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Stable Channels

by

A. E. Bretting

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"Model Experiments in Questions of Navigation" by J. J. Leendertse
"The Effects of Drainage on the Stability of Sea Walls" by F. E. Richart, Jr. and J. H. Schmertmann
"Dune Formation and Stabilization by Vegetation and Planting" by John H. Davis

Reprints of these papers are available upon request from the Florida Engineering and Industrial Experiment Station.

Coastal engineering papers published earlier in the Engineering Progress Series include:

Leaflet No. 66 May, 1955	– "Beach Profiles and Development of Plans for Coastal Protection," by Per Bruun
Bulletin No. 76 November, 1955	– "Coastal Development and Coastal Protection," by Per Bruun
Special Report to the Florida State Legislature	– "Studies and Recommendations for the Control of Beach Erosion in Florida," by the Coastal Engineering Staff
Technical Progress Report No. 5, March, 1957	– "Coastal Engineering Investigations at Jupiter Island," by the Coastal Engineering Staff
Leaflet No. 99 July, 1958	– "Florida's Coastal Engineering Wave Tank," by P. Bruun, J. J. Leendertse, and L. W. Cover

Other publications planned for the near future include a report on a coastal engineering study at Fort Pierce Beach; an analysis of tidal inlets, and a manual on the design of coastal engineering structures.

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STABLE CHANNELS

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a	- - - - A_φ	6.5
a_0	auxiliary notation in formula for A_p	6.3
a_1, a_2, a_3	constants	6.4
A_φ	coefficients being functions of φ (φ second subscript)	6 et seqq.
A_p	coefficients being functions of p (p second subscript)	6 et seqq.
b	$= a \cot^2\varphi = (1 + \frac{\cot\varphi}{c}) \cdot \cot^2\varphi$	3.1-3.2
b	constant in formula for A_p	6.3
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ds	length of element of perimeter of bottom	4.2
dF	area of surface element	4.2
e	basis of natural logarithms	9-11.1.2.2
e_p, e_p^o	deviations from power formulae	6.3
f	as first subscript in $A_{f\varphi}$, $B_{f\varphi}$ and A_{fp} to denote formulae pertaining to F'	7 et seqq.

(to be continued)

LIST OF SYMBOLS (continued)

		Sec.
f	resistance number	11.1.2
F	area of <u>half</u> profile	4.1 et seqq.
F'	$= \frac{F}{y_{max}^2}$	4.3 et seqq.
g	acceleration of gravity	1 et seqq.
$g(m)$, $g_1(m)$, $g_2(m)$, $g_3(m)$	auxiliary functions for calculating A_φ	6.5
i	as first subscript in $A_{i\varphi}$, $B_{i\varphi}$ and A_{ip} to denote formulae pertaining to I'	7 et seqq.
I	slope	4.5 et seqq.
I'	$= \frac{g\,F\,I}{y_{max} \cdot (\frac{\tau_{max}}{\varrho})}$	4.5 et seqq.
k	hydraulic equivalent sand roughness of bottom	1 et seqq.
K	total shearing force	4.5
$\ln$	$= \log_e$, natural logarithm	1 et seqq.
L	= hydrodynamic lift force per unit area	2.1-2.2
m	$= 1 - \frac{\cot\varphi}{c}$	3.1-3.2
m	as subscript signifies mean value	4.2 et seqq.
m	$= c + a \log^2 b$, abbreviation by calculation of A_p	6.3
m	$= \cot\varphi$, used in formulae for A_φ and ξ	6.5 et seqq.
max	as subscript signifies maximum value	1 et seqq.
M	Manning's coefficient	11.1.2.1
n	Kutter's n	11.1.2.1
n	number of ordinate η	3.3
n	$= 2 a \log b$, abbreviation by calculation of A_p	6.3
p	$= \frac{y_{max}}{k}$	5 et seqq.
p	relative distance from bottom	11.1.2.2
q	- - - -	11.1.2.2
q	as first subscript in $A_{q\varphi}$, $B_{q\varphi}$ and A_{qp} to denote formulae pertaining to Q'	7 et seqq.
Q	discharge of <u>half</u> section	4.1 et seqq.
Q'	$= \frac{Q}{2.5 \cdot \sqrt{\frac{\tau_{max}}{\varrho}} \cdot y_{max}^2}$	4.3 et seqq.

(to be continued)

LIST OF SYMBOLS (continued)

		Sec.
r	radius of curvature	4.2
r	as first subscript in $A_{r\varphi}$, $B_{r\varphi}$ and A_{rp} to denote formulae pertaining to R'	7 et seqq.
r	$= \frac{v_{z,\xi}}{v_m}$	9
R	hydraulic radius of section	4.5 et seqq
R'	$= \frac{R}{y_{max}}$	4.5 et seqq.
s	see ds	4.2
S_0, $S_{1,1}$, $S_{2,1}$, $S_{3,1}$, $S_{1,2}$, $S_{2,2}$, $S_{3,2}$, S_1, S_2, S_3	notations defined in Sec. 4.3	4.3 et seqq.
U	wetted perimeter of <u>half</u> section	4.1 et seqq.
U'	$= \frac{U}{y_{max}}$	4.3
v	angle of slope in the longitudinal axis of the channel	1
v	as first subscript in $A_{v\varphi}$, $B_{v\varphi}$ and A_{vp} to denote formulae pertaining to v'_m	7
v	velocity	4.2 et seqq.
v_z	- at distance z from bottom	1-4.2 et seqq.
$v_{m,\xi}$	mean velocity of area element where $z_0 = \xi \cdot y_{max}$	4.2 et seqq.
v_*	shearing velocity	1-4.2 et seqq.
$v_{*,\xi}$	- - where $z_0 = \xi \cdot y_{max}$	4.2 et seqq.
v'_m	$= \frac{v_m}{2.5 \cdot \sqrt{\frac{\tau_{max}}{\rho}}}$	4.5 et seqq.
v'_*	$= \sqrt{\frac{\left(\frac{\tau_{max}}{\rho}\right)}{g\,R\,I}}$	4.5 et seqq.
$v_{z,\xi}$	velocity at distance z from bottom where the normal has length $\xi \cdot y_{max}$	9

(to be continued)

LIST OF SYMBOLS (continued)

		Sec.
v_p	velocity at relative distance from bottom p	11.1.2.2
v_q	- - - - - - - q	11.1.2.2
W	submerged weight of sand grains per unit area of the bottom	2.1
x	abscissa reckoned from bank (Fig. 3)	2.1 et seqq.
y	depth of water at abscissa x (Fig. 3)	1 et seqq.
z	distance from bottom measured at right angles to the bottom	1 et seqq.
z_0	length of normal to the bottom between this and the water surface (Fig. 4)	4.2 et seqq.
z'	stands in general for Q', F', v'_m, B', R' and I'	6-6.1-6.2
z'	$= \frac{z}{y_{max}}$	9
$*$	as first subscript in $A_{*\varphi}$, $B_{*\varphi}$ and A_{*p} to denote formulae pertaining to v'_*	7 et seqq.
α	angle of inclination of bottom in relation to horizontal plane	1 et seqq.
β	$= \xi_{1,0}$, relative abscissa for $\eta = 1$, (relative width of "bank-part")	3.3 et seqq.
β_0	relative width of "middle-part"	4.1 et seqq.
β	as subscript referring to "bank-part"	4.1 et seqq.
β_0	as subscript referring to "middle-part"	4.1 et seqq.
γ	specific weight of water	1 et seqq.
Δ	indicates finite difference (e.g. $\Delta\xi$)	3.3 et seqq.
Δ	$= 1 - \eta$	3.2
Δ_3	auxiliary notation	6.4
ε	$= 1.2225\,\varkappa - 1$	13.3
ζ	$= \frac{z_0}{y_{max}}$, relative length of normal to bottom between this and water surface	4.2 et seqq.
η	$= \frac{y}{y_{max}}$, relative depth	2.1 et seqq.
$\varkappa$	$= \frac{F}{B\,y_{max}}$, "degree of fullness of cross section"	4.5 et seqq.
ξ	$= \frac{x}{y_{max}}$, relative abscissa	2.1 et seqq.
ρ	$= \frac{\gamma}{g}$, density of water	1 et seqq.

(to be continued)

LIST OF SYMBOLS (continued)

		Sec.
τ	shearing stress at bottom	1 et seqq.
φ	angle of internal friction	1 et seqq.
ψ	auxiliary function	4.4

In Sec. 2.2 the symbols of [1] are used.

References are given in [ ].

Formula numbers are given in ( ) .

Numbers of formulae of special importance are framed.

Table numbers are given in / /.

Example: 7.1/4a/b/c/d/ indicates the four Tables 4a, 4b, 4c and 4d which appear in Sec. 7.1.

LIST OF FIGURES

All figures appear on the last four pages of the paper.

All tables appear in the text.

ABSTRACT

The form and size of a channel in cohesionless material, stable against erosion for a definite discharge, Q, are studied.

The angle of internal friction φ and the limiting tractive force τ_{max} are taken as known. Distribution of shearing stresses τ is assumed to be such that they are proportional to the distance between bottom and water surface, measured at right angles to the bottom. In addition to the action of gravity and shearing stress τ the grains are acted upon by a hydrodynamic lift force, proving to be proportional to τ. The differential equation of the bottom form is established and integrated numerically; the form depends on φ.

Based on the logarithmic law of velocity distribution and the assumed distribution of shearing stresses, the velocities in all parts of the cross section can be found, and the total discharge is found by numerical integration.

A profile consisting of the curved "bank-part" of the above mentioned cross section and a "middle-part" of indefinite width and of constant depth y_{max} would be stable for the same tractive force. On the assumption however that nature will produce that cross section which has a minimum of area, only one definite solution, viz. the equilibrium profile, is found. The dimensions depend not on φ alone but also on the relative roughness of the bottom $\frac{k}{y_{max}}$. Provided that the hydraulic roughness k is assumed to be in conformity with that of natural watercourses, it is found that the area of the equilibrium profile varies slowly with φ and must be proportional to

$$\left(\frac{Q}{\sqrt{\tau_{max}}}\right)^{0.9} .$$

The above assumptions are checked by calculation of a complete set of isovels.

Further three model tests, carried out in Vienna in 1916, are studied and compared with profiles calculated according to this theory. The values of φ are found to be varying from 14° to 20°.

On the same basis a study is finally made of the relation between mean and maximum velocities, $\frac{v_m}{v_{max}}$, resulting in a simple diagram giving $\frac{v_m}{v_{max}}$ as a function of $\varkappa$, the "degree of fullness" of the profile, and also as a function of $\frac{y_{max}}{k}$, the reciprocal relative roughness.

Methods for estimating k are given.

STABLE CHANNELS

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1. INTRODUCTION

The term "stable channels" has generally been used as relating to the cross section of a channel which will be stable for a definite discharge both against erosion and against sedimentation. In this paper only the question of stability against erosion has been treated, and only as far as cohesionless bottom material is concerned.

The subject of this investigation is the cross section ultimately created when nature itself, at a definite discharge Q (m^3/sec), excavates a channel bed in sand material with a definite limiting tractive force τ_{max} (kg/m^2) just able to move the sand grains on a horizontal bottom and with an angle of internal friction of φ degrees.

The specific weight of the water is γ (kg/m^3), and the equivalent sand-roughness of the bottom of the channel is k (m); this roughness is taken to be uniform over the whole width of the channel.

It is further assumed that the velocity v_z at every point P of the cross section can be found according to the usual logarithmic law of velocity distribution

$$\frac{v_z}{v_*} = 8.48 + 2.5 \cdot \ln\left(\frac{z}{k}\right),$$

where k is the roughness of the bottom (m),

z is the distance from the bottom (m) measured in the direction perpendicular to the bottom,

$v_* = \sqrt{\frac{\tau}{\rho}}$ (m/sec) ,

τ = the shearing stress (kg/m^2) at the bottom in a line through the point P perpendicular to the bottom,

$\rho = \frac{\gamma}{g}$ $\begin{cases} \gamma, \text{ specific weight of the water (kg/m}^3\text{)} \\ g, \text{ acceleration of gravity} = 9.81 \text{ m/sec}^2. \end{cases}$

To find the velocities, a definite law for the distribution of the shearing stresses along the bottom contour must be assumed.

In the middle part of the cross section, where the bottom line is horizontal, the shearing stress will have its maximum value τ_{max} equal to the limiting tractive force of the sand grains in question.

This limiting tractive force will strictly speaking depend on the longitudinal slope of the channel.

The relation between the limiting tractive forces τ_v for a plane bottom sloping in the direction of the flow at an angle v and τ_h for a horizontal bottom, can be shown to be:

$$\frac{\tau_v}{\tau_h} = \cos v(1 - \operatorname{tg} v \cot\varphi),$$

where φ is the angle of internal friction of the sand.

Since the longitudinal slope = sin v will generally be small we take the limiting tractive force to be independent of the slope.

The following law for the variation of τ is assumed:

$$\boxed{(1)} \qquad \frac{\tau}{\tau_{max}} = \frac{y}{y_{max} \cdot \cos\alpha} ,$$

where y is the depth of water at the point in question, and α is the inclination of the bottom with a horizontal plane. (Fig. 1). The shearing stress will consequently be proportional to the length of a normal to the bottom reckoned between the bottom and the water surface. The shearing stress on the slopes will be somewhat greater than that found by using the hypothesis that τ varies proportionally with the depth y , an assumption previously used. Transfer of shearing forces from the middle of the section against the banks is thus to a certain degree taken into account.

A justification of this assumption is later found by calculation of the corresponding complete set of isovels (Fig.7), which are in fair accordance with experience from actual measurements.

2. ESTABLISHMENT OF THE DIFFERENTIAL EQUATION FOR THE SHAPE OF THE PROFILE.

2.1. EQUATION ON THE ASSUMPTION $L = c \cdot \tau$.

The profile sought should just be in equilibrium at every point of the bottom, given the assumed distribution of shearing stresses.

On the bottom an element of one unit area will be stressed by the force τ in the direction of the flow. The longitudinal slope of the channel is considered insignificant. The surface element has the inclination α^{o} with a horizontal plane.

The submerged weight of sand grains per unit area stressed by τ is designated as W. (Fig. 2). This force is acting vertically downwards and is resolved into the forces $W \cdot \cos\alpha$ in the direction of the normal to the element, and $W \cdot \sin\alpha$ acting in the plane of the element in the direction of its transversal slope.

The grains of the element are further acted upon by the hydrodynamic lifting force L, which is upwards directed in the normal to the plane of the element. The resulting force in this direction will consequently be $(W \cdot \cos\alpha - L)$, acting downwards.

The total stress on the element in question will be the resultant of the three above-mentioned forces :

1) τ in the plane of the element and in the direction of the flow,
2) $(W \cdot \cos\alpha - L)$ in the direction of the normal to the element,
3) $W \cdot \sin\alpha$ in the plane of the element perpendicular to the direction of flow.

The resultant of forces 1) and 3) is $\sqrt{\tau^2 + W^2 \cdot \sin^2\alpha}$ acting in the plane of the element, whereas force 2) is perpendicular to the said resultant.

If the angle of friction of the sand is taken to be φ degrees, it is a condition of equilibrium for the sand grains of this element that

(2) $$\tan\varphi = \frac{\sqrt{\tau^2 + W^2 \cdot \sin^2\alpha}}{W \cdot \cos\alpha - L}$$

The magnitude of the hydrodynamic uplift L will be studied below.

We put provisionally

(3) $$L = c \cdot \tau ,$$

where c will prove to be a constant (Section 2.2).

From equation (2) we get :

(2a) $$W^2 \cdot \cos^2\alpha - 2 \cdot L \cdot W \cdot \cos\alpha + L^2 = (\tau^2 + W^2 \cdot \sin^2\alpha) \cdot \cot^2\varphi$$

From equation 1.(1) we get :

$$\tau = \frac{\tau_{max}}{\cos \alpha} \cdot \frac{y}{y_{max}}$$

For $y = y_{max}$ we have $\alpha = 0^o$, $\cos \alpha = 1$, $\sin \alpha = 0$

$\tau = \tau_{max}$, $L = c \cdot \tau_{max}$, which, inserted in (2a), gives :

$W^2 - 2 c \tau_{max} \cdot W + c^2 \cdot \tau^2_{max} = \cot^2 \varphi \cdot \tau^2_{max}$; taking the square root of both sides we obtain :

$$W - c \tau_{max} = \cot \varphi \cdot \tau_{max}$$

(4) $$W = \tau_{max} (c + \cot \varphi)$$

This value of W is inserted in (2a) together with L from (3), which gives :

(5) $$(\frac{\tau}{\tau_{max}})^2 \cdot (\frac{c - \cot \varphi}{c}) - (\frac{\tau}{\tau_{max}}) \cdot 2 \cos \alpha + (\frac{c + \cot \varphi}{c}) \cdot \left[(1 + \cot^2 \varphi) \cdot \cos^2 \alpha - \cot^2 \varphi\right] = 0$$

Substituting 1.(1) :

$\frac{\tau}{\tau_{max}} = \frac{y}{y_{max}} \cdot \frac{1}{\cos \alpha}$ we get from (5) :

(5a) $$(\frac{y}{y_{max}})^2 \cdot \frac{1}{\cos^2 \alpha} \cdot (\frac{c - \cot \varphi}{c}) - 2(\frac{y}{y_{max}}) + (\frac{c + \cot \varphi}{c}) \cdot \left[(1 + \cot^2 \varphi) \cdot \cos^2 \alpha - \cot^2 \varphi\right] = 0$$

The following substitutions are used :

(6a) $\frac{y}{y_{max}} = \eta$; (6b) $\frac{x}{y_{max}} = \xi$; (6c) $\tan \alpha = \frac{dy}{dx} = \frac{d\eta}{d\xi}$;

(6d) $\frac{1}{\cos^2 \alpha} = 1 + \tan^2 \alpha = 1 + (\frac{d\eta}{d\xi})^2$

(6a) – (6d) are inserted in (5a) :

$$\eta^2 \cdot \left[1 + (\frac{d \eta}{d \xi})^2 \right] \cdot (1 - \frac{\cot \varphi}{c}) - 2 \eta + (1 + \frac{\cot \varphi}{c}) \cdot \left[\frac{1 + \cot^2 \varphi}{1 + (\frac{d \eta}{d \xi})^2} - \cot^2 \varphi \right] = 0$$

$$(7)\qquad \eta^2\cdot\left[1+\left(\frac{d\eta}{d\xi}\right)^2\right]^2\cdot\left(1-\frac{\cot\varphi}{c}\right) - \left[1+\left(\frac{d\eta}{d\xi}\right)^2\right]\cdot\left[2\eta+\cot^2\varphi\cdot\left(1+\frac{\cot\varphi}{c}\right)\right] + \left(1+\cot^2\varphi\right)\cdot\left(1+\frac{\cot\varphi}{c}\right)=0$$

This is the correct differential equation in η and ξ for the profile in question. It can be written in the form

$$\boxed{(8)}\qquad \left(\frac{d\eta}{d\xi}\right)^2 = \frac{\left(1+\frac{\cot\varphi}{c}\right) - 2\eta + \eta^2\left(1-\frac{\cot\varphi}{c}\right)\cdot\left[1+\left(\frac{d\eta}{d\xi}\right)^4\right]}{\cot^2\varphi\cdot\left(1+\frac{\cot\varphi}{c}\right)+2\eta-2\eta^2\cdot\left(1-\frac{\cot\varphi}{c}\right)}$$

2.2 DETERMINATION OF THE FACTOR c IN FORMULA 2.1 (3)

In the paper by H. A. Einstein [Ref. 1] information is found bearing upon this subject. The notations of Einstein's paper are used (in this section only). Formula (36), page 31 runs

$$p_L = c_L \cdot s_f \cdot \frac{u^2}{2}, \qquad \text{where}$$

p_L = average lift pressure per unit of area,

c_L = 0.178 (dimensionless) ,

s_f = density of the fluid = $\frac{\gamma_f}{g}$,

γ_f = specific weight of fluid,

g = acceleration of gravity ,

u = the flow velocity at a distance $0.35 \cdot D_{35}$ from the theoretical bed,

D_{35} = sieve size of the grains of which 35 percent are finer.

It is indicated that the pressure fluctuations due to turbulence in their duration follow the normal error law, the standard deviation being 0.364 of the average lift.

A deviation from the mean of 2.75 times "standard deviation", viz.

$$2.75 \cdot 0.364 \cdot p_L = 1.0 \cdot p_L$$

will only have a statistical probability of 6 per thousand to be exceeded and corresponds to a practical maximum value of the hydrodynamic lift force per area :

$L = 2\,p_L$, which is very seldom exceeded. For greater values of Reynolds' numbers, which are exclusively considered, is found

$$u = u_* \cdot 5.75 \cdot \log\left(30.2\,\frac{y}{\Delta}\right), \qquad \text{where}$$

$y = 0.35\,(0.77 \cdot k_s)$,
$\Delta = k_s$, the equivalent sand roughness,

$u_* = \sqrt{\dfrac{\tau}{s_f}}$, where

τ = shearing stress along the bottom

$u = u_* \cdot 5.75 \log (30.2 \cdot 0.35 \cdot 0.77)$

$u = 5.24 \cdot u_*$

The mean value p_L according to [1] formula (36)

$$p_L = 0.178 \cdot s_f \cdot \frac{1}{2}(5.24\, u_*)^2$$

$$p_L = 2.45 \cdot s_f \cdot u_*^2 = 2.45 \cdot s_f \cdot \frac{\tau}{s_f} = 2.45\,\tau$$

We consequently find the maximum value of the hydrodynamic lift force, defined as above,

$$L = 2\,p_L = 4.9\,\tau$$

The constant c in equation (3) is found to be

$$c = 4.9 \tag{9}$$

3. SOLUTION OF THE DIFFERENTIAL EQUATION 2.1(8)

In the numerical calculations of $\frac{d\eta}{d\xi}$ for varying values of η, the factor $\left[1 + \left(\frac{d\eta}{d\xi}\right)^4\right]$ in the last term of the numerator can be omitted from the first approximation, whereupon the said last term is corrected.

3.1. FORMULAE FOR NUMERICAL INTEGRATION

The following substitutions are used :

2.2.(9) $c = 4.9$

(10) $a = 1 + \dfrac{\cot\varphi}{c}$

(11) $b = \cot^2\varphi \cdot \left(1 + \dfrac{\cot\varphi}{c}\right) = a \cdot \cot^2\varphi$

(12) $m = 1 - \dfrac{\cot\varphi}{c}$,

an l the differential equation (8) takes the following form :

$$\left(\frac{d\eta}{d\xi}\right)^2 = \frac{a - 2\eta + m\,\eta^2 \cdot \left[1 + \left(\frac{d\eta}{d\xi}\right)^4\right]}{b + 2\eta - 2m \cdot \eta^2} \tag{13}$$

For a given value of φ the quantities a, b and m are fixed constants, and the numerical values of $\frac{d\eta}{d\xi}$ can be found corresponding to chosen values of η.

The integration of the differential equation (13) is made numerically for values of $\varphi = 15^\circ$, 20°, 25° and 30°.

The corresponding values of the constants a, b and m are found from formulae (9), (10), (11) and (12), and the following special forms of (13) are found :

$\boxed{\varphi = 15^\circ}$

$$(13a)\qquad \left(\frac{d\eta}{d\xi}\right)^2 = \frac{1.761643 - 2\eta + 0.238357\cdot\eta^2\cdot\left[1 + \left(\frac{d\eta}{d\xi}\right)^4\right]}{24.536516 + 2\eta - 2\cdot 0.238357\cdot\eta^2}$$

$\boxed{\varphi = 20^\circ}$

$$(13b)\qquad \left(\frac{d\eta}{d\xi}\right)^2 = \frac{1.56071 - 2\eta + 0.43929\cdot\eta^2\cdot\left[1 + \left(\frac{d\eta}{d\xi}\right)^4\right]}{11.78125 + 2\eta - 2\cdot 0.43929\cdot\eta^2}$$

$\boxed{\varphi = 25^\circ}$

$$(13c)\qquad \left(\frac{d\eta}{d\xi}\right)^2 = \frac{1.43766 - 2\eta + 0.56234\cdot\eta^2\cdot\left[1 + \left(\frac{d\eta}{d\xi}\right)^4\right]}{6.61168 + 2\eta - 2\cdot 0.56234\,\eta^2}$$

$\boxed{\varphi = 30^\circ}$

$$(13d)\qquad \left(\frac{d\eta}{d\xi}\right)^2 = \frac{1.35348 - 2\eta + 0.64652\cdot\eta^2\cdot\left[1 + \left(\frac{d\eta}{d\xi}\right)^4\right]}{4.06044 + 2\eta - 2\cdot 0.64652\cdot\eta^2}$$

As values of η are taken :
0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7,
0.75, 0.8, 0.85, 0.9, 0.95, 0.98, 1.0.
For each value of φ and these values of η the values of $\left(\frac{d\eta}{d\xi}\right)^2$ are calculated from (13a) b) c) d), and the corresponding values of $\left(\frac{d\eta}{d\xi}\right)$, $\left(\frac{d\xi}{d\eta}\right)$, $\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}$ and $\xi = \eta\cdot\sqrt{1+\left(\frac{d\eta}{d\xi}\right)^2}$ are compiled in Tables 3.1./1a/b/c/d/ for $\varphi = 15^\circ$, 20°, 25° and 30° resp.

$\boxed{\varphi = 15^{\circ}.}$ Table 3.1/1a/

η	$\left(\frac{d\eta}{d\xi}\right)^2$	$\frac{d\eta}{d\xi}$	$\frac{d\xi}{d\eta}$	$\sqrt{1+\left(\frac{d\eta}{d\xi}\right)^2}$	ξ
0	0.071797	0.267949	3.73205	1.035276	0.000000
0.1	0.063240	0.251476	3.976529	1.031135	0.103114
0.2	0.055030	0.234585	4.262854	1.027147	0.205430
0.3	0.047149	0.217139	4.605354	1.023303	0.306991
0.4	0.039582	0.198952	5.026360	1.019599	0.407840
0.5	0.032312	0.179756	5.563085	1.016028	0.508014
0.6	0.025328	0.159148	6.283474	1.012585	0.607551
0.7	0.018616	0.136440	7.329257	1.009265	0.706485
0.75	0.015358	0.123927	8.069238	1.007650	0.755737
0.8	0.012164	0.110291	9.066962	1.006064	0.804851
0.85	0.009032	0.095039	10.521951	1.004506	0.853830
0.9	0.005962	0.077215	12.950913	1.002977	0.902679
0.95	0.002952	0.054329	18.406283	1.001475	0.951401
0.98	0.001174	0.034259	29.189274	1.000587	0.980575
1.0	0.000000	0.000000	∞	1.000000	1.000000

$\boxed{\varphi = 20^{\circ}}$ Table 3.1/1b/

η	$\left(\frac{d\eta}{d\xi}\right)^2$	$\frac{d\eta}{d\xi}$	$\frac{d\xi}{d\eta}$	$\sqrt{1+\left(\frac{d\eta}{d\xi}\right)^2}$	ξ
0	0.13247	0.36396	2.74748	1.06418	0.000000
0.1	0.114025	0.33768	2.96142	1.05547	0.105547
0.2	0.097023	0.31149	3.21042	1.04739	0.209478
0.3	0.081328	0.28518	3.50655	1.03987	0.311961
0.4	0.066822	0.25850	3.86848	1.03287	0.413148
0.5	0.053404	0.23109	4.32726	1.02635	0.513175
0.6	0.040988	0.20245	4.93937	1.02029	0.612174
0.7	0.029500	0.17176	5.82222	1.01464	0.710248
0.75	0.024083	0.15519	6.44384	1.01197	0.758978
0.8	0.018875	0.13739	7.27874	1.009393	0.807514
0.85	0.013869	0.11777	8.49136	1.006910	0.855874
0.9	0.009057	0.095168	10.50770	1.004518	0.904066
0.95	0.004436	0.066606	15.01359	1.002216	0.952105
0.98	0.001752	0.041862	23.888195	1.000876	0.980858
1.0	0.000000	0.000000	∞	1.000000	1.000000

$\boxed{\varphi = 25^{\circ}.}$ Table 3.1/1c/

η	$(\frac{d\eta}{d\xi})^2$	$\frac{d\eta}{d\xi}$	$\frac{d\xi}{d\eta}$	$\sqrt{1+(\frac{d\eta}{d\xi})^2}$	ξ
0	0.21744	0.46631	2.14451	1.10338	0.000000
0.1	0.182852	0.42761	2.33857	1.08759	0.108759
0.2	0.152249	0.39019	2.56285	1.07343	0.214686
0.3	0.125035	0.35360	2.82803	1.06068	0.318204
0.4	0.100742	0.31740	3.15061	1.04916	0.419664
0.5	0.079002	0.28107	3.55780	1.03875	0.519375
0.6	0.059514	0.24395	4.09912	1.02933	0.617598
0.7	0.042047	0.20505	4.87677	1.02081	0.714567
0.75	0.034008	0.18441	5.42262	1.01686	0.762645
0.8	0.026404	0.16249	6.15411	1.01312	0.810496
0.85	0.019216	0.13862	7.21387	1.00956	0.858126
0.9	0.012429	0.11149	8.96978	1.00620	0.905580
0.95	0.006027	0.07763	12.88099	1.00301	0.952860
0.98	0.002367	0.04865	20.55420	1.00118	0.981156
1.0	0.000000	0.000000	∞	1.00000	1.000000

$\boxed{\varphi = 30^{o}.}$ Table 3.1/1d/

η	$(\frac{d\eta}{d\xi})^2$	$\frac{d\eta}{d\xi}$	$\frac{d\xi}{d\eta}$	$\sqrt{1+(\frac{d\eta}{d\xi})^2}$	ξ
0	0.333333	0.577350	1.73205	1.154703	0.000000
0.1	0.273203	0.522687	1.91319	1.128362	0.112836
0.2	0.222426	0.471620	2.12035	1.105634	0.221127
0.3	0.179031	0.423121	2.36339	1.08583	0.325749
0.4	0.141608	0.376308	2.65740	1.06846	0.427384
0.5	0.109141	0.330365	3.02696	1.053158	0.526579
0.6	0.080864	0.284366	3.51659	1.039647	0.623788
0.7	0.056200	0.237065	4.21825	1.027715	0.719401
0.75	0.045082	0.212325	4.709756	1.022293	0.766720
0.8	0.034710	0.186306	5.36750	1.017207	0.813766
0.85	0.025046	0.158259	6.31874	1.012446	0.860579
0.9	0.016059	0.126724	7.89116	1.007998	0.907198
0.95	0.007718	0.087851	11.38297	1.003851	0.953659
0.98	0.003015	0.054905	18.21316	1.001506	0.981476
1.0	0.000000	0.000000	∞	1.000000	1.000000

3.2. CALCULATION OF THE INTEGRAL, ESPECIALLY $\int_{1-\Delta}^{1} \frac{d\xi}{d\eta} \cdot d\eta$.

Calculation of the integral $\xi = \int_0^{\eta} f(\eta) \cdot d\eta$ is made by the trapezoidal rule except for the last interval $0.98 \leqq \eta \geqq 1$, where this rule is not usable because $f(\eta)$ tends to infinity for $\eta = 1$.

