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Extended Observer Forms for Submersive Discrete-Time Systems

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***Abstract*—This technical note addresses the problem of transforming a single-input–single-output discrete-time system into the extended observer form, which comprise a linear time-invariant observable component, and a nonlinear injection term, which depends on the input, output, and their forward shifts up to a finite order. Intrinsic necessary and sufficient conditions are provided for obtaining the extended observer form via a parametrized state transformation. The conditions are formulated directly in terms of the state equations and do not rely on input–output equations as in the earlier papers. Further, an algorithm for obtaining the required transformation is presented. Unlike the existing results on observer forms, the results are not restricted to reversible systems but to more general submersive systems, i.e., to systems, which are reversible via static state feedback.**

***Index Terms*—Discrete-time system, extended observer form, geometric methods, nonlinear observers.**

I. Introduction

This technical note is devoted to transformation of discrete-time state equations

$$x^+ = F(x,u), \quad y = h(x) \tag{1}$$

via a parametrized state transformation $z = \Phi(x,\bar{u})$ into the extended observer form with degrees (l,s)

$$z^+ = Az + \Gamma(\tilde{z},\tilde{u}), \quad y = Cz \tag{2}$$

where for simplicity we omit the time argument $k \in \mathbb{Z}$ and use the shorter notations $\nu := \nu(k)$, $\nu^+ := \nu(k+1)$, $\nu^{[r]} := \nu(k+r)$ for an integer r, and $\nu := \nu^{[0]}$. Moreover, we assume that x belongs to an n dimensional smooth (C^∞) manifold M, $u \in \mathbb{R}$, and $F : M \times \mathbb{R} \to M, h : M \to \mathbb{R}$ are smooth functions, whereas $\Phi(x,\bar{u})$ is a diffeomorhpism on M, parametrized by $\bar{u} := (u,\ldots,u^{[l-1]}) \in \mathbb{R}^l$ and defined in the neighborhood $\mathcal{O}$ of an equilibrium point $(x_{eq},\bar{u}_{eq}) \in \bar{M} := M \times \mathbb{R}^l$, with a left inverse $\Phi^\dagger$ defined in the neighborhood of $(0,0) \in \mathbb{R}^n \times \mathbb{R}^l$, such that $\Phi^\dagger(\Phi(x,\bar{u}),\bar{u}) = x$ for $(x,\bar{u}) \in \mathcal{O}$. Note that for any fixed $\bar{u}$, $\Phi(\cdot,\bar{u})$ is assumed to be a local diffeomorphism. Furthermore, in (2)

$$z := [z_1 \cdots z_n]^\mathsf{T},$$

$$A := \begin{bmatrix} O_{n-1,1} & I_{n-1} \\ 0 & O_{1,n-1} \end{bmatrix},$$

$$C := \begin{bmatrix} 1 & O_{1,n-1} \end{bmatrix}, \quad \Gamma(\tilde{z},\tilde{u}) := \begin{bmatrix} O_{s,1} \\ \gamma_{s+1}(\tilde{z},\tilde{u}) \\ \vdots \\ \gamma_n(\tilde{z},\tilde{u}) \end{bmatrix} \tag{3}$$

where $\tilde{z} := (z_1,\ldots,z_{s+1})$, $\tilde{u} := (\bar{u},u^{[l]})$, $O_{i,j}$ denotes $i \times j$ zero matrix, and I_i is $i \times i$ identity matrix.

Assumption 1: The system (1) is submersive, i.e., rank $\partial F(x,u)/\partial(x,u) = n$ (see [1]), and nonreversible, i.e., rank $\partial F(x,u)/\partial x = n-1$.

If the system (1) admits the form (2) and one can measure/estimate $\tilde{z} = (y,\ldots,y^{[s]})$, and $\tilde{u}$, then one can construct a state observer $\hat{z}^+ = A\hat{z} + \Gamma(\tilde{z},\tilde{u}) + L(C\hat{z} - Cz)$. The dynamics of the observer error $e := \hat{z} - z$ is obtained as $e^+ = (A+LC)e$. Since the pair (A,C) is in the observer canonical form, the matrix L may be chosen such that the eigenvalues of $(A+LC)$ are within the unit circle of the complex plane, thereby guaranteeing that the error asymptotically decays to zero, and the original state may be recovered as $\hat{x} = \Phi^\dagger(\hat{z},\bar{u})$, which asymptotically converges to x due to continuity of Φ.

Various generalizations of the observer forms can be found in [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], and [12]. However, in these works one may identify two main problems that remain to be addressed. First, most of the results (those which are coordinate independent and based on state equations) assume that the system is reversible. Second, the works which consider input injection (such as [2], [6], [7], [8], [9], [10], [11], and [12]) either require one to check an infinite number of conditions (intractable to verify), or allow the linear part of the observer form to be input dependent (which complicates the design of an observer with guaranteed global stability), or require to find the input–output equation of the system as an intermediate step. Note that [4] and [5] addressed only input-free case. Therefore, the contribution of this technical note is the following.

1) Finite number of coordinate independent, differential geometric conditions for transforming system (1) into the form (2). The conditions extend the input-free case from [13].
2) Extension of the results to systems, which are not necessarily reversible, but only submersive (i.e., reversible via static state feedback).
3) A constructive algorithm for finding the parametrized state transformation, requiring to solve a certain nonlinear PDE.

We refer the reader to [14] for preliminaries on differentiable manifolds and [15] for geometric control paradigm.

Throughout this article we use the iterated function notation, i.e., for a function f denote $f^0 :=$ id, $f^1 := f$, and $f^k := f \circ f^{k-1}$ for $k > 0$. In what follows, with a slight abuse of notations, $u^{[i]}$ denotes both the input shifts (the values of a discrete time function) and the extended state variables or parameters of the parametrized state transformation. Obviously, $(u^{[i]})^{[1]} = u^{[i+1]}$.

