eq. (6) is not exact. I now briefly discuss the possible consequences of this for the self-duality. It can be argued[24] that the fact that the vortex charge on a plaquette is restricted to -1, 0, or 1 does not significantly affect the self-duality between Cooper pairs and vortices at $E_C/E_J = \pi^2/2$. Equation (6) does not describe the spin waves of the phases ϕ. On the other hand, Eq. (6) has solutions which correspond to spin waves of the phase ϑ. These, however, are not there in the actual system, described by Eq. (1). The presence of this asymmetry means that the critical resistivity may shift away from $h/4e^2$. This is also discussed by Fazio and Schön.[25] They also discuss the effect of quasiparticle dissipation on the phase-transitions in the array, in the regime where the normal-state resistance of the array is comparable to, or smaller than, $h/4e^2$.

In real systems the induced charges will be distributed randomly between 0 and $2e$. In this case the duality may still occur, provided that the fluxes Φ_i are also randomly distributed between 0 and $h/2e$. Such a system may be fabricated by an array in a uniform magnetic field, but with random area of the plaquettes, such that the fluctuation of the flux Φ_i is larger than Φ_0.

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[10]When the nearest-neighbor and non-nearest-neighbor capacitances are of the same order of magnitude, the CP charges do not interact logarithmically, and there is no exact duality.

[11]It is assumed that the magnetic fluxes created by the currents in the array are much smaller than the applied fluxes Φ_i. This requires that the magnetic screening length $\lambda = h/(4\pi^2\mu_0 I_c)$ is much larger than the array size N.

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