In this region $\left(\frac{d\eta}{d\xi}\right)^4$ in the formula 3.1.(13) can be completely neglected, and we get

$$\frac{d\xi}{d\eta} = \sqrt{\frac{b + 2\eta - 2m \cdot \eta^2}{a - 2\eta + m \cdot \eta^2}} \tag{14}$$

We substitute :

$\eta = 1 - \Delta$, $\eta^2 = 1 - 2\Delta + \Delta^2$, where Δ is small in relation to 1,

$d\eta = - d\Delta$ and find

$$\frac{d\xi}{d\eta} = \sqrt{\frac{b + 2(1-m) - 2\Delta(1-2m) - 2m \cdot \Delta^2}{(a - 2 + m) + 2\Delta(1-m) + m \cdot \Delta^2}}$$

According to 3.1.(10) and 3.1.(12) :

$a - 2 + m = 0$, consequently

$$\frac{d\xi}{d\eta} = \sqrt{\frac{b+2(1-m)}{2\Delta(1-m)}} \cdot \sqrt{\frac{1 - 2\Delta \frac{1-2m}{b+2(1-m)} - 2\Delta^2 \frac{m}{b+2(1-m)}}{1 + \frac{1}{2} \cdot \Delta \cdot \frac{m}{1-m}}} \tag{15}$$

In (15) the latter square root is expanded into a series, and considering that Δ is small compared with 1 we get, neglecting terms with Δ^3 and higher powers :

$$\frac{d\xi}{d\eta} = \sqrt{\frac{b+2(1-m)}{2(1-m)}} \cdot \Delta^{-1/2} \cdot \left\{ 1 - \frac{1}{4} \cdot \Delta \cdot \left[\frac{m}{1-m} + 4 \cdot \frac{1-2m}{b+2(1-m)}\right] + \frac{1}{8} \cdot \Delta^2 \cdot \left[0.75 \cdot \left(\frac{m}{1-m}\right)^2 + 2 \cdot \frac{m}{1-m} \cdot \frac{1-2m}{b+2(1-m)} - 4 \cdot \frac{m}{b+2(1-m)} \cdot \left(2 + \frac{(1-2m)^2}{m(b+2(m-1))}\right)\right]\right\}.$$

By integration we find :

$$\boxed{(16)}\quad \int_{1-\Delta}^{1} \frac{d\xi}{d\eta} \cdot d\eta = \int_{0}^{\Delta} \frac{d\xi}{d\eta} \cdot d\Delta$$

$$= 2 \cdot \sqrt{\Delta} \cdot \sqrt{\frac{b+2(1-m)}{2(1-m)}} \cdot \left\{ 1 - \frac{1}{12} \cdot \Delta \cdot \left[\frac{m}{1-m} + 4 \cdot \frac{1-2m}{b+2(1-m)} \right] \right.$$

$$+ \frac{1}{40} \cdot \Delta^{2} \cdot \left[0.75\left(\frac{m}{1-m}\right)^2 + 2 \cdot \frac{m}{1-m} \cdot \frac{1-2m}{b+2(1-m)} \right.$$

$$\left.\left. - 4 \cdot \frac{m}{b+2(1-m)} \cdot \left(2 + \frac{(1-2m)^2}{m(b+2(m-1))} \right) \right] \right\}.$$

The constants b and m are known for each value of φ from the above-mentioned calculations (compare the formula 3.1.(13) with 3.1.(13a), 3.1.(13b), 3.1.(13c) and 3.1.(13d).

For $\boxed{\varphi = 15^0}$, $b = 24.536516$, $m = 0.238357$ we find :

$$(16a)\quad \int_{1-\Delta}^{1} \frac{d\xi}{d\eta} \cdot d\eta = 8.272272 \cdot \sqrt{\Delta} \cdot \left\{ 1 - 0.032773\Delta + 0.000281\Delta^2 \right\}.$$

For $\Delta = 0.02$

$$(16aa)\quad \int_{0.98}^{1} \frac{d\xi}{d\eta} \cdot d\eta = 1.169876 \cdot \left\{ 1 - 0.00065546 + 0.000000112 \right\}$$

$$= \underline{1.169109.}$$

For $\boxed{\varphi = 20^0}$, $b = 11.78125$, $m = 0.43929$ we find

$$(16b)\quad \int_{1-\Delta}^{1} \frac{d\xi}{d\eta} \cdot d\eta = 6.78400 \cdot \sqrt{\Delta} \cdot \left\{ 1 - 0.068427\Delta + 0.0050592\Delta^2 \right\}.$$

For $\Delta = 0.02$

$$(16bb)\quad \int_{0.98}^{1} \frac{d\xi}{d\eta} \cdot d\eta = 0.959400 \cdot \left\{ 1 - 0.00136854 + 0.00000202 \right\}$$

$$= \underline{0.958089.}$$

For $\boxed{\varphi = 25^0}$, $b = 7.48700$, $m = 0.56234$ we find

$$(16c)\quad \int_{1-\Delta}^{1} \frac{d\xi}{d\eta} \cdot d\eta = 5.84926 \cdot \sqrt{\Delta} \cdot \left\{ 1 - 0.101523\Delta + 0.014835\Delta^2 \right\}.$$

For Δ = 0.02

$$\text{(16cc)} \quad \int_{0.98}^{1} \frac{d\xi}{d\eta} \cdot d\eta = 0.827208 \cdot \left\{ 1 - 0.00203046 + 0.00000593 \right\}$$

$$= \underline{\underline{0.825533.}}$$

For $\boxed{\varphi = 30^{\circ}}$, b = 4.06044 , m = 0.64652 we find

$$\text{(16d)} \quad \int_{1-\Delta}^{1} \frac{d\xi}{d\eta} \cdot d\eta = 5.19364 \cdot \sqrt{\Delta} \cdot \left\{ 1 - 0.1319\,\Delta + 0.0296\,\Delta^{2} \right\}.$$

For Δ = 0.02

$$\text{(16dd)} \quad \int_{0.98}^{1} \frac{d\xi}{d\eta} \cdot d\eta = 0.734490 \cdot \left\{ 1 - 0.02638 + 0.00001184 \right\}$$

$$= \underline{\underline{0.732561.}}$$

3.3. NUMERICAL INTEGRATION. COMPUTATION OF η, ξ VALUES FOR φ = 15°, 20°, 25° and 30°.

The shape of the bank-part of the bottom, i.e. between $\eta = 0$ and $\eta = 1$, where $y = y_{max}$, can now be computed (Tables 3.3./2a/b/c/d/ for φ = 15°, 20°, 25° and 30° respectively).

For each interval in η the mean value of the two neighbouring values of $\frac{d\xi}{d\eta}$ is found, and

$$\Delta \xi_{n,n+1} = \frac{1}{2} \cdot \left\{ \left(\frac{d\xi}{d\eta}\right)_n + \left(\frac{d\xi}{d\eta}\right)_{n+1} \right\} \cdot \Delta\eta = \left(\frac{d\xi}{d\eta}\right)_m \cdot \Delta\eta$$

For the last interval in η from 0.98 to 1.0

$$\Delta \xi_{13,14} = \int_{0.98}^{1} \frac{d\xi}{d\eta} \cdot d\eta \text{ , which are found in}$$

formulae 3.2. (16aa), (16bb), (16cc) and (16dd).

The values of ξ_n are found by successive addition of $\Delta\xi$.

$\boxed{\varphi = 15^{\circ}.}$ Table 3.3/2a/

η	$\frac{d\xi}{d\eta}$	$\left(\frac{d\xi}{d\eta}\right)_m$	$\Delta\xi$	ξ
0.	3.73205			0
		3.85429	0.385429	
0.1	3.97653			0.385429
		4.11969	0.411969	
0.2	4.26285			0.797398
		4.43410	0.443410	
0.3	4.60535			1.240808
		4.81586	0.481586	
0.4	5.02636			1.722394
		5.29472	0.529472	
0.5	5.56309			2.251866
		5.92328	0.592328	
0.6	6.28347			2.844194
		6.80637	0.680637	
0.7	7.32926			3.524831
		7.69925	0.384963	
0.75	8.06924			3.909794
		8.56810	0.428405	
0.8	9.06696			4.338199
		9.79445	0.489723	
0.85	10.52195			4.827922
		11.73643	0.586821	
0.9	12.95091			5.414743
		15.67860	0.783930	
0.95	18.40628			6.198673
		23.79777	0.713933	
0.98	29.18927			6.912606
			1.169109	
1.0				β = 8.081715

$\boxed{\varphi = 20^{\circ}}$ Table 3.3/2b/

η	$\frac{d\xi}{d\eta}$	$\left(\frac{d\xi}{d\eta}\right)_m$	$\Delta\xi$	ξ
0.	2.74748			0
		2.85445	0.285445	
0.1	2.96142			0.285445
		3.08592	0.308592	
0.2	3.21042			0.594037
		3.35849	0.335849	
0.3	3.50655			0.929886
		3.68752	0.368752	
0.4	3.86848			1.298638
		4.09787	0.409787	
0.5	4.32726			1.708425
		4.63332	0.463332	
0.6	4.93937			2.171757
		5.38080	0.538080	
0.7	5.82222			2.709837
		6.13303	0.306652	
0.75	6.44384			3.016489
		6.86129	0.343065	
0.8	7.27874			3.359554
		7.88505	0.394253	
0.85	8.49136			3.753807
		9.49953	0.474977	
0.9	10.50770			4.228784
		12.76065	0.638033	
0.95	15.01359			4.866817
		19.45090	0.583527	
0.98	23.88820			5.450344
			0.958089	
1.0				$\beta = 6.408433$

$\boxed{\varphi = 25^{\circ}.}$ Table 3.3/2c/

η	$\frac{d\xi}{d\eta}$	$\left(\frac{d\xi}{d\eta}\right)_m$	$\Delta\xi$	ξ
0	2.14451			0
		2.24154	0.224154	
0.1	2.33857			0.224154
		2.45071	0.245071	
0.2	2.56285			0.469225
		2.69544	0.269544	
0.3	2.82803			0.738769
		2.98932	0.298932	
0.4	3.15061			1.037701
		3.35421	0.335421	
0.5	3.55780			1.373122
		3.82846	0.382846	
0.6	4.09912			1.755968
		4.48795	0.448795	
0.7	4.87677			2.204763
		5.14970	0.257485	
0.75	5.42262			2.462248
		5.78837	0.289419	
0.8	6.15411			2.751667
		6.68399	0.334200	
0.85	7.21387			3.085867
		8.09183	0.404592	
0.9	8.96978			3.490459
		10.92539	0.546270	
0.95	12.88099			4.036729
		16.71760	0.501528	
0.98	20.55420			4.538257
			0.825533	
1.0				β = 5.363790

$\boxed{\varphi = 30^{\circ}.}$ Table3. 3/2d/

η	$\frac{d\xi}{d\eta}$	$\left(\frac{d\xi}{d\eta}\right)_m$	$\Delta\xi$	ξ
0	1.73205			0
		1.82262	0.182262	
0.1	1.91319			0.182262
		2.01677	0.201677	
0.2	2.12035			0.383939
		2.24187	0.224187	
0.3	2.36339			0.608126
		2.51040	0.251040	
0.4	2.65740			0.859166
		2.84218	0.284218	
0.5	3.02696			1.143384
		3.27178	0.327178	
0.6	3.51659			1.470562
		3.86742	0.386742	
0.7	4.21825			1.857304
		4.46401	0.223200	
0.75	4.70976			2.080504
		5.03863	0.251932	
0.8	5.36750			2.332436
		5.84312	0.292156	
0.85	6.31874			2.624592
		7.10495	0.355248	
0.9	7.89116			2.979840
		9.63707	0.481854	
0.95	11.38297			3.461694
		14.79807	0.443942	
0.98	18.21316			3.905636
			0.732561	
1.0				β = 4.638197

4. DETERMINATION OF PERIMETER, AREA AND DISCHARGE FOR "BANK-PART" AND "MIDDLE-PART". PRINCIPLE OF MINIMUM WORK.

4.1. INTRODUCTION.

The profile determined in the foregoing section should be stable against erosion for every value of η between zero and one as long as the limiting tractive force is not exceeded. The relative width of the curved sloping bottom at one side of the axis of the channel is called $\beta = \xi_{1.0}$.

The absolute width of this part of the cross section is $\beta \cdot y_{max}$, and the corresponding parts of the perimeter, area and discharge are called U_β, F_β and Q_β respectively. This part of the cross section is called the "bank-part" . (Fig. 3). The shearing stress in this part varies between $\tau = 0$ for ($\xi = 0$, $\eta = 0$) and $\tau = \tau_{max}$ for ($\xi = \beta$, $\eta = 1$), and it is evident that another half cross section consisting of the said "bank-part" and a "middle-part" with constant depth y_{max} and an arbitrary width $\beta_0 \cdot y_{max}$ between the "bank-part" and the axis should be equally stable. The perimeter, area and discharge of such a "middle-part" are called $U_{\beta o}$, $F_{\beta o}$ and $Q_{\beta o}$ respectively.

So far an infinity of possible solutions seem to exist for fixed values of the total discharge of the half-section $Q = Q_\beta + Q_{\beta o}$ and for τ_{max} and φ.

If we imagine, however, that the profile is eroded gradually in a uniform mass of sand at a constant discharge Q, the intensity of the erosion will steadily diminish, and the profile must asymptotically approach the state of equilibrium commensurate with the sand grains in all parts of the bottom being in a state of incipient motion.

It thus seems natural to assume the principle of minimum of work and assume that nature will produce the profile that requires a minimum of erosion, i.e. a minimum of the cross section $F = F_\beta + F_{\beta o}$. According to this principle only one definite equilibrium profile can exist for fixed values of Q, τ_{max} and φ .

We therefore proceed to find the cross sections F_β and $F_{\beta o}$ and the discharges Q_β and $Q_{\beta o}$, whereupon the indeterminate quantity β_o is fixed in such a way that it causes $F = F_\beta + F_{\beta o}$ to be a minimum for a fixed value of $Q = Q_\beta + Q_{\beta o}$, or the total discharge in the half cross section, and for fixed values of τ_{max} and φ .

4.2. DERIVATION OF FORMULAE FOR MEAN VELOCITY, AREA AND DISCHARGE OF AN ELEMENT OF THE CROSS SECTION.

As previously mentioned, it is assumed that the logarithmic law of velocity distribution is valid, and that the velocity at a point P, situated at the distance z from the bottom, and measured perpendicularly to the bottom element ds, will depend exclusively on the shearing stress τ on this element and on the distance z.

The length of the normal to the water surface is (Fig. 4)

(17) $$z_0 = \frac{y}{\cos\alpha} = \xi \cdot y_{max}$$

From equation 1.(1) we get by means of (17) :

(1a) $$\frac{\tau}{\tau_{max}} = \frac{y}{y_{max} \cdot \cos\alpha} = \frac{z_0}{y_{max}} = \xi$$

The values of ξ are compiled in Tables 3.1/1a/b/c/d/

We consequently have :

(18) $$v_{*,\xi} = \sqrt{\frac{\tau}{\varrho}} = \sqrt{\frac{\xi \cdot \tau_{max}}{\varrho}} .$$

(19) $$\frac{v_z}{v_{*,\xi}} = 8.48 + 2.5 \ln \frac{z}{k} ,$$

where k is the equivalent sand roughness of the bottom supposed to be constant for the whole width of the channel.

If the element of the cross section shown in Fig. 4 had a constant width equal to the base ds, the mean velocity in such an element could be found as the velocity at a point $0.3679 \cdot z_0$ from the bottom. The curvature of the bottom is slight, so that it is considered permissible to disregard this and find, at the same distance, the mean velocity in the actually wedge-shaped element $v_{m,\xi}$; but for the computation of the element area and its discharge the wedge-shape must be taken into account.

We consequently get :

$$v_{m,\xi} = \sqrt{\xi} \cdot \sqrt{\frac{\tau_{max}}{\varrho}} \cdot \left\{ 8.48 + 2.5 \ln \left(0.3679 \cdot \xi \cdot \frac{y_{max}}{k} \right) \right\}$$

$\boxed{(20)}$ $$v_{m,\xi} = 2.5 \sqrt{\frac{\tau_{max}}{\varrho}} \cdot \sqrt{\xi} \cdot \left\{ \ln \left(\frac{11\, y_{max}}{k} \right) + \ln \xi \right\}$$

The area of the surface-element (Fig. 4), with base ds, measured between the bottom and the water-surface, and when r is the radius of curvature of the bottom, will be

(21) $$dF = \frac{1}{2}\cdot z_0 \cdot ds\cdot\left(1 + \frac{r-z_o}{r}\right) = z_0 \cdot ds\cdot\left(1 - \frac{z_0}{2r}\right)$$

For the radius of curvature r we have

$$\frac{1}{r} = -\frac{\left(\frac{d^2y}{dx^2}\right)}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$ or from 2.1.(6a)b)c)d)

(22) $$\frac{1}{\left(\frac{r}{y_{max}}\right)} = -\frac{\left(\frac{d^2\eta}{d\xi^2}\right)}{\left[1 + \left(\frac{d\eta}{d\xi}\right)^2\right]^{3/2}}$$

(23) $$ds = \frac{dx}{\cos\alpha} = \frac{y_{max}\cdot d\xi}{\cos\alpha} = y_{max}\cdot(d\xi)\cdot\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}$$

(17) $$z_0 = \xi\cdot y_{max}$$

From (21), (22), (23) and (17) we finally get

$$dF = \xi\cdot y_{max}^2\cdot(d\xi)\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}\cdot\left\{1 + \frac{1}{2}\,\frac{\xi\cdot\left(\frac{d^2\eta}{d\xi^2}\right)}{\left[1 + \left(\frac{d\eta}{d\xi}\right)^2\right]^{3/2}}\right\}$$

(24) $$\frac{dF}{y_{max}^2} = \xi\cdot(d\xi)\cdot\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2} + \frac{1}{2}\cdot\frac{\xi^2\cdot d\left(\frac{d\eta}{d\xi}\right)}{1 + \left(\frac{d\eta}{d\xi}\right)^2}$$

By the following calculations with finite differences we introduce the mean values of ξ and $\left[1 + \left(\frac{d\eta}{d\xi}\right)^2\right]$ for the sides of the element in question and put $\Delta\xi$ for $d\xi$ and $\Delta\left(\frac{d\eta}{d\xi}\right)$ for $d\left(\frac{d\eta}{d\xi}\right)$. (24) then takes the form

$$\boxed{(24a)} \qquad \frac{\Delta F}{y_{max}^2} = \xi_m \cdot (\Delta \xi) \cdot \left(\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}\right)_m + \frac{\xi_m^2 \cdot \Delta\left(\frac{d\eta}{d\xi}\right)}{2 \cdot \left(1 + \left(\frac{d\eta}{d\xi}\right)^2\right)_m}$$

where subscript m signifies mean values.

Accordingly the mean velocity in the element considered is taken from (20)

$$\boxed{(20a)} \qquad v_{m,\xi m} = 2.5\sqrt{\frac{\tau_{max}}{\rho}} \cdot \sqrt{\xi_m} \cdot \left\{\ln\left(\frac{11\, y_{max}}{k}\right) + \ln \xi_m\right\}$$

The discharge ΔQ in the element ΔF is :

$$\boxed{(25)} \qquad \frac{\Delta Q}{2.5\sqrt{\frac{\tau_{max}}{\rho}}\, y_{max}^2} = \left[\ln \frac{11\, y_{max}}{k}\right]\left\{(\Delta \xi) \cdot \left(\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}\right)_m \cdot \xi_m^{3/2}\right.$$

$$\left. + \frac{\Delta\left(\frac{d\eta}{d\xi}\right)}{2\left(1 + \left(\frac{d\eta}{d\xi}\right)^2\right)_m} \cdot \xi_m^{5/2}\right\} + \left\{(\Delta \xi) \cdot \left(\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}\right)_m \cdot \xi_m^{3/2} \cdot \ln \xi_m\right.$$

$$\left. + \frac{\Delta\left(\frac{d\eta}{d\xi}\right)}{2\left(1 + \left(\frac{d\eta}{d\xi}\right)^2\right)_m} \cdot \xi_m^{5/2} \cdot \ln \xi_m\right\}.$$

4.3. FORMULAE FOR THE QUANTITIES S AND FOR DIMENSIONLESS VALUES OF PERIMETER, AREA AND DISCHARGE U', F' AND Q' RESPECTIVELY, VALID FOR THE "BANK-PART", "MIDDLE-PART" AND FOR THE TOTAL HALF-SECTION.

The following notations are introduced :

$$(26) \qquad S_0 = \sum_{\eta=0}^{\eta=1} (\Delta \xi) \cdot \left(\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}\right)_m$$

$$(27) \qquad S_{1,1} = \sum_{\eta=0}^{\eta=1} (\Delta \xi) \cdot \left(\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}\right)_m \cdot \xi_m$$

$$(28) \qquad S_{2,1} = \sum_{\eta=0}^{\eta=1} (\Delta \xi) \cdot \left(\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}\right)_m \cdot \xi_m^{3/2}$$

$$(29)\qquad S_{3,1} = \sum_{\eta=0}^{\eta=1} (\Delta \xi) \cdot \left(\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}\right)_m \cdot \xi_m^{3/2} \cdot \ln \xi_m$$

$$(30)\qquad S_{1,2} = -\sum_{\eta=0}^{\eta=1} \xi_m^2 \cdot \frac{\Delta\left(\frac{d\eta}{d\xi}\right)}{2\left(1 + \left(\frac{d\eta}{d\xi}\right)^2\right)_m}$$

$$(31)\qquad S_{2,2} = -\sum_{\eta=0}^{\eta=1} \xi_m^{5/2} \cdot \frac{\Delta\left(\frac{d\eta}{d\xi}\right)}{2\left(1 + \left(\frac{d\eta}{d\xi}\right)^2\right)_m}$$

$$(32)\qquad S_{3,2} = -\sum_{\eta=0}^{\eta=1} \xi_m^{5/2} \cdot \frac{\Delta\left(\frac{d\eta}{d\xi}\right)}{2\left(1 + \left(\frac{d\eta}{d\xi}\right)^2\right)_m} \cdot \ln \xi_m$$

$$(33)\qquad S_1 = S_{1,1} - S_{1,2}$$

$$(34)\qquad S_2 = S_{2,1} - S_{2,2}$$

$$(35)\qquad S_3 = S_{3,1} - S_{3,2}$$

For the "bank-part" of the cross section we find the following dimensionless expressions for perimeter, area and discharge.

$$\boxed{(36)}\qquad \frac{U_\beta}{y_{max}} = S_0$$

$$\boxed{(37)}\qquad \frac{F_\beta}{y_{max}^2} = S_{1,1} - S_{1,2} = S_1$$

$$\boxed{(38)}\qquad \frac{Q_\beta}{2.5\sqrt{\frac{\tau_{max}}{\rho}} \cdot y_{max}^2} = (S_{2,1} - S_{2,2}) \ln\left(\frac{11\, y_{max}}{k}\right) + S_{3,1} - S_{3,2} = S_2 \ln\left(\frac{11\, y_{max}}{k}\right) + S_3 .$$

For the "middle-part" of the section we have

$\frac{d\eta}{d\xi} = 0 \quad , \quad \Delta(\frac{d\eta}{d\xi}) = 0 \quad , \quad \xi_m = 1$

and thereby

(26a) $S_0 = \beta_0$

(27a) $S_{1,1} = \beta_0$ (30a) $S_{1,2} = 0$

(28a) $S_{2,1} = \beta_0$ (31a) $S_{2,2} = 0$

(29a) $S_{3,1} = 0$ (32a) $S_{3,2} = 0$

$\boxed{(36a)}$ $\frac{U_{\beta,0}}{y_{max}} = \beta_0$

$\boxed{(37a)}$ $\frac{F_{\beta,0}}{y_{max}^2} = \beta_0$

$\boxed{(38a)}$ $\frac{Q_{\beta,0}}{2.5\sqrt{\frac{\tau_{max}}{\rho}} \cdot y_{max}^2} = \beta_0 \ln(\frac{11\,y_{max}}{k}).$

We put for the total half-section of the channel

(39) $U = U_\beta + U_{\beta,0}$

(40) $F = F_\beta + F_{\beta,0}$

(41) $Q = Q_\beta + Q_{\beta,0}$, and find from (36), (37), (38), (36a) (37a), (38a)

$\boxed{(42)}$ $U' = \frac{U}{y_{max}} = S_0 + \beta_0$

$\boxed{(43)}$ $F' = \frac{F}{y_{max}^2} = S_1 + \beta_0$

$\boxed{(44)}$ $Q' = \frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\rho}} \cdot y_{max}^2} = (S_2 + \beta_0)\ln(\frac{11\,y_{max}}{k}) + S_3 .$

4.4. DETERMINATION OF β_0 CORRESPONDING TO MINIMUM OF CROSS SECTION.

For a fixed value of φ the quantities S_1, S_2 and S_3 are constants, and if further Q, τ_{max}, ϱ and k are constant, equation 4.3.(44) represents a relation between β_0 and y_{max} which must be satisfied :

$$\text{(44a)} \qquad \psi = \frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\varrho}} \cdot y_{max}^2} - (S_2 + \beta_0)\ln\left(\frac{11\, y_{max}}{k}\right) - S_3 = 0$$

together with

$$\text{(43a)} \qquad F = (S_1 + \beta_0)\, y_{max}^2 .$$

F is a function of β_0 and y_{max}, and to obtain a minimum value of F simultaneously with $\psi = 0$, Lagrange's method gives the following condition :

$$\frac{\left(\frac{\partial F}{\partial \beta_0}\right)}{\left(\frac{\partial \psi}{\partial \beta_0}\right)} = \frac{\left(\frac{\partial F}{\partial y_{max}}\right)}{\left(\frac{\partial \psi}{\partial y_{max}}\right)} , \qquad \text{hence}$$

$$\frac{y_{max}^2}{-\ln\left(\frac{11\, y_{max}}{k}\right)} = \frac{2(S_1 + \beta_0)\, y_{max}}{-\left(\frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\varrho}}}\right) \cdot \frac{2}{y_{max}^3} - \frac{S_2 + \beta_0}{y_{max}}} \qquad \text{or}$$

$$\frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\varrho}} \cdot y_{max}^2} - (S_1 + \beta_0)\ln\left(\frac{11\, y_{max}}{k}\right) + \frac{1}{2}(S_2 + \beta_0) = 0.$$

When deducting (44a) the result is :

$$\boxed{(45)} \qquad \beta_0 = 2(S_1 - S_2)\ln\left(\frac{11\, y_{max}}{k}\right) - (S_2 + 2\, S_3) ,$$

which gives the value of β_0 corresponding to a minimum of F.

4.5. FORMULAE VALID FOR THE OPTIMAL PROFILE FOR DIMENSIONLESS VALUES OF PERIMETER, AREA, DISCHARGE, MEAN VELOCITY, HYDRAULIC RADIUS, WIDTH OF WATER SURFACE, SLOPE, SHEARING VELOCITY AND FULLNESS OF THE PROFILE, ALL EXPRESSED BY THE QUANTITIES S AND β.

It follows from the foregoing section 4.4. that

(46) $$S_1 + \beta_0 = 2(S_1 - S_2)\ln\left(\frac{11\,y_{max}}{k}\right) + (S_1 - S_2 - 2\,S_3) \text{ and}$$

(47) $$S_2 + \beta_0 = 2(S_1 - S_2)\ln\left(\frac{11\,y_{max}}{k}\right) - 2\,S_3.$$

4.4.(45), (46) and (47) are introduced in 4.4.(42), 4.4.(43) and 4.4.(44) resulting in :

$\boxed{(48)}$ $$U' = \frac{U}{y_{max}} = 2(S_1 - S_2)\ln\left(\frac{11\,y_{max}}{k}\right) + (S_0 - S_2 - 2\,S_3)$$

$\boxed{(49)}$ $$F' = \frac{F}{y_{max}^{2}} = 2(S_1 - S_2)\ln\left(\frac{11\,y_{max}}{k}\right) + (S_1 - S_2 - 2\,S_3)$$

$\boxed{(50)}$ $$Q' = \frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\varrho}}\cdot y_{max}^{2}} = 2(S_1 - S_2)\ln^2\left(\frac{11\,y_{max}}{k}\right) - 2\,S_3\ln\left(\frac{11\,y_{max}}{k}\right) + S_3.$$

Equations (48), (49) and (50) are valid for the profile with a minimum of cross sectional area.

For the mean velocity

$v_m = \frac{Q}{F}$ we get, dividing (50) by (49) :

$$v_m' = \frac{v_m}{2.5\sqrt{\frac{\tau_{max}}{\varrho}}\cdot y_{max}} = \frac{2(S_1 - S_2)\ln^2\left(\frac{11\,y_{max}}{k}\right) - 2\,S_3\ln\left(\frac{11\,y_{max}}{k}\right) + S_3}{2(S_1 - S_2)\ln\left(\frac{11\,y_{max}}{k}\right) + (S_1 - S_2 - 2\,S_3)}. \qquad \boxed{(51)}$$

We want to introduce the slope of the water surface I (uniform motion assumed) and the hydraulic radius R of the cross section and find that the total shearing force K for the half cross section F must be :

(52) $$K = \gamma F I = \int_0^U \tau \, ds;$$ we further have

4.2(23a) $$\Delta s = \Delta \xi \cdot \left(\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}\right)_m \cdot y_{max}$$

4.2(1a) $$\tau = \zeta_m \cdot \tau_{max}$$

$$\int_0^U \tau \, ds \cong y_{max} \cdot \tau_{max} \left[\sum_{\eta=0}^{\eta=1} \Delta \xi \cdot \left(\sqrt{1 + \left(\frac{d\eta}{d\xi}\right)^2}\right)_m \cdot \zeta_m + \beta_0\right]$$

and from 4.3(27)

(53) $$\int_0^U \tau \cdot ds \cong (S_{1,1} + \beta_0) \cdot y_{max} \cdot \tau_{max} \; .$$

As $\gamma = \varrho \cdot g$ (52) gives with β_0 from 4.4.(45)

$\boxed{(54)}$ $$I' = \frac{g F I}{y_{max} \cdot \left(\frac{\tau_{max}}{\varrho}\right)} = S_{1,1} + \beta_0 = 2(S_1 - S_2) \ln\left(\frac{11 \, y_{max}}{k}\right) + (S_{1,1} - S_2 - 2 \, S_3) .$$

For the hydraulic radius R we find by 4.3.(42)

$$R' = \frac{R}{y_{max}} = \frac{F}{U \cdot y_{max}} = \frac{F}{y_{max}^2 \, (S_0 + \beta_0)} = \frac{F'}{S_0 + \beta_0}$$

and by (49) and β_0 from 4.4.(45)

$\boxed{(55)}$ $$R' = \frac{2(S_1 - S_2) \ln\left(\frac{11 \, y_{max}}{k}\right) + (S_1 - S_2 - 2 \, S_3)}{2(S_1 - S_2) \ln\left(\frac{11 \, y_{max}}{k}\right) + (S_0 - S_2 - 2 \, S_3)} \; .$$

For the shearing velocity v_* we have

$\boxed{(56)}$ $$v'_* = \sqrt{\frac{\left(\frac{\tau_{max}}{\varrho}\right)}{g R I}} = \sqrt{\frac{S_0 + \beta_0}{S_{1,1} + \beta_0}} = \sqrt{\frac{2(S_1 - S_2) \ln\left(\frac{11 \, y_{max}}{k}\right) + (S_0 - S_2 - 2 \, S_3)}{2(S_1 - S_2) \ln\left(\frac{11 \, y_{max}}{k}\right) + (S_{1,1} - S_2 - 2 \, S_3)}} \; .$$

For the width B of the half profile at the water surface we find

$$\boxed{(57)} \quad B' = \frac{B}{y_{max}} = (\beta + \beta_0) = 2(S_1 - S_2) \ln(\frac{11\, y_{max}}{k}) + (\beta - S_2 - 2\, S_3).$$

We later apply the term $\varkappa$, "degree of fullness" of the cross section, which is defined by

$$\boxed{(58)} \quad \varkappa = \frac{F}{B \cdot y_{max}} = \frac{F'}{B'} = \frac{S_1 + \beta_0}{\beta + \beta_0} = \frac{2(S_1 - S_2) \ln(\frac{11\, y_{max}}{k}) + (S_1 - S_2 - 2S_3)}{2(S_1 - S_2) \ln(\frac{11\, y_{max}}{k}) + (\beta - S_2 - 2S_3)}$$

found by using (49) and (57).

4.6. EVALUATION OF NUMERICAL VALUES OF THE QUANTITIES S.

It remains to compute the numerical values of S_0, S_1, S_2 and S_3 for $\varphi = 15^o$, 20^o, 25^o and 30^o. The calculations are given in Tables 4.6./3a/b/c/d/. The fundamental values of $\sqrt{1 + (\frac{d\,\eta}{d\,\xi})^2}$, ξ, $\frac{d\,\eta}{d\,\xi}$ and $\left(1 + (\frac{d\,\eta}{d\,\xi})^2\right)$ are to be found in Tables 3.2./1a/b/c/d/, whereas $\Delta\,\xi$ is found in Tables 3.4./2a/b/c/d/. The calculations are made from the formulae 4.3.(26) through 4.3.(35).

The results are compiled in Table 4.6./3e/.

