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Stability of backward stochastic differential equations: the general Lipschitz case*

Antonis Papapantoleon[†] Dylan Possamai[‡] Alexandros Saplaouras[§]

Abstract

In this paper, we obtain stability results for backward stochastic differential equations with jumps (BSDEs) in a very general framework. More specifically, we consider a convergent sequence of standard data, each associated to their own filtration, and we prove that the associated sequence of (unique) solutions is also convergent. The current result extends earlier contributions in the literature of stability of BSDEs and unifies several frameworks for numerical approximations of BSDEs and their implementations.

Keywords: BSDE; stability; nonlinear martingale representations; processes with jumps; stochastically discontinuous martingales; random time horizon; stochastic Lipschitz generator.

MSC2020 subject classifications: 60F15; 60F17; 60F25; 60G05; 60G07; 60G44; 60G048; 60G57; 60H05; 60H20.

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1 Introduction

The goal of this paper is to provide a suitable framework under which the stability—sometimes also called robustness—property of well-posed backward stochastic differential equations with jumps (BSDEs for short) is valid. More precisely, consider the *standard data* $\mathcal{D} := (\mathbb{G}, \tau, (X^\circ, X^\natural), C, \xi, f)$ of a BSDE—i.e. assuming that a probability space is given, \mathbb{G} is a filtration, τ is the stopping time serving as terminal horizon for the BSDE, X° and X^\natural are square-integrable martingales which are the integrators of the corresponding stochastic integrals, C is an increasing predictable process playing the role of Lebesgue–Stieltjes integrator, ξ is the terminal condition of the BSDE, and f its generator—see Papapantoleon, Possamaï, and Saplaouras [86, Definitions 3.2 and 3.16], then we know from [86, Theorems 3.5 and 3.23] that the BSDE

$$Y_t = \xi + \int_{(t, \tau]} f(s, Y_{s-}, Z_s, U(s, \cdot)) dC_s - \int_t^\tau Z_s dX_s^\circ - \int_t^\tau \int_{\mathbb{R}^n} U(s, x) \tilde{\mu}^{X^\natural}(ds, dx) - \int_t^\tau dN_s, \quad (1.1)$$

is well-posed, i.e. it has a unique solution $\mathcal{S} := (Y, Z, U, N)$ in appropriate spaces. Assuming now that we are given a sequence of standard data $(\mathcal{D}^k)_{k \in \mathbb{N} \cup \{\infty\}}$, whose associated solutions form the sequence $(\mathcal{S}^k)_{k \in \mathbb{N} \cup \{\infty\}}$, we address the following question:

‘How should the sequence $(\mathcal{D}^k)_{k \in \mathbb{N} \cup \{\infty\}}$ converge,
so that the sequence $(\mathcal{S}^k)_{k \in \mathbb{N} \cup \{\infty\}}$ converges as well?’

Our aim is to make precise the way(s) convergence should be understood in the previous statement, and to provide a framework which is as general as possible, while the conditions are as weak as possible. Naturally, the existing stability results in the literature should then be recovered as special cases. In particular, numerical schemes—and in a loose sense their implementations—for BSDEs with stochastic Lipschitz generators will be considered as sub-cases of our main results (see Theorem 3.1). We would like to point out the fact that the techniques we use are purely probabilistic, so that the numerical schemes that can potentially be derived from our results will not rely on the connection of BSDEs to partial integro-differential equations (PIDEs for short). An important consequence of this fact is that our results potentially provide stochastic numerical methods that could be an alternative to finite difference or finite element schemes for the solution of semi-linear PIDEs. This could be particularly interesting for high-dimensional non-linear PIDEs arising, for example, in the presence of valuation adjustments in option pricing; see *e.g.* Crépey [26, 27] or Bichuch, Capponi, and Sturm [9].

Let us start by providing some historical references for the problem we are interested in. When the limit-BSDE—that is, the one that corresponds to the standard data \mathcal{D}^∞ —is solely driven by a Brownian motion, the articles of Briand, Delyon, and Mémin [18, 19] provide a suitable framework for the stability property to hold. It is noteworthy that in these articles, the filtration \mathbb{G}^k is neither required to coincide with \mathbb{G}^∞ , nor to be a discretisation of \mathbb{G}^∞ , which, as far as we know, makes [19] the most general result for the stability of Brownian-driven BSDEs. More precisely, in [18], respectively in [19], the authors approximate the Brownian motion driving the BSDE by a sequence of scaled random walks, respectively by square-integrable martingales. In both articles, the sequence of filtrations associated to the Itô integrators weakly converges to the Brownian filtration, thus allowing for the aforementioned generality. The earlier work of Hu and Peng [59] falls in the category where $\mathbb{G}^k = \mathbb{G}^\infty$, for every $k \in \mathbb{N}$, and the stability is investigated with respect to the pair (ξ, f) only. The articles of Ma, Protter, San Martín, and Torres [76], Toldo [92], Geiss, Labart, and Luoto [40], and Briand, Geiss, Geiss, and Labart [20], follow the spirit of [18], *i.e.*, the Brownian motion is approximated by a sequence of random walks, and respectively study numerical schemes, BSDEs with random horizon, Hölder-continuous terminal conditions, and rates of convergence. Remaining in the Brownian framework for the limit-BSDE, Jusselin and Mastrolia [66] provide stability results when the limit-BSDE is driven by a Brownian martingale and is approximated by scaled point processes.

When the sequence $(\mathbb{G}^k)_{k \in \mathbb{N}}$ corresponds to refined discretizations of \mathbb{G}^∞ , then we are essentially dealing with an Euler scheme for the BSDE itself. It is then noteworthy that the corresponding sequence of discrete filtrations weakly converges to the original filtration; see Coquet, Mémin, and Słomiński [25, Proposition 3 or Proposition 4.A]. In this line of literature, the majority of the articles consider the case where the BSDE is driven by a Brownian motion, and only a few deal with a more general case, *e.g.* BSDEs driven by Lévy processes. Notably, only the articles of Bouchard and Élie [11], Aazizi [1] (in the pure jump case), Lejay, Mordecki, and Torres [71], Geiss and Labart [39] (in [39, 71] the jump part of the driving martingale is a Poisson process), Kharroubi and Lim [68], with a jump process depending on the Brownian motion itself, Madan, Pistorius, and Stadje [77] (which follows in spirit the approach of [18]) and Dumitrescu and Labart [35] (where the jump part of the driving martingale is a Poisson process and the authors actually consider reflected BSDEs) consider BSDEs which include stochastic integrals with respect to an integer-valued measure, associated to the jumps of a Lévy process. See also Khedher and Vanmaele [69] for BSDEs driven by càdlàg martingales.

When the martingale driving the BSDE is only an Itô integral, the literature is very rich

and there are mainly two approaches: one that relies on purely probabilistic techniques, and another that relies on the connection of the BSDEs driven by Brownian motion with second-order semi-linear parabolic PDEs. In the former approach, the first work that dealt with the stability of BSDEs is due to Bally [4], who provides an approximation scheme for a BSDE (driven by a Brownian motion) based on a time-discretisation scheme constructed on a Poisson net. Contemporary contributions are due to Chevance [22] and Coquet, Mackevičius, and Mémin [24]. With the exception of some of the aforementioned articles, the majority of numerical approximations for BSDEs are implementations of the standard Euler scheme. Indicatively, and with no claim of comprehensiveness, we can mention Bender and Denk [6], Bender and Moseler [7], Bender and Zhang [8], Bouchard and Menozzi [12], Bouchard and Touzi [13], Bouchard, Élie, and Touzi [14], Briand and Labart [17], Chassagneux and Crisan [21], Crisan and Manolarakis [28–30], Crisan, Manolarakis, and Touzi [31], Gobet and Labart [45], Gobet and Lemor [46], Gobet and Makhlouf [47], Gobet and Pagliarani [48], Gobet and Turkedjiev [49, 50], Gobet, Lemor, and Warin [51], Hu, Nualart, and Song [61], Pagès and Sagna [84], Zhang [93, 94]. Numerical solutions of BSDEs via their connection to branching processes have also been recently explored, and correspond to an alternative incarnation of the probabilistic approach, see *e.g.* Bouchard, Tan, and Warin [16], Bouchard, Tan, Warin, and Zou [15], Henry-Labordère [55], Henry-Labordère, Oudjane, Tan, Touzi, and Warin [57], Henry-Labordère, Tan, and Touzi [56].

In the latter approach, the work of Ma, Protter, and Yong [75], which deals with the existence and uniqueness of the solution of a (coupled) forward–backward stochastic differential equation (FBSDE) as well as the stability of a parametrised family, with its proposed ‘four-step scheme’ constitutes the cornerstone of the PDE approach. The papers by Bouchard and Touzi [13], Douglas, Ma, and Protter [33], Gobet and Labart [43, 44], Milstein and Tretyakov [83], are implementations of numerical schemes based either on the four-step scheme or PDE arguments.

Let us also mention that recently, new implementations have been proposed based on modern techniques, such as the use of deep neural networks, *e.g.*, see E, Han, and Jentzen [36], E, Hutzenthaler, Jentzen, and Kruse [37], Beck, E, and Jentzen [5], Huré, Pham, and Warin [62, 63], Germain, Pham, and Warin [41, 42], and of parallel programming, *e.g.*, see Gobet, López-Salas, Turkedjiev, and Vázquez [52] or Gobet, López-Salas, and Vázquez [53] and also Abbas-Turki, Crépey, and Diallo [2] for an application to XVA computations.

A closer examination of the existing literature indicates that there exists no general (at least in the spirit of Briand et al. [19]) stability result for processes with jumps, even in the Lévy case. Our paper aspires to fill this gap, and to describe and study a framework for the stability property for a sequence of BSDEs driven by square-integrable martingales, covering thus also the subclass of Lévy drivers. To this end, we mainly have two mathematical tools at hand: the Moore–Osgood theorem, and a general result for the stability of martingale representations. The former was introduced and successfully used by Briand, Delyon, and Mémin [18, 19] in order to efficiently control the doubly indexed sequence obtained by the sequence of Picard schemes; we will follow the same approach. The latter is presented in Papapantoleon, Possamaï, and Saplaouras [87]. It comes as no surprise that the stability of martingale representations comes into play, since the existence and uniqueness of the solution of the BSDE (1.1) with stochastically Lipschitz generator is obtained by means of a martingale representation argument. After all, the BSDE (1.1) is frequently referred to as the *f*-nonlinear martingale representation of ξ . Additionally to these main tools, our toolbox includes Theorem A.8 and its offsprings. Theorem A.8 provides a characterisation of weak convergence of measures to an atomless measure on the real (half-)line and its corollaries are going to guarantee the

convergence of the Lebesgue–Stieltjes integrals when the integrators are regular enough. In view of the above brief comments, it is natural to combine the frameworks of [86, 87]¹ and to properly enrich them in order to obtain the framework suitable for our purpose. We postpone the detailed description and the required technicalities until Subsection 3.1. However, we would like to underline that if we were to derive the unique solution of BSDE (1.1) by means of a different method compared to the two referred above, we can, *mutatis mutandis*, adapt the arguments and derive the stability property as soon as the alternative method is based on a fixed-point argument, *i.e.* a Picard scheme. This would require to introduce appropriate (uniform) integrability conditions related to the existence and uniqueness of the solution of BSDE (1.1).

Backward SDEs have been used for modelling a vast array of random phenomena in the areas of economics, finance, game theory, and others, and their numerical solutions is a topic of significant importance both in theory and in practice. In particular, allowing for random, and possibly infinite horizon, has found applications in optimal control and optimal stopping theory, see Hu and Schweizer [60], or even more recently in contract theory, see Sannikov [91], Pagès and Possamaï [85], Lin, Ren, Touzi, and Yang [73]. Notice that it is typical with unbounded horizon to allow for stochastic Lipschitz conditions for integrability purposes, but we also emphasise that such BSDEs also naturally arises when one considers Malliavin derivatives of the solutions (see for instance Imkeller, Réveillac, and Richter [64]), the latter being useful to study for instance existence of densities, see Mastrolia, Possamaï, and Réveillac [78, 79]. Another important application of the results of this paper is in the area of numerical schemes for BSDEs, where Theorem 3.1 provides a flexible and general framework for automatically deducing the convergence of such numerical schemes. These can be viewed as generalisations of the celebrated Donsker’s theorem. In that case, the martingales of the sequence that approximates the limit-BSDE have to be defined with respect to their own filtrations, hence the requirement of the convergence of filtrations becomes necessary. This creates additional technical difficulties in the presence of jumps in the limit-BSDE, *e.g.* for the convergence of Lebesgue–Stieltjes integrals as well as for the required uniform integrability, that have to be carefully handled.

The structure of the paper is as follows. In Section 2 we provide the set of conditions a sequence of standard data should satisfy for existence and uniqueness of the solution alongside notation and some helpful comments. In Section 3 the theorem for the stability of BSDEs is stated, while in Subsection 3.1 we provide the sketch of the proof by describing the main arguments. There, it will become evident that the stability of martingale representations plays an important role in obtaining the stability property of BSDEs, as we have already stated. In Subsection 3.4 and Subsection 3.5 the required technical lemmata, which verify our claims in Subsection 3.1, are presented. In Subsection 3.6 we briefly discuss the nature of the imposed conditions. Finally, in Section 4 we present some examples that demonstrate the generality and applicability of Theorem 3.1. In Appendix A several auxiliary results are proved, and we expand upon the notation used in the paper.

General notation

Let \mathbb{R}_+ denote the set of non-negative real numbers, $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$ and $|x|$ denote the absolute value of a real number $x \in \mathbb{R}$. We denote the set of positive natural numbers by \mathbb{N} and we define $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. For two arbitrary $(p, q) \in \mathbb{N} \times \mathbb{N}$, we identify $\mathbb{R}^{p \times q}$ with the set of matrices with p rows and q columns and real entries. The transpose of $v \in \mathbb{R}^{p \times q}$ will be denoted by $v^\top \in \mathbb{R}^{q \times p}$. The element at the i -th row and j -th column

¹To the best of our knowledge, the most general well-posedness results for Lipschitz BSDEs are given in [86].

of v will be denoted by v^{ij} , for $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$, and it will be called the (i, j) -element of v . However, the notation needs some care when we deal with sequences of elements of $\mathbb{R}^{p \times q}$, e.g. if $(v^k)_{k \in \mathbb{N}} \subset \mathbb{R}^{p \times q}$, then we will denote by $v^{k,ij}$ the (i, j) -element of v^k , for $i \in \{1, \dots, p\}$, $j \in \{1, \dots, q\}$, and $k \in \mathbb{N}$. The trace of a square matrix $w \in \mathbb{R}^{p \times p}$ is given by $\text{Tr}[w] := \sum_{i=1}^p w^{ii}$. We endow $\mathbb{R}^{p \times q}$ with the norm defined for any $z \in \mathbb{R}^{p \times q}$ by $\|z\|_2^2 := \text{Tr}[z^\top z]$ and remind the reader that this norm derives from the inner product defined for any $(z, u) \in \mathbb{R}^{p \times q} \times \mathbb{R}^{p \times q}$ by $\text{Tr}[z^\top u]$. We will additionally endow the space $\mathbb{R}^{p \times q}$ with the norm $\|\cdot\|_1$, which is defined by $\|v\|_1 := \sum_{i=1}^p \sum_{j=1}^q |v^{ij}|$. We identify \mathbb{R}^p with $\mathbb{R}^{p \times 1}$, i.e. an arbitrary $x \in \mathbb{R}^p$ will be identified as a column vector of length p . If $(x^k)_{k \in \mathbb{N}} \subset \mathbb{R}^p$, then $x^{k,i}$ denotes the i -th element of x^k , for $i \in \{1, \dots, p\}$ and $k \in \mathbb{N}$.

We abuse notation and denote by 0 the neutral element in the group $(\mathbb{R}^{p \times q}, +)$. Furthermore, for any finite dimensional topological space E , $\mathcal{B}(E)$ will denote the associated Borel σ -algebra. In addition, for any other finite dimensional space F , and for any non-negative measure ν on $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E))$, we will denote the Lebesgue–Stieltjes integrals of any measurable map $f : (\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)) \rightarrow (F, \mathcal{B}(F))$, by

$$\int_{(u,t] \times A} f(s, x) \nu(ds, dx), \text{ for any } (t, A) \in \mathbb{R}_+ \times \mathcal{B}(E),$$

and

$$\int_{(u,\infty) \times A} f(s, x) \nu(ds, dx), \text{ for any } A \in \mathcal{B}(E),$$

where the integrals are to be understood in a component-wise sense. Finally, the letters $p, q, i, j, k, l, \ell, m$, and n are reserved to denote arbitrary positive integers. Specifically, the calligraphic letter ℓ will denote the dimension of the state space of a solution of a BSDE, m will denote the dimension of the state space of an Itô integrator, and n will denote the dimension of the state space of a process associated to an integer-valued random measure.

Let us define the maps $\mathbb{R}^n \ni x \xrightarrow{\text{Id}} x \in \mathbb{R}^n$ and $\mathbb{R}^n \ni x \xrightarrow{q} xx^\top \in \mathbb{R}^{n \times n}$, where we suppress the dependence on the dimension n for notational simplicity. The space of functions defined on \mathbb{R}_+ and with state space $\mathbb{R}^{p \times q}$, which are right-continuous with left-limits (càdlàg) will be denoted by $\mathcal{D}(\mathbb{R}_+; \mathbb{R}^{p \times q})$. The metric induced by the J_1 -topology will be denoted by δ_{J_1} , while the supremum norm will be denoted by $\|\cdot\|_\infty$. Finally, we will denote the locally uniform convergence of the sequence $(\alpha^k)_{k \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}_+; \mathbb{R}^{p \times q})$ by $\alpha^k \xrightarrow[k \rightarrow \infty]{\text{lu}} \alpha^\infty$.

In order to simplify the presentation and minimise the introductory remarks, we will adopt the notation and definitions from [87]. Moreover, definitions related to BSDEs from [86] are also adopted. Nonetheless, for ease of reference, we present in Appendix A the main ones that will be used throughout this work. For example, given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the expectation under \mathbb{P} will be denoted by $\mathbb{E}[\cdot]$. Another example is the following: given a filtration \mathbb{F} , a stopping time τ and $X \in \mathcal{H}^2(\mathbb{F}, \tau; \mathbb{R})$, the space of Itô integrands is

$$\mathcal{H}^2(X, \mathbb{F}, \tau; \mathbb{R}^{\ell \times m}) := \left\{ Z : (\Omega \times \mathbb{R}_+, \mathcal{P}^{\mathbb{F}}) \longrightarrow (\mathbb{R}^{\ell \times m}, \mathcal{B}(\mathbb{R}^{\ell \times m})) : \right. \\ \left. \mathbb{E} \left[\int_{(0, \tau]} \text{Tr} \left[Z_t \frac{d\langle X \rangle_t^{\mathbb{F}}}{dC_t^X} Z_t^\top \right] dC_t^X \right] < \infty \right\}.$$

In a nutshell, we have adopted the notation from [87] and adapted the dimension of the state space; the results of [87] remain valid for multi-dimensional martingale representations.

2 Framework

Let us fix the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ for the remainder of this article, as well as a sequence $(\mathcal{D}^k)_{k \in \overline{\mathbb{N}}}$ where, for any $k \in \overline{\mathbb{N}}$, $\mathcal{D}^k := (\overline{X}^k, \mathbb{G}^k, \tau^k, C^k, \xi^k, f^k)$ are standard data under (a sufficiently large) $\hat{\beta} > 0^2$ in the sense of Papapantoleon, Possamaï, and Saplaouras [86, Section 3.1]. More precisely, for every $k \in \overline{\mathbb{N}}$, they satisfy the following conditions:

- (F1) The martingale $\overline{X}^k := (X^{k,\circ}, X^{k,\natural})$ belongs to $\mathcal{H}_{\hat{\beta}}^{2,k}(\mathbb{R}^m) \times \mathcal{H}_{\hat{\beta}}^{2,k}(\mathbb{R}^n)$, with $X^{k,\natural}$ being purely discontinuous, and (\overline{X}^k, C^k) satisfies Assumption A.1 (see Appendix A.1).
- (F2) The terminal condition satisfies $\xi^k \in \mathbb{L}_{\hat{\beta}}^{2,k}$.
- (F3) The generator $f^k : \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times m} \times \mathfrak{H}^k \longrightarrow \mathbb{R}^d$ is such that for any $(y, z, u) \in \mathbb{R}^d \times \mathbb{R}^{d \times m} \times \mathfrak{H}^k$, the map

$$(t, \omega) \longmapsto f^k(t, \omega, y, z, u_t(\omega; \cdot)) \text{ is } \mathcal{G}_t^k \otimes \mathcal{B}([0, t])\text{-measurable.}$$

Moreover, f^k satisfies a stochastic Lipschitz condition, i.e. there exist

$$\begin{aligned} r^k : (\mathbb{R}_+ \times \Omega, \mathcal{P}) &\longrightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \text{ and} \\ \vartheta^k = (\theta^{k,\circ}, \theta^{k,\natural}) : (\mathbb{R}_+ \times \Omega, \mathcal{P}) &\longrightarrow (\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2)), \end{aligned}$$

such that, for $dC^k \otimes d\mathbb{P} - a.e.$ $(t, \omega) \in \mathbb{R}_+ \times \Omega$

$$\begin{aligned} &|f^k(t, \omega, y, z, u_t(\omega; \cdot)) - f^k(t, \omega, y', z', u'_t(\omega; \cdot))|^2 \leq r_t^k(\omega) |y - y'|^2 \\ &+ \theta_t^{k,\circ}(\omega) \|c_t(\omega)(z - z')\|^2 + \theta_t^{k,\natural}(\omega) \left(\|u_t(\omega; \cdot) - u'_t(\omega; \cdot)\|_{k,t}(\omega) \right)^2. \end{aligned} \quad (2.1)$$

- (F4) Let $(\alpha^k)^2 := \max\{\sqrt{r^k}, \theta^{k,\circ}, \theta^{k,\natural}\}$ and define the increasing, \mathbb{G}^k -predictable and càdlàg process

$$A^k := \int_0^\cdot (\alpha_s^k)^2 dC_s^k. \quad (2.2)$$

Then there exists $\Phi^k > 0$ such that

$$\Delta A_t^k(\omega) \leq \Phi^k, \text{ for } dC^k \otimes d\mathbb{P} - a.e. (t, \omega) \in \mathbb{R}_+ \times \Omega. \quad (2.3)$$

- (F5) It holds

$$\mathbb{E} \left[\int_0^T e^{\hat{\beta} A_t^k} \frac{|f^k(t, 0, 0, \mathbf{0})|^2}{(\alpha_t^k)^2} dC_t^k \right] < \infty,$$

where $\mathbf{0}$ denotes the null application from \mathbb{R}^n to \mathbb{R} .

These conditions guarantee that the BSDE (2.5) has a unique solution for every $k \in \overline{\mathbb{N}}$; see also Remark 2.3 below for more detailed statements.

At this point let us clarify that the processes associated to the stochastic basis $(\Omega, \mathcal{G}, \mathbb{G}^k, \mathbb{P})$ will be stopped at τ^k , for each fixed $k \in \overline{\mathbb{N}}$. If, for example, L^k is a \mathbb{G}^k -adapted process, for some $k \in \overline{\mathbb{N}}$, then it will be assumed to be stopped at time τ^k . In particular, the process \overline{X}^k is stopped at τ^k , for every $k \in \overline{\mathbb{N}}$.

Notation 2.1. Some notational simplifications and rules are in order:

² $\hat{\beta}$ is independent of $k \in \overline{\mathbb{N}}$. We will refer to it as the common value.

- we simplify the notation of [87] associated to integer-valued measures. The integer-valued measure $\mu^{X^{k,\natural}}$ will be denoted by $\mu^{k,\natural}$, its \mathbb{G}^k -compensator $\nu^{(X^{k,\natural}, \mathbb{G}^k)}$ will be denoted by $\nu^{k,\natural}$ and the compensated integer-valued measure $\tilde{\mu}^{(X^{k,\natural}, \mathbb{G}^k)}$ will be denoted by $\tilde{\mu}^{k,\natural}$, for every $k \in \overline{\mathbb{N}}$;
- let $k \in \overline{\mathbb{N}}$ and Y be a \mathbb{G}^k -semimartingale. In the notation of its quadratic covariation $[Y]^{\mathbb{G}^k}$, resp. its predictable quadratic covariation $\langle Y \rangle^{\mathbb{G}^k}$, we will suppress the dependence on the filtration, and we will simply write $[Y]$, resp. $\langle Y \rangle$;
- for every $k \in \overline{\mathbb{N}}$, the notation for the spaces of [86, Section 2.3] associated to the standard data \mathcal{D}^k has been the simplified notations introduced in [86, page 16] extended as follows: the index k will be affixed, succeeding the number 2 (if any) and preceding any other symbol (if any), e.g., $H_{\beta}^{2,k,\natural}$, $\mathfrak{H}_{t,\omega}^k$, and so on. The rule for the norms makes the index k a subscript which precedes the value β or t , e.g., $\|\cdot\|_{\star,k,\beta}$, $\|\cdot\|_{k,t}$.

Notation 2.2. We introduce some auxiliary notation for two subsets of $E := \mathbb{R}_+ \times \mathbb{R}^n$ as well as for sets of continuous functions with compact support. Let $(F, \|\cdot\|_2), (G, \|\cdot\|_2)$ be Euclidean finite-dimensional spaces.

- Set $E_0 := (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}^n)$ and $\tilde{E} := (\mathbb{R}_+ \times \mathbb{R}^n) \setminus E_0$.
- Let $f : (F, \|\cdot\|_2) \rightarrow (G, \|\cdot\|_2)$. The support of the function f is the set $\text{supp}(f) := \overline{\{s \in F : f(s) \neq 0\}}^{\|\cdot\|_2}$, where $\overline{A}^{\|\cdot\|_2}$ denotes the closure of the set A under the metric associated to the norm $\|\cdot\|_2$.
- $C_c(F; G) := \{f : (F, \|\cdot\|_2) \rightarrow (G, \|\cdot\|_2) : f \text{ is continuous with compact support}\}$.
- $C_{c\tilde{E}}(E; \mathbb{R}^\ell) := \{f \in C_c(E; \mathbb{R}^\ell) : \text{supp}(f) \subset \tilde{E}\}$.
- $D^{\circ, \ell \times m} \subset C_c(\mathbb{R}_+; \mathbb{R}^{\ell \times m})$ is a fixed, countable set, possessing the property that it is dense in $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \langle X^{\infty, \circ} \rangle(\omega))$ for \mathbb{P} -a.e. $\omega \in \Omega$; for the existence of such a set the reader may consult Lemma A.14.
- $D^\natural \subset C_{c\tilde{E}}(E; \mathbb{R}^\ell)$ is a fixed, countable set, which is dense in $L^2(E, \mathcal{B}(E), \nu^{\infty, \natural}(\omega); \mathbb{R}^\ell)$ for \mathbb{P} -a.e. $\omega \in \Omega$ (see Lemma A.15), keeping in mind that $\nu^{\infty, \natural}(E_0) = 0$, \mathbb{P} -a.s., as well as $\int_E \|x\|_2^2 \nu^{\infty, \natural}(ds, dx) < \infty$, \mathbb{P} -a.s.

We proceed now with the description of the conditions for the sequence $(\mathcal{D}^k)_{k \in \overline{\mathbb{N}}}$ to be convergent, i.e., we make precise in which sense the convergence mentioned in the introduction should be understood. We remind the reader that we have already mentioned that the stability for martingale representations as given in [87, Corollary 3.10] will be central to our approach. Therefore, it is natural to complement the conditions of that corollary with conditions that ensure the convergence of the Lebesgue-Stieltjes integrals associated to the generators of the BSDEs. This is exactly the role of Conditions (S7)–(S9).

- (S1) The filtration \mathbb{G}^∞ is quasi-left-continuous and the process $X^{\infty, \circ}$ is continuous; the process $X^{\infty, \natural}$ is \mathbb{G}^∞ -quasi-left-continuous, because of its martingale property.
- (S2) The pair $(X^{k, \circ}, X^{k, \natural}) \in \mathcal{H}^{2,k}(\mathbb{R}^m) \times \mathcal{H}^{2,d}(\mathbb{G}^k, \infty; \mathbb{R}^\ell)$ with $M_{\mu_{X^{k, \natural}}}[\Delta X^{k, \circ} | \tilde{\mathcal{P}}^{\mathbb{G}^k}] = 0$ for every $k \in \overline{\mathbb{N}}$, such that in addition $X^k = X^{k, \circ} + X^{k, \natural}$, and

$$(X_\infty^{k, \circ}, X_\infty^{k, \natural}) \xrightarrow{L^2(\Omega, \mathbb{G}, \mathbb{P}; \mathbb{R}^{\ell \times 2})} (X_\infty^{\circ, c}, X_\infty^{\infty, d}). \quad (2.4)$$

(S3) The martingale X^∞ possesses the \mathbb{G}^∞ -predictable representation property.

(S4) The filtrations converge weakly, i.e. $\mathbb{G}^k \xrightarrow{w} \mathbb{G}^\infty$.

(S5) The random variable $\xi^k \in \mathbb{L}^2(\Omega, \mathcal{G}_\infty^k, \mathbb{P}; \mathbb{R})$, for every $k \in \overline{\mathbb{N}}$, and $\xi^k \xrightarrow{\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R})} \xi^\infty$.

(S6) The sequence

$$\left(\int_{(0, \tau^k)} e^{\hat{\beta} A_t^k} \frac{\|f^k(t, 0, 0, \mathbf{0})\|^2}{(\alpha_t^k)^2} dC_t^k \right)_{k \in \overline{\mathbb{N}}},$$

is uniformly integrable, where $\hat{\beta}$ is the common value from Footnote 2, while $\mathbf{0}$ denotes the null application from \mathbb{R}^n to \mathbb{R}^ℓ .