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From here onward, unless stated otherwise, all assertions are local, i.e., valid in the neighborhood of an equilibrium point $(x_{\text{eq}}, u_{\text{eq}}) \in M$ or $(x_{\text{eq}}, \bar{u}_{\text{eq}}) \in \bar{M}$, depending on the context.

Definition 1 (see [16]): Given a vector field X on M and a diffeomorphism $\Theta : M \to M$, define $Ad_\Theta X = (\partial\Theta/\partial x) \circ (\Theta^{-1}(x))X \circ (\Theta^{-1}(x))$ and recursively $Ad^0_\Theta X(x) = X(x)$, $Ad^{k+1}_\Theta X(x) = Ad_\Theta Ad^k_\Theta X(x)$, $k \geq 0$.

The Ad operator is coordinate invariant [17], i.e., given diffeomorphisms Θ and Ω

$$Ad_\Omega Ad_\Theta X = Ad_{\Omega\circ\Xi\circ\Omega^{-1}} Ad_\Omega X. \tag{4}$$

II. Extended System With Input Prolongation

Below we will construct, under Assumption 1, an extended system with input prolongations, associated with dynamics (1) so that the input and its forward shifts $\bar{u}$, which appear in the nonlinear injection term $\Gamma(\tilde{z}, \tilde{u})$ in (2), will be considered as extended state coordinates. Extension will allow to replace the parametrized state transformation $\Phi(x, \bar{u})$, having only a pseudoinverse, by an ordinary diffeomorphism $\Phi_e(\xi)$. Our extended system is a bit different from the standard one where the forward shift of $u^{[l-1]}$ is taken to be equal to v, a new input, whereas we take it equal to $h(x) + v$. The purpose for combining prolongation with output feedback $u^{[l]} = h(x) + v$ is to correct the rank deficiency and guarantee reversibility of the map $\bar{F}(\xi, v)$, as proven in Lemma 3 below. This allows later to use the definition of Ad operator for the map $\bar{F}(\xi, v)$, when $v = 0$.

Denote $\xi = (x, \bar{u}) \in \bar{M}$ and $\bar{F}(\xi, v) := [F^{\mathsf{T}}(x, u)\ u^{[1]} \cdots u^{[l-1]}\ h(x) + v]^{\mathsf{T}}$. The equations of the extended system on $\bar{M}$ are

$$\xi^+ = \bar{F}(\xi, v), \quad y = h_e(\xi) = h(x). \tag{5}$$

Remark 1: If the original system (1) is reversible, then one has to define the last extended state equation as $(u^{[l-1]})^{[1]} = u + v$ in order to counteract rank deficiency with respect to u and to ensure reversibility of $\bar{F}$. However, in this article we do not focus on this case.

We compute $x^{[-1]}$ as the first n components of $\xi^{[-1]} = \bar{F}^{-1}(\xi)$.

Lemma 1: For a vector field X on $\bar{M}$ and an iterated function f^i, where $f : \bar{M} \to \mathbb{R}$, the following holds $\langle \mathrm{d}(f^i), Ad^j_f X\rangle(\xi) = \langle \mathrm{d}(f^{i+j}), X\rangle(\xi^{[-j]})$.

Proof: Relies on the chain rule and Definition 1. ■

Lemma 2: System (1) can be locally transformed into the extended observer form (2) via a parametrized state transformation $z = \Phi(x, \bar{u})$ if and only if the extended system (5) can be locally transformed, via an ordinary diffeomorphism

$$\zeta = \Phi_e(\xi) = \left(\Phi^{\mathsf{T}}(x, \bar{u}), \bar{u}^{\mathsf{T}}\right)^{\mathsf{T}} \tag{6}$$

into the form

$$\zeta^+ = [Az + \Gamma(\tilde{z}, \bar{u}, v)\ u^{[1]} \cdots u^{[l-1]}\ z_1 + v]^{\mathsf{T}},\ y = Cz \tag{7}$$

where $\zeta = (z, \bar{u})$ and A, C, and Γ are defined as in (3).

Note that in (7) we have incorporated the feedback $u^{[l]} = h(x) + v$, which allows us to replace the argument $\tilde{u}$ in (2) by $(\bar{u}, v)$.

Proof: Given $\Phi(x, \bar{u})$ in $\mathcal{O}$, that maps (1) into (2), we construct a map Φ_e in $\mathcal{O}$ by (6). Since $\Phi(\cdot, \bar{u})$ is invertible for any fixed $\bar{u}$, one can easily verify that Φ_e is a diffeomorphism on $\mathcal{O}$, with a smooth inverse $\Phi_e^{-1} : \Phi(\mathcal{O}) \to \mathcal{O}$, given by $\Phi_e^{-1}(z, \bar{u}) = (\Phi^{\dagger}(z, \bar{u}), \bar{u})$. From (2) and the fact that the last l components of Φ_e are identity functions, one can immediately conclude that the dynamics of $\zeta = \Phi_e(\xi)$ is in the form (7), after applying the feedback $u^{[l]} = h(x) + v$.

Conversely, given a local diffeomorphism (6), it is obvious that $\Phi(x, \bar{u})$ is locally invertible with respect to x and, because of (7), transforms system (1) into the form (2). ■

Lemma 3: If the system (1) is nonreversible, submersive, and can be transformed into the extended observer form (2), then the extended system (5) is reversible.

Proof: Note that (non-)reversibility and submersivity are invariant under state transformation. Furthermore, under the hypothesis of this lemma and according to Lemma 2, the system (5) can be transformed into the form (7). Therefore, it suffices to prove that nonreversibility and submersivity of (2) implies reversibility of (7).