$\boxed{\varphi = 15^{o}.}$ Table 4.6/3a/

η	$\sqrt{1+(\frac{d\eta}{d\xi})^2}$	ξ	ξ_m	$\Delta\xi$	$\left(\sqrt{1+(\frac{d\eta}{d\xi})^2}\right)_m$
0	1.035276	0			
			0.051557	0.385429	1.033206
0.1	1.031135	0.103114			
			0.154272	0.411969	1.029141
0.2	1.027147	0.205430			
			0.256211	0.443410	1.025225
0.3	1.023303	0.306991			
			0.357416	0.481586	1.021451
0.4	1.019599	0.407840			
			0.457927	0.529472	1.017814
0.5	1.016028	0.508014			
			0.557783	0.592328	1.014307
0.6	1.012585	0.607551			
			0.657018	0.680637	1.010925
0.7	1.009265	0.706485			
			0.731111	0.384963	1.008458
0.75	1.007650	0.755737			
			0.780294	0.428405	1.006857
0.8	1.006064	0.804851			
			0.829341	0.489723	1.005285
0.85	1.004506	0.853830			
			0.878255	0.586821	1.003742
0.9	1.002977	0.902679			
			0.927040	0.783930	1.002226
0.95	1.001475	0.951401			
			0.965988	0.713933	1.001031
0.98	1.000587	0.980575			
			0.990288	1.169109	1.000294
1.00	1.000000	1.000000			

To be continued

$\boxed{\varphi = 15^{\circ}.}$ Table 4.6/3a/ (continued)

η	$\Delta s' = \Delta\xi\left(\sqrt{1+\left(\frac{d\eta}{d\xi}\right)^2}\right)_m$	$\Delta s' \cdot \xi_m$	$\sqrt{\xi_m}$	$\Delta s' \cdot \xi_m^{3/2}$	$-\ln \xi_m$
0					
	0.398228	0.020531	0.227062	0.004662	2.96507
0.1					
	0.423974	0.065407	0.392775	0.025690	1.86904
0.2					
	0.454595	0.116472	0.506173	0.058955	1.36176
0.3					
	0.491917	0.175819	0.597843	0.105112	1.02886
0.4					
	0.538904	0.246779	0.676703	0.166996	0.78104
0.5					
	0.600802	0.335117	0.746849	0.250282	0.58379
0.6					
	0.688073	0.452076	0.810566	0.366437	0.42004
0.7					
	0.388219	0.283831	0.855050	0.242690	0.31319
0.75					
	0.431343	0.336574	0.883343	0.297310	0.24808
0.8					
	0.492311	0.408294	0.910682	0.371826	0.18713
0.85					
	0.589017	0.517307	0.937153	0.484796	0.12982
0.9					
	0.785675	0.728352	0.962829	0.701278	0.07576
0.95					
	0.714669	0.690362	0.982847	0.678520	0.03461
0.98					
	1.169453	1.158095	0.995132	1.152457	0.00976
1.00					
	+8.167180	+5.535016		+4.907011	
	$= S_0$	$= S_{1,1}$		$= S_{2,1}$	

To be continued

$\varphi = 15^{o}.$ Table 4.6/3a/ (continued)

η	$-\Delta s\cdot\xi_m^{3/2}\cdot\ln\xi_m$	$\frac{d\eta}{d\xi}$	$-\Delta(\frac{d\eta}{d\xi})$	$1+(\frac{d\eta}{d\xi})^2$	$2\cdot[1+(\frac{d\eta}{d\xi})^2]_m$
0		0.267949		1.071797	
	0.013823		0.016473		2.135037
0.1		0.251476		1.063240	
	0.048016		0.016891		2.118270
0.2		0.234585		1.055030	
	0.080283		0.017446		2.102179
0.3		0.217139		1.047149	
	0.108146		0.018187		2.086731
0.4		0.198952		1.039582	
	0.130431		0.019196		2.071894
0.5		0.179756		1.032312	
	0.146112		0.020608		2.057640
0.6		0.159148		1.025328	
	0.153918		0.022708		2.043944
0.7		0.136440		1.018616	
	0.076008		0.012513		2.033974
0.75		0.123927		1.015358	
	0.073757		0.013636		2.027522
0.8		0.110291		1.012164	
	0.069580		0.015252		2.021196
0.85		0.095039		1.009032	
	0.062936		0.017824		2.014994
0.9		0.077215		1.005962	
	0.053129		0.022886		2.008914
0.95		0.054329		1.002952	
	0.023484		0.020070		2.004126
0.98		0.034259		1.001174	
	0.011248		0.034259		2.001174
1.00		0.000000		1.000000	

-1.050871

$= S_{3,1}$

To be continued

$\boxed{\varphi = 15^{\circ}.}$ Table 4.6/3a/ (continued)

η	$-\dfrac{\Delta(\frac{d\eta}{d\xi})}{2\cdot[1+(\frac{d\eta}{d\xi})^2]_m}$	ξ_m^2	$-$ $\xi_m^2\cdot\dfrac{\Delta(\frac{d\eta}{d\xi})}{2\cdot[1+(\frac{d\eta}{d\xi})^2]_m}$	$-$ $\xi_m^{5/2}\cdot\dfrac{\Delta(\frac{d\eta}{d\xi})}{2\cdot[1+(\frac{d\eta}{d\xi})^2]_m}$	$+$ $\xi_m^{5/2}\cdot\dfrac{\Delta(\frac{d\eta}{d\xi})}{2\cdot[1+(\frac{d\eta}{d\xi})^2]_m}\cdot\ln\xi_m$
0					
	0.007716	0.002658	0.000021	0.000005	0.000015
0.1					
	0.007974	0.023800	0.000190	0.000075	0.000140
0.2					
	0.008299	0.065644	0.000545	0.000276	0.000376
0.3					
	0.008716	0.127746	0.001113	0.000665	0.000684
0.4					
	0.009265	0.209697	0.001943	0.001315	0.001027
0.5					
	0.010015	0.311122	0.003116	0.002327	0.001358
0.6					
	0.011110	0.431673	0.004796	0.003887	0.001633
0.7					
	0.006152	0.534523	0.003288	0.002811	0.000880
0.75					
	0.006725	0.608859	0.004095	0.003617	0.000897
0.8					
	0.007546	0.687806	0.005190	0.004726	0.000884
0.85					
	0.008846	0.771332	0.006823	0.006394	0.000830
0.9					
	0.011392	0.859403	0.009790	0.009426	0.000714
0.95					
	0.010014	0.933133	0.009344	0.009184	0.000318
0.98					
	0.017119	0.980670	0.016788	0.016706	0.000163
1.00					
			+0.067042	+0.061414	-0.009919
			$= S_{1,2}$	$= S_{2,2}$	$= S_{3,2}$

$\boxed{\varphi = 20^{\circ}.}$ Table 4.6/3b/

η	$\sqrt{1+(\frac{d\eta}{d\xi})^2}$	ξ	ξ_m	$\Delta\xi$	$\left(\sqrt{1+(\frac{d\eta}{d\xi})^2}\right)_m$
0	1.06418	0			
			0.052774	0.285445	1.05983
0.1	1.05547	0.105547			
			0.157513	0.308592	1.05143
0.2	1.04739	0.209478			
			0.260720	0.335849	1.04363
0.3	1.03987	0.311961			
			0.362555	0.368752	1.03637
0.4	1.03287	0.413148			
			0.463162	0.409787	1.02961
0.5	1.02635	0.513175			
			0.562675	0.463332	1.02332
0.6	1.02029	0.612174			
			0.661211	0.538080	1.01747
0.7	1.01464	0.710248			
			0.734613	0.306652	1.01331
0.75	1.01197	0.758978			
			0.783246	0.343065	1.01068
0.8	1.009393	0.807514			
			0.831694	0.394253	1.008152
0.85	1.006910	0.855874			
			0.879970	0.474977	1.005714
0.9	1.004518	0.904066			
			0.928086	0.638033	1.003367
0.95	1.002216	0.952105			
			0.966482	0.583527	1.001546
0.98	1.000876	0.980858			
			0.990429	0.958089	1.000438
1.00	1.000000	1.000000			

To be continued

$\boxed{\varphi = 20^o.}$ Table 4.6/3b/ (continued)

η	$\Delta s' = \Delta\xi\cdot\left(\sqrt{1+\left(\frac{d\eta}{d\xi}\right)^2}\right)_m$	$\Delta s'\cdot\xi_m$	$\sqrt{\xi_m}$	$\Delta s'\cdot\xi_m^{3/2}$	$-\ln\xi_m$
0					
	0.302523	0.015965	0.229726	0.003668	2.94173
0.1					
	0.324463	0.051107	0.396879	0.020283	1.84824
0.2					
	0.350502	0.091383	0.510607	0.046661	1.34431
0.3					
	0.382164	0.138555	0.602125	0.083427	1.01458
0.4					
	0.421921	0.195418	0.680560	0.132994	0.76968
0.5					
	0.474137	0.266785	0.750117	0.200120	0.57505
0.6					
	0.547480	0.362000	0.813149	0.294360	0.41368
0.7					
	0.310734	0.228269	0.857096	0.195648	0.30841
0.75					
	0.346729	0.271574	0.885012	0.240346	0.24430
0.8					
	0.397467	0.330571	0.911973	0.301472	0.18429
0.85					
	0.477691	0.420354	0.938067	0.394320	0.12786
0.9					
	0.640181	0.594143	0.963372	0.572381	0.07463
0.95					
	0.584429	0.564840	0.983098	0.555293	0.03408
0.98					
	0.958509	0.949335	0.995203	0.944781	0.00961
1.00					
	+ 6.518930	+4.480299		+3.985754	
	$= S_0$	$= S_{1,1}$		$= S_{2,1}$	

To be continued

$\boxed{\varphi = 20^{o}.}$ Table 4.6/3b/ (continued)

η	$-\Delta s' \cdot \xi_m^{3/2} \cdot \ln \xi_m$	$\frac{d\eta}{d\xi}$	$-\Delta\left(\frac{d\eta}{d\xi}\right)$	$1+\left(\frac{d\eta}{d\xi}\right)^2$	$2\cdot\left[1+\left(\frac{d\eta}{d\xi}\right)^2\right]_m$
0		0.36396		1.13247	
	0.010790		0.02628		2.246495
0.1		0.33768		1.114025	
	0.037488		0.02619		2.211048
0.2		0.31149		1.097023	
	0.062727		0.02631		2.178351
0.3		0.28518		1.081328	
	0.084643		0.02668		2.148150
0.4		0.25850		1.066822	
	0.102363		0.02741		2.120226
0.5		0.23109		1.053404	
	0.115079		0.02864		2.094392
0.6		0.20245		1.040988	
	0.121771		0.03069		2.070488
0.7		0.17176		1.029500	
	0.060340		0.01657		2.053583
0.75		0.15510		1.024083	
	0.058717		0.01780		2.042958
0.8		0.13739		1.018875	
	0.055558		0.01962		2.032744
0.85		0.11777		1.013869	
	0.050418		0.022602		2.022926
0.9		0.095168		1.009057	
	0.042717		0.028562		2.013493
0.95		0.066606		1.004436	
	0.018924		0.024744		2.006188
0.98		0.041862		1.001752	
	0.009079		0.041862		2.001752
1.00		0.000000		1.000000	
	-0.830614				
	$= S_{3,1}$				

To be continued

$\boxed{\varphi = 20^{o}.}$ Table 4.6/3b/ (continued)

η	$-\dfrac{\Delta(\frac{d\eta}{d\xi})}{2\cdot[1+(\frac{d\eta}{d\xi})^2]_m}$	ξ_m^2	$-$ $\xi_m^2 \cdot \dfrac{\Delta(\frac{d\eta}{d\xi})}{2\cdot[1+(\frac{d\eta}{d\xi})^2]_m}$	$-$ $\xi_m^{5/2} \cdot \dfrac{\Delta(\frac{d\eta}{d\xi})}{2\cdot[1+(\frac{d\eta}{d\xi})^2]_m}$	$+$ $\xi_m^{5/2} \cdot \dfrac{\Delta(\frac{d\eta}{d\xi})}{2\cdot[1+(\frac{d\eta}{d\xi})^2]_m} \cdot \ln \xi_m$
0					
	0.011698	0.002785	0.000033	0.000008	0.000024
0.1					
	0.011845	0.024810	0.000294	0.000117	0.000216
0.2					
	0.012078	0.067975	0.000821	0.000419	0.000563
0.3					
	0.012420	0.131446	0.001633	0.000983	0.000997
0.4					
	0.012928	0.214519	0.002773	0.001887	0.001452
0.5					
	0.013675	0.316603	0.004330	0.003248	0.001868
0.6					
	0.014823	0.437200	0.006481	0.005270	0.002180
0.7					
	0.008069	0.539656	0.004354	0.003732	0.001151
0.75					
	0.008713	0.613474	0.005345	0.004730	0.001156
0.8					
	0.009652	0.691715	0.006676	0.006088	0.001122
0.85					
	0.011173	0.774347	0.008651	0.008115	0.001038
0.9					
	0.014185	0.861344	0.012218	0.011770	0.000878
0.95					
	0.012334	0.934087	0.011521	0.011326	0.000386
0.98					
	0.020913	0.980950	0.020515	0.020417	0.000196
1.00					
			+0.085645	+0.078110	-0.013227
			$= S_{1,2}$	$= S_{2,2}$	$= S_{3,2}$

$\boxed{\varphi = 25^{\circ}.}$ Table 4.6/3c/

η	$\sqrt{1+\left(\frac{d\eta}{d\xi}\right)^2}$	ξ	ξ_m	$\Delta\xi$	$\left(\sqrt{1+\left(\frac{d\eta}{d\xi}\right)^2}\right)_m$
0	1.10338	0			
			0.054380	0.224154	1.09549
0.1	1.08759	0.108759			
			0.161723	0.245071	1.08051
0.2	1.07343	0.214686			
			0.266445	0.269544	1.06706
0.3	1.06068	0.318204			
			0.368934	0.298932	1.05492
0.4	1.04916	0.419664			
			0.469520	0.335421	1.04396
0.5	1.03875	0.519375			
			0.568487	0.382846	1.03404
0.6	1.02933	0.617598			
			0.666083	0.448795	1.02507
0.7	1.02081	0.714567			
			0.738606	0.257485	1.01884
0.75	1.01686	0.762645			
			0.786571	0.289419	1.01499
0.8	1.01312	0.810496			
			0.834311	0.334200	1.01134
0.85	1.00956	0.858126			
			0.881853	0.404592	1.00788
0.9	1.00620	0.905580			
			0.929220	0.546270	1.00461
0.95	1.00301	0.952860			
			0.967008	0.501528	1.00210
0.98	1.00118	0.981156			
			0.990578	0.825533	1.00059
1.00	1.00000	1.000000			

To be continued

$\boxed{\varphi = 25^{\circ}.}$ Table 4.6/3c/ (continued)

η	$\Delta s' = \Delta\xi \cdot \left(\sqrt{1+\left(\frac{d\eta}{d\xi}\right)^2}\right)_m$	$\Delta s' \cdot \xi_m$	$\sqrt{\xi_m}$	$\Delta s' \cdot \xi_m^{3/2}$	$-\ln \xi_m$
0					
	0.245558	0.013353	0.233195	0.003113	2.91180
0.1					
	0.264802	0.042825	0.402148	0.017222	1.82187
0.2					
	0.287620	0.076635	0.516183	0.039558	1.32259
0.3					
	0.315349	0.116343	0.607399	0.070667	0.99714
0.4					
	0.350166	0.164410	0.685215	0.112656	0.75604
0.5					
	0.395878	0.225051	0.753981	0.169684	0.56477
0.6					
	0.460046	0.306429	0.816139	0.250089	0.40634
0.7					
	0.262336	0.193763	0.859422	0.166524	0.30299
0.75					
	0.293757	0.231061	0.886888	0.204925	0.24007
0.8					
	0.337990	0.281989	0.913406	0.257570	0.18115
0.85					
	0.407780	0.359602	0.939070	0.337691	0.12573
0.9					
	0.548788	0.509945	0.963961	0.491567	0.07341
0.95					
	0.502581	0.486000	0.983366	0.477916	0.03355
0.98					
	0.826020	0.818237	0.995278	0.814373	0.00947
1.00					
	+5.498671 $= S_0$	+3.825643 $= S_{1,1}$		+3.413555 $= S_{2,1}$	

To be continued

$\boxed{\varphi = 25^{o}.}$ Table 4.6/3c/ (continued)

η	$-\Delta s \cdot \xi_m^{3/2} \ln \xi_m$	$\frac{d\eta}{d\xi}$	$-\Delta\left(\frac{d\eta}{d\xi}\right)$	$1+\left(\frac{d\eta}{d\xi}\right)^2$	$2\cdot\left[1+\left(\frac{d\eta}{d\xi}\right)^2\right]_m$
0		0.46631		1.217447	
	0.009064		0.03870		2.400299
0.1		0.42761		1.182852	
	0.031376		0.03742		2.335101
0.2		0.39019		1.152249	
	0.052319		0.03659		2.277284
0.3		0.35360		1.125035	
	0.070465		0.03620		2.225777
0.4		0.31740		1.100742	
	0.085172		0.03633		2.179744
0.5		0.28107		1.079002	
	0.095832		0.03712		2.138516
0.6		0.24395		1.059514	
	0.101621		0.03890		2.101561
0.7		0.20505		1.042047	
	0.050455		0.02064		2.076055
0.75		0.18441		1.034008	
	0.049196		0.02192		2.060412
0.8		0.16249		1.026404	
	0.046659		0.02387		2.045620
0.85		0.13862		1.019216	
	0.042458		0.02713		2.031645
0.9		0.11149		1.012429	
	0.036086		0.03386		2.018456
0.95		0.07763		1.006027	
	0.016034		0.02898		2.008394
0.98		0.04865		1.002367	
	0.007712		0.04865		2.002367
1.00		0.00000		1.000000	
	-0.694449				
	$= S_{3,1}$				

To be continued

$\boxed{\varphi = 25^{\circ}.}$ Table 4.6/3c/ (continued)

η	$-\frac{\Delta(\frac{d\eta}{d\xi})}{2\cdot[1+(\frac{d\eta}{d\xi})^2]_m}$	ξ_m^2	$-$ $\xi_m^2\cdot\frac{\Delta(\frac{d\eta}{d\xi})}{2\cdot[1+(\frac{d\eta}{d\xi})^2]_m}$	$-$ $\xi_m^{5/2}\cdot\frac{\Delta(\frac{d\eta}{d\xi})}{2\cdot[1+(\frac{d\eta}{d\xi})^2]_m}$	$+$ $\xi_m^{5/2}\cdot\frac{\Delta(\frac{d\eta}{d\xi})}{2\cdot[1+(\frac{d\eta}{d\xi})^2]_m}\cdot\ln\xi_m$
0					
	0.016123	0.002957	0.000048	0.000014	0.000041
0.1					
	0.016025	0.026154	0.000419	0.000169	0.000308
0.2					
	0.016067	0.070993	0.001141	0.000589	0.000779
0.3					
	0.016264	0.136112	0.002214	0.001345	0.001341
0.4					
	0.016667	0.220449	0.003674	0.002517	0.001903
0.5					
	0.017358	0.323177	0.005610	0.004230	0.002389
0.6					
	0.018510	0.443667	0.008212	0.006702	0.002723
0.7					
	0.009942	0.545539	0.005424	0.004662	0.001413
0.75					
	0.010639	0.618694	0.006582	0.005837	0.001401
0.8					
	0.011669	0.696075	0.008122	0.007419	0.001344
0.85					
	0.013354	0.777665	0.010385	0.009752	0.001226
0.9					
	0.016775	0.863450	0.014484	0.013962	0.001025
0.95					
	0.014429	0.935104	0.013493	0.013269	0.000445
0.98					
	0.024296	0.981245	0.023840	0.023727	0.000225
1.00					
			+0.103648	+0.094194	-0.016563
			$= S_{1,2}$	$= S_{2,2}$	$= S_{3.2}$

$\boxed{\varphi = 30^{o}}$ Table 4.6/3d/

η	$\sqrt{1+(\frac{d\eta}{d\xi})^2}$	ξ	ξ_m	$\Delta\xi$	$\left(\sqrt{1+(\frac{d\eta}{d\xi})^2}\right)_m$
0	1.15470	0			
			0.056418	0.182262	1.14153
0.1	1.12836	0.112836			
			0.166982	0.201677	1.11700
0.2	1.10563	0.221127			
			0.273438	0.224187	1.09573
0.3	1.08583	0.325749			
			0.376567	0.251040	1.07715
0.4	1.06846	0.427384			
			0.476982	0.284218	1.060809
0.5	1.053158	0.526579			
			0.575184	0.327178	1.046403
0.6	1.039647	0.623788			
			0.671595	0.386742	1.033681
0.7	1.027715	0.719401			
			0.743061	0.223200	1.025004
0.75	1.022293	0.766720			
			0.790243	0.251932	1.019750
0.8	1.017207	0.813766			
			0.837173	0.292156	1.014827
0.85	1.012446	0.860579			
			0.883889	0.355248	1.010222
0.9	1.007998	0.907198			
			0.930429	0.481854	1.005925
0.95	1.003851	0.953659			
			0.967568	0.443942	1.002679
0.98	1.001506	0.981476			
			0.990738	0.732561	1.000753
1.00	1.000000	1.000000			

To be continued

$\boxed{\varphi = 30^{\circ}.}$ Table 4.6/3d/ (continued)

η	$\Delta s' = \Delta\xi \cdot \left(\sqrt{1+\left(\frac{d\eta}{d\xi}\right)^2}\right)_m$	$\Delta s' \cdot \xi_m$	$\sqrt{\xi_m}$	$\Delta s' \cdot \xi_m^{3/2}$	$-\ln \xi_m$
0					
	0.208058	0.011738	0.237525	0.002788	2.87500
0.1					
	0.225273	0.037617	0.408634	0.015372	1.78987
0.2					
	0.245648	0.067169	0.522913	0.035124	1.29668
0.3					
	0.270408	0.101827	0.613651	0.062486	0.97666
0.4					
	0.301501	0.143811	0.690639	0.099321	0.74028
0.5					
	0.342360	0.196920	0.758409	0.149346	0.55307
0.6					
	0.399768	0.268482.	0.819509	0.220023	0.39810
0.7					
	0.228781	0.169998	0.862010	0.146540	0.29697
0.75					
	0.256908	0.203020	0.888956	0.180476	0.23541
0.8					
	0.296488	0.248212	0.914972	0.227107	0.17773
0.85					
	0.358879	0.317209	0.940154	0.298225	0.12343
0.9					
	0.484709	0.450987	0.964587	0.435016	0.07211
0.95					
	0.445131	0.430695	0.983650	0.423653	0.03297
0.98					
	0.733113	0.726323	0.995358	0.722951	0.00930
1.00					
	+4.797025	+3.374008		+3.018428	
	$= S_0$	$= S_{1,1}$		$= S_{2,1}$	

To be continued

$\boxed{\varphi = 30^{\circ}}$ Table 4.6/3d/ (continued)

η	$-\Delta s' \cdot \xi_m^{3/2} \cdot \ln \xi_m$	$\frac{d\eta}{d\xi}$	$-\Delta\left(\frac{d\eta}{d\xi}\right)$	$1+\left(\frac{d\eta}{d\xi}\right)^2$	$2\cdot\left[1+\left(\frac{d\eta}{d\xi}\right)^2\right]_m$
0		0.577350		1.333333	
	0.008016		0.054663		2.606535
0.1		0.522687		1.273202	
	0.027514		0.051067		2.495628
0.2		0.471620		1.222426	
	0.045545		0.048499		2.401457
0.3		0.423121		1.179031	
	0.061028		0.046813		2.320639
0.4		0.376308		1.141608	
	0.073525		0.045943		2.250749
0.5		0.330365		1.109141	
	0.082599		0.045999		2.190005
0.6		0.284366		1.080864	
	0.087592		0.047301		2.137064
0.7		0.237065		1.056200	
	0.043518		0.024740		2.101282
0.75		0.212325		1.045082	
	0.042486		0.026019		2.079792
0.8		0.186306		1.034710	
	0.040364		0.028047		2.059756
0.85		0.158259		1.025046	
	0.036810		0.031535		2.041105
0.9		0.126724		1.016059	
	0.031369		0.038873		2.023777
0.95		0.087851		1.007718	
	0.013968		0.032946		2.010733
0.98		0.054905		1.003015	
	0.006723		0.054905		2.003015
1.00		0.000000		1.000000	
	-0.601058				
	$= S_{3,1}$				

To be continued

$\boxed{\varphi = 30^{\circ}.}$ Table 4.6/3d/ (continued)

η	$-\frac{\Delta\left(\frac{d\eta}{d\xi}\right)}{2\cdot\left[1+\left(\frac{d\eta}{d\xi}\right)^2\right]_m}$	ξ_m^2	$-$ $\xi_m^2 \cdot \frac{\Delta\left(\frac{d\eta}{d\xi}\right)}{2\cdot\left[1+\left(\frac{d\eta}{d\xi}\right)^2\right]_m}$	$-$ $\xi_m^{5/2} \cdot \frac{\Delta\left(\frac{d\eta}{d\xi}\right)}{2\cdot\left[1+\left(\frac{d\eta}{d\xi}\right)^2\right]_m}$	$+$ $\xi_m^{5/2} \cdot \frac{\Delta\left(\frac{d\eta}{d\xi}\right)}{2\cdot\left[1+\left(\frac{d\eta}{d\xi}\right)^2\right]_m} \cdot \ln \xi_m$
0					
	0.020972	0.003183	0.000067	0.000016	0.000046
0.1					
	0.020463	0.027883	0.000571	0.000233	0.000417
0.2					
	0.020196	0.074768	0.001510	0.000790	0.001024
0.3					
	0.020172	0.141803	0.002860	0.001755	0.001714
0.4					
	0.020412	0.227512	0.004644	0.003207	0.002374
0.5					
	0.021004	0.330837	0.006949	0.005270	0.002915
0.6					
	0.022135	0.451040	0.009984	0.008182	0.003257
0.7					
	0.011774	0.552140	0.006501	0.005604	0.001664
0.75					
	0.012510	0.624484	0.007812	0.006945	0.001635
0.8					
	0.013617	0.700859	0.009544	0.008732	0.001552
0.85					
	0.015450	0.781260	0.012070	0.011348	0.001401
0.9					
	0.019208	0.865698	0.016628	0.016039	0.001157
0.95					
	0.016385	0.936188	0.015339	0.015088	0.000497
0.98					
	0.027411	0.981562	0.026906	0.026781	0.000249
1.00					
			+0.121385	+0.109990	− 0.019902
			$= S_{1,2}$	$= S_{2,2}$	$= S_{3,2}$

$\boxed{\varphi = 15^{\circ}.}$ Table 4.6/3e/

		$S_{1,1}$ =	+5.535016
β	= +8.081715	$-S_{1,2}$ =	-0.067042
S_0	= +8.167180	S_1	= +5.467974
$S_{2,1}$ =	+4.907011	$S_{3,1}$ =	-1.050871
$-S_{2,2}$ =	-0.061414	$-S_{3,2}$ =	+0.009919
S_2	= +4.845597	S_3	= -1.040952

$\boxed{\varphi = 20^{\circ}.}$

		$S_{1,1}$ =	+4.480299
β	= +6.408433	$-S_{1,2}$ =	-0.085645
S_0	= +6.518930	S_1	= +4.394654
$S_{2,1}$ =	+3.985754	$S_{3,1}$ =	-0.830614
$-S_{2,2}$ =	-0.078110	$-S_{3,2}$ =	+0.013227
S_2	= +3.907644	S_3	= -0.817387

$\boxed{\varphi = 25^{\circ}.}$

		$S_{1,1}$ =	+3.825643
β	= +5.363790	$-S_{1,2}$ =	-0.103648
S_0	= +5.498671	S_1	= +3.721995
$S_{2,1}$ =	+3.413555	$S_{3,1}$ =	-0.694449
$-S_{2,2}$ =	-0.094194	$-S_{3,2}$ =	+0.016563
S_2	= +3.319361	S_3	= -0.677886

$\boxed{\varphi = 30^{\circ}.}$

		$S_{1,1}$ =	+3.374008
β	= +4.638197	$-S_{1,2}$ =	-0.121385
S_o	= +4.797025	S_1	= +3.252623
$S_{2,1}$ =	+3.018428	$S_{3,1}$ =	-0.601058
$-S_{2,2}$ =	-0.109990	$-S_{3,2}$ =	+0.019902
S_2	= +2.908438	S_3	= -0.581156

5. FORMULAE FOR Q', F', B', R' AND I' AS FUNCTIONS OF $p = \frac{y_{max}}{k}$ FOR $\varphi = 15^o, 20^o, 25^o$ AND 30^o.

The fundamental formulae derived in section 4.5 for F' and Q' make it possible by simple division to find the formula for v'_m.

Equally the formula for R' can be found by division of the formulae for F' and U'.

We therefore give the formulae for Q' and F', R' and I' with the numerical values found for the quantities S and β for the different values of φ.

We further need the formulae for $B' = (\beta + \beta_o)$.

From these the formula for v_* can be found as

$$v'_* = \sqrt{\frac{F'}{I' \cdot R'}} \quad ,$$

and the formula for $\varkappa$ as

$$\varkappa = \frac{F'}{B'} \,.$$

The formulae are as follows :

$$p = \frac{y_{max}}{k} \quad .$$

$\boxed{\varphi = 15^o}$

(59a) $$Q' = \frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\varrho}} \cdot y_{max}^{\ 2}} = 1.244754 \ln^2(11\,p) + 2.081904 \ln(11\,p) - 1.040952$$

(60a) $$F' = \frac{F}{y_{max}^{\ 2}} = 1.244754 \ln(11\,p) + 2.704281$$

(61a) $$B' = \beta + \beta_0 = 1.244754 \ln(11\,p) + 5.318022$$

(62a) $$R' = \frac{R}{y_{max}} = \frac{1.244754 \ln(11\,p) + 2.704281}{1.244754 \ln(11\,p) + 5.403487}$$

(63a) $$I' = \frac{g\,F\,I}{y_{max} \cdot \left(\frac{\tau_{max}}{\varrho}\right)} = 1.244754 \ln(11\,p) + 2.771323$$

$\boxed{\varphi = 20^{\circ}}$

(59b) $$Q' = \frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\rho}} \cdot y_{max}^{2}} = 0.974020 \ln^{2}(11\,p) + 1.634774 \ln(11\,p) - 0.817387$$

(60b) $$F' = \frac{F}{y_{max}^{2}} = 0.974020 \ln(11\,p) + 2.121784$$

(61b) $$B' = \beta + \beta_0 = 0.974020 \ln(11\,p) + 4.135563$$

(62b) $$R' = \frac{R}{y_{max}} = \frac{0.974020 \ln(11\,p) + 2.121784}{0.974020 \ln(11\,p) + 4.246060}$$

(63b) $$I' = \frac{g\,F\,I}{y_{max} \cdot (\frac{\tau_{max}}{\rho})} = 0.974020 \ln(11\,p) + 2.207429\,.$$

$\boxed{\varphi = 25^{\circ}}$

(59c) $$Q' = \frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\rho}} \cdot y_{max}^{2}} = 0.805268 \ln^{2}(11\,p) + 1.355772 \ln(11\,p) - 0.677886$$

(60c) $$F' = \frac{F}{y_{max}^{2}} = 0.805268 \ln(11\,p) + 1.758406$$

(61c) $$B' = \beta + \beta_0 = 0.805268 \ln(11\,p) + 3.400201$$

(62c) $$R' = \frac{R}{y_{max}} = \frac{0.805268 \ln(11\,p) + 1.758406}{0.805268 \ln(11\,p) + 3.535082}$$

(63c) $$I' = \frac{g\,F\,I}{y_{max} \cdot (\frac{\tau_{max}}{\rho})} = 0.805268 \ln(11\,p) + 1.862054.$$

$\boxed{\varphi = 30^o}$

(59d) $$Q' = \frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\varrho}} \cdot y_{max}^2} = 0.688370 \ln^2(11\ p)$$
$$+ 1.162312 \ln(11\ p) - 0.581156$$

(60d) $$F' = \frac{F}{y_{max}^2} = 0.688370 \ln(11\ p) + 1.506497$$

(61d) $$B' = \beta + \beta_0 = 0.688370 \ln(11\ p) + 2.891576$$

(62d) $$R' = \frac{R}{y_{max}} = \frac{0.688370 \ln(11\ p) + 1.506497}{0.688370 \ln(11\ p) + 3.050899}$$

(63d) $$I' = \frac{g\ F\ I}{y_{max} \cdot \left(\frac{\tau_{max}}{\varrho}\right)} = 0.688370 \ln(11\ p) + 1.627882.$$

Numerical values of Q', F', $(\beta + \beta_0)$, R' and I' for fixed values of
$$p = \frac{y_{max}}{k}$$
will be given later in the tables of section 7.1, viz. Tables 7.1./4a/b/c/d/ through 7.1./8a/b/c/d/.

