Force Network Analysis using Complementary Energy

Andrew BORGART
Assistant Professor
Delft University of Technology
Delft, The Netherlands
A.Borgart@tudelft.nl

Yannick LIEM
Student
Delft University of Technology
Delft, The Netherlands
yannickliem@gmail.com

Summary

The method presented solves statically indeterminate force networks, which are graphical representations of forces (force polygons) of a structure, by using complementary energy. Statically indeterminate force networks, which have nodes where four or more members come together, have for each node multiple possible force polygons that make equilibrium. This holds for force networks of structures in one plane, such as trusses, as for geometric three-dimensional structures, such as shells. In the case of the latter the surface of thrust of the shell is discretized into a network of forces (equivalent to the thrust line of an arch), with discrete loads at its vertices.

Keywords: *graphic statics, force network analysis, force density, statically indeterminate trusses, shell structures and complementary energy.*

1. Introduction

The force path of an applied load on a shell structure is not easy to determine, because shells are statically indeterminate. This means that to determine a force network for analysing a shell's structural behaviour several possible load paths have to be considered [1]. The best result is numerous possible states of equilibriums within the thickness of the shells surface or defined envelope, fig 1.1. By adding to the force network analysis the concept of duality between geometry and the in-plane internal forces of networks, similar to the duality of the force polygon and form diagram for line structures (cable and arch, fig. 1.1), it is possible to have a direct graphical relation between the force network and its possible solutions [2]. However, the "exact" solution is not provided for by this method, because of the highly indeterminate nature of shell structures, which makes it impossible without the finite element method to calculate the correct internal forces given in a particular situation.

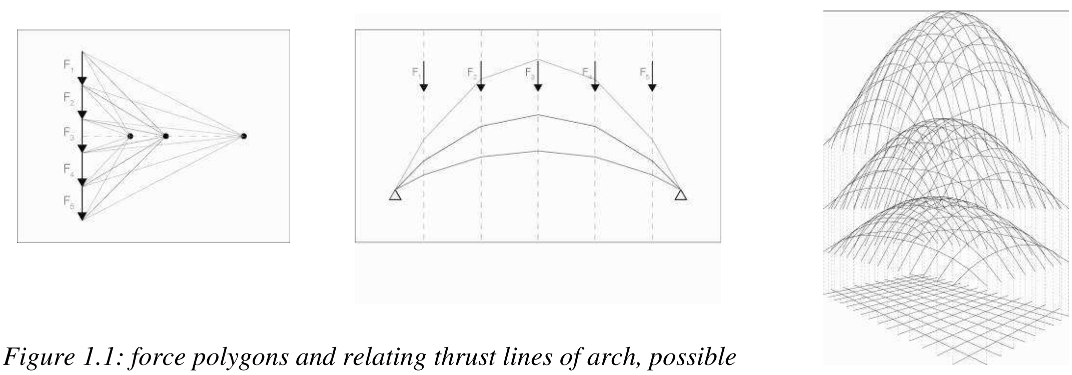


Figure 1.1: force polygons and relating thrust lines of arch, possible solutions of force network for shell structures

From the force network for each node its reciprocal diagram, the force polygon is drawn. This force polygon is scaled by keeping the assumed force density of the node constant, to find the force polygon with the lowest complementary energy. The result for each node is assembled to form the force diagram of the entire structure, the reciprocal diagram of the force network.

2. Complementary energy

The potential energy accumulated in an elastic body is called strain energy. The area under the stress-strain curve is the strain energy (E_v), the area above the curve is the complementary energy (E_c) see fig. 2.1. [3]

complementary energy is expressed in stresses:

$$E_{compl} = \frac{1}{2}\frac{\sigma^2}{E}$$

the complementary energy per unit of bar length equals:

$$E_{compl} = \int_V \frac{1}{2}\frac{\sigma^2}{E} dV = \frac{1}{2}\frac{N^2}{EA}$$

per bar with lenght l:

$$E_{compl} = \frac{1}{2}\frac{N^2}{EA} l$$

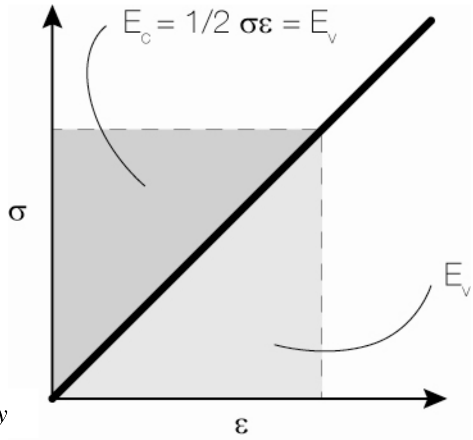


Figure 2.1: strain and complementary energy

3. Force networks for statically indeterminate trusses

Trusses are internally statically determinate if each node has three members connected to it (three-valent system, fig. 3.1). The truss is then entirely composed of triangles. Statically determinate trusses can be analysed by using graphic static, such as Cremona diagrams.

Trusses are internally statically indeterminate if each node has four or more members connected to it (four-valent or higher-valent systems, fig. 3.1). Statically determinate and indeterminate trusses can be analysed by using a matrix method such as the direct stiffness method, the flexibility method, the finite element method or by using complementary energy.

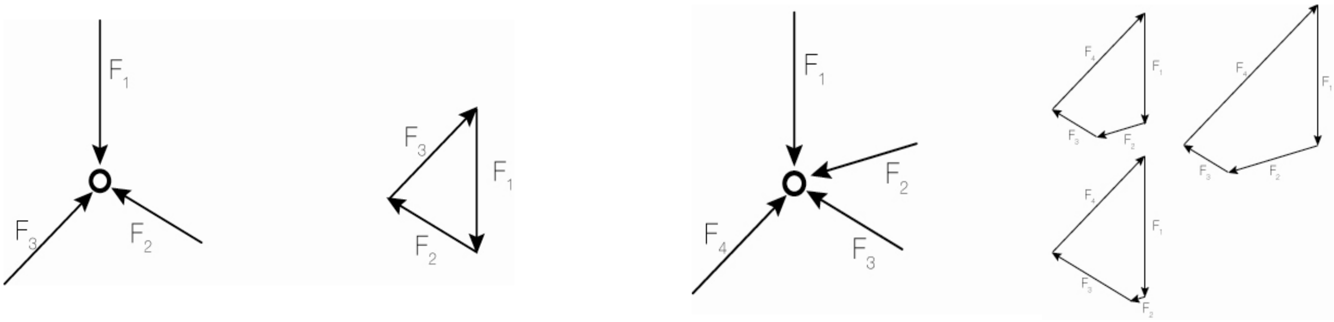


Figure 3.1: three- and four-valent nodes and their respective force polygons

Fig. 3.2 shows an example of a statically indeterminate truss. Although it is composed of triangles because in one node four forces come together the structure is statically indeterminate. This can be solved by making the structure statically determinate by making the force in a bar redundant (in fig 3.2; the unknown statically indeterminate ϕ) and solving this by minimizing the total complementary energy of the structure (fig. 3.3).

