Inducing contractions of the mother of all continued fractions

Dajani, Karma; Kraaikamp, Cor; Sanderson, Slade

DOI
10.1016/j.jnt.2025.05.015

Publication date
2026

Document Version
Final published version

Published in
Journal of Number Theory

Journal of Number Theory 278 (2026) 816–874

Contents lists available at ScienceDirect

Journal of Number Theory

journal homepage: www.elsevier.com/locate/jnt

General Section

Inducing contractions of the mother of all continued fractions

Karma Dajani [a], Cor Kraaikamp [b], Slade Sanderson [a,*]

[a] *Department of Mathematics, Utrecht University, P.O. Box 80010, 3508 TA Utrecht, the Netherlands*
[b] *Delft University of Technology, EWI (DIAM), Mekelweg 4, 2628 CD Delft, the Netherlands*

A R T I C L E   I N F O

Article history:
Received 16 April 2025
Accepted 7 May 2025
Available online 19 June 2025
Communicated by S.J. Miller

MSC:
primary 11A55
secondary 37A05, 37A44

Keywords:
Continued fractions
Ergodic theory
Natural extension
Induced transformation
Contraction

A B S T R A C T

We introduce a new, large class of continued fraction algorithms producing what are called *contracted Farey expansions*. These algorithms are defined by coupling two acceleration techniques—*induced transformations* and *contraction*—in the setting of Shunji Ito's ([19]) natural extension of the Farey tent map, which generates 'slow' continued fraction expansions. In addition to defining new algorithms, we also realise several existing continued fraction algorithms in our unifying setting. In particular, we find regular continued fractions, the second-named author's S-expansions, and Nakada's parameterised family of α-continued fractions for all $0 < \alpha \le 1$ as examples of contracted Farey expansions. Moreover, we give a new description of a planar natural extension for each of the α-continued fraction transformations as an explicit induced transformation of Ito's natural extension.

* Corresponding author.
E-mail addresses: k.dajani1@uu.nl (K. Dajani), c.kraaikamp@tudelft.nl (C. Kraaikamp), s.b.sanderson@uu.nl (S. Sanderson).

https://doi.org/10.1016/j.jnt.2025.05.015

1. Introduction

In 1855, Seidel ([40]) introduced a seemingly overlooked,[1] arithmetic procedure, called *contraction*, which—under mild assumptions—allows one to produce from a given generalised continued fraction (GCF) a new GCF whose convergents are any prescribed subsequence of the original GCF-convergents. Nearly ninety years later, in 1943, Kakutani introduced in [21] *induced transformations*, which accelerate a given dynamical system by only observing the dynamics within a subregion of the domain. In 1989, Shunji Ito ([19]) gave an explicit natural extension of what has been called[2] 'the mother of all continued fractions'—the *Farey tent map*—which generates 'slow' continued fraction expansions (*Farey expansions*) whose convergents (*Farey convergents*) consist of all regular continued fraction (RCF) convergents and so-called *mediant convergents* (see §2.3 below). In this article, we obtain a broad, unifying theory for various continued fraction expansions by 'inducing contractions of the mother of all continued fractions.'

More formally, we use induced transformations of Ito's natural extension to govern contractions of Farey expansions. This coupling of inducing and contracting defines a large class of continued fraction algorithms—producing what we call *contracted Farey expansions*—which are parameterised by measurable subregions of the domain of Ito's natural extension. Within this collection of algorithms we find several well-studied examples. In particular, contracted Farey expansions contain the theory of the second-named author's S-expansions, which themselves contain the theory of RCFs, Minkowski's diagonal continued fractions, Bosma's optimal continued fractions and more ([23]). The collection of S-expansions also partially contains Nakada's parameterised family of α-continued fractions: this latter family is defined for $0 \leq \alpha \leq 1$, but only those for which $\alpha \geq 1/2$ are realised as S-expansions. Our theory of contracted Farey expansions contains Nakada's α-continued fractions for all $0 < \alpha \leq 1$—thus providing a unifying framework within which to view these two partially overlapping families—and gives a new description of the natural extension of each of the α-continued fraction transformations as an induced transformation of Ito's natural extension (cf. [24]).

In [13], the authors use a one-to-one correspondence between certain forward orbits determined by irrationals $x \in (0, 1)$ under Ito's natural extension map and the sequence of all Farey convergents (RCF-convergents and mediants) of x. With this correspondence, certain subregions of the domain of Ito's natural extension 'announce' certain types of Farey convergents. By considering induced transformations on these subregions, the authors obtain unified and simple proofs of results from, e.g., [3,6,19,20], old and classical results of Legendre and Koksma, and various new results such as generalisations of Lévy's

[1] Contraction *is* used in the analytic theory of continued fractions, but usually only for subsequences of odd or even integers ([27]). See also [5], where the more general contraction procedure is used on the continued fraction expansion of the golden mean, $(\sqrt{5} + 1)/2$.

[2] This is true 'up to isomorphism.' The maternal moniker was originally applied to the *Lehner map*, which is isomorphic to the Farey tent map ([12]); see also §4.2 below.

constant and of the Doeblin–Lenstra conjecture to subsequences of RCF-convergents and mediants.