Denote the right-hand sides of the state equations in (2) and (7) by $G(z, \tilde{u})$ and $\bar{G}(\zeta, v)$, respectively. Compute

$$\partial G(z,\tilde{u})/\partial(z,u) = \left[\partial G(z,\tilde{u})/\partial z \,\middle|\, \partial G(z,\tilde{u})/\partial u\right]$$
$$= \left[\begin{array}{ccc|c} O_{s,1} & I_s & O_{s,n-1-s} & O_{s,1} \\ \partial\alpha/\partial z_1 & \partial\alpha/\partial\beta & I_{n-1-s} & \partial\alpha/\partial u \\ \partial\gamma_n/\partial z_1 & \partial\gamma_n/\partial\beta & O_{1,n-1-s} & \partial\gamma_n/\partial u \end{array}\right]$$

where $\alpha := [\gamma_{s+1} \cdots \gamma_{n-1}]^{\mathsf{T}}$ and $\beta := (z_2, \ldots, z_{s+1})$. Note that nonreversibility of (2) implies $\mathrm{rank}(\partial G(z,\tilde{u})/\partial z) < n$, which is only possible when $\partial\gamma_n/\partial z_1 \equiv 0$. At the same time the submersivity assumption requires that $\mathrm{rank}(\partial G(z,\tilde{u})/\partial(z,u)) = n$, resulting in $\partial\gamma_n/\partial u \not\equiv 0$. Next, compute

$$\partial\bar{G}(\zeta, v)/\partial\zeta$$
$$= \begin{bmatrix} O_{s,1} & I_s & O_{s,n-1-s} & O_{s,1} & O_{s,l-1} \\ \partial\alpha/\partial\zeta_1 & \partial\alpha/\partial\sigma & I_{n-1-s} & \partial\alpha/\partial\zeta_{n+1} & \partial\alpha/\partial\varsigma \\ \partial\gamma_n/\partial\zeta_1 & \partial\gamma_n/\partial\sigma & O_{1,n-1-s} & \partial\gamma_n/\partial\zeta_{n+1} & \partial\gamma_n/\partial\varsigma \\ O_{l-1,1} & O_{l-1,s} & O_{l-1,n-1-s} & O_{l-1,1} & I_{l-1} \\ 1 & O_{1,s} & O_{1,n-1-s} & O_{1,1} & O_{1,l-1} \end{bmatrix}$$

where $\sigma := (\zeta_2, \ldots, \zeta_{s+1})$ and $\varsigma := (\zeta_{n+2}, \ldots, \zeta_{n+l})$, whereas the fourth horizontal and the last vertical partitions of the block matrix appear only if $l > 1$. Taking into account the definition of the ζ coordinates, one may conclude that $\partial\gamma_n/\partial u \not\equiv 0$ implies $\partial\gamma_n/\partial\zeta_{n+1} \not\equiv 0$. Note that if one moves the last row to the top of the matrix, then its main diagonal will contain 1's everywhere, except for $\partial\gamma_n/\partial\zeta_{n+1}$, which cannot be 0. Moreover, in this case the matrix has block lower-triangular structure, where its upper left-hand block is $(s+1) \times (s+1)$ identity matrix and lower right-hand block is $(n-s+l-1) \times (n-s+l-1)$ upper triangular matrix. Consequently, the matrix has full rank, which is $n + l$. Therefore, the system (7) (and (5) as a consequence) is reversible. ■

Remark 2: Note that Lemma 3 essentially claims that systems, which admit extended observer forms, can necessarily be made reversible by special output feedback.

III. Intrinsic Conditions for Extended Observer Forms

In this section we give coordinate-free, constructive necessary and sufficient conditions for transformability of the system (1) into the extended observer form (2).

In what follows, we denote functions of ξ depending only on x component as $h(\xi) = h(x)$, with a slight abuse of notation. Moreover, denote $F_0(\xi) := \bar{F}(\xi, 0)$, $F_v(\xi) = \bar{F}(\xi, v)$. Furthermore, $\delta_{k,n}$ stands for the Kronecker delta. Note that the statement of the following theorem holds locally in the neighborhood of an equilibrium point.

Theorem 1: Under Assumption 1 the nonlinear system (1) is locally transformable by a parametrized state transformation $z = \Phi(x, \bar{u})$ into the extended observer form (2) if and only if the following conditions are satisfied.

(A) The one-forms $\mathrm{d}h$, $\mathrm{d}(h\circ F_0)$, ..., $\mathrm{d}(h\circ F_0^{n-1})$, $\mathrm{d}\xi_{n+1}$, ..., $\mathrm{d}\xi_{n+l}$ are linearly independent.

(B) The vector field g on $\bar{M}=M\times\mathbb{R}^l$, uniquely defined by

$$\langle \mathrm{d}\xi_{n+i}, g\rangle = 0, \qquad i=1,\dots,l \tag{8a}$$

$$\left\langle \mathrm{d}\left(h\circ F_0^{k-1}\right), g\right\rangle = \delta_{k,n}, \qquad k=1,\dots,n \tag{8b}$$

satisfies

$$\left[Ad_{F_0}^r g, Ad_{F_0}^q g\right]=0; \quad r,q=0,\dots,n-s-1. \tag{9}$$

(C) The output h satisfies $\partial h^{[k]}/\partial v=0$, $k=0,\dots,s$.

(D) The vector fields $\eta_i := Ad_{F_0}^{i-1}g$, $i=1,\dots,n-s-1$ satisfy $\partial(Ad_{F_v}\eta_i)/\partial v=0$.

Proof: From Lemma 2 it suffices to show that the conditions of the theorem are necessary and sufficient for transforming (5) into (7) via a diffeomorphism $\zeta=\Phi_e(\xi)$. Moreover, from Lemma 3, the map F_v is invertible for any v, and therefore the map F_0 is invertible and thereby the operator Ad_{F_0} and vector fields in condition (B) and (D) are well defined. Further from condition (A) and (8) we see that the vector field g is uniquely defined.

