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Random correlation matrices generated via partial correlation C-vines

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ABSTRACT

The method for generating random $d \times d$ correlation matrices with a partial correlation C-vine is extended so that each correlation can have a distribution that is asymmetric on $(-1, 1)$ or on $(0, 1)$. With the recursion formulas from the partial correlation C-vine to the correlation matrix, first and second moments can be derived, in the case of the same distribution for each partial correlation in tree ℓ of the vine $(1 \le \ell < d)$. Algorithms and conditions are given so that, after a permutation step, all random correlations have a common mean and second moment. The algorithms can be useful for simulation experiments to generate random correlation matrices that cover the whole space or with the restriction that each correlation is positive.

1. Introduction

Let $\mathcal{R}_d$ denote the space of $d \times d$ positive definite correlation matrices. In [1] and [2] methods are presented for generating random d-dimensional correlation matrices, either uniformly over $\mathcal{R}_d$ or with density proportional to $[\det(\boldsymbol{R})]^{\alpha-1}$ for $\boldsymbol{R} \in \mathcal{R}_d$, where $\alpha > 0$, known as the LKJ distribution. A key property of this approach is that each individual correlation has a marginal distribution that is a symmetric Beta distribution on $(-1, 1)$, with variance controlled by the parameter α.

Other methods for generating correlation matrices include those based on random Cholesky matrices, or on random orthogonal matrices and eigenvalues, followed by conversion from covariance to correlation matrices; see [3,4] and [5]. More recently, a method based on the matrix exponential was proposed in [6]. Random correlation matrices generated via this approach are analyzed and compared with other methods in [7].

Since the publication of [2], there has been growing interest in generating correlation matrices with specific properties, such as: (a) asymmetric marginal distributions on $(-1, 1)$, (b) strictly positive correlations, (c) a targeted average correlation value, (d) a bounded smallest eigenvalue, (e) the Perron–Frobenius property, etc. See [7–10] and references therein for various applications (e.g. random clusters with different orientations, finance, simulation studies and sensitivity analysis) involving constrained random correlation matrices that are better matches to the type of correlation matrices occurring in the researcher's application area.

In this article, we develop a method aimed at producing correlation matrices where each correlation has support on either $(0, 1)$ or $(-1, 1)$. One primary motivation is to apply this extended LKJ distribution as a hyperprior in hierarchical models — allowing control not only over the variance but also the expectation of each correlation. We refer to [11] for background on hierarchical models and the application of LKJ distribution in this setting. Another motivation is for simulation studies with many or all positive correlations. This is important in, for example, financial stress testing or sensitivity analysis of statistical methods with clustered data

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where some inference results are based on a simplifying assumption of within cluster exchangeability. See [12] for more discussions on applications and reasons for generating customized correlation matrices.

Our method for dimension d, uses the partial correlation C-vine because (a) the parametrization has $d(d-1)/2$ algebraically independent correlations and partial correlations, of which $d-1$ are correlations, and (b) recursions to obtain $\boldsymbol{R} \in \mathcal{R}_d$ do not require matrix inversions. Having such recursions, first and second moments of the correlations in different C-vine trees can be tuned. The terminology of vines and partial correlation vines come from [13,14] and [15].

Consider $\boldsymbol{R} = (\rho_{st})_{1\le s,t\le d} \in \mathcal{R}_d$ which is parametrized using a partial correlation C-vine, with the algebraically independent parameters:

$$\begin{array}{llllll} \rho_{12} & \rho_{13} & \cdots & \rho_{1d} & & \text{tree 1;} \\ & \rho_{23;1} & \cdots & \rho_{2d;1} & & \text{tree 2;} \\ & \cdots & & & & \\ & & & \rho_{d-1,d;1\dots d-2} & & \text{tree } d-1. \end{array}$$

Here, the notation $\rho_{ij;S}$ refers to a partial correlation of variables indexed with $i,j \in \{1,\dots,d\}$ given variables indexed in $S \subset \{1,\dots,d\}\backslash\{i,j\} \neq \emptyset$ and to a correlation if $S = \emptyset$.

To generate a random correlation matrix in $\mathcal{R}_d$, one approach is that parameters are considered as independent random variables $\{R_{ij;S}\}$

Case 1: all have support on $(-1,1)$,

Case 2: all be positive on $(0,1)$.

Let the result be a correlation matrix $\boldsymbol{R} \in \mathcal{R}_d$ obtained after transforming from the partial correlation vine (see algorithm in [2] and Section 2 for the recursion). In general the marginal distributions of correlations in different rows of resulting correlation matrix are not the same. Hence a random permutation $\boldsymbol{\pi}$ with $d \times d$ permutation matrix $\boldsymbol{\Pi}$ can be generated by permuting the rows/columns of $\boldsymbol{R}$. This leads to correlation matrix $\boldsymbol{R}^{\boldsymbol{\pi}} = \boldsymbol{\Pi}\boldsymbol{R}\boldsymbol{\Pi}^\top$ such that each correlation has the same marginal distribution. As there is no general approach that controls a given marginal distribution for each correlation, the proposed new methods are based on controlling the first and/or second moment of each correlation.

In practice it is often realistic to assume that all correlations are positive (e.g., for the purpose of stress testing analysis). Let $\mathcal{R}_d^+$ be the subset of $\mathcal{R}_d$ for which all correlations are positive. The goal of Case 2 of our procedure is to generate a random correlation matrix in $\mathcal{R}_d^+$. The d-dimensional partial correlation C-vine has $d(d-1)/2$ algebraically independent parameters in $(-1,1)$ that lead to a 1–1 and onto mapping to the $d(d-1)/2$ correlations in $\mathcal{R}_d$. We show that $d(d-1)/2$ algebraically independent partial correlations on a vine in $(0,1)$, do not give a 1–1 and onto mapping to the space of $d(d-1)/2$ correlations in $\mathcal{R}_d^+$. This means that Case 1 is handled well with a C-vine algorithm with a specified mean for random correlations, but for Case 2, it turns out that the C-vine algorithm with independent positive partial correlations cannot generate the entire class of correlation matrices in $\mathcal{R}_d^+$ for $d \ge 4$.

The C-vine partial correlation parametrization allows as to compute the first and second moments of random correlations. The derivations of these two moments are shown in Sections 2 and 3, for the case that random partial correlations in tree ℓ $(1 \le \ell < d)$ are independent and have the same distribution. As far as we can see this goal cannot be achieved with other parameterizations of correlation matrices. For other methods analytical forms of moments are not derivable, even if the methods can be modified to restrict to positive correlations. Also, it may be difficult to prove or disprove the claim that all possible matrices in $\mathcal{R}_d^+$ can be covered.

To generate random matrices in $\mathcal{R}_d^+$, the recursion results in Section 3 are still useful for a C-vine algorithm where random partial correlations in tree ℓ $(2 \le \ell < d)$ are conditionally dependent on partial correlations in previous trees. The resulting correlations are all positive, but some partial correlations are allowed to be negative. Without restricting all partial correlations in the C-vine to be positive, neighborhoods of all matrices in $\mathcal{R}_d^+$ are probabilistically possible.

Section 4 contains a result that is used later for fixing the first moment of the random correlations. Section 5 has numerical results to show how large the dimension can be if one wants to fix the first and/or second moments of random correlations. This applies to cover all of $\mathcal{R}_d$ (Case 1) and a subset of $\mathcal{R}_d^+$ (Case 2) when all partial correlations in the C-vine are respectively in $(-1,1)$ or $(0,1)$. Section 6 has an algorithm to generate correlation matrices at random from all of $\mathcal{R}_d^+$. In Section 7 we have summarized the results. Some auxiliary derivations and results are in Appendix.

2. Generation via partial correlation C-vine

This section contains the details how the C-vine partial correlations parametrization is used to get a matrix $\boldsymbol{R} \in \mathcal{R}_d$. Moreover, the first and second moments of correlations in $\boldsymbol{R}$ based on random partial correlations are computed. Section 2.2 shows the mapping for one other partial correlation vine to $\boldsymbol{R} \in \mathcal{R}_d$ to indicate the resulting correlations have a form that is much more difficult to analyze. Moreover, Section 2.3 has the expectations for Beta distributions on $(0,1)$ or $(-1,1)$ in order to apply to the equations in Section 2.1, when random partial correlations are independent and have Beta distributions. Finally, in Section 2.4 the joint density for the random correlation matrix and the Jacobian for the transform in Section 2.1 is presented. We show that it leads to a simple result only in the cases considered in [2].

2.1. Recursion equations from partial correlation C-vine to correlation matrix

The algorithms to generate a random correlation matrix $\boldsymbol{R} \in \mathcal{R}_d$ based on the partial correlation C-vine (canonical vine) use the equations given in this section where partial correlations are transformed to the correlation matrix. They allow for expectations of first and second moments of random correlations to be derived via recursion equations. Other partial correlation vines in [1,2] involve matrix inversions to transform to the correlation matrix and hence are not amenable to deriving moments of random correlations.

- Row 2 for $d > 2$ and $j \geq 3$

$$R_{2j} = R_{2j;1}\sqrt{1-R_{12}^2}\sqrt{1-R_{1j}^2} + R_{12}R_{1j}. \tag{1}$$

- Row 3 for $d > 3$, $j \geq 4$,

$$R_{3j} = R_{3j;12}\sqrt{1-R_{23;1}^2}\sqrt{1-R_{2j;1}^2}\sqrt{1-R_{13}^2}\sqrt{1-R_{1j}^2} + R_{23;1}R_{2j;1}\sqrt{1-R_{13}^2}\sqrt{1-R_{1j}^2} + R_{13}R_{1j}.$$

- Row 4 for $d > 4$, $j \geq 5$,

$$\begin{aligned} R_{4j} &= R_{4j;123}\sqrt{1-R_{34;12}^2}\sqrt{1-R_{3j;12}^2}\sqrt{1-R_{24;1}^2}\sqrt{1-R_{2j;1}^2}\sqrt{1-R_{14}^2}\sqrt{1-R_{1j}^2} \\ &+ R_{34;12}R_{3j;12}\sqrt{1-R_{24;1}^2}\sqrt{1-R_{2j;1}^2}\sqrt{1-R_{14}^2}\sqrt{1-R_{1j}^2} \\ &+ R_{24;1}R_{2j;1}\sqrt{1-R_{14}^2}\sqrt{1-R_{1j}^2} + R_{14}R_{1j}. \end{aligned}$$

- General: Row ℓ and $j > \ell$ for $d > \ell$:

$$R_{\ell j;1\ldots\ell-2} = R_{\ell j;1\ldots\ell-1}\sqrt{1-R_{\ell-1,\ell;1\ldots\ell-2}^2}\sqrt{1-R_{\ell-1,j;1\ldots\ell-2}^2} + R_{\ell-1,\ell;1\ldots\ell-2}R_{\ell-1,j;1\ldots\ell-2}.$$

  For $1 \leq k < \ell-2$,

$$R_{\ell j;1\ldots k} = R_{\ell j;1\ldots k+1}\sqrt{1-R_{k+1,\ell;1\ldots k}^2}\sqrt{1-R_{k+1,j;1\ldots k}^2} + R_{k+1,\ell;1\ldots k}R_{k+1,j;1\ldots k}$$

  and

$$R_{\ell j} = R_{\ell j;1}\sqrt{1-R_{1\ell}^2}\sqrt{1-R_{1j}^2} + R_{1\ell}R_{1j}.$$

  Hence we get:

$$\begin{aligned} R_{\ell j} &= \left(\sum_{i=1}^{\ell-1} R_{i\ell;1\ldots i-1}R_{ij;1\ldots i-1}\prod_{k=1}^{i-1}\sqrt{1-R_{k\ell;1\ldots k-1}^2}\sqrt{1-R_{kj;1\ldots k-1}^2}\right) \\ &+ \quad R_{\ell j;1\ldots\ell-1}\left(\prod_{k=1}^{\ell-1}\sqrt{1-R_{k\ell;1\ldots k-1}^2}\sqrt{1-R_{kj;1\ldots k-1}^2}\right). \end{aligned} \tag{2}$$

A few observations that will be useful later can be drawn from these formulas.

1. Correlation $R_{\ell j}$ where $l > 1$, depends on one partial correlation in tree level ℓ and exactly two partial correlations in each tree level $\ell-1,\ldots,1$. Each partial correlation that appears in the formula for $R_{\ell j}$ has either ℓ or j in its conditioned set.
2. Two correlations in the same row $R_{\ell j}$ and $R_{\ell k}$, where $j \neq k$, have exactly one common partial correlation in tree level $\ell-1,\ldots,1$.
3. Two correlations in different rows $R_{\ell j}$ and R_{kj}, where $k < \ell$, have exactly one common partial correlation in tree levels $1,\ldots,k$.
4. If R_{1j} for $j \geq 1$ are positive and all $R_{\ell j;1\ldots\ell-1}$ for $j > \ell$ and $\ell \geq 2$ are positive, then all correlations $R_{\ell j}$ with $\ell \geq 2$ and $j > \ell$ are positive based on (2).
5. Let $\boldsymbol{\Pi}$ be a random permutation matrix applied to the correlation matrix derived via the above recursion. If $\{R_{1j} : j \geq 1\}$ and all of the partial correlations $\{R_{\ell j;1\ldots\ell-1} : j > \ell, \ell \geq 2\}$ are positive, then for $d \geq 4$, not all correlation matrices in $\mathcal{R}_d^+$ are achievable; see Appendix A.2.
6. There are no problems in achieving all possible correlation matrices if $\{R_{1j} : j \geq 1\}$ and all of the partial correlations $\{R_{\ell j;1\ldots\ell-1} : j > \ell, \ell \geq 2\}$ have support on $(-1,1)$ because the mapping in (2) is 1–1 and onto $\mathcal{R}_d$.

