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EFFICIENT DOMAIN DECOMPOSITION METHOD FOR ACOUSTIC SCATTERING IN MULTI-LAYERED MEDIA

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Abstract. *An efficient domain decomposition method is developed for discretized Helmholtz equations resulting from acoustical geological survey problems. A Schwarz-type multiplicative preconditioner is constructed using a fast direct method in subdomains by embedding them to rectangular domains. The GMRES iterations are reduced to a subspace corresponding to interfaces. Numerical experiments with several two-dimensional problems demonstrate the effectiveness and scalability of the proposed method.*

1 INTRODUCTION

Acoustical geological survey problems are often modeled using the Helmholtz equation with varying wave number. The discretization of this equations requires sufficiently many nodes per wavelength[7] and due to this the resulting system of linear equations is very large. It is usual that grids have from 500 to a few thousands grid points in each direction. Thus, the resulting problem can have several million unknowns for a two-dimensional problem and several billion unknowns for a three-dimensional problem. The system matrix is indefinite and complex-valued. Thus, many efficient solution procedures available for positive definite problems are not applicable.

Iterative methods without a good preconditioner converge slowly for Helmholtz problems which leads to too high computational cost. Furthermore, while direct methods like LU decomposition might be applicable for two-dimensional problems they are computationally too expensive for three-dimensional problems. Thus, the research on efficient solvers for the Helmholtz equation with varying wave number has concentrated on developing good preconditioners. The main techniques employed to construct these preconditioners are domain decomposition methods, multigrid methods, and fast direct solvers. Plessix and Mulder[13] studied tensor product form preconditioners based on fast direct solvers. While for low frequency problems they were very effective their performance deteriorated severely when the frequency approached practically interesting level. The authors[6,8] used a similar approach with good success for almost perfectly vertically layered media. Larson and Holmgren[12] developed a domain decomposition preconditioner employing a fast direct solver and tested it on a parallel computer. Erlangga, Oosterlee, and Vuik[3] developed a multigrid preconditioner which is fairly effective for geological survey problems.

In this paper, we develop a Schwarz-type multiplicative domain decomposition preconditioner. The subdomain preconditioners employ a fast direct solver via embedding the subdomains into rectangles. By fast direct methods, we mean direct methods based on FFT or cyclic reduction, for example, which require order of $m \log m$ or $m(\log m)^2$ operations to solve a system with m unknowns. With a piecewise constant wave number, the preconditioner can be made to match the system matrix in the interior of the subdomains. This has two important consequences: The residual vanished in the interior of the subdomains and due to this the iterations can be reduced on the interfaces and their near-by grid points. Thus, the solution procedure can be considered as a preconditioned iterative method on a sparse subspace[9,10]. Due to this the memory requirement of the GMRES method[15] is greatly reduced. Secondly, the conditioning of the preconditioned system is improved due to the matching in the interior of the subdomains. The numerical examples demonstrate the effectiveness of the developed approach.

2 MODEL PROBLEMS

In the rectangular computational domain Π, the pressure field p satisfies the Helmholtz partial differential equation

$$-\Delta p - k^2 p = g, \tag{1}$$

where k is the wave number given by $k = \frac{\omega}{c}$, ω is the angular frequency, c is the speed of sound, and g corresponds to a sound source. We assume the speed of sound and, thus, also the wave number to be piecewise constant functions. On the boundaries $\partial\Pi$, we impose a second-order absorbing boundary condition[1,8]

$$\begin{aligned} \frac{\partial p}{\partial n} &= ikp + i\frac{1}{2}\frac{\partial}{\partial s}\frac{1}{k}\frac{\partial p}{\partial s} \quad \text{on } \partial\Pi \\ \frac{\partial p}{\partial n} &= i\frac{3k}{4}p \quad \text{at } C, \end{aligned} \tag{2}$$

where n and s denote the unit outward normal and tangent vectors, respectively, and C denotes the set of the corner points of $\partial\Pi$.