6. DERIVATION OF FORMULAE OF THE TYPE $z' = A_\varphi \cdot p^{B_\varphi} \cdot A_p$, WHERE $p = \frac{y_{max}}{k}$, AND A_φ AND B_φ ARE FUNCTIONS OF φ ALONE, WHEREAS A_p IS FUNCTION OF p ALONE.

6.1. INTRODUCTION.

In formulae 5(59a)b)c)d) through 5(63a)b)c)d) the dimensionless quantities Q', F', B', R' and I' are expressed as functions of $p = \frac{y_{max}}{k}$, the numerical coefficients varying with φ.

When z' in general stands for any of these quantities we try to adjust formulae of the type

$\boxed{(64)}$ $$z' = A_\varphi \cdot p^{B_\varphi} \cdot A_p$$

to the numerical values of z', which are computed for fixed values of p by the above-mentioned formulae for each value of φ.

Putting in the first approximation $A_p = 1$, we find for each value of φ a simple power formula giving certain numerical values of A_φ and B_φ.

We then compute

(65) $\log A_p = \log z' - (\log A_\varphi + B_\varphi \cdot \log p)$,

giving the errors of the power formula varying with p.

It will then appear that these values of $\log A_p$ are practically independent of φ.

Consequently we are now able to express A_φ and B_φ as functions of φ alone and A_p as function of p alone.

6.2. ($A_p = 1$). SIMPLE POWER FORMULA.

For the first approximation with $A_p = 1$ we tabulate for each value of φ the corresponding values of z' for the fixed values: p = 10, 20, 30, 50, 100, 200, 300, 500 and 1000 and take logarithms in (64)

(66) $\log z' = \log A_\varphi + B_\varphi \cdot \log p$.

This form will on double logarithmic paper represent a straight line (Fig. 5).

Its direction is taken parallel to the chord through the points z'_{50} and z'_{200}, and it is placed so that the errors e_p for p = 100 and p = 500 will have the same absolute value but opposite signs, which in the logarithmic representation corresponds to the same relative deviations for p = 100 and p = 500.

It will prove that the devation for $p \cong 24$ will be nearly the same as for p = 500, and that this maximum deviation will not be exceeded between these limits.

We find :

$$\boxed{(67)} \qquad B_\varphi = \frac{\log z'_{200} - \log z'_{50}}{\log 4} = 1.660964 \, (\log z'_{200} - \log z'_{50})$$

and

$$\boxed{(68)} \qquad \log A_\varphi = \frac{1}{2} (\log z'_{500} + \log z'_{100}) - 3.902411 (\log z'_{200} - \log z'_{50})$$

The numerical calculations are given later in the tables of section 7.1.

6.3. DERIVATION OF FORMULA FOR A_p.

From the tables of section 7.1 the numerical values of $\log A_p$ according to 6.1.(65) are known for each value of φ.

They prove to be almost constant for fixed values of p, wherefore the means of log A_p are taken and used. Consequently log A_p is considered to be independent of φ and only varying with p.

The deviation for p = 100 is greatest for Q', where it is about 2.07%, and since no absolute accuracy is required the function A_p is sought in a form as simple as possible.

As ln 10 = 2.30259 we put

$$\boxed{(69)} \qquad A_p = 1 + 2.30259\,(\log A_p)_{(p=100)} \cdot (c + a \log^2 (\tfrac{p}{b}) + \Delta e_p),$$

where c, a and b are constants. By putting $\Delta e_p = 0$ this expression gives approximately A_p.

Considering that A_p as a maximum deviates only about 2% from unity we get, taking natural logarithms on both sides of (69)

$$\ln A_p \cong 2.30259\,(\log A_p)_{(p=100)} \cdot (c + a \log^2 (\tfrac{p}{b}) + \Delta e_p) \quad \text{or}$$

$$(70) \qquad e_p = \frac{\log A_p}{(\log A_p)_{(p=100)}} = \left[c + a \log^2 (\tfrac{p}{b})\right] + \Delta e_p = e_p^o + \Delta e_p\,,$$

$$\boxed{(70a)} \qquad e_p^o = c + a \log^2 (\tfrac{p}{b})$$

The left side of (70) are the relative values of the log A_p's measured by the value for p = 100; it will consequently be +1.0 for p = 100 and about −1.0 for p = 500. We determine the constants c, a and b by the requirement that e_p^o must have the correct values (i.e. $e_p^o = e_p$) for p = 20, 100 and 500. (Fig. 6).

This gives the following formulae:

$$(71) \qquad a = 1.023417\,(e_{500} - e_{20}) - 2.046836\,(1 - e_{20})$$

$$(72) \qquad \log b = \frac{n}{2a}$$

$$(73) \qquad c = m - a \log^2 b$$

where

$$\begin{cases} (74a) \quad n = 3.30103\,a - 1.430677\,(1 - e_{20}) \\ (74b) \quad n = 4.00000\,a - 0.715338\,(e_{500} - e_{20}) \end{cases}$$

and

$$\begin{cases} m = e_{20} + 1.30103\,n - 1.69268\,a & (75a) \\ m = e_{100} + 2.00000\,n - 4.00000\,a & (75b) \\ m = e_{500} + 2.69897\,n - 7.28444\,a & (75c) \end{cases}$$

The formulae (74a) and (74b) must give the same value of n, and the three formulae (75) must give the same value of m, whereupon b and c are easily found by (72) and (73).

Finally it follows from (69) with $\Delta e_p = 0$:

$$\boxed{(76a)} \quad A_p = c_0 + a_0 \log^2 (\tfrac{p}{b}) \quad , \quad \text{where}$$

$$(76b) \quad c_0 = 1 + 2.30259\, c\, (\log A_p)_{(p=100)}$$

$$(76c) \quad a_0 = 2.30259\, a\, (\log A_p)_{(p=100)}$$

When the values of e_p^o by 6.3.(70a) are deducted from the correct values of e_p by 6.3.(70) we find Δe_p and consequently the relation between the errors from using the formulae (76a)b)c) and the errors produced for p = 100 by using the corresponding simple power formula.

6.4. DERIVATION OF EXPRESSIONS FOR a_3, b_3 AND Δ_3 IN THE FORMULA

$$\boxed{(77)} \quad \frac{1 - a_1 \log^2 (\frac{p}{b_1})}{1 - a_2 \log^2 (\frac{p}{b_2})} = 1 + \Delta_3 - a_3 \log^2 (\frac{p}{b_3}) .$$

It will later be useful to have a formula like that above valid for values of b_1 and b_2, which do not differ much.

The values of a_3, b_3 and Δ_3 are to be expressed by the known quantities a_1, b_1, a_2 and b_2. We find:

$$(78) \quad a_3 = a_1 - a_2$$

$$(79) \quad \log b_3 = \frac{a_1 \log b_1 - a_2 \log b_2}{a_1 - a_2}$$

$$(80) \quad \Delta_3 = \frac{a_1 \cdot a_2}{a_1 - a_2} \cdot \frac{(\log b_1 - \log b_2)^2}{1 - a_2 \cdot \log^2 (\frac{p}{b_2})} .$$

In the practical application of (77), when b_1 and b_2 differ only slightly, Δ_3 will often be insignificant, and the variation with p of the denominator $\left[1 - a_2 \log^2 (\frac{p}{b})\right]$, which is near 1, can be neglected, so that

$$\Delta_3 \cong \frac{a_1 \cdot a_2}{a_1 - a_2} (\log b_1 - \log b_2)^2 . \tag{80a}$$

6.5. DERIVATION OF FORMULAE FOR A_φ AND ABSCISSAE ξ OF THE FORM

$$A_\varphi = a + b \cot\varphi + c \cot^2\varphi + d \cot^3\varphi . \tag*{$\boxed{(81)}$}$$

The coefficients A_φ for our different dimensionless quantities will be known for the four different values of φ.

The same is true of the abscissae ξ which are known for fixed values of η and the four values of φ.

All these quantities can conveniently be expressed in formulae of the form (81), i.e. series with four members in increasing powers of the argument $m = \cot\varphi$. Since the intervals in φ are equal, the intervals in m will be unequal. To calculate the coefficients a, b, c and d so that the correct values of the function are obtained for $\varphi = 15^o$, 20^o, 25^o and 30^o, Newton's Method of Interpolation is used.

From the function

$$g(m) = a + bm + cm^2 + dm^3 \tag{81a}$$

the values are known for the arguments m_0, m_1, m_2 and m_3 (equal to cot 15^o, cot 20^o, cot 25^o and cot 30^o).

We define:

$$g_1(m) = \frac{g(m) - g(m_0)}{m - m_0} \tag{82}$$

$$g_2(m) = \frac{g_1(m) - g_1(m_1)}{m - m_1} \tag{83}$$

$$g_3(m) = \frac{g_2(m) - g_2(m_2)}{m - m_2} \tag{84}$$

and calculate the special values $g_1(m_1)$, $g_2(m_2)$ and $g_3(m_3)$; this is done in the usual way in a table.

For the coefficients we then find the following expressions:

(85) $a = g(m_0) + m_0 \cdot \left\{- g_1(m_1) + m_1 \cdot \left[+ g_2(m_2) - m_2 \cdot g_3(m_3)\right]\right\}$

(86) $b = g_1(m_1) - (m_0 + m_1) \cdot g_2(m_2) + (m_0\, m_1 + m_0\, m_2 + m_1\, m_2) \cdot g_3(m_3)$

(87) $c = g_2(m_2) - (m_0 + m_1 + m_2) \cdot g_3(m_3)$

(88) $d = g_3(m_3)$.

7. NUMERICAL DETERMINATION OF A_φ, B_φ AND A_p FOR Q', F', v_m', R', $(\beta+\beta_0)$, I' AND v_*'.

In the following sections the coefficients A_φ, B_φ and A_p for each of the dimensionless quantities Q', F', v_m', R', $(\beta+\beta_0)$, I' and v_*' are denoted by the first subscripts q, f, v, r, b, i and $*$ respectively, for Q' e.g. by $A_{q\varphi}$, $B_{q\varphi}$ and A_{qp}.

7 1. NUMERICAL CALCULATION OF A_φ, B_φ AND log A_p.

For the quantities Q', F', R', $(\beta+\beta_0)$ and I' these calculations are made for the four values of φ and for

Q'	in Tables	7.1/4a/b/c/d/
F'	- -	7.1/5a/b/c/d/
R'	- -	7.1/6a/b/c/d/
$(\beta+\beta_0)$	-	7.1/7a/b/c/d/
I'	-	7.1/8a/b/c/d/.

For v_m' and v_*' the corresponding values of A_φ, B_φ and log A_p can easily be found by combination of the above results.

$\boxed{\varphi = 15^\circ}$ Table 7.1/4a/

$Q' = 1.244754 \ln^2(11\,p) + 2.081904 \cdot \ln(11\,p) - 1.040952$

$A_{q\varphi} = 21.8184$ $B_{q\varphi} = 0.26247$ $\log A_{q\varphi} = 1.33882$

p	Q'	log Q'	log p	$B_{q\varphi} \cdot \log p$	$\log A_{q\varphi} + B_{q\varphi} \cdot \log p$	$\log A_{qp}$
10	36.2472	1.55928	1.00000	0.26247	1.60129	-0.04201
20	46.3995	1.66652	1.30103	0.34148	1.68030	-0.01378
30	52.8926	1.72339	1.47712	0.38770	1.72652	-0.00313
50	61.6557	1.78995	1.69897	0.44593	1.78475	+0.00520
100	74.5852	1.87265	2.00000	0.52494	1.86376	+0.00889
200	88.7109	1.94797	2.30103	0.60395	1.94277	+0.00520
300	97.5282	1.98913	2.47712	0.65017	1.98899	+0.00014
500	109.2195	2.03830	2.69897	0.70840	2.04722	-0.00892
1000	126.1224	2.10080	3.00000	0.78741	2.12623	-0.02543

$A_{qp} = 1$: $Q' = 21.8184 \cdot p^{0.26247} \pm 2.07\%$ for $24.6 \leqq p \leqq 500$.

$\boxed{\varphi = 20^\circ.}$ Table 7.1/4b/

$Q' = 0.974020 \ln^2(11\,p) + 1.634774 \cdot \ln(11\,p) - 0.817387$

$A_{q\varphi} = 17.094$ $B_{q\varphi} = 0.262349$ $\log A_{q\varphi} = 1.23283$

p	Q'	log Q'	log p	$B_{q\varphi} \cdot \log p$	$\log A_{q\varphi} + B_{q\varphi} \cdot \log p$	$\log A_{qp}$
10	28.3873	1.45313	1.00000	0.26235	1.49518	-0.04205
20	36.3354	1.56033	1.30103	0.34132	1.57415	-0.01382
30	41.4186	1.61720	1.47712	0.38752	1.62035	-0.00315
50	48.2786	1.68376	1.69897	0.44572	1.67855	+0.00521
100	58.3999	1.76641	2.00000	0.52470	1.75753	+0.00888
200	69.4572	1.84171	2.30103	0.60367	1.83650	+0.00521
300	76.3590	1.88286	2.47712	0.64987	1.88270	+0.00016
500	85.5104	1.93202	2.69897	0.70807	1.94090	-0.00888

For $A_{qp} = 1$: $Q' = 17.094 \cdot p^{0.262349} \pm 2.07\%$ for $24.6 \leqq p \leqq 500$.

$\boxed{\varphi = 25^{o}.}$ Table 7.1/4c/

$$Q' = 0.805268 \ln^2(11\,p) + 1.355772 \ln(11p) - 0.677886$$

$A_{q\varphi} = 14.143$ $B_{q\varphi} = 0.26230$ $\log A_{q\varphi} = 1.15055$

p	Q'	log Q'	log p	$B_{q\varphi} \cdot \log p$	$\log A_{q\varphi} + B_{q\varphi} \cdot \log p$	$\log A_{qp}$
10	23.4869	1.37082	1.00000	0.26230	1.41285	-0.04203
20	30.0609	1.47800	1.30103	0.34126	1.49181	-0.01381
30	34.2651	1.53485	1.47712	0.38745	1.53800	-0.00315
50	39.9387	1.60140	1.69897	0.44564	1.59619	+0.00521
100	48.3094	1.68403	2.00000	0.52460	1.67515	+0.00888
200	57.4539	1.75932	2.30103	0.60356	1.75411	+0.00521
300	63.1617	1.80045	2.47712	0.64975	1.80030	+0.00015
500	70.7297	1.84960	2.69897	0.70794	1.85849	-0.00889

For $A_{qp} = 1$: $Q' = 14.143 \cdot p^{0.26230} \pm 2.07\%$ for $24.6 \leqq p \leqq 500$.

$\boxed{\varphi = 30^{o}.}$ Table 7.1/4d/

$$Q' = 0.688370 \ln^2(11\,p) + 1.162312 \ln(11\,p) - 0.581156$$

$A_{q\varphi} = 12.099$ $B_{q\varphi} = 0.262249$ $\log A_{q\varphi} = 1.08276$

p	Q'	log Q'	log p	$B_{q\varphi} \cdot \log p$	$\log A_{q\varphi} + B_{q\varphi} \cdot \log p$	$\log A_{qp}$
10	20.0915	1.30301	1.00000	0.26225	1.34501	-0.04200
20	25.7135	1.41016	1.30103	0.34119	1.42395	-0.01379
30	29.3087	1.46700	1.47712	0.38737	1.47013	-0.00313
50	34.1605	1.53353	1.69897	0.44555	1.52831	+0.00522
100	41.3183	1.61614	2.00000	0.52450	1.60726	+0.00888
200	49.1376	1.69142	2.30103	0.60344	1.68620	+0.00522
300	54.0182	1.73254	2.47712	0.64962	1.73238	+0.00016
500	60.4893	1.78168	2.69897	0.70780	1.79056	-0.00888

For $A_{qp} = 1$: $Q' = 12.099 \cdot p^{0.262249} \pm 2.07\%$ for $24.6 \leqq p \leqq 500$.

$\boxed{\varphi = 15^o.}$ Table 7.1/5a/

$F' = 1.244754 \ln(11\,p) + 2.704281$

$A_{f\varphi} = 6.8585$ $B_{f\varphi} = 0.109208$ $\log A_{f\varphi} = 0.83623$

p	F'	log F'	log p	$B_{f\varphi} \cdot \log p$	$\log A_{f\varphi} + B_{f\varphi} \cdot \log p$	$\log A_{fp}$
10	8.55522	0.93223	1.00000	0.10921	0.94544	-0.01321
20	9.41802	0.97396	1.30103	0.14208	0.97831	-0.00435
30	9.92272	0.99663	1.47712	0.16131	0.99754	-0.00091
50	10.55858	1.02360	1.69897	0.18554	1.02177	+0.00183
100	11.42138	1.05772	2.00000	0.21842	1.05465	+0.00307
200	12.28418	1.08935	2.30103	0.25129	1.08752	+0.00183
300	12.78888	1.10683	2.47712	0.27052	1.10675	+0.00008
500	13.42474	1.12791	2.69897	0.29475	1.13098	-0.00307
1000	14.28754	1.15496	3.00000	0.32762	1.16385	-0.00889

For $A_{fp} = 1$: $F' = 6.8585 \cdot p^{0.109208} \pm 0.71\%$
for $23.7 \leqq p \lesssim 500$.

$\boxed{\varphi = 20^o.}$ Table 7.1/5b/

$F' = 0.974020 \ln(11\,p) + 2.121784$

$A_{f\varphi} = 5.3728$ $B_{f\varphi} = 0.10911$ $\log A_{f\varphi} = 0.73020$

p	F'	log F'	log p	$B_{f\varphi} \cdot \log p$	$\log A_{f\varphi} + B_{f\varphi} \cdot \log p$	$\log A_{fp}$
10	6.70015	0.82608	1.00000	0.10911	0.83931	-0.01323
20	7.37529	0.86778	1.30103	0.14195	0.87215	-0.00437
30	7.77021	0.89043	1.47712	0.16117	0.89137	-0.00094
50	8.26777	0.91739	1.69897	0.18537	0.91557	+0.00182
100	8.94291	0.95148	2.00000	0.21822	0.94842	+0.00306
200	9.61806	0.98308	2.30103	0.25106	0.98126	+0.00182
300	10.01298	1.00056	2.47712	0.27027	1.00047	+0.00009
500	10.51054	1.02162	2.69897	0.29448	1.02468	-0.00306

For $A_{fp} = 1$: $F' = 5.3728 \cdot p^{0.10911} \pm 0.707\%$
for $23.8 \leqq p \leqq 500$

$\boxed{\varphi = 25^{o}.}$ Table 7.1/5c/

$F' = 0.805268 \ln(11\ p) + 1.758406$

$A_{f\varphi} = 4.4456$ $B_{f\varphi} = 0.10906$ $\log A_{f\varphi} = 0.64793$

p	F'	log F'	log p	$B_{f\varphi}.\log p$	$\log A_{f\varphi} + B_{f\varphi}.\log p$	$\log A_{fp}$
10	5.54355	0.74378	1.00000	0.10906	0.75699	-0.01321
20	6.10172	0.78545	1.30103	0.14189	0.78982	-0.00437
30	6.42823	0.80809	1.47712	0.16109	0.80902	-0.00093
50	6.83958	0.83503	1.69897	0.18529	0.83322	+0.00181
100	7.39775	0.86910	2.00000	0.21812	0.86605	+0.00305
200	7.95593	0.90069	2.30103	0.25095	0.89888	+0.00181
300	8.28243	0.91816	2.47712	0.27015	0.91808	+0.00008
500	8.69379	0.93921	2.69897	0.29435	0.94228	-0.00307

For $A_{fp} = 1$: $F' = 4.4456\ .\ p^{0.10906} \pm 0.70\%$
for $23.8 \leqq p \leqq 500.$

$\boxed{\varphi = 30^{o}.}$ Table 7.1/5d/

$F' = 0.688370 \ln(11\ p) + 1.506497$

$A_{f\varphi} = 3.8034$ $B_{f\varphi} = 0.10899$ $\log A_{f\varphi} = 0.58017$

p	F'	log F'	log p	$B_{f\varphi}.\log p$	$\log A_{f\varphi} + B_{f\varphi}.\log p$	$\log A_{fp}$
10	4.74217	0.67598	1.00000	0.10899	0.68916	-0.01318
20	5.21931	0.71761	1.30103	0.14180	0.72197	-0.00436
30	5.49842	0.74023	1.47712	0.16099	0.74116	-0.00093
50	5.85006	0.76716	1.69897	0.18517	0.76534	+0.00182
100	6.32720	0.80121	2.00000	0.21798	0.79815	+0.00306
200	6.80434	0.83278	2.30103	0.25079	0.83096	+0.00182
300	7.08345	0.85025	2.47712	0.26999	0.85016	+0.00009
500	7.43509	0.87129	2.69897	0.29417	0.87434	-0.00305

For $A_{fp} = 1$: $F' = 3.8034\ .\ p^{0.10899} \pm 0.707\%$
for $23.8 \leqq p \leqq 500.$

$\boxed{\varphi=15^{o}.}$ Table 7.1/6a/

$$R' = \frac{1.244754 \ln(11\ p) + 2.704281}{1.244754 \ln(11\ p) + 5.403487}$$

$A_{r\varphi} = 0.73272$ $B_{r\varphi} = 0.020945$ $\log A_{r\varphi} = 9.86494$

p	R'	log R'	log p	$B_{r\varphi}.\log p$	$\log A_{r\varphi} + B_{r\varphi}.\log p$	$\log A_{rp}$
10	0.760165	9.88091	1.00000	+0.02095	9.88589	-0.00498
20	0.777242	9.89056,	1.30103	+0.02725	9.89219	-0.00163
30	0.786149	9.89550	1.47712	+0.03094	9.89588	-0.00038
50	0.796406	9.90113	1.69897	+0.03558	9.90052	+0.00061
100	0.808846	9.90786	2.00000	+0.04189	9.90683	+0.00103
200	0.819853	9.91374	2.30103	+0.04820	9.91314	+0.00060
300	0.825724	9.91683	2.47712	+0.05188	9.91682	+0.00001
500	0.832597	9.92043	2.69897	+0.05653	9.92147	-0.00103
1000	0.841099	9.92485	3.00000	+0.06284	9.92778	-0.00293

For $A_{rp} = 1$: $R' = 0.73272 \,.\, p^{0.020945} \pm 0.237\ \%$ for $24.8 \leqq p \leqq 500$.

$\boxed{\varphi=20^{o}.}$ Table 7.1/6b/

$$R' = \frac{0.974020 \,.\, \ln(11\ p) + 2.121784}{0.974020 \,.\, \ln(11\ p) + 4.246060}$$

$A_{r\varphi} = 0.73172$ $B_{r\varphi} = 0.02103$ $\log A_{r\varphi} = 9.86434$

p	R'	log R'	log p	$B_{r\varphi}.\log p$	$\log A_{r\varphi} + B_{r\varphi}.\log p$	$\log A_{rp}$
10	0.759273	9.88040	1.00000	0.02103	9.88537	-0.00497
20	0.776382	9.89008	1.30103	0.02736	9.89170	-0.00162
30.	0.785307	9.89504	1.47712	0.03106	9.89540	-0.00036
50	0.795586	9.90068	1.69897	0.03573	9.90007	+0.00061
100	0.808056	9.90744	2.00000	0.04206	9.90640	+0.00104
200	0.819092	9.91334	2.30103	0.04839	9.91273	+0.00061
300	0.824979	9.91644	2.47712	0.05209	9.91643	+0.00001
500	0.831871	9.92006	2.69897	0.05675	9.92109	-0.00103

For $A_{rp} = 1$: $R' = 0.73172 \,.\, p^{0.02103} \pm 0.240\ \%$ for $24.6 \leqq p \leqq 500$.

$\boxed{\varphi = 25^{o}.}$ Table 7.1/6c/

$$R' = \frac{0.805268 \,.\, \ln (11\ p) + 1.758406}{0.805268 \,.\, \ln (11\ p) + 3.535082}$$

$A_{r\varphi} = 0.72968$ $B_{r\varphi} = 0.02118$ $\log A_{r\varphi} = 9.86313$

p	R'	log R'	log p	$B_{r\varphi}.\log p$	$\log A_{r\varphi} + B_{r\varphi}.\log p$	$\log A_{rp}$
10	0.757292	9.87927	1.00000	0.02118	9.88431	-0.00504
20	0.774488	9.88901	1.30103	0.02755	9.89068	-0.00167
30	0.783462	9.89402	1.47712	0.03128	9.89441	-0.00039
50	0.793800	9.89971	1.69897	0.03598	9.89911	+0.00060
100	0.806345	9.90652	2.00000	0.04235	9.90548	+0.00104
200	0.817451	9.91246	2.30103	0.04873	9.91186	+0.00060
300	0.823376	9.91560	2.47712	0.05246	9.91559	+0.00001
500	0.830315	9.91925	2.69897	0.05716	9.92029	-0.00104
For $A_{rp} = R' = 0.72968 \,.\, p^{0.02118} \pm 0.240\ \%$ for $24.9 \leqq p \leqq 500$.						

$\boxed{\varphi = 30^{o}.}$ Table 7.1/6d/

$$R' = \frac{0.688370 \,.\, \ln (11\ p) + 1.506497}{0.688370 \,.\, \ln (11\ p) + 3.050899}$$

$A_{r\varphi} = 0.72638$ $B_{r\varphi} = 0.021460$ $\log A_{r\varphi} = 9.86117$

p	R'	log R'	log p	$B_{r\varphi}.\log p$	$\log A_{r\varphi} + B_{r\varphi}.\log p$	$\log A_{rp}$
10	0.754333	9.87756	1.00000	0.02146	9.88263	-0.00507
20	0.771664	9.88743	1.30103	0.02792	9.88909	-0.00166
30	0.780713	9.89249	1.47712	0.03170	9.89287	-0.00038
50	0.791141	9.89825	1.69897	0.03646	9.89763	+0.00062
100	0.803801	9.90515	2.00000	0.04292	9.90409	+0.00106
200	0.815014	9.91117	2.30103	0.04938	9.91055	+0.00062
300	0.820998	9.91434	2.47712	0.05316	9.91433	+0.00001
500	0.828008	9.91803	2.69897	0.05792	9.91909	-0.00106
For $A_{rp} = 1$: $R' = 0.72638 \,.\, p^{0.021460} \pm 0.244\ \%$ for $24.7 \leqq p \leqq 500$.						

$\boxed{\varphi = 15^{o}.}$ Table 7.1/7a/

$\beta + \beta_0 = 1.244754 \ln(11\,p) + 5.318022$

$A_{b\varphi} = 9.2804$ $B_{b\varphi} = 0.088795$ $\log A_{b\varphi} = 0.96757$

p	$\beta+\beta_0$	$\log(\beta+\beta_0)$	log p	$B_{b\varphi}$ .log p	$\log A_{b\varphi}$ + $B_{b\varphi}$. log p	$\log A_{bp}$
10	11.16896	1.04801	1.00000	0.08880	1.05637	-0.00836
20	12.03177	1.08033	1.30103	0.11552	1.08309	-0.00276
30	12.53646	1.09818	1.47712	0.13116	1.09873	-0.00055
50	13.17232	1.11967	1.69897	0.15086	1.11843	+0.00124
100	14.03512	1.14722	2.00000	0.17759	1.14516	+0.00206
200	14.89792	1.17313	2.30103	0.20432	1.17189	+0.00124
300	15.40262	1.18759	2.47712	0.21996	1.18753	+0.00006
500	16.03848	1.20516	2.69897	0.23966	1.20723	-0.00207
1000	16.90128	1.22792	3.00000	0.26639	1.23396	-0.00604

For $A_{bp} = 1$: $\beta + \beta_0 = 9.2804 \,.\, p^{0.088795} \pm 0.475\,\%$ for $23.2 \leqq p \lessgtr 500$.

$\boxed{\varphi = 20^{o}.}$ Table 7.1/7b/

$\beta + \beta_0 = 0.974020 \ln(11\,p) + 4.135563$

$A_{b\varphi} = 7.2375$ $B_{b\varphi} = 0.08901$ $\log A_{b\varphi} = 0.85959$

p	$\beta+\beta_0$	$\log(\beta+\beta_0)$	log p	$B_{b\varphi}$.log p	$\log A_{b\varphi}$ + $B_{b\varphi}$. log p	$\log A_{bp}$
10	8.713925	0.94022	1.00000	0.08901	0.94860	-0.00838
20	9.389066	0.97262	1.30103	0.11581	0.97540	-0.00278
30	9.783993	0.99052	1.47712	0.13148	0.99107	-0.00055
50	10.281551	1.01206	1.69897	0.15123	1.01082	+0.00124
100	10.956693	1.03968	2.00000	0.17802	1.03761	+0.00207
200	11.631835	1.06565	2.30103	0.20482	1.06441	+0.00124
300	12.026761	1.08015	2.47712	0.22049	1.08008	+0.00007
500	12.524320	1.09775	2.69897	0.24024	1.09983	-0.00208

For $A_{bp} = 1$: $\beta+\beta_0 = 7.2375 \,.\, p^{0.08901} \pm 0.477\%$ for $23.2 \leqq p \lessgtr 500$

$\boxed{\varphi = 25^{o}.}$ Table 7.1/7c/

$\beta + \beta_0 = 0.805268 \ln(11\,p) + 3.400201$

$A_{b\varphi} = 5.9660$ $B_{b\varphi} = 0.08919$ $\log A_{b\varphi} = 0.77568$

p	$\beta+\beta_0$	$\log(\beta+\beta_0)$	log p	$B_{b\varphi}.\log p$	$\log A_{b\varphi} + B_{b\varphi}.\log p$	$\log A_{bp}$
10	7.185347	0.85645	1.00000	0.08919	0.86487	- 0.00842
20	7.743519	0.88894	1.30103	0.11604	0.89172	- 0.00278
30	8.070023	0.90687	1.47712	0.13175	0.90743	- 0.00056
50	8.481378	0.92847	1.69897	0.15154	0.92722	+0.00125
100	9.039549	0.95615	2.00000	0.17839	0.95407	+0.00208
200	9.597721	0.98217	2.30103	0.20524	0.98092	+0.00125
300	9.924225	0.99670	2.47712	0.22094	0.99662	+0.00008
500	10.335580	1.01433	2.69897	0.24073	1.01641	- 0.00208
For $A_{bp} = 1$: $\beta+\beta_0 = 5.9660 \cdot p^{0.08919}$ $\pm$ o.48% for $23.2 \leqq p \leqq 500$.						

$\boxed{\varphi = 30^{o}.}$ Table 7.1/7d/

$\beta + \beta_0 = 0.688370 \ln(11\,p) + 2.891576$

$A_{b\varphi} = 5.0864$ $B_{b\varphi} = 0.08936$ $\log A_{b\varphi} = 0.70641$

p	$\beta+\beta_0$	$\log(\beta+\beta_0)$	log p	$B_{b\varphi}.\log p$	$\log A_{b\varphi} + B_{b\varphi}.\log p$	$\log A_{bp}$
10	6.12774	0.78730	1.00000	0.089360	0.79577	- 0.00847
20	6.60488	0.81986	1.30103	0.116260	0.82267	-0.00281
30	6.88399	0.83784	1.47712	0.131995	0.83841	- 0.00057
50	7.23562	0.85948	1.69897	0.151820	0.85823	+0.00125
100	7.71277	0.88721	2.00000	0.178720	0.88513	+0.00208
200	8.18992	0.91328	2.30103	0.205620	0.91203	+0.00125
300	8.46902	0.92783	2.47712	0.221355	0.92777	+0.00006
500	8.82067	0.94551	2.69897	0.241180	0.94759	- 0.00208
For $A_{bp} = 1$: $\beta+\beta_0 = 5.0864 \cdot p^{0.08936}$ $\pm$ 0.48% for $23.3 \leqq p \leqq 500$.						