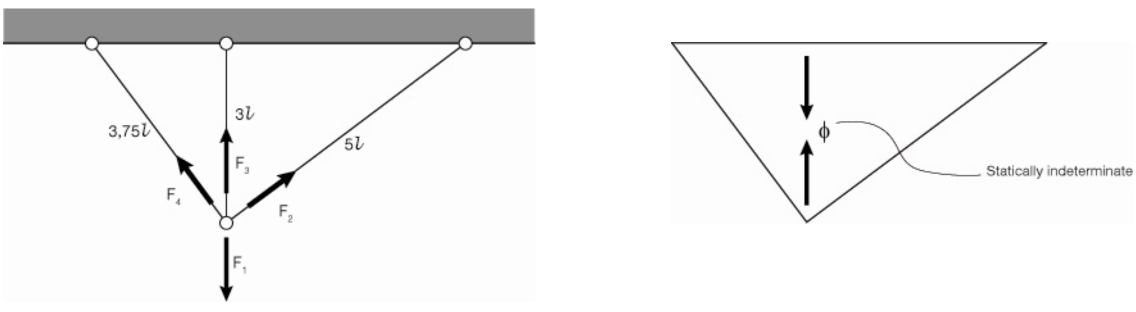


Figure 3.2: internally statically indeterminate truss

$$E_{compl} = \frac{1}{2}\frac{N^2}{EA}l$$
$$E_{compl,tot} = 2{,}1F^2 - 4{,}2F\phi + 3{,}6\phi^2$$
$$\frac{\partial E_{compl.tot}}{\partial \phi} = 0 \Rightarrow \phi = \frac{7}{12}F$$

member	N_i	l_i	E_c
1	$0{,}8(F-\phi)$	3,75l	$1{,}2\frac{l}{EA}(F-\phi)^2$
2	$0{,}6(F-\phi)$	5l	$0{,}9\frac{l}{EA}(F-\phi)^2$
3	ϕ	3l	$1{,}5\frac{l}{EA}\phi^2$

Figure 3.3: solving the statically indeterminate ϕ from the complementary energy of the members

For statically indeterminate trusses with numerous bars solving the therefore equally numerous redundants is numerically somewhat cumbersome. This problem can also be solved by searching for a closed force polygon (which is the reciprocal diagram of the topology of the structure, the force network) with the lowest total complementary energy of the structure. To be able to objectively compare all the possible force polygons with each other, and thus searching for solutions with the same magnitude of the applied load *F*, the total sum of the force density of the structure for each solution has to be kept constant, fig. 3.4.

- *search for a force polygon with the lowest complementary energy:*

$$E_{compl,tot} = F_1^2 l_1 + F_2^2 l_2 + F_3^2 l_3 = minimum$$

- *by keeping, for each solution,*
  the total force density (q) of the structure constant:

$$q_{tot} = \frac{F_1}{l_1} + \frac{F_2}{l_2} + \frac{F_3}{l_3} = constant$$

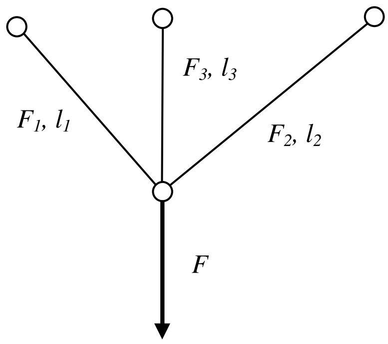


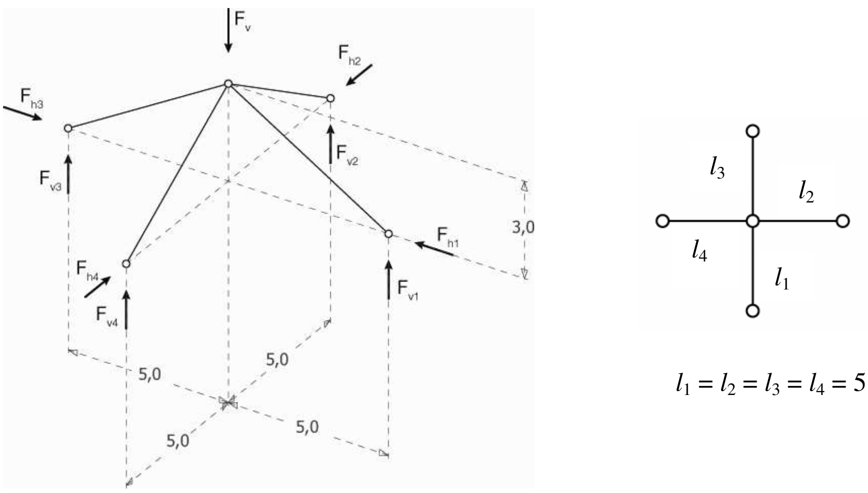
Figure 3.4: minimizing the total complementary energy of the structure, the force network

This procedure can be done by using a computation tool such as Grasshopper. This was used to solve this example. Because in the force network all members have the same properties, when calculating the complementary energy the stiffness *EA* is omitted from the equations.

