Wave reflections in a semi infinite string due to nonlinear energy sinks at the boundary

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1 Abstract

Vibrations or oscillations can be caused in overhead cable lines or bridge cables due to strong rain and winds, making the structure unstable.These vibrations can be mathematically described as a string like initial boundary value problem with non-classical boundary conditions. In this thesis, we consider a nonlinear attachment at the boundary which consists of a mass, nonlinear spring and a damper attached to a semi infinite string. In particular, we consider a weak nonlinearity and damping. In this study we used the D'Alembert solution and the multiple time scales perturbation method to obtain bounded solutions of the initial boundary value problem. We assumed travelling wave initial conditions, and obtained special cases and conducted detuning around these special cases to further study the reflected waves at the boundary and the stability of our solutions. Our main objective is to study the reflection of the incident wave on the boundary and compute how much energy is dissipated at the boundary due to the weak dissipative forces present at the boundary

Figure 1: Fred Hartman bridge, Houston, Texas

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2 Introduction

This paper explores the domain of non-linear energy sinks. More specifically, we study the vibrations on a string attached to a spring mass damper system. The spring in the system under consideration exerts a non-linear force (cubic non-linearity) on the mass. We would like to study how much energy is dissipated at the boundary due to the nonlinear attachment at $x = 0$ and how this impacts the amplitude of the reflected waves in the string.

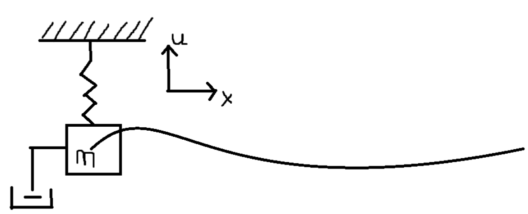
This research has many applications in the real world. One such example is that of overhead cables. External forces such as wind cause the cables to oscillate, thus making them unstable, as they cannot cope with those weather conditions. Usually in such cases, dampers(oil or mass) are attached to the cables to make the system more stable. Due to the presence of a non-linearity and a damper in our system, energy can be taken out of the system, thus making their amplitudes of oscillation reduce.

Figure (1) shows an image I took of the Fred Hartman bridge, located in Houston, Texas. A lot of research was conducted on this bridge by the universities in the US. Soon after its construction, a lot of the users noticed high amplitude oscillations of the cables attached to the bridge. This was caused due to rain and winds.

My research is guided keeping these applications in mind. The report is structured as follows: In Section 3, we formulate the problem and make it dimensionless in order to reduce it to a mathematical system which we can work with, Section 4 discusses an analytic approach used to obtain a solution to our initial boundary value problem, Section 5 deals with obtaining the solutions for the leading order equations, some special cases and the detuned cases and the energy decay of the system, Section 6 discusses some results and avenues of future research and the last section mentions all the literature and references used during the course of the research.

3 Formulation of the problem

Figure 2: Mass-spring-string-damper system



In this paper we study the vibrations in a semi infinite string attached to a non-linear spring mass damper system. To begin with, gravity and other external forces are ignored, thus obtaining a 1-D wave equation with initial and boundary conditions as given below,

$$(\frac{\partial^2 u}{\partial t^2}) = c^2(\frac{\partial^2 u}{\partial x^2}); x > 0, t > 0 \tag{1}$$

with initial conditions,

$$\begin{cases} u(x,0) = f(x), x > 0 \\ \frac{\partial u}{\partial t} = g(x), x > 0 \end{cases} \tag{2}$$

and boundary condition,

$$m\frac{\partial^2 u}{\partial t^2}(0,t) = T\frac{\partial u}{\partial x}(0,t) - \alpha\frac{\partial u}{\partial t}(0,t) - au(0,t) - bu^3(0,t); m > 0, t > 0. \tag{3}$$

Here, u is the vertical displacement in meters, c is the wave velocity, x is the horizontal displacement, m is mass, t is the time in seconds, f is the initial vertical displacement at $t = 0$, g is the initial velocity of the wave, T is the tension in the string, α is the damping coefficient and a and b are the coefficients of the linear and non-linear spring force. Next, we must make the above variables dimensionless before solving the equations. This is done to facilitate scaling up the problem to real conditions and to ease up the process of identifying when to apply familiar mathematical techniques. Before we make the variables dimensionless, it is useful to identify the dimensions of all the quantities present in our problem.

$$[\rho] = [\frac{kg}{m^3}]$$

$$[u] = [m]$$

$$[x] = [m]$$

$$[t] = [s]$$

$$[T] = [\frac{kgm}{s^2}]$$

$$[A] = [m^2]$$

$$[c] = [\frac{m}{s}]$$

$$[M] = [kg]$$

$$[\alpha] : damping\ factor = [\frac{kg}{s}]$$

$$[a] : linear\ coefficient\ in\ spring\ force\ = [\frac{kg}{s^2}]$$

$$[b] : cubic\ coefficient\ in\ spring\ force\ = [\frac{kg}{m^2 s^2}]$$

In order to do this, let us consider the following transformations,

$$x = \tilde{x}L$$

$$u = \tilde{u}L$$

$$t = \tilde{t}\frac{c}{L}$$

Substituting these transformations into the initial boundary value problem, we get the following IBVP,

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} = \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} \tag{4}$$

$$\begin{cases} \tilde{u}(\tilde{x}, 0) = \tilde{f}(\tilde{x}), \tilde{x} > 0 \\ \frac{\partial \tilde{u}}{\partial \tilde{t}} = \tilde{g}(\tilde{x}), \tilde{x} > 0 \end{cases} \tag{5}$$

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2}(0,t) = A\frac{\partial \tilde{u}}{\partial \tilde{x}}(0,t) - B\frac{\partial \tilde{u}}{\partial \tilde{t}}(0,t) - C\tilde{u}(0,t) - D\tilde{u}^3(0,t), m > 0 \tag{6}$$

where,

$$A = \frac{TL}{mc^2}$$

$$B = \frac{\alpha L}{cm}$$

$$C = \frac{aL^2}{mc^2}$$

$$D = \frac{bL}{mc^2}$$

Each of the parameters, A,B,C and D are dimensionless and this has been verified.

Next, we attempt to reduce the number of parameters in our problem in order to reduce the number of numerical simulations we need to run. Fewer the parameters, the more easier it is to control the numerical simulations. We can do so by assuming the following transformation and trying to eliminate 1 or 2 parameters(ideally keep the parameters corresponding to the damping and the non-linear spring force),

$$\tilde{x} = \lambda_1 x$$

$$\tilde{t} = \lambda_2 t$$

$$\tilde{u} = \lambda_3 u$$

When we substitute these transformed variables into (4), we get that,

$$\lambda_1 = \lambda_2$$

Upon substitution into (6) we get,

$$\frac{\lambda_3}{\lambda_1^2} u_{tt} = A\frac{\lambda_3}{\lambda_1} u_x - B\frac{\lambda_3}{\lambda_1} u_t - C\lambda_3 u - D\lambda_3^3 u^3$$

Dividing both sides by λ_3 and multiplying both sides with λ_1^2, we get,

$$u_{tt} = A\lambda_1 u_x - B\lambda_1 u_t - C\lambda_1^2 u - D\lambda_3^2\lambda_1^2 u^3 \tag{7}$$

If we set $\lambda_1 = \frac{1}{A}$, the coefficient of u_x is reduced to 1. Substituting this into (7) we get,

$$u_{tt} = u_x - \frac{B}{A}u_t - \frac{C}{A^2}u - \frac{D}{A^2}\lambda_3^2 u^3$$

$$\implies u_{tt} = \left[u_x - \frac{C}{A^2}u\right] - \frac{B}{A}u_t - \frac{D}{A^2}\lambda_3^2 u^3$$

If we take the damping and the cubic non-linearity to have the same coefficient, we get that $\lambda_3 = \pm\sqrt{\frac{AB}{D}}$. Depending on whether D takes positive/negative values the spring is either a hardening/softening spring.

When $D > 0 : \lambda_3 = \sqrt{\frac{AB}{D}} \implies$ the boundary condition will have the term $-\hat{q}u^3$.

When $D < 0 : \lambda_3 = \sqrt{\frac{AB}{-D}} \implies$ the boundary condition will have the term $+\hat{q}u^3$.

The damping and cubic term have a common coefficient $\frac{B}{A}$ and the linear spring force has the coefficient $\frac{C}{A^2}$. For convinience, let, $\hat{p} = \frac{C}{A^2}$ and $\hat{q} = \frac{B}{A}$.

The IBVP reduces to,

$$\begin{cases} PDE: u_{tt} - u_{xx} = 0; x > 0; t > 0 \\ IC: u(x,0) = f(x); x > 0; t = 0 \\ IC: u_t(x,0) = g(x); x > 0; t = 0 \\ BC: u_{tt}(0,t) = [u_x(0,t) - \hat{p}u(0,t)] - \hat{q}[u_t(0,t) \pm u^3(0,t)]; x = 0; t > 0 \end{cases}$$

where $\hat{p} > 0$ and $\hat{q}$ is very small(weak damping and non-linearity) and $\hat{q} > 0$.

For our analysis, we will consider only a hardening spring. So our IBVP under consideration is,

$$\begin{cases} PDE: u_{tt} - u_{xx} = 0; x > 0; t > 0 \\ IC: u(x,0) = f(x); x > 0; t = 0 \\ IC: u_t(x,0) = g(x); x > 0; t = 0 \\ BC: u_{tt}(0,t) = [u_x(0,t) - \hat{p}u(0,t)] - \hat{q}[u_t(0,t) + u^3(0,t)]; x = 0; t > 0 \end{cases} \tag{8}$$

4 Methods of analysis

The multiple time scales method is a global perturbation method. It is usually used in situations where the mechanical system consists of weak dissipating forces. Since the dissipating forces are weak, we may not be able to observe the effect they have on a short time scale, but become very significant on a longer time scale.In our system, we have a weak damping and non-linearity, thus making multiple time scales an appropriate fit. Any other perturbation method may result in resonance which gives rise to secular terms. Secular terms are present in the ordered equations which become unbounded as time passes, thus making the solution and system unstable. We can identify them as functions of the solutions to the homogeneous system which are present in the right hand side of the non-homogeneous system. Eliminating the secular terms will ensure that the solution remains bounded. Physically, this means that, when oscillations are introduced into our system through the string, the reflected waves would have a lower/or equal to amplitude than the incoming wave as time passes.
In order to do this, we must show that the effect of the weak dissipating forces can be seen in the energy of the system. If the energy of the system reduces, it means that our mechanical system has gone from a highly excited state to a much more stable state, which is more desirable.

5 Analysis of the problem

After the original problem has been made dimensionless and the number of parameters have been reduced, we get (8) given by,

$$\begin{cases} PDE: u_{tt} - u_{xx} = 0; x > 0; t > 0 \\ IC: u(x,0) = f(x); x > 0; t = 0 \\ IC: u_t(x,0) = g(x); x > 0; t = 0 \\ BC: u_{tt}(0,t) = [u_x(0,t) - \hat{p}u(0,t)] - \hat{q}[u_t(0,t) + u^3(0,t)]; x = 0; t > 0 \end{cases}$$

This is the one dimensional wave equation. The D'Alembert solution is a well known solution for the one dimensional wave equation with classical boundary conditions. In (1), the author considers the D'Alembert solution to construct an explicit solution of the boundary value problem. We can assume that the D'Alembert solution satisfies the PDE and the Initial conditions as we do not have classical boundary conditions.

The D'Alembert solution is given by,

(1) For $(x-t)>0$,

$$u(x,t)=\frac{f(x+t)}{2}+\frac{f(x-t)}{2}+\frac{\int_{x-t}^{x+t}g(s)ds}{2} \tag{9}$$

(2) For $(x-t)<0$,

$$u(x,t)=\frac{f(x+t)}{2}+\frac{f(x-t)}{2}+\frac{\int_{0}^{x+t}g(s)ds}{2}+\frac{\int_{x-t}^{0}g(s)ds}{2} \tag{10}$$

The formula for $(x-t)<0$ is not defined for negative arguments, as the initial conditions are only defined for positive values, as we can see from (8). So, for (10), $u(x,t)$ can be expressed as,

$$u(x,t)=h(x+t)+z(t-x) \tag{11}$$

where, $h(x+t)=\frac{f(x+t)}{2}+\frac{\int_{0}^{x+t}g(s)ds}{2}$ is a known function and $z(t-x)=\frac{f(x-t)}{2}+\frac{\int_{x-t}^{0}g(s)ds}{2}$ is an unknown function.We can substitute (11) into the boundary condition in (8).

The boundary condition is given by,

$$u_{tt}(0,t)=[u_x(0,t)-\hat{p}u(0,t)]-\hat{q}[u_t(0,t)+u^3(0,t)] \tag{12}$$

Upon substitution, we get the following expressions that need to be put into the BC,

$$u_{tt}(0,t)=h''(t)+z''(t)$$

$$u_x(0,t)=h'(t)-z'(t)$$

$$\hat{p}u(0,t)=\hat{p}h(t)+\hat{p}z(t)$$

$$\hat{q}u_t(0,t)=\hat{q}[h'(t)+z'(t)]$$

$$\hat{q}u^3(0,t)=\hat{q}(h(t)+z(t))^3=\hat{q}[h(t)^3+z(t)^3+3h(t)z(t)^2+3z(t)h(t)^2]$$

When we substitute the above expressions in (12) and rearrange the terms by putting all the known functions on the RHS and the unknown functions on the LHS, we get,

$$z''(t)+z'(t)+\hat{p}z(t)+\hat{q}z'(t)+\hat{q}z(t)^3+3\hat{q}h(t)z(t)^2+3\hat{q}z(t)h(t)^2=-h''(t)+h'(t)-\hat{p}h(t)-\hat{q}h'(t)-\hat{q}h(t)^3 \tag{13}$$

5.1 Initial conditions

We assume the following initial conditions,

$$\begin{cases} u(x,0)=f(x)=Asin(\omega x), x>0 \\ u_t(x,0)=g(x)=A\omega cos(\omega x), x>0 \end{cases} \tag{14}$$

We know that, $h(x+t)=\frac{f(x+t)}{2}+\frac{\int_{0}^{x+t}g(s)ds}{2}$ and $z(t-x)=\frac{f(x-t)}{2}+\frac{\int_{x-t}^{0}g(s)ds}{2}$, when $x=0$, this reduces to,

$$h(t) = \frac{f(t)}{2} + \frac{\int_0^t g(s)ds}{2} \tag{15}$$

$$z(t) = \frac{f(-t)}{2} + \frac{\int_{-t}^0 g(s)ds}{2} \tag{16}$$

When we substitute (14) in (15) , we get,

$$h(t) = \frac{Asin(\omega t)}{2} + \frac{A\omega}{2}\int_0^t cos(\omega s)ds = Asin(\omega t) \tag{17}$$

As a result we also get that,

$$h'(t) = A\omega cos(\omega t)$$

and

$$h''(t) = -A\omega^2 sin(\omega t)$$

From a physical standpoint, we know that the reflected wave is given by $z(-t)$. We know that at $t = 0$, no reflected wave exists. This implies that at $t = 0$, there is no position or velocity corresponding to the reflected wave. Hence, we get that,

$$\begin{cases} z(0) = 0 \\ z'(0) = 0 \end{cases}$$

We find that we reduce our problem from a PDE to an ODE, in terms of $z(t)$. The initial value problem is,

$$\begin{cases} z''(t) + z'(t) + \hat{p}z(t) + \hat{q}z'(t) + \hat{q}z(t)^3 + 3\hat{q}h(t)z(t)^2 + 3\hat{q}z(t)h(t)^2 = -h''(t) + h'(t) - \hat{p}h(t) - \hat{q}h'(t) - \hat{q}h(t)^3 \\ z(0) = 0, z'(0) = 0 \end{cases} \tag{18}$$

where, $h(t) = Asin(\omega t)$.

5.2 Method of Multiple scales

Our mechanical system has reduced to the analysis of equation (18). In (2), the author calculated the reflection coefficients first and then calculated the natural frequency of the system. However, we will consider the natural frequency of the system and obtain the reflected wave using the multiple time scales method to avoid resonance. For this initial value problem, we will use the multiple time scales method. We consider two separate time scales, one regular time scale of $O(1)$ and one slower time scale of $O(\frac{1}{\hat{q}})$.