2.2. Other partial correlation vines

For all other partial correlation vines, the mapping of the set of partial correlations to $\boldsymbol{R} \in \mathcal{R}_d$ involves some matrix inversions; see Algorithm 28 in Chapter 6 of [16]. This means that moment equations similar to those in Section 3 are not tractable.

The difficulty is indicated below for dimension $d = 4$. In this case there is a class of vines, other than the C-vine, called the D-vine, that uses a different set of partial correlations.

As given in the Appendix Appendix A.1, the partial correlation D-vine has parameters $\rho_{12}, \rho_{23}, \rho_{34}$, $\rho_{13;2}, \rho_{24;3}, \rho_{14;23}$. Since $\rho_{14;23} = (\rho_{14} - a)/\sqrt{bc}$, where

$$a = (\rho_{12}\rho_{24} + \rho_{13}\rho_{34} - \rho_{13}\rho_{23}\rho_{24} - \rho_{12}\rho_{23}\rho_{34})/(1-\rho_{23}^2),$$

$$b = 1-(\rho_{12}^2+\rho_{13}^2-2\rho_{12}\rho_{13}\rho_{23})/(1-\rho_{23}^2),$$
$$c = 1-(\rho_{24}^2+\rho_{34}^2-2\rho_{23}\rho_{24}\rho_{34})/(1-\rho_{23}^2),$$

the correlations $\rho_{13}, \rho_{24}, \rho_{14}$ are derived as:

$$\rho_{13} = \rho_{12}\rho_{23}+\rho_{13;2}\sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)}, \quad \rho_{24}=\rho_{34}\rho_{23}+\rho_{24;3}\sqrt{(1-\rho_{34}^2)(1-\rho_{23}^2)},$$
$$\rho_{14} = a+\rho_{14;23}\sqrt{bc}$$

Similar issues appear in higher dimensions when sequence of partial correlations is $\rho_{jk;j+1,\ldots,k-1}$ for $k-j\geq 2$. The exact formula for ρ_{jk} for $k-j\geq 2$ involves matrix inversion, leading to fractions with square root terms in the denominator.

If all partial correlations in the D-vine are positive, it is possible for the resulting correlation matrix to have a negative value. The following example illustrates that: if $\rho_{12}=0.11, \rho_{23}=0.87, \rho_{34}=0.47$, $\rho_{13;2}=0.78, \rho_{24;3}=0.90, \rho_{14;23}=0.95$; then $\rho_{13}=0.478$, $\rho_{24}=0.801$, $\rho_{14}=-0.041$.

The above results explain why the partial correlation C-vine is applied to derive tractable analytic results.

2.3. Beta distributions

It may be convenient to use (asymmetric or symmetric) Beta distributions with support on $(-1,1)$ or $(0,1)$ as a distribution of partial correlations in C-vine. The relevant moments of such distributions that are needed to determine the first two moments of correlations in the equations in Section 2 are shown here.

In the notation below, $B(a,b)$ is the beta function with two positive arguments.

Case 1. The support on $(-1,1)$

The density derived from $Z=2W-1$ with W having a beta distribution on $(0,1)$. Let $Z\sim\text{Beta}(a,b)$ on $(-1,1)$ with $a>0, b>0$. The density (used in [1]) is:

$$f(z;a,b)=\frac{1}{2B(a,b)}\left(\frac{1+z}{2}\right)^{a-1}\left(\frac{1-z}{2}\right)^{b-1}, \quad -1<z<1.$$

Four expectations are needed:

$$\mu = \mathrm{E}(Z)=2\frac{a}{a+b}-1=\frac{a-b}{a+b}, \quad \nu=\mathrm{E}(Z^2)=4\frac{ab}{(a+b)^2(a+b+1)}+\mu^2,$$
$$\gamma = \mathrm{E}(\sqrt{1-Z^2})=\frac{2B(a+0.5,b+0.5)}{B(a,b)}, \quad \eta=\mathrm{E}(Z\sqrt{1-Z^2})=\frac{4B(a+1.5,b+0.5)-2B(a+0.5,b+0.5)}{B(a,b)}=\gamma(a-b)/(a+b-1).$$

The proof for γ and η is as follows. Let $Y\sim\text{Beta}(a,b)$ on $(-1,1)$. Then

$$\mathrm{E}(\sqrt{1-Z^2}) = \frac{1}{2B(a,b)}\int_{-1}^{1}(1-z^2)^{1/2}\left(\frac{1+z}{2}\right)^{a-1}\left(\frac{1-z}{2}\right)^{b-1}\mathrm{d}z=\frac{1}{B(a,b)}\int_{-1}^{1}\left(\frac{1+z}{2}\right)^{a+0.5-1}\left(\frac{1-z}{2}\right)^{b+0.5-1}\mathrm{d}z$$
$$= 2B(a+0.5,b+0.5)/B(a,b); \quad \text{because } z=(1+z)-1=2\{(1+z)/2\}-1;$$
$$\mathrm{E}(Z\sqrt{1-Z^2}) = \frac{1}{2B(a,b)}\int_{-1}^{1}z(1-z^2)^{1/2}\left(\frac{1+z}{2}\right)^{a-1}\left(\frac{1-z}{2}\right)^{b-1}\mathrm{d}z=\frac{1}{B(a,b)}\int_{-1}^{1}z\left(\frac{1+z}{2}\right)^{a+0.5-1}\left(\frac{1-z}{2}\right)^{b+0.5-1}\mathrm{d}z$$
$$= 2[2B(a+1.5,b+0.5)-B(a+0.5,b+0.5)]/B(a,b). \quad \square$$

Case 2. The support on $(0,1)$.

Let $W\sim\text{Beta}(a,b)$ on $(0,1)$ with $a>0, b>0$. The density is:

$$f(w;a,b)=\frac{1}{B(a,b)}w^{a-1}(1-w)^{b-1}, \quad 0<w<1.$$

Four expectations that we will require later are:

$$\mu = \mathrm{E}(W)=\frac{a}{a+b}, \quad \nu=\mathrm{E}(W^2)=\frac{ab}{(a+b)(a+b+1)},$$
$$\gamma = \mathrm{E}(\sqrt{1-W^2})=\int_0^1(1-w^2)^{1/2}f(w;a,b)\,\mathrm{d}w, \quad \eta=\mathrm{E}(W\sqrt{1-W^2})=\int_0^1 w(1-w^2)^{1/2}f(w;a,b)\,\mathrm{d}w.$$

2.4. Jacobian

The partial correlation C-vine representation is numerically most efficient for converting to the correlation matrix $\boldsymbol{R}\in\mathcal{R}_d$ because the recursions do not involve any submatrix inversions. The density of the randomly generated $\boldsymbol{R}$ can be obtained via the Jacobian, but it is not illuminating other than the special case in [2].

Consider the partial correlation C-vine with mutually independent random variables $\{R_{1j}: 2\leq j\leq d\}$ for tree 1, $\{R_{\ell j;1\cdots\ell-1}: \ell+1\leq j\leq d\}$ for trees $\ell\in\{2,\ldots,d-1\}$. The simplified notation can be used where the partial correlation is a correlation for $\ell=1$ where conditioning set $\{1,\ldots,\ell-1\}$ is an empty set. Let the set of random partial correlations over the $d-1$ trees in a

Table 1
Standard deviation of each correlation in correlation matrix $\boldsymbol{R}$ when $\boldsymbol{R}$ has uniform distribution over $\mathcal{R}_d$.

d	2	4	6	8	10	12	14	16	18	20
$(d+1)^{-1/2}$	0.577	0.447	0.378	0.333	0.302	0.277	0.258	0.243	0.229	0.218

vine be denoted as $\boldsymbol{P}$. The mapping algorithm from C-vine partial correlations to $\boldsymbol{R}$ is valid for any probability distribution on each correlation or partial correlation in the partial correlation vine parametrization.

There is a 1–1 mapping from $\boldsymbol{P}$ onto $\boldsymbol{R}$, if the components of $\boldsymbol{P}$ have support $(-1,1)$. It was shown in [2], that the density of correlation matrix based on the C-vine is:

$$f_{\boldsymbol{R}}(\boldsymbol{r}) = \prod_{\ell=1}^{d-1}\prod_{j=\ell+1}^{d} f_{R_{\ell j;1\cdots\ell-1}}(r_{\ell j;1\cdots\ell-1}) \cdot \prod_{\ell=1}^{d-1}\prod_{j=\ell+1}^{d}(1-r^2_{\ell j;1\cdots\ell-1})^{-[d-\ell-1]/2}. \tag{3}$$

The above simplifies only by choosing appropriate symmetric Beta densities on $(-1,1)$ for the $R_{\ell j;1\cdots\ell-1}$ (with parameter $\alpha_l = \alpha + (d-l-1)/2$ for some $\alpha > 0$), leading to:

$$f_{\boldsymbol{R}}(\boldsymbol{r}) \propto \left\{\prod_{\ell=1}^{d-1}\prod_{j=\ell+1}^{d}(1-r^2_{\ell j;1\cdots\ell-1})\right\}^{\alpha-1} = \{\det(\boldsymbol{r})\}^{\alpha-1}.$$

With all positive partial correlations, (3) is complicated because the density is 0 over a region that is not hyperrectangular.

For the case of uniform distribution over $\mathcal{R}_d$ ($\alpha = 1$), from [1], the marginal distribution of each correlation is Beta$(d/2, d/2)$ on $(-1,1)$ with a standard deviation of $(d+1)^{-1/2}$. In Table 1 these values are shown to provide context and comparisons for the numerical results in Section 5.

Permuting rows and columns after getting $\boldsymbol{R}$ from a vine is mentioned on page 2183 of [1]. For $d = 3$, (3) becomes

$$f_{\boldsymbol{R}}(\boldsymbol{r}) = f_{R_{12}}(r_{12}) f_{R_{13}}(r_{13}) f_{R_{23;1}}(r_{23;1}) \cdot (1-r_{12}^2)^{1/2}(1-r_{13}^2)^{1/2}.$$

Converting from the partial correlation using (1), the marginal cumulative distribution function of R_{23} is:

$$\Pr(R_{23} \le r_{23}) = \int_{-1}^{1}\int_{-1}^{1} F_{R_{23;1}}\left(\frac{r_{23}-r_{12}r_{13}}{(1-r_{12}^2)^{1/2}(1-r_{13}^2)^{1/2}}\right) f_{R_{12}}(r_{12}) f_{R_{13}}(r_{13})\, \mathrm{d}r_{12}\, \mathrm{d}r_{13}$$

and

$$f_{R_{23}}(r_{23}) = \int_{-1}^{1}\int_{-1}^{1}(1-r_{12}^2)^{-1/2}(1-r_{13}^2)^{-1/2} f_{R_{23;1}}\left(\frac{r_{23}-r_{12}r_{13}}{(1-r_{12}^2)^{1/2}(1-r_{13}^2)^{1/2}}\right) f_{R_{12}}(r_{12}) f_{R_{13}}(r_{13})\, \mathrm{d}r_{12}\, \mathrm{d}r_{13}, \tag{4}$$

If after generating random $R_{12}, R_{13}, R_{23;1}$ and computing R_{23}, the three correlations are permuted, then the marginal density of each correlation in $\boldsymbol{R}$ is

$$f_R(r) = [f_{R_{12}}(r) + f_{R_{13}}(r) + f_{R_{23}}(r)]/3,$$

with $f_{R_{23}}$ in Eq. (4). Even for dimension $d = 3$, densities are not tractable. A simple example is given in Section 2.1 of [1] when $R_{12}, R_{13}, R_{23;1}$ are independent and uniform on $(-1,1)$ random variables.

However recursion equations can be developed for the first and second moments, if $f_{R_{\ell j;1\ldots\ell-1}}$ is common for all $\ell < j \le d$ for any $\ell \in \{2,\ldots,d-1\}$. To generate $\boldsymbol{R} \in \mathcal{R}_d^+$, the recursion equations are valid if each $R_{\ell j;1\ldots\ell-1}$ has support on $(0,1)$ even if the Jacobian is not useful. The recursions are given in the next Section 3.

3. Recursion of first and second moments

In this section, assume that the $R_{\ell j;1\ldots\ell-1}$ for $j > \ell$ are independent and have a distribution that depends only on the tree level ℓ of the C-vine. Using equations developed in Section 2, the first and second moments of $R_{\ell j}$ are obtained for $2 \le \ell < d-1$ and $j > \ell$ in Sections 3.1 and 3.2. Section 3.3 has recursive forms of the first and second moments that can be used for writing code. Section 3.4 contains the expectation and variance of R_{ij} after a random permutation of rows/columns is applied.