3 NUMERICAL METHOD

3.1 Discretization

We discretize the Helmholtz equation on a uniform grid with the grid step size h. The discretization stencils for interior grid points, boundary grid points, and corner grid points are given by Figure 1, where the diagonal weights are defined by

$$
\begin{aligned}
d &= 4 - \frac{1}{4} \sum \left(k(x \pm h/2, y \pm h/2)\, h \right)^2, \\
b &= 2 + \frac{1}{2} \sum \left(\frac{i}{k(x, y \pm h/2)\, h} - k(x, y \pm h/2)\, h \right) - \frac{1}{4} \sum \left(k(x + h/2, y \pm h/2)\, h \right)^2, \\
a &= 1 + \frac{1}{2} \left(\frac{i}{k(x, y + h/2)\, h} - k(x, y + h/2)\, h + \frac{i}{k(x + h/2, y)\, h} - k(x + h/2, y)\, h \right) \\
&\quad - \frac{1}{4} \left(k(x + h/2, y + h/2)\, h \right)^2 + \frac{3}{4}.
\end{aligned}
\tag{3}
$$

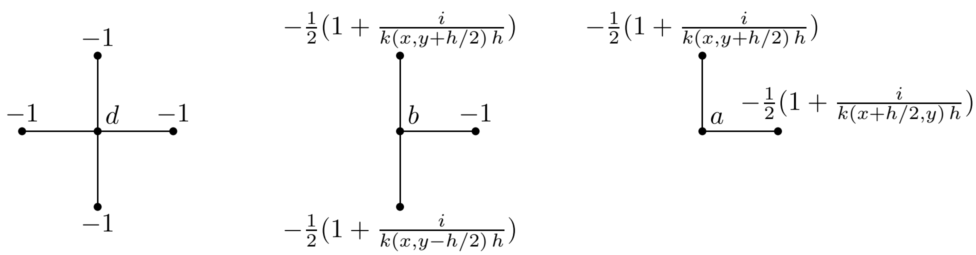

Figure 1: The discretization stencil at (x, y) for an interior grid point (left), a grid point on the left boundary (middle), and the left lower corner point (right). The weights d, b, and a are given by (3).

The discretization leads to a system of linear equations

$$
Ap = b. \tag{4}
$$

3.2 Iterative solution procedure

Instead of solving the system of linear equations (4), we solve iteratively using the GMRES method[15] a left preconditioned system

$$
AB^{-1}q = b, \tag{5}
$$

where B is the preconditioner. Once we have obtained q the solution of the original problem is given by $p = B^{-1}q$.

In order to describe our domain decomposition preconditioner B, we define each subdomain as the union of cells in which the speed of sound is a given constant. More precisely, let c_j, $j = 1, \ldots, n$, denote the different speeds of sound. The closure of the jth subdomain Ω_j is defined by

$$
\bar{\Omega}_j = \bigcup_{\substack{c(x,y) = c_j \\ (x,y) \in M}} [x - h/2, x + h/2] \times [y - h/2, y + h/2], \tag{6}
$$

where M is the set of cell midpoints. Let the rectangular matrix R_j correspond to the restriction operator into the subdomain Ω_j. Multiplying a vector by it results a vector containing the components associated to the subdomain Ω_j and its boundary. We denote the preconditioner in Ω_j by B_j and it will be defined in Section 3.3.

We employ a Schwarz-type multiplicative domain decomposition preconditioner $B = P_n$ which is defined recursively as

$$P_j^{-1} = P_{j-1}^{-1} + R_j^T B_j^{-1} R_j (I - A P_{j-1}^{-1}), \qquad j = 2, \dots, n, \tag{7}$$

where $P_1^{-1} = R_1^T B_1^{-1} R_1$. Thus, we obtain $y = y^{(n)} = B^{-1}x$ by performing the following sequence of operations:

$$\begin{aligned} y^{(1)} &= R_1^T B_1^{-1} R_1 x, \\ y^{(2)} &= y^{(1)} + R_2^T B_2^{-1} R_2 (x - A y^{(1)}), \\ &\vdots \\ y^{(n)} &= y^{(n-1)} + R_n^T B_n^{-1} R_n (x - A y^{(n-1)}). \end{aligned} \tag{8}$$

3.3 Subdomain preconditioners

Systems of linear equations corresponding to general shaped subdomains cannot be solved using a fast direct solver. For this reason, we embed each subdomain into a larger rectangular domain and we defined the subdomain preconditioner as a Schur complement matrix. In order to describe this more precisely, we define the minimum and maximum values of the coordinates for each subdomain Ω_j as

$$\begin{aligned} x_{\min}^j &= \operatorname*{argmin}_{(x,y)\in\bar{\Omega}_j} x, & x_{\max}^j &= \operatorname*{argmax}_{(x,y)\in\bar{\Omega}_j} x, \\ y_{\min}^j &= \operatorname*{argmin}_{(x,y)\in\bar{\Omega}_j} y, \quad \text{and} & y_{\max}^j &= \operatorname*{argmax}_{(x,y)\in\bar{\Omega}_j} y. \end{aligned} \tag{9}$$