We begin by introducing a slow time scale, τ, such that,

$$\tau = \hat{q}t \tag{19}$$

This gives us an asymptotic expansion of $z(t)$ as follows,

$$z(t) = \tilde{z}(t,\tau) = [z_0(t,\tau) + \hat{q}z_1(t,\tau) + \hat{q}^2 z_2(t,\tau) + ...] \tag{20}$$

Above, $z(t)$ is expressed as a function of two time scales, t and τ. The derivatives with respect to t are given by,

$$\begin{cases} \frac{d}{dt} = \frac{\partial}{\partial t} + \hat{q}\frac{\partial}{\partial \tau} \\ \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\hat{q}\frac{\partial^2}{\partial t \partial \tau} + \hat{q}^2\frac{\partial^2}{\partial \tau^2} \end{cases} \tag{21}$$

Next, we must substitute (21) into (18) in terms of the unknown function, z.

$$z''(t) = \left[\frac{\partial^2 z_o}{\partial t^2} + 2\hat{q}\frac{\partial^2 z_0}{\partial t \partial \tau} + \hat{q}^2\frac{\partial^2 z_0}{\partial \tau^2}\right] + \hat{q}\left[\frac{\partial^2 z_1}{\partial t^2} + 2\hat{q}\frac{\partial^2 z_1}{\partial t \partial \tau} + \hat{q}^2\frac{\partial^2 z_1}{\partial \tau^2}\right] + ...$$

$$\implies z''(t) = \frac{\partial^2 z_o}{\partial t^2} + \hat{q}\left[2\frac{\partial^2 z_0}{\partial t \partial \tau} + \frac{\partial^2 z_1}{\partial t^2}\right] + O(\hat{q}^2)$$

$$z'(t) = \left[\frac{\partial z_0}{\partial t} + \hat{q}\frac{\partial z_0}{\partial \tau}\right] + \hat{q}\left[\frac{\partial z_1}{\partial t} + \hat{q}\frac{\partial z_1}{\partial \tau}\right] + ...$$

$$\implies z'(t) = \frac{\partial z_0}{\partial t} + \hat{q}\left[\frac{\partial z_0}{\partial \tau} + \frac{\partial z_1}{\partial t}\right] + O(\hat{q}^2)$$

Substituting these expressions in (18), we get,

$$\frac{\partial^2 z_o}{\partial t^2} + \hat{q}\left[2\frac{\partial^2 z_0}{\partial t \partial \tau} + \frac{\partial^2 z_1}{\partial t^2}\right] + O(\hat{q}^2)$$

$$+\frac{\partial z_0}{\partial t} + \hat{q}\left[\frac{\partial z_0}{\partial \tau} + \frac{\partial z_1}{\partial t}\right] + O(\hat{q}^2) + \hat{p}\left[z_0 + \hat{q}z_1 + O(\hat{q}^2) + \hat{q}\left[\frac{\partial z_0}{\partial t} + O(\hat{q}^2)\right]\right]$$

$$+\hat{q}\left[z_0^3 + O(\hat{q}^2)\right] + \hat{q}\left[3h^2(t)z_0 + O(\hat{q}^2)\right] + \hat{q}\left[3h(t)z_0^2 + O(\hat{q}^2)\right] = -h''(t) + h'(t) - \hat{p}h(t) - \hat{q}h'(t) - \hat{q}h(t)^3$$

$$z(0) = 0 \implies z_0(0,0) + \hat{q}z_1(0,0) + ... = 0$$

$$z'(0) = 0 \implies \left((\frac{\partial}{\partial t} + \hat{q}\frac{\partial}{\partial \tau})(z_0(0,0) + \hat{q}z_1(0,0) + ...\right) = 0$$

Next, we collect the like powers of $\hat{q}$. The leading order IVP is given below, followed by the $O(\hat{q})$ IVP.

$$\begin{cases} O(1) : \frac{\partial^2 z_o}{\partial t^2} + \frac{\partial z_0}{\partial t} + \hat{p}z_0 = -h''(t) + h'(t) - \hat{p}h(t) = h_0(t) \\ z_0(0,0) = 0 \\ \frac{\partial z_0}{\partial t}(0,0) = 0 \end{cases} \tag{22}$$

$$\begin{cases} O(\hat{q}) : \frac{\partial^2 z_1}{\partial t^2} + \frac{\partial z_1}{\partial t} + \hat{p}z_1 + 2\frac{\partial^2 z_0}{\partial t \partial \tau} + \frac{\partial z_0}{\partial \tau} + \frac{\partial z_0}{\partial t} + z_0^3 + 3h^2(t)z_0 + 3h(t)z_0^2 = -h'(t) - h^3(t) = h_1(t) \\ z_1(0,0) = 0 \\ \left(\frac{\partial z_1}{\partial t}(0,0) + \frac{\partial z_0}{\partial \tau}(0,0)\right) = 0 \end{cases} \tag{23}$$

5.3 Leading order equation

We will first solve the leading order equation in the initial value problem, (22).

$$\frac{\partial^2 z_o}{\partial t^2} + \frac{\partial z_0}{\partial t} + \hat{p}z_0 = -h''(t) + h'(t) - \hat{p}h(t) = h_0(t) \tag{24}$$

Equation (24) is a non-homogeneous ODE. It has a fundamental set of solutions and a particular solution, denoted as, $z_0 = z_{0,c} + z_{0,p}$.

The fundamental set $z_{0,c}$ are solutions to the homogeneous ODE, i.e., $\frac{\partial^2 z_o}{\partial t^2} + \frac{\partial z_0}{\partial t} + \hat{p}z_0 = 0$ and the particular solution $z_{0,p}$ is a solution to the non-homogeneous problem, i.e., $\frac{\partial^2 z_o}{\partial t^2} + \frac{\partial z_0}{\partial t} + \hat{p}z_0 = h_0(t)$, where $h_0(t) = (A\omega)cos(\omega t) + (A\omega^2 - A\hat{p})sin(\omega t)$.

5.3.1 Particular solution

We start by calculating the particular solution. The RHS of (24) is $h_0(t) = -h''(t) + h'(t) - \hat{p}h(t)$. From (17), we can see that,

$$h_0(t) = (A\omega)cos(\omega t) + (A\omega^2 - A\hat{p})sin(\omega t)$$

If we assume the particular solution to be of the form $z_{0,p} = k_1 cos(\omega t) + k_2 sin(\omega t)$ and substitute it in the LHS of equation (24), then we get sine and cosine functions. In the RHS we already have sine and cosine terms, so we have to determine the coefficients of these expressions.

We have that $z_{0,p} = k_1 cos(\omega t) + k_2 sin(\omega t)$.

So, $z'_{0,p} = -k_1\omega sin(\omega t) + k_2\omega cos(\omega t)$ and $z''_{0,p} = -k_1\omega^2 cos(\omega t) - k_2\omega^2 sin(\omega t)$.

We get the following set of equations that need to solved simultaneously to determine k_1 and k_2.

$$\begin{cases} (\hat{p}k_1 + k_2\omega - k_1\omega^2) = A\omega \\ (\hat{p}k_2 - k_1\omega - k_2\omega^2) = A\omega^2 - A\hat{p} \end{cases} \tag{25}$$

From (25), we have,

$$\implies k_2\omega + (\hat{p} - \omega^2)k_1 = A\omega$$

$$\implies k_2 = \left[A - (\frac{\hat{p} - \omega^2}{\omega})k_1\right]$$

$$-k_1\omega + (\hat{p} - \omega^2)\left[A - (\frac{\hat{p} - \omega^2}{\omega})k_1\right] = A\omega^2 - A\hat{p}$$

$$\implies -k_1\omega + \hat{p}A - \frac{\hat{p}^2 k_1}{\omega} + 2\hat{p}k_1\omega - A\omega^2 - k_1\omega^3 = A\omega^2 - A\hat{p}$$

$$\implies -k_1\omega - \frac{\hat{p}^2 k_1}{\omega} + 2\hat{p}k_1\omega - k_1\omega^3 = 2(A\omega^2 - A\hat{p})$$

$$k_1\left[-\omega^2 - (\omega^4 - 2\hat{p}\omega^2 + \hat{p}^2)\right] = 2A\omega(\omega^2 - \hat{p})$$

$$k_1(-\omega^2 - (\omega^2 - \hat{p})^2) = 2A\omega(\omega^2 - \hat{p})$$
$$\implies k_1 = \left[\frac{-2A\omega(\omega^2 - \hat{p})}{\omega^2 + (\omega^2 - \hat{p})^2}\right]$$

Now that we have an expression for k_1, the expression for k_2 can easily be found upon substitution into $k_2 = \left[A - (\frac{\hat{p}-\omega^2}{\omega})k_1\right]$. We get k_2 as,

$$k_2 = \left[\frac{A(\omega^2 - (\omega^2 - \hat{p})^2)}{\omega^2 + (\omega^2 - \hat{p})^2}\right]$$

Therefore, we have found the particular solution of (22), given as ,

$$z_{0,P}(t) = \left[\frac{-2A\omega(\omega^2 - \hat{p})}{\omega^2 + (\omega^2 - \hat{p})^2}\right]cos(\omega t) + \left[\frac{A(\omega^2 - (\omega^2 - \hat{p})^2)}{\omega^2 + (\omega^2 - \hat{p})^2}\right]sin(\omega t) \tag{26}$$

5.3.2 Fundamental solution

Now that we have computed the particular solution of the leading order equation. We now have to compute the fundamental sets of solution of the leading order equation, i.e., $z_{0,c}(t,\tau)$.

Our leading order equation can be written as,

$$z_0'' + z_0' + \hat{p}z_0 = 0$$

The characteristic form of the equation is,

$$r^2 + r + \hat{p} = 0$$

This is a regular quadratic equation,whose roots are, $r_{1,2} = \frac{-1\pm\sqrt{1-4\hat{p}}}{2}$. The nature of the roots depends on the value of $\sqrt{1-4\hat{p}}$. We have three scenarios,

The solution of the ODE is of the form,

$$z_0(t,\tau) = A_0(\tau)z_{0,c_1}(t) + B_0(\tau)z_{0,c_2}(t)$$

where,

$$\begin{cases} Case1 : \left(\hat{p} < \frac{1}{4}\right) & : z_{0,c_1}(t) = e^{r_1 t}\,; z_{0,c_2}(t) = e^{r_2 t} \\ Case2 : \left(\hat{p} = \frac{1}{4}\right) & : z_{0,c_1}(t) = e^{rt}\,; z_{0,c_2}(t) = te^{rt} \\ Case3 : \left(\hat{p} > \frac{1}{4}\right) & : z_{0,c_1}(t) = e^{\lambda t}cos(\mu t)\,; z_{0,c_2}(t) = e^{\lambda t}sin(\mu t) \end{cases} \tag{27}$$

where, $r_1 = \frac{-1+\sqrt{1-4\hat{p}}}{2}$, $r_2 = \frac{-1-\sqrt{1-4\hat{p}}}{2}$, $r = -\frac{1}{2}$, $\lambda = \frac{-1}{2}$ and $\mu = \frac{\sqrt{4\hat{p}-1}}{2}$.

Since $z_0(t,\tau)$ depends on both slow and fast timescales, i.e., τ and t. We see that functions z_{0,c_i} are dependent on t and $A_0(\tau)$ and $B_0(\tau)$ are functions of τ, i.e., the slow time scale.

In the next three sections we proceed with the detailed stability analysis of each of the three solutions of the leading order equation.

The three cases are as follows:

$$\begin{cases} (i) z_0(t,\tau) = A_0(\tau)e^{r_1 t} + B_0(\tau)e^{r_2 t} + \left[\frac{-2A\omega(\omega^2-\hat{p})}{\omega^2+(\omega^2-\hat{p})^2}\right] cos(\omega t) + \left[\frac{A(\omega^2-(\omega^2-\hat{p})^2)}{\omega^2+(\omega^2-\hat{p})^2}\right] sin(\omega t), \\ where, r_1 = \frac{-1+\sqrt{1-4\hat{p}}}{2}, r_2 = \frac{-1-\sqrt{1-4\hat{p}}}{2} \\ \\ (ii) z_0(t,\tau) = A_0(\tau)e^{rt} + B_0(\tau)te^{rt} + \left[\frac{-2A\omega(\omega^2-\hat{p})}{\omega^2+(\omega^2-\hat{p})^2}\right] cos(\omega t) + \left[\frac{A(\omega^2-(\omega^2-\hat{p})^2)}{\omega^2+(\omega^2-\hat{p})^2}\right] sin(\omega t), \\ where,\ r = -\frac{1}{2} \\ \\ (iii) z_0(t,\tau) = A_0(\tau)e^{\lambda t} cos(\mu t) + B_0(\tau)e^{\lambda t} sin(\mu t) + \left[\frac{-2A\omega(\omega^2-\hat{p})}{\omega^2+(\omega^2-\hat{p})^2}\right] cos(\omega t) + \left[\frac{A(\omega^2-(\omega^2-\hat{p})^2)}{\omega^2+(\omega^2-\hat{p})^2}\right] sin(\omega t), \\ where, \lambda = \frac{-1}{2}\ and\ \mu = \frac{\sqrt{4\hat{p}-1}}{2} \end{cases}$$

5.4 Case 1 of leading order equation

Now we consider Case 1 of the leading order equation, i.e., $O(1)$ equation.

In order to find the constants involved, we need to solve for $z_0(t,\tau)$ using the initial conditions $z_0(0,0) = 0$ and $\frac{\partial z_0}{\partial t}(0,0) = 0$.

$$z_0(t,\tau) = A_0(\tau)e^{r_1 t} + B_0(\tau)e^{r_2 t} + \left[\frac{-2A\omega(\omega^2-\hat{p})}{\omega^2+(\omega^2-\hat{p})^2}\right] cos(\omega t) + \left[\frac{A(\omega^2-(\omega^2-\hat{p})^2)}{\omega^2+(\omega^2-\hat{p})^2}\right] sin(\omega t)$$

For convenience, let's write $z_0(t,\tau)$ as,

$$z_0(t,\tau) = A_0(\tau)e^{r_1 t} + B_0(\tau)e^{r_2 t} + k_1 cos(\omega t) + k_2 sin(\omega t) \tag{28}$$

When $t = 0 \implies \tau = 0$,

$$z_0(0,0) = A_0 + B_0 + k_1 = 0$$

$$z_0'(0,0) = A_0 r_1 + B_0 r_2 + k_2\omega = 0$$

Simultaneously solving both the equations we get,

$$B_0 = -k_1 - A_0$$

$$\implies A_0 r_1 + (-k_1 - A_0) r_2 + k_2\omega = 0$$

$$\implies A_0(r_1 - r_2) - k_1 r_2 + k_2\omega = 0$$

$$\begin{cases} A_0(0) = \left(\frac{k_1 r_2 - k_2\omega}{r_1 - r_2}\right) \\ \\ B_0(0) = \left(\frac{k_2\omega - k_1 r_1}{r_1 - r_2}\right) \end{cases} \tag{29}$$

We have now obtained the initial conditions for the case 1. Next, we must make sure that the solution stays bounded. In order to check whether this, we must check the presence of secular terms in the RHS of the $O(\hat{q})$ equation.

$$O(\hat{q}) : \frac{\partial^2 z_1}{\partial t^2} + \frac{\partial z_1}{\partial t} + \hat{p} z_1 + 2\frac{\partial^2 z_0}{\partial t \partial \tau} + \frac{\partial z_0}{\partial \tau} + \frac{\partial z_0}{\partial t} + z_0^3 + 3h^2(t) z_0 + 3h(t) z_0^2 = -h'(t) - h^3(t)$$

$$O(\hat{q}) : z_1'' + z_1' + \hat{p} z_1 = \left(-2\frac{\partial^2 z_0}{\partial t \partial \tau} - \frac{\partial z_0}{\partial \tau} - \frac{\partial z_0}{\partial t} - z_0^3 - 3h^2(t) z_0 - 3h(t) z_0^2 - h'(t) - h^3(t) \right)$$

The homogeneous form of the $O(\hat{q})$ equation is, $z_1'' + z_1' + \hat{p} z_1 = 0$. This ODE has the same fundamental solutions as the $O(1)$ equation. We need to make sure that the RHS of the $O(\hat{q})$ equation does not contain secular terms. In order to do this, we need to choose $A_0(\tau)$ and $B_0(\tau)$, such that, the coefficients of the secular terms become zero. We start by substituting $z_0(t, \tau)$ in the RHS and finding the conditions for eliminating secular terms.

$$O(\hat{q}) : z_1'' + z_1' + \hat{p} z_1 = -\left(2\frac{\partial^2 z_0}{\partial t \partial \tau} + \frac{\partial z_0}{\partial \tau} + \frac{\partial z_0}{\partial t} + z_0^3 + 3h^2(t) z_0 + 3h(t) z_0^2 + h'(t) + h^3(t) \right) \quad (30)$$

We consider each of the terms on the RHS and state whether the term contains secular terms or not. Please note that at this stage we are not considering special values of $\hat{p}$ for which we obtain additional secular terms. We will expand on this in a later stage.