The subsequences from [13] of Farey convergents announced by a subregion of the domain of Ito's natural extension are also of central importance in the current article: via contraction, these subsequences form the convergents of our new contracted Farey expansions. That is, we fix a subregion R of the domain Ω of Ito's natural extension and consider the subsequence of the forward orbit of a point $(x, y) \in \Omega$ which enters R under the natural extension map. Via the aforementioned one-to-one correspondence between orbits and Farey convergents, we obtain a subsequence of Farey convergents of x and use contraction to produce a new GCF-expansion of x whose convergents are precisely this subsequence. The digits of these new GCF-expansions may be described in terms of the dynamics of the induced transformation of Ito's natural extension on the subregion R, and hence we obtain a large collection of continued fraction algorithms parameterised by these subregions.

While the present article is informed by [13], these two works may be read independently. We remark, however, that the ideas of both articles may also be combined: in [39], the third-named author exploits results of [13] and the present article to generate new, *superoptimal continued fraction* algorithms producing GCF-expansions which have arbitrarily good approximation properties and converge arbitrarily fast.

This article is organised as follows. In §2 we set definitions and notation for generalised, semi-regular and regular continued fractions that are used throughout, and in §3 we recall several continued fraction algorithms: the Gauss map and its natural extension, Nakada's α-continued fractions and the second-named author's S-expansions. We recall the Farey tent map, Farey expansions and Farey convergents in §4. In §5 we describe Ito's natural extension of the Farey tent map, and, moreover, define and set notation for induced transformations of it (§5.2). In §6.1 we describe contraction in the abstract setting of generalised continued fractions and in §6.2 use induced transformations of Ito's natural extension to govern contractions of Farey expansions. Furthermore, in §6.3 we define a dynamical system which acts essentially as a two-sided shift on contracted Farey expansions. Section 7 realises each of the examples from §3 within our theory of contracted Farey expansions.

Acknowledgments. We thank the anonymous referee who reviewed this article. This work is part of project number 613.009.135 of the research programme Mathematics Clusters which is financed by the Dutch Research Council (NWO).

2. Generalised, semi-regular and regular continued fractions

2.1. Generalised continued fractions

A *generalised continued fraction* (GCF) is a formal (infinite or finite) expression of the form

$$[\beta_0/\alpha_0;\alpha_1/\beta_1,\alpha_2/\beta_2,\dots] = \cfrac{\alpha_{-1}}{\beta_{-1}+\cfrac{\alpha_0}{\beta_0+\cfrac{\alpha_1}{\beta_1+\cfrac{\alpha_2}{\beta_2+\ddots}}}}, \tag{1}$$

where $(\alpha_{-1},\beta_{-1}) := (1,0)$ and for $n \geq 0$, $\alpha_n,\beta_n \in \mathbb{C}$ with $\alpha_n \neq 0$.

Remark 2.1. Notice that for $\alpha_0,\beta_0,x \in \mathbb{C}$ with α_0 nonzero,

$$\cfrac{1}{0+\cfrac{\alpha_0}{\beta_0+x}} = \frac{1}{\alpha_0}(\beta_0+x),$$

with the convention that $c/0=\infty$ and $c/\infty = 0$ for $c \in \mathbb{C}\setminus\{0\}$. Thus—although at this point it is a strictly formal expression—a GCF should be thought of as $1/\alpha_0$ 'multiplied' with the expression

$$\beta_0+\cfrac{\alpha_1}{\beta_1+\cfrac{\alpha_2}{\beta_2+\ddots}}$$

(hence the choice of notation $[\beta_0/\alpha_0;\alpha_1/\beta_1,\dots]$ rather than $[\alpha_0/\beta_0;\alpha_1/\beta_1,\dots]$). Besides allowing for this inversion of α_0, our inclusion of α_{-1} and β_{-1} in (1) also prevents us from needing to treat the index 0 as a special case in the matrix notation introduced below.

The digits α_n and β_n are called *partial numerators* and *partial denominators*, respectively. When a GCF has only finitely many partial numerators and partial denominators, the expression on the right-hand side of (1) may be evaluated to give a number in $\widehat{\mathbb{C}} := \mathbb{C}\cup\{\infty\}$. Define for each integer $n \geq -2$ (with the obvious restriction in the finite case) the n^{th} *tail* of $[\beta_0/\alpha_0;\alpha_1/\beta_1,\alpha_2/\beta_2,\dots]$ to be the GCF

$$[0/1;\alpha_{n+1}/\beta_{n+1},\alpha_{n+2}/\beta_{n+2},\alpha_{n+3}/\beta_{n+3},\dots].$$

For each integer $n \geq -1$, set

$$B_n = B_n([\beta_0/\alpha_0;\alpha_1/\beta_1,\dots]) := \begin{pmatrix} 0 & \alpha_n \\ 1 & \beta_n \end{pmatrix},$$

and for integers $-1 \leq m \leq n$, let

$$B_{[m,n]} = B_{[m,n]}([\beta_0/\alpha_0;\alpha_1/\beta_1,\dots]) := B_m B_{m+1}\cdots B_n.$$

Notice that $\det B_{[m,n]} = (-1)^{n-m+1}\alpha_m\alpha_{m+1}\dots\alpha_n \neq 0$. For a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2\mathbb{C}$, denote by $A \cdot z := \frac{az+b}{cz+d}$, $z \in \widehat{\mathbb{C}}$, the action of A as a Möbius transformation. (We remark that for any $r \in \mathbb{C}\backslash\{0\}$, $(rA)\cdot z = A\cdot z$; this fact will be used repeatedly throughout.) Writing the entries of $B_{[m,n]}$ as $\left(\begin{smallmatrix} R_{[m,n]} & P_{[m,n]} \\ S_{[m,n]} & Q_{[m,n]}\end{smallmatrix}\right)$, we have

$$\begin{aligned}\frac{P_{[m,n]}}{Q_{[m,n]}} = B_{[m,n]}\cdot 0 &= \cfrac{\alpha_m}{\beta_m + \cfrac{\alpha_{m+1}}{\beta_{m+1} + \cfrac{\alpha_{m+2}}{\ddots + \cfrac{\alpha_n}{\beta_n}}}} \\ &= [0/1; \alpha_m/\beta_m, \alpha_{m+1}/\beta_{m+1}, \dots, \alpha_n/\beta_n] \in \widehat{\mathbb{C}}.\end{aligned}$$