(S7) The sequence $(A_\infty^k)_{k \in \overline{\mathbb{N}}}$ is bounded by a constant $\overline{A} > 0$.³

(S8) The generators $(f^k)_{k \in \overline{\mathbb{N}}}$ possess additionally the following properties:

(i) for every $k \in \overline{\mathbb{N}}$, $W \in \mathbb{D}(\mathbb{R}_+; \mathbb{R}^\ell)$, $Z \in D^{\circ, \ell \times m}$ and $U \in D^\natural$, it holds that⁴

$$(f^k(t, W_t, Z_t, U(t, \cdot)))_{t \in \mathbb{R}_+} \in \mathbb{D}(\mathbb{R}_+; \mathbb{R}^\ell), \text{ P-a.s.};$$

(ii) for every $Z \in D^{\circ, \ell \times m}$, $U \in D^\natural$, if $(W^k)_{k \in \overline{\mathbb{N}}}$ is a sequence of càdlàg maps such that $W^k \xrightarrow{J_1} W^\infty$ then

$$(f^k(t, W_t^k, Z_t, U(t, \cdot)))_{t \in \mathbb{R}_+} \xrightarrow[k \rightarrow \infty]{J_1(\mathbb{R}^\ell)} (f^\infty(t, W_t^\infty, Z_t, U(t, \cdot)))_{t \in \mathbb{R}_+}, \text{ P-a.s.}$$

Besides, if $\sup_{k \in \overline{\mathbb{N}}} \|W^k(\omega)\|_\infty < \infty$, P-a.s., then

$$\sup_{k \in \overline{\mathbb{N}}} \|(f^k(t, W_t^k, Z_t, U(t, \cdot)))_{t \in \mathbb{R}_+}\|_\infty < \infty, \text{ P-a.s..}$$

(S9) The sequences $(C^k)_{k \in \overline{\mathbb{N}}}$ and $(\Phi^k)_{k \in \overline{\mathbb{N}}}$ satisfy

(i) $C^k \xrightarrow{(J_1(\mathbb{R}), \mathbb{P})} C^\infty$;

(ii) $C_\infty^k := \lim_{t \rightarrow \infty} C_t^k \in \mathbb{R}_+$, P-a.s., with $C_\infty^k \xrightarrow{\mathbb{P}} C_\infty^\infty$;

(iii) $\Phi^k \xrightarrow{|\cdot|} \Phi^\infty := 0$;

(S10) The stopping time τ^∞ is finite and $\tau^k \xrightarrow[k \rightarrow \infty]{\mathbb{P}} \tau^\infty$.

Remark 2.3. We include below some brief remarks on the nature of the conditions assumed above. A more detailed discussion will be provided in Subsection 3.6.

³This allows to write Condition (S6) without the exponential functions.

⁴In Remark 2.4 we verify that we are allowed to use elements of $C_{c|\bar{E}}(E; \mathbb{R}^\ell)$ as elements of \mathfrak{H}^k . Clearly, the elements of $D^{\circ, \ell \times m}$ also possess the suitable measurability properties as deterministic processes which are $\mathcal{B}(\mathbb{R}_+) \setminus \mathcal{B}(\mathbb{R}^{\ell \times m})$ -measurable.

- (i) We have assumed that $(\mathcal{D}^k)_{k \in \overline{\mathbb{N}}}$ are standard data under (the common value) $\hat{\beta}$, and that $\hat{\beta}$ is sufficiently large such that the quantities $M^{\Phi^k}(\hat{\beta})$, and $M_{\star}^{\Phi^k}(\hat{\beta})$ as defined in [86, Lemma 3.4] are sufficiently close to $18e\Phi^k$. The above in conjunction with (S9).(iii) and [86, Corollary 3.6] ensure that for *all but finitely many* $k \in \overline{\mathbb{N}}$ there exists a unique quadruplet

$$(Y^k, Z^k, U^k, N^k) \in \mathbb{H}_{\hat{\beta}}^{2,k} \times \mathbb{H}_{\hat{\beta}}^{2,k,\circ} \times \mathbb{H}_{\hat{\beta}}^{2,k,\natural} \times \mathbb{H}_{\hat{\beta}}^{2,k,\perp}$$

or

$$(Y^k, Z^k, U^k, N^k) \in \mathcal{S}_{\hat{\beta}}^{2,k} \times \mathbb{H}_{\hat{\beta}}^{2,k,\circ} \times \mathbb{H}_{\hat{\beta}}^{2,k,\natural} \times \mathbb{H}_{\hat{\beta}}^{2,k,\perp},$$

such that for any $t \in \llbracket 0, \tau^k \rrbracket$

$$\begin{aligned} Y_t^k &= \xi^k + \int_t^{\tau^k} f^k(s, Y_s^k, Z_s^k, U^k(s, \cdot)) dC_s^k \\ &\quad - \int_t^{\tau^k} Z_s^k dX_s^{k,\circ} - \int_t^{\tau^k} \int_{\mathbb{R}^n} U^k(s, x) \tilde{\mu}^{k,\natural}(ds, dx) - \int_t^{\tau^k} dN_s^k, \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (2.5)$$

We will assume, without loss of generality, that BSDE (2.5) admits a unique solution, for every $k \in \overline{\mathbb{N}}$. The unique solution associated to the standard data \mathcal{D}^k will be denoted by S^k , for every $k \in \overline{\mathbb{N}}$.

- (ii) The assumption $\Phi^\infty = 0$ is compatible with the assumption that \overline{X}^∞ is a quasi-left-continuous martingale; recall Condition (S1). Indeed, in this case $X^{\infty,\circ}$ (which has been assumed continuous; see (S1)) has a continuous angle bracket process, and $\mu^{\infty,\natural}$ has an atomless compensator; see Jacod and Shiryaev [65, Corollary II.1.19].
- (iii) Condition (S3) states that the pair $(X^{\infty,\circ}, X^{\infty,\natural})$ possesses the \mathbb{G}^∞ -martingale predictable representation property. Therefore, the element $N \in \mathcal{H}_{\beta}^{2,\infty,\perp}$ of the solution Y^∞ has to be the zero process. For the sake of a unified notation, even if $k = \infty$, we will write the orthogonal martingale N in BSDE (2.5).
- (iv) Condition (S7) states that the sequence $(A_\infty^k)_{k \in \overline{\mathbb{N}}}$ has to be bounded. On the one hand, this condition implies the equivalence of the norms in the weighted spaces indexed by β with the respective non-weighted norms, i.e. when $\beta = 0$. On the other hand, Condition (S7) does not necessarily imply that the sequences $(\alpha^k)_{k \in \overline{\mathbb{N}}}$ and $(C^k)_{k \in \overline{\mathbb{N}}}$ are bounded. In (S9).(ii) we do assume, however, that C^k is \mathbb{P} -almost surely finite, for every $k \in \overline{\mathbb{N}}$.

- (v) In view of Condition (S7) again, Condition (S6) is equivalent to the sequence

$$\left(\int_{(0,\infty)} \frac{\|f^k(t, 0, 0, \mathbf{0})\|_2^2}{(\alpha_t^k)^2} dC_t^k \right)_{k \in \overline{\mathbb{N}}},$$

being uniformly integrable. Moreover, Condition (S5) states that $\xi^k \xrightarrow[k \rightarrow \infty]{\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}^\ell)} \xi^\infty$.

In particular $(\|\xi^k\|_2^2)_{k \in \overline{\mathbb{N}}}$ is uniformly integrable, which implies that the sequence $(\xi^k)_{k \in \overline{\mathbb{N}}}$ is $\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}^\ell)$ -bounded

$$\sup_{k \in \overline{\mathbb{N}}} \|\xi^k\|_{\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}^\ell)} < \infty, \quad \text{or, equivalently by Condition (S7),} \quad \sup_{k \in \overline{\mathbb{N}}} \|\xi^k\|_{\mathbb{L}_{\hat{\beta}}^{2,k}} < \infty,$$

where $\hat{\beta}$ is the common value (see Footnote 2).

Remark 2.4. Let $U \in C_{c|\tilde{E}}(E; \mathbb{R}^\ell)$ and $k \in \mathbb{N}$. We have that $\text{supp}(U) \cap E_0 = \emptyset$, which in conjunction with the compactness of $\text{supp}(U)$ allows⁵ us to write $\inf \{\|x\|_2 : x \in$

⁵We use that every Euclidean space with its usual topology is a normal Hausdorff space.

$\text{supp}(U)\} > 0$. Then, it follows

$$\mathbb{E} \left[\int_{(0,\infty) \times \mathbb{R}^n} \|U(t, x)\|_2^2 \mu^{k, \mathfrak{h}}(dt, dx) \right] = \mathbb{E} \left[\sum_{t>0} \|U(t, \Delta X_t^{k, \mathfrak{h}})\|_2^2 \right] < \infty,$$

where we also used the boundedness of the function U and that $\mathbb{E} \left[\sum_{t>0} \|\Delta X_t^{k, \mathfrak{h}}\|_2^2 \right] < \infty$; recall that $X^{k, \mathfrak{h}} \in \mathcal{H}^{2,d}(\mathbb{G}^k, \tau^k; \mathbb{R}^n)$. Additionally, by Jacod and Shiryaev [65, Theorem II.1.8] we have

$$\mathbb{E} \left[\int_{(0,\infty) \times \mathbb{R}^n} \|U(t, x)\|_2^2 \nu^{k, \mathfrak{h}}(dt, dx) \right] < \infty. \quad (2.6)$$

Therefore, the stochastic integral $U \star \tilde{\mu}^{k, \mathfrak{h}}$ is well-defined. Moreover, by He, Wang, and Yan [54, Theorem 11.21.3] and by means of Property (2.6), we deduce that

$$\langle U \star \tilde{\mu}^{k, \mathfrak{h}} \rangle_s = U^2 \star \nu^{k, \mathfrak{h}}_s - \sum_{s \leq \cdot} \int_{\mathbb{R}^n} U(t, x) \nu^{k, \mathfrak{h}}(\{t\} \times dx) \int_{\mathbb{R}^n} U^\top(t, x) \nu^{k, \mathfrak{h}}(\{t\} \times dx), \quad \mathbb{P}\text{-a.s.}$$

The last identity further allows us to write for every $t \in \mathbb{R}_+$

$$(\|U(t, \cdot)\|_k(\omega))^2 = \widehat{K}_t^k(\|U(t, \cdot)\|_2^2)(\omega) - \Delta C_t^k(\omega) \|\widehat{K}_t^k(U(t, \cdot))(\omega)\|_2^2, \quad \mathbb{P}\text{-a.s.} \quad (2.7)$$

The finiteness of $\|U(t, \cdot)\|_k(\omega)$ is a result of the compact support of U and the σ -finiteness of the transition kernel $\widehat{K}_t^k(\omega)$ (see [65, Proposition II.2.9.(iv)]). In particular, observe that $U(t, \cdot) \in \mathfrak{H}_{\omega, t}^k$ for $C^k \otimes \mathbb{P}$ -a.e. $(t, \omega) \in \mathbb{R}_+ \times \Omega$. Therefore, seeing U as a deterministic process, we can conclude that $D^{\mathfrak{h}} \subset \mathfrak{H}^k$, for every $k \in \overline{\mathbb{N}}$. Consequently we are allowed to evaluate elements of $D^{\mathfrak{h}}$ in the last argument of the generator f^k , for every $k \in \overline{\mathbb{N}}$.⁶

3 Stability of backward stochastic differential equations

We start this section with the statement of the main theorem of this article. In Subsection 3.1 we outline the strategy we will follow in order to prove Theorem 3.1, which is based on Moore–Osgood’s theorem, see Theorem A.2 and the references therein, while the remaining technical parts will be presented in Subsection 3.4 and Subsection 3.5.

Theorem 3.1. *Let Conditions (S1)–(S10) hold, and for every $k \in \overline{\mathbb{N}}$, denote by $S^k := (Y^k, Z^k, U^k, N^k)$ the unique solution of the BSDE (2.5) associated to the standard data \mathcal{D}^k . Then*

$$(Y^k, Z^k \cdot X^{k, \circ} + U^k \star \tilde{\mu}^{k, \mathfrak{h}}, N^k) \xrightarrow{(J_1(\mathbb{R}^{\ell \times 3}), \mathbb{L}^2)} (Y^\infty, Z^\infty \cdot X^{\infty, \circ} + U^\infty \star \tilde{\mu}^{\infty, \mathfrak{h}}, 0), \quad (3.1)$$

$$\begin{aligned} & ([Y^k], [Z^k \cdot X^{k, \circ} + U^k \star \tilde{\mu}^{k, \mathfrak{h}}], [N^k], [Y^k, X^{k, \circ}], [Y^k, X^{k, \mathfrak{h}}], [Y^k, N^k]) \\ & \xrightarrow{(J_1(\mathbb{R}^{\ell \times 6\ell}), \mathbb{L}^1)} ([Y^\infty], [Z^\infty \cdot X^{\infty, \circ} + U^\infty \star \tilde{\mu}^{\infty, \mathfrak{h}}], 0, [Y^\infty, X^{\infty, \circ}], [Y^\infty, X^{\infty, \mathfrak{h}}], 0), \end{aligned} \quad (3.2)$$

$$\begin{aligned} & (\langle Y^k \rangle, \langle Z^k \cdot X^{k, \circ} \rangle, \langle U^k \star \tilde{\mu}^{k, \mathfrak{h}} \rangle, \langle N^k \rangle, \langle Y^k, X^{k, \circ} \rangle, \langle Y^k, X^{k, \mathfrak{h}} \rangle, \langle Y^k, N^k \rangle) \\ & \xrightarrow{(J_1(\mathbb{R}^{\ell \times 7\ell}), \mathbb{L}^1)} (\langle Y^\infty \rangle, \langle Z^\infty \cdot X^{\infty, \circ} \rangle, \langle U^\infty \star \tilde{\mu}^{\infty, \mathfrak{h}} \rangle, 0, \langle Y^\infty, X^{\infty, \circ} \rangle, \langle Y^\infty, X^{\infty, \mathfrak{h}} \rangle, 0), \end{aligned} \quad (3.3)$$

where 0 denotes the zero process whose state space is a finite-dimensional Euclidean space.

⁶The reader may recall the domain of the generator; see Condition (F3).

3.1 Outline of the proof

The main strategy for the proof can be visualised in Table 1. Conditions (S1)–(S10) ensure the convergence of the standard data; this corresponds to the first column of the table, and the respective convergence is denoted with a solid arrow. Using Papapantoleon, Possamai, and Saplaouras [86, Corollary 3.15], we can associate to the standard data \mathcal{D}^k , for each $k \in \overline{\mathbb{N}}$, the sequence of Picard iterations $(S^{k,(p)})_{p \in \mathbb{N} \cup \{0\}}$ (where $S^{k,(0)}$ is the zero element of the respective product space), which converges to the unique solution S^k ; this corresponds to the k -row of the table. Our aim is to prove the convergence $S^k \xrightarrow{k \rightarrow \infty} S^\infty$, which corresponds to the convergence in the last column of the table, and is denoted by a wiggly arrow.

Table 1: The doubly-indexed Picard scheme.

\mathcal{D}^1	$S^{1,(0)}$	$S^{1,(1)}$	$S^{1,(2)}$	\dots	$S^{1,(p)}$	$\xrightarrow{p \rightarrow \infty}$	S^1
\mathcal{D}^2	$S^{2,(0)}$	$S^{2,(1)}$	$S^{2,(2)}$	\dots	$S^{2,(p)}$	$\xrightarrow{p \rightarrow \infty}$	S^2
\mathcal{D}^3	$S^{3,(0)}$	$S^{3,(1)}$	$S^{3,(2)}$	\dots	$S^{3,(p)}$	$\xrightarrow{p \rightarrow \infty}$	S^3
\vdots	\vdots	\vdots	\vdots		\vdots		\vdots
\mathcal{D}^k	$S^{k,(0)}$	$S^{k,(1)}$	$S^{k,(2)}$	\dots	$S^{k,(p)}$	$\xrightarrow{p \rightarrow \infty}$	S^k
\downarrow	\downarrow	\downarrow	\downarrow		\downarrow		\downarrow
\mathcal{D}^∞	$S^{\infty,(0)}$	$S^{\infty,(1)}$	$S^{\infty,(2)}$	\dots	$S^{\infty,(p)}$	$\xrightarrow{p \rightarrow \infty}$	S^∞

Strictly speaking, one can generally construct the elements of the doubly-indexed sequence $(S^{k,(p)})_{k \in \overline{\mathbb{N}}, p \in \mathbb{N}}$ and not the elements of $(S^k)_{k \in \overline{\mathbb{N}}}$. Consequently, in order to achieve our aim, we will apply Moore–Osgood’s theorem on $(S^{k,(p)})_{k \in \overline{\mathbb{N}}, p \in \mathbb{N}}$. The aforementioned theorem provides a sufficient framework for the existence of the (unconditional) limit of a doubly-indexed sequence. In our case we will obtain

$$\lim_{(k,p) \rightarrow (\infty, \infty)} \|S^{k,(p)} - S^\infty\| = \lim_{k \rightarrow \infty} \lim_{p \rightarrow \infty} \|S^{k,(p)} - S^\infty\| = \lim_{p \rightarrow \infty} \lim_{k \rightarrow \infty} \|S^{k,(p)} - S^\infty\| = 0.$$

Moore–Osgood’s theorem requires the uniform convergence in one direction (here, say horizontally) and the pointwise convergence in the other direction (here, say vertically). Theorem 3.2 guarantees the finally⁷ uniform in k convergence of the sequence of Picard approximations $\{(S^{k,(p)})_{p \in \mathbb{N}}\}_{k \in \overline{\mathbb{N}}}$. In other words, the first condition of Moore–Osgood’s theorem is relatively effortlessly satisfied. In Table 1, we have denoted these convergences with a solid arrow. The second condition of Moore–Osgood’s theorem amounts to proving the convergence

$$\lim_{k \rightarrow \infty} \|S^{k,(p)} - S^{\infty,(p)}\| = 0, \text{ for every } p \in \mathbb{N}. \quad (3.4)$$

In Table 1, we have denoted these convergences with a dashed arrow. Naturally, we will prove the required pointwise convergence by means of induction. However, a series of helpful comments will reduce the complexity of the proof of Convergence (3.4); the details are postponed until Subsection 3.3. The first step of the induction will be proved in Subsection 3.4 and the p -th step of the induction in Subsection 3.5.

⁷We are going to use the following convention: whenever we write that a sequence $(\alpha^{k,p})_{k,p \in \overline{\mathbb{N}}}$ ‘converges finally uniformly in k ’ (under the metric δ), we mean that there exists $k_0 \in \mathbb{N}$ such that $\sup_{k \geq k_0} \delta(\alpha^{k,p}, \alpha^{k,\infty}) \xrightarrow{p \rightarrow \infty} 0$.

3.2 Uniform *a priori* BSDE estimates

The next result provides uniform *a priori* estimates for the tail of the Picard approximations, see Papapantoleon, Possamaï, and Saplaouras [86, Corollary 3.15]. In particular, Theorem 3.2 generalises Briand, Delyon, and Mémmin [19, Corollary 10].

Proposition 3.2. *For every $k \in \overline{\mathbb{N}}$, we associate to the standard data \mathcal{D}^k the sequence of Picard iterations $(S^{k,(p)})_{p \in \mathbb{N} \cup \{0\}}$, where $S^{k,(0)}$ is the zero element of $S_{k,\hat{\beta}}^2 \times \mathbb{H}_{\hat{\beta}}^{2,k,\circ} \times \mathbb{H}_{\hat{\beta}}^{2,k,\natural} \times \mathbb{H}_{\hat{\beta}}^{2,k,\perp}$. There exists $k_{*,0} \in \mathbb{N}$ s.t.*

$$\lim_{p \rightarrow \infty} \sup_{k \geq k_{*,0}} \|S^k - S^{k,(p)}\|_{*,k,\hat{\beta}}^2 = 0.$$

In particular, $\sup_{k \geq k_{*,0}} \|S^k\|_{*,k,\hat{\beta}}^2 < \infty$.

Proof. We choose $k_{*,0}$ as the one determined by Lemma A.16. Essentially, we have constructed contractions associated to the standard data $(\mathcal{D}^k)_{k \geq k_{*,0}}$ whose constant is smaller than $1/\sqrt{2}$. Then, for any integer $k \geq k_{*,0}$, we have

$$\begin{aligned} \|S^k - S^{k,(p)}\|_{*,k,\hat{\beta}}^2 &\leq \sum_{n=0}^{\infty} 2^{n+1} \|S^{k,(p+n)} - S^{k,(p+n+1)}\|_{*,k,\hat{\beta}}^2 \\ &\leq \sum_{n=0}^{\infty} \frac{2^{n+1}}{2^{p+n}} \|S^{k,(1)}\|_{*,k,\hat{\beta}}^2 = 4^{1-p} \|S^{k,(1)}\|_{*,k,\hat{\beta}}^2, \end{aligned} \quad (3.5)$$

where in the second inequality we have used [86, Inequality (3.42)] and that in Lemma A.16 $M_{\star}^{\Phi^k}(\hat{\beta}) < \frac{1}{4}$ for $k \geq k_{*,0}$. Since $S^{k,(0)}$ is the zero element, we obtain by [86, Lemma 3.8]

$$\|S^{k,(1)}\|_{*,k,\hat{\beta}}^2 \leq \tilde{\Pi}_{\star}^{\hat{\beta},\Phi^k} \|\xi^k\|_{\mathbb{L}_{\hat{\beta}}^2}^2 + \Pi_{\star}^{\Phi^k}(\bar{\gamma}_{\star}^{\Phi^k}, \hat{\beta}) \left\| \frac{f^k(\cdot, 0, 0, \mathbf{0})}{\alpha^k} \right\|_{\mathbb{H}_{k,\hat{\beta}}^2}^2.$$

Again by Theorem A.16, we derive the uniform bound

$$\begin{aligned} \sup_{k \geq k_0^*} \|S^{k,(1)}\|_{*,k,\hat{\beta}}^2 \\ \leq \tilde{\Pi}_{\star}^{\hat{\beta},\Phi^{k_{*,0}}} \sup_{k \geq k_0^*} \|\xi^k\|_{\mathbb{L}_{\hat{\beta}}^2(\mathcal{G}_{\tau^k}^k)}^2 + \Pi_{\star}^{\Phi^k}(\bar{\gamma}_{\star}^{\Phi^{k_{*,0}}}, \hat{\beta}) \sup_{k \geq k_0^*} \left\| \frac{f^k(\cdot, 0, 0, \mathbf{0})}{\alpha^k} \right\|_{\mathbb{H}_{*,k,\hat{\beta}}^2}^2 < \infty, \end{aligned} \quad (3.7)$$

which implies the desired result in conjunction with Inequality (3.5) and Condition (S6).

For the second statement, we combine Inequalities (3.5) and (3.7) and we obtain

$$\begin{aligned} \sup_{k \geq k_{*,0}} \|S^k\|_{*,k,\hat{\beta}}^2 \\ \leq 2(1 + \tilde{\Pi}_{\star}^{\hat{\beta},\Phi^{k_{*,0}}}) \sup_{k \geq k_0^*} \|\xi^k\|_{\mathbb{L}_{\hat{\beta}}^2}^2 + 2(1 + \Pi_{\star}^{\Phi^k}(\bar{\gamma}_{\star}^{\Phi^{k_{*,0}}}, \hat{\beta})) \sup_{k \geq k_0^*} \left\| \frac{f^k(\cdot, 0, 0, \mathbf{0})}{\alpha^k} \right\|_{\mathbb{H}_{\hat{\beta}}^{2,k}}^2 < \infty, \end{aligned}$$

where $\tilde{\Pi}_{\star}^{\hat{\beta},\Phi^{k_{*,0}}}$ is defined in [86, Lemma 3.8] as follows

$$\mathbb{R}_+ \times \mathbb{R}_+ \ni (\delta, \Phi) \longmapsto 17 + 9e^{\delta\Phi} \in \mathbb{R}_+. \quad \square$$

Corollary 3.3. *Let $M_{\infty}^{k,(p)} := Z^{k,(p)} \cdot X_{\infty}^{k,\circ} + U^{k,(p)} \star \tilde{\mu}_{\infty}^{k,\natural} + N_{\infty}^{k,(p)}$, for $k \in \overline{\mathbb{N}}$, $p \in \mathbb{N} \cup \{0\}$ and $M_{\infty}^k := Z^k \cdot X_{\infty}^{k,\circ} + U^k \star \tilde{\mu}_{\infty}^{k,\natural} + N_{\infty}^k$, for $k \in \overline{\mathbb{N}}$. Then, there exists $\bar{k}_{*,0} \in \mathbb{N}$ such that*

$$\lim_{p \rightarrow \infty} \sup_{k \geq \bar{k}_{*,0}} \|M_{\infty}^{k,(p)} - M_{\infty}^k\|_{\mathbb{L}^2} = 0,$$

and

$$\lim_{p \rightarrow \infty} \sup_{k \geq \bar{k}_{*,0}} \mathbb{E} \left[\left\| \text{Var} \left[\int_{(0,\infty)} \delta f_s^{k,(p)} dC_s^k \right] \right\|_2^2 \right] = 0,$$

where, for every $s \in [0, \infty)$,

$$\delta f_s^{k,(p)} := f^k(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s, \cdot)) - f^k(s, Y_s^k, Z_s^k, U^k(s, \cdot)).$$

Proof. The first limit is immediate from Theorem 3.2, Itô's isometry, the orthogonality of the respective spaces (see [86, Corollary 2.7]) and the inequalities

$$\|M_\infty^{k,(p)} - M_\infty^k\|_{\mathbb{L}^2} \leq \|M_\infty^{k,(p)} - M_\infty^k\|_{\mathbb{L}_{\beta}^{2,k}} \leq \|S^{k,(p)} - S^k\|_{*,k,\hat{\beta}}.$$

The second limit is again immediate from Theorem 3.2, Cauchy-Schwarz's inequality (as applied in [86, Inequality 3.16]), the Lipschitz property of the generator f^k for every $k \in \overline{\mathbb{N}}$, the inequalities

$$\frac{r^k}{(\alpha^k)^2} \leq (\alpha^k)^2, \quad \frac{\theta^{k,\circ}}{(\alpha^k)^2}, \quad \frac{\theta^{k,\natural}}{(\alpha^k)^2} \leq 1, \quad \text{for every } k \in \overline{\mathbb{N}},$$

and Condition (S7), i.e., the boundedness of $(A^k)_{k \in \overline{\mathbb{N}}}$. More precisely, we have

$$\begin{aligned} & \sup_{k \geq \bar{k}_{*,0}} \mathbb{E} \left[\left\| \text{Var} \left[\int_{(0,\infty)} \delta f_s^{k,(p)} dC_s^k \right] \right\|_2^2 \right] \\ & \leq \sup_{k \geq \bar{k}_{*,0}} \mathbb{E} \left[\sum_{i=1}^{\ell} \left(\int_{(0,\infty)} |\delta f_s^{k,(p),i}| dC_s^k \right)^2 \right] \leq \frac{1}{\beta} \sup_{k \geq \bar{k}_{*,0}} \mathbb{E} \left[\int_{(0,\infty)} e^{\hat{\beta} A_s^k} \frac{\|\delta f_s^{k,(p)}\|_2^2}{(\alpha^k)^2} dC_s^k \right] \\ & \leq \frac{1}{\hat{\beta}} \sup_{k \geq \bar{k}_{*,0}} \left\{ \bar{A} \|Y^{k,(p)} - Y^k\|_{\mathbb{S}_{\beta}^{2,k}}^2 + \|(Z^{k,(p)} - Z^k) \cdot X^{k,\circ} + (U^{k,(p)} - U^k) \star \tilde{\mu}^{k,\natural}\|_{\mathcal{H}_{\beta}^{2,k}}^2 \right\} \\ & \leq \frac{\bar{A} \vee 1}{\hat{\beta}} \sup_{k \geq \bar{k}_{*,0}} \|S^{k,(p)} - S^k\|_{*,k,\hat{\beta}} \xrightarrow{p \rightarrow \infty} 0. \end{aligned} \quad \square$$

3.3 Reducing the complexity of the induction steps and proving Theorem 3.1

The purpose of the current subsection, is to explain the strategy that will allow us to reduce the complexity of Convergence (3.4) and to prove Theorem 3.1.

Notation 3.4. In order to use as compact notation as possible, we introduce the following

- $L_t^{k,(p)} := \int_{(0,t]} f^k(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s, \cdot)) dC_s^k$, for $t \in [0, \infty]$,⁸ $k \in \overline{\mathbb{N}}$ and $p \in \mathbb{N} \cup \{0\}$. For $i \in \{1, \dots, \ell\}$, we denote the i -element of $L_t^{k,(p)}$ by $L_t^{k,(p),i}$.
- $L_t^k := \int_{(0,t]} f^k(s, Y_s^k, Z_s^k, U^k(s, \cdot)) dC_s^k$, for $t \in [0, \infty]$, $k \in \overline{\mathbb{N}}$ and $p \in \mathbb{N} \cup \{0\}$. For $i \in \{1, \dots, \ell\}$, we denote the i -element of L_t^k by $L_t^{k,i}$.
- $\Gamma^{k,(p)} := \int_{(0,\infty)} \frac{\|f^k(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s, \cdot))\|_2^2}{(\alpha_s^k)^2} dC_s^k$, for $k \in \overline{\mathbb{N}}$ and $p \in \mathbb{N} \cup \{0\}$.

⁸For $t = \infty$, we have abused notation and we understand the interval $(0, \infty]$ as $(0, \infty)$. Actually, the processes C^k , for $k \in \overline{\mathbb{N}}$, are (left) continuous at $t = \infty$ and therefore there is no difference on which integral we consider.

$$\bullet \Delta^{k,(p)} := \text{Tr}[\langle Z^{k,(p)} \cdot X^{k,\circ} + U^{k,(p)} \star \tilde{\mu}^{k,\natural} \rangle_\infty], \text{ for } k \in \overline{\mathbb{N}} \text{ and } p \in \mathbb{N}.$$

After the introduction of these helpful notations, we focus on the aim of the current subsection. We claim that Convergence (3.4) is equivalent to proving the validity of the following two convergences

$$L_\infty^{k,(p)} \xrightarrow[k \rightarrow \infty]{\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}^\ell)} L_\infty^{\infty,(p)}, \quad (\text{LS}_\infty^{(p)})$$

$$L_\cdot^{k,(p)} \xrightarrow[k \rightarrow \infty]{(\mathbb{J}_1(\mathbb{R}^\ell), \mathbb{L}^2)} L_\cdot^{\infty,(p)}, \quad (\text{LS}^{(p)})$$

for every $p \in \mathbb{N} \cup \{0\}$, in conjunction with

$$\text{the sequence } (\Gamma^{k,(p)})_{k \in \overline{\mathbb{N}}} \text{ is uniformly integrable,} \quad (\text{UI}^{(p)})$$

for every $p \in \mathbb{N} \cup \{0\}$. As should be expected, they will be proven by induction.