Necessity: Assume that an appropriate transformation Φ_e exists. Denote F_0, g, h in the ζ coordinates as $\tilde{F}_0=\Phi_e\circ F_0\circ\Phi_e^{-1}(\zeta)$, $\tilde{g}=Ad_{\Phi_e}g$, $\tilde{h}(\zeta)=h\circ\Phi_e^{-1}(\zeta)$.

(A) Using (7) for $v\equiv 0$, one may obtain

$$\tilde{h}^{[k-1]}=z_k+\begin{cases}0, & k=1,\dots,s+1\\ \rho_k(z_1,\dots,z_{k-1},\bar{u}), & k=s+2,\dots,n\end{cases}$$

where ρ_k are certain sums of $\gamma_{s+1},\dots,\gamma_{n-1}$ and their forward shifts. Relying on the structure of $\tilde{h}^{[k-1]}$, it is easy to verify that $\mathrm{d}(\tilde{h}^{[k-1]})$, $k=1,\dots,n$, and $\mathrm{d}u,\dots,\mathrm{d}u^{[l-1]}$ form a linearly independent set of $n+l$ one-forms. The identity $\tilde{h}^{[k]}=\tilde{h}\circ\tilde{F}_0^k$ and the fact that Φ_e is a diffeomorphism then imply condition (A).

(B) Taking into account (7) for $v\equiv 0$, the definition (8) in the ζ coordinates leads to $\tilde{g}=\partial/\partial z_n$. The direct computations yield $Ad_{\tilde{F}_0}^k\tilde{g}=\partial/\partial z_{n-k}$, $k=0,\dots,n-s-1$, using which one obtains $[Ad_{\tilde{F}_0}^r\tilde{g}, Ad_{\tilde{F}_0}^q\tilde{g}]=[\partial/\partial z_{n-r},\partial/\partial z_{n-q}]=0$ for $r,q=0,\dots,n-s-1$. Thus, (9) is satisfied, since Φ_e is a diffeomorphism and the Lie-bracket is invariant under diffeomorphisms.

(C) Follows directly from (7).

(D) Denote the transformed dynamics as $\tilde{F}_v:=\Phi_e\circ F_v\circ\Phi_e^{-1}$, and $\tilde{\eta}_i:=Ad_{\Phi_e}\eta_i$. Since the definition of g and η_i are coordinate independent, we observe from the transformed dynamics (7) that $\tilde{\eta}_i=\partial/\partial z_{n-i+1}$, $i=1,\dots,n-s-1$. Therefore, $Ad_{\tilde{F}_v}\tilde{\eta}_i = Ad_{\Phi_e\circ F_v\circ\Phi_e^{-1}}Ad_{\Phi_e}\eta_i = Ad_{\Phi_e}Ad_{F_v}\eta_i=\partial/\partial z_{n-i}$. Since the right-hand side is independent of v, condition (D) follows trivially from the fact that Φ_e does not depend on v.

Sufficiency: We first consider the case $v\equiv 0$ and show that conditions (A) and (B) of the theorem are sufficient for constructing a diffeomorphism Φ_e as in (6), to obtain the partially linear form (7). Next, conditions (C) and (D) are used to establish that the trivial integrator form of the first s equations and the linear dependence on $z_{s+2},\dots,z_n$ are invariant with respect to the new control variable v.

The case $v\equiv 0$: Compute, by Lemma 1 and (8b)

$$\begin{aligned}\langle \mathrm{d}(h\circ F_0^i), Ad_{F_0}^j g\rangle(\xi) &= \langle \mathrm{d}(h\circ F_0^{i+j}), g\rangle(\xi^{[-j]})\\ &=\delta_{i+j+1,n}, \quad i+j=0,\dots,n-1\end{aligned} \tag{10}$$

from which one may make a conclusion that the matrix $\mathcal{A}:=[\mathrm{d}h\ \mathrm{d}(h\circ F_0)\ \cdots\ \mathrm{d}(h\circ F_0^{n-1})]^\mathsf{T}[g\ Ad_{F_0}g\ \cdots\ Ad_{F_0}^{n-1}g]$ has zeros above the antidiagonal and ones on the antidiagonal. Thus, $\operatorname{rank}\mathcal{A}=n$ and, as a consequence, the vector fields $Ad_{F_0}^j g$, $j=0,\dots,n-1$ are linearly independent.

From condition (9), using [15, Th. 2.36], we know that there exists a change of coordinates $\mathbb{R}^{n+l}\ni w=\Psi_1(\xi)$ such that

$$Ad_{\Psi_1}Ad_{F_0}^k g=\partial/\partial w_{n-k}, \quad k=0,\dots,n-s-1. \tag{11}$$

The system dynamics are written in the w coordinates as

$$w^+=\tilde{F}(w) \tag{12}$$

where

$$\tilde{F}=\Psi_1\circ F_0\circ\Psi_1^{-1}. \tag{13}$$

In order to determine the structure of $\tilde{F}$, one may use (4), Definition 1, and (13) to rewrite (11) as $Ad_{\tilde{F}}Ad_{\Psi_1}Ad_{F_0}^{k-1}g=\partial/\partial w_{n-k}$, $k=0,\dots,n-s-1$, where $Ad_{\Psi_1}Ad_{F_0}^{k-1}g$ can be replaced according to (11) for $k\neq 0$. This results in $Ad_{\tilde{F}}(\partial/\partial w_{n-k+1})=\partial/\partial w_{n-k}$, $k=1,\dots,n-s-1$, which can be rewritten as

$$\partial\tilde{F}/\partial w_{n-k+1}=\partial/\partial w_{n-k}, \quad k=1,\dots,n-s-1. \tag{14}$$