- $h'(t) = A\omega cos(\omega t)$ - Not secular
- $h^3(t) = A^3 sin^3(\omega t)$ - Not secular
- $3h^2(t) z_0(t, \tau) = 3A^2 sin^2(\omega t)[A_0(\tau) e^{r_1 t} + B_0(\tau) e^{r_2 t} + k_1 cos(\omega t) + k_2 sin(\omega t)]$

  $\implies \frac{3A^2}{2}(1 - cos(2\omega t))[A_0(\tau) e^{r_1 t} + B_0(\tau) e^{r_2 t} + k_1 cos(\omega t) + k_2 sin(\omega t)]$ - **Secular**
- $3h(t) z_0^2(t, \tau) = 3A sin(\omega t)[A_0(\tau) e^{r_1 t} + B_0(\tau) e^{r_2 t} + k_1 cos(\omega t) + k_2 sin(\omega t)][A_0(\tau) e^{r_1 t} + B_0(\tau) e^{r_2 t} + k_1 cos(\omega t) + k_2 sin(\omega t)]$

  $\implies 2(3Ak_2 sin^2(\omega t))[A_0 e^{r_1 t} + B_0 e^{r_2 t}] = 3Ak_2(1 - cos(2\omega t))[A_0 e^{r_1 t} + B_0 e^{r_2 t}]$ - **Secular**
- $z_0^3 = [A_0(\tau) e^{r_1 t} + B_0(\tau) e^{r_2 t} + k_1 cos(\omega t) + k_2 sin(\omega t)][A_0(\tau) e^{r_1 t} + B_0(\tau) e^{r_2 t} + k_1 cos(\omega t) + k_2 sin(\omega t)][A_0(\tau) e^{r_1 t} + B_0(\tau) e^{r_2 t} + k_1 cos(\omega t) + k_2 sin(\omega t)]$

  $3(\frac{k_1^2 A_0 e^{r_1 t}}{2}) + 3(\frac{k_2^2 A_0 e^{r_1 t}}{2}) + 3(\frac{k_1^2 B_0 e^{r_2 t}}{2}) + 3(\frac{k_2^2 B_0 e^{r_2 t}}{2})$ - **Secular**
- $\frac{\partial z_0}{\partial t} = \frac{\partial}{\partial t}[A_0(\tau) e^{r_1 t} + B_0(\tau) e^{r_2 t} + k_1 cos(\omega t) + k_2 sin(\omega t)]$

  $(A_0 r_1) e^{r_1 t} + (B_0 r_2) e^{r_2 t}$ - **Secular**
- $\frac{\partial z_0}{\partial \tau} = \frac{\partial}{\partial \tau}[A_0(\tau) e^{r_1 t} + B_0(\tau) e^{r_2 t} + k_1 cos(\omega t) + k_2 sin(\omega t)]$

  $(\frac{\partial A_0}{\partial \tau}) e^{r_1 t} + (\frac{\partial B_0}{\partial \tau}) e^{r_2 t}$ - **Secular**
- $2\frac{\partial^2 z_0}{\partial t \partial \tau} = 2\frac{\partial^2}{\partial t \partial \tau}[A_0(\tau) e^{r_1 t} + B_0(\tau) e^{r_2 t} + k_1 cos(\omega t) + k_2 sin(\omega t)]$

  $(2r_1 \frac{\partial A_0}{\partial \tau}) e^{r_1 t} + (2r_2 \frac{\partial B_0}{\partial \tau}) e^{r_2 t}$ - **Secular**

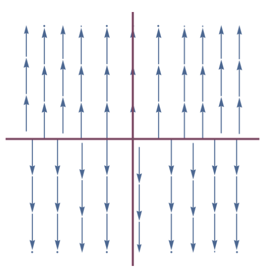
We need to find $A_0(\tau)$ and $B_0(\tau)$ by equating the coefficients of $e^{r_1 t}$ and $e^{r_2 t}$ to zero, as the follows,

$$\begin{cases} \left(\frac{dA_0}{d\tau} + \frac{c_1+2r_1}{2+4r_1} A_0 \right) = 0 \\ \left(\frac{dB_0}{d\tau} + \frac{c_1+2r_2}{2+4r_2} B_0 \right) = 0 \end{cases} \tag{31}$$

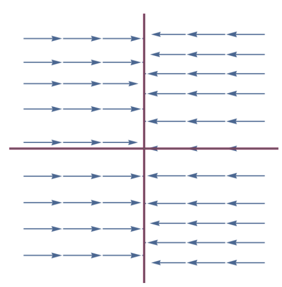
where $c_1 = \frac{3(A+k_2)^2+3k_1^2}{2} \geq 0$

The system in (31) has a unique equilibrium point at $(A_0, B_0) = (0, 0)$ if $c_1 \neq -2r_1$ and $c_1 \neq -2r_2$.

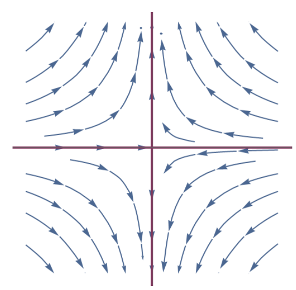
When $c_1 = -2r_1$, we get infinitely many equilibrium points along the A_0 axis, with the phase lines pointing away from the equilibrium points. A series of source nodes.
When $c_1 = -2r_2$, we get infinitely many equilibrium points along the B_0 axis, with phase lines pointing towards the critical points. A series of sink nodes.
When $-2r_2 < c_1 < -2r_1$, we get a saddle at the unique equilibrium point $(A_0, B_0) = (0, 0)$.
When $c_1 > -2r_2$, we get a stable sink node.



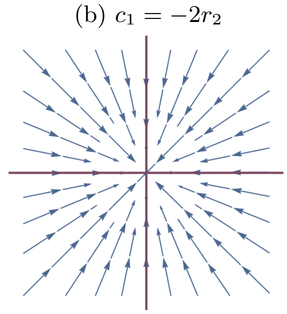
(a) $c_1 = -2r_1$



(b) $c_1 = -2r_2$



(c) $-2r_2 < c_1 < -2r_1$



(d) $c_1 > -2r_2$

Figure 3: Bifurcations of the system- Case 1 leading order

We solve (31) using (29), and get the functions,

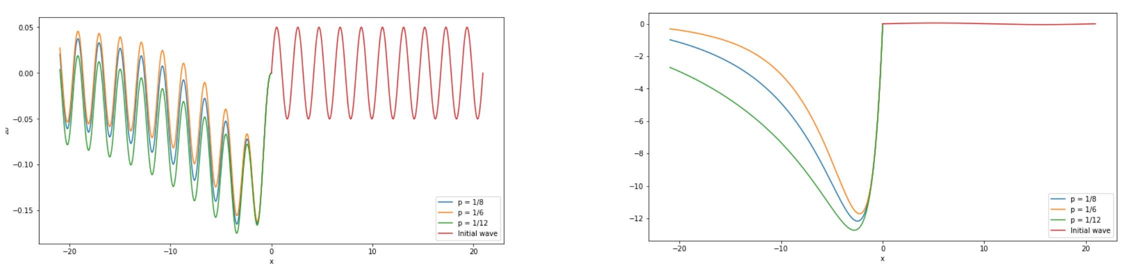
$$A_0(\tau) = \left(\frac{k_1 r_2 - k_2 \omega}{r_1 - r_2} \right) e^{-\frac{\tau(3A^2+6Ak_2+3k_1^2+3k_2^2+2r_1)}{4r_1+2}}$$

$$B_0(\tau) = \left(\frac{k_2\omega - k_1 r_1}{r_1 - r_2}\right) e^{-\frac{\tau(3A^2+6Ak_2+3k_1^2+3k_2^2+2r_2)}{4r_2+2}}$$

These are the functions which avoid secular terms in the $O(\hat{q})$ equation, $\implies z_1(t,\tau)$ is bounded, so our approximation of $z(t) = \tilde{z}(t,\tau) = [z_0(t,\tau) + \hat{q}z_1(t,\tau) + \hat{q}^2 z_2(t,\tau) + ...]$ is,

$$z(t) = z_0(t,\tau) + O(\hat{q})$$

Below we plot the initial incoming wave and the reflected wave, i.e., $u(x,0)$ and $z_0(t,\tau)$ respectively. We consider three sample $\hat{p}$ values such that $\hat{p}_i < \frac{1}{4}$. We also consider the effect of ω on the amplitudes of the oscillations for $\hat{q} = 0.00001$.



(a) Low omega, $\omega = 3$

(b) High omega, $\omega = 50$

Figure 4: When $\hat{q} = 0.00001$

5.5 Case 2 of leading order equation

Now we consider Case 2 of the leading order equation, i.e., $O(1)$. In Case 2, the solution of the ODE takes the form,

$$z_{0,C}(t,\tau) = A_0(\tau)e^{rt} + B_0(\tau)te^{rt}$$

In order to find the constants invloved above, we need to solve for $z_0(t,\tau)$ using the initial conditions $z(0,0) = 0$ and $z'(0,0) = -A\omega$.

$$z_0(t,\tau) = A_0(\tau)e^{rt} + B_0(\tau)te^{rt} + \left[\frac{-2A\omega(\omega^2 - \hat{p})}{\omega^2 + (\omega^2 - \hat{p})^2}\right] cos(\omega t) + \left[\frac{A(\omega^2 - (\omega^2 - \hat{p})^2)}{\omega^2 + (\omega^2 - \hat{p})^2}\right] sin(\omega t)$$

For convenience, lets write $z_0(t,\tau)$ as,

$$z_0(t,\tau) = A_0(\tau)e^{rt} + B_0(\tau)te^{rt} + k_1 cos(\omega t) + k_2 sin(\omega t) \tag{32}$$

When $t = 0 \implies \tau = 0$,

$$z_0(0,0) = A_0 + 0 + k_1 = 0$$
$$z_0'(0,0) = A_0 r + B_0 + k_2\omega = 0$$

From the first condition, we see that $A_0(0,0) = -k_1$. Substituting this in the second condition we get that,

$$B_0(0,0) = (k_1 r - k_2\omega)$$

$$\begin{cases} A_0(0,0) = -k_1 \\ B_0(0,0) = (k_1 r - k_2\omega) \end{cases} \tag{33}$$

We have now obtained the initial conditions for the case 2 constants. Next, we must consider the $O(\hat{q})$ equation.

$$O(\hat{q}) : \frac{\partial^2 z_1}{\partial t^2} + \frac{\partial z_1}{\partial t} + \hat{p} z_1 + 2\frac{\partial^2 z_0}{\partial t \partial \tau} + \frac{\partial z_0}{\partial \tau} + \frac{\partial z_0}{\partial t} + z_0^3 + 3h^2(t) z_0 + 3h(t) z_0^2 = -h'(t) - h^3(t)$$

$$O(\hat{q}) : z_1'' + z_1' + \hat{p} z_1 = \left(-2\frac{\partial^2 z_0}{\partial t \partial \tau} - \frac{\partial z_0}{\partial \tau} - \frac{\partial z_0}{\partial t} - z_0^3 - 3h^2(t) z_0 - 3h(t) z_0^2 - h'(t) - h^3(t) \right)$$

We proceed in the same way as for Case 1 and try to eliminate the secular terms present in,

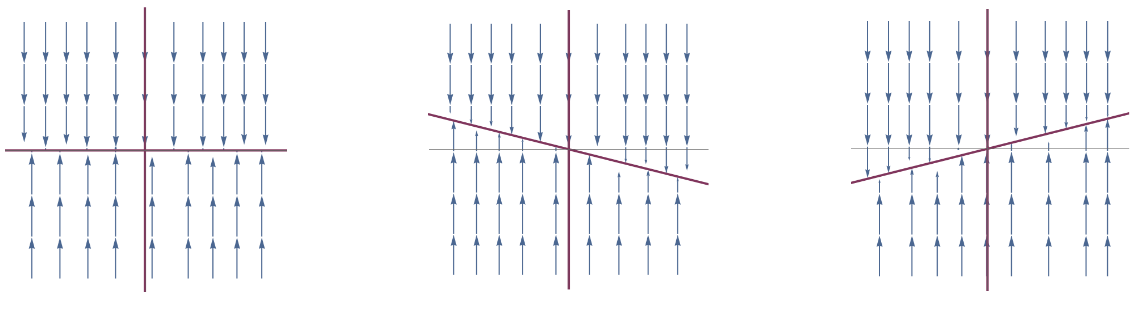
$$O(\hat{q}) : z_1'' + z_1' + \hat{p} z_1 = -\left(2\frac{\partial^2 z_0}{\partial t \partial \tau} + \frac{\partial z_0}{\partial \tau} + \frac{\partial z_0}{\partial t} + z_0^3 + 3h^2(t) z_0 + 3h(t) z_0^2 + h'(t) + h^3(t) \right)$$

Upon examining each of the terms on the RHS of the $O(\hat{q})$ equation and collecting the coefficients of e^{rt} and te^{rt} and equating them to zero, we get the expressions,

$$\begin{cases} \left(\frac{dA_0}{d\tau} = 0 \right. \\ \left(2\frac{dB_0}{d\tau} = -B_0 - (c_1 - \frac{1}{2}) A_0 \right. \end{cases} \tag{34}$$

where, $c_1 = \frac{3(A+k_2)^2 + 3k_1^2}{2}$.

The system in (34) does not have unique equilibrium points. When $c_1 = 1/2$, we get infinitely many equilibrium points along the A_0 axis with the phase lines directed towards the equilibrium points, implying stability. When $c_1 > 1/2$, we get infinitely many equilibrium points along an oblique axis with a decreasing slope and the phase lines imply stability. When $c_1 < 1/2$, we again get infinitely many equilibrium points along an oblique axis with an increasing slope with phase lines implying stability. The behaviour is shown below.



(a) $c_1 = 1/2$ (b) $c_1 > 1/2$ (c) $c_1 < 1/2$

Figure 5: Bifurcations of the system- Case 2 leading order

When we solve (34) using (33), we get,

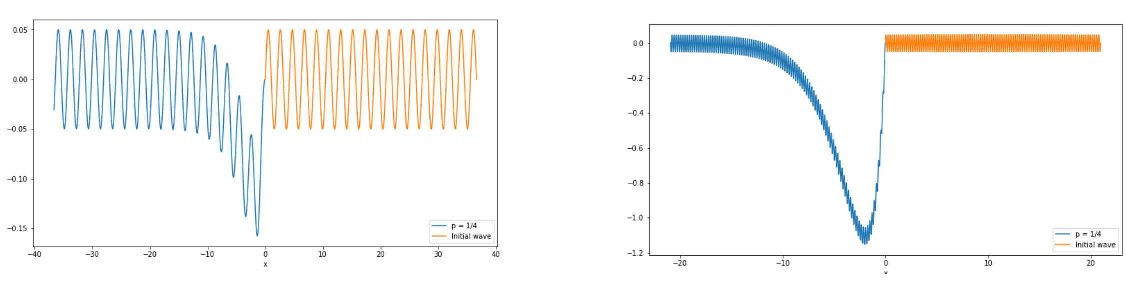
$$A_0(\tau) = -k_1$$

$$B_0(\tau) = \frac{1}{2}e^{-\tau/2}\Bigg[k_1\Bigg(3A^2(e^{\tau/2}-1) + 6Ak_2(e^{\tau/2}-1) + (3k_2^2-1)$$
$$e^{\tau/2} - 3k_2^2 + 2r + 1\Bigg) + 3k_1^3(e^{\tau/2}-1) - 2k_2\omega)\Bigg]$$

These are the functions which avoid secular terms in the $O(\hat{q})$ equation, $\implies z_1(t,\tau)$ is bounded, so our approximation of $z(t) = \tilde{z}(t,\tau) = [z_0(t,\tau) + \hat{q}z_1(t,\tau) + \hat{q}^2 z_2(t,\tau) + ...]$ is,

$$z(t) = z_0(t,\tau) + O(\hat{q})$$

We have the plots of our initial wave $Asin(\omega t)$ and our reflected wave $z_0(t,\tau)$ for different values of ω and $\hat{p} = 1/4$.