To compute moments of $R_{\ell j}$'s when we assume that all partial correlation in the same tree of the C-vine have the same distributions, we can denote partial correlations on C-vine appearing in the formula for $R_{\ell j}$ as $X_i = R_{i\ell;1\ldots i-1}$ and $Y_i = R_{ij;1\ldots i-1}$, where i is the tree level and j,ℓ are suppressed. Then,

$$R_{\ell j} = Y_\ell \prod_{k=1}^{\ell-1}\sqrt{1-X_k^2}\sqrt{1-Y_k^2} \quad + \sum_{i=1}^{\ell-1} X_i Y_i \prod_{k=1}^{i-1}\sqrt{1-X_k^2}\sqrt{1-Y_k^2}.$$

3.1. First moments

Let $\mu_i = \mathrm{E}(X_i)$, $\nu_i = \mathrm{E}(X_i^2)$, $\gamma_i = \mathrm{E}\left(\sqrt{1-X_i^2}\right)$, $\eta_i = \mathrm{E}\left(X_i\sqrt{1-X_i^2}\right)$ and similarly for Y_i. Then the expectation of $R_{\ell j}$ for $j > \ell$ is:

$$\mathrm{E}(R_{\ell j}) = \mu_l \prod_{k=1}^{\ell-1}\gamma_k^2 + \sum_{i=1}^{\ell-1}\mu_i^2 \prod_{k=1}^{i-1}\gamma_k^2.$$

3.2. Second moments

- Row 2: $j \in \{3,\dots,d\}$

$$R_{2j}^2 = Y_2^2(1-X_1^2)(1-Y_1^2) + X_1^2Y_1^2 + 2Y_2 \cdot X_1\sqrt{1-X_1^2} \cdot Y_1\sqrt{1-Y_1^2}\,.$$

  Thus

$$\mathrm{E}(R_{2j}^2) = \nu_2(1-\nu_1)^2 + \nu_1^2 + 2\mu_2\eta_1^2.$$

- Row 3 : $j \in \{4,\dots,d\}$.

$$\begin{aligned} R_{3j}^2 &= Y_3^2(1-X_2^2)(1-Y_2^2)(1-X_1^2)(1-Y_1^2) + X_2^2Y_2^2(1-X_1^2)(1-Y_1^2) + X_1^2Y_1^2 \\ &\quad + 2Y_3 \cdot X_2\sqrt{1-X_2^2} \cdot Y_2\sqrt{1-Y_2^2} \cdot (1-X_1^2)(1-Y_1^2) \\ &\quad + 2Y_3 \cdot \sqrt{1-X_2^2}\,\sqrt{1-Y_2^2} \cdot X_1\sqrt{1-X_1^2} \cdot Y_1\sqrt{1-Y_1^2} + 2X_2Y_2 \cdot X_1\sqrt{1-X_1^2} \cdot Y_1\sqrt{1-Y_1^2}\,. \end{aligned}$$

  Hence,

$$\mathrm{E}(R_{3j}^2) = \nu_3(1-\nu_2)^2(1-\nu_1)^2 + \nu_2^2(1-\nu_1)^2 + \nu_1^2 + 2\mu_3\eta_2^2(1-\nu_1)^2 + 2\mu_3\gamma_2^2\eta_1^2 + 2\mu_2^2\eta_1^2.$$

- Row 4 : $j \in \{5,\dots,d\}$.

$$\begin{aligned} R_{4j}^2 &= Y_4^2(1-X_3^2)(1-Y_3^2)(1-X_2^2)(1-Y_2^2)(1-X_1^2)(1-Y_1^2) \\ &\quad + X_3^2Y_3^2(1-X_2^2)(1-Y_2^2)(1-X_1^2)(1-Y_1^2) + X_2^2Y_2^2(1-X_1^2)(1-Y_1^2) + X_1^2Y_1^2 \\ &\quad + 2Y_4 \cdot X_3\sqrt{1-X_3^2} \cdot Y_3\sqrt{1-Y_3^2} \cdot (1-X_2^2)(1-Y_2^2)(1-X_1^2)(1-Y_1^2) \\ &\quad + 2Y_4 \cdot \sqrt{1-X_3^2}\,\sqrt{1-Y_3^2} \cdot X_2\sqrt{1-X_2^2} \cdot Y_2\sqrt{1-Y_2^2}\,(1-X_1^2)(1-Y_1^2) \\ &\quad + 2Y_4 \cdot \sqrt{1-X_3^2}\,\sqrt{1-Y_3^2} \cdot \sqrt{1-X_2^2}\,\sqrt{1-Y_2^2} \cdot X_1\sqrt{1-X_1^2} \cdot Y_1\sqrt{1-Y_1^2} \\ &\quad + 2X_3Y_3 \cdot X_2\sqrt{1-X_2^2} \cdot Y_2\sqrt{1-Y_2^2}\,(1-X_1^2)(1-Y_1^2) \\ &\quad + 2X_3Y_3\sqrt{1-X_2^2}\,\sqrt{1-Y_2^2} \cdot X_1\sqrt{1-X_1^2} \cdot Y_1\sqrt{1-Y_1^2} + 2X_2Y_2 \cdot X_1\sqrt{1-X_1^2} \cdot Y_1\sqrt{1-Y_1^2}\,. \end{aligned}$$

  This leads to

$$\begin{aligned} \mathrm{E}(R_{4j}^2) &= \nu_4(1-\nu_3)^2(1-\nu_2)^2(1-\nu_1)^2 + \nu_3^2(1-\nu_2)^2(1-\nu_1)^2 + \nu_2^2(1-\nu_1)^2 + \nu_1^2 \\ &\quad + 2\mu_4\eta_3^2(1-\nu_2)^2(1-\nu_1)^2 + 2\mu_4\gamma_3^2\eta_2^2(1-\nu_1)^2 + 2\mu_4\gamma_3^2\gamma_2^2\eta_1^2 + 2\mu_3^2\eta_2^2(1-\nu_1)^2 + 2\mu_3^2\gamma_2^2\eta_1^2 + 2\mu_2^2\eta_1^2. \end{aligned}$$

- Hence for row $\ell \in \{2,\dots,d-1\}$, with $j \in \{\ell+1,\dots,d\}$:

$$\begin{aligned} \mathrm{E}(R_{\ell j}^2) &= \nu_\ell \prod_{k=1}^{\ell-1}(1-\nu_k)^2 + \sum_{i=1}^{\ell-1}\nu_i^2\prod_{k=1}^{i-1}(1-\nu_k)^2 \\ &\quad + 2\mu_\ell\sum_{k=1}^{\ell-1}\left(\prod_{s=k+1}^{\ell-1}\gamma_s^2\right)\cdot\eta_k^2\cdot\left(\prod_{t=1}^{k-1}(1-\nu_t)^2\right) \quad + 2\sum_{i=2}^{\ell-1}\mu_i^2\left[\sum_{k=1}^{i-1}\left(\prod_{s=k+1}^{i-1}\gamma_s^2\right)\cdot\eta_k^2\cdot\left(\prod_{t=1}^{k-1}(1-\nu_t)^2\right)\right]. \end{aligned}$$

3.3. Algorithmic form for code

In this section, recursions for the first and second moments are given so that they can be coded. The first moment is simple as all elements of products are independent. The second moment is more complex as the tree level increases because there are cross-product terms based on ℓ terms in the sum for $R_{\ell j}$ for tree ℓ.

The pattern for first moments is as follows:

$$\begin{aligned} \mathrm{E}(R_{1j}) &= \mu_1, \quad \mathrm{E}(R_{2j}) = \mu_1^2 + \mu_2\gamma_1^2, \quad \mathrm{E}(R_{3j}) = \mu_1^2 + \mu_2^2\gamma_1^2 + \mu_3\gamma_1^2\gamma_2^2, \\ \mathrm{E}(R_{4j}) &= \mu_1^2 + \mu_2^2\gamma_1^2 + \mu_3^2\gamma_1^2\gamma_2^2 + \mu_4\gamma_1^2\gamma_2^2\gamma_3^2, \quad \mathrm{E}(R_{5j}) = \mu_1^2 + \mu_2^2\gamma_1^2 + \mu_3^2\gamma_1^2\gamma_2^2 + \mu_4^2\gamma_1^2\gamma_2^2\gamma_3^2 + \mu_5\gamma_1^2\gamma_2^2\gamma_3^2\gamma_4^2, \end{aligned}$$

The recursion for the first moment $\mathrm{E}(R_{\ell,\ell+1}) = \mathrm{E}(R_{\ell j})$ with $j > \ell$ is:

$$\begin{aligned} \mathrm{E}(R_{\ell j}) &= \mathrm{E}(R_{\ell-1,j}) + (\mu_{\ell-1}^2 - \mu_{\ell-1})\prod_{i=1}^{\ell-2}\gamma_i^2 + \mu_\ell\prod_{i=1}^{\ell-1}\gamma_i^2 \\ &= \mathrm{E}(R_{\ell-1,j}) + \left([\mu_{\ell-1}^2 - \mu_{\ell-1}] + \mu_\ell\gamma_{\ell-1}^2\right)\prod_{i=1}^{\ell-2}\gamma_i^2, \quad \ell \ge 2, \end{aligned} \tag{5}$$

starting from $\mathrm{E}(R_{12}) = \mu_1$. This essentially changes one term with $[\mu_{\ell-1}^2 - \mu_{\ell-1}]$ and adds one term with μ_ℓ.

For second moments $\mathrm{E}(R_{\ell j}^2)$,

$$\mathrm{E}(R_{2j}^2) = \nu_1^2 + \nu_2(1-\nu_1)^2 + 2\mu_2\eta_1^2$$

and there is a recursion for trees 3 and higher that can handle all of the cross-product terms. This is shown in steps.

Step 1. Recursion for the square terms in the expressions for the second moment in Section 3.2. With analogy to first moment with μ changed to ν and γ changed to $(1-\nu)$, this leads to

$$D_\ell = D_{\ell-1} + \left([\nu_{\ell-1}^2 - \nu_{\ell-1}] + \nu_\ell(1-\nu_{\ell-1})^2\right)\prod_{i=1}^{\ell-2}(1-\nu_i)^2, \quad \ell \geq 2,$$

starting from $D_1 = \nu_1$. This essentially changes one term with $[\nu_{\ell-1}^2 - \nu_{\ell-1}]$ and adds one term with ν_ℓ.

Step 2. Recursion for the cross-product terms in the expressions for second moment $\mathrm{E}(R_{kj}^2)$ with $j > k$; an upper triangular matrix A is created. Set A as a large square matrix of 0s to initialize.

For $k = 2$, consider the multiplier of $2\eta_1^2$ terms for $\ell \geq 2$: the starting multiplier is $A_{12} = \mu_2$ and then the recursion is:

$$A_{1\ell} = A_{1,\ell-1} + \left([\mu_{\ell-1}^2 - \mu_{\ell-1}] + \mu_\ell\gamma_{\ell-1}^2\right)\prod_{i=2}^{\ell-2}\gamma_i^2, \quad \ell \geq 3.$$

For $k = 3$, consider the multiplier of $2\eta_2^2(1-\nu_1)^2$ terms for $\ell \geq 3$: the starting multiplier is $A_{23} = \mu_3$ and then the recursion is:

$$A_{2\ell} = A_{2,\ell-1} + \left([\mu_{\ell-1}^2 - \mu_{\ell-1}] + \mu_\ell\gamma_{\ell-1}^2\right)\prod_{i=3}^{\ell-2}\gamma_i^2, \quad \ell \geq 4.$$

For $k = 4$, consider the multiplier of $2\eta_3^2(1-\nu_1)^2(1-\nu_2)^2$ terms for $\ell \geq 4$: the starting multiplier is $A_{34} = \mu_4$ and then the recursion is:

$$A_{3\ell} = A_{3,\ell-1} + \left([\mu_{\ell-1}^2 - \mu_{\ell-1}] + \mu_\ell\gamma_{\ell-1}^2\right)\prod_{i=4}^{\ell-2}\gamma_i^2, \quad \ell \geq 5.$$

The pattern continues.

For $k \geq 5$, consider the multiplier of $2\eta_{k-1}^2\prod_{i=1}^{k-2}(1-\nu_i)^2$ terms for $\ell \geq k$: the starting multiplier is $A_{k-1.k} = \mu_k$ and then the recursion is:

$$A_{k-1,\ell} = A_{k,\ell-1} + \left([\mu_{\ell-1}^2 - \mu_{\ell-1}] + \mu_\ell\gamma_{\ell-1}^2\right)\prod_{i=k}^{\ell-2}\gamma_i^2, \quad \ell \geq k+1.$$

For $\mathrm{E}(R_{\ell,j}^2)$ with $j > \ell$, one extra term is added for $\eta_1^2, \ldots, \eta_{\ell-1}^2$,

Step 3. Multiply row i of the A matrix by $2\eta_i^2\prod_{s=1}^{i-1}(1-\nu_s)^2$.

Step 4. Fill in the diagonal of the A matrix with $A_{\ell\ell} = D_\ell$ for $\ell \geq 1$. Now A is upper diagonal with a positive diagonal and 0s in the lower triangle.

Step 5. Take column sums of the A matrix (or sum each column up to the diagonal entry). This vector consists of second moments $\mathrm{E}(R_{\ell j}^2)$ $(j > \ell)$ for $\ell \in \{1, 2, \ldots\}$ (up to the dimension of A used in the recursions).