Similarly, we define $x_{\min}$, $x_{\max}$, $y_{\min}$, and $y_{\max}$ for the computational domain Π. Then the extended rectangular subdomains are defined by

$$\begin{aligned} \widehat{\Omega}_j = &\left[\max\{x_{\min}^j - lh,\, x_{\min}\},\ \min\{x_{\max}^j + lh,\, x_{\max}\}\right] \\ &\times \left[\max\{y_{\min}^j - lh,\, y_{\min}\},\ \min\{y_{\max}^j + lh,\, y_{\max}\}\right], \end{aligned} \tag{10}$$

where l is a nonnegative integer.

The jth subdomain preconditioner is based on the Helmholtz equation

$$-\Delta p - k_j^2 p = g \quad \text{in } \widehat{\Omega}_j, \tag{11}$$

where $k_j = \omega / c_j$ and the absorbing boundary conditions (2) are posed on the boundaries $\partial\widehat{\Omega}_j$. By using the same grid and discretization for the problem (11), we obtain a matrix C_j which has a block form

$$C_j = \begin{pmatrix} C_{j,dd} & C_{j,de} \\ C_{j,ed} & C_{j,ee} \end{pmatrix} \tag{12}$$

once the unknowns corresponding to the subdomain $\bar{\Omega}_j$ are numbered first (subscript d) and the unknowns corresponding to the extension $\widehat{\Omega}_j \backslash \bar{\Omega}_j$ are numbered second (subscript e). The preconditioner B_j for the jth subdomain is given by a Schur complement matrix

$$B_j = C_{j,dd} - C_{j,de} C_{j,ee}^{-1} C_{j,ed}. \tag{13}$$

A system of linear equations $B_j y_j = x_j$ can solved as

$$y_j = \begin{pmatrix} 1 & 0 \end{pmatrix} C_j^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_j \tag{14}$$

using a fast direct solver like the cyclic reduction method[5,14]. This solution requires order of $m_j \log m_j$ floating point operations, where m_j is the size of C_j.

3.4 Sparse subspace iterations

The GMRES method[15] forms and stores basis vectors for a Krylov subspace which after l iterations is

$$\mathcal{K}_l(v) = \left\{ v,\, AB^{-1}v,\, (AB^{-1})^2 v,\, \ldots,\, (AB^{-1})^l v \right\}, \tag{15}$$

where v is the initial residual. For example, if the initial guess for solution in the GMRES method is zero then $v = b$. In general case, storing the basis vector requires a vast amount of memory for large problems unless the iterations converge already after a few iterations. The vectors v^j, $j = 1, \ldots, l$, defining $\mathcal{K}_l(v)$ are given by

$$v^j = AB^{-1} v^{j-1} = \left[(A - B) + B \right] B^{-1} v^{j-1} = (A - B) B^{-1} v^{j-1} + v^{j-1}, \tag{16}$$

where we have denoted $v^0 = v$. It is easy to see from (16) using induction that $v^{j+1} \in X \cup \{v\}$, where

$$X = \text{range}(A - B). \tag{17}$$

The subspace X corresponds to the rows of the matrices A and B which differ. Due to the construction of the preconditioner, these rows are associated to the interfaces between subdomains and their near-by grid points[4,9,10]. Thus, the subspace X is very sparse. For a two-dimensional problem, the number of nonzero components in vectors on X is of order $\sqrt{m}$, where m is the size of the system of linear equations. If the initial guess is chosen to be b then the initial residual v belongs to X and, thus, $\mathcal{K}_l(v) \subset X$, that is, the Krylov subspaces are sparse. Alternatively, $X \cup \{v\}$ is sparse when b is sparse and the initial guess is zero. These observations reduce the memory usage by a large factor and allows us to use the GMRES method without restarts. Furthermore, the so-called partial solution technique[2,11] can be used to solve the systems of linear equations with B on the sparse subspace[4,9].

4 NUMERICAL RESULTS

In all experiments, the stopping criterion for the GMRES method was that the norm of the residual is reduced by the factor 10^{-6}. In all embeddings, we extended the subdomains by one mesh step, that is, $l = 1$ in (10). The sound source is always a point source at the midpoint of the upper boundary. The initial guess for the solution in the GMRES method is zero for all problems. All experiments are performed without GMRES restarts.