From (14) one may conclude that (12) possesses the following structure:

$$\begin{aligned}w_i^+ &= \tilde{\gamma}_i(w_1,\dots,w_{s+1},w_{n+1},\dots,w_{n+l})\\ &\quad+\begin{cases}0, & i=1,\dots,s,n,\dots,n+l\\ w_{i+1}, & i=s+1,\dots,n-1\end{cases}\end{aligned} \tag{15}$$

where $\tilde{\gamma}_1,\dots,\tilde{\gamma}_{n+l}$ are some smooth functions. Denote $\tilde{h}(w):=h\circ\Psi_1^{-1}(w)$ and next, when $s>0$ define the functions

$$q_i=\tilde{h}\circ\tilde{F}^{i-1}(w), \quad i=1,\dots,s. \tag{16}$$

Notice from condition (A) that $\mathrm{d}q_i$, $i=1,\dots,s$, and $\mathrm{d}u^{[j]}$, $j=0,\dots,l-1$ form a linearly independent set of one-forms. Further, using (10) and (11), observe that for $i=1,\dots,s$ and $k=0,\dots,n-s-1$

$$\begin{aligned}\langle \mathrm{d}q_i, \partial/\partial w_{n-k}\rangle(w) &= \langle \mathrm{d}(\tilde{h}\circ\tilde{F}^{i-1}), Ad_{\Psi_1}Ad_{F_0}^k g\rangle\\ &=\langle \mathrm{d}(h\circ F_0^{i-1}), Ad_{F_0}^k g\rangle\circ\Psi_1^{-1}(w)=0.\end{aligned} \tag{17}$$

Moreover, using (8a), the following can be deduced in a similar manner as above for $k=0,\dots,n-s-1$

$$\left\langle \mathrm{d}\bar{u}, Ad_{F_0}^k g\right\rangle = \left\langle \mathrm{d}\bar{u}\circ\Psi_1^{-1}, \partial/\partial w_{n-k}\right\rangle=0. \tag{18}$$

Note that with a slight abuse of notation we consider $\bar{u}\circ\Psi_1^{-1}$ as a function of w, i.e., we express $\bar{u}$ in w coordinates.

From (17) and (18) we deduce that the independent functions $q_1,\dots,q_s,u^{[0]},\dots,u^{[l-1]}$ may depend nontrivially on $w_1,\dots,w_s,w_{n+1},\dots,w_{n+l}$ only. Complete (16) by

$$q_k=\begin{cases}w_k, & k=s+1,\dots,n\\ u^{[k-n-1]} & k=n+1,\dots,n+l\end{cases} \tag{19}$$

and define $q=\Psi_2(w)$, which we claim constitutes a new system of local coordinates. Indeed, after simultaneously rearranging w and q into $\tilde{q}$ and $\tilde{w}$ such that the last $n-s$ indices are $s+1,\dots,n$, we derive the following from (17) and (18):

$$\mathrm{d}\tilde{q}=\begin{bmatrix}B & O_{s+l,n-s}\\ O_{n-s,s+l} & I_{n-s}\end{bmatrix}\mathrm{d}\tilde{w} \tag{20}$$

where the $(s+l)\times(s+l)$ matrix B is locally invertible by local observabilty condition (A) and independent of $\tilde{w}_{s+1},\dots,\tilde{w}_n$.

By (16) we immediately obtain for $i=1,\dots,s-1$ that $q_i^+=q_{i+1}$. Using (10) compute for $k=s+1,\dots,n$

$$\begin{aligned}\partial q_s^+/\partial w_k &= \langle \mathrm{d}(\tilde{h}\circ\tilde{F}^s), \partial/\partial w_k\rangle(w)\\ &=\langle \mathrm{d}(h\circ F_0^s), Ad_{F_0}^{n-k}g\rangle(\xi)=\delta_{s+1,k}.\end{aligned} \tag{21}$$

Therefore, q_s^+ does not depend on $q_{s+2}=w_{s+2},\dots,q_n=w_n$, and $\partial q_s^+/\partial w_{s+1}=1$. Moreover, from (20) we observe that $w_1,\dots,w_{s+1},w_{n+1},w_{n+l}$ may be expressed only in terms of

$q_1, \ldots, q_{s+1}, q_{n+1}, q_{n+l}$. Hence, using (15), (16), (19), and (21), one gets in $\hat{z}$ coordinates $q_i^+ = q_{i+1}$ for $i = 1, \ldots, s-1$; $q_s^+ = q_{s+1} + \hat{\gamma}_s(q_1, \ldots, q_s, \bar{u})$; $q_i^+ = q_{i+1} + \hat{\gamma}_i(q_1, \ldots, q_{s+1}, \bar{u})$ for $i = s+1, \ldots, n-1$; $q_n^+ = \hat{\gamma}_n(q_1, \ldots, q_{s+1}, \bar{u})$, and the dynamics of $(q_{n+1}, \ldots, q_{n+l}) = \bar{u}$ is in the trivial integrator chain form. Finally, define $\zeta = \Psi_3(q)$ by

$$\zeta_i = \Psi_{3,i}(q) = q_i, \quad i \neq s+1$$
$$\zeta_{s+1} = \Psi_{3,s+1}(q) = q_{s+1} + \hat{\gamma}_s(q_1, \ldots, q_s, \bar{u}) \tag{22}$$

and notice that the system dynamics in ζ coordinates are in the extended observer form (7), with $v = 0$ and $z = (\zeta_1, \ldots, \zeta_n)$.

When $s = 0$, one cannot explicitly set the first coordinate as the output. However, (10) and (22) implicitly ensures that $y = q_1 + \hat{\gamma}_0(\bar{u}) = \zeta_1$.