(Notice that if each $\alpha_j, \beta_j \in \mathbb{Z}$, then $P_{[m,n]}/Q_{[m,n]} \in \mathbb{Q}\cup\{\infty\}$, but in general, $\gcd(P_{[m,n]}, Q_{[m,n]}) \neq 1$.) When $m = -1$, we use the suppressed notation

$$\begin{pmatrix} R_n & P_n \\ S_n & Q_n\end{pmatrix} := \begin{pmatrix} R_{[-1,n]} & P_{[-1,n]} \\ S_{[-1,n]} & Q_{[-1,n]}\end{pmatrix} = B_{[-1,n]}$$

and call P_n/Q_n the n^{th} *convergent* of[3] $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \dots]$. If a GCF is finite and evaluates to $x \in \mathbb{C}$, or if it is infinite and $x = \lim_{n\to\infty} P_n/Q_n \in \mathbb{C}$ exists, we call $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \dots]$ a *GCF-expansion* of x, write $x = [\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \dots]$ and—when the expansion $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \dots]$ is understood—refer to the convergents P_n/Q_n as convergents of x.

Notice for any integers $-1 \leq m \leq n$ that

$$\begin{aligned}\begin{pmatrix} R_{[m,n+1]} & P_{[m,n+1]} \\ S_{[m,n+1]} & Q_{[m,n+1]}\end{pmatrix} &= B_{[m,n]}B_{n+1} = \begin{pmatrix} R_{[m,n]} & P_{[m,n]} \\ S_{[m,n]} & Q_{[m,n]}\end{pmatrix}\begin{pmatrix} 0 & \alpha_{n+1} \\ 1 & \beta_{n+1}\end{pmatrix} \\ &= \begin{pmatrix} P_{[m,n]} & \beta_{n+1}P_{[m,n]} + \alpha_{n+1}R_{[m,n]} \\ Q_{[m,n]} & \beta_{n+1}Q_{[m,n]} + \alpha_{n+1}S_{[m,n]}\end{pmatrix}.\end{aligned}$$

In particular, $R_{[m,n+1]} = P_{[m,n]}$ and $S_{[m,n+1]} = Q_{[m,n]}$. Setting $(P_{[m,m-1]}, Q_{[m,m-1]}) := (0, 1)$ for all $m \geq -1$, this gives

$$B_{[m,n]} = \begin{pmatrix} P_{[m,n-1]} & P_{[m,n]} \\ Q_{[m,n-1]} & Q_{[m,n]}\end{pmatrix}, \quad -1 \leq m \leq n,$$

and we obtain the following recurrence relations for all $-1 \leq m \leq n$:

$$\begin{aligned} P_{[m,n+1]} &= \beta_{n+1}P_{[m,n]} + \alpha_{n+1}P_{[m,n-1]}, & P_{[m,m-1]} &= 0,\ P_{[m,m]} = \alpha_m, \\ Q_{[m,n+1]} &= \beta_{n+1}Q_{[m,n]} + \alpha_{n+1}Q_{[m,n-1]}, & Q_{[m,m-1]} &= 1,\ Q_{[m,m]} = \beta_m.\end{aligned} \tag{2}$$

[3] Note that $[0/1; \alpha_{-1}/\beta_{-1}, \alpha_0/\beta_0, \dots, \alpha_n/\beta_n] = [\beta_0/\alpha_0; \alpha_1/\beta_1, \dots, \alpha_n/\beta_n]$.

Let $(P_{-2}, Q_{-2}) := (0, 1)$. When $m = -1$, the observations above give, for $n \geq -1$,

$$B_{[-1,n]} = \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix}$$

and the recurrence relations

$$\begin{aligned} P_{n+1} &= \beta_{n+1}P_n + \alpha_{n+1}P_{n-1}, \qquad && P_{-2} = 0,\ P_{-1} = 1, \\ Q_{n+1} &= \beta_{n+1}Q_n + \alpha_{n+1}Q_{n-1}, \qquad && Q_{-2} = 1,\ Q_{-1} = 0. \end{aligned} \tag{3}$$

Remark 2.2. The quantities P_n, Q_n are defined in terms of the partial numerators and partial denominators α_n, β_n of a GCF. Conversely, since $\det(B_{[-1,n]}) \neq 0$, the digits α_n, β_n are also determined by the quantities P_n, Q_n. In particular, the recurrence relations (3) imply

$$\begin{pmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{pmatrix} = B_{[-1,n]}^{-1} \begin{pmatrix} P_{n+1} \\ Q_{n+1} \end{pmatrix}, \qquad n \geq -1.$$

Remark 2.3. It shall sometimes be useful to allow for infinite partial denominators $\beta_n = \infty$ for some $n \geq 1$ in a GCF $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \ldots]$. In this case, letting $n_0 \geq 0$ denote smallest index for which $\beta_{n_0+1} = \infty$, the GCF $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \ldots]$ is interpreted to be the finite GCF $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \ldots, \alpha_{n_0}/\beta_{n_0}]$.