4.1 Perfectly vertically layered medium

These test problems are defined in the unit square and we use a 1001×1001 grid to solve these problems. In the first test problem, the speed of sound is constant $c = 1$ in the whole domain. However, we divide the unit square into two and four equally sized subdomains which are extended according to (10) for preconditioning. The number of GMRES iterations for the resulting preconditioned systems are given by the third and fourth column of Table 1. The next two problems have perfectly vertically layered media shown in Figure 2. The number of GMRES iterations for these problems are given by the two last columns of Table 1. The real part of the solution at the frequency $f = 25$ is shown by Figure 2.

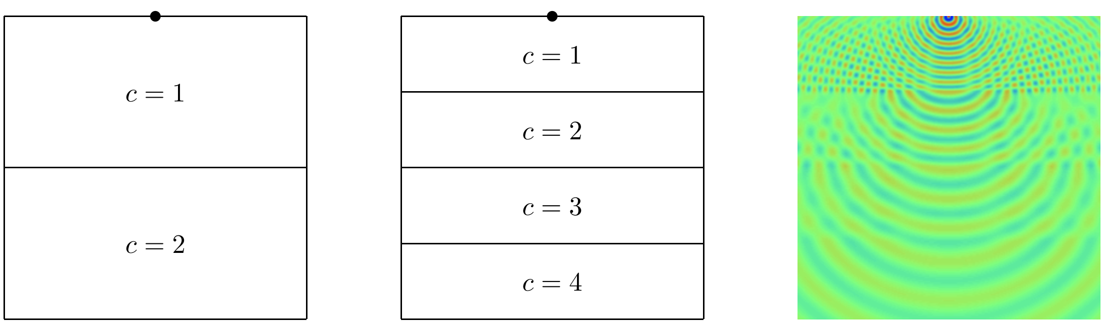

Figure 2: Layered test problems in the unit square and the solution for the four layer problem at $f = 25$.

		homog.	homog.	layered	layered
subdomains		2	4	2	4
f	grid	iter.	iter.	iter.	iter.
25	1001×1001	3	5	32	36
50	1001×1001	3	6	26	29
100	1001×1001	4	7	22	26

Table 1: The number of iterations for different test problems and frequencies f.

4.2 Wedge problem

The speed of sound for the wedge test problem[3,13] is shown in Figure 3. The domain decomposition is given by Figure 4 together with the one grid step extensions for the subdomains. Table 2 gives the number of GMRES iterations and CPU times in seconds at different frequencies on a PC with 3.8 GHz Xeon processor. The real parts of the solutions at $f = 30$ Hz and 50 Hz are plotted in Figure 3.

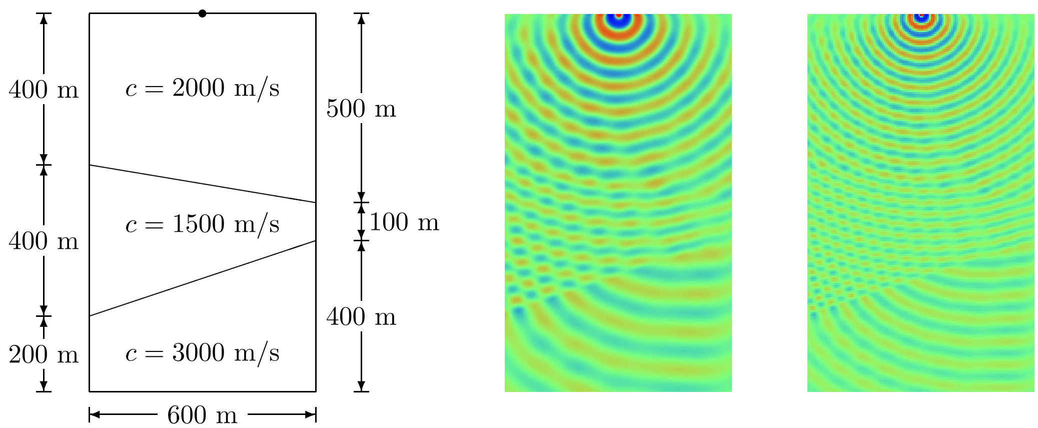

Figure 3: The wedge test problem and the solutions at $f = 30$ Hz and 50 Hz.