(a) Low omega, $\omega = 3$ (b) High omega, $\omega = 50$

Figure 6: When $\hat{q} = 0.00001$

5.6 Case 3 of the leading order equation

Now we consider Case 3 of the leading order equation, i.e., $O(1)$. In Case 3, the solution of the ODE takes the form,

$$z_{0,C}(t,\tau) = A_0(\tau)e^{\lambda t}cos(\mu t) + B_0(\tau)e^{\lambda t}sin(\mu t)$$

In order to find the constants involved, we need to solve for $z_0(t,\tau)$ using the initial conditions $z_0(0,0) = 0$ and $\frac{\partial z_0}{\partial t}(0,0) = 0$.

$$z_0(t,\tau) = A_0(\tau)e^{\lambda t}cos(\mu t)+B_0(\tau)e^{\lambda t}sin(\mu t)+\left[\frac{-2A\omega(\omega^2-\hat{p})}{\omega^2+(\omega^2-\hat{p})^2}\right]cos(\omega t)+\left[\frac{A(\omega^2-(\omega^2-\hat{p})^2)}{\omega^2+(\omega^2-\hat{p})^2}\right]sin(\omega t)$$

For convenience, lets write $z_0(t,\tau)$ as,

$$z_0(t,\tau) = A_0(\tau)e^{\lambda t}cos(\mu t) + B_0(\tau)e^{\lambda t}sin(\mu t) + k_1cos(\omega t) + k_2sin(\omega t) \tag{35}$$

When $t = 0 \implies \tau = 0$,

$$z_0(0,0) = A_0 + k_1 = 0 \implies A_0(0) = -k_1$$

$$z_0'(0,0) = A_0\lambda + B_0\mu + k_2\omega = 0$$

From the first condition, $A_0(0) = -k_1$. Putting this value in the second condition,

$$B_0\mu - k_1\lambda + k_2\omega = 0 \implies B_0(0) = \left(\frac{k_1\lambda - k_2\omega}{\mu}\right)$$

$$\begin{cases} A_0(0) = -k_1 \\ B_0(0) = \left(\frac{k_1\lambda - k_2\omega}{\mu}\right) \end{cases} \tag{36}$$

We have now obtained the initial conditions for the case 3 constants. Next, we must consider the $O(\hat{q})$ equation.

$$O(\hat{q}) : \frac{\partial^2 z_1}{\partial t^2} + \frac{\partial z_1}{\partial t} + \hat{p}z_1 + 2\frac{\partial^2 z_0}{\partial t\partial\tau} + \frac{\partial z_0}{\partial\tau} + \frac{\partial z_0}{\partial t} + z_0^3 + 3h^2(t)z_0 + 3h(t)z_0^2 = -h'(t) - h^3(t)$$

$$O(\hat{q}) : z_1'' + z_1' + \hat{p}z_1 = \left(-2\frac{\partial^2 z_0}{\partial t\partial\tau} - \frac{\partial z_0}{\partial\tau} - \frac{\partial z_0}{\partial t} - z_0^3 - 3h^2(t)z_0 - 3h(t)z_0^2 - h'(t) - h^3(t)\right)$$

We proceed in the same way as for $Case1$ and try to eliminate the secular terms present in,

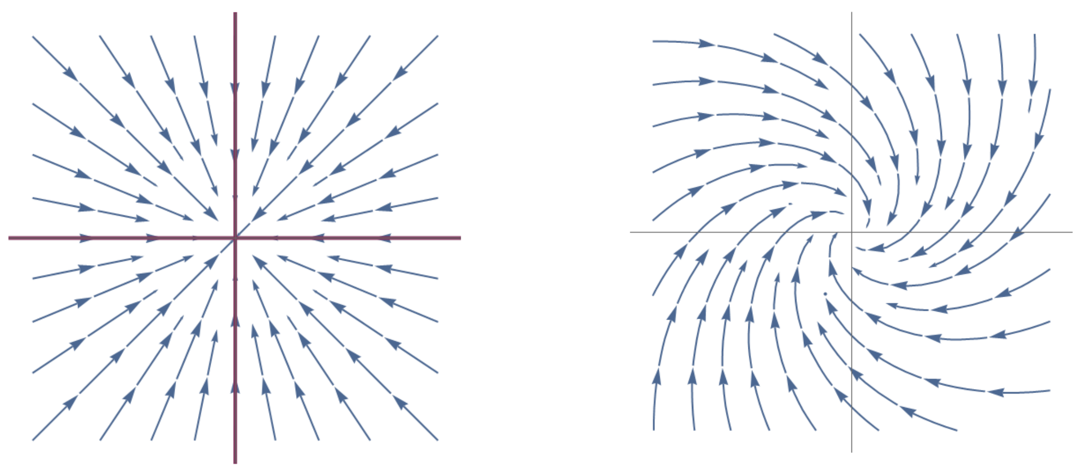
$$O(\hat{q}) : z_1'' + z_1' + \hat{p}z_1 = -\left(2\frac{\partial^2 z_0}{\partial t\partial\tau} + \frac{\partial z_0}{\partial\tau} + \frac{\partial z_0}{\partial t} + z_0^3 + 3h^2(t)z_0 + 3h(t)z_0^2 + h'(t) + h^3(t)\right)$$

Upon examining each of the terms on the RHS of the $O(\hat{q})$ equation and collecting the coefficients of $e^{\lambda t}cos(\mu t)$ and $e^{\lambda t}sin(\mu t)$ and equating them to zero, we get the expressions,

$$\begin{cases} \left(2\frac{dA_0}{d\tau} = -A_0 + B_0(\frac{c_1 - \frac{1}{2}}{\mu}) \right. \\ \\ \left(2\frac{dB_0}{d\tau} = -B_0 - A_0(\frac{c_1 - \frac{1}{2}}{\mu}) \right. \end{cases} \tag{37}$$

where $c_1 = \frac{3(A+k_2)^2 + 3k_1^2}{2} \geq 0$.

The system in (37) has a unique equilibrium at $(A_0, B_0) = (0, 0)$.When $c_1 = 1/2$, we get a stable node at $(0, 0)$. If $c_1 \neq 1/2$, we get a spiral sink at the point $(0, 0)$. This is consistent with the fact that the system has complex eigenvalues with negative real parts.



(a) Low omega, $c_1 = 1/2$

(b) $c_1 \neq 1/2$

Figure 7: Bifurcations of the system - Case 3 leading order

We solve (37) using (36), we get complex roots and the solutions of the system of linear ODEs are given below,

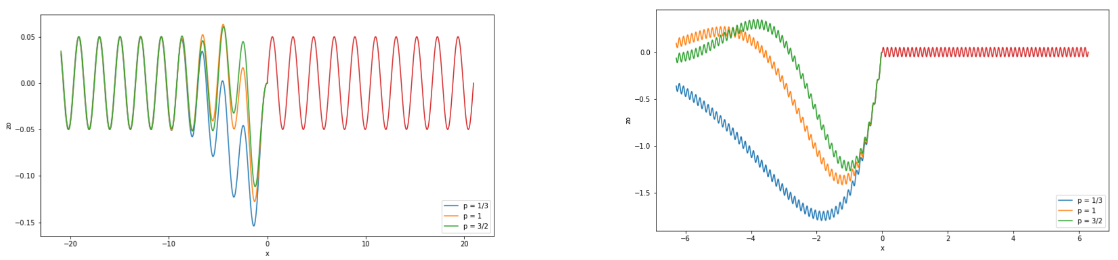
$$A_0(\tau) = -k_1 e^{-\tau} cos(\sqrt{\left(\frac{c_1 - \frac{1}{2}}{\mu}\right)^2}\tau)$$

$$B_0(\tau) = \frac{(k_1\lambda - k_2\omega)}{\mu} e^{-\tau} cos(\sqrt{\left(\frac{c_1 - \frac{1}{2}}{\mu}\right)^2}\tau)$$

These are the functions which avoid secular terms in the $O(\hat{q})$ equation, $\implies z_1(t, \tau)$ is bounded, so our approximation of $z(t) = \tilde{z}(t, \tau) = [z_0(t, \tau) + \hat{q}z_1(t, \tau) + \hat{q}^2 z_2(t, \tau) + ...]$ is,

$$z(t) = z_0(t, \tau) + O(\hat{q})$$

We have the plots of our initial wave $Asin(\omega t)$ and our reflected wave $z_0(t, \tau)$ for different values of ω. We take three sample $\hat{p}$ values such that $\hat{p}_i > 1/4$.



(a) Low omega, $\omega = 3$

(b) High omega, $\omega = 50$

Figure 8: When $\hat{q} = 0.00001$

5.7 Special $\hat{p}$ values

During the course of the research conducted in the previous section, we find that there is a possibility of obtaining additional secular terms for specific values of $\hat{p}$ in each of the intervals in Case 1 of the leading order equation and Case 3 of the leading order equation.

Let us first consider equation (30). We examine each of the terms present on the RHS of the equation. The term z_0^3 has a few possibilities.

Consider,

$$z_0^3 = [A_0(\tau)e^{r_1 t} + B_0(\tau)e^{r_2 t} + k_1 cos(\omega t) + k_2 sin(\omega t)]^3$$

This can also be written as,

$$z_0^3 = \Bigg[(A_0(\tau)e^{r_1 t} + B_0(\tau)e^{r_2 t})^3 + (k_1 cos(\omega t) + k_2 sin(\omega t))^3$$

$$+3(A_0(\tau)e^{r_1 t}+B_0(\tau)e^{r_2 t})^2(k_1 cos(\omega t)+k_2 sin(\omega t))+3(A_0(\tau)e^{r_1 t}+B_0(\tau)e^{r_2 t})(k_1 cos(\omega t)+k_2 sin(\omega t))^2\Bigg]$$

The second and third terms do not prodeuce any secular terms and the last term has already been included in our leading order case. The first term can be expanded as follows,

$$(A_0(\tau)e^{r_1 t} + B_0(\tau)e^{r_2 t})^3 = A_0^3 e^{3r_1 t} + B_0^3 e^{3r_2 t} + 3A_0^2 B_0 e^{(2r_1+r_2)t} + 3A_0 B_0^2 e^{(r_1+2r_2)t}$$

This term can produce additional secular terms if $3r_1 = r_1$ or $3r_1 = r_2$ or $3r_2 = r_2$ or $3r_2 = r_1$ or $2r_1 + r_2 = r_1$ or $2r_1 + r_2 = r_2$ or $r_1 + 2r_2 = r_1$ or $r_1 + 2r_2 = r_2$.

Given that, $r_1 = \frac{-1+\sqrt{1-4\hat{p}}}{2}$, $r_2 = \frac{-1-\sqrt{1-4\hat{p}}}{2}$. We substitute these r_1 and r_2 values to find that at $\hat{p} = 0$ and at $\hat{p} = \frac{3}{16}$ we get additional secular terms. This implies that we need to analyse these special cases separately to get a better understanding of our system.

For Case 3 of the leading order equation, we again find that the z_0^3 term on the RHS of equation (30) has a possibility for additional secular terms.

Consider,

$$z_0^3 = (A_0(\tau)e^{\lambda t}cos(\mu t) + B_0(\tau)e^{\lambda t}sin(\mu t) + k_1 cos(\omega t) + k_2 sin(\omega t))^3$$

In this expansion we consider the term $3(A_0(\tau)e^{\lambda t}cos(\mu t) + B_0(\tau)e^{\lambda t}sin(\mu t))(k_1 cos(\omega t) + k_2 sin(\omega t))^2$. This can be expanded further as,

$$(A_0(\tau)e^{\lambda t}cos(\mu t) + B_0(\tau)e^{\lambda t}sin(\mu t))\left(\frac{k_1^2 + k_2^2}{2} + \frac{k_1^2 - k_2^2}{2}cos(2\omega t) + k_1 k_2 sin(2\omega t)\right)$$

We see that in addition to the secular terms that have already been obtained we also get terms like,

$$\frac{k_1^2 - k_2^2}{4} A_0 e^{\lambda t}\left(cos(\mu + 2\omega)t + cos(\mu - 2\omega)t\right)$$

This can result in additional secular terms if $cos(\mu + 2\omega)t = cos(\mu)t$. This can happen only if $\omega = 0$ or $\omega = \pm\mu$. However, ω is the frequency and it should always be positive. Hence, $\mu = \pm\omega$.

Given that $\mu = \frac{\sqrt{4\hat{p}-1}}{2}$, we have that $\hat{p} = \frac{1+4\omega^2}{4}$.

Therefore,

For Case 1 of the leading order equation :

$$\hat{p} = 0$$
$$\hat{p} = \frac{3}{16}$$

For Case 2 of the leading order equation : There are no special cases, as $\hat{p}$ already take a specific value.

For Case 3 of the leading order equation :

$$\hat{p} = \frac{1 + 4\omega^2}{4}$$

5.7.1 When $\hat{p} = 0$

First we consider the following case,

- **When $\hat{p} = 0$**

  In the case 1 of (27), we have r_1 and r_2, given by, $r_1 = \frac{-1+\sqrt{1-4\hat{p}}}{2}$, $r_2 = \frac{-1-\sqrt{1-4\hat{p}}}{2}$. When $\hat{p} = 0$, $r_1 = 0$ and $r_2 = -1$. Putting this into the equation for $z_0(t, \tau)$, we get,

  $$z_0(t, \tau) = A_0(\tau) + B_0(\tau)e^{-t} + k_1 cos(\omega t) + k_2 sint(\omega t)$$

  When $t = 0$ and $\tau = 0$, we have, $z_0(0, 0) = 0$ and $\frac{\partial z_0}{\partial t}(0, 0) = 0$. This gives us the initial conditions,

$$\begin{cases} A_0(0) = -k_1 - k_2\omega \\ B_0(0) = k_2\omega \end{cases} \tag{38}$$

For this case we need to find and eliminate the secular terms. We do so by considering (23).

$$O(\hat{q}) : z_1'' + z_1' + \hat{p}z_1 = \left(-2\frac{\partial^2 z_0}{\partial t \partial \tau} - \frac{\partial z_0}{\partial \tau} - \frac{\partial z_0}{\partial t} - z_0^3 - 3h^2(t)z_0 - 3h(t)z_0^2 - h'(t) - h^3(t) \right)$$

We calculate each of the terms present in the RHS and collect the coefficients of the functions $e^{0t} = 1$ and e^{-t}.

- $2\frac{\partial^2 z_0}{\partial t \partial \tau} = (\frac{2\partial B_0}{\partial \tau})e^{-t} + N.S.T$
- $\frac{\partial z_0}{\partial \tau} = \frac{\partial A_0}{\partial \tau} + \frac{\partial B_0}{\partial \tau}e^{-t} + N.S.T$
- $\frac{\partial z_0}{\partial t} = B_0 e^{-t} + N.S.T$
- $z_0^3 = (A_0^3 + \frac{3A_0 k_1^2}{2} + \frac{3A_0 k_2^2}{2}) + (3A_0^2 B_0 + \frac{3B_0 k_1^2}{2} + \frac{3B_0 k_2^2}{2})e^{-t} + N.S.T$
- $3A^2 sin^2(\omega t) z_0 = (\frac{3A^2 A_0}{2}) + (\frac{3A^2 B_0}{2})e^{-t} + N.S.T$
- $3A sin(\omega t) z_0^2 = (3Ak_2 A_0) + (3Ak_2 B_0)e^{-t} + N.S.T$
- $h'(t)$ and $h^3(t)$ are Non secular terms.

The coefficients of e^{0t} and e^{-t} need to be equated to zero. So we have,

$$\left[\frac{dA_0}{d\tau} + A_0(C + A_0^2) \right] = 0$$

$$\left[\frac{dB_0}{d\tau} + B_0(1 - C - 3A_0^2) \right] = 0$$

where, $C = (\frac{3(A+k_2)^2 + 3k_1^2}{2}) \implies C > 0$.
We can convert the above equations into a system of nonlinear ODEs as below,

$$\begin{cases} \frac{dA_0}{d\tau} = -A_0(C + A_0^2) = F, \\ \frac{dB_0}{d\tau} = -B_0(1 - C - 3A_0^2) = G \end{cases} \tag{39}$$

When $-A_0(C + A_0^2) = 0 \implies A_0 = 0$. When $A_0 = 0$, from $\frac{\partial B_0}{\partial \tau} = 0$, we have $B_0 = 0$ *or* $C = 1$. When $C \neq 1$ and $C > 0$, the system has one equilibrium point $(A_0, B_0) = (0, 0)$. We need to linearise (39). We have to calculate the Jacobian of the system to linearise it.

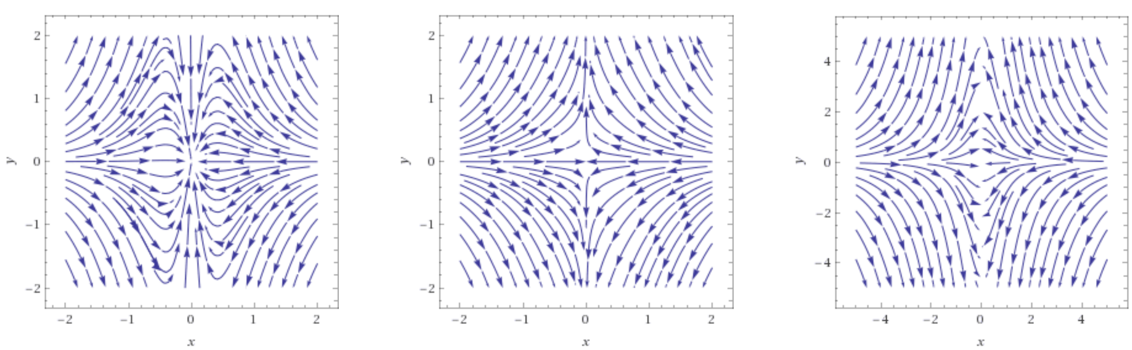
$$J = \begin{bmatrix} (\frac{\partial F}{\partial A_0}) & (\frac{\partial F}{\partial B_0}) \\ (\frac{\partial G}{\partial A_0}) & (\frac{\partial G}{\partial B_0}) \end{bmatrix}$$

$$J_{[0,0]} = \begin{bmatrix} -C & 0 \\ & \\ 0 & C-1 \end{bmatrix}$$

When $C \neq 1$ and $C > 0$, the system reduces to,

$$\begin{bmatrix} \frac{dA_0}{d\tau} \\ \frac{dB_0}{d\tau} \end{bmatrix} = \begin{bmatrix} -C & 0 \\ 0 & C-1 \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} \tag{40}$$

We observe that the eigenvalues of (40) are $-C$ and $C-1$.
When $0 < C < 1$, both the eigenvalues are negative $\implies \lambda_1 < 0, \lambda_2 < 0$. This means we obtain an **attractor stable node** at $(A_0, B_0) = (0,0)$ for the linearised system. When perturbed, the stability of this node does not change for the nonlinear system, it is stable even for the nonlinear system.
When $C > 1$, we observe that the eigenvalues ar eof opposite signs, i.e., $\lambda_1 < 0, \lambda_2 > 0$. This would give us an **unstable saddle** at $(A_0, B_0) = (0,0)$. This would also result in a saddle for the nonlinear system.
When $C = 1 \implies$ that we obtain a singular matrix J. Here we have $\lambda_1 < 0, \lambda_2 = 0$. This situation leads to a scenario with **infinitely many critical points along the same direction as the second eigenvector** $V_2 = (0, p)$. $\lambda_1 < 0 \implies$ that the phase trajectories are going into the critical points and are parallel to the A_0 axis. This would result in inifinitely many sink nodes for the linear system and for the nonlinear system it remains unchanged.