3.4. Moments after random permutation

After a random permutation $\boldsymbol{\pi}$ (matrix $\boldsymbol{\Pi}$) of $\boldsymbol{R}$ to get $\boldsymbol{R}^{\boldsymbol{\pi}} = \boldsymbol{\Pi}\boldsymbol{R}\boldsymbol{\Pi}^\top$, the following hold for any non-diagonal element $R_{ij}^{\boldsymbol{\pi}}$:

$$\mathrm{E}\left(R_{ij}^{\boldsymbol{\pi}}\right) = \frac{2}{d(d-1)}\sum_{\ell=1}^{d-1}(d-\ell)\mathrm{E}[R_{\ell,\ell+1}],\ \mathrm{E}\left((R_{ij}^{\boldsymbol{\pi}})^2\right) = \frac{2}{d(d-1)}\sum_{\ell=1}^{d-1}(d-\ell)\mathrm{E}\left(R_{\ell,\ell+1}^2\right),\ \mathrm{Var}\left(R_{ij}^{\boldsymbol{\pi}}\right) = \mathrm{E}\left((R_{ij}^{\boldsymbol{\pi}})^2\right) - \left\{\mathrm{E}\left(R_{ij}^{\boldsymbol{\pi}}\right)\right\}^2.$$

4. Fixing the first moment of each random correlation

The results in this section are based on a common distribution F_j applied to each partial correlations tree level of the C-vine for $j \in \{1, \ldots, d-1\}$. Let μ_j and γ_j be respectively $\mathrm{E}(X)$ and $\mathrm{E}\left(\sqrt{1-X^2}\right)$ for $X \sim F_j$.

Using the recursive formula for the expectation of the correlation in row ℓ and assuming that these expectations coincide with μ_1, that is $\mathrm{E}(R_{2j}) = \mu_1^2 + \mu_2\gamma_1^2 = \mu_1$, leads to

$$\mu_2 = \mu_1\left[\frac{1-\mu_1}{\gamma_1^2}\right]. \tag{6}$$

Since from (5), $\mathrm{E}(R_{3j}) = \mathrm{E}(R_{2j}) + \left[\mu_2(\mu_2-1) + \mu_3\gamma_2^2\right]\gamma_1^2$, then assuming that $\mathrm{E}(R_{3j}) = \mathrm{E}(R_{2j}) = \mu_1$ leads to $\mu_2(\mu_2-1) + \mu_3\gamma_2^2 = 0$ or $\mu_3 = \mu_2\left[\frac{1-\mu_2}{\gamma_2^2}\right]$. The assumption $\mathrm{E}(R_{\ell j}) = \mathrm{E}(R_{\ell-1,j}) = \mu_1$ and the recursive formula (5) further leads to:

$$\left[\mu_{\ell-1}(\mu_{\ell-1}-1) + \mu_\ell\gamma_{\ell-1}^2\right]\prod_{i=1}^{\ell-2}\gamma_i^2 = 0,$$

or

$$\mu_\ell = \mu_{\ell-1}\left[\frac{1-\mu_{\ell-1}}{\gamma^2_{\ell-1}}\right], \quad \ell \in \{3,\dots,d-1\}. \tag{7}$$

For F_j with support on $(0,1)$, start with $\mu_1 \in (0,1)$. From (6) and (7), recursions can reach $k\times k$ correlation matrices if $\mu_{k-1}(1-\mu_{k-1})/\gamma^2_{k-1} < 1$.

For F_j with support on $(-1,1)$, start with $\mu_1 \in (-1,1)$. From (6) and (7), recursions can reach $k\times k$ correlation matrices if $-1 < \mu_{k-1}(1-\mu_{k-1})/\gamma^2_{k-1} < 1$.

Results with Beta distributions on $(0,1)$ or $(-1,1)$ for F_j are given numerically in Section 5 and proven below. Fig. 1 shows that multiplicative factor for the next μ_ℓ when $F_{\ell-1}$ is a Beta(a,b) distribution on $(0,1)$.

Case 1. F_ℓ corresponds to Beta(a,b_ℓ) in $(-1,1)$.

For $X_\ell \sim$ Beta(a,b_ℓ) on $(-1,1)$, let $\mu_{a,b_\ell} = \mu_\ell = (a-b_\ell)/(a+b_\ell)$ and let $\gamma_{a,b_\ell} = \gamma_\ell = \mathrm{E}\left(\sqrt{1-X_\ell^2}\right)$. When $b>a$ then $\mu_{a,b}<0$ and we show below that $(1-\mu_{a,b})/\gamma^2_{a,b} \ge 1$. Hence, it is possible for the recursion to reach a limit with $\mu_{a,b_\ell} \le -1$ for some ℓ.

Proposition 1. *Let $X \sim$ Beta(a,b) on $(-1,1)$ with $a>0, b>0$. For all $a \le b$,*

$$(1-\mu_{a,b})/\gamma^2_{a,b} \ge 1,$$

where $\mu_{a,b} = (b-a)/(a+b)$ and $\gamma_{a,b} = \mathrm{E}\left(\sqrt{1-X_\ell^2}\right) = 2B(a+0.5,b+0.5)/B(a,b)$.

Proof: We find a lower bound for the factor

$$\frac{1-\mu_{a,b}}{\gamma^2_{a,b}} = \frac{2b}{a+b} \Bigg/ \left\{\left(\frac{\Gamma(a+0.5)}{\Gamma(a)}\right)^2\left(\frac{\Gamma(b+0.5)}{\Gamma(b)}\right)^2\frac{4}{(a+b)^2}\right\}. \tag{8}$$

Gautschi's inequality [17] states that for all $x>0$ and $s\in(0,1)$ the following holds:

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}.$$

Since $\Gamma(x+1) = x\Gamma(x)$ then this can be rewritten as:

$$x^{-s} < \frac{\Gamma(x)}{\Gamma(x+s)} < \left(\frac{x+1}{x^{1/(1-s)}}\right)^{1-s}.$$

For $s=0.5$, this becomes:

$$\sqrt{\frac{x^2}{x+1}} < \frac{\Gamma(x+0.5)}{\Gamma(x)} < \sqrt{x}.$$

Applying this to (8) leads to:

$$\frac{1-\mu_{a,b}}{\gamma^2_{a,b}} = \frac{b(a+b)}{2\left(\frac{\Gamma(a+0.5)}{\Gamma(a)}\right)^2\left(\frac{\Gamma(b+0.5)}{\Gamma(b)}\right)^2} > \frac{b(a+b)}{2ab} = \frac{a+b}{2a}.$$

Observing that $(a+b)/2a \ge 1$ for $b\ge a$ concludes the proof. □

Case 2. F_ℓ corresponds to Beta(a,b_ℓ) in $(0,1)$.

For every fixed $a>0$, we show below that factor $(1-\mu_\ell)/\gamma_\ell^2$ stays below 1 for large enough b. This allows us to conclude that the initial choice b_1 determines whether it is possible to make choices of b_ℓ to obtain the same expectation of correlations computed from partial correlations in higher order trees of C-vine. For $X_\ell \sim$ Beta(a,b_ℓ), let $\mu_{a,b} = \mu_\ell = a/(a+b_\ell)$ and let $\gamma_{a,b_\ell} = \gamma_\ell = \mathrm{E}\left(\sqrt{1-X_\ell^2}\right)$. If we choose b_1 such that $\mu_{a,b_1}/\gamma^2_{a,b_1} \le 1$ then $0\le\mu_2\le\mu_1$ and $b_1 \le b_2$. Hence X_2 is stochastically smaller than X_1, $\gamma_{a,b_1} \le \gamma_{a,b_2}$ and $\mu_{a,b_2}/\gamma^2_{a,b_2} \le 1$. By iterating, $b_1\le b_2\le b_3$ etc.

Proposition 2. *Let $X\sim$ Beta(a,b) on $(0,1)$ with $a>0, b>0$. Let $\mu_{a,b} = a/(a+b)$ and let $\gamma_{a,b} = \mathrm{E}\left(\sqrt{1-X_\ell^2}\right)$. For all $a>0$ there exists $b_a>0$ such that for all $b\ge b_a$,*

$$(1-\mu_{a,b})/\gamma^2_{a,b} \le 1.$$

Proof. For $x\in(0,1)$, $1-x^2\in(0,1)$ and $\sqrt{1-x^2}\ge 1-x^2$. Incorporating this bound in the formula for γ, one gets

$$\gamma_{a,b} \ge 1 - \frac{B(a+2,b)}{B(a,b)} = 1 - \frac{a(a+1)}{(a+b+1)(a+b)} =: g(a,b).$$

Since $0\le(1-\mu_{a,b})/\gamma^2_{a,b} \le \{b/(a+b)\}/g^2(a,b)$ then it suffices to show that $\{b/(a+b)\}/g^2(a,b)\le 1$ for sufficiently large b. Note that $\{b/(a+b)\}/g^2(a,b) = \{b(a+b)(a+b+1)^2\}/\{(a+b+1)(a+b)-a(a+1)\}^2$. We investigate conditions for b for which the difference of the denominator and the numerator is non-negative. After factoring out ab, the difference reduces to the quadratic inequality:

$$b^2+ab-(a+1)^2 \ge 0.$$

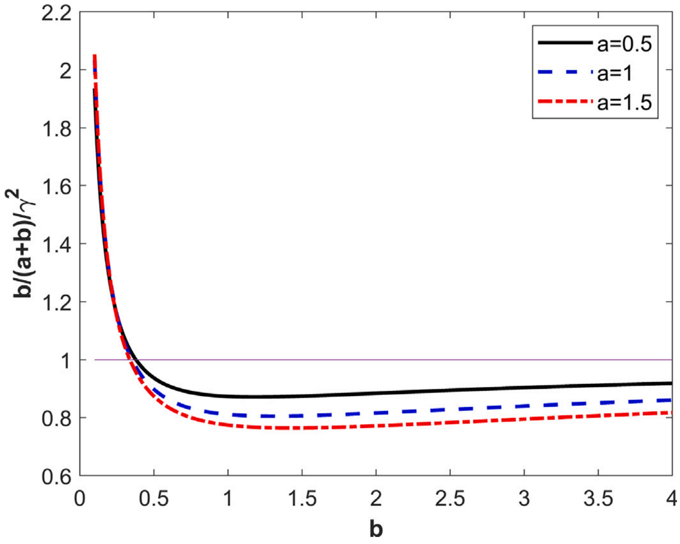

Fig. 1. Factor $(1-\mu_{a,b})/\gamma^2_{a,b}$, with $\mu_{a,b}=\mu_{\ell-1}=a/(a+b)$, $\gamma_{\ell-1}=\gamma_{a,b}$, $a=a_{\ell-1}$ and $b=b_{\ell-1}$ from Case 2 in Section 2.3, as a function of b for $a=0.5,1,1.5$. For $\ell=2$, if a is fixed, then the recursion can continue if $\mu_{a,b_1}/\gamma^2_{a,b_1}\le 1$ or $b\ge b_{min}$ such that $\mu_{a,b_{min}}/\gamma^2_{a,b_{min}}=1$.

There are two real solutions: $b_1=\{-a-\sqrt{a^2+4(a+1)^2}\}/2$ and $b_2=\{-a+\sqrt{a^2+4(a+1)^2}\}/2$. Since b_1 is negative, then $b\ge\{-a+\sqrt{a^2+4(a+1)^2}\}/2$ implies that $\{b/(a+b)\}/g^2(a,b)\le 1$. □

Remark. Note that the above proposition asserts that when $a=1$, $(1-\mu_{a,b})/\gamma^2_{a,b}\le 1$ when $b\ge(-1+\sqrt{17})/2\approx 1.56$, but in Fig. 1 it can be observed that the quantity of interest is below 1 already for $b>0.352$.

5. Results and numerical summaries based on Section 3.3

The results in this section are based on a common Beta distributions applied to each tree level of the C-vine. The results in the preceding Sections 3 and 4 can be applied for other tree-level distributions by deriving the four expectations in Sections 3 and 2.3.

We refer to $R_{\ell,\ell+1}$ (in place of $R_{\ell,j}$ with $j>\ell$) for simpler notation because having a common distribution for each tree of the C-vine implies that the moments of $R_{\ell,j}$ for $j>\ell$ are the same.

Starting from R_{1j} as independent Beta(a_1,b_1) on $(0,1)$ or Beta(a_1,b_1) on $(-1,1)$, consider trying to do the following.

Case I. Make $\mathrm{E}(R_{\ell,\ell+1})$ constant for all $\ell\ge 1$ with Beta(a_ℓ,b_ℓ) for partial correlations in tree ℓ.

Case II. Make $\mathrm{E}(R_{\ell,\ell+1})$ and $E(R^2_{\ell,\ell+1})$ constant for all $\ell\ge 1$, with Beta(a_ℓ,b_ℓ) for partial correlations in tree ℓ.

Case I with $a_\ell=a_1$ has simpler code for solving roots of equation. For case (I) for positive correlations, provided b_1 is large enough (with lower bound depending on a_1), and a_ℓ fixed at a_1, the b_ℓ values increase with ℓ, and the variance of the partial correlation decrease as ℓ increase. This leads to constant $\mathrm{E}(R_{\ell,\ell+1})$ and decreasing $\mathrm{Var}(R_{\ell,\ell+1})$. Hence $\mathrm{Var}(R_{12})$ provides an upper bound on random correlations after the row/column permutation.

The lower bound is derived below in Section 5.1.

Case II: a_ℓ, b_ℓ decrease as ℓ increases, and one of these values might head to 0 (meaning no solution). This means for Case II to work in higher dimensions, the starting values of a_1,b_1 should be larger (and smaller variance); compare Table 1.

These observations are consistent with the result for uniformly distributed random correlation matrices in dimension d: the marginal distribution of each correlation is Beta$(d/2,d/2)$ and partial correlations in increasing tree level are beta distributed, symmetric ($a=b$) and their parameters decrease in decrements of $1/2$ as ℓ increases.

For comparison, the values of mean and standard deviation for uniform distribution over the space of correlation matrices with all positive entries are presented in Table 2. These were obtained via simulations written in Fortran90 code using the rejection method: generate $d(d-1)/2$ values in $(0,1)$ for the off-diagonal elements of the symmetric matrix and keep those matrices that are positive definite.