Next, we will assume within the current subsection that the aforementioned convergences are true for every $p \in \mathbb{N} \cup \{0\}$, together with the uniform *a priori* estimates of subsection 3.2, and we will prove Theorem 3.1 under these assumptions. Then, we will prove in the upcoming subsections the validity of these assumptions, i.e. the validity of convergences $(\text{LS}_\infty^{(p)})$, $(\text{LS}^{(p)})$ and $(\text{UI}^{(p)})$, for every $p \in \mathbb{N} \cup \{0\}$.

3.3.1 Convergence (3.1) is true

Unsurprisingly, we are going to transform the BSDEs associated to the Picard schemes into martingale representations and then use the stability already proved for the latter; see [87, Corollary 3.10]. Recall that for every $k \in \overline{\mathbb{N}}$, we have stopped the processes indexed by k at time τ^k . Therefore, we can substitute for every $k \in \overline{\mathbb{N}}$ the terminal time τ^k by ∞ and we will do so for notational convenience. The reader may also keep in mind that $N^{\infty,(p)} = 0$, for every $p \in \mathbb{N}$.

For every $k \in \overline{\mathbb{N}}$ and every $p \in \mathbb{N} \cup \{0\}$ it is true (by construction) that

$$\begin{aligned} Y_t^{k,(p+1)} &= \xi^k + \int_{(t, \tau^k]} f^k(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s, \cdot)) dC_s^k \\ &\quad - \int_t^{\tau^k} Z_s^{k,(p+1)} dX_s^{k,\circ} - \int_t^{\tau^k} \int_{\mathbb{R}^n} U^{k,(p+1)}(s, x) \tilde{\mu}^{k,\natural}(ds, dx) - \int_t^{\tau^k} dN_s^{k,(p+1)} \\ &= \xi^k + \int_{(t, \infty)} f^k(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s, \cdot)) dC_s^k \\ &\quad - \int_t^\infty Z_s^{k,(p+1)} dX_s^{k,\circ} - \int_t^\infty \int_{\mathbb{R}^n} U^{k,(p+1)}(s, x) \tilde{\mu}^{k,\natural}(ds, dx) - \int_t^\infty dN_s^{k,(p+1)}. \end{aligned} \quad (3.8)$$

By Theorem 3.2, we get that $\mathcal{S}^{k,(p)} \xrightarrow[p \rightarrow \infty]{\|\cdot\|_{*,k,\hat{\beta}}} \mathcal{S}^k$ finally uniformly in k . In particular,

this convergence implies $Y^{k,(p)} \xrightarrow[p \rightarrow \infty]{\mathcal{S}_{k,\hat{\beta}}^2} Y^k$ finally uniformly in k , which in turn implies

$Y^{k,(p)} \xrightarrow[p \rightarrow \infty]{(\mathbb{J}_1(\mathbb{R}^\ell), \mathbb{L}^2)} Y^k$ finally uniformly in k . Hence, by orthogonality of the respective parts, Itô's isometry and Doob's inequality, see [86, Section 3.5], we obtain

$$(Y^{k,(p)}, Z^{k,(p)} \cdot X^{k,\circ}, U^{k,(p)} \star \tilde{\mu}^{k,\natural}, N^{k,(p)}) \xrightarrow[p \rightarrow \infty]{(\|\cdot\|_\infty, \mathbb{L}^2)} (Y^k, Z^k \cdot X^{k,\circ}, U^k \star \tilde{\mu}^{k,\natural}, N^k),$$

finally uniformly in k . At this point, we will combine two facts in order to rewrite the above convergence under the \mathbb{J}_1 -topology. The first one is that the convergence under

the $\|\cdot\|_\infty$ -norm allows us to conclude the convergence of the sum of two convergent sequences. The second is that every $\|\cdot\|_\infty$ -convergent sequence is also J_1 -convergent; see [65, Proposition VI.1.17]. The latter argument was also used a few lines above. Therefore

$$(Y^{k,(p)}, Z^{k,(p)} \cdot X^{k,\circ} + U^{k,(p)} \star \tilde{\mu}^{k,\natural}, N^{k,(p)}) \xrightarrow[p \rightarrow \infty]{(J_1(\mathbb{R}^{\ell \times 3}), \mathbb{L}^2)} (Y^k, Z^k \cdot X^{k,\circ} + U^k \star \tilde{\mu}^{k,\natural}, N^k), \quad (3.9)$$

finally uniformly in k . Consequently, in order to apply Moore–Osgood’s theorem, see Theorem A.2, it is sufficient to prove that for every $p \in \mathbb{N} \cup \{0\}$, we have

$$(Y^{k,(p)}, Z^{k,(p)} \cdot X^{k,\circ} + U^{k,(p)} \star \tilde{\mu}^{k,\natural}, N^{k,(p)}) \xrightarrow[k \rightarrow \infty]{(J_1(\mathbb{R}^{\ell \times 3}), \mathbb{L}^2)} (Y^{\infty,(p)}, Z^{\infty,(p)} \cdot X^{\infty,\circ} + U^{\infty,(p)} \star \tilde{\mu}^{\infty,\natural}, N^{\infty,(p)}). \quad (3.10)$$

To this end, let us relate BSDE (3.8) to appropriate martingales.⁹ The aforementioned transformation will allow us to use the stability of martingale representations; see [87, Corollary 3.10]. For fixed $k \in \overline{\mathbb{N}}$, $p \in \mathbb{N} \cup \{0\}$ we define for $t \in [0, \infty]$

$$\begin{aligned} M_t^{k,(p)} &:= Y_t^{k,(p+1)} + \int_{(0,t]} f^k(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s, \cdot)) dC_s^k \\ &\stackrel{(3.8)}{=} \xi^k + \int_{(0,\infty)} f^k(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s, \cdot)) dC_s^k \\ &\quad - \int_t^\infty Z_s^{k,(p+1)} dX_s^{k,\circ} - \int_t^\infty \int_{\mathbb{R}^n} U^{k,(p+1)}(s, x) \tilde{\mu}^{k,\natural}(ds, dx) - \int_t^\infty dN_s^{k,(p+1)} \\ &\stackrel{(3.8)}{=} Y_0^{k,(p+1)} \\ &\quad + \int_0^\infty Z_s^{k,(p+1)} dX_s^{k,\circ} + \int_0^\infty \int_{\mathbb{R}^n} U^{k,(p+1)}(s, x) \tilde{\mu}^{k,\natural}(ds, dx) + \int_0^\infty dN_s^{k,(p+1)} \\ &\quad - \int_t^\infty Z_s^{k,(p+1)} dX_s^{k,\circ} - \int_t^\infty \int_{\mathbb{R}^n} U^{k,(p+1)}(s, x) \tilde{\mu}^{k,\natural}(ds, dx) - \int_t^\infty dN_s^{k,(p+1)} \\ &= Y_0^{k,(p+1)} + \int_0^t Z_s^{k,(p+1)} dX_s^{k,\circ} \\ &\quad + \int_0^t \int_{\mathbb{R}^n} U^{k,(p+1)}(s, x) \tilde{\mu}^{k,\natural}(ds, dx) + \int_0^t dN_s^{k,(p+1)}. \end{aligned} \quad (3.11) \quad (3.12)$$

Hence $M^{k,(p)} \in \mathcal{H}^2(\mathbb{G}^k, \tau^k; \mathbb{R}^\ell)$ for every $k \in \overline{\mathbb{N}}$ and every $p \in \mathbb{N} \cup \{0\}$. At this point, for fixed $p \in \mathbb{N} \cup \{0\}$ we can obtain the convergence

$$(Z^{k,(p+1)} \cdot X^{k,\circ} + U^{k,(p+1)} \star \tilde{\mu}^{k,\natural}, N^{k,(p+1)}) \xrightarrow[k \rightarrow \infty]{(J_1(\mathbb{R}^{\ell \times 2}), \mathbb{L}^2)} (Z^{\infty,(p+1)} \cdot X^{\infty,\circ} + U^{\infty,(p+1)} \star \tilde{\mu}^{\infty,\natural}, 0), \quad (3.13)$$

if we apply [87, Corollary 3.10] to the sequence $(M^{k,(p)})_{k \in \overline{\mathbb{N}}}$. In view of Conditions (S1), (S2), (S3) and (S4), we need only to prove the convergence

$$M_\infty^{k,(p)} \xrightarrow[k \rightarrow \infty]{\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}^\ell)} M_\infty^{\infty,(p)}, \quad (3.14)$$

for every $p \in \mathbb{N} \cup \{0\}$, in order to apply [87, Corollary 3.10].

⁹The same technique was used in the proof of [86, Theorem 3.5] in order to use finally the orthogonal decomposition of square-integrable martingales.

However, in view of Condition (S5), which states that the sequence $(\xi^k)_{k \in \overline{\mathbb{N}}}$ is \mathbb{L}^2 -convergent, and recalling that for the Picard schemes holds $Y_\infty^{k,(p+1)} = \xi^k$, for every $k \in \overline{\mathbb{N}}$ and $p \in \mathbb{N} \cup \{0\}$, we immediately obtain from Identity (3.11) that (for the same p)

$$L_\infty^{k,(p)} \xrightarrow[k \rightarrow \infty]{\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}^\ell)} L_\infty^{\infty,(p)}, \quad (\text{LS}_\infty^{(p)})$$

is equivalent to Convergence (3.14).

Assume for the following that $(\text{LS}_\infty^{(p)})$ is valid for every $p \in \mathbb{N} \cup \{0\}$, i.e. we can apply [87, Corollary 3.10] for the martingale sequence $(M^{k,(p)})_{k \in \overline{\mathbb{N}}}$. Then, we obtain the convergence

$$\begin{aligned} (M^{k,(p)}, Z^{k,(p+1)} \cdot X^{k,\circ} + U^{k,(p+1)} \star \tilde{\mu}^{k,\natural}, N^{k,(p+1)}) &\xrightarrow{(\mathbb{J}_1(\mathbb{R}^{\ell \times 3}), \mathbb{L}^2)} \\ (M^{\infty,(p)}, Z^{\infty,(p+1)} \cdot X^{\infty,\circ} + U^{\infty,(p+1)} \star \tilde{\mu}^\infty, 0), \end{aligned} \quad (3.15)$$

for every $p \in \mathbb{N} \cup \{0\}$. Comparing now Convergence (3.15) with Convergence (3.10) associated to the $(p+1)$ -Picard step, we realise that they differ only in the first element. Recall the definition of $M^{k,(p)}$, for $k \in \overline{\mathbb{N}}$; see (3.11). It is immediate¹⁰ that we can obtain the convergence

$$(M^{k,(p)}, Y^{k,(p+1)}) \xrightarrow{(\mathbb{J}_1(\mathbb{R}^2), \mathbb{L}^2)} (M^{\infty,(p)}, Y^{\infty,(p+1)}), \text{ for every } p \in \mathbb{N} \cup \{0\}, \quad (3.16)$$

if the convergence

$$L_\cdot^{k,(p)} \xrightarrow{(\mathbb{J}_1(\mathbb{R}^\ell), \mathbb{L}^2)} L_\cdot^{\infty,(p)} \quad (\text{LS}^{(p)})$$

holds for every $p \in \mathbb{N} \cup \{0\}$. Hence, our claim as stated in the title of the current section is valid.

Corollary 3.5. *If the convergences $(\text{LS}_\infty^{(p)})$ and $(\text{LS}^{(p)})$ are true for every $p \in \mathbb{N} \cup \{0\}$, then*

$$M_\infty^{k,(p)} \xrightarrow[(k,p) \rightarrow (\infty, \infty)]{\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}^\ell)} M_\infty^\infty, \quad (3.17)$$

$$L_\infty^{k,(p)} \xrightarrow[(k,p) \rightarrow (\infty, \infty)]{\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}^\ell)} L_\infty^\infty. \quad (3.18)$$

In both cases, the iterated limits exist and are equal to the respective right-hand side.

Proof. In view of Theorem 3.3 and Equation (3.14), Moore–Osgood’s theorem (see Theorem A.2) ensures the convergence in (3.17). For the convergence in (3.18), one uses (3.11) and (3.17) in conjunction with the convergence

$$\xi^k \xrightarrow[k \rightarrow \infty]{\mathbb{L}^2(\mathcal{G}; \mathbb{R}^\ell)} \xi^\infty;$$

see Condition (S5). □

¹⁰Since we are using the \mathbb{J}_1 -topology, we have to be careful with arguments like this. However, the continuity of the process corresponding to the Lebesgue–Stieltjes integrals with respect to C^∞ (recall that $\Phi^\infty = 0$) allows us to proceed.

3.3.2 Convergence (3.2) is true

The reader should recall that the convergences $(LS_\infty^{(p)})$ and $(LS^{(p)})$ are assumed true for every $p \in \mathbb{N} \cup \{0\}$. We will apply Moore–Osgood’s theorem for the doubly-indexed sequence $([S^{k,(p)}])_{k \in \overline{\mathbb{N}}, p \in \mathbb{N} \cup \{0\}}$. It is well understood that the convergence of a sequence of (special) semimartingales does not guarantee the convergence of the associated square brackets. However, if the sequence is P-UT (see Jacod and Shiryaev [65, Definition VI.6.1]), then [65, Theorem VI.6.26] ensures that we have the desired convergence. To this end, we will prove that the P-UT property holds in a proper sense; the details will be provided below, but one could describe it as ‘the sequence $(S^{k,(p)})_{k \in \overline{\mathbb{N}}, p \in \mathbb{N} \cup \{0\}}$ is P-UT finally uniformly in k ’.

In this paragraph we focus initially on the sequences corresponding to the martingales of the Picard schemes, where we have also adjoined the sequence $(M^{k,(p)})_{k \in \overline{\mathbb{N}}, p \in \mathbb{N} \cup \{0\}}$. In this case, an integrability condition, namely [65, Corollary VI.6.30], is sufficient for the P-UT property.¹¹ Moreover, from Convergence (3.17) we have, in particular, that the sequence $(\|M_\infty^{k,(p)}\|_2^2)_{k \in \overline{\mathbb{N}}, p \in \mathbb{N} \cup \{0\}}$ is uniformly integrable. Therefore, in view of [65, Corollary VI.6.30], we can conclude that the sequence of martingales is P-UT and the (joint) convergence of the square brackets is obtained.

In this paragraph we turn our attention to the sequence $(Y^{k,(p)})_{k \in \overline{\mathbb{N}}, p \in \mathbb{N} \cup \{0\}}$. The reader should recall that $Y^{k,(p)} = M^{k,(p)} + L^{k,(p)}$ for every $k \in \overline{\mathbb{N}}, p \in \mathbb{N} \cup \{0\}$. We have already argued about the P-UT property of the martingale sequence $(M^{k,(p)})_{k \in \overline{\mathbb{N}}, p \in \mathbb{N} \cup \{0\}}$ and the convergence of the associated square bracket sequence. The continuity of L^∞ , hence the continuity of $[L^\infty] = 0$, allows us to simply derive the convergence of $([L^{k,(p)}])_{k \in \overline{\mathbb{N}}, p \in \mathbb{N} \cup \{0\}}$ to the zero process. Then, we can have the joint convergence of the square bracket sequences, and the polarisation identity will allow us to conclude the desired Convergence (3.2). We leave these details to the reader. Hence, it is left to prove the convergence

$$[L^{k,(p)}] \xrightarrow[(k,p) \rightarrow (\infty, \infty)]{(J_1(\mathbb{R}^{\ell \times \ell}, \mathbb{L}^1))} 0.$$

We will use Moore–Osgood’s theorem once again, this time for $([L^{k,(p)}])_{k \in \overline{\mathbb{N}}, p \in \mathbb{N} \cup \{0\}}$. On the one hand, from [65, Comment VI.6.6], the sequence $(L^{k,(p)})_{p \in \mathbb{N} \cup \{0\}}$ is P-UT, for every $k \in \overline{\mathbb{N}}$, if the sequence

$$\left(\text{Tr} \left[\text{Var} [L^{k,(p)}]_t \right] \right)_{p \in \mathbb{N} \cup \{0\}},$$

is tight in \mathbb{R} , for every $t \in \mathbb{R}_+$, where the total variation is calculated element-wise. Using Theorem 3.3 and Theorem 3.2, we have that for the arbitrary $\varepsilon > 0$, there exists $p_0 \in \mathbb{N}$, which depends only on ε and not on k , such that

$$\begin{aligned} & \sup_{p \geq p_0} \sup_{k \geq k_{*,0}} \mathbb{E} \left[\left\| \text{Var} [L^{k,(p)}]_t \right\|_2^2 \right] \\ & \leq 2 \sup_{p \geq p_0} \sup_{k \geq k_{*,0}} \mathbb{E} \left[\left\| \text{Var} [L^{k,(p)} - L^k]_t \right\|_2^2 \right] + 2 \sup_{k \geq k_{*,0}} \mathbb{E} \left[\left\| \text{Var} [L^k]_t \right\|_2^2 \right] \\ & \leq 2\varepsilon + 2 \sup_{k \geq k_{*,0}} \mathbb{E} \left[\left\| \text{Var} [L^k]_t \right\|_2^2 \right] \\ & \leq \varepsilon + \sup_{k \geq k_{*,0}} \|S^k\|_{*,k,\hat{\beta}} + \sup_{k \geq k_{*,0}} \left\| \frac{f^k(\cdot, 0, 0, \mathbf{0})}{\alpha^k} \right\|_{H_{\hat{\beta}}^{2,k}} < \infty. \end{aligned}$$

¹¹The reader can immediately verify from Convergence (3.17) that the martingale sequences are \mathbb{L}^2 -bounded, a property that implies Condition 6.31 of [65, Corollary 6.30].

For the derivation of the upper bound of the total variation of L^k we have used arguments analogous to Theorem 3.3. Now, Markov's inequality implies the boundedness in probability (uniformly in k) of the desired sequence. Hence, for every $k \geq k_{*,0}$, the sequence $(L^{k,(p)})_{p \in \mathbb{N} \cup \{0\}}$ is P-UT and consequently

$$[L^{k,(p)}] \xrightarrow[p \rightarrow \infty]{(\|\cdot\|_\infty, \mathbb{L}^1)} [L^k], \text{ for every } k \geq k_{*,0},$$

by means of [65, Theorem VI.6.26]. We can derive the convergence of the square brackets finally uniform in k from the following inequality

$$\begin{aligned} \text{Tr}[[L^{k,(p)} - L^k]_\infty] &= \sum_{i=1}^{\ell} \sum_{t \geq 0} (\Delta(L^{k,(p)}, i) - L^{k,i})_t^2 \leq \sum_{i=1}^{\ell} \left(\sum_{t \geq 0} |\Delta(L^{k,(p)}, i) - L^{k,i}|_t \right)^2 \\ &\leq \|\text{Var}[L^{k,(p)} - L^k]\|_2^2, \end{aligned}$$

which is true for every k . On the other hand, the induction hypothesis for every $p \in \mathbb{N} \cup \{0\}$ allows us to apply [87, Corollary 3.10], which in particular provides

$$[L^{k,(p)}] \xrightarrow[k \rightarrow \infty]{(J_1(\mathbb{R}^{\ell \times \ell}), \mathbb{L}^1)} [L^{\infty,(p)}], \text{ for every } p \in \mathbb{N} \cup \{0\}.$$

Overall, the conditions of Moore–Osgood's theorem are satisfied, hence we have

$$[L^{k,(p)}] \xrightarrow[(k,p) \rightarrow (\infty, \infty)]{(J_1(\mathbb{R}^{\ell \times \ell}), \mathbb{L}^1)} [L^\infty] = 0.$$

Both iterated limits of the sequence $([L^{k,(p)}])_{k \in \overline{\mathbb{N}}, p \in \mathbb{N} \cup \{0\}}$ exist and are equal to 0. The continuity of the limit allows us to derive the (joint) convergence (3.2).

3.3.3 Convergence (3.3) is true

The convergence of its martingale parts can be justified because of [87, Corollary 3.10], whose validity implied the convergence of the martingale parts of (3.1). For the parts associated to the square-integrable \mathbb{G}^k -special semimartingales $(Y^k)_{k \in \overline{\mathbb{N}}}$, we observe that for every $k \in \overline{\mathbb{N}}$ the process $[Y^k]$ is a \mathbb{G}^k -special semimartingale with canonical decomposition $[Y^k] = ([Y^k] - \langle Y^k \rangle) + \langle Y^k \rangle$. Moreover, the arguments presented above allow us to conclude that the sequence $(\text{Tr}[Y^k]_\infty)_{k \in \overline{\mathbb{N}}}$ is uniformly integrable. This further implies¹³ the uniform integrability of $(\text{Tr}[\langle Y^k \rangle_\infty])_{k \in \overline{\mathbb{N}}}$. Hence, the sequence associated to the angle brackets is also tight in \mathbb{R}^ℓ . Therefore, we can apply¹⁴ Mémin [81, Theorem 11] for the sequence $([Y^k])_{k \in \overline{\mathbb{N}}}$.

3.4 The first step of the induction is valid

Recall that in the proof of Theorem 3.2 we have set $\Upsilon^{k,(0)} = (0, 0, 0)$, for every $k \in \overline{\mathbb{N}}$. Now we provide some useful lemmata that we will then use for proving the first step of the induction in Theorem 3.9.

Lemma 3.6. *The sequences $(C^k)_{k \in \overline{\mathbb{N}}}$ and $(L^{k,(0)})_{k \in \overline{\mathbb{N}}}$ possess the P-UT property.*

¹²For the sequence spaces $\ell^1(\mathbb{N})$ and $\ell^2(\mathbb{N})$ it is true that $\ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$, i.e., for a sequence of real numbers $x := (x_p)_{p \in \mathbb{N}}$ it holds $\|x\|_{\ell^2(\mathbb{N})} := (\sum_{p \in \mathbb{N}} |x_p|^2)^{\frac{1}{2}} \leq \sum_{p \in \mathbb{N}} |x_p| = \|x\|_{\ell^1(\mathbb{N})}$.

¹³See Lenglart, Lépine, and Pratelli [72, Théorème 3.2.1]) and use the de La Vallée-Poussin criterion

¹⁴Along with standard arguments and the polarisation identity since the aforementioned theorem is stated in the real-valued case.

Proof. First, recall the fact that $(C^k)_{k \in \overline{\mathbb{N}}}$ is a sequence of increasing processes. Therefore, $\text{Var}(C^k) = C^k$, for every $k \in \overline{\mathbb{N}}$. Secondly, Condition (S9).(i) implies that $(C^k)_{k \in \overline{\mathbb{N}}}$ is tight in $\mathbb{D}(\mathbb{R}_+; \mathbb{R})$, which in turn implies that $\text{Var}(C^k)_t$ is tight in \mathbb{R} for every $t \in \mathbb{R}_+$. Now, we can conclude by [65, Proposition VI.6.12].

For the sequence $(L^{k,(0)})_{k \in \overline{\mathbb{N}}}$ it is sufficient to prove that $(\text{Var}[L^{k,(0)}]_\infty)_{k \in \overline{\mathbb{N}}}$ is tight in \mathbb{R}^ℓ ; see [65, Remark VI.6.6]. To this end, we will prove that the sequence is \mathbb{L}^2 -bounded. Indeed, by the following inequality (here $f^{k,i}$ denotes the i -element of f^k , for every $k \in \overline{\mathbb{N}}$, $i \in \{1, \dots, \ell\}$)

$$\text{Var}[L^{k,(0),i}]_\infty \leq \int_{(0,\infty)} |f^{k,i}(s, Y_s^{k,(p)}, Z_s^{k,(p)}, U^{k,(p)}(s, \cdot))| dC_s^k,$$

for every $i \in \{1, \dots, \ell\}$ and by Cauchy–Schwarz’s inequality, applied as in [86, Inequality (3.16)], we derive

$$\sup_{k \in \overline{\mathbb{N}}} \mathbb{E} \left[\|\text{Var}[L^{k,(0)}]_\infty\|_2^2 \right] \leq \frac{1}{\tilde{\beta}} \sup_{k \in \overline{\mathbb{N}}} \mathbb{E} [\Gamma^{k,(0)}] \stackrel{\text{(S6)}}{<} \infty. \quad \square$$

In the remainder of this article, we denote by μ_{C^i} the (random) measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ associated to the increasing and càdlàg process C^i , for $i \in \overline{\mathbb{N}}$.

Lemma 3.7. *For any subsequence $(C^{k_{l_i}})_{i \in \mathbb{N}}$ there exists a further subsequence $(C^{k_{l_i}})_{i \in \mathbb{N}}$ such that*

$$C^{k_{l_i}} \xrightarrow{i \rightarrow \infty} C^\infty, \text{ P-a.s.}, \text{ as well as } C_\infty^{k_{l_i}} \xrightarrow{i \rightarrow \infty} C_\infty^\infty, \text{ P-a.s.}$$

Moreover, $\mu_{C^{k_{l_i}}} \xrightarrow{i \rightarrow \infty} \mu_{C^\infty}$, P-a.s., and $\sup_{i \in \mathbb{N}} \mu_{C^{k_{l_i}}}(\mathbb{R}_+) < \infty$, P-a.s.

Proof. The first statement is direct by Dudley [34, Theorem 9.2.1]. Indeed, the fact that $(\mathbb{D}(\mathbb{R}_+; \mathbb{R}), \delta_{J_1})^{15}$ and $(\mathbb{R}, \delta_{|\cdot|})$ are both Polish spaces, together with Conditions (i), (ii) of (S9) allow us to verify the statement. Passing possibly to a further subsequence we can assume without loss of generality that both convergent sequences are indexed by $(k_{l_i})_{i \in \mathbb{N}}$. The second statement is also true in view of Condition (S9).(iii) (the condition $\Phi^\infty = 0$ implies that C^∞ is continuous), Theorem A.7 and Theorem A.6. \square

Lemma 3.8. *The sequences $(\sup_{t \in \mathbb{R}_+} \|L_t^{k,(0)}\|_2^2)_{k \in \overline{\mathbb{N}}}$ and $(\|L_\infty^{k,(0)}\|_2^2)_{k \in \overline{\mathbb{N}}}$ are uniformly integrable.*

Proof. Using Cauchy–Schwarz’s inequality as in [86, Inequality 3.16], we can obtain for every $t \in [0, \infty)$ and $k \in \overline{\mathbb{N}}$

$$\|L_t^{k,(0)}\|_2^2 \leq \frac{1}{\tilde{\beta}} \int_{(0,t]} e^{\hat{\beta} A_s^k} \frac{\|f^k(s, 0, 0, \mathbf{0})\|_2^2}{(\alpha_s^k)^2} dC_s^k \leq \frac{1}{\tilde{\beta}} \int_{(0,\infty)} e^{\hat{\beta} A_s^k} \frac{\|f^k(s, 0, 0, \mathbf{0})\|_2^2}{(\alpha_s^k)^2} dC_s^k,$$

as well as

$$\|L_\infty^{k,(0)}\|_2^2 \leq \frac{1}{\tilde{\beta}} \int_{(0,\infty)} e^{\hat{\beta} A_s^k} \frac{\|f^k(s, 0, 0, \mathbf{0})\|_2^2}{(\alpha_s^k)^2} dC_s^k.$$

In view of (S6), which states that the right-hand side is uniformly integrable, we obtain the required result by He, Wang, and Yan [54, Theorem 1.7.1]. \square

¹⁵The metric compatible with the J_1 -convergence will be denoted by δ_{J_1} and the metric associated to a norm $\|\cdot\|$ will be denoted by $\delta_{\|\cdot\|}$.

Proposition 3.9. *The first step of the induction is valid, that is*

$$L_{\cdot}^{k,(0)} \xrightarrow[k \rightarrow \infty]{(J_1(\mathbb{R}^\ell, \mathbb{L}^2))} L_{\cdot}^{\infty,(0)} \quad (\text{LS}^{(0)})$$

and

$$L_{\infty}^{k,(0)} \xrightarrow[k \rightarrow \infty]{\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}^\ell)} L_{\infty}^{k,(0)}. \quad (\text{LS}_{\infty}^{(0)})$$

Proof. Before we present the arguments, let us remind the reader that by definition $(Y^{k,(0)}, Z^{k,(0)}, U^{k,(0)}) := (0, 0, 0)$ for every $k \in \overline{\mathbb{N}}$. We are going to apply Vitali's theorem, i.e. we will prove initially Convergence (LS⁽⁰⁾) and (LS_∞⁽⁰⁾) in probability and then that the sequences of the respective $\|\cdot\|_2^2$ -norms are uniformly integrable. The latter has been proved in Theorem 3.8. The former will be proved by means of Dudley [34, Theorem 9.2.1].