(a) When $0 < C < 1$ (b) When $C > 1$ (c) When $C = 1$

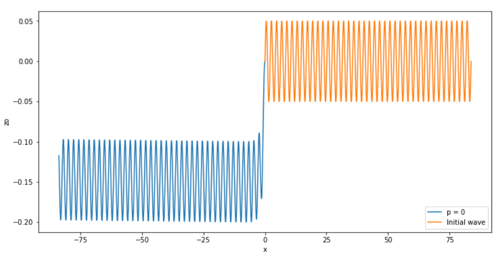
Figure 9: Bifurcations of special case 1

We can obtain a closed form solution of the system in (39) using the initial conditions in (38) as, where, $a = \frac{(k_1+k_2\omega)^2}{C+(k_1+k_2\omega)^2}$.

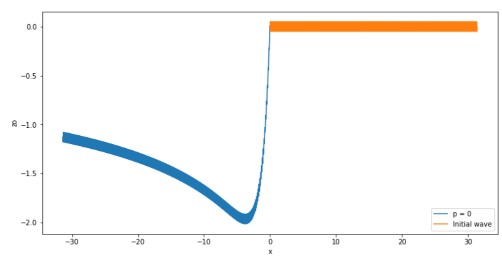
$$A_0(\tau) = \sqrt{\frac{aCe^{-2C\tau}}{1 - ae^{-C\tau}}}$$

$$B_0(\tau) = \frac{\left(k_2\omega(e^{-(2C+1)\tau})(e^{2C\tau} - a)\right)^{3/2}}{(1-a)^{3/2}}$$

The reflected wave and the incoming wave are plotted using the solution for $z_0(t,\tau)$.



(a) Low omega, $\omega = 3$



(b) High omega, $\omega = 50$

Figure 10: Reflected waves- Special case 1 and $\hat{q} = 0.00001$

5.7.2 When $\hat{p} = \frac{3}{16}$

Next we consider the following case,

- **When** $\hat{p} = \frac{3}{16}$

In the case 1 solution of the $O(1)$ equation, we have r_1 and r_2, given by, $r_1 = \frac{-1+\sqrt{1-4\hat{p}}}{2}$, $r_2 = \frac{-1-\sqrt{1-4\hat{p}}}{2}$.

When $\hat{p} = \frac{3}{16}$, $r_1 = \frac{-1}{4}$ and $r_2 = \frac{-3}{4}$. Putting this into the equation for $z_0(t,\tau)$, we get,

$$z_0(t,\tau) = A_0(\tau)e^{-t/4} + B_0(\tau)e^{-3t/4} + k_1 cos(\omega t) + k_2 sint(\omega t)$$

When $t = 0$ and $\tau = 0$, the initial conditions are given by,

$$\begin{cases} A_0(0) = -\frac{3k_1+4k_2\omega}{2} \\ B_0(0) = \frac{k_1+4k_2\omega}{2} \end{cases} \tag{41}$$

For this case we need to find and eliminate the secular terms. We do so by considering the $O(\hat{q})$ equation.

$$O(\hat{q}) : z_1'' + z_1' + \hat{p}z_1 = \left(-2\frac{\partial^2 z_0}{\partial t \partial \tau} - \frac{\partial z_0}{\partial \tau} - \frac{\partial z_0}{\partial t} - z_0^3 - 3h^2(t)z_0 - 3h(t)z_0^2 - h'(t) - h^3(t) \right)$$

We calculate each of the terms present in the RHS and collect the coefficients of the functions $e^{-t/4}$ and $e^{-3t/4}$.

- $2\frac{\partial^2 z_0}{\partial t \partial \tau} = (\frac{-1}{2}\frac{\partial A_0}{\partial \tau})e^{-t/4}(\frac{-3}{2}\frac{\partial B_0}{\partial \tau})e^{-3t/4} + N.S.T$
- $\frac{\partial z_0}{\partial \tau} = \frac{\partial A_0}{\partial \tau}e^{-t/4} + \frac{\partial B_0}{\partial \tau}e^{-3t/4} + N.S.T$

* $\frac{\partial z_0}{\partial t} = (\frac{-1A_0}{4})e^{-t/4} + (\frac{-3B_0}{4})e^{-3t/4} + N.S.T$

* $z_0^3 = (\frac{3A_0k_1^2}{2} + \frac{3A_0k_2^2}{2})e^{-t/4} + (\frac{3B_0k_1^2}{2} + \frac{3B_0k_2^2}{2} + A_0^3)e^{-3t/4} + N.S.T$

* $3A^2sin^2(\omega t)z_0 = (\frac{3A^2A_0}{2})e^{-t/4} + (\frac{3A^2B_0}{2})e^{-3t/4} + N.S.T$

* $3Asin(\omega t)z_0^2 = (3Ak_2A_0)e^{-t/4} + (3Ak_2B_0)e^{-3t/4} + N.S.T$

* $h'(t)$ and $h^3(t)$ are Non secular terms.

The coefficients of $e^{-t/4}$ and $e^{-3t/4}$ need to be equated to zero. We get the following expressions,

$$\left[\frac{dA_0}{d\tau} + A_0(C - \frac{1}{2})\right] = 0$$

$$\left[\frac{dB_0}{d\tau} - B_0(C - \frac{3}{2}) - 2A_0^3\right] = 0$$

where, $C = (\frac{6(A+k_2)^2+6k_1^2}{2}) \implies C > 0$.
We can convert the above equations into a system of nonlinear ODEs as below,

$$\begin{cases} \frac{dA_0}{d\tau} = -A_0(C - \frac{1}{2}) = F, \\ \frac{dB_0}{d\tau} = B_0(C - \frac{3}{2}) + 2A_0^3 = G \end{cases} \tag{42}$$

We have to equate $F = 0$. When we do so we find that $A_0 = 0$ or $C = \frac{1}{2}$. When $A_0 = 0$, we have from $G = 0$, $B_0 = 0$ or $C = \frac{3}{2}$. So when $C \neq \frac{1}{2}$ and $C \neq \frac{3}{2}$ and $C > 0$, the system has one equilibrium point $(A_0, B_0) = (0, 0)$. In order to analyse the stability of this equilibrium point for various intervals of C, we need to linearise the system. We start by calculating the Jacobian of the system as follows,

$$J = \begin{bmatrix} (\frac{\partial F}{\partial A_0}) & (\frac{\partial F}{\partial B_0}) \\ (\frac{\partial G}{\partial A_0}) & (\frac{\partial G}{\partial B_0}) \end{bmatrix}$$

$$J_{[0,0]} = \begin{bmatrix} (\frac{1}{2} - C) & 0 \\ 0 & (C - \frac{3}{2}) \end{bmatrix}$$

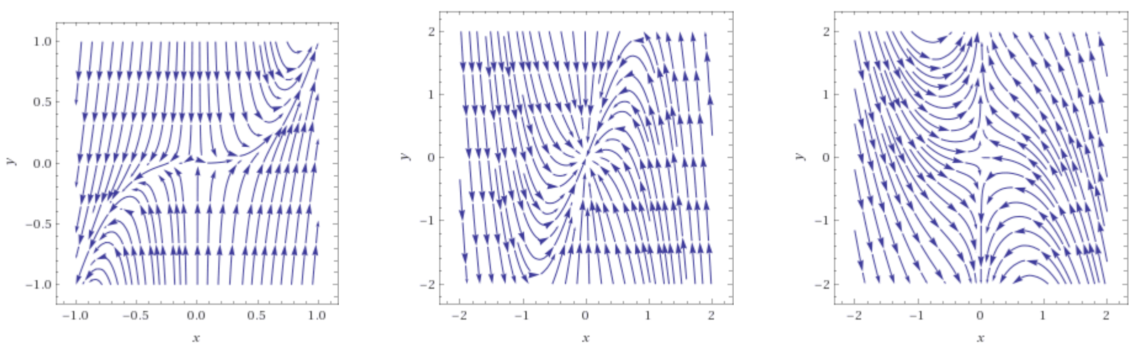
When $C \neq \frac{1}{2}$ and $C \neq \frac{3}{2}$ $C > 0$, the system reduces to,

$$\begin{bmatrix} \frac{dA_0}{d\tau} \\ \frac{dB_0}{d\tau} \end{bmatrix} = \begin{bmatrix} (\frac{1}{2} - C) & 0 \\ 0 & (C - \frac{3}{2}) \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} \tag{43}$$

We observe that the eigenvalues of (43) are $\lambda_1 = (\frac{1}{2} - C)$ and $\lambda_2 = (C - \frac{3}{2})$.

When $0 < C < \frac{1}{2}$, we have that $\lambda_1 > 0$ and $\lambda_2 < 0$. This means that we obtain an **unstable saddle** at $(A_0, B_0) = (0, 0)$ for the linearised system. This means that when perturbed slightly, it remains a saddle point for the nonlinear system as well.
When $\frac{1}{2} < C < \frac{3}{2}$, we have that $\lambda_1 < 0$ and $\lambda_2 < 0$. This means we obtain an **attractor or sink** at $(A_0, B_0) = (0, 0)$ for the linearised system. When perturbed, it will continue

to be a stable node for the nonlinear system.
When $C > \frac{3}{2}$, we obtain $\lambda_1 < 0$ and $\lambda_2 > 0$. This $\implies$ that we get an **unstable saddle point** at $(A_0, B_0) = (0, 0)$ for the linearised system. This continues to be an unstable saddle when perturbed for the nonlinear system.
When $C = \frac{1}{2}$, we obtain a singular matrix J. Here $\lambda_1 = 0$ and $\lambda_2 < 0$. This means that we obtain **infinitely many equilibrium points along the direction of the eigenvector V_1 of the form** $(A_0, B_0) = (p, 0)$. Since $\lambda_2 < 0$, the phase trajectories are directed towards the equlibrium points and are parallel to the B_0 axis. We get **stable nodes** for the linearised and the nonlinear system as well.
When $C = \frac{3}{2}$, we obtain a singular matrix J. Here $\lambda_1 < 0$ and $\lambda_2 = 0$. This means that we obtain **infinitely many equilibrium points along the direction of the eigenvector V_2 of the form** $(A_0, B_0) = (0, p)$. Since $\lambda_1 < 0$, the phase trajectories are directed towards the equlibrium points and are parallel to the A_0 axis. We get **stable nodes** for the linearised and the nonlinear system as well.



(a) When $0 < C < \frac{1}{2}$

(b) When $\frac{1}{2}C < \frac{3}{2}$

(c) When $C > \frac{3}{2}$

Figure 11

(a) When $C = 1/2$

(b) When $C = 3/2$

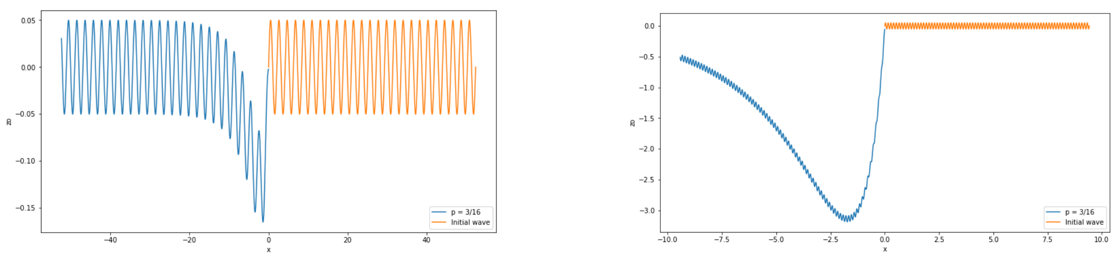
Figure 12: Bifurcations of special case 2

We solve the system in (42) using the initial conditions in (41), we get the following solution,

$$A_0(\tau) = -\frac{3k_1 + 4k_2\omega}{2} e^{\tau/2 - C\tau}$$

$$B_0(\tau) = \frac{e^{(C-3/2)\tau}}{(4(4C-3))}\Bigg(2k_1\Big(72k_2^2\omega^2(e^{(3-4C)\tau} - 1) + 4C - 3\Big)$$
$$+8k_2\omega\Big(8k_2^2\omega^2(e^{(3-4C)\tau}-1)+4C-3\Big)+108k_2k_1^2\omega(e^{(3-4C)\tau}-1)+27k_1^3(e^{(3-4C)\tau}-1)\Bigg)$$

We can now plot the solution,i.e., the reflected waves and the incoming waves,



(a) Low omega, $\omega = 3$

(b) High omega, $\omega = 50$

Figure 13: Reflected waves- Special case 2 and $\hat{q} = 0.00001$

5.7.3 When $\hat{p} = (\frac{1+4\omega^2}{4})$

This case arises as a special $\hat{p}$ value for Case 3. We have that,

$$z_0(t,\tau) = A_0(\tau)e^{\lambda t}cos(\mu t) + B_0(\tau)e^{\lambda t}sin(\mu t) + k_1cos(\omega t) + k_2sin(\omega t) \tag{44}$$

where $\lambda = \frac{-1}{2}$ and $\mu = \frac{\sqrt{4\hat{p}-1}}{2}$
Substituting the special $\hat{p}$ value into μ, we get that $\mu = \pm\omega$.

- **When** $\mu = \omega$

  We substitute $\mu = \omega$ into the $z_0(t,\tau)$ to obtain,

  $$z_0(t,\tau) = A_0(\tau)e^{-t/2}cos(\omega t) + B_0(\tau)e^{-t/2}sin(\omega t) + k_1cos(\omega t) + k_2sin(\omega t)$$

  The initial codnitions are given by,

  $$\begin{cases} A_0(0) = -k_1 \\ B_0(0) = -\frac{k_1+2k_2\omega}{2\omega} \end{cases} \tag{45}$$

  Using this $z_0(t,\tau)$, we compute the RHS of the $O(\epsilon)$ equation as follows, where N.S.T stands for non-secular terms,

$$\frac{\partial z_0}{\partial t} = (\frac{-A_0}{2} + B_0\mu)e^{-t/2}cos(\omega t) + (-A_0\mu - \frac{-B_0}{2})e^{-t/2}sin(\omega t) + N.S.T$$

$$\frac{\partial z_0}{\partial \tau} = (\frac{dA_0}{d\tau})e^{-t/2}cos(\omega t) + (\frac{dB_0}{d\tau})e^{-t/2}sin(\omega t)$$

$$2\frac{\partial^2 z_0}{\partial t \partial \tau} = (\frac{-dA_0}{d\tau} + 2\mu\frac{dB_0}{d\tau})e^{-t/2}cos(\omega t) + (-2\mu\frac{-dA_0}{d\tau} + \frac{-dB_0}{d\tau})e^{-t/2}sin(\omega t) + N.S.T$$

$$3hz_0^2 = (\frac{3AB_0k_1}{2} + 3AA_0k_2 - \frac{3AA_0k_2}{2})e^{-t/2}cos(\omega t) + (\frac{3AA_0k_1}{2} + 3AB_0k_2 + \frac{3AB_0k_2}{2})e^{-t/2}sin(\omega t) + N.S.T$$

$$3h^2z_0 = (\frac{3A^2A_0}{2} - \frac{3A^2A_0}{4})e^{-t/2}cos(\omega t) + (\frac{3A^2B_0}{2} + \frac{3A^2B_0}{4})e^{-t/2}sin(\omega t) + N.S.T$$

$$z_0^3 = (\frac{3A_0(k_1^2+k_2^2)}{2} + \frac{3A_0(k_1^2-k_2^2)}{4} + \frac{3B_0k_1k_2}{2})e^{-t/2}cos(\omega t) +$$
$$(\frac{3B_0(k_1^2+k_2^2)}{2} - \frac{3B_0(k_1^2-k_2^2)}{4} + \frac{3A_0k_1k_2}{2})e^{-t/2}sin(\omega t)$$

Collecting all the coefficients of $e^{-t/2}cos(\omega t)$ and $e^{-t/2}sin(\omega t)$, we get the following system,

$$\begin{cases} 2\omega\frac{dA_0}{d\tau} = A_0(c_1 - \omega) + B_0(3c_2 + c_3 - \frac{1}{2}) = F, \\ 2\omega\frac{dB_0}{d\tau} = -A_0(c_2 + 3c_3 - \frac{1}{2}) - B_0(c_1 + \omega) = G \end{cases} \tag{46}$$

Here, $c_1 = \frac{3}{2}(Ak_1 + k_1k_2)$, $c_2 = \frac{3}{4}(A + k_2)^2$ and $c_3 = \frac{3}{4}k_1^2$. The equilibrium point is $(A_0, B_0) = (0, 0)$.

We need to compute the characteristic equation of (46). Doing so will help us compute the eigenvalues of the system. Depending on the eigenvalues, we can determine the stability of the phase trajectories as t tends to infinity.

The characteristic equation is computed by, $|J - \lambda I| = 0$, where J is the Jacobian of the system and λ is the eigenvalues of the system.

$$|J - \lambda I| = 0$$
$$\begin{vmatrix} (c_1 - \omega - \lambda) & (3c_2 + c_3 - \frac{1}{2}) \\ (-c_2 - 3c_3 + \frac{1}{2}) & (-c_1 - \omega - \lambda) \end{vmatrix} = 0$$

Upon calculating the determinant, we get the following quadratic characteristic equation. When we solve the equation below for λ, we obtain the two eigenvalues of the system.

$$\lambda^2 + 2\omega\lambda + \left(\omega^2 + 3(c_2 + c_3)^2 - 2(c_2 + c_3) + \frac{1}{4}\right) = 0 \tag{47}$$

The roots of this equation are:

$$\lambda_{1,2} = -\omega \pm \sqrt{2(c_2 + c_3) - 3(c_2 + c_3)^2 - \frac{1}{4}}$$

$$\implies \lambda_{1,2} = -\omega \pm i\sqrt{Q_0} \tag{48}$$

Given the constraints of our physical system, i.e., $\omega > 0$ and $A > 0$,and the fact that the values of k_1, k_2 and c_1, c_2, c_3 are dependent on ω and A. We find that Q_0 always takes negative values, which leads to complex conjugate eigenvalues, which would result in a spiral sink always as the real part of our eigenvalues is negative.