5.1. Beta(a_ℓ,b_ℓ) on $(0,1)$, Case I

Let correlations in tree 1 of the C-vine have independent Beta(a_1,b_1) distribution on $(0,1)$.

Table 2
Mean and standard deviation (SD) of each correlation for random correlation matrices in $\mathcal{R}_d^+$. Fraction refers to the probability that symmetric matrix with all off-diagonal values in $(0,1)$ is positive definite; a, b are parameters of Beta distribution with the mean and SD.

dimension	fraction	mean	SD	a	b	$a+b$
2	1.000	0.500	0.289	1	1	2
3	0.800	0.466	0.268	1.14	1.31	2.45
4	0.426	0.438	0.252	1.27	1.62	2.89
5	0.132	0.415	0.238	1.36	1.92	3.29
6	0.021	0.396	0.226	1.46	2.20	3.66
7	0.0017	0.381	0.217	1.54	2.51	4.05
8	0.000059	0.366	0.208	1.56	2.70	4.26

Table 3
For random $\boldsymbol{R} \in \mathcal{R}_d^+$ with independent Beta(a, b_ℓ) distributions for partial correlations in tree ℓ. $b_{min}(a)$ is the smallest value of b_1 so that b_ℓ is increasing (μ_ℓ is decreasing) and $\mathrm{E}(R_{\ell,\ell+1})$ is constant. Compare with Fig. 1.

a	0.5	1.0	1.5	2.0	3.0	4.0	5.0
$b_{min}(a)$	0.379	0.352	0.338	0.330	0.320	0.314	0.310

Consider $a_\ell = a_1 = a$ for $\ell \geq 2$, with independent Beta(a, b_ℓ) distributions on $(0,1)$ for the all partial correlations in tree ℓ of the C-vine. That is, fix a and find b_ℓ for $\ell \in \{2,\ldots,d-1\}$ so that the mean of the marginal correlation $R_{\ell,\ell+1}$ is constant and equal to $a/(a+b_1)$.

From previous sections, $\mu_\ell = \mathrm{E}(R_{\ell j;1\cdots\ell-1}) = a/(a+b_\ell)$ for $\ell \geq 2$ and let $\gamma_\ell = \gamma_\ell(a, b_\ell)$ be from Case 2 in Section 2.3. From (7) in Section 3.1, recursively,

$$\mu_\ell = \mu_{\ell-1}(1-\mu_{\ell-1})/\gamma_{\ell-1}^2(a, b_{\ell-1}) = a/(a+b_\ell),$$

and then $b_\ell = a(1-\mu_\ell)/\mu_\ell$. This can continue to dimension ℓ provided $0 < \mu_\ell < 1$.

If $b_1 \geq b_{min}(a)$, then $R_{\ell,\ell+1;1\cdots\ell-1}$ can be chosen with Beta(a, b_ℓ) distribution for $\ell \geq 2$ with b_ℓ increasing in ℓ with $\mathrm{E}(R_{\ell,\ell+1}) = a/(a+b_1)$. Note that b_ℓ in increasing implies μ_ℓ is decreasing.

If $0 < b_1 < b_{min}(a)$, then $R_{\ell,\ell+1;1\ldots\ell-1}$ can be chosen with Beta(a, b_ℓ) distribution for $\ell \geq 2$ with b_ℓ decreasing in ℓ up to a dimension $d = d(a, b_1)$.

The value of $b_{min}(a)$ (rounded up in third decimal place) is tabulated in Table 3 for some values of a.

Some numerical examples are shown in Table 4 with $a = 1.5$ and $b_1 \in \{0.3, 0.338, 0.5\}$. It turns out that the resulting standard deviation of $R_{\ell,\ell+1}$ decreases in ℓ with Beta(a, b_ℓ) chosen in this way.

5.2. Beta(a_ℓ, b_ℓ) on $(0,1)$, Case II

From the preceding subsection, with fixed $a_\ell = a_1$ for $\ell \geq 2$, the first moment of $\mathrm{E}(R_{\ell,\ell+1})$ can be held constant (in some cases with no upper bound on the dimension d) but $SD(R_{\ell,\ell+1})$ decreases.

To make the second moment $\mathrm{E}(R_{\ell,\ell+1}^2)$ constant, this suggests that one needs to choose Beta(a_ℓ, b_ℓ) distributions for the partial correlations $R_{\ell,j;1\ldots\ell-1}$ in tree ℓ so that a_ℓ, b_ℓ are decreasing in ℓ. Hence the largest dimension d for which this works will depend on a_1, b_1; a larger dimension can be achieved when a_1, b_1 are larger. Some examples are shown in Table 5.

5.3. Beta(a_ℓ, b_ℓ) on $(-1,1)$, Case I

Requiring $a_\ell = a_1 = a$ for $\ell > 1$ tends to work only if $a_1 \geq b_1$. Some examples are shown in Table 6. In some cases, b_ℓ goes towards 0 or ∞ for degenerate distributions at 1 or -1, in which case the largest reachable dimension d is small.

From the previous sections, $\mu_\ell = \mathrm{E}(R_{\ell j;1\ldots\ell-1}) = 2a/(a+b_\ell) - 1$ for $\ell \geq 2$ and let $\gamma_\ell = \gamma_\ell(a, b_\ell)$ be from Case 1 in Section 2.3. From (7) in Section 3.1, recursively,

$$\mu_\ell = \mu_{\ell-1}(1-\mu_{\ell-1})/\gamma_{\ell-1}^2(a, b_{\ell-1}) = 2a/(a+b_\ell) - 1,$$

and then $b_\ell = a(1-\mu_\ell)/(\mu_\ell+1)$. This can continue to dimension ℓ provided $-1 < \mu_\ell < 1$.

5.4. Beta(a_ℓ, b_ℓ) on $(-1,1)$, Case II

Similar comments to Section 5.2 apply.

For the symmetric Beta case, if $a_1 = b_1$, then this matches the theory in [2] with the joint density $f_{\boldsymbol{R}}(\boldsymbol{r})$ being proportional to a power of $\det(\boldsymbol{R})$ and $a_\ell = b_\ell = a_1 - (\ell-1)/2$; this works for dimensions d such that $a_1 - (d-1)/2 > 0$. Each correlation has Beta$(d/2, d/2)$ distribution in the case of correlation matrices being uniform in dimension d.

Other cases are given in Table 7 to show that a_ℓ, b_ℓ decrease as ℓ increase.

Table 4

Random $\boldsymbol{R} \in \mathcal{R}^+_{\ell+1}$: some examples of expectations and standard deviations (SD) of random correlations with independent Beta(a_ℓ, b_ℓ) distributions for partial correlations in tree ℓ and fixing $a_\ell = a_1$ for $\ell > 1$ and making $\mathrm{E}[R_{\ell,\ell+1}]$ to be constant over ℓ. The last column comes from Section 3.4: it is for the SD of a random correlation for $\boldsymbol{R} \in \mathcal{R}_{\ell+1}$ after a random permutation. After row/column permutation, based on QQ-plots, the marginal distribution of each correlation is close to Beta(a^*, b^*), where a^*, b^* are Beta parameters that match with the mean and SD of $R_{\ell,\ell+1}$ in this table. For example, for $a = 1.5, b_1 = 0.5$, $d = 6, \ell = 5$, a mean of $\mu = 0.75$ and an SD of $\sigma = 0.219$, corresponds to $a^* = 2.17, b^* = 0.72$ (via $\theta = a^* + b^* = \mu(1-\mu)/\sigma^2 - 1$, $a^* = \mu\theta$, $b^* = \theta - a^*$).

ℓ	a_ℓ	b_ℓ	$\mathrm{E}(R_{\ell,\ell+1})$	$\mathrm{E}(R^2_{\ell,\ell+1})$	SD	SD_π
$a = 1, b_1 = 0.3$, works up to $\ell = 2$						
1	1.5	0.300	0.833	0.744	0.223	0.223
2	1.5	0.211	0.833	0.733	0.195	0.214
3	1.5	0.001	0.832	0.730	0.195	0.209
4	1.5	0.001	0.832	0.730	0.195	0.207
$a = 1.5, b_1 = 0.338$						
1	1.5	0.3380	0.816	0.719	0.230	0.230
2	1.5	0.3384	0.816	0.706	0.199	0.220
3	1.5	0.3398	0.816	0.705	0.196	0.215
4	1.5	0.3443	0.816	0.705	0.196	0.211
5	1.5	0.3585	0.816	0.704	0.196	0.209
6	1.5	0.4032	0.816	0.704	0.196	0.207
7	1.5	0.5356	0.816	0.704	0.196	0.206
8	1.5	0.8737	0.816	0.704	0.196	0.205
20	1.5	15.63	0.816	0.704	0.196	0.200
50	1.5	59.32	0.816	0.704	0.196	0.198
50	1.5	133.5	0.816	0.704	0.196	0.197
$a = 1.5, b_1 = 0.5$						
1	1.5	0.500	0.750	0.625	0.250	0.250
2	1.5	0.789	0.750	0.606	0.208	0.237
3	1.5	1.383	0.750	0.602	0.200	0.229
4	1.5	2.274	0.750	0.601	0.197	0.223
5	1.5	3.355	0.750	0.601	0.197	0.219
6	1.5	4.546	0.750	0.601	0.196	0.216
7	1.5	5.804	0.750	0.601	0.196	0.214
8	1.5	7.106	0.750	0.601	0.196	0.212
20	1.5	23.92	0.750	0.601	0.196	0.203
50	1.5	67.90	0.750	0.601	0.196	0.199
100	1.5	142.2	0.750	0.601	0.196	0.198

6. C-vine algorithm to cover all of $\mathcal{R}^+_d$

This section has an algorithm to generate all matrices in $\mathcal{R}^+_d$. Some random partial correlations that are negative are required. To ensure that all resulting correlations are positive, the partial correlations in tree ℓ $(2 \le \ell < d)$ are conditionally dependent on the partial correlations in previous trees. Because of the conditional dependence, the steps in the algorithm can sometimes lead to an impossibility (no way to generate $R_{\ell j;1\ldots\ell-1}$ in $(-1,1)$) so the algorithm is an acceptance–rejection method with a high acceptance rate.

Using theory from earlier sections, we can aim for a target mean and SD. For a reasonable target for the mean of the random correlation, the acceptance rate seems to be above 0.97 for $4 \le d \le 7$ and does not decrease quickly like acceptance–rejection method leading to Table 2.

The main idea of the algorithm is based on the observation that for $\ell > 1$, from (2), the form of the correlation based on previous rows is:

$$R_{\ell j} = I_{\ell j} + R_{\ell j;1\ldots\ell-1} M_{\ell j}, \quad j > \ell,$$

where $M_{\ell j} > 0$. In order $R_{\ell j} > 0$ the partial correlation $R_{\ell j;1\ldots\ell-1} \in \big(\max\{-I_{\ell j}/M_{\ell j}, -1\}, 1\big)$.

This leads to the following recursion for $I_{\ell j}$ and $M_{\ell j}$, where lower case r is used. Let

$$\begin{aligned}
s_{1j} &= \sqrt{1-r^2_{1j}}, & t_{1j} &= r_{1j}, \\
s_{2j} &= \sqrt{1-r^2_{1j}}\sqrt{1-r^2_{2j;1}} = s_{1j}\sqrt{1-r^2_{2j;1}}, & t_{2j} &= \sqrt{1-r^2_{1j}}\, r_{2j;1} = s_{1j} r_{2j;1}, \\
s_{3j} &= s_{2j}\sqrt{1-r^2_{3j;12}}, & t_{3j} &= s_{2j} r_{3j;12}, \\
s_{\ell j} &= s_{\ell-1,j}\sqrt{1-r^2_{\ell j;1\ldots\ell-1}}, & t_{\ell j} &= s_{\ell-1,j} r_{\ell j;1\ldots\ell-1}.
\end{aligned}$$

Then

$$M_{\ell j} = s_{\ell-1,\ell} s_{\ell-1,j}, \quad \ell \in \{2,\ldots,d-1\}.$$

Table 5
Random $\boldsymbol{R} \in \mathcal{R}^+_{\ell+1}$: some examples of expectations and standard deviations (SD) of random correlations with independent Beta(a_ℓ, b_ℓ) distributions for partial correlations in tree ℓ and making $\mathrm{E}(R_{\ell,\ell+1})$ and $\mathrm{E}(R^2_{\ell,\ell+1})$ each to be constant over ℓ. After row/column permutation, based on QQ-plots, the marginal distribution of each correlation is close to Beta(a_1, b_1). As d increases beyond 10, the marginal distribution is closer to Beta(a^*, b^*) with slightly smaller mean and slightly larger SD.