To this end, let us consider a subsequence $(L_{\infty}^{k_l,(0)})_{l \in \mathbb{N}}$. By Theorem 3.7 there exists a further subsequence $(k_{l_i})_{i \in \mathbb{N}}$ such that $\mu_{C^{k_{l_i}}} \xrightarrow[i \rightarrow \infty]{w} \mu_{C^{\infty}}$, \mathbb{P} -almost surely. Consequently, we have also that $\sup_{i \in \mathbb{N}} \mu_{C^{k_{l_i}}}(\mathbb{R}_+) < \infty$, \mathbb{P} -almost surely. In view of Condition (S8).(ii), we can apply Theorem A.9 as well as Theorem A.11. Therefore, the subsequence $(L_{\infty}^{k_{l_i},(0)})_{i \in \mathbb{N}}$ converges \mathbb{P} -a.s. to $L_{\infty}^{\infty,(0)}$, and the subsequence $(L_{\cdot}^{k_{l_i},(0)})_{i \in \mathbb{N}}$ converges under the J_1 -topology, \mathbb{P} -a.s., to $(L_{\cdot}^{\infty,(0)})_{i \in \mathbb{N}}$. \square

Corollary 3.10. *The convergences*

$$(Y^{k,(1)}, Z^{k,(1)} \cdot X^{k,\circ} + U^{k,(1)} \star \tilde{\mu}^{k,\natural}, N^{k,(1)}) \xrightarrow{(J_1(\mathbb{R}^{\ell \times 3}, \mathbb{L}^2))} (Y^{\infty,(1)}, Z^{\infty,(1)} \cdot X^{\infty,\circ} + U^{\infty,(1)} \star \tilde{\mu}^{\infty,\natural}, 0),$$

for

$$E^{k,(1)} := ([Y^{k,(1)}], [Z^{k,(1)} \cdot X^{k,\circ} + U^{k,(1)} \star \tilde{\mu}^{k,\natural}], [N^{k,(1)}], [Y^{k,(1)}, X^{k,\circ}], [Y^{k,(1)}, X^{k,\natural}], [Y^{k,(1)}, N^{k,(1)}]),$$

$$E^{k,(1)} \xrightarrow{(J_1(\mathbb{R}^{\ell \times (3\ell+m+n+\ell)}, \mathbb{L}^1))} ([Y^{\infty,(1)}], [Z^{\infty,(1)} \cdot X^{\infty,\circ} + U^{\infty,(1)} \star \tilde{\mu}^{\infty,\natural}], 0, [Y^{\infty,(1)}, X^{\infty,\circ}], [Y^{\infty,(1)}, X^{\infty,\natural}], 0),$$

and for

$$F^{k,(1)} := (\langle Y^{k,(1)} \rangle, \langle Z^{k,(1)} \cdot X^{k,\circ} \rangle, \langle U^{k,(1)} \star \tilde{\mu}^{k,\natural} \rangle, \langle N^{k,(1)} \rangle, \langle Y^{k,(1)}, X^{k,\circ} \rangle, \langle Y^{k,(1)}, X^{k,\natural} \rangle, \langle Y^{k,(1)}, N^{k,(1)} \rangle),$$

$$F^{k,(1)} \xrightarrow{(J_1(\mathbb{R}^{\ell \times (4\ell+m+n+\ell)}, \mathbb{L}^1))} (\langle Y^{\infty,(1)} \rangle, \langle Z^{\infty,(1)} \cdot X^{\infty,\circ} \rangle, \langle U^{\infty,(1)} \star \tilde{\mu}^{\infty,\natural} \rangle, 0, \langle Y^{\infty,(1)}, X^{\infty,\circ} \rangle, \langle Y^{\infty,(1)}, X^{\infty,\natural} \rangle, 0),$$

are valid, where 0 denotes the zero process whose state space is a finite-dimensional Euclidean space.

Proof. Apply [87, Corollary 3.10] for the sequence $(M^{k,(0)})_{k \in \overline{\mathbb{N}}}$, which allows us to conclude. \square

Remark 3.11. (i) Observe that $Y_{\infty}^{k,(1)}$ is well-defined \mathbb{P} -almost surely, for every $k \in \overline{\mathbb{N}}$. Indeed, by (3.11)

$$Y_t^{k,(1)} = M_t^{k,(0)} - \int_{(0,t]} f^k(s, Y_s^{k,(0)}, Z_s^{k,(0)}, U_s^{k,(0)}(s, \cdot)) dC_s^k, \quad \text{for every } k \in \overline{\mathbb{N}}.$$

The limit as $t \rightarrow \infty$ of the martingale part exists due to its square integrability and the limit of the Lebesgue–Stieltjes integral exists \mathbb{P} -almost surely; see Theorem 3.6.

- (ii) Although we have accomplished the aim of this section, we will need to obtain an additional result. Namely, we need to complement our induction hypothesis on the p -step with the assumption that the uniform integrability of the sequence $(\Gamma^{k,(p+1)})_{k \in \overline{\mathbb{N}}}$ (recall Notation 3.4) is inherited by the uniform integrability of the sequences $(\Gamma^{k,(p)})_{k \in \overline{\mathbb{N}}}$ and of the standard data. The following lemma serves this aim.

Lemma 3.12. *The sequence $(\Gamma^{k,(1)})_{k \in \overline{\mathbb{N}}}$ is uniformly integrable.*

Proof. We will use the Lipschitz property of the generator f^k as well as the fact that

$$\frac{r^k}{(\alpha^k)^2} \leq (\alpha^k)^2, \quad \frac{\theta^{k,\circ}}{(\alpha^k)^2} \leq 1, \quad \text{and} \quad \frac{\theta^{k,\natural}}{(\alpha^k)^2} \leq 1; \quad (3.19)$$

see the definition of α^k in (F4). Let $k \in \overline{\mathbb{N}}$. By definition of $\Gamma^{k,(1)}$, the Lipschitz property of the generator f^k and the definition of A^k

$$\begin{aligned} \Gamma^{k,(1)} &= \int_{(0,\infty)} \frac{\|f^k(s, Y_s^{k,(1)}, Z_s^{k,(1)}, U_s^{k,(1)}(s, \cdot))\|_2^2}{(\alpha_s^k)^2} dC_s^k \\ &\leq \int_{(0,\infty)} \left[(\alpha_s^k)^2 \|Y_s^{k,(1)}\|_2^2 + \|C_s^k Z_s^{k,(1)}\|_2^2 + \|U_s^{k,(1)}(\cdot)\|_{k,s}^2 \right] dC_s^k \\ &\quad + \int_{(0,\infty)} \frac{\|f^k(s, 0, 0, \mathbf{0})\|_2^2}{(\alpha_s^k)^2} dC_s^k \\ &= \int_{(0,\infty)} \|Y_s^{k,(1)}\|_2^2 dA_s^k + \int_{(0,\infty)} d\text{Tr} \left[\langle Z^{k,(1)} \cdot X^{k,\circ} + U^{k,(1)} \star \tilde{\mu}^{k,\natural} \rangle_s \right] + \Gamma^{k,(0)} \\ &\leq 2\overline{A} \sup_{t \in \mathbb{R}_+} \|\mathbb{E}[\xi^k | \mathcal{G}_t^k]\|_2^2 + 2\overline{A} \sup_{t \in \mathbb{R}_+} \left\| \mathbb{E} \left[\int_{(t,\infty)} f^k(u, 0, 0, \mathbf{0}) dC_u^k \middle| \mathcal{G}_t^k \right] \right\|_2^2 \\ &\quad + \text{Tr} \left[\langle M^{k,(0)} \rangle_\infty \right] + \Gamma^{k,(0)}. \end{aligned} \quad (3.21)$$

In the first inequality, we used the Lipschitz property and the definition of α^k (see (F4)) as well as Inequalities (3.19). In the second equality, we used the definition of A^k, C^k and $\|\cdot\|_{k,t}$; see also [86, Identity (2.9)]. In the second Inequality, we used Identity (3.8), i.e.,

$$Y_t^{k,(1)} = \mathbb{E} \left[\xi^k + \int_{(t,\infty)} f^k(s, 0, 0, \mathbf{0}) dC_s^k \middle| \mathcal{G}_t^k \right].$$

We only need to observe now that the summands on the right-hand side of (3.21) form a uniformly integrable sequence, for $k \in \overline{\mathbb{N}}$, as sum of elements of uniformly integrable random variables; see He et al. [54, Corollary 1.10]. Indeed,

- for the sequence associated to the first summand we have that $(\|\xi^k\|_2^2)_{k \in \overline{\mathbb{N}}}$ is uniformly integrable by Vitali's theorem. Then, we can conclude the uniform integrability of the required sequence by Theorem A.17;
- for the sequence associated to the second summand, i.e.,

$$\left(\sup_{t \in \mathbb{R}_+} \left\| \mathbb{E} \left[\int_{(t,\infty)} f^k(s, 0, 0, \mathbf{0}) dC_s^k \middle| \mathcal{G}_t^k \right] \right\|_2^2 \right)_{k \in \overline{\mathbb{N}}}, \quad (3.22)$$

we can prove by means of the conditional Jensen inequality and the conditional Cauchy-Schwarz inequality, that for all $k \in \overline{\mathbb{N}}$ holds

$$\left\| \mathbb{E} \left[\int_{(t,\infty)} f^k(s, 0, 0, \mathbf{0}) dC_s^k \middle| \mathcal{G}_t^k \right] \right\|_2^2 \leq \left\| \mathbb{E} \left[\int_{(0,\infty)} |f^k(s, 0, 0, \mathbf{0})| dC_s^k \middle| \mathcal{G}_t^k \right] \right\|_2^2$$

$$\leq \mathbb{E}[\Gamma^{k,(0)} | \mathcal{G}_t^k].$$

Since $(\Gamma^{k,(0)})_{k \in \overline{\mathbb{N}}}$ is uniformly integrable (see Condition (S6)) we can conclude the uniform integrability of (3.22) by He et al. [54, Theorem 1.7, Theorem 1.8]. Then, we can conclude the uniform integrability of the required sequence by Theorem A.17;

- for the sequence associated to the third summand we use the fact (which is true in view of the validity of (LS⁽⁰⁾)) that

$$M_{\infty}^{k,(0)} \xrightarrow[k \rightarrow \infty]{\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P})} M_{\infty}^{\infty,(0)},$$

which implies¹⁶ the uniform integrability of $(\text{Tr}[\langle M^{k,(0)} \rangle_{\infty}])_{k \in \overline{\mathbb{N}}}$;

- finally, the sequence associated to the last summand is uniformly integrable by (S6). \square

Remark 3.13. The validity of Theorem 3.12 allows us indeed to complement our induction step with the statement

$$\text{the sequence } (\Gamma^{k,(p)})_{k \in \overline{\mathbb{N}}} \text{ is uniformly integrable.} \quad (\text{UI}^{(p)})$$

The reader may observe that the above property for $p = 1$ transfers the uniform integrability to the sequence

$$\left(\sup_{t \in \mathbb{R}_+} \|L_t^{k,(1)}\|_2^2 \right)_{k \in \overline{\mathbb{N}}}. \quad (3.23)$$

This is immediate by applying Cauchy–Schwarz’s inequality to $(\|L_t^{k,(1)}\|_2^2)_{k \in \overline{\mathbb{N}}}$ (as in [86, Inequality (3.16)] and observing that $(\Gamma^{k,(1)})_{k \in \overline{\mathbb{N}}}$ dominates the sequence $(\|L_t^{k,(1)}\|_2^2)_{k \in \overline{\mathbb{N}}}$, for every $t \in \mathbb{R}_+$.

3.5 The p -th step of the induction is valid

In this sub-section, we assume that Convergences

$$L^{k,(p-1)} \xrightarrow[k \rightarrow \infty]{(\mathbb{J}_1(\mathbb{R}^{\ell}), \mathbb{L}^2)} L^{\infty,(p-1)}, \quad (\text{LS}^{(p-1)})$$

$$L_{\infty}^{k,(p-1)} \xrightarrow[k \rightarrow \infty]{\mathbb{L}^2(\mathcal{G}; \mathbb{R})} L_{\infty}^{\infty,(p-1)}, \quad (\text{LS}_{\infty}^{(p-1)})$$

as well as the statement

$$\text{the sequence } (\Gamma^{k,(p-1)})_{k \in \overline{\mathbb{N}}} \text{ is uniformly integrable,} \quad (\text{UI}^{(p-1)})$$

are true for some arbitrary but fixed $p \in \mathbb{N}$. Then, we will prove that Convergences (LS^(p)) and (LS_∞^(p)), as well as the statement

$$\text{the sequence } (\Gamma^{k,(p)})_{k \in \overline{\mathbb{N}}} \text{ is uniformly integrable} \quad (\text{UI}^{(p)})$$

are also true.

Compared to the first step of the induction, the p -th step is more involved. Let us thus briefly explain the approach we are going to follow in order to reduce the complexity.

¹⁶The uniform integrability of $(\text{Tr}[\langle M^{k,(0)} \rangle_{\infty}])_{k \in \overline{\mathbb{N}}}$ can also be deduced.

In view of Vitali's theorem, it is sufficient to prove initially that Convergences $(LS^{(p)})$ and $(LS_\infty^{(p)})$ hold in probability and then we have to prove that the sequences are (sufficiently) uniformly integrable. In order to obtain the aforementioned convergence in probability, we are going to use that $(\mathbb{D}(\mathbb{R}^\ell), \delta_{J_1(\mathbb{R}^\ell)})$ and $(\mathbb{R}^\ell, \|\cdot\|_2)$ are Polish spaces, as we did in the first step of the induction. Therefore, in view of Dudley [34, Theorem 9.2.1], it is sufficient to prove that from every subsequence $(L^{k_{l_i},(p)})_{i \in \mathbb{N}}$, resp. $(L_\infty^{k_{l_i},(p)})_{i \in \mathbb{N}}$, we can extract a further subsequence $(L^{k_{l_i},(p)})_{i \in \mathbb{N}}$, resp. $(L_\infty^{k_{l_i},(p)})_{i \in \mathbb{N}}$, such that

$$L^{k_{l_i},(p)} \xrightarrow{i \rightarrow \infty} L^{\infty,(p)}, \quad \mathbb{P}\text{-a.s.}, \quad (3.24)$$

resp.

$$L_\infty^{k_{l_i},(p)} \xrightarrow{i \rightarrow \infty} L_\infty^{\infty,(p)}, \quad \mathbb{P}\text{-a.s.} \quad (3.25)$$

Equivalently, for the given subsequence $(k_{l_i})_{i \in \mathbb{N}}$, there exists a set $\tilde{\Omega} \in \mathcal{G}$ with $\mathbb{P}[\tilde{\Omega}] = 1$ such that

$$\limsup_{i \rightarrow \infty} \left\{ \omega \in \tilde{\Omega} : \delta_{J_1(\mathbb{R}^\ell)}(L^{k_{l_i},(p)}(\omega), L^{\infty,(p)}(\omega)) > \varepsilon \right\} = \emptyset, \quad \forall \varepsilon > 0,$$

resp.

$$\limsup_{i \rightarrow \infty} \left\{ \omega \in \tilde{\Omega} : \|L_\infty^{k_{l_i},(p)}(\omega) - L_\infty^{\infty,(p)}(\omega)\|_2 > \varepsilon \right\} = \emptyset, \quad \forall \varepsilon > 0.$$

To this end, let us consider Convergence (3.25), fix an $\varepsilon > 0$ and assume that for \mathbb{P} -almost every $\omega \in \Omega$ there exist $\tilde{Z}^{\varepsilon,(p)}(\omega, \cdot) \in D^{o,\ell \times m}$ and $\tilde{U}^{\varepsilon,(p)}(\omega, \cdot) \in D^{\mathfrak{d}}$ such that

$$\left\| \int_{(0,\infty)} f^\infty(s, Y_s^{\infty,(p)}, \tilde{Z}_s^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}(s, \cdot)) dC_s^\infty - L_\infty^{\infty,(p)} \right\|_2 < \frac{\varepsilon}{3}, \quad \mathbb{P}\text{-a.s.} \quad (3.26)$$

Then, using the set inclusion, where F^k are assumed to be \mathbb{R}^ℓ -valued random functions for $k \in \{1, 2, 3\}$

$$\left\{ \omega \in \Omega : \left\| \sum_{k=1}^3 F^k(\omega) \right\|_2 > \varepsilon \right\} \subseteq \bigcup_{k=1}^3 \left\{ \omega \in \Omega : \|F^k(\omega)\|_2 > \frac{\varepsilon}{3} \right\},$$

we can obtain Convergence (3.25) if we can find an $\tilde{\Omega} \subset \Omega$ with $\mathbb{P}[\tilde{\Omega}] = 1$ such that¹⁷

$$\limsup_{i \rightarrow \infty} \left\{ \omega \in \tilde{\Omega} : \left\| L_\infty^{i,(p)}(\omega) - \left(\int_{(0,\infty)} f^i(s, Y_s^{i,(p)}, \tilde{Z}_s^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}(s, \cdot)) dC_s^i \right)(\omega) \right\|_2 > \frac{\varepsilon}{3} \right\} = \emptyset, \quad (3.27)$$

and

$$\begin{aligned} \limsup_{i \rightarrow \infty} \left\{ \omega \in \tilde{\Omega} : \left\| \left(\int_{(0,\infty)} f^i(s, Y_s^{i,(p)}, \tilde{Z}_s^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}(s, \cdot)) dC_s^i \right)(\omega) \right. \right. \\ \left. \left. - \left(\int_{(0,\infty)} f^\infty(s, Y_s^{\infty,(p)}, \tilde{Z}_s^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}(s, \cdot)) dC_s^\infty \right)(\omega) \right\|_2 > \frac{\varepsilon}{3} \right\} = \emptyset. \end{aligned} \quad (3.28)$$

Hence, we prove (3.25) if (3.26), (3.27) and (3.28) are true. An analogous decomposition can be done for (3.24), where the distance is measured by the $\delta_{J_1(\mathbb{R}^\ell)}$ -metric. Returning now to the uniform integrability that the sequences should satisfy, we will need to prove that the family $(\sup_{t \in \mathbb{R}_+} \|L_t^{k,(p)}\|_2^2)_{k \in \mathbb{N}}$ is uniformly integrable, which is a sufficient condition for concluding both the Convergence $(LS^{(p)})$ and $(LS_\infty^{(p)})$.

¹⁷For notational convenience, we index the k_{l_i} -element in the next expression simply by i .

Intermediate Assumption. From now on we fix an arbitrary subsequence $(L^{k_l, (p)})_{l \in \mathbb{N}}$, resp. $(L_\infty^{k_l, (p)})_{l \in \mathbb{N}}$.

Let us conclude the description of our strategy by collecting all the information we have available for the next subsections. We will state them as a remark so that they are easily referred to. Moreover, for notational convenience, we can assume that the sequence for which the forthcoming convergences are obtained \mathbb{P} -almost surely is indexed by $(k_{l_i})_{i \in \mathbb{N}}$. This can be done without loss of generality, since we can pass to a further subsequence finitely many times.

Remark 3.14. (i) By Theorem 3.7, there exist a $\delta_{J_1(\mathbb{R}^\ell)}$ -convergent (\mathbb{P} -a.s.) subsequence $(C^{k_{l_i}})_{i \in \mathbb{N}}$ as well as a $\delta_{|\cdot|}$ -convergent (\mathbb{P} -a.s.) subsequence $(C_\infty^{k_{l_i}})_{i \in \mathbb{N}}$.

(ii) By Condition (S8).(ii), we can assume in particular that

$$f^{k_{l_i}}(\cdot, 0, 0, \mathbf{0}) \xrightarrow{i \rightarrow \infty} f^\infty(\cdot, 0, 0, \mathbf{0}), \quad \mathbb{P}\text{-a.s.}$$

(iii) The convergences in $(LS^{(p-1)})$ and $(LS_\infty^{(p-1)})$, which are assumed true (this is the induction assumption), allow us to obtain that

$$L^{k_{l_i}, (p-1)} \xrightarrow{i \rightarrow \infty} L^{\infty, (p-1)}, \text{ as well as } L_\infty^{k_{l_i}, (p-1)} \xrightarrow{i \rightarrow \infty} L_\infty^{\infty, (p-1)}, \quad \mathbb{P}\text{-a.s.}$$

(iv) In view of the discussion made in the outline of the proof of Theorem 3.1, see in particular on Page 16, the validity of the convergences in $(LS^{(p-1)})$ and $(LS_\infty^{(p-1)})$ allows us to apply [87, Corollary 3.10] for the martingale sequence $(M^{k, (p-1)})_{k \in \mathbb{N}}$. More precisely, the convergences

$$(Y^{k, (p)}, Z^{k, (p)} \cdot X^{k, \circ} + U^{k, (p)} \star \tilde{\mu}^{k, \natural}, N^{k, (p)}) \xrightarrow{(J_1(\mathbb{R}^{\ell \times 3}), \mathbb{L}^2)} (Y^{\infty, (p)}, Z^{\infty, (p)} \cdot X^{\infty, \circ} + U^{\infty, (p)} \star \tilde{\mu}^{\infty, \natural}, 0),$$

for

$$E^{k, (p)} := ([Y^{k, (p)}], [Z^{k, (p)} \cdot X^{k, \circ} + U^{k, (p)} \star \tilde{\mu}^{k, \natural}], [N^{k, (p)}], [Y^{k, (p)}, X^{k, \circ}], [Y^{k, (p)}, X^{k, \natural}], [Y^{k, (p)}, N^{k, (p)}]),$$

$$E^{k, (p)} \xrightarrow{(J_1(\mathbb{R}^{\ell \times (3\ell + m + n + \ell)}), \mathbb{L}^1)} ([Y^{\infty, (p)}], [Z^{\infty, (p)} \cdot X^{\infty, \circ} + U^{\infty, (p)} \star \tilde{\mu}^{\infty, \natural}], 0, [Y^{\infty, (p)}, X^{\infty, \circ}], [Y^{\infty, (p)}, X^{\infty, \natural}], 0),$$

and for

$$F^{k, (p)} := (\langle Y^{k, (p)} \rangle, \langle Z^{k, (p)} \cdot X^{k, \circ} \rangle, \langle U^{k, (p)} \star \tilde{\mu}^{k, \natural} \rangle, \langle N^{k, (p)} \rangle, \langle Y^{k, (p)}, X^{k, \circ} \rangle, \langle Y^{k, (p)}, X^{k, \natural} \rangle, \langle Y^{k, (p)}, N^{k, (p)} \rangle),$$

$$F^{k, (p)} \xrightarrow{(J_1(\mathbb{R}^{\ell \times (4\ell + m + n + \ell)}), \mathbb{L}^1)} (\langle Y^{\infty, (p)} \rangle, \langle Z^{\infty, (p)} \cdot X^{\infty, \circ} \rangle, \langle U^{\infty, (p)} \star \tilde{\mu}^{\infty, \natural} \rangle, 0, \langle Y^{\infty, (p)}, X^{\infty, \circ} \rangle, \langle Y^{\infty, (p)}, X^{\infty, \natural} \rangle, 0),$$

are valid. Here 0 denotes the zero process whose state space is a finite-dimensional Euclidean space. For later reference, we state only the results we are going to make use of: for $k \in \mathbb{N}$

$$H^{k, (p)} := (\langle Z^{k, (p)} \cdot X^{k, \circ} \rangle, \langle U^{k, (p)} \star \tilde{\mu}^{k, \natural} \rangle, \langle Z^{k, (p)} \cdot X^{k, \circ}, X^{k, \circ} \rangle, \langle U^{k, (p)} \star \tilde{\mu}^{k, \natural}, X^{k, \natural} \rangle) \\ H^{k, (p)} \xrightarrow[k \rightarrow \infty]{(J_1(\mathbb{R}^{\ell \times (2\ell + m + n)}), \mathbb{L}^1)} H^{\infty, (p)} \quad (3.29)$$

Additionally, we can apply Theorem A.17 for the sequence $(\|M_\infty^{k,(p-1)}\|_2^2)_{k \in \overline{\mathbb{N}}}$ in order to obtain that

$$\left(\sup_{t \in \mathbb{R}_+} \|M_t^{k,(p-1)}\|_2^2 \right)_{k \in \overline{\mathbb{N}}} \text{ is uniformly integrable.} \quad (3.30)$$

(v) The sequence $(\Delta^{k,(p)})_{k \in \overline{\mathbb{N}}}$ is uniformly integrable, since it is strongly majored¹⁸ by $(\text{Tr}[\langle M^{k,(p-1)} \rangle_\infty])_{k \in \overline{\mathbb{N}}}$ which is uniformly integrable. The uniform integrability of the latter sequence is derived by the uniform integrability of the sequence $(\|M_\infty^{k,(p-1)}\|_2^2)_{k \in \overline{\mathbb{N}}}$.

(vi) In view of Conditions (S1)–(S5), we can apply Mémin [81, Corollary 12]. Therefore, we may assume that

$$(\overline{X}^i, [\overline{X}^i], \langle \overline{X}^i \rangle) \xrightarrow[i \rightarrow \infty]{J_1(\mathbb{R}^{(m+n) \times \{1+2(m+n)\}})} (\overline{X}^\infty, [\overline{X}^\infty], \langle \overline{X}^\infty \rangle), \text{ P-a.s.} \quad (3.31)$$

In particular $X^{i,\natural} \xrightarrow[i \rightarrow \infty]{J_1(\mathbb{R}^n)} X^{\infty,\natural}$, P-a.s., which, in conjunction with Jacod and Shiryaev [65, Corollary VI.2.8], further implies

$$g * \mu^{i,\natural} \xrightarrow[i \rightarrow \infty]{J_1(\mathbb{R}^\ell)} g * \mu^{\infty,\natural}, \text{ P-a.s., for every } g \in D^\natural.$$

On the other hand, for every $i \in \overline{\mathbb{N}}$ holds

$$g \star \tilde{\mu}^{i,\natural} = g * \mu^{i,\natural} - g * \nu^{i,\natural}, \text{ P-a.s., for every } g \in D^\natural.$$

Since D^\natural is countable, we can assume that this is true for every $\omega \in \overline{\Omega}$, for some $\overline{\Omega} \in \mathcal{G}$ with $\mathbb{P}[\overline{\Omega}] = 1$. Finally, by the last convergence and Mémin [81, Theorem 11, Corollary 12], we have that for every $g \in D^\natural$ it holds P-a.s.

$$(g \star \tilde{\mu}^{i,\natural}, g * \nu^{i,\natural}, \langle g \star \tilde{\mu}^{i,\natural} \rangle) \xrightarrow[i \rightarrow \infty]{J_1(\mathbb{R}^{\ell \times (2+\ell)}} (g \star \tilde{\mu}^{\infty,\natural}, g * \nu^{\infty,\natural}, \langle g \star \tilde{\mu}^{\infty,\natural} \rangle). \quad (3.32)$$

(vii) In view of (iv) and (vi) we can P-a.s. have that

$$\begin{aligned} (M^{i,(p-1)}, \langle Z^{i,(p)} \cdot X^{i,\circ} \rangle, \langle U^{i,(p)} \star \tilde{\mu}^i \rangle) &\xrightarrow[i \rightarrow \infty]{J_1(\mathbb{R}^{\ell \times (1+2\ell)}} \\ (M^{\infty,(p-1)}, \langle Z^{\infty,(p)} \cdot X^{\infty,\circ} \rangle, \langle U^{\infty,(p)} \star \tilde{\mu}^{\infty,\natural} \rangle), \end{aligned} \quad (3.33)$$

$$\begin{aligned} (\langle Z^{k_{l_i},(p)} \cdot X^{k_{l_i},\circ}, X^{k_{l_i},\circ} \rangle, \langle U^{k_{l_i},(p)} \star \tilde{\mu}^{k_{l_i},\natural}, g \star \tilde{\mu}^{k_{l_i},\natural} \rangle) &\xrightarrow[i \rightarrow \infty]{J_1(\mathbb{R}^{\ell \times (m+n)}} \\ (\langle Z^{\infty,(p)} \cdot X^{\infty,\circ}, X^{\infty,\circ} \rangle, \langle U^{\infty,(p)} \star \tilde{\mu}^{\infty,\natural}, g \star \tilde{\mu}^{\infty,\natural} \rangle). \end{aligned} \quad (3.34)$$

(viii) Recall (3.11), i.e. $Y^{k,(p)} = M^{k,(p-1)} - L^{k,(p-1)}$ for every $k \in \overline{\mathbb{N}}$. We claim that

$$\left(\sup_{t \in \mathbb{R}_+} \|Y_t^{k,(p)}\|_2^2 \right)_{k \in \overline{\mathbb{N}}} \text{ is uniformly integrable.} \quad (3.35)$$

This can be concluded as follows. We have

$$\sup_{t \in \mathbb{R}_+} \|Y_t^{k,(p)}\|_2^2 \leq 2 \sup_{t \in \mathbb{R}_+} \|M_t^{k,(p-1)}\|_2^2 + 2 \sup_{t \in \mathbb{R}_+} \|L_t^{k,(p-1)}\|_2^2, \text{ for every } k \in \overline{\mathbb{N}}.$$

¹⁸See [65, Definition VI.3.34].

For the sequence $(\sup_{t \in \mathbb{R}_+} \|L_t^{k, (p-1)}\|_2^2)_{k \in \mathbb{N}}$, we can conclude its uniform integrability by arguing analogously to Equation (3.23), since $(\Gamma^{k, (p-1)})_{k \in \mathbb{N}}$ has been assumed uniformly integrable; see $(UI^{(p-1)})$. Now, we can conclude our initial statement by (3.30).

- (ix) In view of Condition (S9).(iii), which implies that $L^{\infty, (p-1)}$ is a continuous process, parts (iii) and (vii) above and Jacod and Shiryaev [65, Proposition VI.1.123] we can conclude that

$$Y^{k_{l_i}, (p)} \xrightarrow{i \rightarrow \infty} Y^{\infty, (p)}, \mathbb{P}\text{-a.s.} \quad (3.36)$$

- (x) The sets D° and D^\sharp are countable. Therefore, in view of Condition (S8).(ii) we can assume that

$$\left(f^{k_{l_i}}(t, Y_t^{k_{l_i}, (p)}, \tilde{Z}_t^{\varepsilon, (p)}, \tilde{U}^{\varepsilon, (p)}(t, \cdot)) \right)_{t \in \mathbb{R}_+} \xrightarrow{i \rightarrow \infty} \left(f^\infty(t, Y_t^{\infty, (p)}, \tilde{Z}_t^{\varepsilon, (p)}, \tilde{U}^{\varepsilon, (p)}(t, \cdot)) \right)_{t \in \mathbb{R}_+}, \mathbb{P}\text{-a.s.}$$

- (xi) In view of (S10), we may assume that $\tau^{k_{l_i}} \rightarrow \tau^\infty$, \mathbb{P} -a.s. The finiteness of τ^∞ implies that $\tau^{k_{l_i}}$ is finally finite \mathbb{P} -a.s.

Remark 3.15. The purpose of the above remarks was not only to collect all the available information, but also to provide us with a set $\Omega_{\text{sub}} \in \mathcal{G}$ with $\mathbb{P}[\Omega_{\text{sub}}] = 1$ such that the above properties hold for every $\omega \in \Omega_{\text{sub}}$. In the next subsections, whenever we say that a property holds \mathbb{P} -almost surely for a subsequence indexed by $(k_{l_i})_{i \in \mathbb{N}}$, the reader should understand that it holds for every $\omega \in \Omega_{\text{sub}}$. Moreover, we will index the elements of the subsequence $(k_{l_i})_{i \in \mathbb{N}}$ simply by $i \in \mathbb{N}$; when we use the index $k \in \mathbb{N}$ we will refer to the initial sequence.