The case $v \neq 0$: From condition (C), we observe that the first s derivatives of $\zeta_1 = q_1$ are independent of v, and therefore the corresponding trivial integrator form of the dynamics of the first s coordinates is preserved.

Finally we need to show that the quasi-linear form obtained for $v = 0$ is preserved, under the same transformation, for arbitrary v; i.e., we need to show that the transformed map in the ζ coordinates i.e., $\tilde{F}_v$, depends linearly on z_i, $i = s+2, \ldots, n$.

For $i = s+2, \ldots, n$, $\partial \tilde{F}_v(z)/\partial z_i = Ad_{\tilde{F}_v} \tilde{\eta}_{n-i+1}(z^+) = Ad_{\Phi_e \circ F_v \circ \Phi_e^{-1}} Ad_{\Phi_e} \eta_i(z^+) = Ad_{\Phi_e} Ad_{F_v} \eta_i(\Phi_e(x^+))$. Since Φ_e does not depend on v, from condition (D) of the theorem, we can conclude that $\partial(\partial \tilde{F}_v(z)/\partial z_i)/\partial v = \partial(Ad_{\Phi_e} Ad_{F_v} \eta_i(\Phi_e(x^+)))/\partial v = 0$. Therefore, the dependence of $\tilde{F}_v$ on z_i, $i = s+2, \ldots, n$ is independent of v, and in particular coincides with that of $\tilde{F}_0$, i.e. $\partial \tilde{F}_v(z)/\partial z_i = \partial \tilde{F}_0(z)/\partial z_i$ for $i = s+2, \ldots, n$ and therefore the quasi-linear form obtained for the case $v = 0$ is preserved. Note that here we have again made use of the fact that ξ_{eq} is an equilibrium point, and consequently z^+ is within the range of the transformation Φ_e. Thus, we obtain the desired form (7), thereby proving the theorem with the required coordinate transformation given by $\zeta = \Phi_e(\xi) := \Psi_3 \circ \Psi_2 \circ \Psi_1(\xi)$, as in (22) and (16) with (19) and (11). ■

Remark 3: The assumption that we are operating near an equilibrium point is necessary to ensure that the forward state ξ^+ does not escape the domain of the transformation Φ_e. In general, this transformation, locally defined as the vector fields η_i, may not be complete. If in addition to the conditions of the theorem we demand that the vector fields $Ad_{F_0}^k g$, $k = 0, \ldots, n-s-1$ are complete, then we may drop the assumption that ξ is an equilibrium point. We refer the reader to [15] for technical details.

IV. Constructing the Linearizing Transformation

We now provide a constructive procedure for obtaining the transformation $\zeta = \Phi_e(\xi) = (\Phi(x, \bar{u}), \bar{u})$. Since the $\bar{u}$ coordinates are unchanged, we will only construct the transformation $z = \Phi(x, \bar{u})$, where $\bar{u}$ may be treated as constant. For the same reason in this section we consider only the first n components of the transformations Ψ_1, Ψ_2, and Ψ_3, using the same notation.

As shown in (18), $\langle \mathrm{d}\bar{u}, \eta_i \rangle = 0$, $i = 1, \ldots, n-s$, therefore $\eta_i = Ad_{F_0}^{i-1} g$ may be considered as vector fields on M with a slight abuse of notation, as they have no components along the extended coordinates.

Lemma 4: The inverse transformation $x = \Phi^\dagger(z, \bar{u})$ has to satisfy the following constrained PDE:

$$\partial \Phi^\dagger / \partial z_i = \eta_{n-i+1}(\Phi^\dagger, \bar{u}), \qquad i = s+1, \ldots, n \tag{23a}$$
$$z_i = h \circ F_0^{i-1} \circ \Phi^\dagger, \qquad i = 1, \ldots, s. \tag{23b}$$

Proof: Replacing the index k by $n-i$, one may rewrite (11) as $Ad_{\Psi_1} \eta_{n-i+1} = \partial/\partial w_i$ for $i = s+1, \ldots, n$. Taking into account Definition 1, the expression above leads to $Ad_\Phi \eta_{n-i+1} = Ad_{\Psi_3 \circ \Psi_2}(\partial/\partial w_i) = (\partial \Psi_3 \circ \Psi_2(w)/\partial w) \circ (\Psi_2^{-1} \circ \Psi_3^{-1}(z))(\partial/\partial w_i) \circ (\Psi_2^{-1} \circ \Psi_3^{-1}(z))$, $i = s+1, \ldots, n$. Using (16)–(19) and (22) one may conclude that

$$\partial(\Psi_3 \circ \Psi_2(w))/\partial w = \begin{bmatrix} D_1 & O_{s,n-s} \\ D_2 & I_{n-s} \end{bmatrix}$$

where D_1 and D_2 are $s \times s$ and $(n-s) \times s$ matrices, respectively. This results in $Ad_\Phi \eta_{n-i+1} = \partial/\partial z_i$, $i = s+1, \ldots, n$, which, according to Definition 1, can be rewritten as $\eta_{n-i+1}(\Phi^\dagger, \bar{u}) = ((\partial \Phi/\partial x) \circ (\Phi^\dagger, \bar{u})))^{-1} \partial/\partial z_i$. Using the inverse function theorem, i.e., $((\partial \Phi/\partial x) \circ (\Phi^\dagger, \bar{u})))^{-1} = \partial \Phi^\dagger/\partial z$ and the identity $(\partial \Phi^\dagger/\partial z)\partial/\partial z_i = \partial \Phi^\dagger/\partial z_i$ one obtains (23a), whereas (23b) follows directly from (16) and (22). ■

We now find the transformation $\Phi^\dagger$ via the following steps.