$\boxed{\varphi = 15^o.}$ Table 7.1/8a/

$I' = 1.244754 \ln (11\,p) + 2.771323$

$A_{i\varphi} = 6.9199$ $B_{i\varphi} = 0.108561$ $\log A_{i\varphi} = 0.84010$

p	I'	log I'	log p	$B_{i\varphi} \cdot \log p$	$\log A_{i\varphi} + B_{i\varphi} \cdot \log p$	$\log A_{ip}$
10	8.62226	0.93562	1.00000	0.10856	0.94866	-0.01304
20	9.48507	0.97704	1.30103	0.14124	0.98134	-0.00430
30	9.98976	0.99957	1.47712	0.16036	1.00046	-0.00089
50	10.62562	1.02635	1.69897	0.18444	1.02454	+0.00181
100	11.48842	1.06026	2.00000	0.21712	1.05722	+0.00304
200	12.35122	1.09171	2.30103	0.24980	1.08990	+0.00181
300	12.85592	1.10910	2.47712	0.26892	1.10902	+0.00008
500	13.49178	1.13007	2.69897	0.29300	1.13310	-0.00303
1000	14.35458	1.15699	3.00000	0.32568	1.16578	-0.00879
For $A_{ip} = 1$: $I' = 6.9199 \cdot p^{0.108561} \pm 0.702\%$ for $23.69 \leqq p \leqq 5oo$.						

$\boxed{\varphi = 20^o.}$ Table 7.1/8b/

$I' = 0.974020 \ln (11\,p) + 2.207429$

$A_{i\varphi} = 5.4502$ $B_{i\varphi} = 0.108096$ $\log A_{i\varphi} = 0.73642$

p	I'	log I'	log p	$B_{i\varphi} \cdot \log p$	$\log A_{i\varphi} + B_{i\varphi} \cdot \log p$	$\log A_{ip}$
10	6.78579	0.83160	1.00000	0.10810	0.84452	-0.01292
20	7.46093	0.87280	1.30103	0.14064	0.87706	-0.00426
30	7.85586	0.89519	1.47712	0.15967	0.89609	-0.00090
50	8.35342	0.92186	1.69897	0.18365	0.92007	+0.00179
100	9.02856	0.95562	2.00000	0.21619	0.95261	+0.00301
200	9.70370	0.98694	2.30103	0.24873	0.98515	+0.00179
300	10.09863	1.00426	2.47712	0.26777	1.00419	+0.00007
500	10.59619	1.02515	2.69897	0.29175	1.02817	-0.00302
For $A_{ip} = 1$: $I' = 5.4502 \cdot p^{0.108096} \pm 0.695\%$ for $23.72 \leqq p \leqq 500$.						

$\boxed{\varphi = 25^o.}$ Table 7.1/8c/

$I' = 0.805268 \ln (11\ p) + 1.862054$

$A_{i\varphi} = 4.5399$ $B_{i\varphi} = 0.107547$ $\log A_{i\varphi} = 0.65705$

p	I'	log I'	log p	$B_{i\varphi} \cdot \log p$	$\log A_{i\varphi} + B_{i\varphi} \cdot \log p$	$\log A_{ip}$
10	5.64720	0.75183	1.00000	0.10755	0.76460	-0.01277
20	6.20537	0.79277	1.30103	0.13992	0.79697	-0.00420
30	6.53188	0.81504	1.47712	0.15886	0.81591	-0.00087
50	6.94323	0.84156	1.69897	0.18272	0.83977	+0.00179
100	7.50140	0.87512	2.00000	0.21509	0.87214	+0.00298
200	8.05957	0.90631	2.30103	0.24747	0.90452	+0.00179
300	8.38608	0.92355	2.47712	0.26641	0.92346	+0.00009
500	8.79743	0.94435	2.69897	0.29027	0.94732	-0.00297

For $A_{ip} = 1 : I' = 4.5399 \cdot p^{0.107547} \pm 0.688\%$ for $23.70 \leqq p \leqq 500$.

$\boxed{\varphi = 30^o.}$ Table 7.1/8d/

$I' = 0.688370 \ln (11\ p) + 1.627882$

$A_{i\varphi} = 3.9143$ $B_{i\varphi} = 0.106933$ $\log A_{i\varphi} = 0.59266$

p	I'	log I'	log p	$B_{i\varphi} \cdot \log p$	$\log A_{i\varphi} + B_{i\varphi} \cdot \log p$	$\log A_{ip}$
10	4.86355	0.68695	1.00000	0.10693	0.69959	-0.01264
20	5.34070	0.72760	1.30103	0.13912	0.73178	-0.00418
30	5.61980	0.74972	1.47712	0.15795	0.75061	-0.00089
50	5.97144	0.77608	1.69897	0.18168	0.77434	+0.00174
100	6.44859	0.80947	2.00000	0.21387	0.80653	+0.00294
200	6.92573	0.84046	2.30103	0.24606	0.83872	+0.00174
300	7.20484	0.85762	2.47712	0.26489	0.85755	+0.00007
500	7.55648	0.87832	2.69897	0.28861	0.88127	-0.00295

For $A_{ip} = 1 : I' = 3.9143 \cdot p^{0.106933} \pm 0.679\%$ for $23.76 \leqq p \leqq 500$.

7.2. NUMERICAL CALCULATIONS OF FORMULAE FOR A_p AS FUNCTION OF p.

These calculations are compiled in Tables

7.2./9/10/11/12/ for

A_{qp} , A_{fp} , A_{bp} , A_{ip} , respectively.

The values of log A_p are found in the last columns of the tables of section 7.1. It is seen that the values of e.g. A_{qp} are practically equal for the different values of φ , and the same applies to the other A_p's. The values in the tables of section 7.2. are taken as the mean values and denoted by $(\log A_p)_m$.

The corresponding values of e_p from formula 6.3.(70) are next found, and the values e_{20}, e_{100} and e_{500} are introduced in formulae 6.3.(71 through 75), which give the values of a, b and c indicated in the following tables. These values introduced in formulae 6.3.(76a)b)c) thus give the definite formulae for the A_p's.

The errors of these formulae are found by calculation of the values of e_p^o according to 6.3.(70a)

$\boxed{6.3.(70a)}$ $$e_p^o = c + a \log^2\left(\frac{p}{b}\right).$$

The differences $e_p - e_p^o = \Delta e_p$ are proportional to the errors; the absolute values of the errors are found by multiplication by Δe_p of the errors for p = 100 stated in the tables of section 7.1.

$\boxed{A_{qp}.}$

Table 7.2/9/

p	$(\log A_{qp})_m$	$e_p = \frac{\log A_{qp}}{\log A_{q100}}$	e_p^o by 6.3(70a)	Δe_p
10	-0.04202	-4.7320	-4.0575	-0.6745
20	-0.01380	$\boxed{-1.5541}$	-1.5541	0.0000
30	-0.00314	-0.3536	-0.4815	+0.1279
50	+0.00521	+0.5867	+0.4585	+0.1272
100	$\boxed{+0.00888}$	$\boxed{+1.0000}$	+1.0000	0.0000
200	+0.00521	+0.5867	+0.6967	-0.1100
300	+0.00015	+0.0169	+0.1225	-0.1056
500	-0.00889	$\boxed{-1.0011}$	-1.0011	0.0000
1000	-0.02543	-2.8637	-3.2659	+0.4022

By 6.3 (71 through 75) :

a =-4.6619 , b = 110.262 , c = 1.0084 , log b = 2.04243

By 6.3 (76a)b)c) : for ~ 17.5 $\leqq$ p $\leqq$ ~ 622

$$A_{qp} = 1.020619 - 0.095322 \cdot \log^2 \left(\frac{p}{110.262}\right) \pm 0.265\ \%$$

$\boxed{A_{fp}.}$

Table 7.2/10/

p	$(\log A_{fp})_m$	$e_p = \frac{\log A_{fp}}{\log A_{f100}}$	e_p^o by 6.3(70a)	Δe_p
10	-0.01321	-4.3170	-3.8322	-0.4848
20	-0.00436	$\boxed{-1.4248}$	-1.4247	-0.0001
30	-0.00093	-0.3039	-0.3968	+0.0929
50	+0.00182	+0.5948	+0.4980	+0.0965
100	$\boxed{+0.00306}$	$\boxed{+1.0000}$	+1.0002	+0.0002
200	+0.00182	+0.5948	+0.6813	-0.0865
300	+0.00009	+0.02941	+0.1143	-0.0849
500	-0.00306	$\boxed{-1.0000}$	-0.9997	-0.0003
1000	-0.00889	-2.9053	-3.2243	+0.3190

By 6.3 (71 through 75) :

a = -4.52842 , b = 108.035 , c = 1.0053 , log b = 2.03356

By 6.3 (76a)b)c) : for 17.4 $\leqq$ p $\leqq$ 616

$$A_{fp} = 1.007083 - 0.031907 \cdot \log^2 \left(\frac{p}{108.035}\right) \pm 0.068\ \%$$

$\boxed{A_{bp}.}$ Table 7.2/11/

p	$(\log A_{bp})_m$	$e_p = \frac{\log A_{bp}}{\log A_{b100}}$	e_p^o by 6.3(70a)	Δe_p
10	-0.00840	-4.0508	-3.6917	-0.3663
20	-0.00278	$\boxed{-1.3430}$	-1.3431	+0.0001
30	-0.00056	-0.2705	-0.3432	+0.0727
50	+0.00125	+0.6039	+0.5238	+0.0801
100	$\boxed{+0.00207}$	$\boxed{+1.0000}$	+0.9998	+0.0002
200	+0.00125	+0.6039	+0.6694	-0.0655
300	+0.00007	+0.0338	+0.1023	-0.0685
500	-0.00208	$\boxed{-1.0048}$	-1.0050	+0.0002
1000	-0.00604	-2.9179	-3.2079	+0.2900

By 6.3 (71 through 75) :

$a = -4.44962$, $b = 106.46$, $c = +1.0031$, $\log b = +2.02718$

By 6.3 (76a)b)c) : for $17.2 \leqq p \leqq 605$

$A_{bp} = 1.0047812 - 0.0212087 \cdot \log^2 \left(\frac{p}{106.46}\right) \pm 0.0382\ \%$

$\boxed{A_{ip}.}$ Table 7.2/12/

p	$(\log A_{ip})_m$	$e_p = \frac{\log A_{ip}}{\log A_{i100}}$	e_p^o by 6.3(70a)	Δe_p
10	-0.01284	-4.2943	-3.8208	-0.4735
20	-0.00424	$\boxed{-1.4181}$	-1.4182	+0.0001
30	-0.00089	-0.2977	-0.3927	+0.0950
50	+0.00179	+0.5987	+0.5001	+0.0986
100	$\boxed{+0.00299}$	$\boxed{+1.0000}$	+0.9999	+0.0001
200	+0.00178	+0.5953	+0.6801	-0.0848
300	+0.00008	+0.0268	+0.1132	-0.0864
500	-0.00299	$\boxed{-1.0000}$	-1.0002	+0.0002
1000	-0.00879	-2.9398	-3.2229	+0.2829

By 6.3 (71 through 75) :

$a = -4.5216$, $b = 107.914$, $c = +1.0048$, $\log b = +2.03307$

By 6.3 (76a)b)c) : for $17.3 \leqq p \leqq 636$

$A_{ip} = 1.0069177 - 0.031130 \cdot \log^2 \left(\frac{p}{107.914}\right) \pm 0.0679\ \%$

8. GENERAL REVIEW OF FORMULAE FOR Q', F', v_m', R', B', I' AND v_*' AND OF FORMULAE FOR THE ABSCISSAE ξ AS FUNCTIONS OF φ FOR FIXED VALUES OF THE ORDINATES η.

8.1. THE DEFINITIVE FORMULAE FOR THE DIMENSIONLESS QUANTITIES Q', F', v_m', R', B', I' AND v_*'.

These are given in the following tables:

Q'	in Table 8.1./13/,	Formulae (89)	and (89a)b)c)d)e)
F'	- - 8.1./14/,	- (90)	- (90a)b)c)d)e)
v_m'	- - 8.1/15/,	- (91)	- (91a)b)c)d)e)
R'	- - 8.1./16/,	- (92)	- (92a)b)c)d)e)
B' $=\beta+\beta_0$	- - 8.1./17/,	- (93)	- (93a)b)c)d)e)
I'	- - 8.1./18/,	- (94)	- (94a)b)c)d)e)
v_*'	- - 8.1./19/,	- (95)	– (95a)b)c)d)e)

The coefficients $A_{q\varphi}$ etc. are in general functions of φ ; they are given as functions of $m = \cot\varphi$; in some cases the variation with φ is so slight that a mean value near the correct value for $\varphi = 20^{\circ}$ is used.

The exponents B_φ in most cases vary so little with φ that a mean value can be used. The factors A_{qp} etc. are generally functions of $p = \frac{y_{max}}{k}$ as indicated. For R', B' and v_*' these factors can be put equal to unity, because the simple power formulae have already very small errors.

In the tables are given the directly calculated values for A_φ and B_φ for $\varphi = 15^{\circ}, 20^{\circ}, 25^{\circ}$ and 30°.

The tables also state the ranges of validity of the simple power formulae and of the complete formulae, including the factors A_p with the corresponding maximum errors as well.

Table 8.1/13/

$$\text{(89)}\quad Q' = \frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\varrho}}\cdot y_{max}^{2}} = A_{q\varphi}\cdot p^{B_{q\varphi}}\cdot A_{qp}$$

Q for <u>half</u> of total section.

φ	$A_{q\varphi}$	$B_{q\varphi}$
15^{o}	21.8184	0.26247
20^{o}	17.094	0.26235
25^{o}	14.143	0.26230
30^{o}	12.099	0.26225

$$\text{(89c)}\quad A_{q\varphi} \cong 3.1815 \,.\, A_{f\varphi}$$

$$\text{(89d)}\quad B_{q\varphi}\ \text{mean} \cong 0.26234$$

$$\text{(89e)}\quad A_{qp} = 1.020619 - 0.095322\cdot\log^{2}\left(\frac{p}{110.262}\right)$$

For $A_{qp} = 1$ and $25 \leq p \leq 500$:

$$\text{(89a)}\quad Q' = A_{q\varphi}\cdot p^{B_{q\varphi}} \pm 2.07\,\%$$

Complete formula and $17.5 \leq p \leq 622$:

$$\text{(89b)}\quad Q' = A_{q\varphi}\cdot p^{B_{q\varphi}}\,.\,A_{qp} \pm 0.265\,\%$$

Table 8.1/14/

(90) $F' = \frac{F}{y_{max}^2}$	φ	$A_{f\varphi}$	$B_{f\varphi}$
$= A_{f\varphi} \cdot p^{B_{f\varphi}} \cdot A_{fp}$	15^o	6.8585	0.10921
	20^o	5.3728	0.10911
	25^o	4.4456	0.10906
F for <u>half</u> of total section.	30^o	3.8034	0.10899

(90c) $A_{f\varphi} = 1.03153 + 1.63688 \cdot m - 0.02187 \cdot m^2 + 0.000437 \cdot m^3$

(90d) $B_{f\varphi}$ mean $\tilde{=}$ 0.1091

(90e) $A_{fp} = 1.007083 - 0.031907 \cdot \log^2 \left(\frac{p}{108.035}\right)$

For $A_{fp} = 1$ and $25 \leqq p \gtreqless 500$:

(90a) $F' = A_{f\varphi} \cdot p^{B_{f\varphi}} \pm 0.71\ \%$

Complete formula and $17.4 \leqq p \gtreqless 616$.

(90b) $F' = A_{f\varphi} \cdot p^{B_{f\varphi}} \cdot A_{fp} \pm 0.068\ \%$

Table 8.1/15/

	φ	$A_{v\varphi}$	$B_{v\varphi}$
(91) $v'_m = \dfrac{v_m}{2.5\sqrt{\dfrac{\tau_{max}}{\rho}}} = A_{v\varphi} \cdot p^{B_{v\varphi}} \cdot A_{vp}$	15^o	3.1815	0.15324
	20^o	3.1816	0.15324
	25^o	3.1813	0.15324
	30^o	3.1811	0.15326

(91c) $A_{v\varphi}$ mean $\tilde{=}$ 3.1815

(91d) $B_{v\varphi}$ mean $\tilde{=}$ 0.15324

(91e) $A_{vp} = 1.0135645 - 0.0634373 \cdot \log^2 \left(\frac{p}{111.428}\right)$

For $A_{vp} = 1$ and $25 \leqq p \lessapprox 500$:

(91a) $v_m = A_{v\varphi} \cdot p^{B_{v\varphi}} \pm 1.37\ \%$

Complete formula and $17.5 \leqq p \lesseqqgtr 620$

(91b) $v'_m = A_{v\varphi} \cdot p^{B_{v\varphi}} \cdot A_{vp} \pm 0.195\ \%$

Table 8.1/16/

$$(92)\quad R' = \frac{R}{y_{max}} = A_{r\varphi} \cdot p^{B_{r\varphi}} \cdot A_{rp}$$

φ	$A_{r\varphi}$	$B_{r\varphi}$
15^{o}	0.73272	0.02095
20^{o}	0.73172	0.02103
25^{o}	0.72968	0.02118
30^{o}	0.72638	0.02146

(92c) $A_{r\varphi}$ mean $\cong$ 0.732 for $15^{o} \leqq \varphi \leqq 20^{o}$

(92d) $B_{r\varphi}$ mean $\cong$ 0.0210 - $15^{o} \leqq \varphi \leqq 20^{o}$

(92e) $A_{rp} \cong 1$ as simple power-formula sufficient

For $A_{rp} = 1$ and $25 \leqq p \leqq 500$

(92a, b) $R' = A_{r\varphi} \cdot p^{B_{r\varphi}} \pm 0.24\ \%$

Table 8.1/17/

$$(93)\quad B' = \frac{B}{y_{max}} = (\beta + \beta_0)$$

$$= A_{b\varphi} \cdot p^{B_{b\varphi}} \cdot A_{bp}$$

B for <u>half</u> of total section.

φ	$A_{b\varphi}$	$B_{b\varphi}$
15°	9.2804	0.08880
20°	7.2375	0.08901
25°	5.9660	0.08919
30°	5.0864	0.08936

$$(93c)\quad A_{b\varphi} = 1.29443 + 2.23919 \cdot m - 0.03069 \cdot m^2 + 0.00109 \cdot m^3$$

$$(93d)\quad B_{b\varphi} \text{ mean} = 0.0890$$

$$(93e)\quad A_{bp} = 1.0047812 - 0.0212087 \cdot \log^2 \left(\frac{p}{106.46}\right)$$

For $A_{bp} = 1$ and $25 \leqq p \leqq 500$

$$(93a)\quad B' = A_{b\varphi} \cdot p^{B_{b\varphi}} \pm 0.48\ \%$$

Complete formula and $17.2 \leqq p \leqq 605$

$$(93b)\quad B' = A_{b\varphi} \cdot p^{B_{b\varphi}} \cdot A_{bp} \pm 0.038\ \%$$

Table 8.1/18/

$$\text{(94)}\quad I' = \frac{g \cdot F \cdot I}{y_{max} \cdot \left(\frac{\tau_{max}}{\rho}\right)} = A_{i\varphi} \cdot p^{B_{i\varphi}} \cdot A_{ip}$$

φ	$A_{i\varphi}$	$B_{i\varphi}$
15^{o}	6.9199	0.108561
20^{o}	5.4502	0.108096
25^{o}	4.5399	0.107547
30^{o}	3.9143	0.106933

$$\text{(94c)}\quad A_{i\varphi} = 1.27988 + 1.51750 \cdot m + 0.00519 \cdot m^2 - 0.00184 \cdot m^3$$

$$\text{(94d)}\quad B_{i\varphi} \text{ mean} = 0.10833 \text{ for } 15^{o} \leq \varphi \leq 20^{o}$$

$$\text{(94e)}\quad A_{ip} = 1.0069177 - 0.031130 \cdot \log^2 \left(\frac{p}{107.914}\right)$$

For $A_{ip} = 1$ and $25 \leq p \leq 500$

$$\text{(94a)}\quad I' = A_{i\varphi} \cdot p^{B_{i\varphi}} \pm 0.70\ \%$$

Complete formula and $17.3 \leq p \leq 636$

$$\text{(94b)}\quad I' = A_{i\varphi} \cdot p^{B_{i\varphi}} \cdot A_{ip} \pm 0.068\ \%.$$

Table 8.1/19/

	φ	$A_{*\varphi}$	$B_{*\varphi}$
(95) $v'_* = \sqrt{\dfrac{(\frac{\tau_{max}}{\varrho})}{g.R.I}}$			
$= A_{*\varphi} \cdot p^{B_{*\varphi}} \cdot A_{*p}$	15^o	1.1630	- 0.01015
	20^o	1.1604	- 0.00997
	25^o	1.1584	- 0.00983
	30^o	1.1566	- 0.00970

(95c) $A_{*\varphi}$ mean $\cong$ 1.160 for $15^o \leqq \varphi \lesseqqgtr 25^o$

(95d) $B_{*\varphi}$ mean $\cong$ 0.0100 for $15^o \leqq \varphi \lesseqqgtr 25^o$

(95e) $A_{*p} \cong 1$ as simple power-formula sufficient.

For $A_{*p} = 1$ and $25 \leqq p \lesseqqgtr 500$

(95a) b) $v'_* = A_{*\varphi} \cdot p^{B_{*\varphi}} \pm 0.11\ \%$

8.2. FORMULAE FOR THE ABSCISSAE ξ AS FUNCTIONS OF $m = \cot\varphi$.

These formulae are found by the method indicated in Sec. 6.5. and are based on the values of ξ, which are found for fixed values of η and for the four different values of φ as listed in Tables 3.3./2a/b/c/d/.

The numerical coefficients a, b, c and d in the formulae:

(96) $\xi = a + b \cdot m + c \cdot m^2 + d \cdot m^3$,

(96a) where $m = \cot\varphi$.
are given in Table 8.2./20/.

A separate formula for ξ is consequently available for each of the fixed values of η.

For later use we have computed by means of these formulae the numerical values of ξ for $\varphi = 14^\circ$, 16°, 17°, 18° and 19°. These are, together with the previously found values for $\varphi = 15^\circ$, 20°, 25° and 30° (Tables 3.3.(2a)b)c)d)), compiled in the Table 8.2./21/.

Table 8.2/20/

η	ξ		1·	m·	m^2·	m^3·
0	ξ_0	=	0	0	0	0
0.1	$\xi_{0.1}$	=	+0.00723	+0.10041	+0.00049	-0.000065
0.2	$\xi_{0.2}$	=	+0.02898	+0.20256	+0.00179	-0.00024
0.3	$\xi_{0.3}$	=	+0.06579	+0.30833	+0.00365	-0.00051
0.4	$\xi_{0.4}$	=	+0.11622	+0.42315	+0.00455	-0.00070
0.5	$\xi_{0.5}$	=	+0.18331	+0.54901	+0.00447	-0.00082
0.6	$\xi_{0.6}$	=	+0.27246	+0.68819	+0.00360	-0.00090
0.7	$\xi_{0.7}$	=	+0.38875	+0.84935	+0.00049	-0.00078
0.75	$\xi_{0.75}$	=	+0.46333	+0.93745	-0.00083	-0.00078
0.8	$\xi_{0.8}$	=	+0.54583	+1.04230	-0.00555	-0.00039
0.85	$\xi_{0.85}$	=	+0.64936	+1.15793	-0.01000	-0.00007
0.9	$\xi_{0.9}$	=	+0.78211	+1.29522	-0.01590	+0.00039
0.95	$\xi_{0.95}$	=	+0.97200	+1.47634	-0.02432	+0.00107
0.98	$\xi_{0.98}$	=	+1.15392	+1.63955	-0.03239	+0.00175
1.0	$\xi_{1.0}$	=	+1.46574	+1.90032	-0.04441	+0.00274

Table 8.2/21/

η	ξ				
	$\varphi = 14^{o}$	$\varphi = 15^{o}$	$\varphi = 16^{o}$	$\varphi = 17^{o}$	$\varphi = 18^{o}$
0	0	0	0	0	0
0.1	0.41364	0.38543	0.36060	0.33863	0.31901
0.2	0.85471	0.79740	0.74698	0.70227	0.66235
0.3	1.32825	1.24081	1.16382	1.09549	1.03443
0.4	1.84141	1.72240	1.61757	1.52446	1.44123
0.5	2.40426	2.25187	2.11750	1.99816	1.89142
0.6	3.03248	2.84419	2.67807	2.53045	2.39835
0.7	3.75287	3.52483	3.32366	3.14480	2.98468
0.75	4.15957	3.90979	3.68943	3.49342	3.31790
0.8	4.61183	4.33820	4.09672	3.88201	3.68976
0.85	5.12818	4.82792	4.56295	4.32735	4.11634
0.9	5.74634	5.41474	5.12223	4.86212	4.62914
0.95	6.57108	6.19867	5.87020	5.57814	5.31653
0.98	7.32166	6.91261	6.55199	6.23136	5.94415
1.0	8.54989	8.08172	7.66902	7.30216	6.97354

To be continued

Table 8.2/21/ (continued)

η	ξ			
	$\varphi = 19^{\circ}$	$\varphi = 20^{\circ}$	$\varphi = 25^{\circ}$	$\varphi = 30^{\circ}$
0	0	0	0	0
0.1	0.30138	0.28545	0.22415	0.18226
0.2	0.62648	0.59404	0.46923	0.38394
0.3	0.97955	0.92989	0.73877	0.60813
0.4	1.36637	1.29864	1.03770	0.85917
0.5	1.79535	1.70843	1.37312	1.14338
0.6	2.27942	2.17176	1.75597	1.47056
0.7	2.84046	2.70984	2.20476	1.85730
0.75	3.15977	3.01649	2.46225	2.08050
0.8	3.51653	3.35955	2.75167	2.33244
0.85	3.92618	3.75381	3.08587	2.62459
0.9	4.41914	4.22878	3.49046	2.97984
0.95	5.08068	4.86682	4.03673	3.46169
0.98	5.68520	5.45034	4.53826	3.90564
1.0	6.67722	6.40843	5.36379	4.63820

9. DETERMINATION OF ISOVELS IN THE EQUILIBRIUM PROFILE.

The principal assumptions made in the foregoing sections, which made it possible to determine a definite equilibrium profile for fixed values of ϱ, τ_{max}, φ, k and Q, are expressed by the formulae 1.(1), 2.1(3), 2.2(9), 4.2.(19), 4.4.(45), giving the distribution of the shearing stresses, the relation between hydraulic lifting force and shearing stress, the logarithmic law of velocity distribution and the principle of minimum cross section.

On the same assumptions it is possible to calculate the accurate form of the isovels and thus, by comparing these with the results from actual measurements, to get a certain check on the correctness of the assumptions.

We therefore indicate the method of determination of isovels having velocities expressed as multiples of the mean velocity of the total cross section.

The velocity at a point P of the cross section is denoted by $v_{z,\xi}$. Here the subscripts indicate that P is situated at the distance z from the bottom in a normal to this with the length $\xi \cdot y_{max}$ between the bottom and the water surface. By 4,2,(18) and 4.2.(19) we find this velocity to be

$$v_{z,\xi} = 2.5\sqrt{\frac{\tau_{max}}{\varrho}} \cdot \sqrt{\xi} \cdot \left[3.392 + \ln\left(\frac{z}{k}\right)\right] \tag{97}$$

Substituting

$$\frac{v_{z,\xi}}{v_m} = r \tag{98}$$

and

$$\frac{z}{y_{max}} = z' \tag{99}$$

we get, with $\frac{y_{max}}{k} = p$ as usual:

$$r = \frac{2.5}{v_m}\sqrt{\frac{\tau_{max}}{\varrho}} \cdot \sqrt{\xi} \cdot \ln\left[29.725\ p \cdot z'\right] \tag{100}$$

Further we have from Table 8.1./15/

8.1.(91b)c)d) $$\frac{v_m}{2.5\sqrt{\frac{\tau_{max}}{\varrho}}} = 3.1815 \cdot p^{0.15324} \cdot A_{vp}$$

and

8.1.(91e) $A_{vp} = 1.013565 - 0.063437 \log^2 \left(\frac{p}{111,428} \right)$ whence

(101) $$r = \frac{\sqrt{\varsigma} \cdot \ln [29.725\, p \cdot z']}{3.1815 \cdot p^{0.15324} \cdot A_{vp}}$$

This formula gives the velocity at the point P measured with the mean velocity v_m as unity as function of the dimensionless quantities ς, total length of the normal, z', distance from the bottom, and $p = \frac{y_{max}}{k}$.

If we choose the normals ς_m used in the numerical integration, these will vary with φ; but if the normals are chosen as fixed multiples of y_{max} and introduced in a drawing of the equilibrium profile in question, the relative velocities will depend only on p, i.e. the relative roughness of the bottom.

From (101) we get:

$\boxed{(101a)}$ $$z' = \frac{e^{\left(\frac{3.1815 \cdot p^{0.15324} \cdot A_{vp}}{\sqrt{\varsigma}} \cdot r \right)}}{29.725 \cdot p}$$

For a fixed value of p a complete set of isovels, e.g. with relative velocities
r = 1.2 , 1.1 , 1.0 , 0.9 , ...
can be drawn by choosing suitable values of ς.

As an example the isovels are worked out for an equilibrium profile with $\varphi = 30^{\circ}$ and $p = \frac{y_{max}}{k} = 50$.

We get

$29.725 \cdot p = 1486.25$ and from 8.1.(91e)

$A_{vp} = 1.00588$

$3.1815 \cdot p^{0.15324} \cdot A_{vp} = 5.8281$

(101aa) $$z' = \frac{e^{\left(\frac{5.8281}{\sqrt{\varsigma}} \cdot r \right)}}{1486.25}$$

For the values ς we have chosen the values ς_m, for which $\sqrt{\varsigma_m}$ can be found in Table 4.6./3d/.

The results are shown in Fig. 7.

It will be seen that the form of the isovels compares fairly well with the results of actual measurements, and it can consequently be concluded that the assumption made concerning the distribution of shearing stresses must be at least approximately correct.

10. STUDY OF MODELTESTS CARRIED OUT IN VIENNA, 1916. COMPARISON WITH THEORY.

10.1. DESCRIPTION OF MODEL TESTS AND RESULTS.

These model tests are briefly mentioned in [2] ; the complete report is found in [3] .

The sand used had a median grain diameter of about 1.4 mm and $\frac{d_{60\%}}{d_{10\%}} \simeq 1.8$; it was a rather sharp quartz sand with 38% of voids, measured in loose filling, and with a specific weight of 1.64 kg/l.

Three different profiles were shaped by three different constant discharges, and when the profile had become stationary, its size and shape was carefully measured by a photogrammetric method. The limiting tractive force for the sand, τ_{max}, was found by separate tests on a nearly horizontal bottom and turned out to be 0.075 kg/m^2.

For each equilibrium profile the discharge and the slope of the water surface (at uniform movement) were further measured.

The observed results are given in the following Tables 10.1 /22/ and 10.1 /23/.

Table 10.1 /22

Dimensions, Slope and Discharge.

Test No	Width of water surface 2 B (cm)	Slope I ‰	Area of cross section 2 F (cm^2)	Maximum depth y_{max} (cm)	Discharge 2 Q (l/sec)
1	100	1.35	446.2	5.6	13.8
2	198	0.79	1444.0	9.0	53.5
3	280	0.65	2475.0	10.5	87.0

Table 10.1./23/

Co-ordinates x and y. (See Fig. 3)

Test No	Depths y (cm)	Abscissae x (cm)									
		2.5	5	7.5	10	15	20	25	30	40	50
1	y	1.1	2.05	2.8	3.45	4.3	4.8	5.1	5.3	5.5	5.6
2	y	1.2	2.3	3.2	4.0	5.2	6.1	6.7	7.2	7.8	8.2
3	y	1.6	2.8	3.8	4.8	6.2	7.2	7.9	8.3	8.9	9.2
Test No	**Depths y (cm)**	**Abscissae x (cm)**									
		60	70	80	90	100	110	120	130	140	
1	y										
2	y	8.6	8.8	8.9	9.0	9.0					
3	y	9.4	9.6	9.8	10.0	10.1	10.2	10.3	10.4	10.5	

10.2. COMPARISON OF THEORY WITH MODEL TESTS; COMMENTS.

If we compare the results of these tests with the theory developed in the present paper, we should note that the area of the cross section necessary to carry a definite discharge without erosion varies only insignificantly with the angle of friction φ. This will be seen by the formulae given in Sec. 11.