3.5.1 The claim (3.26) is true

Proposition 3.16 (Lusin approximation). *For every $\varepsilon > 0$, there exist $\tilde{Z}^{\varepsilon, (p)} : \Omega_{\text{sub}} \times \mathbb{R}_+ \rightarrow \mathbb{R}^{\ell \times m}$ and $\tilde{U}^{\varepsilon, (p)} : \Omega_{\text{sub}} \times \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^\ell$ (which are defined ω -by- ω) with the following properties*

- (i) $\tilde{Z}^{\varepsilon, (p)}(\omega, \cdot) \in D^{\circ, \ell \times m}$;
- (ii) $\tilde{U}^{\varepsilon, (p)}(\omega, \cdot) \in D^\sharp$;
- (iii) the pair $(\tilde{Z}^{\varepsilon, (p)}, \tilde{U}^{\varepsilon, (p)})$ ¹⁹ satisfies

$$\int_{(0, \infty)} \text{Tr}[(Z_s^{\infty, (p)} - \tilde{Z}_s^{\varepsilon, (p)}) \frac{d\langle X^{\infty, \circ} \rangle_s}{dC_s^\infty} (Z_s^{\infty, (p)} - \tilde{Z}_s^{\varepsilon, (p)})^\top] dC_s^\infty < \frac{\varepsilon^2}{18} \frac{\hat{\beta}}{e^{\hat{\beta}A}}, \quad \forall \omega \in \Omega_{\text{sub}},$$

$$\int_{(0, \infty)} \left\| U^{\infty, (p)}(s, \cdot) - \tilde{U}^{\varepsilon, (p)}(s, \cdot) \right\|_{\infty, s}^2 dC_s^\infty < \frac{\varepsilon^2}{18} \frac{\hat{\beta}}{e^{\hat{\beta}A}}, \quad \forall \omega \in \Omega_{\text{sub}}.$$

In particular, it holds

$$\left\| \int_{(0, \infty)} f^\infty(s, Y_s^{\infty, (p)}, \tilde{Z}_s^{\varepsilon, (p)}, \tilde{U}^{\varepsilon, (p)}(s, \cdot)) dC_s^\infty - L_{\infty}^{\infty, (p)} \right\|_2 < \frac{\varepsilon}{3}, \quad \forall \omega \in \Omega_{\text{sub}}. \quad (3.37)$$

¹⁹We will omit the ω in order to simplify notation. Moreover, we will denote $Z^{\varepsilon, (p)}(\omega, s)$ by $Z_s^{\varepsilon, (p)}$.

Proof. Let us fix an $\varepsilon > 0$. For the first inequality we apply Theorem A.14 and for the second we apply Theorem A.15; recall that

$$\begin{aligned} & \int_{(0,\infty)} \text{Tr} \left[(Z_s^{\infty,(p)} - \tilde{Z}_s^{\varepsilon,(p)}) \frac{d\langle X^{\infty,\circ} \rangle_s}{dC_s^\infty} (Z_s^{\infty,(p)} - \tilde{Z}_s^{\varepsilon,(p)})^\top \right] dC_s^\infty \\ &= \int_{(0,\infty)} \|c_s^\infty(\tilde{Z}_s^{\varepsilon,(p)} - Z_s^{\infty,(p)})\|_2^2 dC_s^\infty, \end{aligned}$$

and

$$\begin{aligned} & \int_{(0,\infty)} \|U^{\infty,(p)}(s, x) - \tilde{U}^{\varepsilon,(p)}(s, x)\|_2^2 \nu^{\infty, \sharp}(ds, dx) \\ &= \int_{(0,\infty)} \left\| U^{\infty,(p)}(s, \cdot) - \tilde{U}^{\varepsilon,(p)}(s, \cdot) \right\|_{\infty, s}^2 dC_s^\infty. \end{aligned}$$

Finally, we can prove the validity of (3.37) in view of the following set inclusions²⁰

$$\begin{aligned} & \left\{ \left\| \int_{(0,\infty)} f^\infty(s, Y_s^{\infty,(p)}, \tilde{Z}_s^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}(s, \cdot)) dC_s^\infty - L_\infty^{\infty,(p)} \right\|_2 \geq \frac{\varepsilon}{3} \right\} \\ &= \left\{ \left\| \int_{(0,\infty)} f^\infty(s, Y_s^{\infty,(p)}, \tilde{Z}_s^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}(s, \cdot)) dC_s^\infty - L_\infty^{\infty,(p)} \right\|_2^2 \geq \frac{\varepsilon^2}{9} \right\} \\ &\subseteq \left\{ \frac{1}{\hat{\beta}} \int_{(0,\infty)} e^{\hat{\beta} A_s^\infty} \left[\|c_s^\infty(\tilde{Z}_s^{\varepsilon,(p)} - Z_s^{\infty,(p)})\|_2^2 \right. \right. \\ &\quad \left. \left. + \left\| \tilde{U}^{\varepsilon,(p)}(s, \cdot) - U^{\infty,(p)}(s, \cdot) \right\|_{\infty, s}^2 \right] dC_s^\infty \geq \frac{\varepsilon^2}{9} \right\}^{22} \\ &\subseteq \left\{ \frac{e^{\hat{\beta} \bar{A}}}{\hat{\beta}} \int_{(0,\infty)} \|c_s^\infty(\tilde{Z}_s^{\varepsilon,(p)} - Z_s^{\infty,(p)})\|_2^2 dC_s^\infty \geq \frac{\varepsilon^2}{18} \right\} \\ &\quad \cup \left\{ \frac{e^{\hat{\beta} \bar{A}}}{\hat{\beta}} \int_{(0,\infty)} \left\| \tilde{U}^{\varepsilon,(p)}(s, \cdot) - U^{\infty,(p)}(s, \cdot) \right\|_{\infty, s}^2 dC_s^\infty \geq \frac{\varepsilon^2}{18} \right\} \subseteq \emptyset. \end{aligned}$$

□

3.5.2 The claim (3.27) is true

Lemma 3.17. *It holds*

$$\limsup_{i \rightarrow \infty} \left\{ \omega \in \tilde{\Omega} : \left\| L_\infty^{i,(p)}(\omega) - \left(\int_{(0,\infty)} f^i(s, Y_s^{i,(p)}, \tilde{Z}_s^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}(s, \cdot)) dC_s^i \right)(\omega) \right\|_2 \geq \frac{\varepsilon}{3} \right\} = \emptyset.$$

Proof. Using the Cauchy–Schwarz’s inequality as in [86, Inequality 3.16], Condition (S7), the Lipschitz property of the generator and Inequalities (3.19), we have

$$\begin{aligned} & \left\| L_\infty^{i,(p)} - \int_{(0,\infty)} f^i(s, Y_s^{i,(p)}, \tilde{Z}_s^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}(s, \cdot)) dC_s^i \right\|_2^2 \\ &\leq \frac{e^{\hat{\beta} \bar{A}}}{\hat{\beta}} \int_{(0,\infty)} \left\| (Z_s^{i,(p)} - \tilde{Z}_s^{\varepsilon,(p)}) c_s^i \right\|_2^2 + \left\| U_s^{i,(p)}(\cdot) - \tilde{U}^{\varepsilon,(p)}(s, \cdot) \right\|_{i, s}^2 dC_s^i \\ &= \frac{e^{\hat{\beta} \bar{A}}}{\hat{\beta}} \int_{(0,\infty)} \text{Tr} \left[(Z_s^{i,(p)} - \tilde{Z}_s^{\varepsilon,(p)}) \frac{d\langle X^{i,\circ} \rangle_s}{dC_s^i} (Z_s^{i,(p)} - \tilde{Z}_s^{\varepsilon,(p)})^\top \right] dC_s^i \end{aligned}$$

²⁰We regard the following sets as subsets of Ω_{sub} .

²²We have used the fact that $\theta^{\infty, c}, \theta^{\infty, d} \leq \alpha^\infty$ (see (F4)) as well as [86, Inequality 3.16].

$$+ \frac{\mathbf{e}^{\hat{\beta}\bar{A}}}{\hat{\beta}} \int_{(0,\infty)} \left\| U_s^{i,(p)}(\cdot) - \tilde{U}^{\varepsilon,(p)}(s, \cdot) \right\|_{i,s}^2 dC_s^i. \quad (3.38)$$

In order to argue about the validity of the aforementioned convergence, we initially return to Convergence (3.31), on which we will apply the usual measure-theoretic arguments in order to derive for every $Z \in D^{\circ, \ell \times m}$

$$\int_{(0,\infty)} Z_s \frac{d\langle X^{i,\circ} \rangle_s}{dC_s^i} Z_s dC_s^i \xrightarrow{i \rightarrow \infty} \int_{(0,\infty)} Z_s \frac{d\langle X^{\infty,\circ} \rangle_s}{dC_s^\infty} Z_s dC_s^\infty, \quad \mathbb{P}\text{-a.s.} \quad (3.39)$$

Let us fix $(p, q) \in \{1, \dots, m\}^2$. The first step towards proving our aim is to assume an open interval $I \subset \mathbb{R}_+$. Then, from Convergence (3.31) we have for the pq -elements of $(\langle X^{i,\circ} \rangle)_{i \in \mathbb{N}}$

$$\int_{(0,\infty)} \mathbb{1}_I(s) d\langle X^{i,\circ,p}, X^{i,\circ,q} \rangle_s \xrightarrow{i \rightarrow \infty} \int_{(0,\infty)} \mathbb{1}_I(s) d\langle X^{i,\circ,p}, X^{i,\circ,q} \rangle_s, \quad \mathbb{P}\text{-a.s.}$$

The reader may recall that $\langle X^{\infty,\circ} \rangle$ is a continuous process, so no doubts are raised on points of discontinuity because they do not exist. The next step is to assume a step function, i.e., a finite family of real numbers $(\alpha^k)_{k \in \{1, \dots, n\}}$ and a finite family of disjoint open intervals $(I^k)_{k \in \{1, \dots, n\}}$, for which

$$\int_{(0,\infty)} \sum_{k=1}^n \alpha^k \mathbb{1}_{I^k}(s) d\langle X^{i,\circ,p}, X^{i,\circ,q} \rangle_s \xrightarrow{i \rightarrow \infty} \int_{(0,\infty)} \sum_{k=1}^n \alpha^k \mathbb{1}_{I^k}(s) d\langle X^{i,\circ,p}, X^{i,\circ,q} \rangle_s, \quad \mathbb{P}\text{-a.s.},$$

is immediate from the previous convergence. The third step is to assume a function Z with compact support which has discontinuities of the first kind, i.e., the left- and right-limits exist. Since every such function is the uniform limit of step-functions, we can conclude from the second step the validity of

$$\int_{(0,\infty)} Z(s) d\langle X^{i,\circ,p}, X^{i,\circ,q} \rangle_s \xrightarrow{i \rightarrow \infty} \int_{(0,\infty)} Z(s) d\langle X^{i,\circ,p}, X^{i,\circ,q} \rangle_s, \quad \mathbb{P}\text{-a.s.}$$

Finally, we recall that $D^{\circ, \ell \times m}$ is countable and consists of continuous functions with compact support. Hence, from the last step we can immediately derive (by summing suitably the respective elements) Convergence (3.39). Next, from Convergence (3.32) for $U \in D^{\natural}$

$$\begin{aligned} [U^\top U] * \nu_\infty^{i,\natural} - \sum_{s \geq 0} \int_{\mathbb{R}^n} (U(s, x))^\top \nu^{i,\natural}(\{s\} \times dx) \int_{\mathbb{R}^n} U(s, x) \nu^{i,\natural}(\{s\} \times dx) \\ \xrightarrow{i \rightarrow \infty} [U^\top U] * \nu_\infty^{\infty,\natural}, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Returning to (3.34), and using again the usual measure-theoretic arguments, we have in particular for $Z \in D^{\circ, \ell \times m}$ ²³

$$\int_{(0,\infty)} Z_s^{i,(p)} \frac{d\langle X^{i,\circ} \rangle_s}{dC_s^i} Z_s dC_s^i \xrightarrow{i \rightarrow \infty} \int_{(0,\infty)} Z_s^{\infty,(p)} \frac{d\langle X^{\infty,\circ} \rangle_s}{dC_s^\infty} Z_s dC_s^\infty,$$

and $U \in D^{\natural}$

$$\begin{aligned} [(U^{i,(p)})^\top U] * \nu_\infty^{i,\natural} - \sum_{s \geq 0} \int_{\mathbb{R}^n} (U^{i,(p)}(s, x))^\top \nu^{i,\natural}(\{s\} \times dx) \int_{\mathbb{R}^n} U(x) \nu^{i,\natural}(\{s\} \times dx) \\ \xrightarrow{i \rightarrow \infty} [(U^{\infty,(p)})^\top U] * \nu_\infty^{\infty,\natural}. \end{aligned}$$

²³We use the fact that Z and U have compact supports and the pointwise convergence on the continuity points of the limit (see [65, Proposition VI.2.1], in order to attain the value of the integral over \mathbb{R}_+ on a compact subinterval).

We return to (3.33) and at this point we use Condition (S10), which implies²⁴

$$\int_{(0,\infty)} Z_s^{i,(p)} \frac{d\langle X^{i,\circ} \rangle_s}{dC_s^i} (Z_s^{i,(p)})^\top dC_s^i \xrightarrow{i \rightarrow \infty} \int_{(0,\infty)} Z_s^{\infty,(p)} \frac{d\langle X^{\infty,\circ} \rangle_s}{dC_s^\infty} (Z_s^{\infty,(p)})^\top dC_s^\infty,$$

$$\langle U^{i,(p)} \star \tilde{\mu}^{i,\infty} \rangle_\infty \xrightarrow{i \rightarrow \infty} \langle U^{\infty,(p)} \star \tilde{\mu}^{\infty,\infty} \rangle_\infty.$$

Overall, from all the above convergence and Inequality (3.38) we derive the desired statement. \square

3.5.3 The claim (3.28) is true

We will apply Theorem A.9. In order to proceed further, we will provide the following preparatory results. Recall that in Theorem 3.15 we have fixed an Ω_{sub} with $\mathbb{P}[\Omega_{\text{sub}}] = 1$. For the remainder of the section we will fix an arbitrarily small $\varepsilon > 0$; we will assume that $\varepsilon < 1$. Moreover, $\tilde{Z}^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}$ will be assumed to satisfy the properties described in Theorem 3.16 and will be hereinafter fixed.

Lemma 3.18. *For every $\omega \in \Omega_{\text{sub}}$, we have $\sup_{i \in \mathbb{N}} \|Y^{i,(p)}(\omega)\|_\infty < \infty$.*

Proof. We will apply Theorem A.13. To this end, we will ensure that the three required conditions are satisfied. The first, resp. second, condition is indeed true in view of (viii), resp. (ix), of Theorem 3.14. Regarding the validity of the last condition, we can argue as follows

$$\begin{aligned} \mathbb{E} \left[\limsup_{(i,t) \rightarrow (\infty,\infty)} \|Y_t^{i,(p)}\|_2 \right] &= \mathbb{E} \left[\lim_{j \rightarrow \infty} \|Y_{t_j}^{i_j,(p)}\|_2 \right]^{25} \leq \mathbb{E} \left[\lim_{j \rightarrow \infty} \sup_{t \in \mathbb{R}_+} \|Y_t^{i_j,(p)}\|_2 \right] \\ &\leq \liminf_{j \rightarrow \infty} \mathbb{E} \left[\sup_{t \in \mathbb{R}_+} \|Y_t^{i_j,(p)}\|_2 \right] \leq \sup_{i \in \mathbb{N}} \mathbb{E} \left[\sup_{t \in \mathbb{R}_+} \|Y_t^{i,(p)}\|_2 \right] < \infty, \end{aligned}$$

where in the last step we used Theorem 3.14.(viii) again. Therefore,

$$\mathbb{P} \left[\left\{ \limsup_{(i,t) \rightarrow (\infty,\infty)} \|Y_t^{i,(p)}\|_2 = \infty \right\} \right] = 0,$$

which allows us to conclude that

$$\limsup_{(i,t) \rightarrow (\infty,\infty)} \|Y^{i,(p)}(\omega)\|_\infty < \infty, \text{ for every } \omega \in \Omega_{\text{sub}}. \quad \square$$

Proposition 3.19. *For every $\omega \in \Omega_{\text{sub}}$ holds*

$$\int_{(0, \cdot]} f^i(s, Y_s^{i,(p)}, \tilde{Z}_s^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}(s, \cdot)) dC_s^i \xrightarrow{i \rightarrow \infty} \int_{(0, \cdot]} f^\infty(s, Y_s^{\infty,(p)}, \tilde{Z}_s^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}(s, \cdot)) dC_s^\infty,$$

$$\int_{(0,\infty)} f^i(s, Y_s^{i,(p)}, \tilde{Z}_s^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}(s, \cdot)) dC_s^i \xrightarrow{\|\cdot\|_2} \int_{(0,\infty)} f^\infty(s, Y_s^{\infty,(p)}, \tilde{Z}_s^{\varepsilon,(p)}, \tilde{U}^{\varepsilon,(p)}(s, \cdot)) dC_s^\infty.$$

Proof. We will apply Theorem A.11 for the former convergence and Theorem A.9 for the latter. For both of these, the required assumptions are satisfied

- by Theorem 3.14 we have the weak convergence for the measures associated to the sequence $(C^i)_{i \in \mathbb{N}}$;
- by (S8) we have that the integrands converge in the Skorokhod topology;
- Theorem 3.18 in conjunction with (S8).(ii) imply that the integrands are uniformly bounded. \square

²⁴The reader may recall Theorem 3.14.(xi).

²⁵The equality holds for some subsequence $((i_j, t_j))_{j \in \mathbb{N}}$, which depends on the path, such that $(i_j, t_j) \rightarrow (\infty, \infty)$.

3.5.4 The claim (3.24) is true

Since the limit process is continuous, the metric δ_{j_1} is identical to the metric induced by the locally uniform convergence; see Jacod and Shiryaev [65, Proposition VI.1.17]. Hence, we need only to prove that for every $N \in \mathbb{N}$

$$\sup_{t \in [0, N]} \|L_t^{i, (p)} - L_t^{\infty, (p)}\|_2 \xrightarrow{i \rightarrow \infty} 0.$$

On the other hand, a closer examination of the properties we have presented in the previous sub-subsections leads us to the required conclusion. Indeed, starting from the choice of $\tilde{Z}^{\varepsilon, (p)}$ and $\tilde{U}^{\varepsilon, (p)}$ in Theorem 3.16 we have for every $N \in \mathbb{N}$

$$\begin{aligned} & \left\| \int_{(0, N]} f^\infty(s, Y_s^{\infty, (p)}, \tilde{Z}_s^{\varepsilon, (p)}, \tilde{U}_s^{\varepsilon, (p)}(s, \cdot)) dC_s^\infty - L_N^{\infty, (p)} \right\|_2^2 \\ & \leq \frac{e^{\hat{\beta} \bar{A}}}{\hat{\beta}} \int_{(0, \infty)} \|c_s^\infty(\tilde{Z}_s^{\varepsilon, (p)} - Z_s^{\infty, (p)})\|_2^2 + \|\tilde{U}_s^{\varepsilon, (p)}(s, \cdot) - U^{\infty, (p)}(s, \cdot)\|_{\infty, s}^2 dC_s^\infty < \frac{\varepsilon}{3}, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

The analogous to Convergence (3.27), resp. (3.28), can be proven in view of the comments in Theorem 3.14 and using the usual measure theoretic arguments, resp. has been proved in Theorem 3.19.

3.5.5 Uniform integrability

In this sub-subsection, we will prove that statement (UI^(p)) is true, i.e. $(\Gamma^{k, (p)})_{k \in \overline{\mathbb{N}}}$ is uniformly integrable. This property will simultaneously ensure that the sequence $(\sup_{t \in \mathbb{R}_+} |L_t^{k, (p)}|^2)_{k \in \overline{\mathbb{N}}}$ is uniformly integrable and the validity of the induction step. Recall that we need the uniform integrability of $(\sup_{t \in \mathbb{R}_+} |L_t^{k, (p)}|^2)_{k \in \overline{\mathbb{N}}}$ in order to apply Vitali's theorem and prove that Convergences (LS^(p)) and (LS_∞^(p)) hold in \mathbb{L}^2 -mean and not only in probability, which we proved in the previous sub-subsections.

Lemma 3.20. *The sequence $(\Gamma^{k, (p)})_{k \in \overline{\mathbb{N}}}$ is uniformly integrable.*

Proof. We essentially follow the same steps as in Theorem 3.12 in order to derive for every $k \in \mathbb{N}$

$$\begin{aligned} \Gamma^{k, (p)} & \leq 2\bar{A} \sup_{t \in \mathbb{R}_+} \|\mathbb{E}[\xi^k | \mathcal{G}_t^k]\|_2^2 + 2\bar{A} \sup_{t \in \mathbb{R}_+} \left\| \mathbb{E}[L_\infty^{k, (p-1)} - L_t^{k, (p-1)} | \mathcal{G}_t^k] \right\|_2^2 \\ & \quad + \text{Tr}[\langle M^{k, (p-1)} \rangle_\infty] + \Gamma^{k, (0)}. \end{aligned}$$

We only need to observe now that the summands of the right hand side form a uniformly integrable sequence, for $k \in \overline{\mathbb{N}}$, as sum of elements of uniformly integrable random variables; see He et al. [54, Corollary 1.10]. Indeed

- for the sequence associated to the first summand, we have that $(\|\xi^k\|_2^2)_{k \in \overline{\mathbb{N}}}$ is uniformly integrable by Vitali's theorem. Then, we can conclude the uniform integrability of the required sequence by Theorem A.17;
- for the sequence associated to the second summand, we can prove by means of the conditional Jensen Inequality and the conditional Cauchy-Schwarz's inequality that for all $k \in \overline{\mathbb{N}}$ holds

$$\|\mathbb{E}[L_\infty^{k, (p-1)} - L_t^{k, (p-1)} | \mathcal{G}_t^k]\|_2^2 \leq 2\mathbb{E}[\Gamma^{k, (p-1)} | \mathcal{G}_t^k].$$

Since $(\Gamma^{k, (p-1)})_{k \in \overline{\mathbb{N}}}$ is uniformly integrable (by the induction assumption) we can conclude the uniform integrability of $(\|\mathbb{E}[L_\infty^{k, (p-1)} - L_t^{k, (p-1)} | \mathcal{G}_t^k]\|_2^2)_{k \in \overline{\mathbb{N}}}$ by He et al. [54, Theorem 1.7] and Theorem A.17;

- for the sequence associated to the third summand we use the fact that (which is true in view of the validity of $(LS^{(p-1)})$)

$$M_{\infty}^{k,(p-1)} \xrightarrow[k \rightarrow \infty]{\mathbb{L}^2(\Omega, \mathcal{G}, \mathbb{P})} M_{\infty}^{\infty,(p-1)},$$

which implies the uniform integrability of $(\text{Tr}[\langle M^{k,(p-1)} \rangle_{\infty}])_{k \in \overline{\mathbb{N}}}$ ²⁶; recall also Theorem 3.14.(v);

- finally, the sequence associated to the last summand is uniformly integrable by (S6). \square

We can now deduce the following.

Lemma 3.21. *The sequence $(\sup_{t \in \mathbb{R}_+} \|L_t^{k,(p)}\|_2^2)_{k \in \overline{\mathbb{N}}}$ is uniformly integrable.*

Proof. Apply Cauchy–Schwarz’s inequality as in [86, Inequality 3.16] and use the previous lemma. \square

3.6 On the nature of the conditions

In this subsection, we would like to discuss the nature of the conditions we have imposed in order to set the framework for Theorem 3.1.

Let us start with Conditions (S1)–(S5), which are required for Papantoleon, Possamai, and Saplaouras [87, Corollary 3.10]. We have repeatedly stated (and it should be clear from the outline of the proof presented in Subsection 3.1) that the stability of martingale representations plays a crucial role here. However, one may wonder about the necessity of the convergence imposed in (S2)

$$\overline{X}_{\infty}^k \xrightarrow[k \rightarrow \infty]{\mathbb{L}^2(\mathcal{G}; \mathbb{R}^{m+n})} \overline{X}_{\infty}^{\infty},$$

which is stronger compared to [87, Condition (M2)]; the latter, assuming that $m = n$, reads as

$$X_{\infty}^{k,\circ} + X_{\infty}^{k,\natural} \xrightarrow[k \rightarrow \infty]{\mathbb{L}^2(\mathcal{G}; \mathbb{R}^n)} X_{\infty}^{\infty,c} + X_{\infty}^{\infty,d}.$$

The answer has two parts. Initially, we would like to allow the driving martingales $X^{k,\circ}$ and $X^{k,\natural}$ to have different dimensions, for $k \in \overline{\mathbb{N}}$. Afterwards, in sub-subsection 3.5.2, we need to use the convergence

$$(\langle X^{i,\circ} \rangle, \langle X^{i,\natural} \rangle) \xrightarrow[i \rightarrow \infty]{J_1(\mathbb{R}^{m \times m} \times \mathbb{R}^{n \times n})} (\langle X^{\infty,c} \rangle, \langle X^{\infty,d} \rangle), \text{ P-a.s.,}$$

as well as the respective convergences from Theorem 3.14.(iv). These convergences are not guaranteed by [87, Theorem 3.3]. Nevertheless, we still have the flexibility to approximate the continuous martingale part of \overline{X}^{∞} ²⁷ with purely discontinuous martingales, a property which is essential when the discussion comes to numerical schemes.

Regarding Condition (S7), we have already provided some comments in Theorem 2.3. Let us provide a more technical remark at this point. The aforementioned condition comes essentially into play for proving the uniform integrability of the sequence $(\Gamma^{k,(p)})_{k \in \overline{\mathbb{N}}}$, for every $p \in \mathbb{N}$. There, we need in particular to prove the uniform integrability of the sequence $(\int_{(0,\infty)} \|Y_s^{k,(p)}\|_2^2 dA_s^k)_{k \in \overline{\mathbb{N}}}$. From Theorem A.17 we can prove the uniform integrability of $(\sup_{t \in \mathbb{R}_+} \|Y_t^{k,(p)}\|_2^2)_{k \in \overline{\mathbb{N}}}$, which is equivalent to the desired

²⁶The uniform integrability of $(\text{Tr}[\langle M^{k,(0)} \rangle_{\infty}])_{k \in \overline{\mathbb{N}}}$ can also be deduced.

²⁷Recall that $X^{\infty,\circ} \in \mathcal{H}^{2,c}(\mathbb{G}^{\infty}, \tau^{\infty}; \mathbb{R}^m)$.

property, once Condition (S7) is enforced. In a more general situation, it is not clear if, or how, the desired property can be obtained.

Let us proceed now to Condition (S8). The first part of this condition provides a regularity property for the paths of the generators under relatively weak assumptions and is in accordance with Briand, Delyon, and Mémin [19, Condition (H3) (ii)], which considers the case where $X^{\infty, \circ}$ is the Brownian motion and $X^{\infty, \natural} = 0$. One could have assumed a weaker regularity property but, as a trade-off, the convergence in the second part should be strengthened, *e.g.*, to pathwise convergence under the supremum norm. For the second part of the condition, similar conditions are considered when we need to guarantee the convergence of compositions under the J_1 -topology, *e.g.* see Kurtz and Protter [70, Lemma 2.1]. In the case where $(f^k)_{k \in \overline{\mathbb{N}}}$ are globally Lipschitz with the same Lipschitz constant, one need not be as abstract as in [70] since one can exploit the Lipschitz property of the generators. For example, in Briand, Delyon, and Mémin [19, Proposition 11] under the aforementioned uniform equi-continuity assumption on the generators, the uniform (in time) convergence *for every fixed point* (y, z) , in our notation, (see [19, Convergence (7)]) is equivalent to the uniform (in time) convergence *on compacts*. Hence, Condition (S8).(ii) is a relaxation of [19, Convergence (7)].

Let us further proceed to the discussion about Condition (S9), which we will analyse part by part.

- (i) The pair (\overline{X}^k, C^k) satisfies [86, Assumption 2.10], for each $k \in \overline{\mathbb{N}}$. The existence of the integrator C^k is ensured by arguments analogous to Jacod and Shiryaev [65, Proposition II.2.9]. In other words, we have set

$$C^k := h^k(\langle X^{k, \circ} \rangle, \text{Id} * \nu^{(X^{\natural}, G^k)}),$$

for some function h^k , for every $k \in \overline{\mathbb{N}}$. On the other hand, in view of Conditions (S1)–(S5), we know by [87, Corollary 3.10] that

$$\langle \overline{X}^k \rangle \xrightarrow[k \rightarrow \infty]{J_1(\mathbb{R}^{(m+n) \times (m+n)}, \mathbb{L}^1)} \langle \overline{X}^\infty \rangle.$$

Hence, we can choose the sequence $(h^k)_{k \in \mathbb{N}}$, which may allow dependence on time, as soon as it respects the J_1 -topology. In particular, we may choose $h^k = h$ for every $k \in \overline{\mathbb{N}}$, where $h : (\mathbb{D}(\mathbb{R}), \delta_{J_1}) \rightarrow (\mathbb{R}_+, |\cdot|)$ is a continuous function.