4. Force networks for shell structures

Shell structures are statically indeterminate, and thus also their surface of thrust, which is the 3D equivalent of the thrust line of an arch. A planar projection of the surface of thrust gives the force network; its reciprocal diagram is the force polygon. We are dealing with a statically indeterminate structure and the network is a four-valent system.

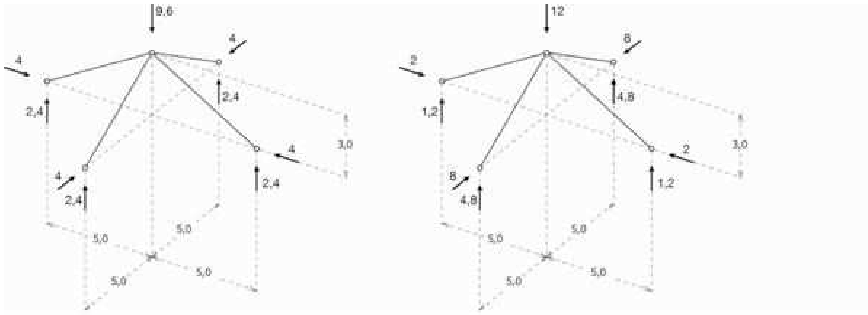
First we will consider a four-valent force network with only one node with an arbitrary load F_v, fig 4.1.



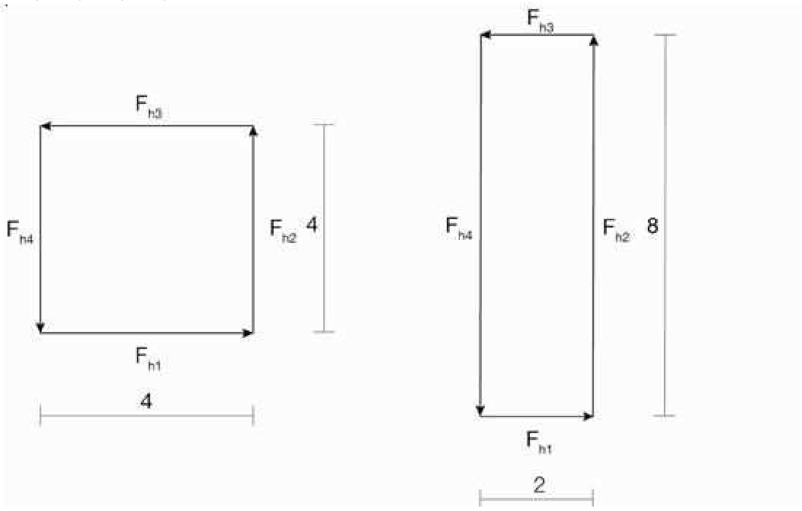
$l_1 = l_2 = l_3 = l_4 = 5$

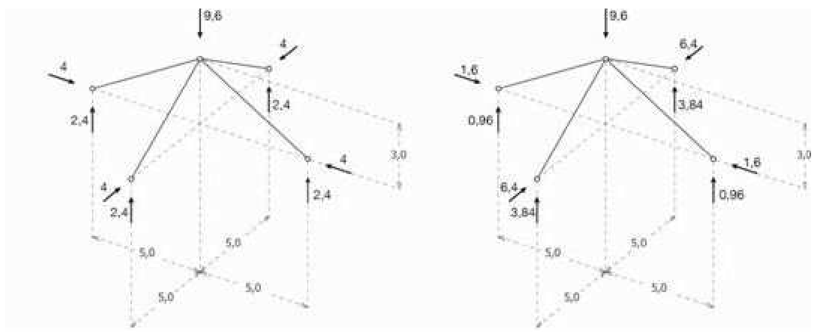
Figure 4.1: one node four-valent force network

As noted before to objectively compare the possible solutions with the same load, the total force density of the node for each solution has to be kept constant, fig. 4.2.