The relative width of the cross section $\frac{B}{y_{max}}$ will, however, vary considerably with φ, the value of which cannot be predicted with any degree of certainty.

We have therefore chosen to use our theory to calculate such profiles of equilibrium as can carry the same discharge as in the test in an area of the same size and with the same τ_{max}. This is done for a number of values for φ and the dimensions and the shape of the calculated profiles are compared with those of the tests. We also calculate for each value of φ the values of $p = \frac{y_{max}}{k}$ and thereby find k and further the value of the slope I, which latter quantity can be compared with the measured values. I is probably the least accurate of the measured values because uniform motion only existed over a length of about 10 m, so that a very strict agreement between calculated and measured values of I can hardly be expexted.

That value of φ is chosen which in the most essential features gives a good agreement in the shape of the profile, and it is at the same time found that in this selection the values of I agree quite well.

It also becomes possible to compare the calculated value of the roughness k at the bottom with the diameter of the sand grains, and a ratio of these quantities can be found which seems to agree quite well with statements from elsewhere. *)

10.3. DERIVATION OF FORMULAE SUITABLE FOR ANALYSIS OF MODEL TEST DATA.

As known quantities are considered

φ, ρ, τ_{max}, Q and F, and we want to find

$p = \frac{y_{max}}{k}$, y_{max}, k, B and I, whereupon the complete cross section can be drawn and compared in detail with the observed one, and the computed value of I can be compared with the measured one.

We have

(102) $v_m = \frac{Q}{F}$ as a known quantity and start from the formulae in Table 8.1./15/

(91),(91c)d) $$\frac{v_m}{2.5\sqrt{\frac{\tau_{max}}{\rho}}} = 3.1815 \cdot p^{0.15324} \cdot A_{vp} \quad \text{, whence}$$

$\boxed{(103)}$ $$p = \frac{y_{max}}{k} = \left[\frac{v_m}{2.5\sqrt{\frac{\tau_{max}}{\rho}} \cdot 3.1815}\right]^{6.5257} \cdot \frac{1}{A_{vp}^{\;6.5257}} \quad \text{, where}$$

(91e) $$A_{vp} = 1.013565 - 0.063437 \log^2 \left(\frac{p}{111,428}\right)$$

From (103) p can be found; according to 8.1.(91e) A_{vp} is a function of p, so we start with $A_{vp} = 1$ and use successive approximation. p is seen to be independent of φ.

When p is found, the other unknown quantities can be found directly:

From Table 8.1./14/ we find:

(90),(90d) $$\frac{F}{y_{max}^{\;2}} = A_{f\varphi} \cdot p^{0.1091} \cdot A_{fp} \quad , \quad \text{whence:}$$

*) For tests Nos. 1, 2 and 3 are found respectively 2.91, 1.84 and 3.00.

$$\boxed{(104)} \qquad y_{max} = \frac{\sqrt{\left(\frac{F}{A_{f\varphi}}\right)}}{p^{0.05455} \cdot \sqrt{A_{fp}}} \quad , \quad \text{where}$$

(90c) $A_{f\varphi} = 1.03153 + 1.63688 \cot\varphi - 0,02187 \cot^2\varphi + 0.000431 \cot^3\varphi$

(90e) $A_{fp} = 1.007083 - 0.031907 \log^2\left(\frac{p}{108.035}\right)$.

y_{max} is seen to vary with φ, because $A_{f\varphi}$ is far from being a constant.

For the hydraulic roughness k we have

$$\boxed{(105)} \qquad k = \frac{y_{max}}{p} .$$

From Table 8.1./17/ we find

$$\boxed{(93),(93d)} \quad B = A_{b\varphi} \cdot p^{0.0890} \cdot A_{bp} \cdot y_{max} \quad , \text{ where}$$

(93c) $A_{b\varphi} = 1.29443 + 2.23919 \cot\varphi - 0.03069 \cot^2\varphi + 0.00109 \cot^3\varphi$

(93e) $A_{bp} = 1.0047812 - 0.0212087 \cdot \log^2\left(\frac{p}{106.46}\right)$.

From Table 8.1./18/ we find

$$\boxed{(94),(94d)} \quad I = \left(\frac{A_{i\varphi}}{F}\right) \cdot \left(\frac{\tau_{max}}{\gamma}\right) \cdot y_{max} \cdot p^{o.10833} \cdot A_{ip} \quad , \text{ where}$$

(94c) $A_{i\varphi} = 1.27988 + 1.51750 \cot\varphi + 0.00519 \cot^2\varphi - 0.00184 \cot^3\varphi$

(94e) $A_{ip} = 1.0069177 - 0.031130 \log^2\left(\frac{p}{107.914}\right)$.

10.4. NUMERICAL CALCULATION OF y_{max}, B, I, and k FOR MODEL TESTS.

The formulae developed in Sec. 10.3. are used. As units we use grams and centimeters.

For all three tests is reckoned

$\tau_{max} = 0.075 \text{ kg/m}^2 = 0.0075 \text{ gr/cm}^2$.

Specific weight $\gamma = 1 \text{ gr/cm}^3$.

Acceleration of gravity $g = 981\ cm/sec^2$.

$\sqrt{\frac{\tau_{max}}{\varrho}} = 2.7125\ cm/sec$, and 10.3 (103) becomes

$$\boxed{(103a)} \quad p = \frac{y_{max}}{k} = \left[\frac{v_m}{21.575}\right]^{6.5257} \cdot \frac{1}{A_{vp}^{\ 6.5257}} ,$$

which formula is valid for all three tests and independent of the choice of φ .

For the calculations are used the other formulae from 10.3 besides (103a).

<u>10.4.1 Test No 1.</u>

Measured: $\tau = 0.0075\ gr/cm^2$,

$$Q = \frac{13.8\ l/sec}{2} = 6900\ cm^3/sec,$$

$$F = \frac{446.2\ cm^2}{2} = 223.1\ cm^2,$$

$I = 1.35\ o/oo = 0.00135$,

$y_{max} = 5.6\ cm$,

$$B = \frac{100\ cm}{2} = 50{,}0\ cm.$$

(102, 1) $$v_m = \frac{6900}{223.1} = 30.928\ cm/sec.$$

$$\frac{v_m}{21.575} = 1.4335$$ and from 10.4(103a)

(103a, 1) $$p = \frac{10\ 45}{A_{vp}^{\ 6.5257}} ,$$

where A_{vp} is taken from 8.1(91e)

$p = 13.58$ gives

$A_{vp} = 0.96059$; $A_{vp}^{\ 6.5257} = 0.7696$

$p = \frac{10.45}{0.7696} = 13.58$, final value for all φ's.

For use in 10.3(104) we find

$p^{0.05455} = 13.58^{0.05455} = 1.1533$ and

$A_{fp} = 1.007083 - 0.031907 \cdot \log^2 \left(\frac{13.58}{108.035}\right)$

$A_{fp} = 0.981200 \quad ; \quad \sqrt{A_{fp}} = 0.99056,$

and 10.3(104), with $F = 223.1\ cm^2$, becomes:

$$y_{max} = \frac{\sqrt{\frac{223.1}{A_{f\varphi}}}}{1.1533 \cdot 0.99056}$$

(104,1) $\quad y_{max} = \frac{13.075}{\sqrt{A_{f\varphi}}}.$

This expression for y_{max} is valid for Test No.1 and all values of φ .

For use in 8.1(93) we find

$p^{0.0890} = 13.58^{0.0890} = 1.2620$ and

$A_{bp} = 1.0047812 - 0.0212087 \log^2 \left(\frac{13.58}{106.46}\right) = 0.987819$

and 8.1(93) becomes

(93,1) $\quad B = 1.24663\ A_{b\varphi} \cdot y_{max}$ *(valid for all φ's).*

For use in 8.1(94) we find $A_{ip} = 0.981693$.

$$\frac{\tau_{max}}{\gamma} = \frac{0.0075}{1} = 0.0075\ cm$$

$F = 223.1\ cm^2$

and 8.1(94) becomes:

$$I = \frac{A_{i\varphi}}{223.1} \cdot 0.0075 \cdot 1.3270 \cdot 0.981693 \cdot y_{max}$$

(94,1) $\quad I = 4.37934 \cdot 10^{-5} \cdot A_{i\varphi} \cdot y_{max}$ (valid for all φ 's).

For the calculation of Test No 1 for different values of φ we now have p = 13.58 and (104,1) , (93,1) and (94,1).

The respective values of $A_{f\varphi}$, $A_{b\varphi}$ and $A_{i\varphi}$ are found by 8.1(90c) , 8.1(93c) and 8.1(94c).

The results found for $\varphi = 18^\circ$, 19° and 20° are compiled in Table 10.4.1/24/, where the calculated and observed values of y_{max}, B and I can be compared, and the calculated value of k is given.

Table 10.4.1/24/

Test No. 1. $p = 13.58$.

$\varphi =$		18^{o}	19^{o}	20^{o}
$\sqrt{A_{f\varphi}}$		2.4238	2.3689	2.3179
$y_{max} = \dfrac{13.075}{\sqrt{A_{f\varphi}}}$	(cm)	5.39	5.52	5.64
y_{max} observed	(cm)		5.6	
$A_{b\varphi}$		7.9270	7.5624	7.2375
$B = 1.24663 \cdot A_{b\varphi} \cdot y_{max}$	(cm)	53.31	52.06	50.90
B observed	(cm)		50	
$A_{i\varphi}$		5.9458	5.6857	5.4502
$I = 4.37934 \cdot 10^{-5} \cdot A_{i\varphi} \cdot y_{max}$	10^{-5}	140.5	137.4	134.6
I observed	10^{-5}		135	
$k = \dfrac{y_{max}}{p}$	(cm)	0.397	0.407	0.415

10.4.2 Test No. 2.

Measured: $\tau = 0.0075 \text{ gr/cm}^2$,

$$Q = \frac{53.6 \text{ l/sec}}{2} = 26800 \text{ cm}^3/\text{sec},$$

$$F = \frac{1444.0 \text{ cm}^2}{2} = 722.0 \text{ cm}^2,$$

$$I = 0.79 \text{ o/oo} = 0.00079,$$

$$y_{max} = 9.0 \text{ cm},$$

$$B = \frac{198 \text{ cm}}{2} = 99.0 \text{ cm}$$

The calculations are made similarly to those applying to Test No. 1.

We find:

(102, 2) $$v_m = \frac{26800}{722.0} = 37.050 \text{ cm/sec}$$

$$\frac{v_m}{21.575} = 1.7173 \quad \text{and from 10.4(103a)}$$

(103a, 2) $$p = \frac{34.08}{A_{vp}^{\;6.5257}}, \qquad \text{whence}$$

$p = 34.698$

(104, 2) $y_{max} = \dfrac{22.148}{\sqrt{A_{f\varphi}}}$ for all φ's.

(93, 2) $B = 1.3709\ A_{b\varphi} \cdot y_{max}$ for all φ's

(94, 2) $I = 1.5244 \cdot 10^{-5} \cdot A_{i\varphi} \cdot y_{max}$ for all φ's.

The results are given in Table 10.4.2/25/

Table 10.4.2/25/

Test No. 2. $p = 34.698$

	$\varphi =$	17^o	18^o	19^o
$\sqrt{A_{f\varphi}}$		2.4833	2.4238	2.3689
$y_{max} = \dfrac{22.148}{\sqrt{A_{f\varphi}}}$	(cm)	8.92	9.14	9.35
y_{max} observed	(cm)		9.0	
$A_{b\varphi}$		8.3283	7.9270	7.5654
$B = 1.3709 \cdot A_{b\varphi} \cdot y_{max}$	(cm)	101.83	99.30	96.97
B observed	(cm)		99	
$A_{i\varphi}$		6.2345	5.9458	5.6857
$I = 1.5244 \cdot 10^{-5} \cdot A_{i\varphi} \cdot y_{max}$	10^{-5}	84.8	82.8	81.0
I observed	10^{-5}		79	
$k = \dfrac{y_{max}}{p}$	(cm)	0.257	0.263	0.270

10.4.3 Test No. 3.

Measured: $\tau = 0.0075\ gr/cm^2$,

$Q = \dfrac{87.0\ 1/sec}{2} = 43500\ cm^3/sec$,

$F = \dfrac{2475\ cm^2}{2} = 1237.5\ cm^2$,

$I = 0.65$ o/oo $= 0.00065$,

$y_{max} = 10.5$ cm,

$$B = \frac{280 \text{ cm}}{2} = 140 \text{ cm}.$$

The calculations are made similarly to those applying to Test No. 1.

We find:

(102, 3) $$v_m = \frac{43500}{1237.5} = 35.152 \text{ cm/sec}$$

$$\frac{v_m}{21.575} = 1.6293 \quad \text{and from 10.4(103a)}$$

(103a, 3) $$p = \frac{24.180}{A_{vp}^{\;6.5257}} \quad \text{whence}$$

$$p = 26.10$$

(104, 3) $$y_{max} = \frac{29.517}{\sqrt{A_{f\varphi}}} \quad \text{for all } \varphi\text{'s}$$

(93, 3) $$B = 1.3332 \cdot A_{b\varphi} \cdot y_{max} \quad \text{for all } \varphi\text{'s}$$

(94, 3) $$I = 0.85874 \cdot 10^{-5} \cdot A_{i\varphi} \cdot y_{max} \quad \text{for all } \varphi\text{'s}.$$

The results are given in Table 10.4.3/26/

Table 10.4.3/26/

Test No. 3. p = 26.10

	$\varphi =$	14^o	15^o	16^o
$\sqrt{A_{f\varphi}}$		2.6969	2.6189	2.5481
$y_{max} = \frac{29.517}{\sqrt{A_{f\varphi}}}$	(cm)	10.94	11.27	11.58
y_{max} observed	(cm)	10.5		
$A_{b\varphi}$		9.8520	9.2804	8.7764
$B = 1.3332 \cdot A_{b\varphi} \cdot y_{max}$	(cm)	143.76	139.45	135.04
B observed	(cm)	140		
$A_{i\varphi}$		7.3310	6.9199	6.5571
$I = 0.85874 \cdot 10^{-5} \cdot A_{i\varphi} \cdot y_{max}$	10^{-5}	68.9	66.9	65.2
I observed	10^{-5}	65		
$k = \frac{y_{max}}{p}$	(cm)	0.420	0.431	0.444

10.5. ADJUSTMENT OF THEORETICAL EQUILIBRIUM PROFILES TO THOSE OF MODEL TESTS.

From Tables 10.4.1/24/, 10.4.2/25/ and 10.4.3/26/ we already get a rough idea of the value of φ that will give the best adjustment of the theoretical equilibrium profiles to the observed ones.

To compare the profiles more closely the theoretical profiles are drawn according to the co-ordinates given in Table 8.2/21/.

The observed profiles, the co-ordinates of which are given in Table 10.1/23/, are then reduced to the same scale as the theoretical ones, and superimposed correctly upon the theoretical profiles, i.e. with coinciding axes and water surfaces.

This has been done for three different values of φ for each tested profile.

That value of φ is chosen which in general gives the best conformity between calculated and observed profile for each test.

The following values of φ were chosen:

Test 1 : $\varphi = 19^\circ$
- 2 : $\varphi = 17^\circ$
- 3 : $\varphi = 14^\circ$.

The results are shown in Figs. 8, 9 and 10.

It is seen from the Tables /24/, /25/ and /26/ that the agreement of I corresponding to this choice is quite good considering the uncertainty of these observations, as mentioned in SEC.10.2.

As seen in the figures the agreement is considerably better for Tests 1 and 2 than for Test 3.

If we study the data for Test 3 we find, however, that its mean velocity v_m is smaller than v_m for Test 2 although y_{max} is ~ 17% greater in the former case. This is in disagreement with the well-known fact that for equal tractive force a greater mean velocity is needed to move the sand grains when the depth is greater. It therefore seems as if some uncertainty must exist in some of the observed data.

It does appear from all three tests that the angle of internal friction φ , which must be introduced in the theory put forward to get a tolerably good agreement, is considerably smaller than would generally be expected.

If we remember, however, the fact that the sand grains in a state of incipient motion hardly exert any normal pressure on the underlying ones and that the angle φ is known to decrease with decreasing density of the sand, it is hardly surprising that φ is found to be small.

In Figs. 8, 9 and 10 the slope of the banks in the model tests is seen everywhere to be greater than for the theoretical profiles, which fact might seem to contradict the reasoning above.

It does seem possible, however, that the capillary tension in the sand above the water surface may be able to give the banks a certain cohesion causing them to be locally steeper than in the case of a true cohesionless material.

A similar effect in a full-size channel or natural watercourse would of course be relatively insignificant.

We, therefore, provisionally conclude that the angle of internal friction φ for a sand material like that used in the model tests should be estimated at 18 to 20°, but it is clear that a further study of this question is highly desirable (see Sec. 14).

11. FORMULAE FOR THE DIMENSIONS OF EQUILIBRIUM PROFILES FOR GIVEN VALUES OF Q, τ_{max}, ϱ, k AND φ.

In the formula 8.1(89) for Q' in Table 8 1/13/ we insert

8.1(89c) $A_{q\varphi} = 3.1815 \cdot A_{f\varphi}$

and

8.1(89d) $B_{q\varphi} = 0.26234 = \text{constant}$

and get

$$\frac{Q}{2.5\sqrt{\dfrac{\tau_{max}}{\varrho}} \cdot k^2} = 3.1815\ A_{f\varphi} \cdot p^{2.26234} \cdot A_{qp}\ ,$$

where

8.1(89e) $A_{qp} = 1.020619 - 0.095322 \log^2 \left(\frac{p}{110.262}\right)$

and where $A_{f\varphi}$ is found from 8.1(90c), Table 8.1/14/.

Solving with respect to p we get

$$\boxed{(106)}\quad p \cdot 1.009114 \cdot \left[1 - 0.041283 \log^2 \left(\frac{p}{110.262}\right)\right] = \left[\frac{Q}{2.5\sqrt{\dfrac{\tau_{max}}{\varrho}} \cdot k^2 \cdot 3.1815 \cdot A_{f\varphi}}\right]^{0.44202}$$

where

8.1(90c) $A_{f\varphi} = 1.03153 + 1.63688\, m - 0.02187\, m^2 + 0.000437\, m^3$

and

$m = \cot \varphi$.

In the first approximation the factor next to p on the left side of (106), viz.: $1.009114 \cdot \left[1 - 0.041283 \log^2 \left(\frac{p}{110.262}\right)\right]$ is neglected, whereby a nearly correct value of p is found; p is then found correctly by successive approximation.

For $20 \leqq p \lessapprox 600$ the said factor will hardly deviate more than 1.25% from unity.

When $p = \frac{y_{max}}{k}$ is found we get directly

$$\boxed{(107)} \quad y_{max} = k \cdot p.$$

For the half width of the water surface, B, we get from Table 8.1/17/, formula 8.1(93) with 8.1(93d)

$$B_{b\varphi} \cong 0.0890 \quad = \text{constant}$$

$$\boxed{(108)} \quad B = k \cdot A_{b\varphi} \cdot p^{1.0890} \cdot A_{bp}$$

where

8.1(93c) $A_{b\varphi} = 1.29443 + 2.23919\ m - 0.03069\ m^2 + 0.00109\ m^3$

and

$m = \cot \varphi$,

further

8.1(93e) $A_{bp} = 1.0047812 - 0.0212087 \log^2 \left(\frac{p}{106.46}\right)$.

As p is known from (106) B can be directly found from (108).

From Table 8.1/14/, formula 8.1(90) we get with $B_{f\varphi} \cong 0.1091$ = constant for the area of the half cross section

(109) $F = k^2 \cdot A_{f\varphi} \cdot p^{2.1091} \cdot A_{fp}$,

where

8.1(90e) $A_{fp} = 1.007083 \left[1 - 0.031682 \log^2 \left(\frac{p}{108.035}\right)\right]$.

p is known from (106) and F can be directly found from (109).

We want, however, to find a more explicit form of the formula for F and therefore insert the expression (106) for p in (109).

We find :

$$\left\{1.009114\cdot\left[1 - 0.041283 \log^2\left(\frac{p}{110.262}\right)\right]\right\}^{2.1091}$$
$$\cong 1.019223\cdot\left[1 - 0.087070 \cdot \log^2\left(\frac{p}{110.262}\right)\right].$$

By using the formulae 6.4(77 through 80a) we further find:

$$\frac{1.019223\cdot\left[1 - 0.087070 \log^2\left(\frac{p}{110.262}\right)\right]}{1.007083\cdot\left[1 - 0.031682 \log^2\left(\frac{p}{108.035}\right)\right]}$$
$$\cong 1.012051\cdot\left[1 - 0.055388 \log^2\left(\frac{p}{111.55}\right)\right],$$

and (109) takes the form :

$$\text{(109a)} \quad F = \frac{k^{0.13547}\cdot A_{f\varphi}^{\;0.067735}\cdot\left[\dfrac{Q}{2.5\sqrt{\dfrac{\tau_{max}}{\varrho}}}\right]^{0.932265}}{2.97877\cdot\left[1 - 0.055388\cdot\log^2\left(\frac{p}{111.55}\right)\right]},$$

where

8.1(90c) $A_{f\varphi} = 1.03153 + 1.63688\, m - 0.02187\, m^2 + 0.000473\, m^3$

and

$m = \cot\varphi$.

In (109a) the factor $A_{f\varphi}^{\;0.067735}$ takes the following values for the four values of φ :

φ	$A_{f\varphi}^{\;0.067735}$
15^o	1.1393
20^o	1.1206
25^o	1.1063
30^o	1.0947,

and it is possible to put with good approximation

$$A_{f\varphi}^{\;0.067735} = 1.0560\cdot\left[1 + 0.02178 \cot\varphi\right],$$

whereupon (109a) takes the following final form :

$$\boxed{\text{(109b)}} \quad F = \frac{0.354509\, k^{0.13547}\cdot\left[1 + 0.02178 \cot\varphi\right]}{1 - 0.055388 \log^2\left(\frac{p}{111.55}\right)}\cdot\left[\frac{Q}{2.5\sqrt{\dfrac{\tau_{max}}{\varrho}}}\right]^{0.932265}.$$

From (109b) it is clearly seen that F is practically independent of $p = \frac{y_{max}}{k}$ for actual channels, where p will probably be of the order of 100 and that it varies only slowly with φ.

For practical calculations it should consequently be advantageous to judge roughly the value of p from (106) and then use (109b) to find F.

From (106) it is seen that p varies approximately proportionally to $k^{-0.88404}$, and thereby it is seen from (107) that y_{max} is approximately proportional to $k^{0.11596}$, whereas (108) and (106) show that B is approximately proportional to

$$\frac{k}{(k^{0.88404})^{1.0890}} = \frac{k}{k^{0.96272}} \sim k^{0.03728}$$

The dimensions proper of the equilibrium profile consequently are only slightly influenced by an uncertainty of k.

11.1. CHOICE OF THE ANGLE OF INTERNAL FRICTION φ AND THE EQUIVALENT SAND ROUGHNESS k.

11.1.1. φ.

As to the choice of φ we refer to the remarks made in Sec. 10.5 .

The value of φ must probably be lower than those values used in calculations of soil mechanics, and the results of the Vienna tests make it probable that $\varphi = 18^o$ to 20^o will be appropriate. Further studies of this question do, however, seem necessary, and we refer to the proposals made in the conclusion, Sec. 14.

11.1.2. k.

As to the value of k, the equivalent sand roughness, which must be introduced in the formulae above, several view points can be advanced.

11.1.2.1. Determination of k by means of the Manning coefficient M in the formula $v_m = M \cdot R^{2/3} \cdot I^{1/2}$ (metric units).

If an empirical knowledge of M for channels of a similar character is available, it is easily proved by comparing the Manning formula with the results of Nikuradse for the resistance number f, that

$$M = \frac{25.84}{\sqrt[6]{k}} \qquad (m^{1/3}/sec)$$

or

(110) $$k = \left(\frac{M}{25.84}\right)^6 \qquad \text{(k in meters)}.$$

If Kutter's n is known we have

(111) $$M = \frac{1}{n}.$$

The above-mentioned results of Nikuradse concerning the resistance number f agree perfectly with the results obtained by pure calculation on the basis of the logarithmic velocity distribution, as it is done in this paper.

11.1.2.2. k found from observation of velocity distribution in a normal to the bottom.

From the logarithmic velocity distribution in a definite normal to the bottom of total length z_0 it is easily found that the velocities v_p and v_q at points $p \cdot z_0$ and $q \cdot z_0$ from the bottom will have the mean value v_m equal to the mean velocity in the normal concerned, provided that

(112) $$p \cdot q = \frac{1}{e^2} = 0.1352$$ (e basis of natural logarithms).

(113) $$v_m = \frac{v_p + v_q}{2}$$

The hydraulic roughness of the bottom in the neighborhood of the foot of the normal is found to be

(114) $$k = \frac{29.7 \cdot p \cdot z_0}{\left[\frac{p}{q}\right]^{\frac{v_p}{v_p - v_q}}} \qquad (p > q).$$

The values $p = 0.90$ and $q = 0.15$ e.g. satisfy (112), and if the velocities are measured at distances $0.90 \cdot z_0$ and $0.15 \cdot z_0$ from the bottom we get from (113) and (114)

(114a) $$k = \frac{26.7 \cdot z_0}{\left[6\right]^{\frac{v_p}{v_p - v_q}}} \qquad \text{for } p = 0.90 \text{ and } q = 0.15.$$

By taking normals distributed over the whole width of the channel we can get an idea of the variation of k transversally to the axis of the channel and obtain a mean value of k.

If empirical values of k in this way are known from channels of varying character, a sufficiently good estimate of k for planing purposes can probably be made.

11.1.2.3. k found by means of $\varkappa$, "degree of fullness", or $\frac{v_m}{v_{max}}$.

Provided that the channel considered is shaped as an equilibrium profile a relation can be shown to exist between $p = \frac{y_{max}}{k}$ and $\varkappa$ as well as between p and $\frac{v_m}{v_{max}}$, the ratio of mean to maximum velocities for the total cross section.

If the cross section is measured in detail $\varkappa = \frac{F}{B \cdot y_{max}}$ can easily be found, and the corresponding value of $p = \frac{y_{max}}{k}$ can be read in a diagram; we then find

$$k = \frac{y_{max}}{p}$$

A possible knowledge of $\frac{v_m}{v_{max}}$ can similarly be used for finding k.

This method is presented in detail in Sec. 12,(see Fig. 11).

11.1.2.4. Assuming the same value of k as empirically found for natural watercourses, viz.: $\frac{k}{R} = \left(\frac{c_k}{R}\right)^{1.56}$, where c_k = 0.425 meters or 1.395 feet.

This method will be developed in detail in the following Sec. 11.2, where formulae for p, y_{max}, B and F are presented that do not directly contain the roughness k.

11.2 DETERMINATION OF DIMENSIONS FOR GIVEN VALUES OF Q, τ_{max}, ϱ AND φ AND FOR k ASSUMED TO CORRESPOND TO ROUGHNESS OF NATURAL WATERCOURSES.

As it will perhaps sometimes be doubtful which value must be assigned to k in the formulae of Sec. 11, we think it useful to develop formulae not containing any value of k, which can be chosen more or less arbitrarily.

From a study of many hundreds of observations of velocities in natural watercourses together with their hydraulic radii R and their slopes, we have found statistically that k must be independent of the slope I and is a function of R alone.

We have found

$$\boxed{(115)} \qquad \frac{k}{R} = \left(\frac{c_k}{R}\right)^{1.56},$$

where

$$\boxed{(115a)} \qquad c_k = 0.425 \text{ meters or } 1.395 \text{ feet.}$$

For small values of R, ($R < 1.0$ m), e.g. for mountain torrents, the dispersion of the observations in relation to the law (115) is considerable, but for larger rivers in plain country the agreement between the observations and (115) is very good.

It must naturally be remembered that k itself varies during floods and that the same value of Q is known to correspond to different depths by rising and falling waters. No great exactitude can consequently be expected.

When we set up a velocity formula based on (115) and on the ordinary assumption that shearing stresses are uniformly distributed along the perimeter we find

$$(116) \qquad v_m = 8.83 \cdot \left[0.750 + \log\left(\frac{R}{c_k}\right)\right] \cdot \sqrt{g \cdot R \cdot I}\,.$$

This result is in good accordance with some of the better formulae for natural watercourses, e.g. those of Hermanek.

This question is further treated in Sec. 12.2.

In the following we therefore use the assumption (115) and (115a).

(115) is written as follows:

$$(115b) \qquad \frac{R}{k} = \left(\frac{c_k}{k}\right)^{\frac{1.56}{0.56}} = \left(\frac{c_k}{k}\right)^{2.785}$$

From Table 8.1/16/ we find

$$8.1(92) \qquad \frac{R}{y_{max}} = A_{r\varphi} \cdot A_{rp} \cdot p^{0.0210}$$

by multiplication of 8.1(92) by $p = \frac{y_{max}}{k}$ we get:

$$\boxed{(117)} \qquad \frac{R}{k} = A_{r\varphi} \cdot A_{rp} \cdot p^{1.0210}, \quad \text{where}$$

$$8.1(92c) \qquad A_{r\varphi} \cong 0.732 \quad \text{and}$$

$$8.1(92e) \qquad A_{rp} \cong 1.000$$

From (115b) and (117) we get

$$\boxed{(118)} \quad \frac{c_k}{k} = (A_{r\varphi} \cdot A_{rp})^{0.358974} \cdot p^{0.3665125}$$

By division of $p = \frac{y_{max}}{k}$ by (118) we get

$$\boxed{(119)} \quad \frac{y_{max}}{c_k} = (A_{r\varphi} \cdot A_{rp})^{-0.358974} \cdot p^{0.6334875}$$

Using (118) and (119) k and y_{max} are now expressed by p.

In Sec. 11 we found :

$$11.(106) \quad p \cdot 1.009114 \cdot \left[1 - 0.041283 \log^2 \left(\frac{p}{110.262}\right)\right]$$

$$= \frac{1}{k^{0.88404}} \cdot \left[\frac{Q}{2.5 \cdot \sqrt{\frac{\tau_{max}}{\varrho}} \cdot 3.1815 \cdot A_{f\varphi}}\right]^{0.44202}$$

(118) is raised to the power 0.88404 and multiplied by 11.(106) and, when solving with respect to p, results in :

$$(120) \quad p \cdot 1.009114^{1.479316} \cdot \left[1 - 0.041283 \log^2 \left(\frac{p}{110.262}\right)\right]^{1.479316}$$

$$= \left[\frac{Q \cdot (A_{r\varphi} \cdot A_{rp})^{0.717948}}{2.5 \cdot \sqrt{\frac{\tau_{max}}{\varrho}} \cdot c_k^2 \cdot 3.1815 \cdot A_{f\varphi}}\right]^{0.653887}$$

Introducing the values 8.1(92c) and 8.1(92e)

$A_{r\varphi} \cong 0.732$ and $A_{rp} \cong 1.000$ we get :

$$\boxed{(120a)} \quad p \cdot \left[1 - 0.06107 \log^2 \left(\frac{p}{110.262}\right)\right]$$

$$= 0.39987 \left[\frac{Q}{2.5 \cdot \sqrt{\frac{\tau_{max}}{\varrho}} \cdot c_k^2 \cdot A_{f\varphi}}\right]^{0.653887}$$

In (120a) p is expressed by known quantities. The parenthesis on the right side is dimensionless. Note that the variation of $A_{f\varphi}$ with φ is considerable.

For <u>rough calculations</u> we take for instance

$\varphi = 20^{\circ}$; $A_{f\varphi} = 5.3728$; $A_{f\varphi}^{0.653887} = 3.000$

and estimate $p \cong 110$; i.e. $\log^2 \cdot \frac{p}{110.262} \cong 0$.

We then get:

$$\boxed{(120b)} \quad p = 0.1333 \cdot \left[\frac{Q}{2.5 \cdot \sqrt{\frac{\tau_{max}}{\varrho}} \cdot c_k^{\,2}} \right]^{0.6539} \qquad (\varphi = 20^{o})$$

valid for every consistent system of units.