- (ii) This is almost analogous to the previous point. Since $\langle \overline{X}^k \rangle_\infty \in \mathbb{L}^1(\mathcal{G})$, for every $k \in \overline{\mathbb{N}}$, the finiteness of C_∞^k , for every $k \in \overline{\mathbb{N}}$, is immediate by the comments above. However, there is a subtle point in the convergence part that we will take care of with the help of Condition (S10). In general, we *cannot* prove that $\langle \overline{X}^k \rangle_\infty \xrightarrow{\mathbb{L}^1(\mathcal{G})} \langle \overline{X}^\infty \rangle_\infty$, although we can prove that $\langle \overline{X}^k \rangle_\infty \xrightarrow{\mathbb{L}^1(\mathcal{G})} \langle \overline{X}^\infty \rangle_\infty$; see Mémin [81, Proof of Corollary 12] for the one dimensional case. However, the process $\langle \overline{X}^\infty \rangle$ is continuous. Hence, for a subsequence $(k_l)_{l \in \mathbb{N}}$ such that

$$\langle \overline{X}^{k_l} \rangle \xrightarrow[l \rightarrow \infty]{J_1(\mathbb{R}^{(m+n) \times (m+n)})} \langle \overline{X}^\infty \rangle, \text{ P-a.s.},$$

it holds for every $t \in \mathbb{R}_+$

$$\langle \overline{X}^{k_l} \rangle_t \xrightarrow[l \rightarrow \infty]{\|\cdot\|_2} \langle \overline{X}^\infty \rangle_t, \text{ P-a.s.}$$

Hence, if Condition (S10) is enforced, we can derive for the diagonal elements of the predictable quadratic covariation processes (which are increasing) that

$$\langle \overline{X}^{k_{l_i}, jj} \rangle_\infty = \langle \overline{X}^{k_{l_i}, jj} \rangle_{\tau^k} \xrightarrow[i \rightarrow \infty]{|\cdot|} \langle \overline{X}^{\infty, jj} \rangle_{\tau^k} = \langle \overline{X}^{\infty, jj} \rangle_\infty, \text{ P-a.s.} \quad (3.40)$$

By the polarisation identity, we can prove the convergence for all the off-diagonal elements as well. Hence, the assumption $C_\infty^k \xrightarrow[k \rightarrow \infty]{\mathbb{P}} C_\infty^\infty$ is reasonable.

(iii) In view of (S7) and the previous comments, this assumption is completely natural.

Finally, let us comment on Condition (S10), which was used only for the proof²⁸ of Theorem 3.17 and for arguing about the reasonableness of Condition (S9).(ii). The finiteness of τ^∞ seems unavoidable given our need for Convergence (3.40). Regarding the convergence of the stopping times, on the one hand it is trivially satisfied when the terminal times are deterministic and not necessarily identical. This case corresponds to the majority of the articles in the literature. On the other hand, when the terminal times are random, the required convergence can be proved for specific debut times of a convergent sequence of processes; see the proof of Jacod and Shiryaev [65, Theorem IX.1.17]. However, there is no general way (to the best of our knowledge) for constructing convergent sequences of stopping times when the associated filtrations change. Answers to this problem in special cases may be offered by Coquet and Toldo [23, Proposition 20] or Kchia [67, Lemma 20]. As a last comment, the convergence we assumed does not require any integrability condition on the stopping times; as a comparison in Toldo [92] a uniform integrability condition is required.

4 Examples and applications

In this section, we present a few examples and applications that demonstrate the power of our framework. We assume that the sequence of processes (\bar{X}^k) for $k \in \mathbb{N}$ are indexed either in a discrete- or in a continuous-time set; in case the processes are indexed in a discrete-time set, they can be embedded in our framework in the obvious way. In other words, the processes $(\bar{X}^k)_{k \in \mathbb{N}}$ could be, for example, random walks or discretisations or even other continuous-time approximations of the driving martingales. In case of random walks, which is the natural choice when considering numerical schemes for BSDEs, each martingale will be defined with respect to its own filtration, and the convergence of these filtrations is a natural requirement. Otherwise, our framework allows great flexibility in choosing a convergent sequence of suitable standard data.

Before we proceed, let us remind the reader that the Skorokhod space endowed with the J_1 -topology is Polish. Consequently, we can use Skorokhod's representation theorem in order to assume, without loss of generality, that the processes converge in probability, instead of in law. Of course, there is a crucial point which needs some care: the weak convergence of the associated filtrations; the reader should recall that this notion requires the filtrations to be defined on the same probability space, which is the case throughout this work.

4.1 Continuous martingales

As a first example, we consider the case where the limit-BSDE is driven by a non-trivial, continuous, square-integrable martingale with independent increments, *i.e.*, $X^{\infty, \natural} = 0$. In this case, we may assume that, for every $k \in \mathbb{N}$, $X^{k, \natural} = 0$ and, consequently, the domain of the generator f^k is $\Omega \times \mathbb{R}_+ \times \mathbb{R}^\ell \times \mathbb{R}^{\ell \times m}$, *i.e.*, there is no dependence on elements of \mathfrak{H}^k . Consider a sequence $(X^{k, \circ})_{k \in \mathbb{N}}$ of square-integrable martingales with

²⁸Actually, we have not used Condition (S10) when an alternative valid argument could have been used, *e.g.*, the use of the compact supports of integrands Z and U .

independent increments such that

$$X^{k,\circ} \xrightarrow[k \rightarrow \infty]{(J_1, \mathbb{P})} X^{\infty,\circ} \quad \text{and} \quad X_{\infty}^{k,\circ} \xrightarrow[k \rightarrow \infty]{L^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}^{\ell})} X_{\infty}^{\infty,\circ}.$$

Then, we obtain the weak convergence of the associated natural filtrations

$$\mathbb{F}^{X^{k,\circ}} \xrightarrow[k \rightarrow \infty]{w} \mathbb{F}^{X^{\infty,\circ}},$$

by Coquet, Mémin, and Słomiński [25, Proposition 2]. Once Conditions (S6)–(S10) are satisfied, this yields a slight, but strict, generalisation of the stability results in Briand, Delyon, and Mémin [19], coming from the fact that we allow for unbounded Lipschitz constants and integrators of Lebesgue–Stieltjes integrals, which suitably combined form a bounded family of processes $(A^k)_{k \in \mathbb{N}}$, and the fact that we allow for unbounded terminal times. The natural candidate for the Itô integrator is Brownian motion, however more general *continuous*, square-integrable martingales can also be considered. This was already observed in Briand, Delyon, and Mémin [19, Section 6].

Remark 4.1. Let us point out that the generality of our framework allows to conclude the convergence for Itô integrals defined with respect to general (*i.e.* non-continuous) martingales, once we assume that $X^{k,\circ} = 0$, that $X^{k,\natural}$ is a square-integrable martingale (with independent increments), and we restrict the class of integrators to $U^k \equiv Z^k \cdot x$, for every $k \in \mathbb{N}$. In this case, Condition (S3) should be replaced by the *strong* predictable representation property; see He, Wang, and Yan [54, Chapter XIII].

4.2 Probabilistic numerical schemes for deterministic equations

As a second example, we are interested in approximating the solution of a (deterministic) integral equation with terminal condition via a sequence of solutions of BSDEs. To this end, for simplicity, we will modify the framework of the previous example by considering the stochastically trivial case for the limit-BSDE, *i.e.*, $X^{\infty,\circ} = X^{\infty,\natural} = 0$. The filtration that we associate to the limit-BSDE is the constantly trivial filtration, *i.e.*, $\mathcal{G}_t^{\infty} := \{\emptyset, \Omega\}$ for every $t \in \mathbb{R}_+$. As a consequence, in the limit-BSDE all the random elements are trivialised. The reader can verify that, for any sequence $(X^{k,\circ})_{k \in \mathbb{N}}$ such that $X_{\infty}^{k,\circ} \xrightarrow[k \rightarrow \infty]{L^2(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{R}^{\ell})} 0$ and for any sequence of filtrations associated to $(X^{k,\circ})_{k \in \mathbb{N}}$, Conditions (S1)–(S5) are trivially satisfied. In particular, Condition (S4) is trivially satisfied because the only ‘random variables’ one can choose from $\mathcal{G}_{\infty}^{\infty}$ are constant numbers. Let us point out, that we can freely choose the integrator C^{∞} of the Lebesgue–Stieltjes integral to be any càdlàg, increasing function. Almost all the notions of convergence are clear, *e.g.*, the sequence of processes converges to a constant function and the sequence of random variables converges to a number. However, the convergence of the generators f^k to the generator f^{∞} of the (deterministic) integral limit-equation needs to be modified. If we want to have a unique solution for the limit-equation, we should choose the domain of f^{∞} to be $\Omega \times \mathbb{R}_+ \times \mathbb{R}^{\ell}$, where of course there is no (true) randomness, *i.e.*, no dependence on $\omega \in \Omega$. Now, one should modify Condition (S8).(ii) as follows: if $(W^k)_{k \in \mathbb{N}}$ is a sequence of càdlàg maps such that $W^k \xrightarrow{J_1} W^{\infty}$, then for every $Z \in D^{\circ, \ell \times m}$

$$(f^k(t, W_t^k, Z_t))_{t \in \mathbb{R}_+} \xrightarrow[k \rightarrow \infty]{J_1(\mathbb{R}^{\ell})} (f^{\infty}(t, W_t^{\infty}))_{t \in \mathbb{R}_+}, \quad \mathbb{P}\text{-a.s.}$$

In other words, one would expect the Z -variable to vanish in the limit. Now, Theorem 3.1 yields that the sequence of solutions (Y^k, Z^k, N^k) converges to $(Y, 0, 0)$, where Y is the solution of the deterministic integral equation. This allows us to design probabilistic

schemes for the numerical solution of deterministic equations, which could be interesting in high-dimensional situations. This example may also be compared to the results obtained in Backhoff-Veraguas, Lacker, Tangpi, et al. [3], where the authors consider the case where the Z -variable does not vanish and, thus, uncover interesting phenomena under suitable scaling.

4.3 PII martingales

Let us turn our interest to more general cases and consider a setting which is not covered, to the best of our knowledge, in the existing literature. More precisely, we assume additionally to (S1)–(S10) that the sequence of integrators consists of martingales with independent increments and the associated filtrations are their natural ones. Since we do not necessarily require the increments to be stationary, we consider a class which is broader than the class of Lévy martingales; these are called PII martingales in Jacod and Shiryaev [65]. We underline that, in view of the independence of the increments, Condition (S4) is satisfied due to Coquet, Mémin, and Słomiński [25, Proposition 2]. Thus, if the sequence $(\bar{X}^k)_{k \in \mathbb{N}}$ consists of processes indexed on a continuous time-set, then Theorem 3.1 allows to conclude the convergence of perturbations of the limit-BSDE when we ‘wiggle’ all the elements of the standard data. On the other hand, if the sequence $(\bar{X}^k)_{k \in \mathbb{N}}$ consists of processes indexed on a discrete time-set, then Theorem 3.1 can be applied to derive the convergence of numerical schemes for the limit-BSDE. The reader may immediately verify that the work of Madan, Pistorius, and Stadje [77] is a special case of this example.

4.4 BSDEs as dual problems

As a final example, let us recall that in many applications BSDEs can provide an alternative characterisation for the solution of the problem at hand; a prominent example is stochastic optimal control problems. The current work covers Lipschitz BSDEs, so one can immediately derive the stability of, say, stochastic optimal control problems via Theorem 3.1, as long as their dual problem corresponds to a BSDE with a Lipschitz generator. Such an approach has been used for the control of population dynamics by Jusselin and Mastrolia [66]. In [66, Theorem 2] the authors use stronger conditions, compared to Theorem 3.1, with the only exception of (S8). More precisely, the notion of convergence for the generators in [66, Theorem 2] requires suitable scaling of the Z -variable in order to obtain the desired limit. Thus, [66, Theorem 2] cannot be recovered by Theorem 3.1. The reader may recall from a previous example that the approach of using a suitable scaling has also been used by Backhoff-Veraguas, Lacker, Tangpi, et al. [3].

A Auxiliary results

A.1 Definitions and complete notation

Assumption A.1. 2.10 of [86] Let $\bar{X} := (X^\circ, X^\natural) \in \mathcal{H}^2(\mathbb{R}^m) \times \mathcal{H}^2(\mathbb{R}^n)$ and C be a predictable, càdlàg and increasing process. The pair (\bar{X}, C) satisfies [86, Assumption 2.10] if each component of $\langle X^\circ \rangle$ is absolutely continuous with respect to C and if the disintegration property given C holds for the compensator ν^{X^\natural} , i.e., there exists a transition kernel $K^{\bar{X}} : (\Omega \times \mathbb{R}_+, \mathcal{P}) \longrightarrow \mathcal{R}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, where $\mathcal{R}(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is the space of Radon measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, such that

$$\nu^{X^\natural}(\omega; dt, dx) = K_t^{\bar{X}}(\omega; dx) dC_t^{\bar{X}}.$$

Following the notation of [86], we define

$$c^k := \left(\frac{d\langle X^{k,\circ} \rangle}{dC^k} \right)^{\frac{1}{2}},$$

in conjunction with the notation introduced in Section 2, for $t \in \mathbb{R}_+$, $V : \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^p$ be such that $\mathbb{R}^n \ni x \mapsto V(\omega, t, x) \in \mathbb{R}^p$ is Borel-measurable

$$\widehat{K}_t^k(V_s)(\omega) := \int_{\mathbb{R}^n} V(\omega, s, x) K_t^k(\omega; dx), \quad (\text{A.1})$$

if $\int_{\mathbb{R}^n} (\|V(\omega, s, x)\|_2^2 \wedge 1) K_t^k(\omega; dx) < \infty$, otherwise $\widehat{K}_t^k(V_s)(\omega) := \infty$;

$$\zeta_t^{k,\natural} := \int_{\mathbb{R}^n} \nu^{k,\natural}(\{t\} \times dx) = \widehat{\nu}_t^{k,\natural}(\mathbb{1}_{\mathbb{R}^n}) \leq 1$$

and

$$\widehat{\nu}_t^{k,\natural}(V_s)(\omega) := \int_{\mathbb{R}^n} V(\omega, s, x) \nu^{k,\natural}(\omega; \{t\} \times dx),$$

if $\int_{\mathbb{R}^n} \|V(\omega, s, x)\|_1 \nu^{k,\natural}(\omega; \{t\} \times dx) < \infty$, otherwise $\widehat{\nu}_t^{k,\natural}(V_s)(\omega) := \infty$; and

$$\left(\|V(t, \cdot)\|_{k,t}(\omega) \right)^2 := \widehat{K}_t^k(\|V_s - \widehat{\nu}_t^{k,\natural}(V_t)\|_2^2)(\omega) + (1 - \zeta_t^{k,\natural}(\omega)) \Delta C_t^k(\omega) \|\widehat{K}_t^k(V_t)(\omega)\|_2^2.$$

Finally, we define the following spaces for $d\mathbb{P} \otimes dC^k$ -a.e. $(\omega, t) \in \Omega \times \mathbb{R}_+$

$$\mathfrak{H}_{\omega,t}^k := \overline{\left\{ \mathcal{U} : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \longrightarrow (\mathbb{R}^\ell, \mathcal{B}(\mathbb{R}^\ell)), \|\mathcal{U}(\cdot)\|_{k,t}(\omega) < \infty \right\}}.$$

as well as

$$\mathfrak{H}^k := \left\{ U : \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^\ell, U(\omega, t, \cdot) \in \mathfrak{H}_{\omega,t}^k, \text{ for } d\mathbb{P} \otimes dC^k\text{-a.e. } (\omega, t) \in \Omega \times \mathbb{R}_+ \right\}.$$

For every $k \in \overline{\mathbb{N}}$ the following are true: the generator $(f^k)_{k \in \overline{\mathbb{N}}} : \Omega \times \mathbb{R}_+ \times \mathbb{R}^\ell \times \mathbb{R}^{\ell \times m} \times \mathfrak{H}^k \longrightarrow \mathbb{R}^\ell$ satisfies Condition (F3), i.e., it is such that for any $(y, z, u) \in \mathbb{R}^\ell \times \mathbb{R}^{\ell \times m} \times \mathfrak{H}^k$, the map

$$(t, \omega) \longmapsto f^k(t, \omega, y, z, u_t(\omega; \cdot)) \text{ is } \mathcal{F}_t^k \otimes \mathcal{B}([0, t])\text{-measurable.}$$

Moreover, f satisfies a stochastic Lipschitz condition, that is to say there exist

$$r^k : (\Omega \times \mathbb{R}_+, \mathcal{P}^k) \longrightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \text{ and } \vartheta^k = (\theta^{k,\circ}, \theta^{k,\natural}) : (\Omega \times \mathbb{R}_+, \mathcal{P}^k) \longrightarrow (\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2)),$$

such that, for $dC \otimes d\mathbb{P}$ -a.e. $(t, \omega) \in \mathbb{R}_+ \times \Omega$

$$\begin{aligned} & \left| f^k(t, \omega, y, z, u_t(\omega; \cdot)) - f^k(t, \omega, y', z', u'_t(\omega; \cdot)) \right|^2 \\ & \leq r_t^k(\omega) |y - y'|^2 + \theta_t^{k,\circ}(\omega) \|c_t^k(\omega)(z - z')\|^2 + \theta_t^{k,\natural}(\omega) \left(\|u_t(\omega; \cdot) - u'_t(\omega; \cdot)\|_{k,t}(\omega) \right)^2. \end{aligned}$$

Moreover, to every generator f^k we associate the predictable processes α^k and A^k , where $\alpha^k := \max\{\sqrt{r^k}, \theta^{k,\circ}, \theta^{k,\natural}\}$, $A^k := ((\alpha^k)^2 \cdot C^k)$, and the bound Φ^k such that $\Delta A^k \leq \Phi^k$. All of them are described in Condition (F4).

The spaces that are necessary for the existence and uniqueness result are introduced in [86, Section 2.3]. For $\beta \geq 0$, $p \in \mathbb{N}$ and for each $k \in \overline{\mathbb{N}}$ we use the following spaces

$$\mathbb{L}_{\beta}^{2,k} := \left\{ \xi : \mathbb{R}^\ell\text{-valued, } \mathcal{G}_{\tau^k}^k\text{-measurable, } \|\xi\|_{\mathbb{L}_{\beta}^{2,k}}^2 := \mathbb{E} \left[e^{\beta A_{\tau^k}^k} \|\xi\|_2^2 \right] < \infty \right\},$$

$$\begin{aligned}
 \mathcal{H}_\beta^{2,k}(\mathbb{R}^p) &:= \left\{ M \in \mathcal{H}^2(\mathbb{G}^k, \tau^k; \mathbb{R}^p) : \|M\|_{\mathcal{H}_\beta^{2,k}(\mathbb{R}^p)}^2 := \mathbb{E} \left[\int_{(0, \tau^k]} e^{\beta A_t^k} d\text{Tr} \left[\langle M \rangle_t^{\mathbb{G}^k} \right] \right] < \infty \right\}, \\
 \mathbb{H}_\beta^{2,k}(\mathbb{R}^p) &:= \left\{ \phi : \mathbb{R}^p\text{-valued } \mathbb{G}^k\text{-optional semimartingale with càdlàg paths and} \right. \\
 &\quad \left. \|\phi\|_{\mathbb{H}_\beta^{2,k}(\mathbb{R}^p)}^2 := \mathbb{E} \left[\int_{(0, \tau^k]} e^{\beta A_t^k} \|\phi_t\|_2^2 dC_t^k \right] < \infty \right\}, \\
 \mathcal{S}_\beta^{2,k}(\mathbb{R}^p) &:= \left\{ \phi : \mathbb{R}^\ell\text{-valued } \mathbb{G}^k\text{-optional semimartingale with càdlàg paths and} \right. \\
 &\quad \left. \|\phi\|_{\mathcal{S}_\beta^{2,k}(\mathbb{R}^p)}^2 := \mathbb{E} \left[\sup_{t \in [0, \tau^k]} e^{\beta A_t^k} \|\phi_t\|_2^2 \right] < \infty \right\}, \\
 \mathbb{H}^{2,k,\circ} &:= \left\{ Z : (\Omega \times \mathbb{R}_+, \mathcal{P}^k) \longrightarrow (\mathbb{R}^{\ell \times m}, \mathcal{B}(\mathbb{R}^{\ell \times m})) : \right. \\
 &\quad \left. \mathbb{E} \left[\int_{(0, \tau^k]} \text{Tr} \left[Z_t \frac{d\langle X \rangle_s}{dC_s} Z_t^\top \right] dC_t \right] < \infty \right\}, \\
 \mathbb{H}_\beta^{2,k,\circ} &:= \left\{ Z \in \mathbb{H}^{2,k,\circ} : \|Z\|_{\mathbb{H}_\beta^{2,k,\circ}}^2 := \mathbb{E} \left[\int_{(0, \tau^k]} e^{\beta A_t^k} d\text{Tr} [\langle Z \cdot X^{k,\circ} \rangle_t^{\mathbb{G}^k}] \right] < \infty \right\}, \\
 \mathbb{H}^{2,k,\natural} &:= \left\{ U : (\tilde{\Omega}, \tilde{\mathcal{P}}^k) \longrightarrow (\mathbb{R}^\ell, \mathcal{B}(\mathbb{R}^\ell)) : \mathbb{E} \left[\int_{(0, \tau^k]} d\text{Tr} [U \star \tilde{\mu}^{k,\natural}_t]^{\mathbb{G}^k} \right] < \infty \right\}, \\
 \mathbb{H}_\beta^{2,k,\natural} &:= \left\{ U \in \mathbb{H}^{2,k,\natural} : \|U\|_{\mathbb{H}_\beta^{2,k,\natural}}^2 := \mathbb{E} \left[\int_{(0, \tau^k]} e^{\beta A_t^k} d\text{Tr} [\langle U \star \tilde{\mu}^{k,\natural}_t \rangle_t^{\mathbb{G}^k}] \right] < \infty \right\}, \\
 \mathcal{H}_\beta^{2,k,\perp} &:= \left\{ M \in \mathcal{H}^{2,k} : \langle X^{k,\circ}, M \rangle = 0, M_{\mu^{k,\natural}}[\Delta M | \tilde{\mathcal{P}}^k] = 0, \text{ and } \|M\|_{\mathcal{H}_\beta^{2,k}}^2 < \infty \right\}.
 \end{aligned}$$

Finally, for $(Y, Z, U, N) \in \mathbb{H}_\beta^{2,k} \times \mathbb{H}_\beta^{2,k,\circ} \times \mathbb{H}_\beta^{2,k,\natural} \times \mathcal{H}_\beta^{2,k,\perp}$ we define

$$\|(Y, Z, U, N)\|_{k,\beta}^2 := \|\alpha Y\|_{\mathbb{H}_\beta^{2,k}}^2 + \|Z\|_{\mathbb{H}_\beta^{2,k,\circ}}^2 + \|U\|_{\mathbb{H}_\beta^{2,k,\natural}}^2 + \|N\|_{\mathcal{H}_\beta^{2,k,\perp}}^2,$$

and for $(Y, Z, U, N) \in \mathcal{S}_\beta^{2,k} \times \mathbb{H}_\beta^{2,k,\circ} \times \mathbb{H}_\beta^{2,k,\natural} \times \mathcal{H}_\beta^{2,k,\perp}$, we define

$$\|(Y, Z, U, N)\|_{*,k,\beta}^2 := \|Y\|_{\mathcal{S}_\beta^{2,k}}^2 + \|Z\|_{\mathbb{H}_\beta^{2,k,\circ}}^2 + \|U\|_{\mathbb{H}_\beta^{2,k,\natural}}^2 + \|N\|_{\mathcal{H}_\beta^{2,k,\perp}}^2.$$

A.2 Moore–Osgood’s theorem

Moore–Osgood’s theorem, whose well-known form is Theorem A.2, provides sufficient conditions for the existence of the limit of a doubly-indexed sequence and can be seen as a special case of Rudin [90, Theorem 7.11]. Here we provide a second form, since the second time²⁹ we need to apply the aforementioned theorem we need to relax the existence of the pointwise limits; compare the second conditions of Theorem A.2 and Theorem A.3. The interested reader should consult Hobson [58, Chapter VI, Sections 336–338] for more details on the existence of the iterated limits and of the joint limit of a doubly-indexed sequence. Specifically for the validity of Theorem A.3 see [58, Chapter VI, Section 337, p. 466] and the reference therein.

Theorem A.2. *Let (Γ, d_Γ) be a metric space and $(\gamma_{k,p})_{k,p \in \mathbb{N}}$ be a sequence such that $\gamma_{\infty,p} := \lim_{k \rightarrow \infty} \gamma_{k,p}$ exists for every $p \in \mathbb{N}$ and $\gamma_{k,\infty} := \lim_{p \rightarrow \infty} \gamma_{k,p}$ exists for every $k \in \mathbb{N}$. If*

$$(i) \quad \lim_{p \rightarrow \infty} \sup_{k \in \mathbb{N}} d_\Gamma(\gamma_{k,p}, \gamma_{k,\infty}) = 0,$$

²⁹The first one is in the proof of Theorem 3.1, while the second is in the proof of Theorem A.9

(ii) $\lim_{k \rightarrow \infty} d_{\Gamma}(\gamma_{k,p}, \gamma_{\infty,p}) = 0$, for all $p \in \mathbb{N}$,

then the joint limit $\lim_{k,p \rightarrow \infty} \gamma_{n,k}$ exists. In particular holds $\lim_{k,p \rightarrow \infty} \gamma_{k,p} = \lim_{p \rightarrow \infty} \gamma_{\infty,p} = \lim_{k \rightarrow \infty} \gamma_{k,\infty}$.

Theorem A.3. Let $(\mathbb{R}, |\cdot|)$ and $(\gamma_{k,p})_{k,p \in \mathbb{N}}$ be a sequence such that $\gamma_{k,\infty} := \lim_{p \rightarrow \infty} \gamma_{k,p}$ exists for every $k \in \mathbb{N}$. If

(i) $\lim_{p \rightarrow \infty} \sup_{k \in \mathbb{N}} |\gamma_{k,p} - \gamma_{k,\infty}| = 0$,

(ii) $\lim_{p \rightarrow \infty} \left\{ \limsup_{k \rightarrow \infty} \gamma_{k,p} - \liminf_{k \rightarrow \infty} \gamma_{k,p} \right\} = 0$,

then the joint limit $\lim_{k,p \rightarrow \infty} \gamma_{k,p}$ exists. In particular, it holds $\lim_{k,p \rightarrow \infty} \gamma_{k,p} = \lim_{p \rightarrow \infty} \lim_{k \rightarrow \infty} \gamma_{k,p} = \lim_{k \rightarrow \infty} \limsup_{p \rightarrow \infty} \gamma_{k,p}$.

A.3 Weak convergence of measures on the positive real line

The aim of the current appendix is to provide a characterization of weak convergence of finite measures³⁰ defined on the positive real line in case the limit measure is atomless. This characterization uses relatively compact sets of the Skorokhod space instead of relatively compact sets of the space of continuous functions defined on \mathbb{R}_+ endowed with the supremum norm $\|\cdot\|_{\infty}$. This result is of independent interest.

Definition A.4. Let $(\mu^k)_{k \in \mathbb{N}}$ be a countable family of measures on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. We will say that the sequence $(\mu^k)_{k \in \mathbb{N}}$ converges weakly to the measure μ^{∞} if for every $f : (\mathbb{R}_+, \|\cdot\|_{\infty}) \rightarrow (\mathbb{R}, |\cdot|)$ continuous and bounded holds

$$\lim_{k \rightarrow \infty} \left| \int_{[0, \infty)} f(x) \mu^k(dx) - \int_{[0, \infty)} f(x) \mu^{\infty}(dx) \right| = 0.$$

We denote the weak convergence of $(\mu^k)_{k \in \mathbb{N}}$ to μ^{∞} by $\mu^k \xrightarrow{w} \mu^{\infty}$.

In this section we will use the set

$$\mathcal{W}_{0,\infty}^+ := \left\{ F \in \mathcal{D}(\mathbb{R}_+) : F \text{ is increasing with } F_0 = 0 \text{ and } \lim_{t \rightarrow \infty} F_t \in \mathbb{R}_+ \right\}.$$

Remark A.5. We can extend every element of $\mathcal{W}_{0,\infty}^+$, name F an arbitrary element, on $[0, \infty]$ such that it is left-continuous at the symbol ∞ by defining $F_{\infty} := \lim_{t \rightarrow \infty} F_t$.

We provide in the following proposition some convenient equivalence results for the weak convergence of finite measures. The statement is tailor-made to our needs, but the interested reader may consult Bogachev [10, Sections 8.1–8.3]. Then we provide in Theorem A.8 a new, to the best of our knowledge, characterisation of weak convergence of finite measures to an atomless measure defined on the positive real line.

Proposition A.6. Let $(\mu^k)_{k \in \mathbb{N}}$ be sequence of finite measures on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ with associated distribution functions $(F^k)_{k \in \mathbb{N}}$, where $F^k \in \mathcal{W}_{0,\infty}^+$, for $k \in \mathbb{N}$.³¹ We assume the following

(i) the sequence $(\mu^k)_{k \in \mathbb{N}}$ is bounded, i.e. $\sup_{k \in \mathbb{N}} \mu^k(\mathbb{R}_+) = \sup_{k \in \mathbb{N}} F_{\infty}^k < \infty$;

(ii) the sequence $(\mu^k)_{k \in \mathbb{N}}$ is tight. In other words, for every $\varepsilon > 0$ there exists $N_{\varepsilon} \in \mathbb{N}$ such that

$$\sup_{k \in \mathbb{N}} \mu^k(\mathbb{R}_+ \setminus [0, N_{\varepsilon}]) = \sup_{k \in \mathbb{N}} |F_{\infty}^k - F_{N_{\varepsilon}}^k| < \varepsilon;$$

³⁰We only consider positive measures, i.e., not signed ones.

³¹For simplicity we assume that $\mu^k(\{0\}) = 0$.

(iii) the limit-measure μ^∞ is atomless, i.e. $\mu^\infty(\{t\}) = 0$ for every $t \in \mathbb{R}_+$. Equivalently, F^∞ is continuous.

Then, the following are equivalent

- (a) $\mu^k \xrightarrow{w} \mu$;
- (b) $F_t^k \xrightarrow[k \rightarrow \infty]{} F_t^\infty$, for every $t \in \mathbb{R}_+$;
- (c) $F^k \xrightarrow{J_1(\mathbb{R})} F^\infty$;
- (d) $F^k \xrightarrow{\text{lu}} F^\infty$, where lu stands for the locally uniform convergence;
- (e) $\sup_{I \in \mathcal{I}_N} |\mu^k(I) - \mu^\infty(I)| \xrightarrow[k \rightarrow \infty]{} 0$, for every $N \in \mathbb{N}$, where $\mathcal{I}_N := \{I \subset [0, N] : I \text{ is interval}\}$.