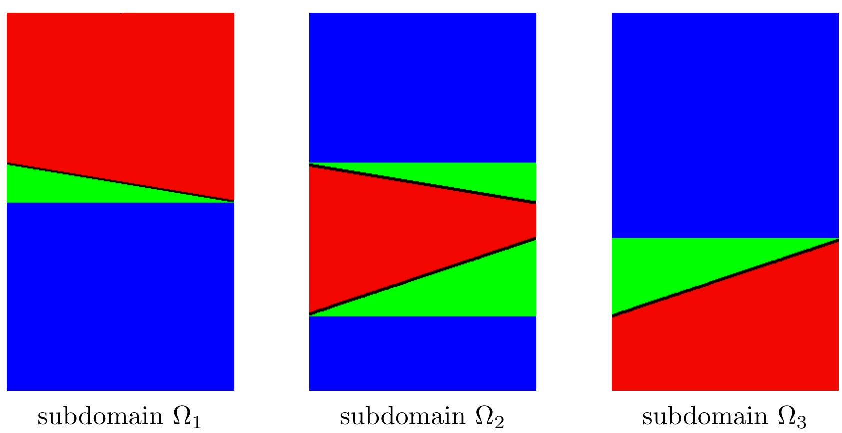

Figure 4: The subdomains Ω_j for the wedge test problem (red and black), their extensions $\widehat{\Omega}_j \backslash \bar{\Omega}_j$ (green), and the associated parts of the sparse subspace $X \cup \{v\}$ (black).

4.3 Salt deposit problem

This test problem mimics a simple salt deposit. The length of the side of the square computational domain is chosen to be 600 meters. The profile for the speed of sound is shown in Figure 5. It is perfectly vertically layered except for the salt disk with the radius

f [Hz]	grid	iter.	time
30	232×386	18	0.78
40	301×501	20	1.49
50	376×626	22	2.62
60	481×801	23	4.50
80	721×1201	26	11.53
100	901×1501	28	18.95
160	1441×2401	34	60.08

Table 2: The number of iterations and CPU times in seconds for different frequencies f for the wedge problem.

of 150 meters and the center at (300 m, 250 m) when the lower left corner is at origin. We consider three frequencies $f = 30$ Hz, 60 Hz, and 120 Hz and three grids 301×301, 601×601, and 1201×1201. Table 3 gives the number of GMRES iterations for all possible combinations of frequencies and grids. Figure 6 shows the real parts of the solutions at the considered frequencies. The domain decomposition is shown in Figure 7.

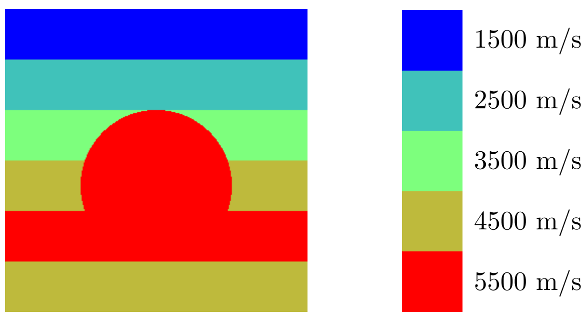

Figure 5: The speed of sound for the salt deposit problem.

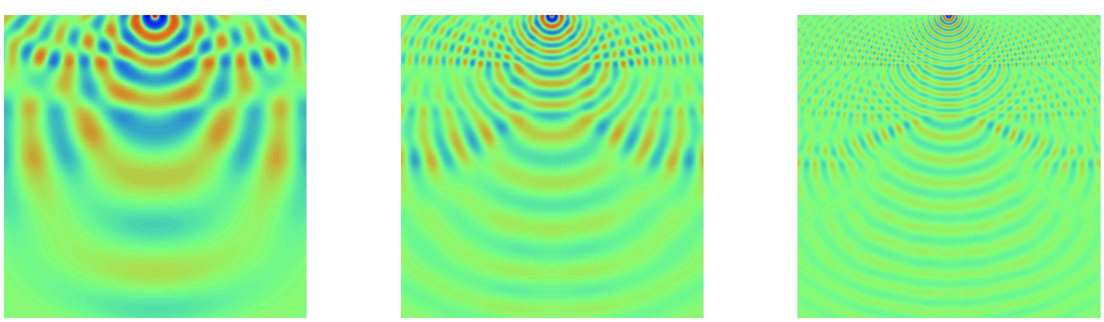

Figure 6: The solutions for the salt deposit problem at $f = 30$ Hz, 60 Hz, and 120 Hz.