1) Define the functions $\varphi_i = h \circ F_0^{i-1}$, $i = 1, \ldots, s$. From condition (A) of the theorem, it is clear that $\mathrm{d}\varphi_i$ are linearly independent over M.
2) Complete the functions $\varphi_1, \ldots, \varphi_s$ into a set of coordinates on M as $\varphi = \Upsilon(x, \bar{u})$. We may then express the transformation in the φ coordinates as $z = \tilde{\Upsilon}(\varphi, \bar{u}) = \Phi(\Upsilon^\dagger(\varphi, \bar{u}), \bar{u})$. Define $\chi(\cdot, \bar{u}) = \tilde{\Upsilon}^{-1}(\cdot, \bar{u})$ such that $\varphi = \chi(z, \bar{u})$. The PDE (23) can now be written in terms of χ as
$$\partial \chi/\partial z_i = \eta_{n-i+1}(\Upsilon^\dagger(\chi, \bar{u}), \bar{u}), \qquad i = s+1, \ldots, n$$
$$z_i = \varphi_i, \qquad i = 1, \ldots, s. \tag{24}$$
3) Let $R(\varphi, \bar{u}) := [\eta_{n-s}(\Upsilon^\dagger(\varphi, \bar{u})\bar{u}) \cdots \eta_1(\Upsilon^\dagger(\varphi, \bar{u}), \bar{u})]$. One may verify that the first s rows of R are zero, i.e.,
$$R(\varphi, \bar{u}) = \begin{bmatrix} O_{s,n-s} \\ \tilde{R}(\varphi, \bar{u}) \end{bmatrix}.$$
The above fact is immediate from (17) where it was shown that the output and its first $s-1$ derivatives annihilate the rectified vector fields $Ad_{F_0}^{k-1} g$, $k = 1, \ldots, n-s$. The constrained PDE (24) can be written now as an unconstrained PDE as
$$\partial \chi/\partial z = \begin{bmatrix} I_s & 0 \\ & \tilde{R}(\varphi, \bar{u}) \end{bmatrix}. \tag{25}$$
4) From (25) we solve for the first s coordinates as $\tilde{\Upsilon}_1 = \varphi_1$, $\ldots$, $\tilde{\Upsilon}_s = \varphi_s$, whereas $\partial(\tilde{\Upsilon}_{s+1}, \ldots, \tilde{\Upsilon}_n)/\partial(\varphi_{s+1}, \ldots, \varphi_n) = \tilde{R}^{-1}(\varphi, \bar{u}) = [\theta_{s+1}(\varphi, \bar{u}) \cdots \theta_n(\varphi, \bar{u})]^\top$. The rows θ_{s+1}, $\ldots$, θ_n of the above matrix may be considered as one-forms over the $\varphi_{s+1}, \ldots, \varphi_n$ space by treating $\bar{u}, \varphi_1, \ldots, \varphi_s$ as constants as these variables have now been fixed. Further, all the one-forms θ_i in $\tilde{R}^{-1}$ will be integrable if the system is transformable into the extended observer form (since the PDE must then be solvable). The remaining coordinates are obtained as $\tilde{\Upsilon}_i = \int \theta_i + \mathcal{C}_i(\varphi_1, \ldots, \varphi_s)$, $i = s+1, \ldots, n$, where $\mathcal{C}_i$ are integration constants. Note that the integration is performed only with respect to $\varphi_{s+1}, \ldots, \varphi_n$. The required coordinate change is now obtained as $z = \Phi(x, \bar{u}) = \tilde{\Upsilon}(\Upsilon(x, \bar{u}), \bar{u})$.

Note 1: Treating $\varphi_1, \ldots, \varphi_s$ as constants of integration can be justified because one may affinely add arbitrary functions depending on $\varphi_1, \ldots, \varphi_s$ to the new coordinates and still preserve the extended observer form, because $\mathrm{d}(\tilde{\Upsilon}_i(\varphi) + f(\varphi_1, \ldots, \varphi_s))/\mathrm{d}t = \dot{\tilde{\Upsilon}}_i + \dot{f}(\varphi_1, \ldots, \varphi_{s+1})$ for any smooth function f, which ensures that the extended observer form is preserved. Indeed, it is natural to expect that the choice of coordinates for the extended observer form with output degree s will have s degrees of freedom.

Note 2: The integration constant $\mathcal{C}_{s+1}$ may be chosen such that (22) is also satisfied, thereby yielding the transformation we want.

V. Example

Consider the system

$$\begin{aligned} x_1^+ &= u + x_3 + u\,(x_1 + (x_2 - 1)(x_4 + x_3)) \\ x_2^+ &= x_4 + x_3, \quad x_3^+ = x_1 x_2, \quad x_4^+ = -x_1 x_2 + x_3 \\ &\quad + u\,(x_1 + (x_2 - 1)(x_4 + x_3)), \quad y = x_2. \end{aligned} \tag{26}$$

One may verify that the system is nonreversible but submersive. The corresponding extended (reversible) system of the form (5) for $l = 1$ is $\xi_1^+ = \xi_5 + \xi_3 + \xi_5(\xi_1 + (\xi_2 - 1)(\xi_4 + \xi_3))$, $\xi_2^+ = \xi_4 + \xi_3$, $\xi_3^+ = \xi_1\xi_2$, $\xi_4^+ = -\xi_1\xi_2 + \xi_3 + \xi_5(\xi_1 + (\xi_2 - 1)(\xi_4 + \xi_3))$, $\xi_5^+ = \xi_2 + v$, and $y = \xi_2$.