Proof. The equivalence between (a) and (b) is a classical result, e.g. see [10, Proposition 8.1.8]. The equivalences between (b), (c) and (d) are provided by Jacod and Shiryaev [65, Theorem VI.2.15.c.(i)]. We obtain the equivalence between (d) and (e) in view of the validity of the following inequalities for every $N \in \mathbb{N}$:

$$\begin{aligned}
 \sup_{t \in [0, N]} |F_t^k - F_t^\infty| &\leq \sup_{I \in \mathcal{I}_N} |\mu^k(I) - \mu^\infty(I)| \\
 &\leq \sup_{s, t \in [0, N]} |F_{t-}^k - F_s^k - F_{t-}^\infty + F_s^\infty| + \sup_{s, t \in [0, N]} |F_t^k - F_s^k - F_t^\infty + F_s^\infty| \\
 &\quad + \sup_{s, t \in [0, N]} |F_{t-}^k - F_{s-}^k - F_{t-}^\infty + F_{s-}^\infty| \\
 &\leq 3 \sup_{t \in [0, N]} |F_t^k - F_t^\infty| + 3 \sup_{t \in [0, N]} |F_{t-}^k - F_{t-}^\infty| \\
 &\leq 3 \sup_{t \in [0, N]} |F_t^k - F_t^\infty| + 3 \sup_{t \in [0, N]} |F_{t-}^k - F_t^k| + 3 \sup_{t \in [0, N]} |F_t^k - F_t^\infty| \\
 &\leq 6 \sup_{t \in [0, N]} |F_t^k - F_t^\infty| + 3 \sup_{t \in [0, N]} |F_{t-}^k - F_t^k|.
 \end{aligned}$$

Then

$$\begin{aligned}
 \limsup_{k \rightarrow \infty} \sup_{I \in \mathcal{I}_N} |\mu^k(I) - \mu^\infty(I)| &\leq 6 \limsup_{k \rightarrow \infty} \sup_{t \in [0, N]} |F_t^k - F_t^\infty| + 3 \limsup_{k \rightarrow \infty} \sup_{t \in [0, N]} |F_{t-}^k - F_t^k| \\
 &\leq 3 \sup_{t \in [0, N]} |F_{t-}^\infty - F_t^\infty| = 0,
 \end{aligned}$$

where we used [65, Lemma VI.2.5] in the last inequality. Hence, for every $N \in \mathbb{N}$,

$$\lim_{k \rightarrow \infty} \sup_{t \in [0, N]} |F_t^k - F_t^\infty| = 0, \text{ if and only if } \lim_{k \rightarrow \infty} \sup_{I \in \mathcal{I}_N} |\mu^k(I) - \mu^\infty(I)| = 0. \quad \square$$

Assume the framework of the previous proposition and that its Part (a) is true. Then, for $\mathbb{1}_{\mathbb{R}_+}$ we get

$$\mu^k(\mathbb{R}_+) \xrightarrow[k \rightarrow \infty]{|\cdot|} \mu^\infty(\mathbb{R}_+),$$

or, alternatively written, $F_\infty^k \xrightarrow[k \rightarrow \infty]{|\cdot|} F_\infty^\infty$. The following lemma allows us to use a stronger convergence for the sequence of distribution functions in the special case we consider, namely the convergence holds under the supremum norm. In other words, we have the convergence of the distribution functions under the Kolmogorov–Smirnov distance. For the sake of completeness we present the details.

Lemma A.7. Let $(F^k)_{k \in \mathbb{N}} \subset \mathcal{W}_{0,\infty}^+$ which satisfies the following properties

- (i) $F^k \xrightarrow[k \rightarrow \infty]{J_1(\mathbb{R})} F^\infty$;
- (ii) $F_\infty^k \xrightarrow[k \rightarrow \infty]{|\cdot|} F_\infty^\infty$;
- (iii) the limit function F^∞ is continuous.

Then it holds $F^k \xrightarrow[k \rightarrow \infty]{\|\cdot\|_\infty} F^\infty$.

Proof. Let us fix an arbitrary $\varepsilon > 0$. We start by exploiting the fact that $F^\infty \in \mathcal{W}_{0,\infty}^+$, i.e., there exists $K > 0$ such that

$$\sup_{t \geq K} \{F_\infty^\infty - F_t^\infty\} < \frac{\varepsilon}{8}. \quad (\text{A.2})$$

By (ii), there exists $k_1 = k_1(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{k \geq k_1} |F_\infty^k - F_\infty^\infty| < \frac{\varepsilon}{8}. \quad (\text{A.3})$$

By (iii) there exists $\bar{s} > K$ such that $F_{\bar{s}}^\infty = \frac{1}{2}(F_K^\infty + F_\infty^\infty)$, i.e., $|F_\infty^\infty - F_{\bar{s}}^\infty| < \frac{\varepsilon}{16}$. By (i), (iii) and Theorem A.6.(d), there exists $k_2 = k_2(\varepsilon, \bar{s})$ such that

$$\sup_{k \geq k_2} \sup_{0 \leq t \leq \bar{s}} |F_t^k - F_t^\infty| < \frac{1}{2}(F_\infty^\infty - F_K^\infty) < \frac{\varepsilon}{16}, \quad (\text{A.4})$$

where the last inequality is valid in view of (A.2), since $\bar{s} > K$. Using the fact that the functions are increasing we derive

$$\begin{aligned} \sup_{k \geq k_1 \vee k_2} \left\{ \sup_{\bar{s} \leq t < \infty} F_\infty^k - F_t^k \right\} &= \sup_{k \geq k_1 \vee k_2} \{F_\infty^k - F_{\bar{s}}^k\} \\ &\leq \sup_{k \geq k_1 \vee k_2} |F_\infty^k - F_\infty^\infty| + \sup_{k \geq k_1 \vee k_2} |F_\infty^\infty - F_{\bar{s}}^\infty| + \sup_{k \geq k_1 \vee k_2} |F_{\bar{s}}^\infty - F_{\bar{s}}^k| \\ &\stackrel{(\text{A.3})}{\leq} \frac{\varepsilon}{8} + \frac{\varepsilon}{16} + \frac{\varepsilon}{16} = \frac{\varepsilon}{4}. \end{aligned} \quad (\text{A.5})$$

Therefore, by combining the above, we obtain for every $k \geq k_1 \vee k_2$

$$\sup_{t \in \mathbb{R}_+} |F_t^k - F_t^\infty| \leq \sup_{0 \leq t \leq \bar{s}} |F_t^k - F_t^\infty| + \sup_{\bar{s} \leq t \leq \infty} |F_t^k - F_t^\infty| \stackrel{(\text{A.4}), (\text{A.5})}{\stackrel{(\text{A.3}), (\text{A.2})}{<}} \frac{\varepsilon}{16} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} < \varepsilon. \quad \square$$

The next step is to provide a characterisation of the aforementioned weak convergence of measures using relatively compact sets of the Skorokhod space $\mathbb{D}(\mathbb{R}_+; \mathbb{R}^p)$. Two remarks are in order. The first remark is that a relatively compact set of the Skorokhod space is, in general, only locally uniformly bounded, see Jacod and Shiryaev [65, Theorem VI.1.14.b]. Therefore, we will restrict ourselves to those relatively compact subsets which are uniformly bounded. The second remark is that proving Theorem A.8 for integrands in $\mathbb{D}(\mathbb{R}_+; \mathbb{R})$, allows us to generalise the statement for integrands in $\mathbb{D}(\mathbb{R}_+; \mathbb{R}^p)$, where the norm in convergence (A.6) should be substituted by $\|\cdot\|_1$. Indeed, one can verify that the modulus w'^{32} takes care of the ‘uniform local behaviour’ of elements of a relatively compact set, independently of the dimension of the state space.

The following theorem can be regarded as an extension of Parthasarathy [88, Theorem II.6.8], in the special case that the limiting measure is an atomless measure on \mathbb{R}_+ . The proof of Theorem A.8 is inspired by the proof of the aforementioned theorem. However,

³²For $\alpha \in \mathbb{D}(\mathbb{R}_+; \mathbb{R}^p)$, $\theta > 0$, $N \in \mathbb{N}$ we use the modulus $w'_N(\alpha, \theta)$ as defined in [65, Section VI.1].

we need to underline that in [88, Theorem II.6.8] the equi-continuity of the integrands is used in order to determine for each arbitrary, but fixed, measure μ a countable cover of \mathbb{R}_+ consisting of μ -continuity sets on which the integrands are uniformly arbitrarily small. This is a property which is not possible to be obtained for relatively compact subsets of $\mathbb{D}(\mathbb{R}_+; \mathbb{R})$ and an arbitrary measure μ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Hence, we will need to properly use the ‘equi-right-continuity’ of the elements of a relatively compact subset of $\mathbb{D}(\mathbb{R}_+; \mathbb{R})$.

Theorem A.8. *Consider the measurable space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, let $(\mu^k)_{k \in \mathbb{N}}$ be a sequence of finite measures such that μ^∞ is an atomless finite measure and $\mathcal{A} \subset \mathbb{D}(\mathbb{R}_+; \mathbb{R})$. Assume that the following conditions are true*

- (i) $\mu^k \xrightarrow{w} \mu^\infty$;
- (ii) \mathcal{A} is uniformly bounded, i.e., $\sup_{\alpha \in \mathcal{A}} \|\alpha\|_\infty < \infty$;
- (iii) \mathcal{A} is uniformly ‘equi-right-continuous’, i.e., $\lim_{\zeta \downarrow 0} \sup_{\alpha \in \mathcal{A}} w'_N(\alpha, \zeta) = 0$ for every $N \in \mathbb{N}$.³³

Then

$$\lim_{k \rightarrow \infty} \sup_{\alpha \in \mathcal{A}} \left| \int_{[0, \infty)} \alpha(x) \mu^k(dx) - \int_{[0, \infty)} \alpha(x) \mu^\infty(dx) \right| = 0. \quad (\text{A.6})$$

Proof. We will decompose initially the quantity whose convergence we intent to prove as follows

$$\begin{aligned} & \sup_{\alpha \in \mathcal{A}} \left| \int_{[0, \infty)} \alpha(x) \mu^k(dx) - \int_{[0, \infty)} \alpha(x) \mu^\infty(dx) \right| \\ & \leq \sup_{\alpha \in \mathcal{A}} \left| \int_{[0, N]} \alpha(x) \mu^k(dx) - \int_{[0, N]} \alpha(x) \mu^\infty(dx) \right| \\ & \quad + \sup_{\alpha \in \mathcal{A}} \left| \int_{[N, \infty)} \alpha(x) \mu^k(dx) \right| + \sup_{\alpha \in \mathcal{A}} \left| \int_{[N, \infty)} \alpha(x) \mu^\infty(dx) \right|, \end{aligned} \quad (\text{A.7})$$

for some $N > 0$ to be determined. Then, the first summand of the right-hand side will be decomposed as follows

$$\begin{aligned} & \sup_{\alpha \in \mathcal{A}} \left| \int_{[0, N]} \alpha(x) \mu^k(dx) - \int_{[0, N]} \alpha(x) \mu^\infty(dx) \right| \\ & \leq \sup_{\alpha \in \mathcal{A}} \left| \int_{[0, N]} \alpha(x) \mu^k(dx) - \int_{[0, N]} \alpha(x) \tilde{\mu}^{k, \alpha}(dx) \right| \\ & \quad + \sup_{\alpha \in \mathcal{A}} \left| \int_{[0, N]} \alpha(x) \tilde{\mu}^{k, \alpha}(dx) - \int_{[0, N]} \alpha(x) \tilde{\mu}^{\infty, \alpha}(dx) \right| \\ & \quad + \sup_{\alpha \in \mathcal{A}} \left| \int_{[0, N]} \alpha(x) \tilde{\mu}^{\infty, \alpha}(dx) - \int_{[0, N]} \alpha(x) \mu^\infty(dx) \right|, \end{aligned} \quad (\text{A.9})$$

where the measures $\tilde{\mu}^{k, \alpha}$, for $k \in \mathbb{N}$ and $\alpha \in \mathcal{A}$, will be constructed given the measure μ^k and the function α . For the second and third summand of (A.7) we will prove that they become arbitrarily small for large $N > 0$. Then, we will conclude once we obtain the convergence to 0 (as $k \rightarrow \infty$) of the summands of (A.9).

To this end, let us fix an $\varepsilon > 0$.

³³The combination of parts (ii) and (iii) guarantee that \mathcal{A} is relatively compact.

■ Condition (i) implies that $\sup_{k \in \overline{\mathbb{N}}} \mu^k(\mathbb{R}_+) < \infty$, as well as the tightness of the sequence since $\mu^k(\mathbb{R}_+) \xrightarrow[k \rightarrow \infty]{} \mu^\infty(\mathbb{R}_+)$. Hence, for $M := \sup_{\alpha \in \mathcal{A}} \|\alpha\|_\infty < \infty$, which is Condition (ii), there exists $N = N(\varepsilon, M) > 0$ such that

$$\sup_{k \in \overline{\mathbb{N}}} \mu^k([N, \infty)) < \frac{\varepsilon}{2M}, \text{ which implies } \sup_{k \in \overline{\mathbb{N}}, \alpha \in \mathcal{A}} \left| \int_{[N, \infty)} \alpha(x) \mu^k(dx) \right| < \frac{\varepsilon}{2}.$$

In other words, we have proven that the second and third summand of the right-hand side of (A.7) become arbitrarily small for large N . In the following N is assumed fixed, but large enough so that the above hold.

■ We proceed now to construct, for every $k \in \overline{\mathbb{N}}$ and $\alpha \in \mathcal{A}$, a suitable measure $\tilde{\mu}^{k, \alpha}$ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that we can obtain the convergence of the summands in (A.9). Define $K_N := \sup_{k \in \overline{\mathbb{N}}} \mu^k([0, N]) < \infty$. By (iii) there exists $\zeta = \zeta(\varepsilon, K_N) > 0$ such that for every $\zeta' < \zeta$ it holds $\sup_{\alpha \in \mathcal{A}} w'_N(\alpha, \zeta') < \frac{\varepsilon}{4K_N}$. By the definition of w' (recall we use [65, Section VI.1]) for every $\alpha \in \mathcal{A}$ and every $\zeta' < \zeta$, there exists a ζ' -sparse set $P_\alpha^{N, \zeta'}$, i.e.

$$P_\alpha^{N, \zeta'} := \left\{ 0 = t(0; P_\alpha^{N, \zeta'}) < t(1; P_\alpha^{N, \zeta'}) < \dots < t(\kappa^\alpha; P_\alpha^{N, \zeta'}) = N : \right. \\ \left. \min_{i \in \{1, \dots, \kappa^\alpha - 1\}} (t(i; P_\alpha^{N, \zeta'}) - t(i-1; P_\alpha^{N, \zeta'})) > \zeta' \right\},$$

such that

$$\max_{i \in \{1, \dots, \kappa^\alpha\}} \sup \left\{ |\alpha(s) - \alpha(u)| : s, u \in [t(i-1; P_\alpha^{N, \zeta'}), t(i; P_\alpha^{N, \zeta'})] \right\} \\ < w'_N(\alpha, \zeta') + \frac{\varepsilon}{4K_N} < \sup_{\alpha \in \mathcal{A}} w'_N(\alpha, \zeta') + \frac{\varepsilon}{4K_N} < \frac{\varepsilon}{2K_N}. \quad (\text{A.10})$$

To sum up, for every $(\alpha, \zeta') \in \mathcal{A} \times (0, \zeta)$ we can find a partition of $[0, N]$, $P_\alpha^{N, \zeta'}$, satisfying (A.10). For the following, given a finite measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, called λ , and a sparse set $P_\alpha^{N, \zeta'}$ we define

- the interval $A(i; P_\alpha^{N, \zeta'}) := [t(i-1; P_\alpha^{N, \zeta'}), t(i; P_\alpha^{N, \zeta'})]$ for $i \in \{1, \dots, \kappa^\alpha\}$ and
- the operator $\lambda \mapsto \tilde{\lambda}^\alpha$ where the finite measure $\tilde{\lambda}^\alpha : (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \rightarrow [0, \infty)$ is defined by

$$\tilde{\lambda}^\alpha(\cdot) := \sum_{i=1}^{\kappa^\alpha} \lambda(A(i; P_\alpha^{N, \zeta'})) \delta_{t(i-1; P_\alpha^{N, \zeta'})}(\cdot).$$

Here, δ_x is the Dirac measure sitting at the point x . Using the notation introduced above, we have

$$\left| \int_{[0, N]} \alpha(x) \lambda(dx) - \int_{[0, N]} \alpha(x) \tilde{\lambda}^\alpha(dx) \right| \\ = \left| \sum_{i=1}^{\kappa^\alpha} \int_{A(i; P_\alpha^{N, \zeta'})} \alpha(x) \lambda(dx) - \sum_{i=1}^{\kappa^\alpha} \int_{A(i; P_\alpha^{N, \zeta'})} \alpha(x) \tilde{\lambda}^\alpha(dx) \right| \\ \leq \sum_{i=1}^{\kappa^\alpha} \left| \int_{A(i; P_\alpha^{N, \zeta'})} \alpha(x) \lambda(dx) - \int_{A(i; P_\alpha^{N, \zeta'})} \alpha(x) \tilde{\lambda}^\alpha(dx) \right| \\ = \sum_{i=1}^{\kappa^\alpha} \left| \int_{A(i; P_\alpha^{N, \zeta'})} \alpha(x) \lambda(dx) - \alpha(t(i-1; P_\alpha^{N, \zeta'})) \lambda([t(i-1; P_\alpha^{N, \zeta'}), t(i; P_\alpha^{N, \zeta'})]) \right|$$

$$\begin{aligned}
 &= \sum_{i=1}^{\kappa^\alpha} \left| \int_{A(i; P_\alpha^{N, \zeta'})} \alpha(x) - \alpha(t(i-1; P_\alpha^{N, \zeta'})) \lambda(dx) \right| \\
 &\leq \sum_{i=1}^{\kappa^\alpha} \int_{A(i; P_\alpha^{N, \zeta'})} |\alpha(x) - \alpha(t(i-1; P_\alpha^{N, \zeta'}))| \lambda(dx) \\
 &\stackrel{(A.10)}{\leq} \frac{\varepsilon}{2K_N} \sum_{i=1}^{\kappa^\alpha} \int_{A(i; P_\alpha^{N, \zeta'})} \lambda(dx) = \frac{\varepsilon}{2K_N} \lambda([0, N]).
 \end{aligned} \tag{A.11}$$

For every $k \in \overline{N}$ and $\alpha \in \mathcal{A}$ we define $\widetilde{\mu}^{k, \alpha} := \widetilde{(\mu^k)}^\alpha$. Now, using the approximation (A.11) for μ^k , for $k \in \overline{N}$, we obtain

$$\sup_{k \in \overline{N}, \alpha \in \mathcal{A}} \left| \int_{[0, N]} \alpha(x) \mu^k(dx) - \int_{[0, N]} \alpha(x) \widetilde{\mu}^{k, \alpha}(dx) \right| < \frac{\varepsilon}{2}. \tag{A.12}$$

To sum up, we have constructed the measures $\widetilde{\mu}^{k, \alpha}$ such that the first and third summand of (A.9) become arbitrarily small uniformly on $k \in \overline{N}$ and on $\alpha \in \mathcal{A}$.

■ We can conclude (A.6) once we obtain the validity of

$$\lim_{k \rightarrow \infty} \sup_{\alpha \in \mathcal{A}} \left| \int_{[0, N]} \alpha(x) \widetilde{\mu}^{k, \alpha}(dx) - \int_{[0, N]} \alpha(x) \widetilde{\mu}^{\infty, \alpha}(dx) \right| = 0.$$

Indeed, for $\zeta' \in (0, \zeta)$,

$$\begin{aligned}
 &\left| \int_{[0, N]} \alpha(x) \widetilde{\mu}^{k, \alpha}(dx) - \int_{[0, N]} \alpha(x) \widetilde{\mu}^{\infty, \alpha}(dx) \right| \\
 &= \left| \sum_{i=1}^{\kappa^\alpha} \int_{A(i; P_\alpha^{N, \zeta'})} \alpha(x) \widetilde{\mu}^{k, \alpha}(dx) - \sum_{i=1}^{\kappa^\alpha} \int_{A(i; P_\alpha^{N, \zeta'})} \alpha(x) \widetilde{\mu}^{\infty, \alpha}(dx) \right| \\
 &= \left| \sum_{i=1}^{\kappa^\alpha} \left(\int_{A(i; P_\alpha^{N, \zeta'})} \alpha(x) \mu^k(dx) - \int_{A(i; P_\alpha^{N, \zeta'})} \alpha(x) \mu^\infty(dx) \right) \right| \\
 &= \left| \sum_{i=1}^{\kappa^\alpha} \alpha(t(i; P_\alpha^{N, \zeta'})) \left[\mu^k(A(i; P_\alpha^{N, \zeta'})) - \mu^\infty(A(i; P_\alpha^{N, \zeta'})) \right] \right| \\
 &\stackrel{(ii)}{\leq} \|\alpha\|_\infty \sum_{i=1}^{\kappa^\alpha} \left| \mu^k(A(i; P_\alpha^{N, \zeta'})) - \mu^\infty(A(i; P_\alpha^{N, \zeta'})) \right| \\
 &\leq \sup_{\alpha \in \mathcal{A}} \|\alpha\|_\infty \cdot \kappa^\alpha \cdot \sup_{i \in \{1, \dots, \kappa^\alpha\}} \left| \mu^k(A(i; P_\alpha^{N, \zeta'})) - \mu^\infty(A(i; P_\alpha^{N, \zeta'})) \right| \\
 &\leq \sup_{\alpha \in \mathcal{A}} \|\alpha\|_\infty \cdot \kappa^\alpha \cdot \sup_{I \in \mathcal{I}_N} |\mu^k(I) - \mu^\infty(I)|.
 \end{aligned} \tag{A.13}$$

Let us now define $m(P_\alpha^{N, \zeta'}) := \min\{t(i; P_\alpha^{N, \zeta'}) - t(i-1; P_\alpha^{N, \zeta'}), i \in \{1, \dots, \kappa^\alpha\}\} > \zeta'$ by the definition of $P_\alpha^{N, \zeta'}$. Therefore,

$$\sup_{\alpha \in \mathcal{A}} \kappa^\alpha \leq \sup_{\alpha \in \mathcal{A}} \left\lceil \frac{N}{m(P_\alpha^{N, \zeta'})} \right\rceil^{34} \leq \left\lceil \frac{N}{\zeta'} \right\rceil < \infty,$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x . Hence, for every fixed

sparse set $P_{\alpha}^{N,\zeta'}$ which satisfies (A.10) we obtain

$$\begin{aligned} \sup_{\alpha \in \mathcal{A}} \left| \int_{[0,N]} \alpha(x) \tilde{\mu}^{k,\alpha}(\mathrm{d}x) - \int_{[0,N]} \alpha(x) \tilde{\mu}^{\infty,\alpha}(\mathrm{d}x) \right| \\ \leq \sup_{\alpha \in \mathcal{A}} \|\alpha\|_{\infty} \times \sup_{\alpha \in \mathcal{A}} \kappa^{\alpha} \times \sup_{I \in \mathcal{I}_N} |\mu^k(I) - \mu(I)| \xrightarrow[k \rightarrow \infty]{(i)} 0, \end{aligned}$$

where we have used Theorem A.6.(e) which is equivalent to (i). \square

We have presented the previous theorem for $\mathcal{A} \subset \mathbb{D}(\mathbb{R}_+; \mathbb{R})$. However, as we mentioned before, the result can be readily adapted for $\mathcal{A} \subset \mathbb{D}(\mathbb{R}_+; \mathbb{R}^p)$. This will be presented in the following proposition.

Proposition A.9. *Consider the measurable space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and let $(\mu^k)_{k \in \overline{\mathbb{N}}}$ be a sequence of finite measures such that $\mu^k(\{0\}) = 0$ for every $k \in \overline{\mathbb{N}}$ and μ^{∞} is atomless. Additionally, let $\mathcal{A} := (\alpha^k)_{k \in \overline{\mathbb{N}}} \subset \mathbb{D}(\mathbb{R}_+; \mathbb{R}^p)$. Assume the following to be true*

- (i) $\mu^k \xrightarrow{w} \mu^{\infty}$;
- (ii) \mathcal{A} is uniformly bounded, i.e. $\sup_{\alpha \in \mathcal{A}} \|\alpha\|_{\infty} < \infty$;
- (iii) \mathcal{A} is $\delta_{J_1(\mathbb{R}^p)}$ -convergent. In other words, $\alpha^k \xrightarrow[k \rightarrow \infty]{J_1(\mathbb{R}^p)} \alpha^{\infty}$.

Then

$$\lim_{k,m \rightarrow \infty} \left\| \int_{[0,\infty)} \alpha^k(x) \mu^m(\mathrm{d}x) - \int_{[0,\infty)} \alpha^{\infty}(x) \mu^{\infty}(\mathrm{d}x) \right\|_1 = 0.$$

Proof. Let us define, for $k \in \overline{\mathbb{N}}$ and $m \in \mathbb{N}$,

$$\gamma^{k,m} := \left\| \int_{[0,\infty)} \alpha^k(x) \mu^m(\mathrm{d}x) - \int_{[0,\infty)} \alpha^k(x) \mu^{\infty}(\mathrm{d}x) \right\|_1.$$

We are going to apply Theorem A.3 in order to obtain the required result. By Theorem A.8 we have that

$$\lim_{m \rightarrow \infty} \sup_{k \in \overline{\mathbb{N}}} |\gamma^{k,m}| = 0.$$

In other words, Condition (i) of Theorem A.3 is satisfied. Let us prove, now, that Condition (ii) of the aforementioned theorem is also satisfied, i.e. we need to prove that

$$\lim_{m \rightarrow \infty} \left(\limsup_{k \rightarrow \infty} \gamma^{k,m} - \liminf_{k \rightarrow \infty} \gamma^{k,m} \right) = 0.$$

However, it is sufficient to prove that $\lim_{m \rightarrow \infty} \limsup_{k \rightarrow \infty} \gamma^{k,m} = 0$, since the elements of the doubly-indexed sequence are positive. To this end, we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} \gamma^{k,m} \\ \leq \limsup_{k \rightarrow \infty} \left\| \int_{[0,\infty)} \alpha^k(x) \mu^m(\mathrm{d}x) - \int_{[0,\infty)} \alpha^{\infty}(x) \mu^m(\mathrm{d}x) \right\|_1 \\ + \limsup_{k \rightarrow \infty} \left\| \int_{[0,\infty)} \alpha^{\infty}(x) \mu^m(\mathrm{d}x) - \int_{[0,\infty)} \alpha^{\infty}(x) \mu^{\infty}(\mathrm{d}x) \right\|_1 \\ + \limsup_{k \rightarrow \infty} \left\| \int_{[0,\infty)} \alpha^{\infty}(x) \mu^{\infty}(\mathrm{d}x) - \int_{[0,\infty)} \alpha^k(x) \mu^{\infty}(\mathrm{d}x) \right\|_1. \end{aligned}$$

³⁴We denote by $\lceil x \rceil$ the least integer greater than or equal to x .

Initially, we are going to prove now that the two last summands of the right-hand side of the above inequality are equal to zero. We start with the second summand and we realise that we need only to use that $\mu^k \xrightarrow{w} \mu^\infty$ and Theorem A.8 for the special case of a singleton. The third summand is also equal to zero as an outcome of the bounded convergence theorem. Indeed, we have that the sequence \mathcal{A} is uniformly bounded and by [65, Proposition VI.2.1] we have the pointwise convergence $\alpha^k(x) \rightarrow \alpha^\infty(x)$ for $k \rightarrow \infty$, at every point x which is a point of continuity of α^∞ . Since the set $\{x \in \mathbb{R}_+ : \Delta\alpha^\infty(x) \neq 0\}$ is at most countable, we can conclude it is an μ^∞ -null set. Therefore, since every element of \mathcal{A} is Borel-measurable, we can conclude.

Finally, we deal with the first summand. Let us provide initially some auxiliary results. Using again [65, Proposition VI.2.1] we have that

$$\limsup_{k \rightarrow \infty} \|\alpha^k(x) - \alpha^\infty(x)\|_1 \leq \|\Delta\alpha^\infty(x)\|_1 \text{ for every } x \in \mathbb{R}_+, \quad (\text{A.14})$$

because the only two possible accumulation points of the sequence $(\alpha^{k,i}(x))_{k \in \mathbb{N}}$ are $\alpha^{\infty,i}(x)$ and $\alpha^{\infty,i}(x-)$, for every $i \in \{1, \dots, p\}$ and every $x \in \mathbb{R}_+$. Now observe that the function $\mathbb{R}_+ \ni x \mapsto \|\Delta\alpha^\infty(x)\|_1 \in \mathbb{R}_+$ is Borel measurable and μ^∞ -almost everywhere equal to the zero function. This observation allows us to apply Mazzone [80, Theorem 1] in order to obtain the convergence

$$\int_{[0,\infty)} \|\Delta\alpha^\infty(x)\|_1 \mu^m(dx) \xrightarrow[m \rightarrow \infty]{|\cdot|} \int_{[0,\infty)} \|\Delta\alpha^\infty(x)\|_1 \mu_\infty(dx) = 0. \quad (\text{A.15})$$

In view of the above and the boundedness of the sequence \mathcal{A} , we can apply (the reverse) Fatou's lemma and we obtain for every $m \in \mathbb{N}$

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \left\| \int_{[0,\infty)} \alpha^k(x) \mu^m(dx) - \int_{[0,\infty)} \alpha^\infty(x) \mu^m(dx) \right\|_1 \\ & \leq \int_{[0,\infty)} \limsup_{k \rightarrow \infty} \|\alpha^k(x) - \alpha^\infty(x)\|_1 \mu^m(dx) \stackrel{(\text{A.14})}{\leq} \int_{[0,\infty)} \|\Delta\alpha^\infty(x)\|_1 \mu^m(dx). \end{aligned}$$

Now, we can conclude that the Condition (ii) of Theorem A.3 is indeed satisfied by combining the above bound with Convergence (A.15). \square

Remark A.10. Let us adopt the notation of Theorem A.8 and Theorem A.9 and assume that the measurable space is $([0, T], \mathcal{B}([0, T]))$, for some $T \in \mathbb{R}_+$. We claim that we can adapt the aforementioned results without loss of generality, which can be justified as follows. The limit measure μ^∞ is atomless, so the generality is not harmed if in Theorem A.8 the integrands α^k , for some $k \in \overline{\mathbb{N}}$, have a jump at point T . Indeed, by the weak convergence $\mu^k \xrightarrow{w} \mu^\infty$ and Theorem A.6 we have that the sequence of distribution functions $(F^k)_{k \in \overline{\mathbb{N}}}$ which is associated to the sequence $(\mu^k)_{k \in \overline{\mathbb{N}}}$ converges uniformly. Therefore, we can easily conclude that

$$\limsup_{k \rightarrow \infty} \|\Delta\alpha^k(T)\|_2 \mu^k(\{T\}) = \limsup_{k \rightarrow \infty} \|\Delta\alpha^k(T)\|_2 \Delta F_T^k \leq \|\Delta\alpha^\infty(T)\|_2 \Delta F_T^\infty = 0.$$

In other words, we can simply assume that the distribution functions are constant after time T in order to reduce the general case to the compact-interval case.