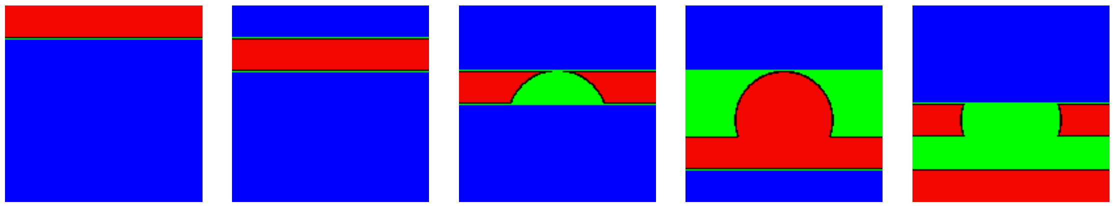

Figure 7: The subdomains Ω_j for the salt deposit problem (red and black), their extensions $\widehat{\Omega}_j \setminus \bar{\Omega}_j$ (green), and the associated parts of the sparse subspace $X \cup \{v\}$ (black).

	grid		
f [Hz]	301 × 301	601 × 601	1201 × 1201
30	18	23	31
60	19	22	31
120	22	24	27

Table 3: The number of iterations for different frequencies f and grids for the salt deposit problem.

5 CONCLUSIONS

We proposed a Schwarz-type multiplicative domain decomposition preconditioner for the Helmholtz equation with a piecewise constant wave number. The computational efficiency results from the use of fast direct solvers for subdomain preconditioning and the good conditioning of the preconditioned system. Furthermore, iterations can be reduced on the subdomain boundaries and their neighboring grid points. The number of GMRES iterations required to reduce the residual by a given factor depends only mildly on the grid step size and frequency. The numerical experiments demonstrated the method to be efficient for two-dimensional problems with realistic frequencies and jumps of the speed of sounds encountered in acoustical geological survey problems. The proposed method can be generalized in a straightforward manner for three-dimensional problems.

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REFERENCES

[1] A. Bamberger, P. Joly, and J E. Roberts, Second-order absorbing boundary conditions for the wave equation: a solution for the corner problem. *SIAM J. Numer. Anal.*, **27**, 323–352, (1990).

[2] A. Banegas, Fast Poisson solvers for problems with sparsity, *Math. Comp.*, **32**, 441–446, (1978).

[3] Y. A. Erlangga, C. W. Oosterlee, and C. Vuik, A novel multigrid based preconditioner for heterogeneous Helmholtz problems. *SIAM J. Sci. Comput.*, **27**, 1471–1492, (2006).

[4] E. Heikkola, Y. A. Kuznetsov, P. Neittaanmäki, and J. Toivanen, Fictitious domain methods for the numerical solution of two-dimensional scattering problems. *J. Comput. Phys.*, **145**, 89–109, (1998).

[5] E. Heikkola, T. Rossi, and J. Toivanen, Fast direct solution of the Helmholtz equation with a perfectly matched layer/an absorbing boundary condition. *Internat. J. Numer. Methods Engrg.*, **57**, 2007–2025, (2003).

[6] Q. Huynh, K. Ito, and J. Toivanen, A fast Helmholtz solver for scattering by a sound-soft target in sediment. *Proceedings of the 16th International Conference on Domain Decomposition Methods*, 2006. To appear.

[7] F. Ihlenburg, *Finite element analysis of acoustic scattering.* Springer-Verlag, New York, (1998).

[8] K. Ito and J. Toivanen, A fast iterative solver for scattering by elastic objects in layered media, *Appl. Numer. Math.* To appear.

[9] K. Ito and J. Toivanen, Preconditioned iterative methods on sparse subspaces, *Appl. Math. Letters.* To appear.

[10] Y. A. Kuznetsov, Numerical methods in subspaces. *Vychislitel'nye Processy i Sistemy II*, Nauka, Moscow, 265–350, (1985). In Russian.

[11] Y. A. Kuznetsov and A. M. Matsokin, On partial solution of systems of linear algebraic equations, *Sov. J. Numer. Anal. Math. Modelling*, **4**, 453–467, (1989).

[12] E. Larsson and S. Holmgren, Parallel solution of the Helmholtz equation in a multilayer domain. *BIT*, **43**, 387–411, (2003).

[13] R. E. Plessix and W. A. Mulder, Separation-of-variables as a preconditioner for an iterative Helmholtz solver. Appl. Numer. Math., **44**, 385–400, (2003).

[14] T. Rossi and J. Toivanen, A parallel fast direct solver for block tridiagonal systems with separable matrices of arbitrary dimension. *SIAM J. Sci. Comput.*, **20**, 1778–1796, (1999).

[15] Y. Saad and M. H. Schultz, GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Statist. Comput.*, **7**, 856–869, (1986).