$$q_{tot} = \frac{4}{5} + \frac{4}{5} + \frac{4}{5} + \frac{4}{5} = 3,2 \qquad q_{tot} = \frac{2}{5} + \frac{8}{5} + \frac{4}{5} + \frac{8}{5} = 4$$





$$q_{tot} = \frac{4}{5} + \frac{4}{5} + \frac{4}{5} + \frac{4}{5} = 3,2 \qquad q_{tot} = \frac{1,6}{5} + \frac{6,4}{5} + \frac{1,6}{5} + \frac{6,4}{5} = 3,2$$

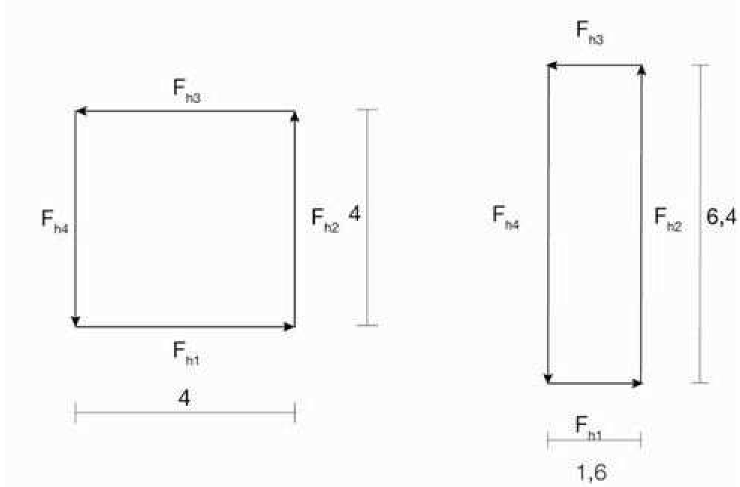


Figure 4.2: one node four-valent force network and possible force polygons

Now for each possible solution the complementary energy is calculated until one solution has the minimum complementary energy. This can be done by scaling the possible force polygons while keeping the force density for each polygon the same. The scaling of the force polygons can be easily done with the following equations, by varying the constant c, fig. 4.3.

$$F_a;\ F_b = cF_a$$

$$e = F_a\sqrt{\left(1+c^2-2c\times\cos\alpha\right)};\ e^* = \sqrt{\left(1+c^2-2c\times\cos\alpha\right)}$$

$$\delta_2 = \delta - \arcsin\left(\frac{c}{e^*}\sin\alpha\right);\ \beta_2 = \beta - \arcsin\left(\frac{1}{e^*}\sin\alpha\right)$$

$$F_c = \frac{\sin\delta_2}{\sin\gamma}e;\ F_d = \frac{\sin\beta_2}{\sin\gamma}e$$

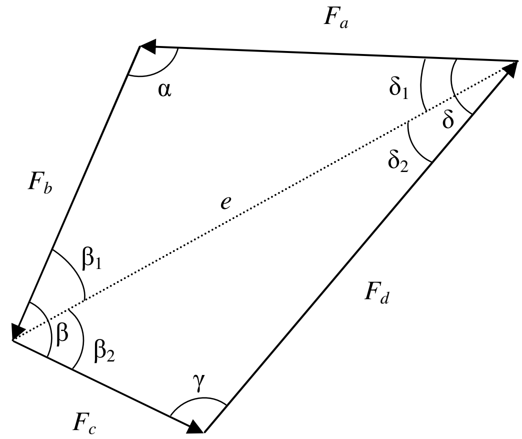


Figure 4.3: one node four-valent force polygon

To summarize the procedure for a one node, four-valent or higher-valent, force network.

- search for the force polygon with the lowest complementary energy by keeping, for each solution, the total force density constant:

$$E_{compl,tot} = \sum_{i=1}^{n} F_i^2 l_i = minimum \qquad q_{tot} = \sum_{i=1}^{n}\frac{F_i}{l_i} = constant$$

We will now look at force networks with multiple nodes, such as surfaces of thrust of shell structures. For a force network to be in equilibrium with its reaction forces its reciprocal diagram, the force polygon has to have a circumference which forms a continuous line. If this is not the case the case the topology of the force network does not represent a force network that is physically possible, see broken red line in fig 4.4.