Compute $\mathrm{d}h = \mathrm{d}\xi_2$, $\mathrm{d}(h \circ F_0) = \mathrm{d}\xi_3 + \mathrm{d}\xi_4$, $\mathrm{d}(h \circ F_0^2) = x_5\mathrm{d}\xi_1 + (1 + (\xi_2 - 1)\xi_5)\mathrm{d}\xi_3 + (\xi_2 - 1)\xi_5\mathrm{d}\xi_4 + (\xi_3 + \xi_4)\xi_5\mathrm{d}\xi_2 + (\xi_1 + (\xi_2 - 1)\,(\xi_3 + \xi_4))\,\mathrm{d}\xi_5$, $\mathrm{d}\,(h \circ F_0^3) = \xi_2\,(1 + (\xi_3 + \xi_4)\,\xi_5)\,\mathrm{d}\xi_1 + (\xi_5 + \xi_1)\,\mathrm{d}\xi_2 + ((\xi_3 + \xi_4)(\xi_3 + \xi_1\xi_5) + (2\xi_2 - 1)\,(\xi_3 + \xi_4)^2\xi_5)\,\mathrm{d}\xi_2 + \xi_2\,(\xi_4 + \xi_1\xi_5 + 2(\xi_3 + (\xi_2 - 1)\,(\xi_3 + \xi_4)\,\xi_5))\,\mathrm{d}\xi_3 + \xi_2(\xi_3 + \xi_1\xi_5 + 2\,(\xi_2 - 1)(\xi_3 + \xi_4)\xi_5)\mathrm{d}\xi_4 + \xi_2(1 + (\xi_3 + \xi_4)(\xi_1 + (\xi_2 - 1)(\xi_3 + \xi_4)))\mathrm{d}\xi_5$. Since the one-forms above and $\mathrm{d}\xi_5$ are linearly independent, condition (A) of Theorem 1 is satisfied. Next, solve the system of (8) to obtain $g = (1/\xi_2)\partial/\partial\xi_1 + \xi_5/\xi_2(\partial/\partial\xi_4 - \partial/\partial\xi_3)$. Compute the vector fields $Ad_{F_0}g = \partial/\partial\xi_3 - \partial/\partial\xi_4$, $Ad_{F_0}^2 g = \partial/\partial\xi_1 + \partial/\partial\xi_4$, $Ad_{F_0}^3 g = (\xi_1 - \xi_3 - \xi_4)\xi_5(\partial/\partial\xi_1 + \partial/\partial\xi_4) + \xi_5(\partial/\partial\xi_3 - \partial/\partial\xi_4) + \partial/\partial\xi_2$ to verify that conditions (9) are satisfied for $s = 1$ and are violated when $s = 0$. Consequently, $s = 1$ is the minimal possible degree of the extended observer form. Taking into account that $h^{[1]} = \xi_3 + \xi_4$ it is easy to verify that conditions (C) are satisfied for $k = 0, 1$. Since by definition $\eta_1 = g$ and $\eta_2 = Ad_{F_0}g$, it is easy to verify that $Ad_{F_v}\eta_1 = Ad_{F_0}g$ and $Ad_{F_v}\eta_2 = Ad_{F_0}^2 g$. Consequently, conditions (D) are fulfilled for $i = 1, 2$.

Next we will follow the steps from Section IV.

1) Define $\varphi_1 := h = x_2$.
2) To complete φ_1 into a set of coordinates on M, one may choose $\varphi_2 := x_1$, $\varphi_3 := x_3$, $\varphi_4 := x_4$.
3) After computation of the vector fields η_3, η_2, η_1 in φ coordinates we obtain

$$\tilde{R}(\varphi, \bar{u}) = \begin{bmatrix} 1 & 0 & 1/\varphi_1 \\ 0 & 1 & -u/\varphi_1 \\ 1 & -1 & u/\varphi_1 \end{bmatrix}.$$

4) The matrix inverse

$$\tilde{R}^{-1}(\varphi, \bar{u}) = \begin{bmatrix} 0 & 1 & 1 \\ u & 1 - u & -u \\ \varphi_1 & -\varphi_1 & -\varphi_1 \end{bmatrix}$$

yields the one-forms $\theta_2 = \mathrm{d}\varphi_3 + \mathrm{d}\varphi_4$, $\theta_3 = u\mathrm{d}\varphi_2 + (1 - u)\varphi_3 - u\varphi_4$, and $\theta_4 = \varphi_1(\mathrm{d}\varphi_2 - \mathrm{d}\varphi_3 - \mathrm{d}\varphi_4)$, whose integrals lead to $\tilde{\Upsilon}_2 = \varphi_3 + \varphi_4$, $\tilde{\Upsilon}_3 = u\varphi_2 + (1 - u)\varphi_3 - u\varphi_4$, and $\tilde{\Upsilon}_4 = \varphi_1(\varphi_2 - \varphi_3 - \varphi_4)$, whereas $\tilde{\Upsilon}_1 = \varphi_1$. As a result, the required transformation is $z_1 = x_2$, $z_2 = x_3 + x_4$, $z_3 = ux_1 + (1 - u)x_3 - ux_4$, $z_4 = x_2(x_1 - x_3 - x_4)$, which brings the system (26) into the extended observer form $z_1^+ = z_2$, $z_2^+ = z_3 + uz_1z_2$, $z_3^+ = z_4 + z_1z_2 + uu^{[1]}$, $z_4^+ = uz_2$, $y = z_1$.

VI. Conclusion

Coordinate independent, differential geometric necessary, and sufficient conditions for transforming a nonlinear discrete-time system into the extended observer form were established. The conditions are directly verifiable from the state equations, and do not rely on i/o equations. Unlike the existing results, the conditions are finite for systems with inputs, when the linear part of the observer form is time-invariant. We explicitly address nonreversible but submersive systems. However, the reversible case can be handled in a similar manner as suggested in Remark 1. A constructive algorithm for obtaining the linearizing parametrized state transformation was presented. The proposed conditions broaden the class of systems, for which exponentially stable observers can be constructed. Two immediate directions for future research are: extending the results to multi-input multi-output systems, and nonsubmersive systems.

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