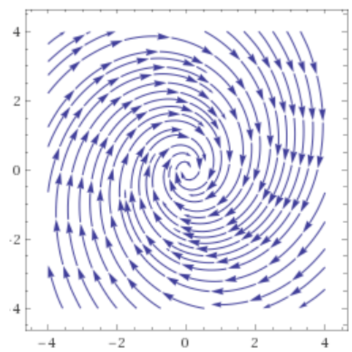


Figure 14: System leads to spiral sinks

- **When** $\mu = -\omega$

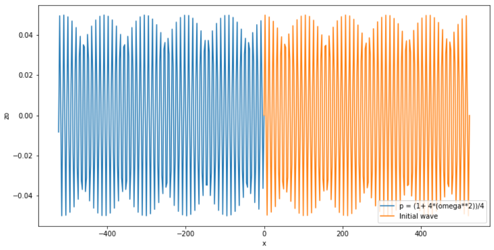
  When we plug in $\mu = -\omega$ into (44), we get the same system as the case where $\mu = \omega$. The inferences concerning stability also remain the same.

  If we solve the system (46) using the initial conditions in (45), we get the following solutions, where $a = c_1 - \omega$ and $b = (3c_2 + c_3 - 1/2)$

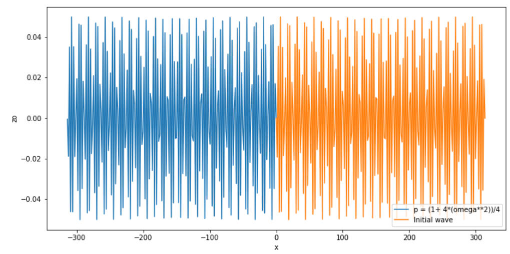
$$A_0(\tau) = -e^{-\omega\tau}\left(k_1 cos(\sqrt{Q_o}t) + \frac{\sqrt{Q_0}}{b} sin(\sqrt{Q_0}t)\right)$$

$$B_0(\tau) = -e^{-\omega\tau}\left(\frac{k_1 + 2k_2\omega}{2\omega} cos(\sqrt{Q_o}t) - \frac{a+\omega}{b} sin(\sqrt{Q_0}t)\right)$$

We can now plot the incoming and reflected wave as follows,



(a) Low omega, $\omega = 3$



(b) High omega, $\omega = 50$

Figure 15: Reflected waves- Special case 3 and $\hat{q} = 0.00001$

5.8 Detuning

Next we move on to the process of detuning. Detuning is the process by which a small perturbation is induced to the special $\hat{p}$ value to observe if that perturbation impacts its stability.

$$\hat{p} = p_{critical} + \delta\hat{q}$$

We will consider the following cases :

- $\hat{p} = 0 + \delta\hat{q}$
- $\hat{p} = \frac{3}{16} + \delta\hat{q}$
- $\hat{p} = (\frac{1+4\omega^2}{4}) + \delta\hat{q}$

5.8.1 Case 1 : $\hat{p} = 0 + \delta\hat{q}$

Firstly, we consider the the special value $\hat{p} = 0$. We perturb $\hat{p} = 0$ by a small value $\delta\hat{q}$ where $\hat{q}$ is a very small quantity and δ is an $O(1)$ constant.

We want to analyse the effect of this perturbation on our solution. In order to do that, we need to substitute $\hat{p} = 0 + \delta\hat{q}$ in the Boundary condition in (8).

We get,

$$u_{tt} = u_x - \delta\hat{q}u - \hat{q}(u_t + u^3) \tag{49}$$

We know that $u(x,t) = h(x+t) + z(t-x)$. Substituting this in (49), we get,

$$h'' + z'' = h' - z' - \delta\hat{q}h - \delta\hat{q}z - \hat{q}h' - \hat{q}z' - \hat{q}(h^3 + z^3 + 3h^2z + 3hz^2)$$

$$z'' + z' + \delta\hat{q}z + \hat{q}z' + \hat{q}(z^3 + 3h^2z + 3hz^2) = h' - h'' - \delta\hat{q}h - \hat{q}h' - \hat{q}h^3$$

Upon using the expansion $z = z_0(t,\tau) + z_1(t,\tau) + ...$, we get,

$$O(1): \frac{\partial^2 z_0}{\partial t^2} + \frac{\partial z_0}{\partial t} = h' - h''$$

$$z_0(t,\tau) = A_0(\tau) + B_0(\tau)e^{-t} + k_1 cos\omega t + k_2 sin\omega t$$

where $k_1 = \frac{-2A\omega}{1+\omega^2}$ and $k_2 = \frac{A(1-\omega^2)}{1+\omega^2}$
The initial conditions are given by,

$$\begin{cases} A_0(0) = -k_1 \\ B_0(0) = k_2\omega \end{cases} \tag{50}$$

$$O(\hat{q}): \; \frac{\partial^2 z_1}{\partial t^2} + \frac{\partial z_1}{\partial t} = \left(\frac{-2\partial^2 z_0}{\partial t \partial \tau} - \frac{\partial z_0}{\partial \tau} - \delta z_0 - \frac{\partial z_0}{\partial t} - z_0^3 - 3h^2 z_0 - 3hz_0^2 - \delta h' - h' - h^3 \right)$$

- $\frac{\partial z_0}{\partial t} = -B_0(\tau)e^{-t} + N.S.T$
- $\delta z_0 = \delta A_0 + \delta B_0 e^{-t} + N.S.T$
- $\frac{\partial z_0}{\partial \tau} = \frac{dA_0}{d\tau} + \frac{dB_0}{d\tau}e^{-t}$
- $\frac{2\partial^2 z_0}{\partial t \partial \tau} = \frac{-2dB_0}{d\tau}e^{-t}$
- $z_0^3 = A_0^3 + (3A_0^2 B_0)e^{-t} + \frac{3(k_1^2+k_2^2)}{2}(A_0 + B_0 e^{-t}) + N.S.T$
- $3h^2 z_0 = \frac{3A^2}{2}(A_0 + B_0 e^{-t} + N.S.T$
- $3hz_0^2 = 3Ak_2(A_0 + B_0 e^{-t}) + N.S.T$

We want to eliminate the secular terms, so we collect the coefficients of 1 and e^{-t} and equate them to zero to obtain a system of ODEs.

$$\left[B_0 - \delta B_0 - \frac{dB_0}{d\tau} + \frac{2dB_0}{d\tau} - 3A_0^2 B_0 - \frac{3(k_1^2 + k_2^2)}{2} B_0 - \frac{3A^2}{2} B_0 - 3Ak_2 B_0 \right] = 0$$

$$\left[-\delta A_0 - \frac{dA_0}{d\tau} - A_0^3 - \frac{3(k_1^2 + k_2^2)}{2} A_0 - \frac{3A^2}{2} A_0 - 3Ak_2 A_0 \right] = 0$$

We obtain the following system of ODEs,

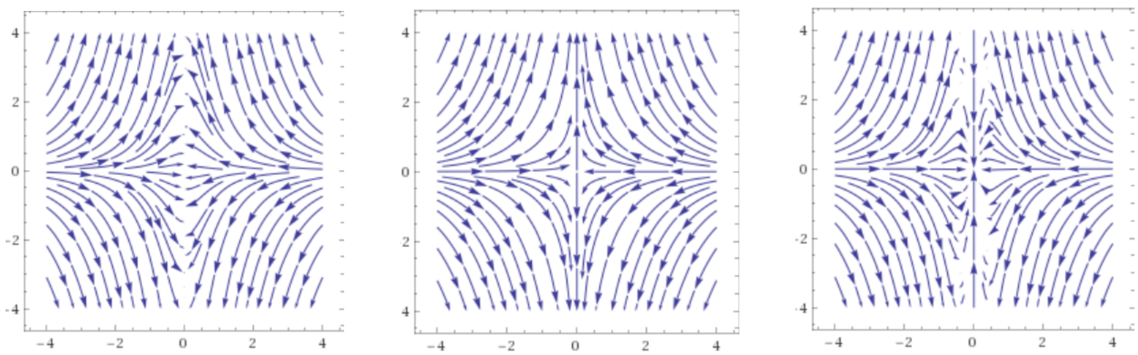
$$\begin{cases} \frac{dA_0}{d\tau} = A_0(-c_1 - A_0^2 - \delta), \\ \frac{dB_0}{d\tau} = B_0(c_1 + 3A_0^2 + \delta - 1) \end{cases} \tag{51}$$

where, $c_1 = \frac{3(A+k_2)^2 + 3k_1^2}{2} \geq 0$

In order to analyse the stability of our $O(1)$ solution, we need to analyse how the eigenvalues of (51) behave around the equilibrium point $(A_0, B_0) = (0, 0)$. We also see that when $A_0 = 0 \implies B_0 = 0$ or $c_1 = 1 - \delta$.

$$J = \begin{vmatrix} -(c_1 + \delta) & 0 \\ 0 & (c_1 + \delta) - 1 \end{vmatrix}$$

(i) When $c_1 = (1 - \delta) \implies \mathrm{c}_1 + \delta = 1$, $\lambda_1 < 0$ and $\lambda_2 = 0$. This means we obtain infinitely many equilibrium points along the B_0 axis with each of the equilibrium points being an attractor(stable).
(ii) When $c_1 > (1 - \delta) \implies \mathrm{c}_1 + \delta > 1$, we get $\lambda_1 < 0$ and $\lambda_2 > 0$. This gives us a saddle for the linearised system and a saddle for the nonlinear system as well.
(iii) When $0 < c_1 + \delta < 1$, $\lambda_1 < 0$ and $\lambda_2 < 0$. This leads to a stable node for the linearised system and it is asymptotically stable for the nonlinear system.



(a) $c_1 + \delta = 1$ (b) $c_1 + \delta > 1$ (c) $0 < c_1 + \delta < 1$

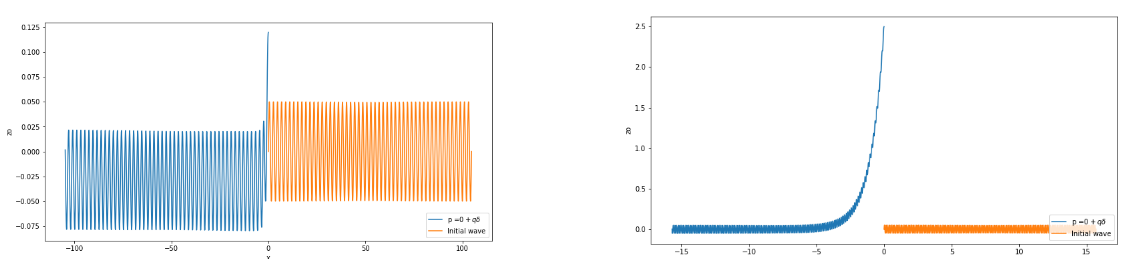
Figure 16: Bifurcations of the system- Case 1 detuned system.

We solve the system in (51) using the initial conditions in (50), to get,

$$A_0(\tau) = \sqrt{\frac{(c_1+\delta)k_1^2 e^{-2(c_1+\delta)\tau}}{(c_1+\delta) + k_1^2(1-e^{-2(c_1+\delta)\tau})}}$$

$$B_0(\tau) = k_2 \omega e^{-\tau(2c_1+2\delta+1)} \left(\frac{(c_1+\delta)e^{2\tau(c_1+\delta)} + k_1^2(e^{2\tau(c_1+\delta)}-1)}{c_1+\delta} \right)^{3/2}$$

We plot the reflected wave and the initial wave for a low and high omega and $\hat{q} = 0.01$ below,



(a) Low omega, $\omega = 3$ (b) High omega, $\omega = 50$

Figure 17: Reflected waves- Case 1 detuned system and $\hat{q} = 0.00001$

5.8.2 Case 2 : $\hat{p} = \frac{3}{16} + \delta\hat{q}$

Next, we consider the the special value $\hat{p} = \frac{3}{16}$. We perturb $\hat{p} = \frac{3}{16}$ by a small value $\delta\hat{q}$ where $\hat{q}$ is a very small quantity and δ is an $O(1)$ constant.

We want to analyse the effect of this perturbation on our solution. In order to do that, we need to substitute $\hat{p} = \frac{3}{16} + \delta\hat{q}$ in the Boundary condition in (8).

We get,

$$u_{tt} = u_x - \frac{3}{16}u - \delta\hat{q}u - \hat{q}(u_t + u^3) \tag{52}$$

We know that $u(x,t) = h(x+t) + z(t-x)$. Substituting this in (52), we get,

$$h'' + z'' = h' - z' - \frac{3}{16}h - \frac{3}{16}z - \delta\hat{q}h - \delta\hat{q}z - \hat{q}h' - \hat{q}z' - \hat{q}(h^3 + z^3 + 3h^2z + 3hz^2)$$

$$z'' + z' + \frac{3}{16}z + \delta\hat{q}z + \hat{q}z' + \hat{q}(z^3 + 3h^2z + 3hz^2) = h' - h'' - \frac{3}{16}h - \delta\hat{q}h - \hat{q}h' - \hat{q}h^3$$

Upon using the expansion $z = z_0(t,\tau) + z_1(t,\tau) + ...$, we get,

$$O(1): z_0'' + z_0' + \frac{3}{16}z_0 = h' - h'' - \frac{3}{16}h$$

$$z_0(t,\tau) = A_0(\tau)e^{-t/4} + B_0(\tau)e^{-3t/4} + k_1 cos\omega t + k_2 sin\omega t$$

where $k_1 = \left(\frac{2(A\omega^2 - \frac{3A}{16})}{\frac{3}{8} - (\frac{3}{16})^2\frac{1}{\omega} - \omega^3}\right)$ and $k_2 = \left(A + \frac{512(A\omega^2 - \frac{3A}{16})}{96\omega - 9 - 256\omega^4}\right)$

The initial conditions are given by,

$$\begin{cases} A_0(0) = -\frac{3k_1 + 4k_2\omega}{2} \\ B_0(0) = \frac{k_1 + 4k_2\omega}{2} \end{cases} \tag{53}$$

$$O(\hat{q}): \frac{\partial^2 z_1}{\partial t^2} + \frac{\partial z_1}{\partial t} + \frac{3}{16}z_1 = \left(\frac{-2\partial^2 z_0}{\partial t \partial \tau} - \frac{\partial z_0}{\partial \tau} - \delta z_0 - \frac{\partial z_0}{\partial t} - z_0^3 - 3h^2 z_0 - 3hz_0^2 - \delta h' - h' - h^3\right)$$

- $\frac{\partial z_0}{\partial t} = \frac{-A_0}{4}e^{-t/4} - \frac{3B_0}{4}e^{-3t/4} + N.S.T$
- $\delta z_0 = \delta A_0 e^{-t/4} + \delta B_0 e^{-3t/4} + N.S.T$
- $\frac{\partial z_0}{\partial \tau} = \frac{dA_0}{d\tau}e^{-t/4} + \frac{dB_0}{d\tau}e^{-3t/4}$
- $\frac{2\partial^2 z_0}{\partial t \partial \tau} = -\frac{1 dA_0}{2d\tau}e^{-t/4} - \frac{3dB_0}{2d\tau}e^{-3t/4}$
- $z_0^3 = \frac{3(k_1^2 + k_2^2)}{2}(A_0 e^{-t/4} + B_0 e^{-3t/4}) + N.S.T$
- $3h^2 z_0 = \frac{3A^2}{2}(A_0 e^{-t/4} + B_0 e^{-3t/4} + N.S.T$
- $3hz_0^2 = 3Ak_2(A_0 e^{-t/4} + B_0 e^{-3t/4}) + N.S.T$

We want to eliminate the secular terms, so we collect the coefficients of $e^{-t/4}$ and $e^{-3t/4}$ and equate them to zero to obtain a system of ODEs.

$$\left[\frac{A_0}{4} - \delta A_0 - \frac{1}{2}\frac{dA_0}{d\tau} - \frac{3(A+k_2)^2 - 3k_1^2}{2}A_0 - \frac{3A^2 A_0}{2} - 3Ak_2 A_0\right] = 0$$

$$\left[\frac{3B_0}{4} - \delta B_0 + \frac{1}{2}\frac{dB_0}{d\tau} - \frac{3(A+k_2)^2 - 3k_1^2}{2}B_0 - \frac{3A^2 B_0}{2} - 3Ak_2 B_0\right] = 0$$

We obtain the following system of ODEs,

$$
\begin{cases}
\frac{1 dA_0}{2 d\tau} = A_0(-(c_1+\delta) - \frac{1}{4}), \\
\\
\frac{1 dB_0}{2 d\tau} = B_0((c_1+\delta) - \frac{3}{4})
\end{cases}
\tag{54}
$$

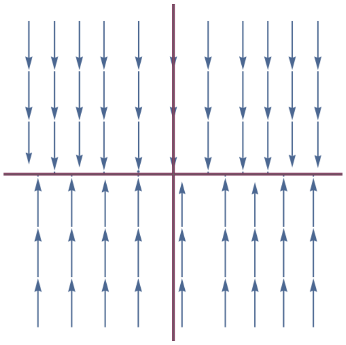
where, $c_1 = \frac{3(A+k_2)^2+3k_1^2}{2} \geq 0$.
We solve the system in (54) using the initial conditions in (53), to get,

$$
A_0(\tau) = -\frac{4k_2\omega + 3k_1}{2} e^{-\frac{\tau}{2}(4c_1+4\delta+1)}
$$

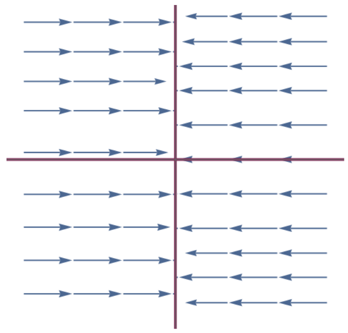
$$
B_0(\tau) = \frac{4k_2\omega + k_1}{2} e^{-\frac{\tau}{2}(3-4c_1-4\delta)}
$$

This system has an equilibrium point when $c_1 \neq \frac{-1}{4}$, $c_1 \neq \frac{3}{4}$, i.e., $(A_0, B_0) = (0, 0)$.