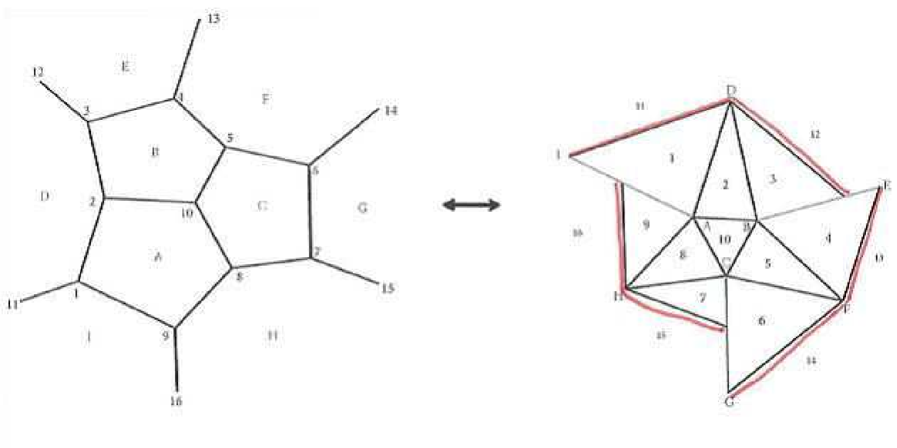


Figure 4.4: multiple node four-valent force network and its reciprocal diagram the force polygon

When searching for the force polygon for a force network with multiple nodes (the reciprocal diagram) with the least total complementary energy of the whole network, only the total sum of the force densities of the boundary (support reaction) members of the network has to remain constant for a given load; the reference force density. The force densities of the intermediate members have no bearing on the solution, fig 4.5.