Letting $T_n := [0/1; \alpha_{n+1}/\beta_{n+1}, \alpha_{n+2}/\beta_{n+2}, \ldots]$, $n \geq 0$, denote the n^{th} tail of the GCF-expansion $x = [\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \ldots]$, one obtains

$$x = \cfrac{\alpha_{-1}}{\beta_{-1} + \cfrac{\alpha_0}{\beta_0 + \cfrac{\alpha_1}{\ddots + \cfrac{\alpha_n}{\beta_n + T_n}}}} = \begin{pmatrix} 0 & \alpha_{-1} \\ 1 & \beta_{-1} \end{pmatrix} \begin{pmatrix} 0 & \alpha_0 \\ 1 & \beta_0 \end{pmatrix} \cdots \begin{pmatrix} 0 & \alpha_n \\ 1 & \beta_n \end{pmatrix} \cdot T_n = B_{[-1,n]} \cdot T_n. \tag{4}$$

Notice also that for any $z \in \widehat{\mathbb{C}}$,

$$B_{[-1,n]}^T \cdot z = \begin{pmatrix} 0 & 1 \\ \alpha_n & \beta_n \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot z = \cfrac{1}{\beta_n + \cfrac{\alpha_n}{\beta_{n-1} + \cfrac{\alpha_{n-1}}{\ddots + \cfrac{\alpha_1}{\beta_0 + \cfrac{\alpha_0}{z}}}}}, \tag{5}$$

or

$$B^T_{[-1,n]} \cdot z = [0/1; 1/\beta_n, \alpha_n/\beta_{n-1}, \dots, \alpha_1/\beta_0, \alpha_0/z], \tag{6}$$

where, in the case that $z = \infty$, the right-hand side is interpreted as $[0/1; 1/\beta_n, \alpha_n/\beta_{n-1}, \dots, \alpha_1/\beta_0]$.

2.2. Semi-regular continued fractions

A *semi-regular continued fraction* (SRCF) is a GCF as in (1) with integral partial numerators and partial denominators $\alpha_n, \beta_n \in \mathbb{Z}$ satisfying

(i) $\alpha_0 = 1$ and $\alpha_n = \pm 1$ for $n \geq 1$,
(ii) $\beta_n > 0$ for $n \geq 1$, and
(iii) $\alpha_{n+1} + \beta_n \geq 1$ for $n \geq 1$.

If there are infinitely many digits, we further require

(iv) $\alpha_{n+1} + \beta_n \geq 2$ infinitely often.

By Tietze's Convergence Theorem (see, say, [37]) the above conditions guarantee that $x = \lim_{n\to\infty} P_n/Q_n \in \mathbb{R}$ always exists, and thus we call $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \dots]$ a SRCF-expansion of x. Notice that the convergents P_n/Q_n of any SRCF-expansion of $x \in \mathbb{R}$ are reduced since

$$|P_{n-1}Q_n - P_nQ_{n-1}| = |\det(B_{[-1,n]})| = |\alpha_{-1}\alpha_0 \cdots \alpha_n| = 1.$$

2.3. Regular continued fractions

A *regular continued fraction* (RCF) is a SRCF with partial numerators $\alpha_n = 1$ for $n \geq 1$. (Note that with this assumption on partial numerators, conditions (iii) and (iv) of SRCFs are trivially satisfied for any choice of integral partial denominators satisfying condition (ii).) A RCF will also be denoted by

$$[a_0; a_1, a_2, \dots] := [a_0/1; 1/a_1, 1/a_2, \dots], \quad a_n \in \mathbb{Z} \quad \text{with} \quad a_n > 0,\ n \neq 0.$$

For a RCF, we use the special notation $p_n := P_n$ and $q_n := Q_n,\ n \geq -2$, so the recurrence relations (3) become

$$\begin{aligned} p_{n+1} &= a_{n+1}p_n + p_{n-1}, & p_{-2} &= 0,\ p_{-1} = 1, \\ q_{n+1} &= a_{n+1}q_n + q_{n-1}, & q_{-2} &= 1,\ q_{-1} = 0. \end{aligned} \tag{7}$$

Since a RCF is a SRCF, the limit $x = \lim_{n\to\infty} p_n/q_n \in \mathbb{R}$ exists for any infinite choice of $a_n,\ n \geq 0$ (this can also be proven directly; see, e.g., [17]), and the odd- and even-indexed

RCF-convergents $(p_{2k-1}/q_{2k-1})_{k\geq 0}$ and $(p_{2k}/q_{2k})_{k\geq 0}$ form strictly decreasing and strictly increasing sequences, respectively (see, e.g., Theorem 4 of [22]). Conversely, every real number x has a RCF-expansion. Moreover, if x is irrational, its RCF-expansion is unique and has infinitely many partial denominators a_n; if x is rational, it has exactly two RCF-expansions,

$$[a_0; a_1, \ldots, a_n] \quad \text{and} \quad [a_0; a_1, \ldots, a_n - 1, 1],$$

where $a_n \geq 2$ if $n \geq 1$ ([17]).