The following corollary is almost evident due to the fact that the limit measure is atomless. However we state it in the form we will need it in Theorem 3.19 and we provide its complete (rather trivial) proof.

Corollary A.11. Consider the measurable space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and let $(\mu^k)_{k \in \overline{\mathbb{N}}}$ be a sequence of finite measures, where $\mu^k(\{0\}) = 0$ for every $k \in \overline{\mathbb{N}}$ and μ^∞ is atomless. Let, moreover, $\mathcal{A} := (\alpha^k)_{k \in \overline{\mathbb{N}}} \subset \mathcal{D}(\mathbb{R}_+; \mathbb{R}^p)$ and the following to be true

- (i) $\mu^k \xrightarrow{w} \mu^\infty$;
- (ii) \mathcal{A} is uniformly bounded, i.e. $\sup_{\alpha \in \mathcal{A}} \|\alpha\|_\infty < \infty$;
- (iii) \mathcal{A} is $\delta_{J_1(\mathbb{R}^p)}$ -convergent.

Then

$$\int_{[0, \cdot]} \alpha^k(x) \mu^k(dx) \xrightarrow[k \rightarrow \infty]{J_1(\mathbb{R}^p)} \int_{[0, \cdot]} \alpha^\infty(x) \mu^\infty(dx). \quad (\text{A.16})$$

Proof. Let F^k denote the distribution function associated to the measure μ^k for every $k \in \overline{\mathbb{N}}$ and

$$\gamma^k(t) := \int_{[0, t]} \alpha^k(x) \mu^k(dx) = \int_{[0, t]} \alpha^k(s) dF_s^k \in \mathbb{D}(\mathbb{R}_+; \mathbb{R}^p) \text{ for } k \in \overline{\mathbb{N}}.$$

We are going to apply Ethier and Kurtz [38, Proposition 3.6.5] in order to prove the required convergence. To this end, let us fix a $t^\infty \in \mathbb{R}_+$ and a sequence $(t^k)_{k \in \mathbb{N}}$ such that $t^k \rightarrow t^\infty$ as $k \rightarrow \infty$. We need to prove that the requirements (a)–(c) of the aforementioned proposition are satisfied. However, due to the continuity of the limit, these three requirements reduce to a single one: for any $t^\infty \in \mathbb{R}_+$ and for any sequence $(t^k)_{k \in \mathbb{N}}$ such that $t^k \xrightarrow[k \rightarrow \infty]{|\cdot|} t^\infty$ holds

$$\lim_{k \rightarrow \infty} \|\gamma^k(t^k) - \gamma^\infty(t^\infty)\|_1 = 0. \quad (\text{A.17})$$

Let us fix a $t^\infty \in \mathbb{R}_+$ and consider a sequence $(t^k)_{k \in \mathbb{N}}$ such that $t^k \xrightarrow[k \rightarrow \infty]{|\cdot|} t^\infty$. Then, by [65, Theorem VI.2.15.c)], we can easily conclude that $F_{t^k \wedge \cdot}^k \xrightarrow[k \rightarrow \infty]{J_1(\mathbb{R}_+; \mathbb{R})} F_{t^\infty \wedge \cdot}^\infty$. We can now immediately verify by means of Theorem A.9 that (A.17) is true. \square

Remark A.12. Let us borrow the notation of Theorem A.9, with F^k the distribution function of the measure μ^k , for $k \in \overline{\mathbb{N}}$, where F^∞ is not necessarily continuous. It is well-known in the literature of the Skorokhod J_1 -convergence that a sufficient condition for the convergence of the integral functions is the joint convergence of the integrands and the integrators, i.e., if $(\alpha^k, F^k) \xrightarrow[k \rightarrow \infty]{J_1} (\alpha^\infty, F^\infty)$, then $\alpha^k \cdot F^k \xrightarrow[k \rightarrow \infty]{J_1} \alpha^\infty \cdot F^\infty$, where \cdot denotes integration here. When F^∞ is continuous, the joint convergence is trivially obtained because of the continuity of the limit integrator F^∞ . In other words, we could have proved Theorem A.11 without using Theorem A.9. However, the purpose of presenting this approach is twofold: firstly to present the characterisation of weak convergence of measures as stated in Theorem A.8 and secondly to prove that under the assumption $(\alpha^k, F^k) \xrightarrow[k \rightarrow \infty]{J_1} (\alpha^\infty, F^\infty)$, it is true that $\alpha^k \cdot F^l \xrightarrow[(k, l) \rightarrow (\infty, \infty)]{J_1} \alpha^\infty \cdot F^\infty$, which is stronger than $\alpha^k \cdot F^k \xrightarrow[k \rightarrow \infty]{J_1} \alpha^\infty \cdot F^\infty$. In particular

$$\lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \alpha^k \cdot F^l = \lim_{l \rightarrow \infty} \left\{ \limsup_{k \rightarrow \infty} \alpha^k \cdot F^l - \liminf_{k \rightarrow \infty} \alpha^k \cdot F^l \right\} = \lim_{k \rightarrow \infty} \alpha^k \cdot F^k = \alpha^\infty \cdot F^\infty.$$

The next natural question is whether a characterisation similar to Theorem A.8 holds if we consider a weakly-convergent sequence of measures to a limit-measure with atoms. In view of Mazzone [80, Theorem 1], one should not expect a characterisation based on arbitrary relatively compact sets of the Skorokhod space.

The following lemma will be helpful in proving that a relatively compact set of $\mathbb{D}(\mathbb{R}_+; \mathbb{R}^p)$ is uniformly bounded. Observe that every relatively compact subset of $\mathbb{D}(\mathbb{R}_+; \mathbb{R}^p)$ satisfies (ii) of the following lemma; see [65, Theorem VI.1.14.b).i)].

Lemma A.13. Let $(\alpha^k)_{k \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}^p)$ such that

- (i) $\alpha^k(\infty) := \lim_{t \rightarrow \infty} \alpha^k(t)$ exists and $\|\alpha^k(\infty)\|_2 < \infty$ for every $k \in \mathbb{N}$;
- (ii) $\sup_{k \in \mathbb{N}} \sup_{t \in [0, N]} \|\alpha^k(t)\|_2 < \infty$ for every $N \in \mathbb{N}$;
- (iii) $\limsup_{(k, t) \rightarrow (\infty, \infty)} \|\alpha^k(t)\|_2 < \infty$.

Then the sequence $(\alpha^k)_{k \in \mathbb{N}}$ is uniformly bounded, i.e. $\sup_{k \in \mathbb{N}} \|\alpha\|_\infty < \infty$.

Proof. Assume that the sequence is unbounded, i.e. for every $M \in \mathbb{N}$ there exists $k(M) \in \mathbb{N}$ such that $\|\alpha^{k(M)}\|_\infty > M$. We can extract a subsequence of $(k(M))_{M \in \mathbb{N}}$, called $(k(M_n))_{n \in \mathbb{N}}$, such that $k(M_l) < k(M_m)$ whenever $l < m$ and $M_n \rightarrow \infty$ as $n \rightarrow \infty$. This can be done as follows. For $n = 1$ we define $k(M_1) := k(1)$ and for $i \in \mathbb{N} \setminus \{1\}$ we set

$$k(M_{i+1}) := \min \left\{ k(M) \in \mathbb{N} : \|\alpha^{k(M)}\|_\infty > \left\lceil \|\alpha^{k(M_i)}\|_\infty \right\rceil \right\},$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x . Since we have assumed that for every $k \in \mathbb{N}$ the value $\|\alpha^k(\infty)\|_2$ is finite, we have in particular (using that $\alpha^k \in \mathcal{D}(\mathbb{R}_+; \mathbb{R}^p)$) that $\|\alpha^k\|_\infty < \infty$. Therefore

$$\left\{ k(M) \in \mathbb{N} : \|\alpha^{k(M)}\|_\infty > \left\lceil \|\alpha^{k(M_i)}\|_\infty \right\rceil \right\} \neq \emptyset,$$

and its minimum exists as \mathbb{N} is a well-ordered set under the usual order. In view of these comments, the subsequence $(k(M_n))_{n \in \mathbb{N}}$ is well-defined and, according to our initial assumption, it has to be unbounded.

By definition of $\alpha^{k(M)}$, there exists $t_{k(M)} \in \mathbb{R}_+$ such that $\|\alpha^{k(M)}(t_{k(M)})\|_\infty > M$, a property which holds in particular for the subsequence indexed by $(k(M_n))_{n \in \mathbb{N}}$. Let us distinguish, now, two cases for the sequence $(t_{k(M_n)})_{n \in \mathbb{N}}$.

- If $(t_{k(M_n)})_{n \in \mathbb{N}}$ is bounded, then (ii) leads to a contradiction, for $N := \sup\{t_{k(M_n)}, n \in \mathbb{N}\}$.
- If $(t_{k(M_n)})_{n \in \mathbb{N}}$ is unbounded, then there exists a subsequence $(t_{k(M_{n_l})})_{l \in \mathbb{N}}$ such that $t_{k(M_{n_l})} \rightarrow \infty$ as $l \rightarrow \infty$. Therefore, we have also

$$\lim_{l \rightarrow \infty} \|\alpha^{k(M_{n_l})}(t_{k(M_{n_l})})\|_\infty \geq \lim_{l \rightarrow \infty} M_{n_l} = \infty.$$

But, then (iii) leads to a contradiction.

Now, we have only to verify that the set $\{\|\alpha^k\|_\infty : k \in \{1, \dots, k(1)\}\}$ is bounded in order to conclude that the sequence $(\alpha^k)_{k \in \mathbb{N}}$ is $\|\cdot\|_\infty$ -bounded. But the above is clear since it is a finite set of finite numbers; use that $\|\alpha^k(\infty)\|_2 < \infty$ and that $\alpha^k \in \mathcal{D}(\mathbb{R}_+; \mathbb{R}^p)$. \square

A.4 Some helpful lemmata for the proof of Theorem 3.1

Lemma A.14. There exists a countable set $D^{\circ, \ell \times m} \subset C_c(\mathbb{R}_+; \mathbb{R}^{\ell \times m})$ such that for every $Z \in \mathcal{H}^{\infty, \circ}$ and for every $\varepsilon > 0$, there exists $\tilde{Z}^\varepsilon \in D^{\circ, \ell \times m}$ with the property

$$\int_{(0, \infty)} \text{Tr} \left[(Z_s - \tilde{Z}_s^\varepsilon) \frac{d\langle X \rangle_s}{dC_s} (Z_s - \tilde{Z}_s^\varepsilon)^\top \right] dC_s < \varepsilon, \text{ P-a.s.}$$

Proof. Since $Z \in \mathcal{H}^{\infty, \circ}$, then

$$\int_{(0, \infty)} \text{Tr} \left[Z_s \frac{d\langle X \rangle_s}{dC_s} Z_s^\top \right] dC_s < \infty, \text{ P-a.s.}$$

Let $\tilde{\Omega} \subset \Omega$ be the set of ω for which the aforementioned property holds and $\langle X \rangle$ is finite. Then, for $\bar{Z}^n := Z \mathbf{1}_{\{\|Z\|_2 \leq n\}}$ holds

$$\int_{(0,\infty)} \text{Tr} \left[(Z_s - \bar{Z}_s^n) \frac{d\langle X \rangle_s}{dC_s} (Z_s - \bar{Z}_s^n)^\top \right] dC_s \xrightarrow{n \rightarrow \infty} 0, \quad \forall \omega \in \tilde{\Omega}.$$

Therefore, for every $\varepsilon > 0$, there exists n_0 (depending on ω) such that

$$\int_{(0,\infty)} \text{Tr} \left[(Z_s - \bar{Z}_s^n) \frac{d\langle X \rangle_s}{dC_s} (Z_s - \bar{Z}_s^n)^\top \right] dC_s < \frac{\varepsilon}{2}, \quad \forall n \geq n_0, \forall \omega \in \tilde{\Omega}. \quad (\text{A.18})$$

The fact that \bar{Z}^n is bounded, allows us to write

$$\int_{(0,\infty)} \text{Tr} \left[\bar{Z}_s^n \frac{d\langle X \rangle_s}{dC_s} \bar{Z}_s^{n\top} \right] dC_s = \sum_{k=1}^{\ell} \sum_{i,j=1}^m \int_{(0,\infty)} \bar{Z}_s^{n,ki} \bar{Z}_s^{n,kj} d\langle X^{\infty,o,i}, X^{\infty,o,j} \rangle_s, \quad \forall \omega \in \tilde{\Omega},$$

where we have denoted by $\bar{Z}_s^{n,ki}$ the ki -element of the matrix \bar{Z}_s^n , for $k \in \{1, \dots, \ell\}$ and $i \in \{1, \dots, m\}$. Kunita-Watanabe's inequality (in conjunction with Young's Inequality) writes for all $k \in \{1, \dots, \ell\}$, $(i, j) \in \{1, \dots, m\}^2$, and $\omega \in \tilde{\Omega}$

$$\begin{aligned} \int_{(0,\infty)} \bar{Z}_s^{n,ki} \bar{Z}_s^{n,kj} d\text{Var}[\langle X^{\infty,o,i}, X^{\infty,o,j} \rangle]_s \\ \leq \frac{1}{2} \int_{(0,\infty)} |\bar{Z}_s^{n,ki}|^2 d\langle X^{\infty,o,i} \rangle_s + \frac{1}{2} \int_{(0,\infty)} |\bar{Z}_s^{n,kj}|^2 d\langle X^{\infty,o,j} \rangle_s. \end{aligned} \quad (\text{A.19})$$

Now we use the fact that $\langle X^{\infty,o,i} \rangle$ is a finite, Borel measure on \mathbb{R}_+ . Therefore, the set

$$\left\{ f : \mathbb{R}_+ \rightarrow \mathbb{R} : f = \sum_{i=1}^p \alpha_i \mathbf{1}_{I_i}, \right.$$

$$\left. \text{with } \alpha_i \in \mathbb{Q}, I_i \text{ bounded interval with rational endpoints } \forall i \in \{1, \dots, p\} \right\},$$

is countable and dense in $\mathbb{L}^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \langle X^{\infty,o,i} \rangle)$, for every $i \in \{1, \dots, m\}$. The regularity of the measure $\langle X^{\infty,o,i} \rangle$ (as finite, Borel defined on a locally compact space for every $\omega \in \tilde{\Omega}$) and Tietze's extension theorem allow³⁵ us to choose a countable subset of $C_c(\mathbb{R}_+; \mathbb{R}^{\ell \times m})$, denoted by D° , which is dense in $\mathbb{L}^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \langle X^{\infty,o,i} \rangle)$ for every $\omega \in \tilde{\Omega}$. We can easily pass to the required set $D^{\circ, \ell \times m}$ by observing now that for every $k \in \{1, \dots, \ell\}$ and for every $i \in \{1, \dots, m\}$, for every $\omega \in \tilde{\Omega}$, we can choose $\tilde{Z}^{\varepsilon, ki}$ ³⁶ such that

$$\int_{(0,\infty)} |\bar{Z}_s^{n_0,ki} - \tilde{Z}_s^{\varepsilon,ki}|^2 d\langle X^{\infty,o,i} \rangle_s < \frac{\varepsilon}{2\ell m^2}, \quad \forall \omega \in \tilde{\Omega},$$

which in conjunction with (A.18) and (A.19) results in

$$\int_{(0,\infty)} \text{Tr} \left[(Z_s - \tilde{Z}_s^\varepsilon) \frac{d\langle X \rangle_s}{dC_s} (Z_s - \tilde{Z}_s^\varepsilon)^\top \right] dC_s < \varepsilon, \quad \forall \omega \in \tilde{\Omega}. \quad \square$$

Lemma A.15. *Let ν be a measure on $(E, \mathcal{B}(E))$ such that $\int_{\mathbb{R}_+ \times \mathbb{R}^n} \|x\|_2^2 \nu(dt, dx) < \infty$ and $\nu(E_0) = 0$. Then, there exists a countable set $D^\natural \subset C_{c|\tilde{E}}(E; \mathbb{R}^\ell)$ such that D^\natural is dense in $\mathbb{L}^2(E, \mathcal{B}(E), \nu)$.*

³⁵The local compactness allows for the following property: for every K compact and U open, there exists open V such that its closure \bar{V} is compact and $K \subset V \subset \bar{V} \subset U$. Then, Tietze's extension theorem guarantees the existence of f in $C_c(\mathbb{R}_+; \mathbb{R})$ such that $\mathbf{1}_K \leq f \leq \mathbf{1}_U$ with $\text{supp}(f)$ compact.

³⁶We suppress the dependence on ω .

Proof. Since $\nu(E_0) = 0$, it is sufficient to prove that there exists a countable subset of $C_c(\tilde{E}; \mathbb{R}^\ell)$ which is dense in $\mathbb{L}^2(\tilde{E}, \mathcal{B}(\tilde{E}), \nu|_{\mathcal{B}(\tilde{E})})$. The integrability assumption implies that $\nu|_{\mathcal{B}(\tilde{E})}$ is finite on every compact subset of \tilde{E} . In view of the above property, $\nu|_{\mathcal{B}(\tilde{E})}$ is a Radon measure³⁷ as a locally finite Borel measure. The reader may observe that it is crucial that \tilde{E} is open, so that we exclude compact subsets of E which intersect E_0 .

Now that we have verified that $\nu|_{\mathcal{B}(\tilde{E})}$ is Radon, we can use the well-known fact that the family of the step functions with rational \mathbb{R}^ℓ -values and supported on rectangles whose vertices' endpoints are rational is \mathbb{L}^2 -dense in the collection of simple functions, which in turn is dense in $\mathbb{L}^2(\tilde{E}, \mathcal{B}(\tilde{E}), \nu|_{\mathcal{B}(\tilde{E})})$. We will denote by \mathcal{Z} the aforementioned family of step functions. Hence, \mathcal{Z} is a countable \mathbb{L}^2 -dense set. Let us define

$\mathcal{Z}_c := \{f \in \mathcal{Z} : \text{supp}(f) \text{ is compact and the rectangles in the representation are closed}\}$, as well as

$\mathcal{Z}_o := \{f \in \mathcal{Z} : \text{the rectangles in the representation are open}\}$.

For every $f_c \in \mathcal{Z}_c$ and for every $f_o \in \mathcal{Z}_o$ such that³⁸ $f_c|_{\text{supp}(f_c)} = f_o|_{\text{supp}(f_c)}$, there exists by Tietze's extension theorem $f \in C_c(\tilde{E}; \mathbb{R}^\ell)$ such that $f_c \leq f \leq f_o$. The collection of all $f \in C_c(\tilde{E}; \mathbb{R}^\ell)$ with the above property is countable and \mathbb{L}^2 -dense; we will denote it by D^\natural . Without destroying the properties of D^\natural , we can assume that the zero function, denoted by 0, lies in D^\natural . Now, we can isometrically identify $C_c(\tilde{E}; \mathbb{R}^\ell)$ and $C_{c|\tilde{E}}(E; \mathbb{R}^\ell)$ and conclude by using the fact that $\nu(E_0) = 0$. \square

Lemma A.16. *Under Conditions (S6) and (S9).(iii), there exists $k_{*,0} \in \mathbb{N}$, such that*

$$\sup_{k \geq k_{*,0}} M_\star^{\Phi^k}(\hat{\beta}) < \frac{1}{4},$$

with $\mathbb{R}_+ \times \mathbb{R}_+ \ni (\beta, \Phi) \mapsto M_\star^\Phi(\beta)$, defined as in [86, Lemma 3.4].

Proof. For the convenience of the reader, we restate the notation of the aforementioned lemma. For $\beta, \Phi > 0$ and $\mathcal{C}_\beta := \{(\gamma, \delta) \in (0, \beta]^2, \gamma < \delta\}$ we define

$$\Pi_\star^\Phi(\gamma, \delta) := \frac{8}{\gamma} + \frac{9}{\delta} + 9\delta \frac{e^{(\delta-\gamma)\Phi}}{\gamma(\delta-\gamma)}, \text{ with } M_\star^\Phi(\beta) := \inf_{(\gamma, \delta) \in \mathcal{C}_\beta} \Pi_\star^\Phi(\gamma, \delta).$$

The infimum is attained at $(\gamma_\star^\Phi(\beta), \beta)$; the exact values are given in [86, Lemma 3.4]. Hence, using Assumption (S9).(iii) and Remark 2.3.(i), we can assume that there exists $\Phi^{k_{*,0}}$ such that

$$M_\star^{\Phi^{k_{*,0}}}(\hat{\beta}) = \Pi_\star^{\Phi^{k_{*,0}}}(\gamma_\star^{\Phi^{k_{*,0}}}, \hat{\beta}) < \frac{1}{4}.$$

We observe now that, when we fix $\gamma, \delta \in (0, \beta]^2$, then the function

$$\mathbb{R}_+ \ni \Phi \mapsto \Pi_\star^\Phi(\gamma, \delta) \in \mathbb{R}_+, \quad (\text{A.20})$$

is decreasing and continuous under the usual topology of \mathbb{R} . Consequently, by [86, Lemma 3.4] and the fact that the function in (A.20) is decreasing, we obtain

$$\begin{aligned} M_\star^{\Phi^k}(\hat{\beta}) &= \Pi_\star^{\Phi^k}(\gamma_\star^{\Phi^k}, \hat{\beta}) \leq \Pi_\star^{\Phi^k}(\gamma_\star^{\Phi^{k_{*,0}}}, \hat{\beta}) \\ &\leq \Pi_\star^{\Phi^{k_{*,0}}}(\gamma_\star^{\Phi^{k_{*,0}}}, \hat{\beta}) = M_\star^{\Phi^{k_{*,0}}}(\hat{\beta}) < \frac{1}{4}, \text{ for } k \geq k_{*,0}. \end{aligned} \quad (\text{A.21})$$

\square

³⁷In other words, it is finite on compact subsets of \tilde{E} , inner-regular on all open sets and outer-regular on every Borel set.

³⁸We denote by $f|_A$ the restriction of the function f on the set A .

Lemma A.17. *Let $(\zeta^k)_{k \in \mathbb{N}}$ be a sequence of \mathbb{R}^p -valued random variables such that $(\|\zeta^k\|_2^2)_{k \in \mathbb{N}}$ is uniformly integrable. Then the sequence $(\sup_{t \in \mathbb{R}_+} \|\mathbb{E}[\zeta^k | \mathcal{G}_t^k]\|_2^2)_{k \in \mathbb{N}}$ is uniformly integrable.*

Proof. Let Φ be a moderate Young function³⁹ for which the sequence $(\|\zeta^k\|_2^2)_{k \in \mathbb{N}}$ satisfies the de La Vallée Poussin–Meyer criterion (see Meyer [82, Lemme]), i.e.

$$\sup_{k \in \mathbb{N}} \mathbb{E} \left[\Phi(\|\zeta^k\|_2^2) \right] < \infty. \quad (\text{A.22})$$

Then, by [87, Proposition A.9], $\Psi := \Phi \circ \text{quad}$ is a moderate Young function. Using the fact that Φ is increasing, we can write (A.22) as

$$M := \sup_{k \in \mathbb{N}} \mathbb{E} \left[\Psi(\|\zeta^k\|_2) \right] < \infty. \quad (\text{A.23})$$

The latter form will be more convenient for later use. Before we proceed to prove the claim of the corollary, we provide some helpful results. In order to ease notation, let us denote the Orlicz norm of Ψ by $\|\cdot\|_\Psi$, which we simplify for the case $\|\|\zeta^k\|_2\|_\Psi$ by $\|\zeta^k\|_{2,\Psi}$ for $k \in \mathbb{N}$; see Rao and Ren [89, Subsection III.3.1 Theorem 3, p. 54]. Observe that $\|\zeta^k\|_{2,\Psi} < \infty$, for every $k \in \mathbb{N}$ because of (A.23). We head to prove that $\sup_{k \in \mathbb{N}} \|\zeta^k\|_{2,\Psi} < \infty$. To this end, observe that

- if $M \leq 1$, then $\sup_{k \in \mathbb{N}} \zeta^k \leq 1$ by the definition of the Orlicz norm;
- if $M > 1$, then using the convexity of Ψ and the fact that $\Psi(0) = 0$ we obtain

$$\mathbb{E} \left[\Psi \left(\frac{\|\zeta^k\|_2}{M} \right) \right] \leq \frac{\mathbb{E}[\Psi(\|\zeta^k\|_2)]}{M} \stackrel{(\text{A.23})}{\leq} 1, \text{ which implies } \|\zeta^k\|_{2,\Psi} \leq M.$$

and consequently

$$\sup_{k \in \mathbb{N}} \|\zeta^k\|_{2,\Psi} \leq 1 \vee M \stackrel{(\text{A.23})}{<} \infty. \quad (\text{A.24})$$

We proceed now to prove the uniform integrability of $(\sup_{t \in \mathbb{R}_+} \|\mathbb{E}[\zeta^k | \mathcal{G}_t^k]\|_2^2)_{k \in \mathbb{N}}$. It suffices to prove that

$$\sup_{k \in \mathbb{N}} \mathbb{E} \left[\Phi \left(\sup_{t \in \mathbb{R}_+} \|\mathbb{E}[\zeta^k | \mathcal{G}_t^k]\|_2^2 \right) \right] < \infty,$$

or equivalently that $\sup_{k \in \mathbb{N}} \mathbb{E}[\Psi(\sqrt{2}S^k)] < \infty$, where $S^k := \sup_{t \in \mathbb{R}_+} \|\mathbb{E}[\zeta^k | \mathcal{G}_t^k]\|_2$. Using the properties of moderate Young functions, e.g. see Long [74, Theorem 3.1.1, p. 82], we can obtain

$$\mathbb{E}[\Psi(\sqrt{2}S^k)] \leq 1 \vee \bar{c}_\Psi^{\|\sqrt{2}S^k\|_\Psi}, \quad (\text{A.25})$$

where \bar{c}_Ψ depends only on Ψ ⁴⁰. Recall that $\Psi = \Phi \circ \text{quad}$ is a moderate Young function. Hence, for Ψ^* the conjugate Young function of Ψ , we have by [87, Proposition A.9] that Ψ^* is also moderate. Therefore, $\bar{c}_\Psi, \bar{c}_{\Psi^*} \in (1, \infty)$; we used that $\bar{c}_{\Psi^*} = \underline{c}_\Psi$. Consequently, we can conclude the required property if we prove that $\sup_{k \in \mathbb{N}} \|S^k\|_\Psi < \infty$, where the $\sqrt{2}$ can be taken out, since $\|\cdot\|_\Psi$ is a norm. To this end we will use Doob's \mathbb{L}^Ψ -inequality. We can obtain now, by standard properties of norms and the fact that

$$\|x\|_2 = \left(\sum_{i=1}^{\ell} |x_i|^2 \right)^{\frac{1}{2}} \leq \sum_{i=1}^{\ell} |x_i| = \|x\|_1,$$

³⁹See [87, Appendix 2].

⁴⁰Actually, for a Young function Θ it is defined $\bar{c}_\Theta := \sup_{x>0} \frac{x\theta(x)}{\Theta(x)}$ for $\theta(x)$ the right derivative of Θ . \bar{c}_Θ is finite if and only if Θ is moderate; see [74, Theorem 3.1.1 (c), p. 82]. Analogously, we define $\underline{c}_\Theta := \inf_{x>0} \frac{x\theta(x)}{\Theta(x)}$.

and consequently $S^k \leq \sup_{t \in \mathbb{R}_+} \|\mathbb{E}[\zeta^k | \mathcal{G}_t^k]\|_1$, the following inequalities

$$\begin{aligned} \|S^k\|_{\Psi} &\leq \left\| \sup_{t \in \mathbb{R}_+} \|\mathbb{E}[\zeta^k | \mathcal{G}_t^k]\|_1 \right\|_{\Psi} \leq \sum_{j=1}^p \left\| \sup_{t \in \mathbb{R}_+} \|\mathbb{E}[\zeta^{k,j} | \mathcal{G}_t^k]\| \right\|_{\Psi} \stackrel{\text{Doob's In.}}{\leq} 2\bar{c}_{\Psi^*} \sum_{j=1}^p \|\zeta^{k,j}\|_{\Psi} \\ &\leq 2p\bar{c}_{\Psi^*} \|\zeta^k\|_{\Psi} = 2p\bar{c}_{\Psi^*} \|\zeta^k\|_{2,\Psi}, \end{aligned}$$

where we applied Doob's inequality for moderate Young functions; see Dellacherie and Meyer [32, Paragraph VI.103, p. 169]. The above inequality and (A.24) imply the desired

$$\sup_{k \in \mathbb{N}} \|S^k\|_{\Psi} \leq 2p\bar{c}_{\Psi^*} \sup_{k \in \mathbb{N}} \|\zeta^k\|_{2,\Psi} < \infty,$$

which, in view of (A.25), implies also the finiteness of $\sup_{k \in \mathbb{N}} \mathbb{E}[\Psi(\sqrt{2}S^k)]$. Therefore, the sequence

$$\left(\sup_{t \in \mathbb{R}_+} \|\mathbb{E}[\zeta^k | \mathcal{G}_t^k]\|_2^2 \right)_{k \in \mathbb{N}},$$

is uniformly integrable since it satisfies the de La Vallée Poussin theorem for the Young function Φ . \square

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