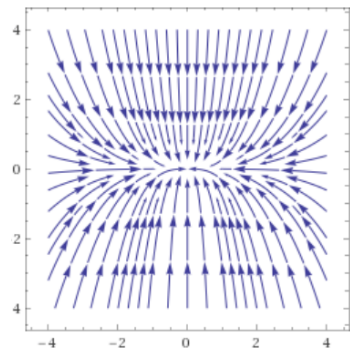
(i) When $c_1 + \delta = \frac{-1}{4}$, $\lambda_1 = 0$ and $\lambda_2 < 0$. We get infinitely many equilibrium points along the A_0 axis with phase lines pointing towards a stable node at each of these equilibria.
(ii) When $c_1 + \delta = \frac{3}{4}$, $\lambda_1 < 0$ and $\lambda_2 = 0$. We get infinitely many equilibrium points along the B_0 axis with phase lines pointing towards a stable node at each of these equilibria.
(iii) When $\frac{-1}{4} < c_1 + \delta < \frac{3}{4}$, $\lambda_1 < 0$ and $\lambda_2 < 0$. This leads to a stable node for the linear system of ODEs we have obtained.
(iv) When $c_1 + \delta > \frac{3}{4}$, $\lambda_1 < 0$ and $\lambda_2 > 0$. This leads to a saddle at the equilibrium point.
(v) When $c_1 + \delta < \frac{-1}{4}$, $\lambda > 0$ and $\lambda_2 < 0$. This leads to a saddle for the linear system of ODEs at the equilibrium point.



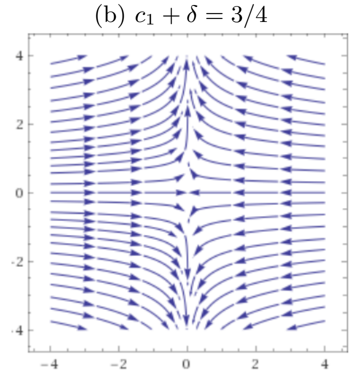
(a) $c_1 + \delta = -1/4$



(b) $c_1 + \delta = 3/4$



(c) $-1/4 < c_1 + \delta < 3/4$



(d) $c_1 + \delta > 3/4$

Figure 18: Bifurcations of the system- Case 2 detuned system

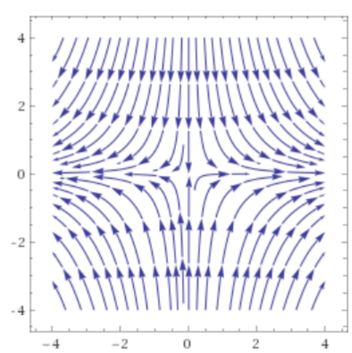
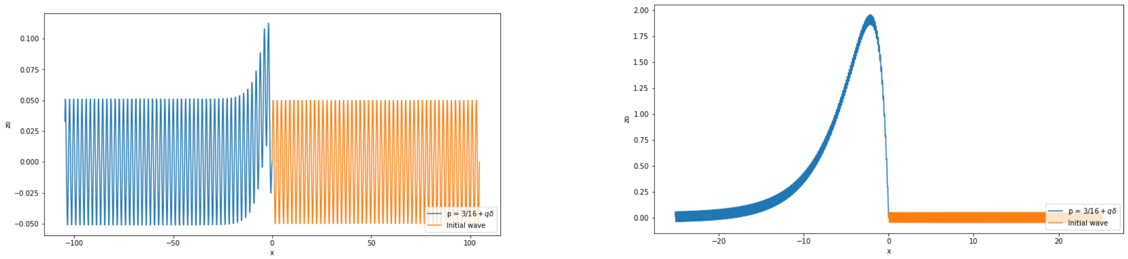


Figure 19: $c_1 + \delta < -1/4$

Next, we plot our initial and reflected wave,similar to the previous cases,



(a) Low omega, $\omega = 3$

(b) High omega, $\omega = 50$

Figure 20: Reflected waves- Case 2 detuned system and $\hat{q} = 0.00001$

5.8.3 Case 3: $\hat{p} = (\frac{1+4\omega^2}{4}) + \delta\hat{q}$

Next, we consider the the special value $\hat{p} = \left(\frac{1}{4} + \omega^2 + \delta\hat{q}\right)$. We perturb $\hat{p} = \left(\frac{1}{4} + \omega^2\right)$ by a small value $\delta\hat{q}$ where $\hat{q}$ is a very small quantity and δ is an $O(1)$ constant.

We want to analyse the effect of this perturbation on our solution. In order to do that, we need to substitute $\hat{p} = \left(\frac{1}{4} + \omega^2 + \delta\hat{q}\right)$ in the Boundary condition in (8).

We get,

$$u_{tt} = u_x - \frac{1}{4}u - \omega^2 u - \delta\hat{q}u - \hat{q}(u_t + u^3) \tag{55}$$

We know that $u(x,t) = h(x+t) + z(t-x)$. Substituting this in (55), we get,

$$h'' + z'' = h' - z' - \frac{h}{4} - \frac{z}{4} - \omega^2 h - \omega^2 z - \hat{q}\delta h - \hat{q}\delta z - \hat{q}h' - \hat{q}z' - \hat{q}(h^3 + z^3 + 3h^2 z + 3hz^2)$$

$$z'' + z' + \frac{z}{4} + \omega^2 z + \hat{q}(\delta z + z' + z^3 + 3h^2 z + 3hz^2) = h' - h'' - \frac{h}{4} - \omega^2 h - \hat{q}(\delta h + h' + h^3)$$

Upon substitution of the expansion $z = z_0(t,\tau) + \hat{q}z_1(t,\tau) + ...$, we get,

$$O(1) : \frac{\partial^2 z_0}{\partial t^2} + \frac{\partial z_0}{\partial t} + (\frac{1}{4} + \omega^2)z_0 = h' - h'' - \frac{h}{4}$$

$$\implies r^2 + r + (\frac{1}{4} + \omega^2) = 0 \implies r_{1,2} = \frac{-1 \pm 2i\omega}{2}$$

$$\implies z_0(t,\tau) = A_0(\tau)e^{-t/2}cos\omega t + B_0(\tau)e^{-t/2}sin\omega t + k_1 cos\omega t + k_2 sin\omega t$$

where, $k_1 = \frac{4A\omega^2 - 4A\omega^3 + 2A\omega}{\frac{1}{4} + 4\omega^2}$ and $k_2 = \frac{5A\omega^2 - \frac{A}{4} - A\omega}{\frac{1}{4} + 4\omega^2}$.
The initial conditions are given by,

$$\begin{cases} A_0(0) = \frac{2(k_2\omega - k_1\omega)}{2\omega+1} \\ B_0(0) = -\frac{(k_1 + 2k_2\omega)}{2\omega+1} \end{cases} \tag{56}$$

$$O(\hat{q}) : \frac{\partial^2 z_1}{\partial t^2} + \frac{\partial z_1}{\partial t} + (\frac{1}{4} + \omega^2)z_1 = \left(\frac{-2\partial^2 z_0}{\partial t \partial \tau} - \frac{\partial z_0}{\partial \tau} - \delta z_0 - \frac{\partial z_0}{\partial t} - z_0^3 - 3h^2 z_0 - 3hz_0^2 - \delta h' - h' - h^3\right)$$

- $\frac{\partial z_0}{\partial t} = (-\frac{A_0}{2} + B_0\omega)e^{-t/2}cos\omega t + (-\frac{B_0}{2} - A_0\omega)e^{-t/2}sin\omega t + N.S.T$
- $\delta z_0 = (\delta A_0)e^{-t/2}cos\omega t + (\delta B_0)e^{-t/2}sin\omega t + N.S.T$
- $\frac{\partial z_0}{\partial \tau} = (\frac{dA_0}{d\tau})e^{-t/2}cos\omega t + (\frac{dB_0}{d\tau})e^{-t/2}sin\omega t$
- $\frac{2\partial^2 z_0}{\partial t \partial \tau} = (-\frac{dA_0}{d\tau} + 2\omega\frac{dB_0}{d\tau})e^{-t/2}cos\omega t + (-\frac{dB_0}{d\tau} - 2\omega\frac{dA_0}{d\tau})e^{-t/2}sin\omega t$
- $z_0^3 = \left(\frac{3(k_1^2+k_2^2)}{2}A_0 + \frac{3(k_1^2-k_2^2)}{4}A_0 + \frac{3k_1k_2}{2}A_0\right)e^{-t/2}cos\omega t + \left(\frac{3(k_1^2+k_2^2)}{2}B_0 + \frac{3(k_1^2-k_2^2)}{4}B_0 - \frac{3k_1k_2}{2}B_0\right)e^{-t/2}sin\omega t + N.S.T$
- $3h^2z_0 = (\frac{3A^2A_0}{2} - \frac{3A^2A_0}{4})e^{-t/2}cos\omega t + (\frac{3A^2B_0}{2} - \frac{3A^2B_0}{4})e^{-t/2}sin\omega t + N.S.T$
- $3hz_0^2 = (3Ak_2A_0 - \frac{3Ak_2A_0}{2})e^{-t/2}cos\omega t + (3Ak_2B_0 - \frac{3Ak_2B_0}{2})e^{-t/2}sin\omega t + N.S.T$

We would like to eliminate the secular terms, so we collect the coefficients of $e^{-t/2}cos\omega t$ and $e^{-t/2}sin\omega t$ and equate them to zero to obtain a system of ODEs.

$$\left[\frac{A_0}{2} - B_0\omega - \delta A_0 - 2\omega\frac{dB_0}{d\tau} - \frac{3(k_1^2+k_2^2)}{2}A_0 - \frac{3(k_1^2-k_2^2)}{4}A_0 - \frac{3k_1k_2}{2}A_0 - \frac{3A^2A_0}{2} + \frac{3A^2A_0}{4} - 3Ak_2A_0 + \frac{3Ak_2A_0}{2}\right] = 0$$

$$\left[\frac{B_0}{2} + A_0\omega - \delta B_0 + 2\omega\frac{dA_0}{d\tau} - \frac{3(k_1^2+k_2^2)}{2}B_0 - \frac{3(k_1^2-k_2^2)}{4}B_0 + \frac{3k_1k_2}{2}B_0 - \frac{3A^2B_0}{2} + \frac{3A^2B_0}{4} - 3Ak_2B_0 + \frac{3Ak_2B_0}{2}\right] = 0$$

We obtain the following system of ODEs,

$$\begin{cases} 2\omega\frac{dA_0}{d\tau} = A_0(-\omega) + B_0(c_1 - c_2 + \delta - \frac{1}{2}), \\ 2\omega\frac{dB_0}{d\tau} = B_0(-\omega) - A_0(c_1 + c_2 + \delta - \frac{1}{2}) \end{cases} \tag{57}$$

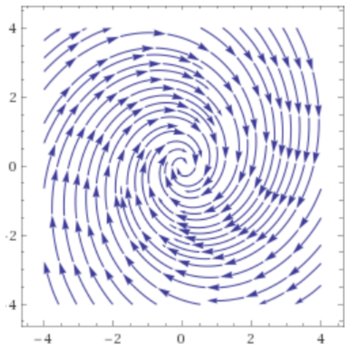
where, $c_1 = \frac{3(A+k_2)^2+9k_1^2}{4}$ and $c_2 = \frac{3k_1k_2}{2}$.

$$J = \begin{vmatrix} -(1/2) & (c_1 - c_2 + \delta - \frac{1}{2})/2\omega \\ (-c_1 - c_2 - \delta + \frac{1}{2})/2\omega & -(1/2) \end{vmatrix}$$

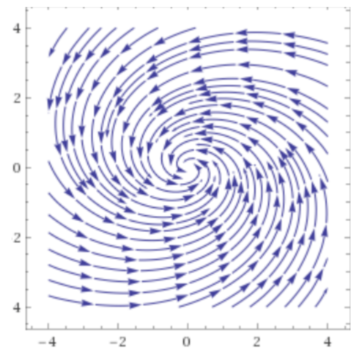
We calculate the characteristic equation using $|J - \lambda I| = 0$, and get,

$$\lambda_{1,2} = -\frac{1}{2} \pm \frac{i\sqrt{Q_0}}{\omega}$$

We take a look at the system in (57). (i) When $c_2 + \frac{1}{2} < c_1 + \delta \implies c_1 - c_2 + \delta - \frac{1}{2} > 0 \implies c_1 + c_2 + \delta - \frac{1}{2} > 0$ we get imaginary eigenvalues with a negative real part. This leads to a stable spiral for the linear system of ODEs.
(ii) When $c_2 + \frac{1}{2} > c_1 + \delta \implies c_1 - c_2 + \delta - \frac{1}{2} < 0$ which $\implies c_1 + c_2 + \delta - \frac{1}{2} < 0$ or $c_1 + c_2 + \delta - \frac{1}{2} > 0$ or $c_1 + c_2 + \delta - \frac{1}{2} = 0$. In each of these cases we get the following behaviour.

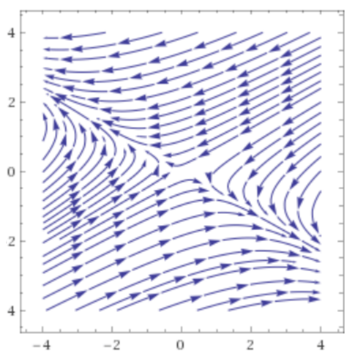


(a) $c_2 + \frac{1}{2} < c_1 + \delta$

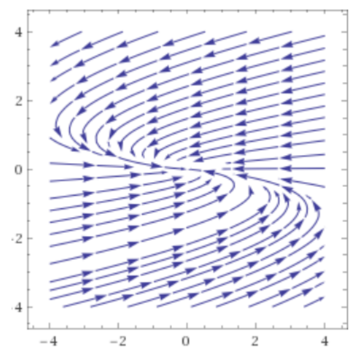


(b) $c_1 + c_2 + \delta - \frac{1}{2} < 0$

Figure 21



(a) $c_1 + c_2 + \delta - \frac{1}{2} > 0$



(b) $c_1 + c_2 + \delta - \frac{1}{2} = 0$

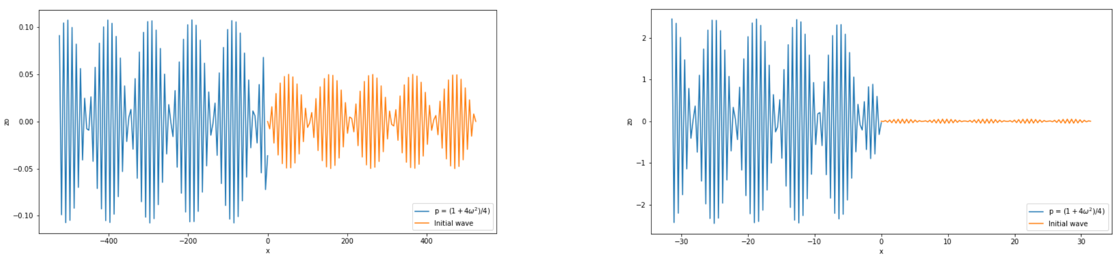
Figure 22: Bifurcations of Case 3 in Detuning

We can solve (57) using (56) to get,

$$A_0(\tau) = \frac{2(k_2\omega - k_1\omega)}{2\omega + 1} e^{-\tau/2} cos(\frac{\sqrt{Q_0}}{\omega}\tau)$$

$$B_0(\tau) = -\frac{k_1 + 2k_2\omega}{2\omega + 1} e^{-\tau/2} cos(\frac{\sqrt{Q_0}}{\omega}\tau)$$

where, $Q_0 = (c_1 + \delta) - (c_1 + \delta)^2 + (c_2^2 - \frac{1}{4})$. Now that we have the solutions, we plot the incoming wave and the reflected wave for a low and high omega value.