2.3.1. Mediant convergents

The fractions

$$\frac{\lambda p_n + p_{n-1}}{\lambda q_n + q_{n-1}} \quad \text{for } \lambda \in \mathbb{N},\ 1 \leq \lambda < a_{n+1},\ n \geq -1, \tag{8}$$

are called the *mediants* (or *mediant convergents*) of $x = [a_0; a_1, a_2, \ldots]$. Notice that if $\lambda = 0$, the expression in (8) gives p_{n-1}/q_{n-1}, while if $\lambda = a_{n+1}$, the expression gives p_{n+1}/q_{n+1} by the recurrence relations (7). Since the mediant $(a+b)/(c+d)$ of two fractions a/c and b/d with positive denominators lies between them in value, monotonicity of the odd-/even-indexed RCF-convergents gives the following relations for all $n \geq 0$ (see §1.4 of [22]):

$$\begin{aligned} x < \frac{p_{2n+1}}{q_{2n+1}} = \frac{a_{2n+1}p_{2n} + p_{2n-1}}{a_{2n+1}q_{2n} + q_{2n-1}} < \frac{(a_{2n+1}-1)p_{2n} + p_{2n-1}}{(a_{2n+1}-1)q_{2n} + q_{2n-1}} < \cdots \\ < \frac{p_{2n} + p_{2n-1}}{q_{2n} + q_{2n-1}} < \frac{p_{2n-1}}{q_{2n-1}} \end{aligned} \tag{9}$$

and

$$\frac{p_{2n}}{q_{2n}} < \frac{p_{2n+1} + p_{2n}}{q_{2n+1} + q_{2n}} < \cdots < \frac{(a_{2n+2}-1)p_{2n+1} + p_{2n}}{(a_{2n+2}-1)q_{2n+1} + q_{2n}} < \frac{a_{2n+2}p_{2n+1} + p_{2n}}{a_{2n+2}q_{2n+1} + q_{2n}} = \frac{p_{2n+2}}{q_{2n+2}} < x. \tag{10}$$

3. Some continued fraction algorithms

In this section we introduce some important continued fraction (CF) algorithms which shall be revisited throughout the paper. In particular, the reader will find in §3.1 the Gauss map, which generates RCF-expansions; in §3.2 Nakada's parameterised family of α-CFs, which generate SRCFs including RCFs, Hurwitz's singular CFs, nearest integer CFs, and Rényi's backward CFs; and in §3.3 the second-named author's S-expansions, which generate SRCFs including Minkowski's diagonal CFs, Bosma's optimal CFs and (a strict subcollection of) Nakada's α-CFs.

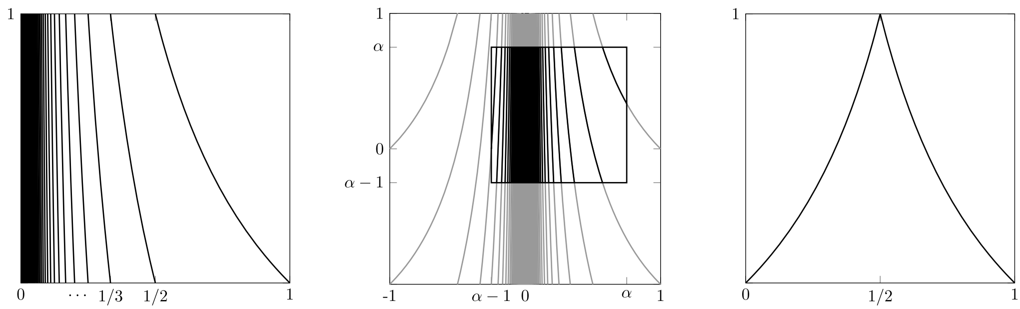


Fig. 1. Graphs of the Gauss map G (left), Nakada's α-CF maps G_α (centre), and the Farey tent map F (right).

3.1. The Gauss map

3.1.1. The Gauss map

The partial denominators a_n of RCF-expansions are generated by the *Gauss map* $G : [0,1] \to [0,1]$ defined by[4] $G(0) = 0$ and $G(x) = 1/x - \lfloor 1/x \rfloor$, $x > 0$; see Fig. 1. Indeed, for $x \in \mathbb{R}$, set $a_0 = a_0(x) := \lfloor x \rfloor$ and $x_0 := x - a_0 \in [0,1)$. Define $a(0) := \infty$, $a(x) := \lfloor 1/x \rfloor$ for $x \neq 0$, and $a_n = a_n(x) := a(G^{n-1}(x_0))$ for $n > 0$. Notice that for any integer $k \geq 1$, $a_n = k$ if and only if $G^{n-1}(x_0) \in (1/(k+1), 1/k]$. One finds that for $G^{n-1}(x_0) \neq 0$,

$$G^n(x_0) = \frac{1}{G^{n-1}(x_0)} - a_n.$$

Rearranging gives

$$G^{n-1}(x_0) = \frac{1}{a_n + G^n(x_0)},$$

which holds for both $G^{n-1}(x_0) \neq 0$ and $G^{n-1}(x_0) = 0$, and which—with repeated applications—in turn gives

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n + G^n(x_0)}}} = [a_0; a_1, \dots, a_{n-1}, a_n + G^n(x_0)].$$

Symbolically, the Gauss map acts as a left-shift on RCF-expansions. That is, if $x = [0; a_1, a_2, \dots] \in (0,1)$, then $G(x) = [0; a_2, a_3, \dots]$.

[4] While G may be defined as a self-map of $[0,1)$, we choose to include the endpoint 1 for later notational purposes.

The dynamical system $([0,1], \mathcal{B}, \nu_G, G)$ is exact (and hence strongly mixing, weakly mixing and ergodic; see [17]), where $\mathcal{B}$ is the Borel σ-algebra[5] on $[0,1]$ and ν_G is the *Gauss measure*, which is the absolutely continuous, G-invariant probability measure with density $1/((\log 2)(1+x))$.