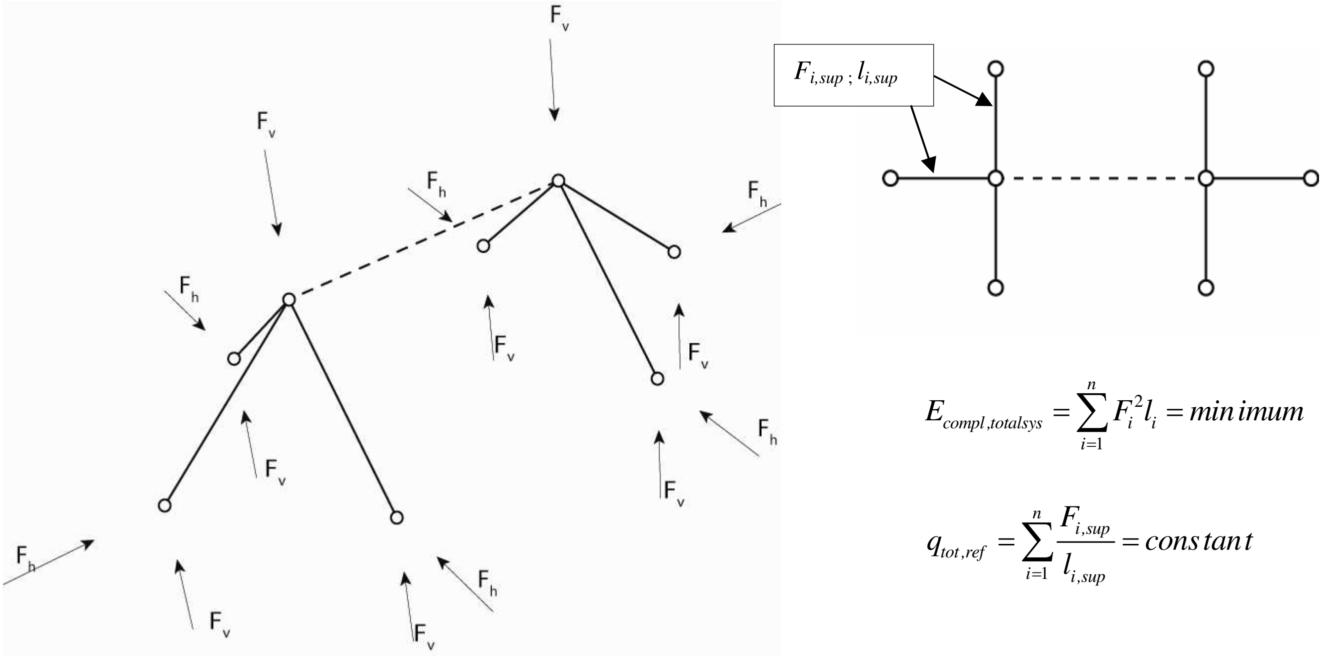


Figure 4.5: multiple node four-valent force network and boundary branches and its solution procedure

The procedure to solve a multiple node, four-valent or higher-valent, force network:

- generate a planar projection of a proposed discretized surface of thrust; the force network or primal grid (the applied loads are discretized as forces acting on the vertices of the discretized surface of thrust), fig. 4.6
- extract relevant information from the primal grid, e.g. length of member, angle between members, member connectivity, to the generate a reciprocal grid; the force polygons (also see [2]) with the least total complementary energy by keeping the total reference force density constant (this can be done for example by using a Generalized Reduced Gradient (GRG) algorithm), fig 4.7

  * search for the force polygon with the lowest complementary energy by keeping, for each solution, the total reference force density constant:

$$E_{compl,totalsys} = \sum_{i=1}^{n} F_i^2 l_i = minimum \qquad q_{tot,ref} = \sum_{i=1}^{n} \frac{F_{i,sup}}{l_{i,sup}} = constant$$

- when the problem has been solved the actual force densities can be determined in all the members, and thus the correct forces in the members and the correct reaction forces
- generate the correct discretized 3D surface of thrust by calculating the z-coordinates of primal grid, using the (actual) force densities in the members, fig. 4.7