(a) Low omega, $\omega = 3$

(b) High omega, $\omega = 50$

Figure 23: Reflected waves- Case 3 detuned system and $\hat{q} = 0.01$

5.9 Energy of the system

In order to check whether the energy of the system reduces due to our system and to what extent it reduces, we must consider the energy of the system as a whole.

$$E_{tot}(t) = E_{string} + E_{spring}$$

The energy of the string is given by the kinetic energy of the string and the potential energy of the string. The upper limit of the integral comes from the fact that

$$c = \lambda/T$$

, where c is the wave speed which is $c = 1$ and the time period of a wave travelling with a frequency ω is $T = 2\pi/\omega$. This $\implies$ that the wavelength, i.e., $\lambda = c * T$, $\implies \lambda = 2\pi/\omega$.

Let us first consider the energy of the string.

$$E_{string}(t) = \int_0^\infty \frac{1}{2}(\dot{u}(x,t)^2 + u'(x,t)^2)dx$$

We examine the rate at which the energy of the string changes,

$$\frac{dE_{str}}{dt} = \frac{d}{dt}\left[\int_0^\infty \frac{1}{2}(\dot{u}(x,t)^2 + u'(x,t)^2)dx\right]$$

$$\frac{dE_{str}}{dt} = \int_0^\infty \Big(u_t u_{tt} + u_x u_{xt}\Big)dx \tag{58}$$

$$\int_0^\infty (u_x u_{xt})dx = u_x(\infty,t)u_t(\infty,t) - u_x(0,t)u_t(0,t) - \int_0^\infty u_{xx}u_t dx$$

We assume that the energy at $x = \infty$ is zero as our interest is only in the waves near the boundary at $x = 0$. This implies that,

$$\int_0^\infty (u_x u_{xt})dx = -u_x(0,t)u_t(0,t) - \int_0^\infty u_{xx}u_t dx \tag{59}$$

Substituting (59) in (58), we get,

$$\frac{dE_{str}}{dt} = \int_0^\infty u_t(u_{tt} - u_{xx})dx - u_x(0,t)u_t(0,t)$$

From (8) we know that $u_{tt} - u_{xx} = 0$, this $\implies$

$$\frac{dE_{str}}{dt} = -u_x(0,t)u_t(0,t)$$

We can substitute the boundary condition in (8) in the place of $u_x(0,t)$, to get,

$$\frac{dE_{str}}{dt} = \Big(- u_t(0,t)u_{tt}(0,t) - \hat{p}u_t(0,t)u(0,t) - \hat{q}u_t(0,t)u^3(0,t) \Big) - \hat{q}u_t^2(0,t)$$

$$\frac{dE_{str}}{dt} = \Big(- \frac{dE_{spr}}{dt} \Big) - -\hat{q}u_t^2(0,t)$$

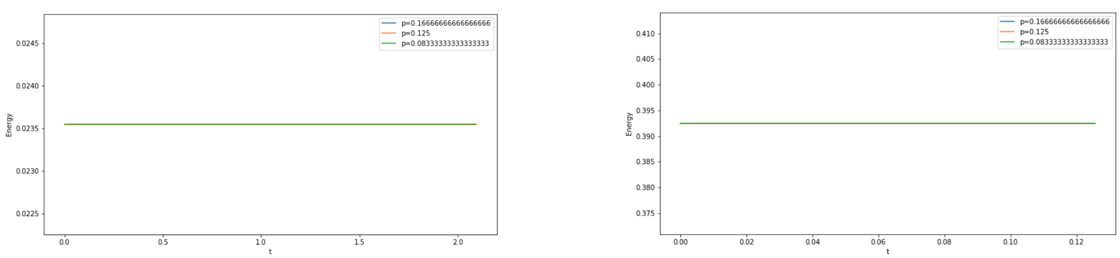
$$\frac{dE_{tot}}{dt} = -\hat{q}u_t^2(0,t) < 0$$

This implies that the energy in our system is decaying for all of our cases. The energy at time t can be calculated using the following expression.

$$E(t) - E(0) = -\hat{q} \int_0^t u_t^2(0,t) \tag{60}$$

where, $E(0)$ is the energy present in our initial wave, which is found to be $E(0) = A^2\omega\pi$, where A is the amplitude of the incoming wave and ω is the frequency of the initial wave.

The energy at $t = 0$ for a low value of omega, $\omega = 3$ and a high omega, $\omega = 50$ are given below,

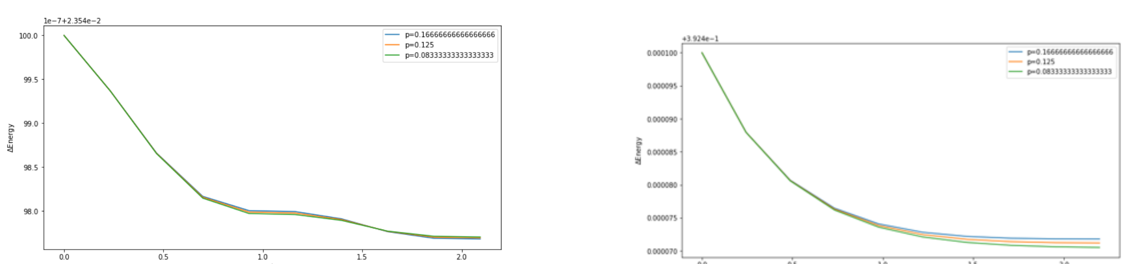


(a) $E(0)$ for low omega, $\omega = 3$

(b) $E(0)$ for high omega, $\omega = 50$

Figure 24: When $\hat{q} = 0$, no damping

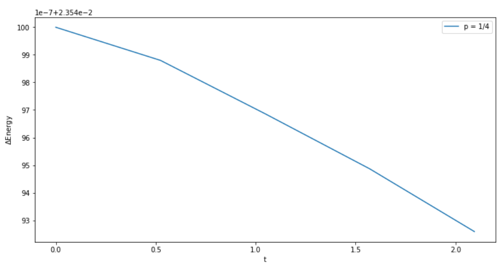
Now, we consider the energy decay for each of the leading order cases. For each of the intervals, we consider sample $\hat{p}$ values and plot the energy decay due to one incoming wave,i.e., the energy between $t = 0$ and $t = \frac{2\pi}{\omega}$.



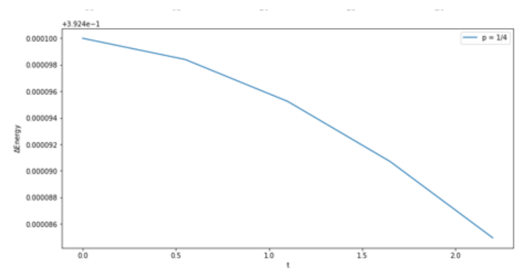
(a) $E(t)$ for low omega, $\omega = 3$

(b) $E(t)$ for high omega, $\omega = 50$

Figure 25: Case 1 leading order for $\hat{p} = 1/6, 1/8, 1/12$ and $\hat{q} = 0.00001$

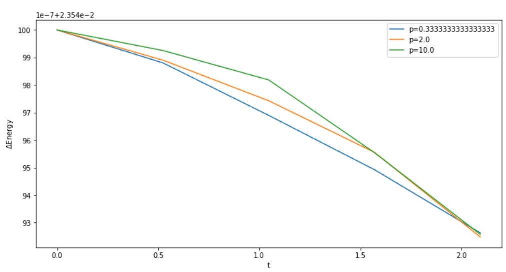


(a) $E(t)$ for low omega, $\omega = 3$

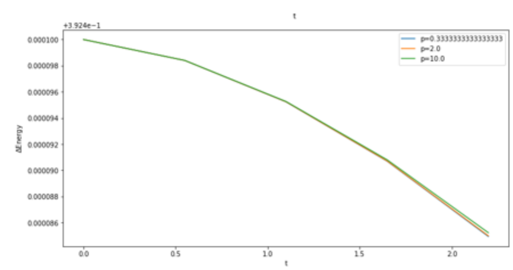


(b) $E(t)$ for high omega, $\omega = 50$

Figure 26: Case 2 leading order for $\hat{p} = 1/4$ and $\hat{q} = 0.00001$



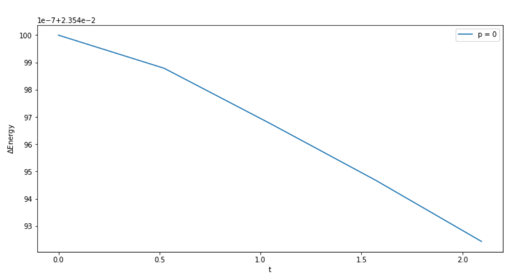
(a) $E(t)$ for low omega, $\omega = 3$



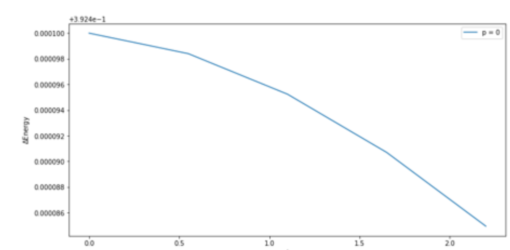
(b) $E(t)$ for high omega, $\omega = 50$

Figure 27: Case 3 leading order for $\hat{p} = 1/3, 1, 2$ and $\hat{q} = 0.00001$

Next, we consider the energy of the system in the special cases.

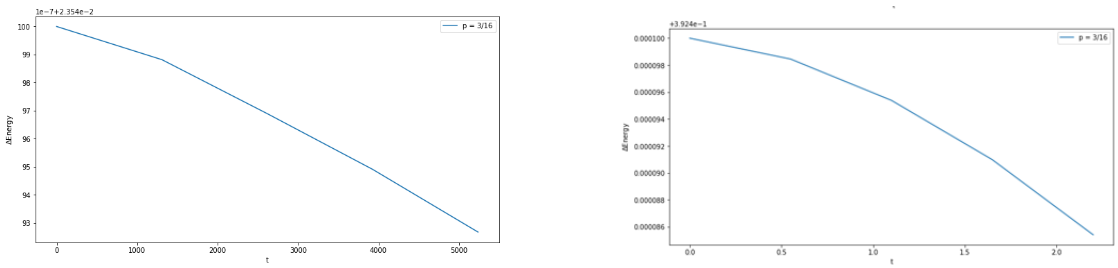


(a) $E(t)$ for low omega, $\omega = 3$



(b) $E(t)$ for high omega, $\omega = 50$

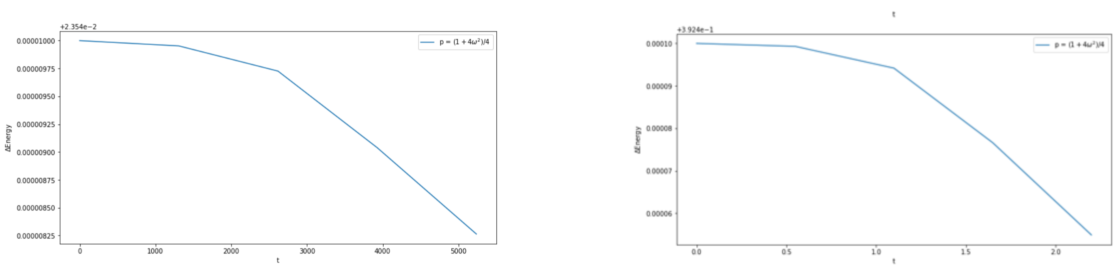
Figure 28: Case 1-special case for $\hat{p} = 0$ and $\hat{q} = 0.00001$



(a) $E(t)$ for low omega, $\omega = 3$

(b) $E(t)$ for high omega, $\omega = 50$

Figure 29: Case 2-special case for $\hat{p} = \frac{3}{16}$ and $\hat{q} = 0.00001$



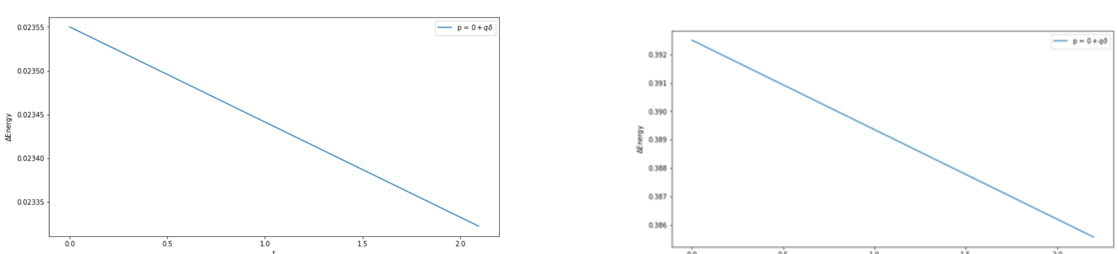
(a) $E(t)$ for low omega, $\omega = 3$

(b) $E(t)$ for high omega, $\omega = 50$

Figure 30: Case 3-special case for $\hat{p} = \frac{1+4\omega^2}{4}$ and $\hat{q} = 0.00001$

In the special cases above, we observe that the energies drop immediately for the cases where $\hat{p} = 0, 3/16$, which is in line with the evolution of the amplitudes given in Figure (10) and Figure (13). In special case 3 however, we see that the energy takes longer to start reducing. At higher frequencies, the energy stays at the same amplitude as the incident wave for a longer period of time, before it starts decaying. Overall the energy decreases over time, consistent with what was proved in (60).

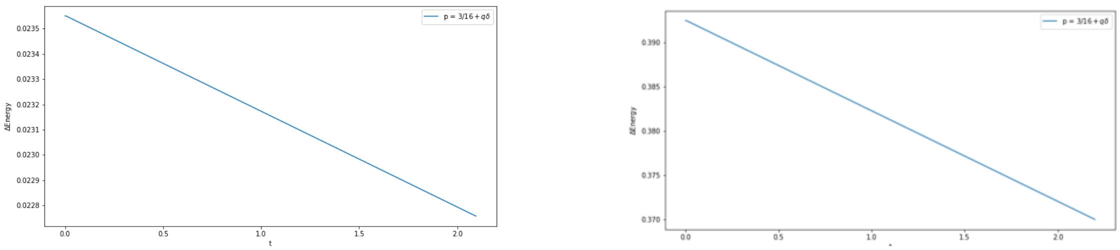
Finally, we consider the change in energy in the Detuned cases.



(a) $E(t)$ for low omega, $\omega = 3$

(b) $E(t)$ for high omega, $\omega = 50$

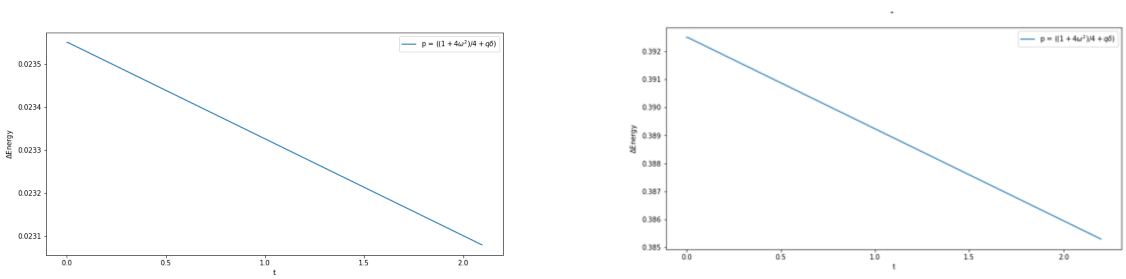
Figure 31: Case 1-detuned case for $\hat{p} = 0 + \hat{q}\delta$ and $\hat{q} = 0.00001$



(a) $E(t)$ for low omega, $\omega = 3$

(b) $E(t)$ for high omega, $\omega = 50$

Figure 32: Case 2-detuned case for $\hat{p} = \frac{3}{16} + \hat{q}\delta$ and $\hat{q} = 0.00001$



(a) $E(t)$ for low omega, $\omega = 3$

(b) $E(t)$ for high omega, $\omega = 50$

Figure 33: Case 3-detuned case for $\hat{p} = \frac{1+4\omega^2}{4} + \hat{q}\delta$ and $\hat{q} = 0.00001$

As we can see in Figures (31),(32) and (33), the energy decreases in the time interval $0 < t < \frac{2\pi}{\omega}$. This is supported by the corresponding reflected waves in Figures (17), (20) and (23). As we can see, there is an initial increase in amplitudes in all three cases and then there is a decay in the amplitude.

(60) was used to plot the decay of energy in the system. The trapezoidal approximation rule was used to calculate the amount of energy taken out of the system by the damper. Trapezoids are much more accurate representations of the area under the curve as opposed to using rectangles, which makes this a good approximation.

6 Conclusions and future research

In the mechanical system, we considered a hardening spring and observed that due to the presence of an initial velocity for the incident wave, for higher ω values, we observe that the reflected waves have an initial impact after which the wave stabilises. From Figures (25),(26),(27),(28),(29),(30),(31),(32) and (33), we see that the energy of the system between $t = 0$ and $t = \frac{2\pi}{\omega}$ is decaying for all the cases that we have discussed.

Our main objective was to show how the incident wave gets reflected at the boundary and to show that the presence of a nonlinear attachment at the boundary is responsible for an overall decay in the energy of the system. The research conducted in this study provides sufficient evidence to prove the same.

This topic can be extended by including new avenues for research such as, considering the case where the nonlinear spring at the boundary is of the softening type, considering different initial conditions, applying variations of the multiple time scales method and also obtaining more accurate approximations of the solutions, i.e., $z(t) = z_0(t,\tau) + \hat{q}z_1(t,\tau) + O(\hat{q}^2)$.

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