3.1.2. The natural extension of the Gauss map

In the late 1970s and early 1980s, Nakada, Ito and Tanaka ([32,33]) introduced an explicit, planar natural extension $(\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$ of the system $([0,1], \mathcal{B}, \nu_G, G)$. Here $\Omega := [0,1]^2$; the map $\mathcal{G} : \Omega \to \Omega$ is defined by $\mathcal{G}(0,y) = (0,y)$ and for $z = (x,y) \in \Omega$ with $x \neq 0$,

$$\mathcal{G}(z) := \left(G(x), \frac{1}{a(x)+y}\right); \tag{11}$$

and the $\mathcal{G}$-invariant probability measure $\bar{\nu}_G$ has density $1/((\log 2)(1+xy)^2)$. Since $([0,1], \mathcal{B}, \nu_G, G)$ is strongly mixing, so is the natural extension $(\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$.

Symbolically, the map $\mathcal{G}$ acts as a two-sided-shift on RCF-expansions. That is, if

$$(x,y) = ([0; a_1, a_2, \dots], [0; b_1, b_2, \dots]) \in \Omega$$

with $x \in (0,1)$, then

$$\mathcal{G}(x,y) = ([0; a_2, a_3, \dots], [0; a_1, b_1, b_2, \dots]). \tag{12}$$

The map $\mathcal{G}$ may also be understood geometrically: setting

$$V_k := \left(\frac{1}{k+1}, \frac{1}{k}\right] \times [0,1] \quad \text{and} \quad H_k := [0,1] \times \left(\frac{1}{k+1}, \frac{1}{k}\right] \tag{13}$$

for each integer $k \geq 1$, one finds that $\mathcal{G}(V_k) = H_k$, up to null sets; see Fig. 2. We call V_k and H_k the k^{th} *vertical* and *horizontal regions*, respectively.

3.2. Nakada's α-continued fractions

In 1981, Nakada ([32]) introduced a one-parameter family of continued fraction algorithms, called *α-CF maps*, each of which generates—in a similar fashion as the Gauss map—SRCF-expansions. For each $\alpha \in [0,1]$, Nakada's α-CF map $G_\alpha : [\alpha-1, \alpha] \to [\alpha-1, \alpha]$ is defined by $G_\alpha(0) := 0$ and for $x \neq 0$,

$$G_\alpha(x) := \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor;$$

[5] Throughout, $\mathcal{B}$ represents the Borel σ-algebra on the appropriate domain.

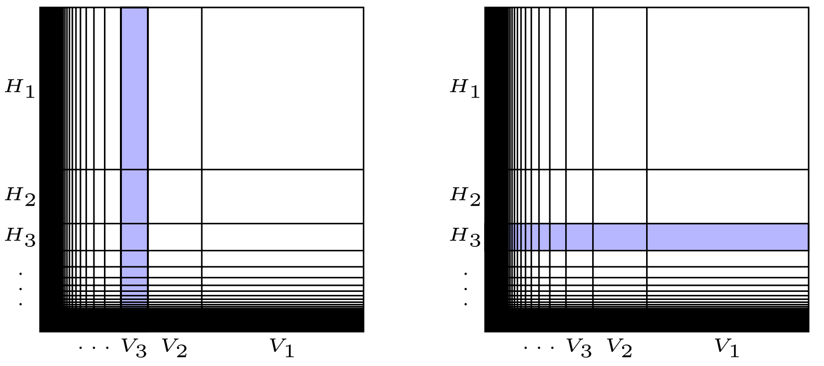


Fig. 2. Up to null sets, the map $\mathcal{G}$ sends the vertical region V_k to the horizontal region H_k.

see Fig. 1. Notice that $G_1 = G$ is the Gauss map. In fact, Nakada's α-CFs contain several other well-studied continued fraction algorithms: when $\alpha = (\sqrt{5}-1)/2$, G_α generates Hurwitz's *singular* CFs ([16]); $\alpha = 1/2$ generates the *nearest integer* CFs introduced by Minnigerode in [30] and studied by Hurwitz in [16]; and $\alpha = 0$ generates Rényi's *backward* CFs ([38]). The latter map G_0 has an infinite, σ-finite, absolutely continuous invariant measure ρ_0 with density $1/(x+1)$, while for $\alpha \in (0,1]$, there is a unique, absolutely continuous invariant probability measure ρ_α ([28]). Moreover, for each $\alpha \in [0,1]$, the dynamical system $([\alpha-1,\alpha], \mathcal{B}, \rho_\alpha, G_\alpha)$ is exact and, hence, ergodic ([28]).

Since Nakada's introduction of the α-CFs, much work has been done to determine explicitly the invariant measures ρ_α and to understand the metric entropy $h(G_\alpha) = h_{\rho_\alpha}(G_\alpha)$ as a function $\alpha \mapsto h(G_\alpha)$ of $\alpha \in (0,1]$. Nakada restricted his initial study in [32] to $\alpha \geq 1/2$ and derived ρ_α by constructing an explicit, planar natural extension of $([\alpha-1,\alpha], \mathcal{B}, \rho_\alpha, G_\alpha)$. However, it was observed at the time that difficulties arose in extending these methods to $\alpha < 1/2$. The second-named author in 1991 ([23]) reobtained Nakada's natural extensions for $\alpha \geq 1/2$ in a simple fashion as special instances within his theory of S-expansions; see §3.3 below. In 1999, Moussa et al. ([31]) determined explicit, absolutely continuous invariant probability measures for a subset of a slightly different family of maps called *folded* α-CF*s*, which are factors of the α-CF maps. From their results one could obtain ρ_α for $\sqrt{2}-1 \leq \alpha < 1/2$ and—using Rohlin's formula—the entropy $h(G_\alpha)$ as a function of $\alpha \in [\sqrt{2}-1, 1]$ (see [28]):