In the Metric System we have

$$\frac{1}{c_k^{\,1.3078}} = \frac{1}{0.425^{1.3078}} = \frac{1}{0.327} \qquad \text{and}$$

$$\boxed{(120c)} \quad p = 0.408 \cdot \left[\frac{Q}{2.5 \cdot \sqrt{\frac{\tau_{max}}{\varrho}}} \right]^{0.6539} \qquad (\varphi = 20^{o})$$

In the FPS-System we have

$$\frac{1}{c_k^{\,1.3078}} = \frac{1}{1.395^{1.3078}} = \frac{1}{1.545}$$

$$\boxed{(120d)} \quad p = 0.0863 \cdot \left[\frac{Q}{2.5 \cdot \sqrt{\frac{\tau_{max}}{\varrho}}} \right]^{0.6539} \qquad (\varphi = 20^{o})$$

From (118) we get

$$(118a) \quad \frac{k}{c_k} = (A_{r\varphi} \cdot A_{rp})^{-0.358974} \cdot p^{-0.3665125}$$

The equation (120) for p is raised to the power -0.3665125 and multiplied by (118a), whereby we get

$$(121) \quad \frac{k}{c_k} = \frac{\left[1.009114 \cdot \left[1 - 0.041283 \cdot \log^2 \left(\frac{p}{110.262} \right) \right] \right]^{0.542189}}{\left[\frac{(A_{r\varphi} \cdot A_{rp})^{0.531036}}{(3.1815\, A_{f\varphi})^{0.239658}} \right] \cdot \left[\frac{Q}{2.5 \cdot \sqrt{\frac{\tau_{max}}{\varrho}} \cdot c_k^{\,2}} \right]^{0.239658}}$$

In formula 11.(109a) we found :

$$11.(109a)\quad F = \frac{k^{0.13547}\cdot A_{f\varphi}^{\;0.067735}\cdot\left[\dfrac{Q}{2.5\cdot\sqrt{\dfrac{\tau_{max}}{\varrho}}}\right]^{0.932265}}{2.97877\cdot\left[1-0.055388\cdot\log^2\left(\frac{p}{111.55}\right)\right]}\;.$$

When (121) above is raised to the power 0.13547 and multiplied by 11.(109a) and observing that
$2 - 0.13547 = 1.86453 = 2\cdot 0.932265$ we get

$$(122)\quad \frac{F}{c_k^2} = \frac{A_{f\varphi}^{0.100201}\cdot\left[\dfrac{Q}{2.5\sqrt{\dfrac{\tau_{max}}{\varrho}}\cdot c_k^2}\right]^{0.899799}\cdot\left[1.009114\left[1-0.041283\log^2\left(\frac{p}{110.262}\right)\right]\right]^{0.073450}}{2.97877\cdot\left[A_{r\varphi}\cdot A_{rp}\right]^{0.071940}\cdot\left[1-0.055388\log^2\left(\frac{p}{111.55}\right)\right]}\;.$$

We put :

$$\left[1-0.041283\log^2\left(\frac{p}{110.262}\right)\right]^{0.073450} = 1-0.00303224\log^2\left(\frac{p}{110.262}\right)$$

and find by formulae 6.4(77 through 80a)

$$\frac{1-0.055388\log^2\left(\frac{p}{111.55}\right)}{1-0.00303224\log^2\left(\frac{p}{110.262}\right)} = 1-0.052356\log^2\left(\frac{p}{111.623}\right),$$

which result is inserted in the denominator of (122).

We further have

$A_{r\varphi} \cong 0.732 \quad , \quad A_{rp} \cong 1.000$

$(A_{r\varphi}\cdot A_{rp})^{0.071940} = 0.97780$

and find the new form of (122)

$$\boxed{(122a)}\quad \frac{F}{c_k^2} = \frac{A_{f\varphi}^{0.100201}\cdot\left[\dfrac{Q}{2.5\sqrt{\dfrac{\tau_{max}}{\varrho}}\cdot c_k^2}\right]^{0.899799}}{2.97877\cdot 0.97780\cdot\left[1-0.052356\log^2\left(\frac{p}{111.623}\right)\right]}\;.$$

The variation of the factor $A_{f\varphi}^{0.100201}$ with φ is only slight, and we get with good accuracy :

(123) $$A_{f\varphi}^{0.100201} = 1.0956 \cdot (\cot\varphi)^{0.0766}$$

With (123) we finally get from (122a)

$\boxed{(122b)}$ $$\frac{F}{c_k^2} = \frac{0.37615(\cot\varphi)^{0.0766} \cdot \left[\dfrac{Q}{2.5\sqrt{\dfrac{\tau_{max}}{\varrho}} \cdot c_k^2}\right]^{0.899799}}{1 - 0.052356 \cdot \log^2\left(\dfrac{p}{111.623}\right)}$$

It is clearly seen that the variation of F with φ is only slight.

For rougher calculations we put

$\varphi = 20^o$, $(\cot\varphi)^{0.0766} \cong 1.0812$ and

$p \sim 112$ i.e. $\log^2\left(\frac{p}{111.623}\right) \cong 0$

and get

$$\frac{F}{c_k^2} \cong 0.407 \cdot \left[\frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\varrho}} \cdot c_k^2}\right]^{0.9} \qquad (\varphi = 20^o)$$

$\boxed{(122c)}$ $$F \cong 0.407 \cdot c_k^{0.2} \cdot \left[\frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\varrho}}}\right]^{0.9}. \qquad (\varphi = 20^o)$$

This expression can be used for every consistent system of units.

In the Metric System

$c_k = 0.425$ m , $c_k^{0.2} = 0.8427$ ($m^{0.2}$)

$\boxed{(122d)}$ $$F \cong 0.343 \cdot \left[\frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\varrho}}}\right]^{0.9} \qquad (\varphi = 20^o)$$

In the FPS-System

$c_k = 1.395$ feet , $c_k^{0.2} = 1.069$ ($\text{feet}^{0.2}$)

$\boxed{(122e)}$ $$F \cong 0.435 \cdot \left[\frac{Q}{2.5\sqrt{\frac{\tau_{max}}{\varrho}}}\right]^{0.9} \qquad (\varphi = 20^o)$$

From (119) we get with

$$A_{r\varphi} \cong 0.732 \quad , \quad A_{rp} = 1.000$$

$$\boxed{(124)} \quad \frac{y_{max}}{c_k} = 1.11852\ p^{0.633488} \quad .$$

When p is found by one of the formulae (120a)b)c)d) y_{max} is directly found by (124) .

From Sec. 11. we have :

$$11.(108) \quad B = k \cdot A_{b\varphi} \cdot p^{1.0890} \cdot A_{bp} \cdot$$

We find by division by (118)

$$\boxed{(125)} \quad \frac{B}{c_k} = 1.11852 \cdot A_{b\varphi} \cdot p^{0.72249} \cdot A_{bp} \cdot$$

The variation of B with φ is considerable so that the correct values of $A_{b\varphi}$ should be taken from Table 8.1./17/, formula 8.1(93c).

The variation of A_{bp} will probably only be small (cf. formula 8.1(93c)).

By the successive use of the formulae

(120a)b)c)d) for $p = \frac{y_{max}}{k}$ and

(122a)b)c)d)e) for F and

(124) for y_{max} and

(125) for B

all the principal dimensions of the equilibrium profile can be found and the complete profile can be drawn using the tables of the co-ordinates η and ξ , Table 8.2/21/.

It must not be forgotten that Q, F and B refer to the half total cross section.

11.3. ILLUSTRATIVE EXAMPLE OF CALCULATION OF DIMENSIONS

The Metric System is used.

Given :

$Q = \frac{800}{2}$ m^3/sec = 400 m^3/sec for half cross section,

$\tau_{max} = 0.15 \text{ kg/m}^2$,

$\gamma = 1026 \text{ kg/m}^3$,

$\varphi = 20°$

and

k ~ as for natural watercourses.

For use in the formulae we calculate :

$$\frac{\tau_{max}}{\varrho} = \frac{\tau_{max}}{\gamma} \cdot g = \frac{0.15 \cdot 9.81}{1026} = 0.001435 \quad (m/sec)^2,$$

$$\sqrt{\frac{\tau_{max}}{\varrho}} = 0.03785 \text{ m/sec},$$

$$\frac{Q}{2.5 \cdot \sqrt{\frac{\tau_{max}}{\varrho}}} = \frac{400}{2.5 \cdot 0.03785} = 4225 \text{ m}^2$$

$$\left[\frac{Q}{2.5 \sqrt{\frac{\tau_{max}}{\varrho}}}\right]^{0.9} = 10^{3.6} \cdot 0.4225^{0.9} = 3980 \cdot 0\ 4606 = 1833$$

and further have

11.2(115a) $\quad c_k = 0.425$ m,

11.2(122d) $\quad F = 0.343 \cdot 1833 = 628 \text{ m}^2$

Total area : $\underline{\underline{2F = 1256 \text{ m}^2}}$

$$\left[\frac{Q}{2.5 \sqrt{\frac{\tau_{max}}{\varrho}}}\right]^{0.6539} = 10^{2.6156} \cdot 0.4225^{0.6539} = 412 \cdot 0.5700 = 234.5$$

11.2(120c) $\quad p = \frac{y_{max}}{k} = 0.408 \cdot 234.5 = 95.7$

$$p^{0.6335} = 95.7^{0.6335} = 17.95$$

11.2(124) $\quad \underline{\underline{y_{max}}} = 0.425 \cdot 1.1185 \cdot 17.95 = \underline{\underline{8.53 \text{ m}}}$

$$k = \frac{y_{max}}{p} = \frac{8.53}{95.7} = 0.0891 \text{ m.}$$

$$p^{0.7225} = 95.7^{0.7225} = 27.00.$$

From Table 8.1/17/ we get

$A_{b\varphi} = 7.2375$, and 8.1(93e) $A_{bp} = 1.004781 - 0.02121 \log^2 (\frac{p}{106.46})$

$\frac{106.46}{95.7} = 1.1125$; $\log 1.1125 = 0.0462$; $\log^2 1.1125 = 0.002135$.

$A_{bp} = 1.004781 - 0.000045 = 1.004736$

11.2(125) $B = 0.425 \cdot 1.1185 \cdot 7.2375 \cdot 27.0 \cdot 1.0047 = 93.2$ m

Width of water surface of total section : $2B = 186.4$ m.

12. FORMULAE FOR THE MEAN VELOCITY v_m IN EQUILIBRIUM PROFILES.

12.1 v_m AS A FUNCTION OF HYDRAULIC RADIUS R. FIXED VALUE OF k.

From Table 8.1/16/ we have

8.1(92)c)d) $$R' = \frac{R}{y_{max}} = 0.732 \cdot p^{0.210} ,$$

$$\frac{R}{k} = 0.732 \cdot p^{1.210}$$

or

(126) $$p = \left(\frac{R}{0.732\, k} \right)^{\frac{1}{1.210}}$$

From Table 8.1/19/ we have

8.1(95)c)d) $$v_*' = \sqrt{\frac{\frac{\tau_{max}}{\varrho}}{g R I}} = 1.160 \cdot p^{0.0100}$$

or

(127) $$\sqrt{\frac{\tau_{max}}{\varrho}} = 1.160\, p^{0.01} \cdot \sqrt{g R I}\, .$$

From Table 8.1/15/ we have

8.1(91b)c)d) $$v_m' = \frac{v_m}{2.5 \cdot \sqrt{\frac{\tau_{max}}{\varrho}}} = 3.1815 \cdot p^{0.15324} \cdot A_{vp} ,$$

where

8.1(91e) $$A_{vp} = 1.0135645 - 0.0634373 \log^2 \left(\frac{p}{111.428} \right) .$$

$$v_m = 2.5 \cdot 3.1815 \cdot 1.160 \cdot p^{0.01} \cdot \sqrt{g\,R\,I} \cdot p^{0.15324} \cdot A_{vp}$$

or

(128) $$v_m = 2.5 \cdot 3.1815 \cdot 1.160 \cdot p^{0.16324} \cdot \sqrt{g\,R\,I} \cdot A_{vp}\,.$$

For the calculation of A_{vp} we find by means of (126)

$$\log\left(\frac{p}{111.428}\right) = \log\left[\frac{1}{111.428}\cdot\left(\frac{R}{0.732\,k}\right)^{\frac{1}{1.210}}\right]$$

$$= \log\left[\left(\frac{R}{0.732 \cdot 300\,k}\right)^{\frac{1}{1.210}}\right] = \frac{1}{1.210}\cdot\log\left(\frac{R}{219.6\,k}\right),$$

from which

$$\log^2\left(\frac{p}{111.428}\right) = \frac{1}{1.4641}\cdot\log^2\left(\frac{R}{219.6\,k}\right)$$

and

$$0.0634373 \cdot \log^2\left(\frac{p}{111.428}\right) = 0.043329 \cdot \log^2\left(\frac{R}{219.6\,k}\right),$$

which inserted in 8.1(91e) gives

(129) $$A_{vp} = 1.013565 - 0.043329 \log^2\left(\frac{R}{219.6\,k}\right).$$

We further have from (126):

(130) $$p^{0.16324} = \left(\frac{R}{0.732\,k}\right)^{\frac{0.16324}{1.210}} = \frac{1}{0.732^{0.134909}}\left(\frac{R}{k}\right)^{0.134909}$$

Inserting (130) in (128) we find

$$v_m = \frac{2.5 \cdot 3.1815 \cdot 1.160}{0.732^{0.134909}}\cdot\left(\frac{R}{k}\right)^{0.134909}\cdot\sqrt{g\,R\,I}\cdot A_{vp}\,,$$

where A_{vp} is taken from (129) and

$$0.732^{0.134909} \cong 0.9588.$$

We finally get

(131a) $$v_m = 9.6228 \cdot \left(\frac{R}{k}\right)^{0.134909}\cdot\sqrt{g\,R\,I}\cdot\left[1.013565 - 0.043329 \log^2\left(\frac{R}{219.6\,k}\right)\right]$$

or

$\boxed{(131b)}$ v_m

$$= \left[\frac{9.6228 \cdot \sqrt{g}}{k^{0.134909}}\right] \cdot R^{0.634909} \cdot I^{1/2} \cdot \left[1.013565 - 0.043329 \log^2\left(\frac{R}{219.6k}\right)\right]$$

(131a)b) is valid for every consistent system of units.

In the Metric System we have

$$\sqrt{g} = \sqrt{9.81} = 3.1321 \ m^{1/2}/\sec ,$$

which gives

$$(131c) \quad v_m = \frac{30.14}{k^{0.13491}} \cdot R^{0.6349} \cdot I^{1/2} \cdot \left[1.0136 - 0.0433 \cdot \log^2\left(\frac{R}{219.6k}\right)\right]$$

Herein the last parenthesis (i.e. A_{vp}) is practically always near unity.

12.2 v_m AS A FUNCTION OF HYDRAULIC RADIUS R, ASSUMING k AS FOR <u>NATURAL WATERCOURSES.</u>

From 11.2(115)a), valid for natural watercourses, we have

$$\left(\frac{R}{k}\right) = \left(\frac{R}{c_k}\right)^{1.56}$$

where

$$11.2(115a) \quad c_k = 0.425 \text{ meters or } 1.395 \text{ feet.}$$

This is inserted in 12.1(131a) and we get

$$(132) \qquad v_m = 9.6228 \cdot \left(\frac{R}{c_k}\right)^{0.21046} \cdot \sqrt{g\,R\,I} \cdot A_{vp}$$

where

$$(132a) \qquad A_{vp} = 1.013565 - 0.043329 \log^2\left[\frac{1}{219.6}\left(\frac{R}{c_k}\right)^{1.56}\right] .$$

We have

$$\log\left[\frac{1}{219.6}\left(\frac{R}{c_k}\right)^{1.56}\right] = \log\left[\left(\frac{R}{31.7\,c_k}\right)^{1.56}\right] = 1.56 \log\left(\frac{R}{31.7\,c_k}\right)$$

and

$$\log^2\left[\frac{1}{219.6} \cdot \left(\frac{R}{c_k}\right)^{1.56}\right] = 2.4336 \cdot \log^2\left(\frac{R}{31.7\,c_k}\right) ,$$

which gives

$$(132b) \qquad A_{vp} = 1.013565 - 0.105445 \log^2\left(\frac{R}{31.7\,c_k}\right).$$

Inserting (132b) in (132) we finally get

(133) $v_m = 9.6228\left(\frac{R}{c_k}\right)^{0.21046} \cdot \sqrt{g\,R\,I} \cdot \left[1.013565 - 0.105445 \log^2\left(\frac{R}{31.7\,c_k}\right)\right]$

or

$\boxed{(133a)}$ v_m

$$= \left[\frac{9.6228 \cdot \sqrt{g}}{c_k^{0.21046}}\right] \cdot R^{0.71046} \cdot I^{1/2} \cdot \left[1.013565 - 0.105445 \log^2\left(\frac{R}{31.7\,c_k}\right)\right]$$

which formulae are valid for an arbitrary consistent system of units.

In the Metric System we have

$$\sqrt{g} = \sqrt{9.81} = 3.1321 \ m^{1/2}/\mathrm{sec},$$

$$c_k = 0.425 \ m \quad ; \quad 31.7 \cdot c_k = 13.48,$$

$$c_k^{0.21046} = 0.425^{0.21046} = 0.8350$$

and (133a) becomes

(133b) $v_m = 36.0953 \cdot R^{0.71046} \cdot I^{1/2} \cdot \left[1.01357 - 0.105445 \cdot \log^2\left(\frac{R}{13.48}\right)\right].$

This formula has much resemblance to the empirical formulae of J. Hermanek.

Hermanek gives for $R \cong$ mean depth of section and $1.5\ m < R < 6.0\ m$ the following formula

(134) $v_m = 34 \cdot R^{0.75} \cdot I^{1/2}$.

The last factor in (133b) will be near unity for such values of R.

12.3 v_m AS A FUNCTION OF y_{max}. FIXED VALUE OF k.

If we want to express the mean velocity by y_{max} instead of by R we insert R from 8.1(92) in 12.1(128) and get

$$v_m = 2.5 \cdot 3.1815 \cdot 1.160 \cdot 0.732^{0.5} \cdot p^{0.17374} \cdot \sqrt{g\,y_{max}\,I} \cdot A_{vp}$$

(135) $v_m = 7.89379 \cdot \left(\frac{y_{max}}{k}\right)^{0.17374} \cdot \sqrt{g\,y_{max} \cdot I} \cdot A_{vp}$,

where

(135a) $A_{vp} = 1.013565 - 0.063437 \cdot \log^2\left(\frac{y_{max}}{111.43\,k}\right)$

or

$$\boxed{(135b)} \quad v_m = \left[\frac{7.89379 \cdot \sqrt{g}}{k^{0.17374}}\right] y_{max}^{0.67374} \cdot I^{1/2} \cdot A_{vp} \,.$$

(135)a)b) are valid for an arbitrary consistent system of units.

In the Metric System we have

$$\sqrt{g} = \sqrt{9.81} = 3.1321 \; m^{1/2}/sec$$

and

$$(135c) \quad v_m = \left[\frac{24.7241}{k^{0.17374}}\right] \cdot y_{max}^{0.67374} \cdot I^{1/2} \cdot A_{vp} \,.$$

A_{vp} will generally be very near unity and the similarity between (135c) and the Manning formula

$$v_m = \left(\frac{25.84}{\sqrt[6]{k}}\right) \cdot R^{2/3} \cdot I^{1/2}$$

becomes striking; it must, however, be remembered that (135c) contains y_{max} and not R.

13. THE RATIO OF MEAN VELOCITY v_m TO MAXIMUM VELOCITY v_{max}.

13.1 $\frac{v_m}{v_{max}}$ AS A FUNCTION OF $p = \frac{y_{max}}{k}$

From Table 8.1/15/ we have

$$8.1(91b)c)d) \quad \frac{v_m}{2.5\sqrt{\frac{\tau_{max}}{\varrho}}} = 3.1815 \cdot p^{0.15324} \cdot A_{vp}$$

where

$$8.1(91e) \quad A_{vp} = 1.013565 - 0.063437 \log^2 \left(\frac{p}{111.428}\right) .$$

For the maximum velocity v_{max} which occurs at the surface where the depth is y_{max} we have directly

$$(136) \quad \frac{v_{max}}{2.5\sqrt{\frac{\tau_{max}}{\varrho}}} = 3.392 + \ln \frac{y_{max}}{k} = 3.392 \left[1 + 0.67883 \cdot \log p\right]$$

We divide 8.1(91b) by (136) and get

$$(137) \quad \frac{v_m}{v_{max}} = \frac{0.950665 \cdot p^{0.15324} \cdot \left[1 - 0.062588 \log^2 \left(\frac{p}{111.428}\right)\right]}{1 + 0.67883 \log p}$$

The ratio of mean to maximum velocities $\frac{v_m}{v_{max}}$ is consequently a function of p alone and quite independent of φ.

The denominator in (137)

$$1 + 0.67883 \log p = \frac{1}{3.392} \cdot \frac{v_{max}}{2.5\sqrt{\frac{\tau_{max}}{\rho}}}$$

can be developed in a form similar to that of the numerator in (137).

The calculation is made in the same way as that used for the functions A_p in Sec. 6.3.

We find as a first approximation

(138a) $$1 + 0.67883 \log p = 1.31155\, p^{0.12535} \pm 0.925\ \%$$
for $24.2 \leq p \leq 500$.

The logarithmic deviations e_p from the simple power formula are then treated as indicated in Sec. 6.3 and we find

(138b) $$1 + 0.67883 \log p$$
$$= 1.323712 \cdot p^{0.12535} \cdot \left[1 - 0.041934 \log^2 \left(\frac{p}{109.277}\right)\right] \pm 0.104\ \%$$
for $\sim 18 < p < \sim 950$.

Inserting (138b) in (137) we get

(137a) $$\frac{v_m}{v_{max}} = \frac{0.718181 \cdot p^{0.02789} \cdot \left[1 - 0.062588 \log^2 \left(\frac{p}{111.428}\right)\right]}{\left[1 - 0.041934 \log^2 \left(\frac{p}{109.277}\right)\right]}$$

The parentheses are divided by means of the formulae 6.4(77 through 80a) and we finally get the following form of (137a)

$\boxed{(137b)}$ $$\frac{v_m}{v_{max}} = 0.718187 \cdot p^{0.02789} \cdot \left[1 - 0.020654 \log^2 \left(\frac{p}{117.230}\right)\right]$$

13.2 DEGREE OF FULLNES $\varkappa$ OF THE CROSS SECTION OF EQUILIBRIUM PROFILES.

We define $\varkappa$ as follows

4.5(58) $$\varkappa = \frac{F}{B \cdot y_{max}} = \frac{\left(\frac{F}{y_{max}^2}\right)}{\left(\frac{B}{y_{max}}\right)} = \frac{F'}{B'} = \frac{F'}{\beta+\beta_0}$$

From Tables 8.1/14/ and 8.1/17/ we get

(90b)d) $F' = A_{f\varphi} \cdot p^{0.1091} \cdot A_{fp}$

(93b)d) $B' = A_{b\varphi} \cdot p^{0.0890} \cdot A_{bp}$

and by division

(139) $$\varkappa = \frac{F'}{B'} = \frac{A_{f\varphi}}{A_{b\varphi}} \cdot p^{0.0201} \cdot \frac{A_{fp}}{A_{bp}} .$$

For the ratio $\frac{A_{f\varphi}}{A_{b\varphi}}$ we get from Tables 8.1/14/17/ the following values

φ	15°	20°	25°	30°
$\frac{A_{f\varphi}}{A_{b\varphi}}$	0.739031	0.742356	0.745156	0.747759

The variation with φ is so small that we take as an average

(139a) $\frac{A_{f\varphi}}{A_{b\varphi}} = 0.7424 = \text{constant}$, nearly corresponding to $\varphi = 20^\circ$.

For A_{fp} and A_{bp} we get from Tables 8.1/14/17/

(90e) $$A_{fp} = 1.007083 \cdot \left[1 - 0.0316826 \log^2 \left(\frac{p}{108.035}\right)\right]$$

(93e) $$A_{bp} = 1.004781 \cdot \left[1 - 0.0211078 \log^2 \left(\frac{p}{106.46}\right)\right]$$

The ratio of the two parentheses is found by means of the formulae 6.4(77 through 80a). We finally get

(139b) $$\frac{A_{fp}}{A_{bp}} = 1.002290 \cdot \left[1 - 0.0105748 \log^2 \left(\frac{p}{111.245}\right)\right]$$

and by inserting (139a) and (139b) (139) takes the following form

$\boxed{(139c)}$ $$\varkappa = 0.7441 \cdot p^{0.0201} \cdot \left[1 - 0.010575 \log^2 \left(\frac{p}{111.245}\right)\right] .$$

13.3 RELATION BETWEEN $\frac{v_m}{v_{max}}$ AND $\varkappa$.

We found

13.1(137b) $\frac{v_m}{v_{max}} = 0.718187 \cdot p^{0.02789} \cdot \left[1 - 0.020654 \log^2\left(\frac{p}{117.230}\right)\right]$

13.2 (139c) $\varkappa = 0.7441 \cdot p^{0.0201} \cdot \left[1 - 0.010575 \log^2\left(\frac{p}{111.245}\right)\right].$

These equations represent $\frac{v_m}{v_{max}}$ and $\varkappa$ as functions of $p = \frac{y_{max}}{k}$; the relationship between $\frac{v_m}{v_{max}}$ and $\varkappa$ is given by Fig. 11 with the parameter p on the curve and the figure can therefore also be used for the determination of p and thereby of k when one of the other two variables is known.

In order to eliminate p between 13.1(137b) and 13.2(139c) by raising the latter to power 1.38756 and dividing we find

$$(140) \quad \frac{\left(\frac{v_m}{v_{max}}\right)}{\varkappa^{1.38756}} = 1.082324 \cdot \frac{1 - 0.020654 \log^2\left(\frac{p}{117.230}\right)}{1 - 0.010575 \log^2\left(\frac{p}{111.245}\right)}$$

In (140) the fraction on the right side is treated according to the formulae of Sec. 6.4 and we get

$$(140a) \quad \frac{\left(\frac{v_m}{v_{max}}\right)}{\varkappa^{1.38756}} = 1.082352 \cdot \left[1 - 0.005981 \log^2\left(\frac{p}{133.321}\right)\right].$$

To eliminate p in (140a) we use

$$13.2(139c) \quad \varkappa = 0.7441 \cdot p^{0.0201} \cdot \left[1 - 0.010575 \log^2\left(\frac{p}{111.245}\right)\right].$$

Taking logarithms in 13.2(139c) and noting that the parenthesis does not deviate much from unity we find

$$(141) \quad \left[\log\left(\frac{p}{111.245}\right)\right]\left[1 - 0.22851 \log\left(\frac{p}{111.245}\right)\right] = 49.751 \log(1.2225\varkappa).$$

We substitute

$$(142) \quad \varepsilon = 1.2225\varkappa - 1$$

and noting that ε will always be small compared to unity, (141) becomes

$$\left[\log\left(\frac{p}{111.245}\right)\right] \cdot \left[1 - 0.22851 \log\left(\frac{p}{111.245}\right)\right] = 21.606 \cdot \varepsilon.$$

Solving this quadratic equation we find

$$\log\left(\frac{p}{111.245}\right) = 2.1881 \cdot \left[1 - \sqrt{1 - 19.479\varepsilon}\right] \qquad \text{or}$$

(141a) $$\log\left(\frac{p}{133.321}\right) = 2.1096 \cdot \left[1 - 1.0372 \cdot \sqrt{1 - 19.479\varepsilon}\right].$$

(141a) is inserted in (140a) and we finally find

$\boxed{(140b)}$ $$\frac{v_m}{v_{max}} = 1.02255 \cdot \varkappa^{1.38756} \cdot \left[1 + 0.5904\varepsilon + 0.05845\sqrt{1 - 19.479\varepsilon}\right]$$

where ε is taken from (142).

(140b) gives directly the relation between $\varkappa$ and $\frac{v_m}{v_{max}}$.
We find for instance

$\varkappa$	$\frac{v_m}{v_{max}}$
0.77	0.7478
0.80	0.7929
0.84	0.8479

It must be remembered that (140b) is exclusively valid for equilibrium profiles. For other channels, e.g. with fixed bottoms, no definite relation between $\varkappa$ and $p = \frac{y_{max}}{k}$ can exist and $\frac{v_m}{v_{max}}$ can vary within considerably wider limits.

For an equilibrium profile it will consequently be possible to determine the mean velocity by measuring the maximum velocity and sounding the complete cross section of the channel, i.e. determining $\varkappa$.

This should be an advantage in tidal channels, where the discharge cannot be considered steady for a time sufficient to carry out velocity measurements at a sufficient number of points of the cross section.

$\varkappa$ depends not alone on the area of the section and its maximum depth which will probably both be well defined, but also on the width of the water surface.

This width will e.g. for natural streams often be influenced by vegetation on the banks and thus be narrower than for the equilibrium profile, even if the cross section as a whole is near the equilibrium profile. The area can then be considered correct and the width of that equilibrium profile which as an average gives the best adjustment to the profile in question can be used by the determination of $\varkappa$. This is what has been done in Sec. 10 by the study of the Vienna model tests.

SEC. 14. CONCLUSIONS AND SUGGESTIONS FOR FURTHER STUDIES.

The topics studied in this paper would be of interest for the problem of stable channels in general, but they do not cover the whole problem.

The shape of the profile which is stable against erosion is found, but only provided that no transport of bed load occurs; neither has the question of silting up of the channel been taken into account.

It is hoped however that the present study will be useful if the problem in all its aspects is to be solved.

The whole study is carried through theoretically on certain assumptions which are clearly stated, and further work in this field should try better to confirm or to correct these assumptions by direct control or indirectly by comparing the results with actual observations.

The most important assumptions made are:

1) Distribution of τ according to formula 1.(1) or 4.2(1a).
2) Hydrodynamic lift L proportional to τ, 2.1(3) and 2.2(9) based on the work of Einstein [1].
3) Logarithmic velocity distribution as indicated in Sec.1; mean velocity in an infinitesimal wedge-shaped element of area assumed equal to mean velocity in normal to bottom at the base of elements, Sec.4.2.
4) Uniform distribution of roughness k, Sec. 4.2.
5) Constant value of angle of internal friction φ along perimeter.
6) Principle of minimum area of cross section, Sec. 4.4.
7) For some of the formulae for the dimensions of the channel and for the mean velocity an empirical relation between hydraulic roughness k and hydraulic radius R, as found for natural watercourses, has been introduced. Thus formulae not containing k have been established.

The isovels studied in Sec. 9 and the example presented in Fig.7 seem to some degree to confirm assumption 1) concerning the distribution of τ.

That this choice is better than the usual assumption taking τ proportional to the depth, seems evident, but the question could be studied further by other methods, and the result possibly be improved; it seems probable however that such further studies will be considerably facilitated when an approximately correct solution is at hand.

Assumption 2) concerning hydrodynamic lift should also receive closer study and the size of the proportionality factor $c = 4.9$ might probably be improved. It seems clear however that this assumption is better than the complete disregard of the hydrodynamic lift, and the large size of c may perhaps explain why the angle φ is found to be so small as is the case (see below).

The logarithmic velocity distribution accepted is probably so well established that it does not need further comments, but the hydraulic roughness in real profiles must obviously be expected to vary along the bottom; the k introduced must therefore be taken as an average value.

As to the principle of minimum of area it will hardly be possible to give any strict proof of this assumption, but it almost seems selfevident and corresponds completely to many similar cases in hydraulics. A comparison of results with observations from nature seems desirable.

The values of φ have been determined for three model tests carried out in Vienna, 1916, giving $\varphi = 19^{\circ}$, 17° and 14° as the most probable values for these tests. The values are even lower than the expected value of about 20°, and it seems highly desirable that further observations should be made. Time has not allowed independent model tests to be carried out, but this ought to be done. Further it seems probable that more insight into the size of φ might be obtained by three-axial tests with sand where the hydrodynamic lift is replaced by an upward stream of water in the cylinder. By such tests the effective grain to grain normal stress could be reduced even to zero. This can possibly also be effected by other means, e.g. using a liquid with a specific weight greater than that of water.

A correct value of φ can thus be found for effective stresses smaller than those corresponding to the weight of the grains proper.