ℓ	a_ℓ	b_ℓ	$\mathrm{E}(R_{\ell,\ell+1})$	$\mathrm{E}(R^2_{\ell,\ell+1})$	SD
$a_1 = 1, b_1 = 0.5$, works up to $\ell = 2$					
1	1.000	0.500	0.667	0.533	0.298
2	0.066	0.044	0.667	0.533	0.298
$a_1 = 4, b_1 = 8$					
1	4.000	8.000	0.333	0.128	0.131
2	2.243	6.528	0.333	0.128	0.131
3	1.425	5.409	0.333	0.128	0.131
4	0.975	4.540	0.333	0.128	0.131
5	0.701	3.848	0.333	0.128	0.131
6	0.525	3.303	0.333	0.128	0.131
10	0.205	1.881	0.333	0.128	0.131
20	0.034	0.451	0.333	0.128	0.131
27	0.017	0.197	0.333	0.128	0.131
$a_1 = 2, b_1 = 8$					
1	2.000	8.800	0.200	0.055	0.121
2	1.395	6.838	0.200	0.055	0.121
3	1.025	5.928	0.200	0.055	0.121
4	0.780	5.185	0.200	0.055	0.121
5	0.610	4.572	0.200	0.055	0.121
6	0.488	4.059	0.200	0.055	0.121
10	0.231	2.642	0.200	0.055	0.121
20	0.059	1.024	0.200	0.055	0.121
38	0.011	0.019	0.200	0.055	0.121

$$I_{2j} = t_{12}t_{1j}, \quad I_{3j} = t_{13}t_{1j} + t_{23}t_{2j}, \quad I_{4j} = t_{14}t_{1j} + t_{24}t_{2j} + t_{34}t_{3j}, \quad I_{\ell j} = \sum_{k=1}^{\ell-1} t_{k\ell}t_{kj}.$$

6.1. Algorithm for generating from $\mathcal{R}^+_d$

The algorithm in this section makes use of the above recursions.

Input d. Initialize $d \times d$ matrices $\boldsymbol{R}, \boldsymbol{P}$, $(I_{\ell j}), (M_{\ell j}), (s_{\ell j}), (t_{\ell j})$ with 0s. Set diagonal of $\boldsymbol{R}$ to 1. The matrix $\boldsymbol{P}$ is for storing the random partial correlations for rows 2 to $d-1$.

Input parameters $a_1, b_1, \mu_1, \ldots, \mu_{d-1}$ and a^*, where $\mu_1 = a_1/(a_1 + b_1)$ is the target mean for all random correlations. For $\ell \geq 2$, μ_ℓ is the specification for the conditional means of partial correlations in trees ℓ and should be smaller than μ_1 based on Section 5. In some numerical experiments, the last parameter a^* seems to have little effect, so results in Section 6.4 use $a^* = a_1$.

- Generate $r_{1j} \sim \mathrm{Beta}(a_1, b_1)$ on $(0,1), j \in \{2, \ldots, d\}$ in row 1, and set $R_{1j} = r_{1j}$; compute s_{1j}, t_{1j}.
- Compute M_{2j}, I_{2j} (note that $I_{2j} > 0$) and $q_2 = \max\{-I_{2j}/M_{2j}, -1\}$.
  Generate $r_{2j;1} \in (q_2, 1)$, $j \in \{3, \ldots, d\}$ in row 2 with parameters μ_2, a^* using algorithm below.
  Let $r_{2j} = I_{2j} + r_{2j;1}M_{2j}$. Set $R_{j2} = r_{2j}$.
  Compute s_{2j}, t_{2j}.
- For $\ell \in \{3, \ldots, d-1\}$: compute $M_{\ell j}, I_{\ell j}$ and $q_\ell = \max\{-I_{\ell j}/M_{\ell j}, -1\}$.
  If $q_\ell \geq 1$, then reject and go to next simulation. Otherwise generate $r_{\ell j;1\ldots\ell-1} \in (q_\ell, 1)$, $j \in \{\ell+1, \ldots, d\}$ in row ℓ with parameters μ_ℓ, a^*.
  Let $r_{\ell j} = I_{\ell j} + r_{\ell j;1\ldots\ell-1}M_{\ell j}$. Set $R_{\ell j} = r_{\ell j}$.
  Compute $s_{\ell j}, t_{\ell j}$.

To generate partial correlation $r_{\ell j;1\ldots\ell-1} \in (q_l, 1)$ where $-1 < q_l < 1$ the following procedure is applied.
To generate $Z \in (q, 1)$ where $-1 < q < 1$ with target mean μ follow

- If $q \geq \mu$, generate Z uniform in the interval $(q, 1)$.
- If $q < \mu$, let $\mu^* = (\mu - q)/(1-q)$ and $b = a(1-\mu^*)/\mu^*$ and a fixed.
  Generate W from Beta(a, b) with mean $a/(a+b) = \mu^*$.
  Then $Z = q + (1-q)W$ be Beta(a, b) in $(q, 1)$ with mean $q + (1-q)\mu^* = \mu$.
- Return Z as the partial correlation.

Table 6
Random $\boldsymbol{R} \in \mathcal{R}_{\ell+1}$: some examples of expectations and standard deviations (SD) of random correlations with independent Beta(a_ℓ, b_ℓ) distributions for partial correlations in tree ℓ and fixing $a_\ell = a_1$ for $\ell > 1$ and making $\mathrm{E}(R_{\ell,\ell+1})$ to be constant over ℓ. The last column comes from Section 3.4: it is for the SD of a correlation after random permutation when the dimension of the correlation matrix is $\ell + 1$. After row/column permutation, based on QQ-plots, the marginal distribution of each correlation is close to Beta(a^*, b^*), where a^*, b^* are Beta parameters that match with the mean and SD of $R_{\ell,\ell+1}$ in this table. For example, for $a = 4, b_1 = 2$, $d = 6, \ell = 5$, a mean of $\mu = 1/3$ and an SD of $\sigma = 0.330$, corresponds to $a^* = 4.77, b^* = 2.38$ (via $\mu^* = (\mu+1)/2$, $\theta = a^* + b^* = 4\mu^*(1-\mu^*)/\sigma^2 - 1$, $a^* = \mu^*\theta$, $b^* = \theta - a^*$).

ℓ	a_ℓ	b_ℓ	$\mathrm{E}(R_{\ell,\ell+1})$	$\mathrm{E}(R^2_{\ell,\ell+1})$	SD	SD_π
$a = 3, b_1 = 2$						
1	3	2.000	0.200	0.200	0.400	0.400
2	3	1.979	0.200	0.178	0.372	0.391
3	3	1.961	0.200	0.164	0.353	0.383
4	3	1.945	0.200	0.155	0.340	0.377
5	3	1.930	0.200	0.150	0.331	0.371
6	3	1.917	0.200	0.146	0.326	0.366
7	3	1.905	0.200	0.144	0.322	0.362
8	3	1.895	0.200	0.142	0.320	0.359
20	3	1.837	0.200	0.140	0.316	0.338
50	3	1.822	0.200	0.140	0.316	0.326
100	3	1.822	0.200	0.140	0.316	0.321
$a = 4, b_1 = 2$						
1	4	2.000	0.333	0.238	0.356	0.356
2	4	2.148	0.333	0.219	0.329	0.348
3	4	2.267	0.333	0.209	0.313	0.341
4	4	2.364	0.333	0.203	0.303	0.335
5	4	2.442	0.333	0.199	0.297	0.330
6	4	2.507	0.333	0.197	0.293	0.326
7	4	2.561	0.333	0.196	0.291	0.323
8	4	2.606	0.333	0.195	0.289	0.320
20	4	2.837	0.333	0.192	0.285	0.303
50	4	2.896	0.333	0.192	0.285	0.293
100	4	2.898	0.333	0.192	0.285	0.289
$a = 6, b_1 = 2$						
1	6	2.000	0.500	0.333	0.289	0.289
2	6	2.613	0.500	0.318	0.260	0.280
3	6	3.066	0.500	0.312	0.248	0.273
4	6	3.401	0.500	0.309	0.243	0.268
5	6	3.654	0.500	0.307	0.239	0.264
6	6	3.851	0.500	0.306	0.237	0.261
7	6	4.007	0.500	0.305	0.235	0.259
8	6	4.133	0.500	0.305	0.234	0.256
20	6	4.735	0.500	0.303	0.231	0.244
50	6	4.929	0.500	0.303	0.230	0.236
100	6	4.944	0.500	0.303	0.230	0.234

Remarks that compare with some auxiliary results in the Appendix are the following.

(i) The "rejection" rate depends on the input parameters $a_1, b_1, \mu_1, \ldots, \mu_{d-1}, a^*$. The rate of rejection is small compared with generating positive symmetric matrices, with diagonal elements of 1 and off-diagonal elements < 1, and accepting those that are positive definite.

(ii) For $d = 4$, there are generated $\boldsymbol{R} \in \mathcal{R}_4^+$ that do not have positive partial correlation C-vine for any permutation.

(iii) For $d = 5$, there are generated $\boldsymbol{R} \in \mathcal{R}_5^+$ that do have neither positive partial correlation C-vine nor positive partial correlation D-vine for any permutation.

6.2. *Example of rejection*

In one run of 5000 cases for $d = 5$, with $a_1 = 3$, $b_1 = 3$, $\mu_\ell = 0.3$ for $\ell \geq 2$, $a^* = 2$, there were 5 rejections. These can occur if enough negative partial correlations are generated in early steps.

An example below fails for r_{45}; there were also cases that failed for r_{34} or r_{35}.

Table 7

Random $\boldsymbol{R} \in \mathcal{R}_{\ell+1}$: some examples of expectations and standard deviations (SD) of random correlations with independent Beta(a_ℓ, b_ℓ) distributions for partial correlations in tree ℓ and making $\mathrm{E}(R_{\ell,\ell+1})$ and $\mathrm{E}(R^2_{\ell,\ell+1})$ each to be constant over ℓ. After row/column permutation, based on QQ-plots, the marginal distribution of each correlation is close to Beta(a_1, b_1).

ℓ	a_ℓ	b_ℓ	$\mathrm{E}(R_{\ell,\ell+1})$	$\mathrm{E}(R^2_{\ell,\ell+1})$	SD
$a_1 = 1, b_1 = 0.6$, works up to $\ell = 2$					
1	1.000	0.600	0.250	0.423	0.600
2	0.356	0.162	0.250	0.423	0.600
$a_1 = 8, b_1 = 5$					
1	8.000	5.000	0.231	0.121	0.260
2	7.083	4.689	0.231	0.121	0.260
3	6.294	4.332	0.231	0.121	0.260
4	5.580	3.944	0.231	0.121	0.260
5	4.914	3.535	0.231	0.121	0.260
6	4.282	3.111	0.231	0.121	0.260
10	1.907	1.322	0.231	0.121	0.260
13	0.794	0.245	0.231	0.121	0.260
$a_1 = 6, b_1 = 4.8$					
1	6.000	4.800	0.111	0.096	0.289
2	5.424	4.350	0.111	0.096	0.289
3	4.856	3.896	0.111	0.096	0.289
4	4.295	3.437	0.111	0.096	0.289
5	3.739	2.975	0.111	0.096	0.289
6	3.186	2.510	0.111	0.096	0.289
10	0.969	0.642	0.111	0.096	0.289
11	0.367	0.193	0.111	0.096	0.289

$$\boldsymbol{R} = \begin{pmatrix} 1.000 & 0.801 & 0.624 & 0.547 & 0.348 \\ 0.801 & 1.000 & 0.753 & 0.053 & 0.736 \\ 0.624 & 0.753 & 1.000 & 0.313 & 0.842 \\ 0.547 & 0.053 & 0.313 & 1.000 & NA \\ 0.348 & 0.736 & 0.842 & NA & 1.000 \end{pmatrix}, \quad \boldsymbol{P} = \begin{pmatrix} - & 0.801 & 0.624 & 0.547 & 0.348 \\ - & - & 0.541 & -0.768 & 0.814 \\ - & - & - & 0.692 & 0.844 \\ - & - & - & - & NA \\ - & - & - & - & - \end{pmatrix}$$

$$(q_{\ell j}) = \begin{pmatrix} - & - & - & - & - \\ - & - & -1 & -0.874 & -0.496 \\ - & - & - & -0.197 & -1.000 \\ - & - & - & - & 1.155 \\ - & - & - & - & - \end{pmatrix}$$

6.3. Distribution of partial correlation C-vine for random matrix in $\mathcal{R}_d^+$

With the algorithm in Section 6.1, the partial correlations for tree 2 and higher take the full range of $(-1, 1)$. This is expected based of the summary given below.

From simulation based on the acceptance–rejection method of Table 2, one can obtain the partial correlation C-vine in dimension d (without any permutation), and get distributions of partial correlations in tree 2, tree 3, etc. Note that, at each tree level, partial correlations have support on the interval $(-1, 1)$ even with all correlations being positive.

In Table 8, Beta distributions have support in $(0, 1)$ in row 1 and in $(-1, 1)$ for rows 2 to $d - 1$. The Beta parameter estimates are based on method of moment estimation. The results in this table suggest that the conditional mean parameters μ_ℓ in Section 6.1 should be decreasing as ℓ increases.

6.4. Numerical results

In this section the numerical results for the algorithm in Section 6.1 are presented (see Table 9).

If the conditional means of partial correlations $\mu_2, \ldots, \mu_{d-1}$ are specified in (7), with γ_ℓ computed from Case 2 in Section 2.3, and $a^* = a_1$, then the mean of each correlation can be quite close to $a_1/(a_1 + b_1)$ when the acceptance rate is close to 1. The formula (7) is useful even if the partial correlation in tree ℓ do not all have the same distribution.

For $d \in \{4, 5, 6, 7\}$, note that we have come close to the mean and SD of correlations from uniform distribution over the space $\mathcal{R}_d^+$, making use of the a, b values in Table 2.