$$h(G_\alpha) = \begin{cases} \frac{\pi^2}{6\log(1+g)}, & \sqrt{2}-1 \leq \alpha \leq g, \\ \frac{\pi^2}{6\log(1+\alpha)}, & g < \alpha \leq 1, \end{cases} \tag{14}$$

where $g := (\sqrt{5}-1)/2$. Following this, the entropy function was conjectured to be monotone increasing and continuous on the remaining subinterval $(0, \sqrt{2}-1)$ ([4]).

It was thus quite surprising when, in 2008, Luzzi and Marmi ([28]) gave numerical evidence suggesting that $h(G_\alpha)$ possessed a seemingly complicated, non-monotone, self-similar structure on $(0, \sqrt{2}-1)$. In the same year, Nakada and Natsui ([34]) confirmed this non-monotonicity by giving countably many non-empty intervals on which the function is increasing, decreasing and constant, respectively. These intervals are determined by a phenomenon called *matching*, where the G_α-orbits of α and $\alpha - 1$ coincide after some finite number of steps. Further numerical data on these so-called *matching intervals* was given in [7], and the authors also exhibited points in the complement of the union of matching intervals at which the entropy function fails to be locally monotone. The matching intervals were completely classified in [8], and their union was shown to have full measure. (These intervals have surprising connections to unimodal maps, the real slice of the boundary of the Mandelbrot set, and the parameter space of a family of maps producing signed binary expansions ([1,9,10]).)

In 2012, Kraaikamp, Schmidt and Steiner ([24]) proved that the entropy function is indeed continuous on $(0, 1]$ (this fact had also been proven in a 2009 preprint of Tiozzo for $\alpha > 0.056\ldots$ and was later improved to Hölder continuity on $(0, 1]$ ([41,42])). In [24], the authors construct a planar natural extension of $([\alpha - 1, \alpha], \mathcal{B}, \rho_\alpha, G_\alpha)$ for each $\alpha > 0$. The domain of this natural extension is first defined theoretically as an orbit closure of a certain planar map; a further detailed analysis of the G_α-orbits of α and $\alpha - 1$ allow for a more explicit description of this domain (see §7 of [24]). Moreover, the authors prove (Theorem 2 of [24]) a conjecture of Luzzi and Marmi ([28]) that the product of the entropy $h(G_\alpha)$ and the measure of the natural extension domain (using density $1/(1 + xy)^2$) is constant—in fact, equal to $\pi^2/6$—as a function of α, and they extend the constant branch of $h(G_\alpha)$ in (14) to the maximal interval $[g^2, g]$. However, even equipped with such machinery, a number of open questions are left at the end of [24]. In particular, the authors ask for explicit values of the entropy $h(G_\alpha)$ for $\alpha < g^2$, and they restate a conjecture of [7] on the explicit form of the density of ρ_α.

3.3. S-expansions

In 1991, the second-named author introduced in [23] a large class of new continued fraction algorithms by coupling two tools: singularisation and induced transformations of the natural extension $(\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$ of the Gauss map. Singularisation is an old, arithmetic procedure—tracing back as early as Lagrange's addendum ([25]) to Euler's *Vollständige Anleitung zur Algebra*—whereby one can sometimes manipulate a SRCF-expansion to produce a new, 'accelerated' SRCF-expansion of the same number.

Indeed, suppose that x has a SRCF-expansion $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \ldots]$ with convergents $(P_n/Q_n)_{n\geq -1}$. Suppose, moreover, that $\beta_{n+1} = \alpha_{n+2} = 1$ for some $n \geq 0$. *Singularisation at position* n replaces this SRCF-expansion with the SRCF-expansion

$$x = [\beta_0/\alpha_0; \alpha_1/\beta_1, \ldots, \alpha_{n-1}/\beta_{n-1}, \alpha_n/(\beta_n + \alpha_{n+1}), -\alpha_{n+1}/(\beta_{n+2} + 1), \alpha_{n+3}/\beta_{n+3}, \ldots].$$

One can show that the convergents $(P'_n/Q'_n)_{n\geq -1}$ of this new expansion satisfy

$$\frac{P'_j}{Q'_j} = \begin{cases} \frac{P_j}{Q_j}, & j < n, \\ \frac{P_{j+1}}{Q_{j+1}}, & j \geq n, \end{cases}$$

i.e., singularisation at position n removes the n^{th} convergent P_n/Q_n; see [23]. By iterating this procedure (possibly infinitely many times), one obtains a new SRCF-expansion of x whose convergents are a subsequence $(P_{n_k}/Q_{n_k})_{k\geq -1}$ of the original convergents.

Beginning from a RCF-expansion $[a_0; a_1, a_2, \dots] = [a_0/1; 1/a_1, 1/a_2, \dots]$ with convergents p_n/q_n, acceleration via singularisation admits two major restrictions:

(i) the convergents p_n/q_n which are removed correspond to partial denominators $a_{n+1} = 1$, and
(ii) consecutive convergents $p_n/q_n,\ p_{n+1}/q_{n+1}$ cannot be removed.