The fundamental results of the study are given in the tables in Sec. 8.1 All quantities here are dimensionless and we have succeeded in expressing Q', F', v_m', R', B', I' and v_x' by the product of two functions, one depending on φ alone and another depending on $p = \frac{y_{max}}{k}$ alone.

The limits of validity and the corresponding maximum deviation of these formulae are given. For many practical purposes the simple power formulae without the correcting factors A_p should probably suffice; but for the application of the formulae for model tests, e.g. where $p = y_{max}/k$ is small, the complete formulae including A_p should be used.

For all derivations in this paper the number of digits in the constants of the formulae might seem exaggerated for practical purposes; considering however the great number of numerical operations that have been needed in deriving the formulae, it has been judged appropriate to retain a surplus number of digits in all derivations, and the values of the constants can then finally be rounded off.

The form of the formulae is such that all practical calculations can easily be carried out by means of a log log slide rule.

The result found in Sec. 11, viz. that the area F of the half cross section of the equilibrium profile varies only slightly with the angle of friction φ and with the relative roughness, but mainly depends solely on the discharge Q, the limiting tractive force τ_{max} and the hydraulic roughness k, might facilitate the comparison of results with observations in nature.

Where the flow in the channel is caused by tides, the use of the results in Sec. 13 might facilitate the observation of Q.

Until more definite knowledge about the roughness k is acquired, the results in Sec. 11.2, where k is taken to be equal to the empirically determined roughness of natural watercourses, might be of use. The results thus obtained should, as far as possible, be compared with observations in nature.

The formulae for the mean velocity found in Sec. 13 also make it possible to find k from available observations.

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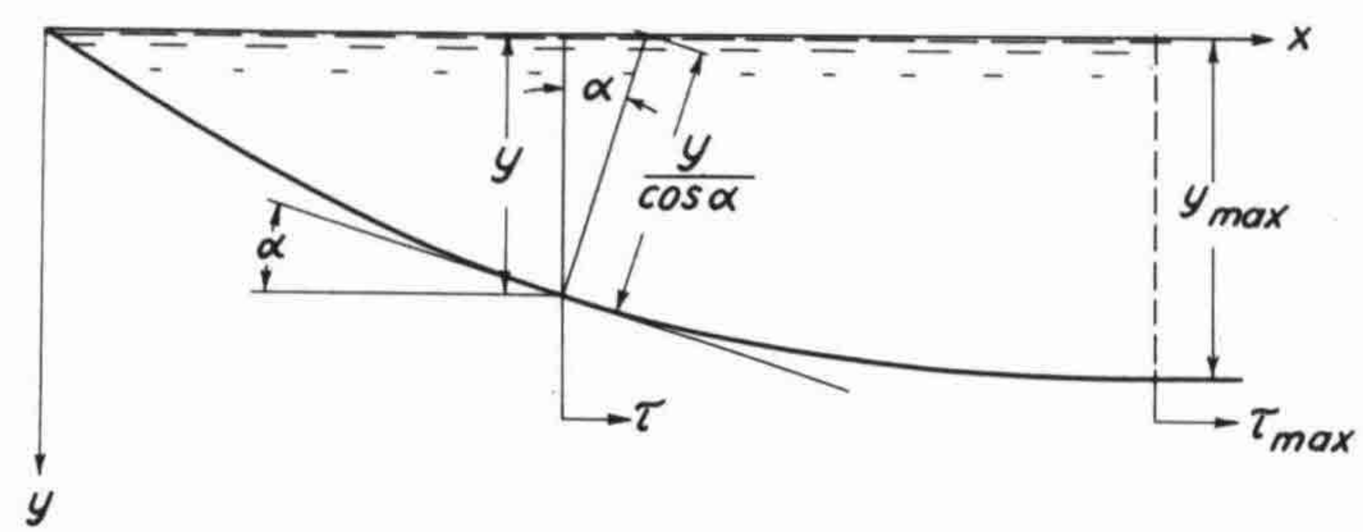


Fig. 1

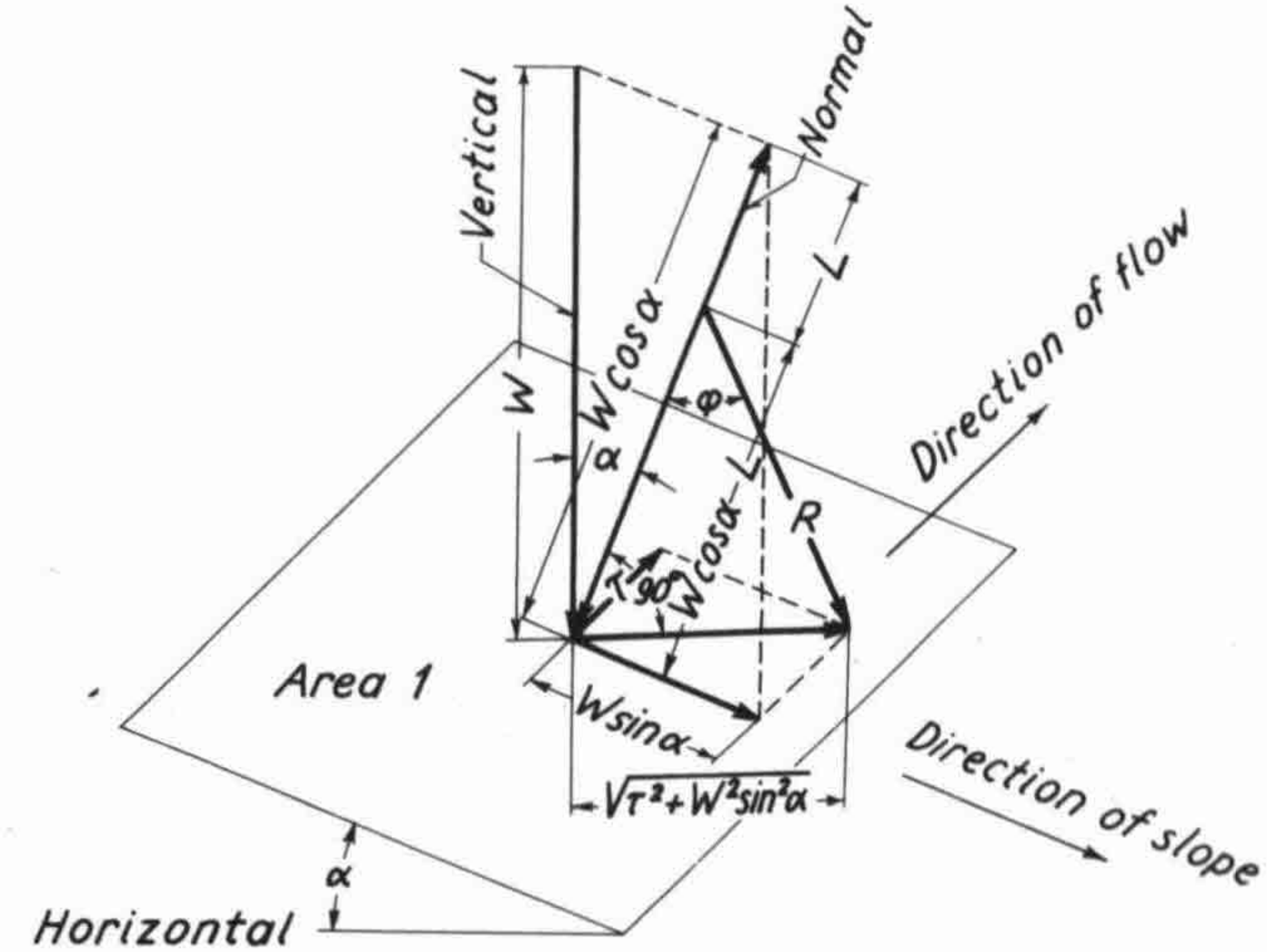


Fig. 2

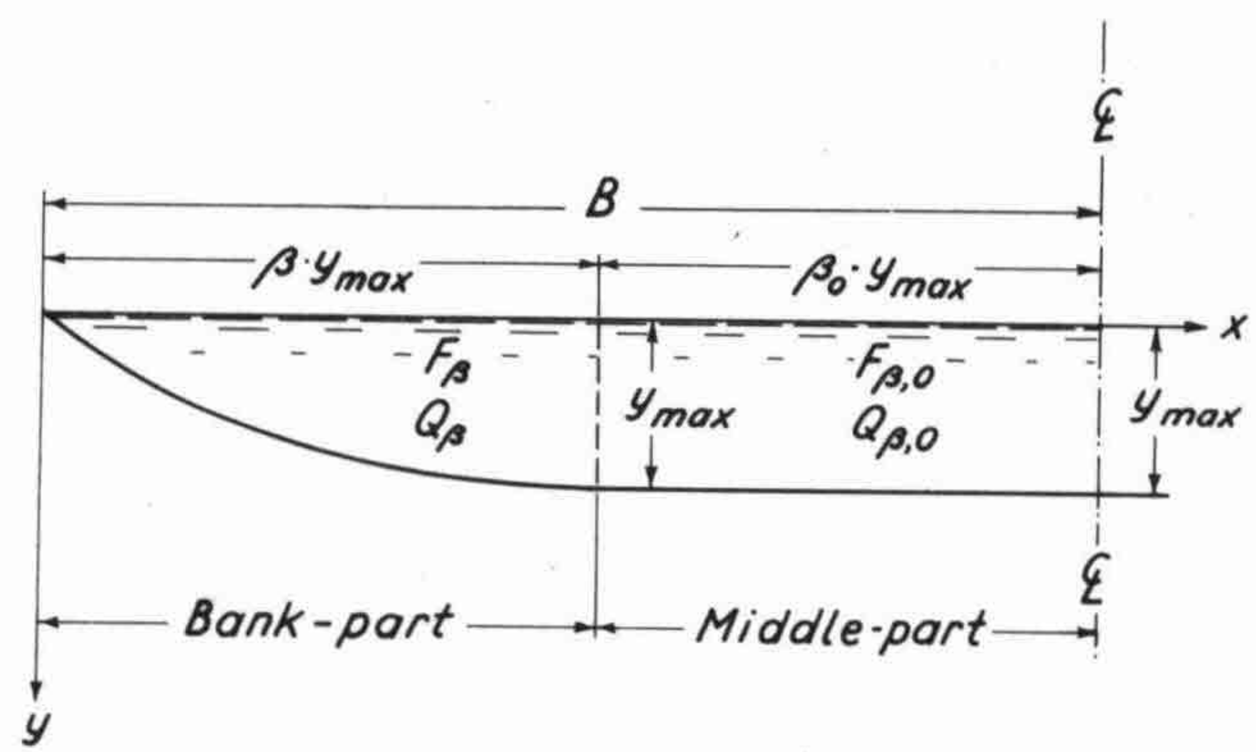


Fig. 3

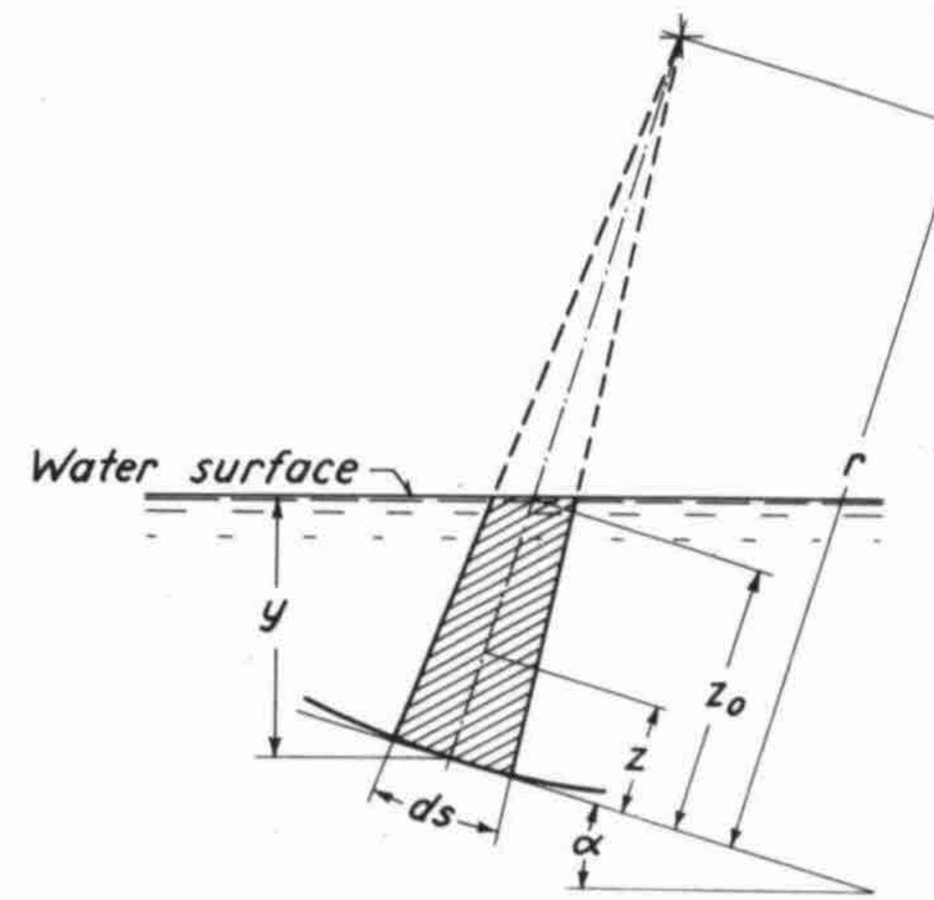
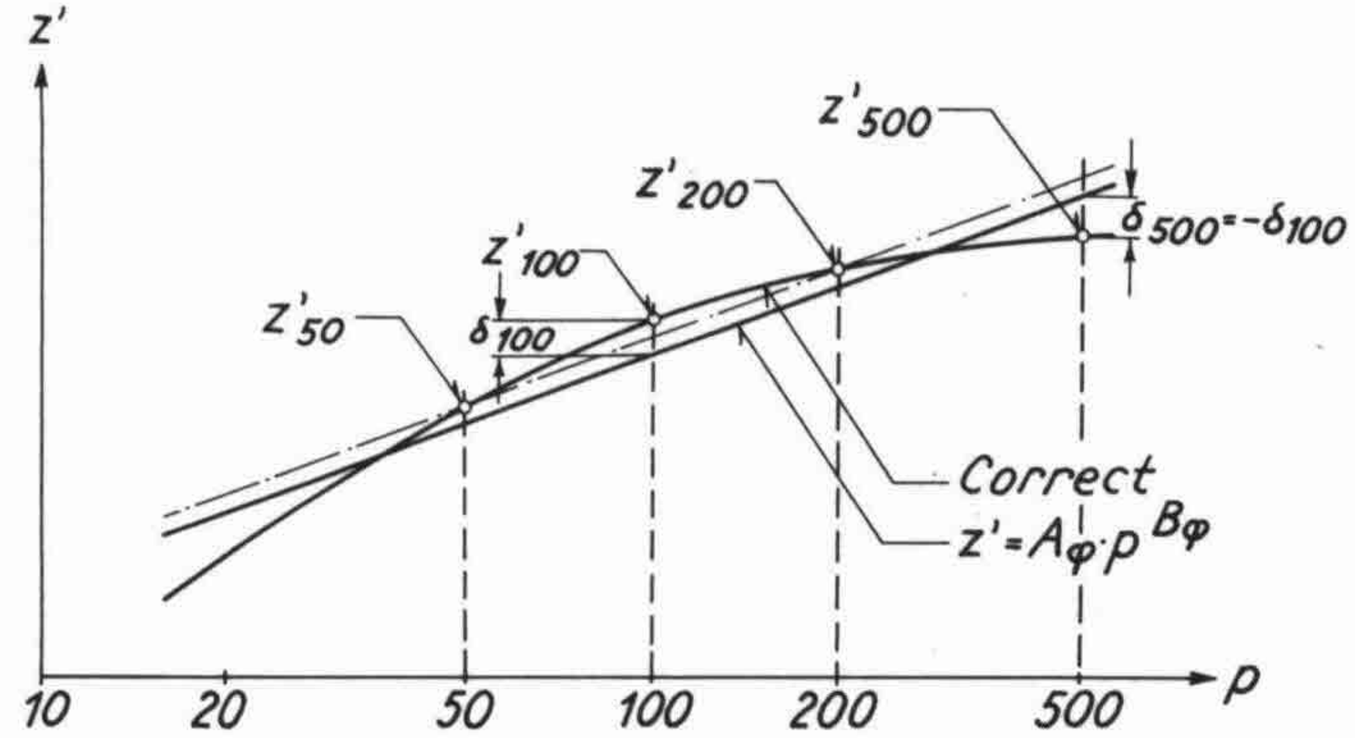
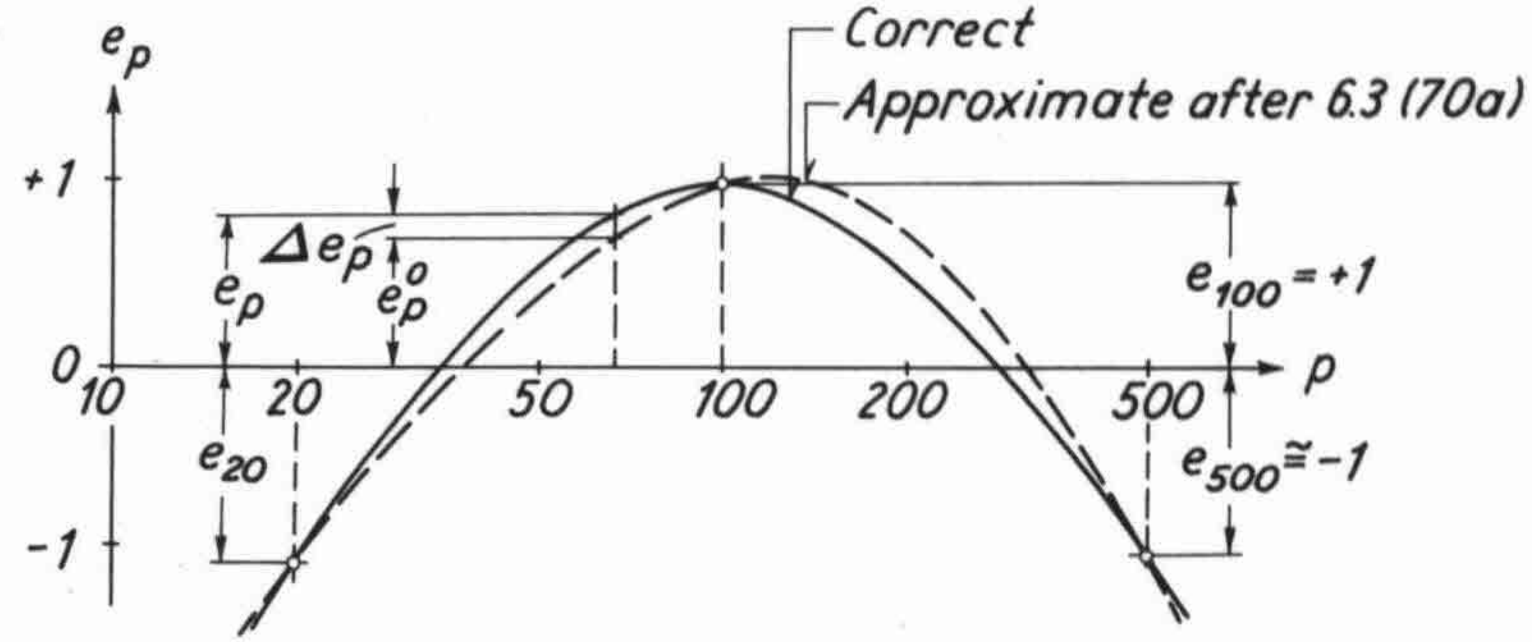


Fig. 4



Double logarithmic paper

Fig. 5



Semi - logarithmic paper

Fig. 6

Isovels

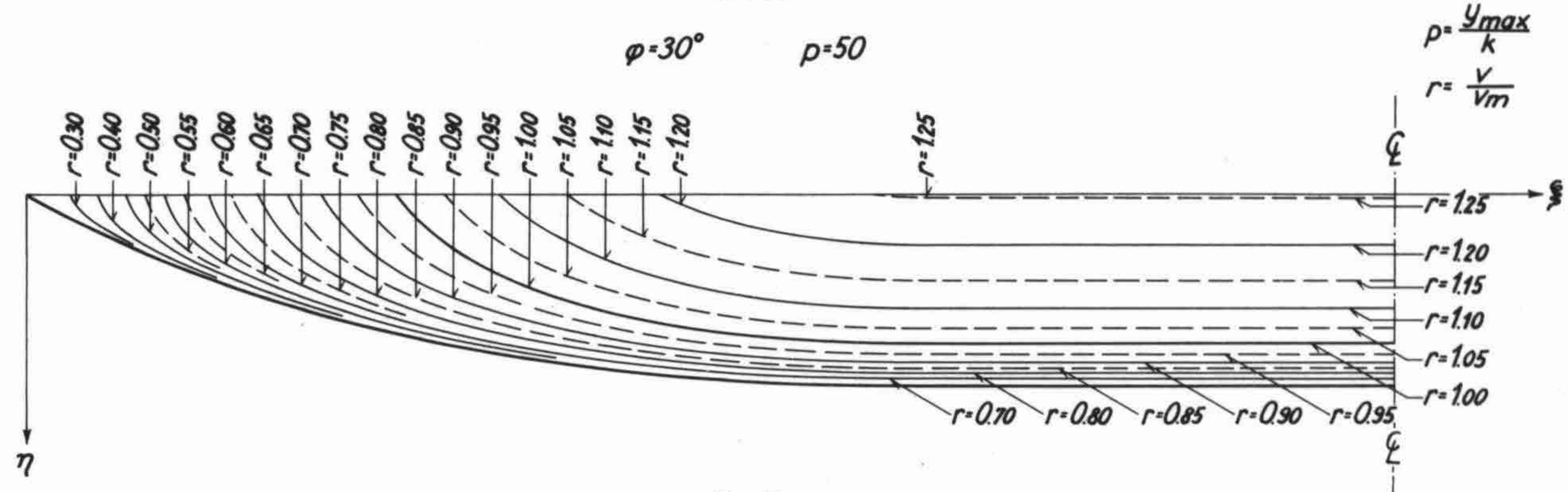


Fig. 7

$\xi = \frac{x}{y_{max,calc.}}$

$\eta = \frac{y}{y_{max,calc.}}$

—•— Observed
— — Calculated

TEST NO. 1

$\varphi = 19°$
$\gamma = 1 \, gr/cm^3$
$\tau_{max} = 0.0075 \, gr/cm^2$
$Q = 6900 \, cm^3/sec$
$F = 223.1 \, cm^2$

		Observed	Calcul.
y_{max}	cm	5.6	5.52
B	cm	50	52.06
I		135×10^{-5}	137.4×10^{-5}
k	cm		0.407

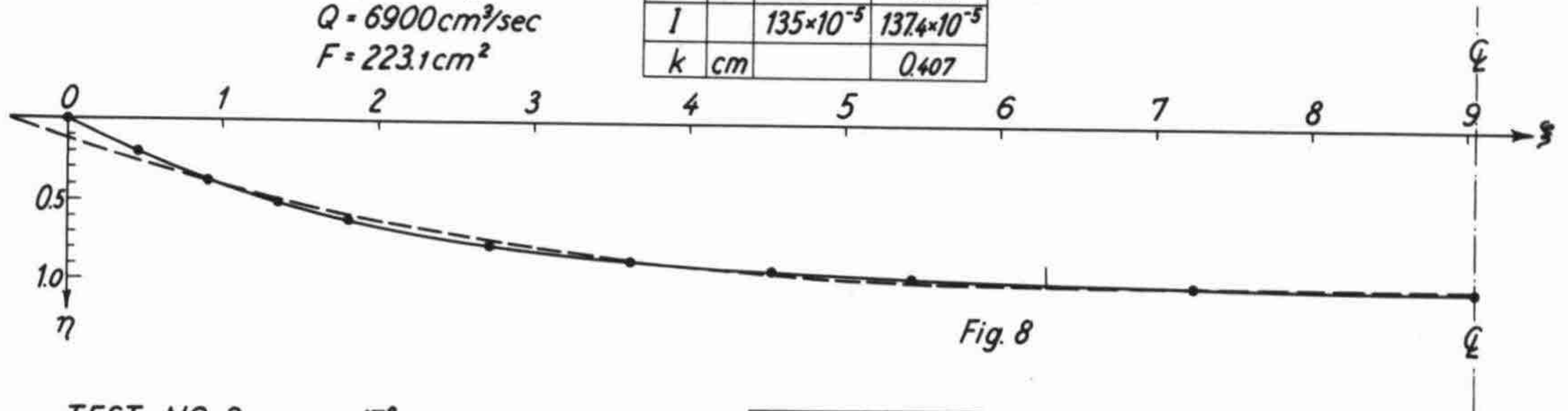


Fig. 8

TEST NO. 2

$\varphi = 17°$
$\gamma = 1 \, gr/cm^3$
$\tau_{max} = 0.0075 \, gr/cm^2$
$Q = 26750 \, cm^3/sec$
$F = 722.0 \, cm^2$

		Observed	Calcul.
y_{max}	cm	9.0	8.92
B	cm	99	101.83
I		79×10^{-5}	84.8×10^{-5}
k	cm		0.257

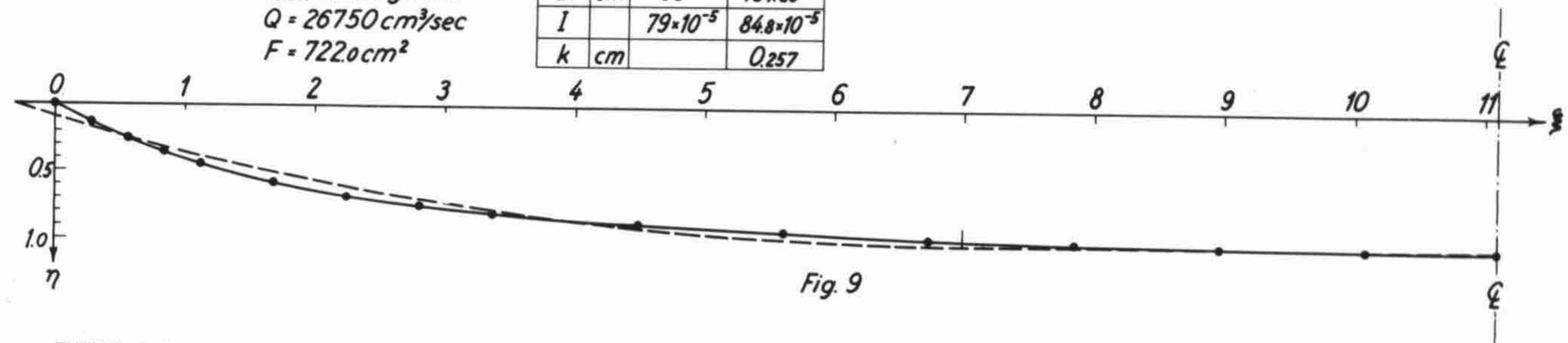


Fig. 9

TEST NO. 3

$\varphi = 14°$
$\gamma = 1 \, gr/cm^3$
$\tau_{max} = 0.0075 \, gr/cm^2$
$Q = 43500 \, cm^3/sec$
$F = 1237.5 \, cm^2$

		Observed	Calcul.
y_{max}	cm	10.5	10.94
B	cm	140	143.76
I		65×10^{-5}	68.9×10^{-5}
k	cm		0.402

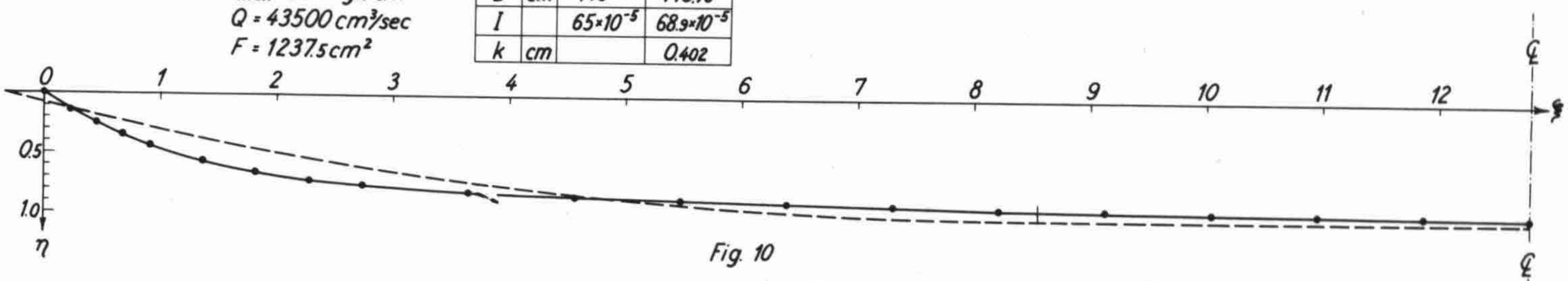


Fig. 10

$$p = \frac{y_{max}}{k}$$

$$\varkappa = \frac{F}{B \cdot y_{max}}$$

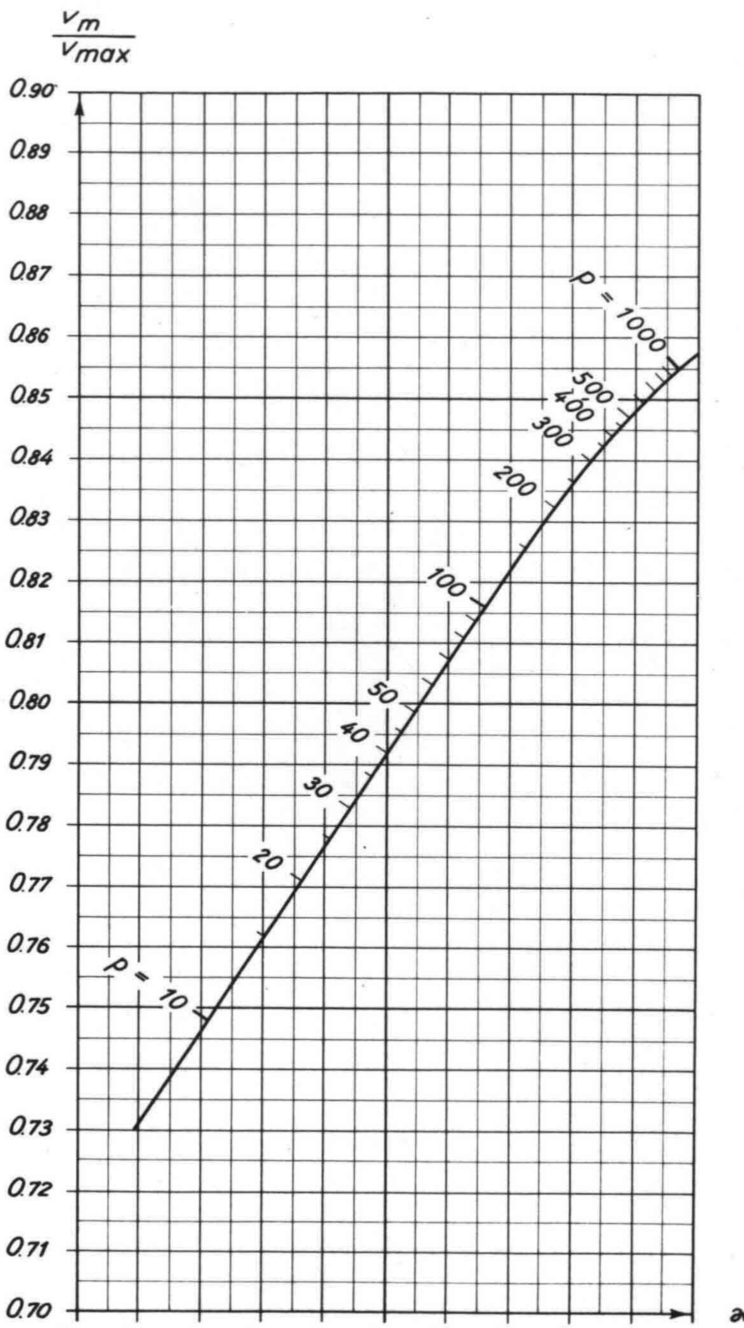


Fig. 11

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