Table 8
Random correlation matrices in $\mathcal{R}_d^+$ based on acceptance–rejection is possible only for small d. Beta(a, b) distributions on $(-1, 1)$ are good fits to the result partial correlations in trees 2 to $d-1$ of the C-vine. For each d, approximately 40000 matrices in $\mathcal{R}_d^+$ were used.

d	R	min	max	mean	SD	a	b
4	R_{1j}	0	1	0.438	0.252	1.27	1.62
4	$R_{2j;1}$	−1	1	0.301	0.383	3.39	1.82
4	$R_{3j;12}$	−1	1	0.225	0.495	1.76	1.12
5	R_{1j}	0	1	0.415	0.238	1.36	1.92
5	$R_{2j;1}$	−1	1	0.291	0.342	4.42	2.43
5	$R_{3j;12}$	−1	1	0.225	0.421	2.67	1.69
5	$R_{4j;123}$	−1	1	0.179	0.518	1.54	1.07
6	R_{1j}	0	1	0.396	0.226	1.46	2.20
6	$R_{2j;1}$	−1	1	0.284	0.311	5.47	3.05
6	$R_{3j;12}$	−1	1	0.221	0.374	3.55	2.26
6	$R_{4j;123}$	−1	1	0.180	0.442	2.33	1.62
6	$R_{5j;1234}$	−1	1	0.145	0.535	1.38	1.03

Table 9
Some summary results for the algorithm in Section 6.1. The marginal distributions of different correlations were similar; the tabulated results assume a random permutation. Each line is based on 10^6 simulations via Fortran90 code. The last column has the acceptance rate rounded downwards to 3 decimal places. The columns $\mathrm{E}(R_{ij})$ and $SD(R_{ij})$ are close to the mean $\mu = a_1/(a_1+b_1)$ and $SD = \sqrt{\mu(1-\mu)/(a_1+b_1+1)}$ when the acceptance rate is closer to 1. After row/column permutation, based on QQ-plots, the marginal distribution of each correlation is close to Beta(a_1, b_1). As d increases beyond 10, the marginal distribution is closer to Beta(a^*, b^*) with slightly smaller mean and slightly larger SD.

d	a_1	b_1	$\mu_2, \ldots, \mu_{d-1}$	a^*	$\mathrm{E}(R_{ij})$	$SD(R_{ij})$	accept
4	1.00	3.00	.210,.180	1.00	.249	.197	.999
5	1.00	3.00	.210,.180,.156	1.00	.248	.198	.998
6	1.00	3.00	.210,.180,.156,.138	1.00	.247	.199	.995
7	1.00	3.00	.210,.180,.156,.138,.123	1.00	.246	.199	.985
4	2.00	2.00	.370,.285	2.00	.498	.228	.999
5	2.00	2.00	.370,.285,.229	2.00	.494	.229	.997
6	2.00	2.00	.370,.285,.229,.191	2.00	.491	.230	.991
7	2.00	2.00	.370,.285,.229,.191,.162	2.00	.488	.230	.981
4	1.27	1.62	.344,.275	1.27	.437	.254	.997
5	1.36	1.92	.324,.260,.214	1.36	.410	.243	.993
6	1.46	2.20	.311,.249,.207,.175	1.46	.391	.233	.986
7	1.54	2.51	.297,.239,.199,.169,.147	1.54	.371	.223	.978
10	1.65	3.13	.345,.221, ...,0.100	1.65	.332	.207	.925
20	1.87	4.58	.290,.233, ...,0.049	1.87	.269	.174	.446
20	2.80	6.87	.290,.230, ...,0.048	2.80	.276	.153	.901
20	3.74	9.16	.290,.229, ...,0.048	3.74	.280	.137	.990
30	1.98	5.58	.261,.213, ...,0.033	1.98	.237	.154	.059
30	3.96	11.16	.262,.211, ...,0.032	3.96	.251	.125	.954
30	5.94	16.75	.262,.210, ...,0.032	5.94	.255	.103	.999
40	3.07	9.55	.243,.199, ...,0.024	3.07	.225	.128	.218
40	4.10	13.74	.230,.189, ...,0.024	4.10	.219	.111	.879
50	3.58	11.97	.230,.189, ...,0.019	3.58	.213	.115	.170
50	4.00	13.38	.230,.189, ...,0.019	4.00	.216	.112	.416
50	4.00	16.00	.200,.168, ...,0.019	4.00	.191	.100	.768
50	4.00	20.00	.167,.144, ...,0.019	4.00	.162	.084	.978
70	2.80	20.00	.123,.110, ...,0.013	2.80	.118	.074	.623
70	2.70	22.00	.109,.099, ...,0.013	2.70	.106	.068	.874
70	2.60	25.00	.094,.086, ...,0.013	2.60	.092	.059	.983
100	1.80	24.00	.070,.065, ...,0.009	1.80	.067	.053	.469
100	1.70	28.00	.057,.054, ...,0.009	1.70	.056	.045	.919

Comparing results in Table 9 with ones in Tables 4 and 5 for larger d values in $\{20, 30, 40, 50, 70, 100\}$ we observe that the mean and SD of random correlations in $\mathcal{R}_d^+$ decrease as d increases. Without including all correlation matrices in $\mathcal{R}_d^+$, the methods in Sections 5.1 and 5.2 can generate random correlation matrices with larger mean correlations. Table 9 shows for $d > 10$ that the choice of a_1, b_1 strongly affect the acceptance rate of the algorithm. To get a higher acceptance rate, the Beta parameters a_1, b_1 for tree 1 should be chosen such that the Beta distribution for each random correlation has smaller mean and SD. This is consistent with Table 1 where uniform distribution in $\mathcal{R}_d$ implies a zero mean and a decreasing SD as d increases, and Table 2 where uniform in $\mathcal{R}_d^+$ implies decreasing mean and SD as d increase. We note that the marginal distribution of each correlation is right-skewed. Larger correlation values can occur with the method of Section 6. For example, in the last case of Table 9 with $d = 100$, $a_1 = 1.7$

and $b_1 = 28$, consider the 90th percentile Q_{90} of the $d(d-1)/2 = 4550$ correlations in the random correlation matrix in $\mathcal{R}^+_{100}$; in one simulation run, the median and 75th percentile of Q_{90} are 0.40 and 0.47 respectively. For a smaller d value, such as $d = 7$, $a_1 = 1.54$ and $b_1 = 2.51$ in row 12 of Table 9, the median and 75th percentile of Q_{90} are 0.81 and 0.87.

7. Discussion

In this paper the partial correlation C-vines algorithms are given for generating random correlation matrices in $\mathcal{R}_d$ for which the expectation of each random correlation is fixed, or for which the first two moments of each random correlation are fixed. Hence one can have skewed distribution for each correlation with a positive mean. The results in Table 1 and Section 5 imply that the achievable range of standard deviations decreases as the dimension d increases.

Essentially the same algorithms can generate random positive correlation matrices in $\mathcal{R}^+_d$ for which the expectation of each positive random correlation is fixed, or for which the first two moments of each random correlation are fixed. However as shown in results in Appendix Appendices A.2 and A.3, not all matrices in $\mathcal{R}^+_d$ can be achieved for $d \geq 4$. The matrices that cannot be achieved are farther away from the exchangeable correlation matrix. The algorithm in Section 6.1 can cover all of $\mathcal{R}^+_d$ but it cannot control the mean and second moment of each random correlation. However the analytic results of the C-vine recursions are useful to get close to a specified mean for each correlation.

We have provided useful algorithms for simulating random correlation matrices in $\mathcal{R}_d$ and $\mathcal{R}^+_d$ with different means and second moments for all correlations. The study of random correlation matrices in $\mathcal{R}^+_d$ is much harder than in $\mathcal{R}_d$. This article is first to indicate all of the extra difficulties. With the C-vine and positive partial correlations, it is guaranteed that the generated correlation matrices are in $\mathcal{R}^+_d$, but for $d \geq 4$, not all matrices in $\mathcal{R}^+_d$ can be generated that way. There are correlation matrices in R^+_d ($d \geq 4$) that contain a negative partial correlation on any C-vine (for every permutation of variables). This means that such matrices will not be able to be generated with the method in Section 5. Hence we designed an algorithm in Section 6 that relaxes the assumption of the independence of partial correlations on C-vines and uses partial correlations on $(-1, 1)$ to generate correlation matrices in $\mathcal{R}^+_d$. This algorithm is an acceptance rejection method but with a high acceptance rate if initial parameters are chosen appropriately.

The methods in this article are extensions of the methods in [2] for the C-vine partial correlation vine. The C-vine recursion cannot handle correlation matrices with structural forms such as block matrices in [7] or the Perron–Frobenius property in [8]. However, it generalizes the LKJ distribution [2], able to control only variances of correlations to one that can also control means. Code will be made available for our algorithms.

A further comparison with [2] involves that dependence of random correlations in different positions of $\boldsymbol{R}$ generated from $\mathcal{R}_d$ and $\mathcal{R}^+_d$. For the correlation matrix distributions in [2], it can be shown from the recursions in Section 2.1 that pairs of correlations in different positions are uncorrelated because the mean of each correlation is 0. More generally, if the targeted mean value of a correlation is not 0, the recursions in Section 2.1 imply that R_{24} does not depend on R_{13} (depends only on R_{12} and R_{14}); R_{35} does not depend on R_{12}, R_{14}, R_{24}; etc. In general, with $\ell < j$, $R_{\ell j}$ does not depend on R_{ik} with $i < k$ with $\{\ell, j\} \cap \{i, k\} = \emptyset$. Consider two positions $\{\ell, j\}$ and $\{i, k\}$. If their intersection is the empty set, then the correlations in these positions are uncorrelated. For the remaining pairs, there is a correlation (see formulas of covariances in Appendix A.5) that increases as the targeted mean value of individual correlations increase in $(0, 1)$. These comments apply to all methods in Section 5. For Section 6 with the acceptance–rejection step, there could be weak dependence for positions that do not share a common subscript.

The method in [6] can generate correlation matrices with only positive correlations but it does not come close to covering $\mathcal{R}^+_d$. In our experiments with this method all the simulated matrices have at least one positive partial correlation C-vine after row/column permutation. Moreover, we observed different behavior of both methods in terms of how the generated correlations are related. The relationships between entries of correlation matrices generated with our method have been discussed above. In our experiments the method in [6] produced matrices on $\mathcal{R}^+_d$ for which entries with an overlapping subscript were positively correlated. More comparisons of two methods are needed to fully describe their similarities and differences.

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Appendix. Auxiliary results

A.1. Partial correlation vines for dimensions 4 and 5

For dimension $d = 3$, each of three vines can be considered as a C-vine or D-vine. For dimension $d = 4$, each regular vine is either a C-vine or a D-vine (two classes). For dimension $d \geq 5$, there are regular vines that are in neither of the two boundary classes of C-vines and D-vines (see [18] where vine equivalence classes are studied). Other vine equivalence classes for $d = 4$ and $d = 5$ are given below.

For $d = 4$, there are two equivalence classes as given below.

C. $\rho_{12}, \rho_{13}, \rho_{14}; \rho_{23;1}, \rho_{24;1}, \rho_{34;12}$,

D. $\rho_{12}, \rho_{23}, \rho_{34}; \rho_{13;2}, \rho_{24;3}, \rho_{14;23}$.

For $d = 5$, Example 6.1 of [16] has representations of the 6 equivalence classes with the non-boundary vines labeled as B0, B1, B2, B3 vines. Representative sets of algebraically independent partial correlations for the partial correlation vines are given below.

C. $\rho_{12}, \rho_{13}, \rho_{14}, \rho_{15}; \rho_{23;1}, \rho_{24;1}, \rho_{25;1}; \rho_{34;12}, \rho_{35;12}; \rho_{45;123}$,
D. $\rho_{12}, \rho_{23}, \rho_{34}, \rho_{45}; \rho_{13;2}, \rho_{24;3}, \rho_{35;4}; \rho_{14;23}, \rho_{25;34}; \rho_{15;234}$,
B0. $\rho_{12}, \rho_{13}, \rho_{14}, \rho_{15}; \rho_{23;1}, \rho_{24;1}, \rho_{35;1}; \rho_{34;12}, \rho_{25;13}; \rho_{45;123}$,
B1. $\rho_{12}, \rho_{13}, \rho_{14}, \rho_{25}; \rho_{23;1}, \rho_{24;1}, \rho_{15;2}; \rho_{34;12}, \rho_{35;12}; \rho_{45;123}$,
B2. $\rho_{12}, \rho_{23}, \rho_{14}, \rho_{15}; \rho_{13;2}, \rho_{24;1}, \rho_{25;1}; \rho_{34;12}, \rho_{35;12}; \rho_{45;123}$,
B3. $\rho_{12}, \rho_{13}, \rho_{14}, \rho_{25}; \rho_{23;1}, \rho_{34;1}, \rho_{15;2}; \rho_{24;13}, \rho_{35;12}; \rho_{45;123}$.

These are referred to in Appendix A.3.

If the indices in the correlations and partial correlation are permuted, the partial correlation vine is different (different set of algebraically independent parameters) but the equivalence class has not changed.

A.2. Examples of positive correlation matrices not achievable from non-negative partial correlation C-vine

Based on simulations of $\boldsymbol{R} \in \mathcal{R}_d^+$ for $d \geq 4$, via the rejection–acceptance method, examples were found where $\boldsymbol{R}$ does not have a non-negative partial correlation C-vine for any permutation.