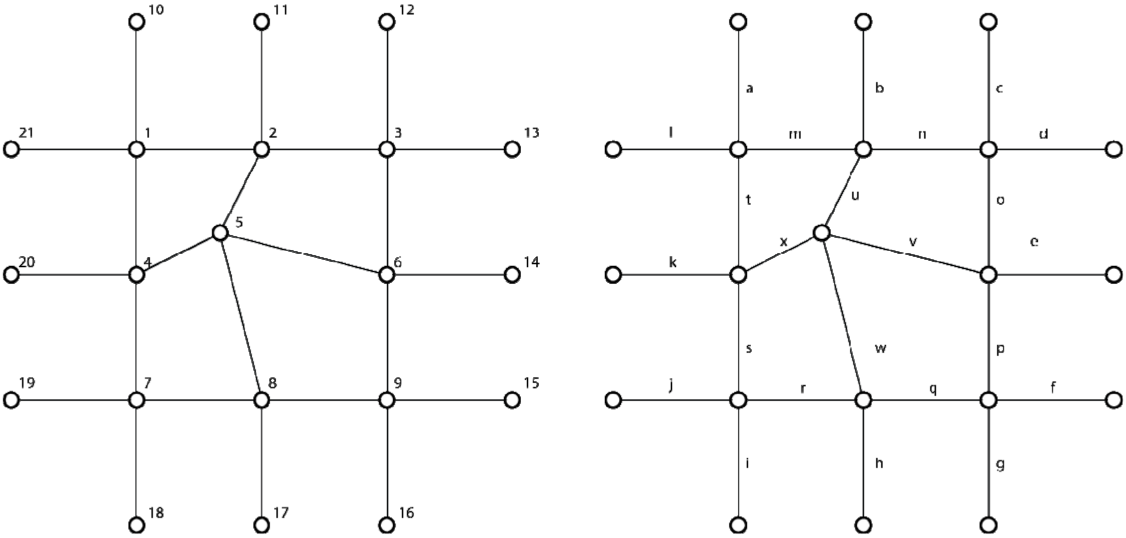
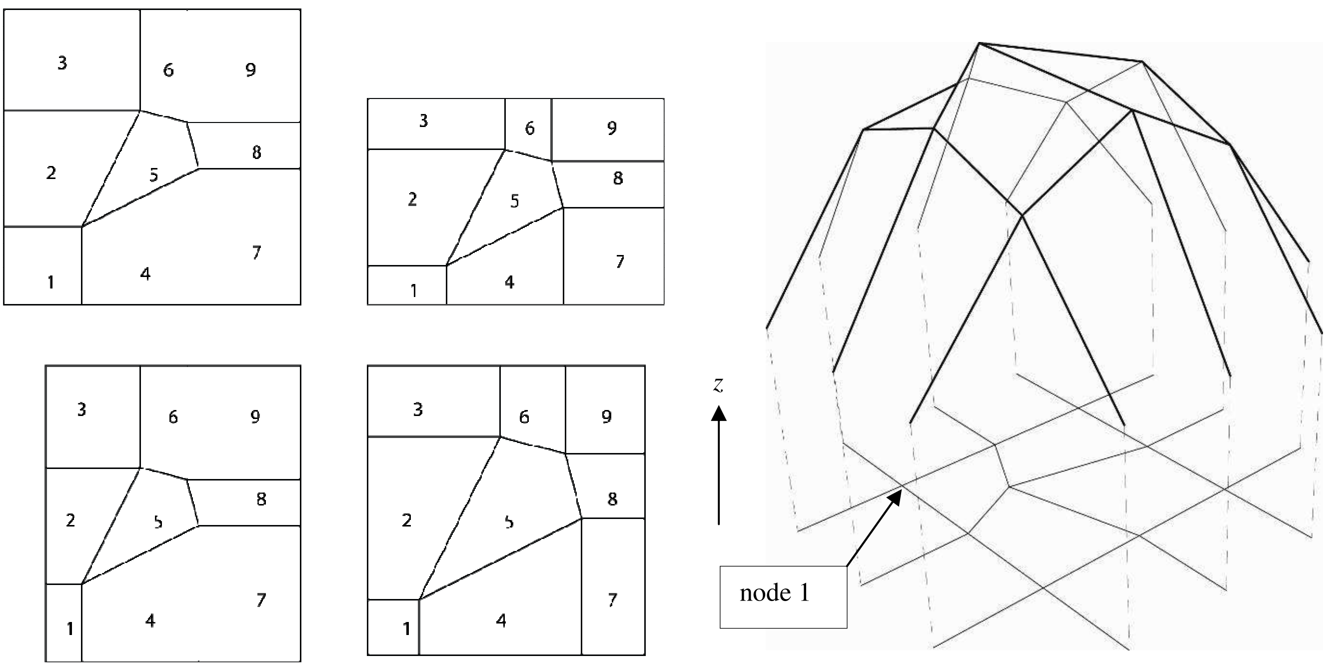


Figure 4.6: primal grid



- equilibrium of internal forces and load, calculcation of z-coordinate to generate correct discretized surface of thrust (example for node 1):

$$\left(z_1 - z_{10}\right)\frac{N_a}{l_a} + \left(z_1 - z_2\right)\frac{N_m}{l_m} + \left(z_1 - z_4\right)\frac{N_t}{l_t} + \left(z_1 - z_{21}\right)\frac{N_l}{l_l} - F_{z,1} = 0$$

Figure 4.7: possible force polygons (reciprocal grids), correct discretized 3D surface of thrust

5. Conclusions

By using complementary energy statically indeterminate structures such as trusses and shell structures can be easily solved. Further research needs to be done for a rigorous mathematical derivation. Also further physical explanations need to be provided for various aspects such as force density. Never the less the in this paper described method is yet a further step in understanding the structural behaviour of shells.

References

[1] O'DWYER D., "Funicular analysis of masonry vaults", *Computers & Structures*, Vol. 73, No. 1-5, 1999, pp. 187-197

[2] BLOCK P., "Thrust Network Analysis, Exploring Three-dimensional Equilibrium", *PhD*, MIT, 2009.

[3] BLAAUWENDRAAD J., "Theory of Elasticity, Energy Principles and Variational Methods", DUT, 2002