For $d = 3$, the three partial correlation C-vines have parameters (i) $\rho_{12}, \rho_{13}, \rho_{23;1}$, (ii) $\rho_{12}, \rho_{23}, \rho_{13;2}$, (iii) $\rho_{13}, \rho_{23}, \rho_{23;1}$. Suppose $\rho_{12}, \rho_{13}, \rho_{23}$ are all positive and let ρ_{jk} be the smallest with $g \neq j$ and $g \neq k$. Then $\rho_{jg;k}$ is equal in sign to $\rho_{jg} - \rho_{jk}\rho_{gk}$ and the latter quantity is positive. Hence any $\boldsymbol{R} \in \mathcal{R}_3^+$ can be obtained via at least one non-negative partial correlation C-vine.

For $d = 4$, there are $\boldsymbol{R} \in \mathcal{R}_4^+$ for which all 12 partial correlation C-vine representations have at least one negative value. Some examples are given below.

Looking at examples where the C-vine fails (negative partial correlation obtained form the matrix with only positive elements), we could ask if the region of failure can be characterized? A partial answer can be given. Let $\boldsymbol{P}_g$ be the partial correlation matrix given variable g for $g \in \{1, 2, 3, 4\}$, in the form:

$$\boldsymbol{P}_g = (\rho_{jk;g})_{(j,k)\in\{1,2,3,4\}} = \left(\frac{\rho_{jk} - \rho_{jg}\rho_{kg}}{\sqrt{(1-\rho_{jg}^2)(1-\rho_{kg}^2)}} \right)_{(j,k)\in\{1,2,3,4\}}.$$

Note that $\rho_{jk;g} = 0$ if $g = j$ or $g = k$. If $\boldsymbol{R}$ is such that $\boldsymbol{P}_g$ has no negative entries, then there is a partial correlation C-vine with g as the root variable.

Proof. Take $g = 1$ without loss of generality and suppose $\boldsymbol{P}_1$ has no negative entries. Then $\rho_{jk;1}, \rho_{hj;1}, \rho_{hk;1} \geq 0$ where (h, j, k) is a permutation of $(2, 3, 4)$. Since $\rho_{ab;1c}$ is equal in sign to $\rho_{ab;1} - \rho_{ac;1}\rho_{bc;1}$, there is a second order partial correlation that is non-negative. The above argument for $d = 3$ can be used. If $\rho_{hj;1k} \geq 0$, then tree 2 has $\rho_{hk;1}$ and $\rho_{jk;1}$, and tree 3 has $\rho_{hj;1k}$. □

This means that a necessary condition for C-vine failure is that each of $\boldsymbol{P}_g$ has at least one negative entry. These tend to occur with (j, k) such that ρ_{jk} is the smallest or second smallest (depending on the conditional variable g), and these two smallest correlations are in different rows/columns.

Examples are the following:

$$\boldsymbol{R}_1 = \begin{pmatrix} 1.000 & 0.593 & 0.268 & 0.785 \\ 0.593 & 1.000 & 0.531 & 0.168 \\ 0.268 & 0.531 & 1.000 & 0.404 \\ 0.785 & 0.168 & 0.404 & 1.000 \end{pmatrix}, \quad \boldsymbol{R}_2 = \begin{pmatrix} 1.000 & 0.816 & 0.244 & 0.025 \\ 0.816 & 1.000 & 0.038 & 0.289 \\ 0.244 & 0.038 & 1.000 & 0.153 \\ 0.025 & 0.289 & 0.153 & 1.000 \end{pmatrix}.$$

$\boldsymbol{R}_1$ has smallest correlations in the (2,4) and (1,3) positions, and $\boldsymbol{R}_2$ has smallest correlations in the (1,4) and (2,3) positions.

Depending on the context for needing random correlation matrices in $\mathcal{R}_d^+$, not including correlation matrices similar to these two examples (in larger dimensions) might be acceptable. The positive correlation matrices not achievable from any non-negative partial correlation C-vine are further away from "exchangeable". If all correlation matrices in $\mathcal{R}_d^+$ are needed, the algorithm in Section 6.1 can be used.

Note that generating matrices in $\mathcal{R}_d^+$ based on random positive $d \times d$ Cholesky decomposition would also not cover all of $\mathcal{R}_d^+$ because these matrices have a partial correlation C-vine representation; see Appendix Appendix A.4.

A.3. Coverage rates

In this section, based on simulation results, fractions of $R \in \mathcal{R}_d^+$ that cannot be achieved from non-negative partial correlation C-vines and non-negative partial correlation D-vines are given for $d \in \{4, 5, 6, 7\}$.

$d = 4$.

The estimated failure rate for C-vine is: 41/2141 = 0.019.

The estimated failure rate for D-vine is: 0/212621 = 0.

$d = 5$.

The estimated failure rate for C-vine is: 570/6446 = 0.088.

The estimated failure rate for D-vine is: 21/6446 = 0.0033.

The estimated failure rate for C-vine but not D-vine is: 560/6446 = 0.087.

The estimated failure rate for D-vine but not C-vine is: 11/6446 = 0.0017.

The estimated failure rate for simultaneous D-vine and C-vine is: 10/6446 = 0.0016.

Examples where C-vine fails, and D-vine does not:

$$\boldsymbol{R}_3 = \begin{pmatrix} 1.000 & 0.759 & 0.137 & 0.225 & 0.065 \\ 0.759 & 1.000 & 0.397 & 0.058 & 0.226 \\ 0.137 & 0.397 & 1.000 & 0.396 & 0.055 \\ 0.225 & 0.058 & 0.396 & 1.000 & 0.670 \\ 0.065 & 0.226 & 0.055 & 0.670 & 1.000 \end{pmatrix}, \quad \boldsymbol{R}_4 = \begin{pmatrix} 1.000 & 0.312 & 0.884 & 0.078 & 0.350 \\ 0.312 & 1.000 & 0.187 & 0.382 & 0.256 \\ 0.884 & 0.187 & 1.000 & 0.270 & 0.517 \\ 0.078 & 0.382 & 0.270 & 1.000 & 0.699 \\ 0.350 & 0.256 & 0.517 & 0.699 & 1.000 \end{pmatrix}.$$

An example where D-vine fails and C-vine does not:

$$\boldsymbol{R}_5 = \begin{pmatrix} 1.000 & 0.486 & 0.647 & 0.380 & 0.133 \\ 0.486 & 1.000 & 0.845 & 0.095 & 0.509 \\ 0.647 & 0.845 & 1.000 & 0.129 & 0.290 \\ 0.380 & 0.095 & 0.129 & 1.000 & 0.348 \\ 0.133 & 0.509 & 0.290 & 0.348 & 1.000 \end{pmatrix}.$$

An example where C-vine and D-vine both fail:

$$\boldsymbol{R}_6 = \begin{pmatrix} 1.000 & 0.382 & 0.178 & 0.448 & 0.163 \\ 0.382 & 1.000 & 0.459 & 0.008 & 0.827 \\ 0.178 & 0.459 & 1.000 & 0.445 & 0.728 \\ 0.448 & 0.008 & 0.445 & 1.000 & 0.064 \\ 0.163 & 0.827 & 0.728 & 0.064 & 1.000 \end{pmatrix}.$$

When the C-vine and D-vine both fail, the B1 and B2 partial correlation vines do not fail to have non-negative partial correlation vines.

$d = 6$.

The estimated failure rate for C-vine is: 257/1100 = 0.234.

The estimated failure rate for D-vine is: 27/1100 = 0.025.

The estimated failure rate for C-vine but not D-vine is: 245/1100 = 0.223.

The estimated failure rate for D-vine but not C-vine is: 15/1100 = 0.014.

The estimated failure rate for simultaneous D-vine and C-vine is: 12/1100 = 0.010.

For cases with D-vine and C-vine failures, there were non-negative partial correlation vines with many of the other 62 regular vines.

$d = 7$.

The estimated failure rate for C-vine is: 359/797 = 0.45.

The estimated failure rate for D-vine is: 76/797 = 0.10.

The estimated failure rate for simultaneous D-vine and C-vine is: 43/797 = 0.05.

Conjecture

One can expect the failure rate of C-vines and D-vines to increase as d increases, because there is an increasing number of non-boundary vines as d increases. The rate is estimable via simulations up to $d = 7$. It is necessary to generate 10^6 to 10^8 positive matrices with 1 on the diagonal in order to assess the failure rate.

We have not proved for $d \geq 4$, that for any $\boldsymbol{R} \in \mathcal{R}_d^+$ there are partial correlation vine representations where all partial correlations are non-negative. So this is left as an open problem.

From results in Appendix A.2, one should put small positive correlations in the first tree of the partial correlation vine and have some smaller partial correlations in the second tree etc. A greedy sequential spanning tree algorithm that finds trees with smallest partial correlation and satisfying the proximity condition of vines is not guaranteed to work as this can lead to a negative partial correlation in the last tree.

A.4. Cholesky matrix via partial correlation C-vine

Let $\boldsymbol{R}$ be an $d \times d$ positive definite correlation matrix. The Cholesky decomposition matrix of $\boldsymbol{R}$ can be expressed in terms of the partial correlations as follows:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ \rho_{12} & \sqrt{1-\rho_{12}^2} & 0 & \cdots & \cdots & 0 \\ \rho_{13} & \rho_{23;1}\sqrt{1-\rho_{13}^2} & \sqrt{1-\rho_{23;1}^2}\sqrt{1-\rho_{13}^2} & \cdots & \cdots & 0 \\ \rho_{14} & \rho_{24;1}\sqrt{1-\rho_{14}^2} & \rho_{34;12}\sqrt{1-\rho_{24;1}^2}\sqrt{1-\rho_{14}^2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{1d} & \rho_{2d;1}\sqrt{1-\rho_{1d}^2} & \rho_{3d;12}\sqrt{1-\rho_{2d;1}^2}\sqrt{1-\rho_{1d}^2} & \cdots & \cdots & \sqrt{(1-\rho_{d-1,d;1\dots d-2}^2)\cdots(1-\rho_{2d;1}^2)(1-\rho_{1d}^2)} \end{pmatrix}.$$

The $(j,1)$ element is ρ_{1j} for $2 \le j \le d$. The (i,i) element, for $2 \le i \le d$, is

$$\prod_{k=1}^{i-1}\sqrt{1-\rho_{k,i;1\dots k-1}^2}.$$

The (i,j) element, for $2 \le j < i \le d$, is:

$$\rho_{ji;1\dots j-1}\prod_{k=1}^{j-1}\sqrt{1-\rho_{k,i;1\dots k-1}^2}.$$

The parameter vector representation in the Cholesky matrix with

$$\rho_{12},\dots,\rho_{1d};\quad \rho_{23;1},\dots,\rho_{2d;1};\quad \cdots\quad ;\rho_{j,j+1;1\dots j-1},\dots,\rho_{j,d;1\dots j-1};\cdots;\rho_{d-1,d;1\dots d-2}$$

corresponds to the partial correlation C-vine. Hence the Cholesky matrix based method in [19] is equivalent, but their hyperspherical parameterization is not useful for the extensions considered in this article.

A.5. Covariance between elements of the correlation matrix for C-vine algorithm before permutations

Correlations in the first row are independent. For correlations in the same row $2 \le j \le d-2$ for $j+1 \le k < \ell \le d$ the covariance is:

$$\mathrm{Cov}(R_{jk}, R_{j\ell}) = \sum_{i=1}^{j-1} \mu_i E_{i\ell}^* + E_{j\ell}^* - \mathrm{E}(R_{jk})\mathrm{E}(R_{j\ell}),$$

where

$$E_{1\ell}^* = \mu_1 v_1 + \sum_{t=2}^{\ell-1} \mu_t^2 \left(\prod_{s=1}^{t-1}\gamma_s^2\right)\frac{\eta_1}{\gamma_1} + \mu_\ell\left(\prod_{s=1}^{\ell-1}\gamma_s^2\right)\frac{\eta_1}{\gamma_1}.$$

For $1 < i < \ell$

$$\begin{aligned} E_{i\ell}^* \;=\; & \mu_i\left(\prod_{s=1}^{i-1}\gamma_s^2\right)\left[\mu_1\frac{\eta_1}{\gamma_1} + \sum_{t=2}^{i-1}\mu_t\frac{\eta_t}{\gamma_t}\left(\prod_{s=1}^{t-1}(1-v_s)\right) + v_i\left(\prod_{s=1}^{i-1}(1-v_s)\right)\right] \\ & + \frac{\eta_i}{\gamma_i}\left(\prod_{s=1}^{i-1}(1-v_s)\right)\sum_{t=i+1}^{\ell-1}\mu_t^2\left(\prod_{s=1}^{t-1}\gamma_s^2\right) + \mu_\ell\left(\prod_{s=1}^{\ell-1}\gamma_s^2\right), \end{aligned}$$

and when $i = \ell$

$$E_{\ell\ell}^* \;=\; \mu_\ell\left(\prod_{s=1}^{\ell-1}\gamma_s^2\right)\left[\mu_1\frac{\eta_1}{\gamma_1} + \sum_{t=2}^{\ell-1}\mu_t\frac{\eta_t}{\gamma_t}\left(\prod_{s=1}^{t-1}(1-v_s)\right) + \mu_\ell\left(\prod_{s=1}^{\ell-1}(1-v_s)\right)\right].$$

Correlations in different rows j and ℓ, $1 \le j < \ell \le d-1$ have covariance equal to:

$$\mathrm{Cov}(R_{j\ell}, R_{\ell k}) = \mathrm{Cov}(R_{jk}, R_{\ell k}) = \sum_{i=1}^{j-1}\mu_i E_{i\ell}^* + E_{j\ell}^* - \mathrm{E}(R_{j\ell})\mathrm{E}(R_{\ell k}).$$

The remaining covariances are zero.

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