Restriction (ii) comes from the fact that in order to remove both p_n/q_n and p_{n+1}/q_{n+1}, one would need to either first singularise the original expansion at position n, then singularise the new expansion again at position n, or first singularise at position $n+1$ and then at position n. However, in either case, the partial denominator corresponding to the second singularisation is strictly greater than 1, contrary to the singularisation requirements. Nevertheless, one may singularise to remove non-consecutive convergents p_n/q_n with $a_{n+1} = 1$ independent of order and, thus, simultaneously; see [23].

In [23], the natural extension $(\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$ of the Gauss map is used to govern the singularisation process, beginning from RCF-expansions. In particular, one fixes a measurable *singularisation area*[6] $S \subset \Omega$ satisfying $\bar{\nu}_G(\partial S) = 0$,

(a) $S \subset V_1$, and
(b) $S \cap \mathcal{G}(S) = \varnothing$,

and considers the $\mathcal{G}$-orbit of the point $(x, 0)$ in Ω with $x = [0; a_1, a_2, \dots]$ irrational. That $\bar{\nu}_G(\partial S) = 0$ is a technical condition, called $\bar{\nu}_G$*-continuity*, guaranteeing that for Lebesgue–a.e. x, the $\mathcal{G}$-orbit of $(x, 0)$ behaves like a '$\bar{\nu}_G$-generic' point; see Remark 4.6.i of [23]. Condition (a) guarantees that if $\mathcal{G}^n(x, 0) \in S$, then $a_{n+1} = 1$, and condition (b) guarantees that two consecutive points in the $\mathcal{G}$ orbit of $(x, 0)$ do not belong to S; cf. restrictions (i) and (ii) above. By simultaneously singularising the RCF-expansion of x at all positions n for which $\mathcal{G}^n(x, 0) \in S$, one obtains a SRCF-expansion of x, called an S*-expansion*, whose convergents are a subsequence of the RCF-convergents of x. Put a different way, S-expansions are SRCFs whose convergents are precisely the subsequence

[6] Technically, condition (a) should be replaced by $S \subset \overline{V_1}$ and (b) by $S \cap \mathcal{G}(S) \subset \{(g, g)\}$ with $g = (\sqrt{5} - 1)/2$; see Definition 4.4 and Remark 4.6.ii of [23]. Moreover, in [23], $\mathcal{G}$ is defined on $[0, 1) \times [0, 1]$ rather than $\Omega = [0, 1]^2$. What follows in §7.2 below can be done with these minor adjustments, but for simplicity we shall omit these details.

of RCF-convergents p_n/q_n for which $\mathcal{G}^n(x,0) \in \Omega \setminus S$ for $n \geq 0$, i.e., S-expansions are governed by the induced transformation of $(\Omega, \mathcal{B}, \bar{\mu}_G, \mathcal{G})$ on $\Omega \setminus S$.

In addition to defining new CF-algorithms, the author shows in [23] that many previously studied CF-algorithms are realised by certain singularisation areas S. Since these arise from induced transformations of a common dynamical system, ergodic properties of the underlying algorithms are easily comparable with one another. The collection of S-expansions includes Minkowski's diagonal CFs ([29]), Bosma's optimal CFs ([2]) and (the natural extensions of) Nakada's α-CFs. However, the α-CFs realised as S-expansions are—somewhat curiously—only those for which $\alpha \geq 1/2$ (cf. [15]). This is explained (for $\alpha \in [\sqrt{2}-1, 1/2)$) by Nakada and Natsui in the introduction of [34], where it is noted that convergents of α-CFs are not necessarily RCF-convergents and, hence, not necessarily S-expansion convergents. In §7.3 below, we exhibit the natural extension of each $([\alpha-1,\alpha], \mathcal{B}, \rho_\alpha, G_\alpha)$, $\alpha \in (0,1]$, as an induced transformation of the natural extension of another, slower continued fraction map—the Farey tent map.

4. The Farey tent map and Farey expansions

In this section we introduce another CF-map—the Farey tent map—whose natural extension (see §5 below) shall be of central importance to us. The Farey tent map generates SRCF-expansions whose convergents consist of all RCF-convergents and mediant convergents; see (8) above and Proposition 4.1 below. Much of this background can be found also in [13], but we include it here for completeness.

4.1. The Farey tent map

Define $\varepsilon : [0,1] \to \{0,1\}$ by

$$\varepsilon(x) := \begin{cases} 0, & x \leq 1/2, \\ 1, & x > 1/2, \end{cases}$$

and for $\varepsilon \in \{0,1\}$, set

$$A_\varepsilon := \begin{pmatrix} 1-\varepsilon & \varepsilon \\ 1 & 1 \end{pmatrix}.$$

The *Farey tent map* $F : [0,1] \to [0,1]$ is defined by

$$F(x) := A_{\varepsilon(x)}^{-1} \cdot x = \begin{cases} x/(1-x), & x \leq 1/2, \\ (1-x)/x, & x > 1/2; \end{cases} \tag{15}$$

see Fig. 1 above. The dynamical system $([0,1], \mathcal{B}, \mu, F)$ is ergodic, where μ is the infinite, σ-finite, absolutely continuous F-invariant measure with density $1/x$ ([14,19,36]). One finds from the definition of F that if $x \in [0,1]$ has RCF-expansion[7] $x = [0; a_1, a_2, a_3, \dots]$, then

$$F(x) = \begin{cases} [0; a_1 - 1, a_2, a_3, \dots], & a_1 > 1, \\ [0; a_2, a_3, a_4, \dots], & a_1 = 1. \end{cases} \tag{16}$$

From this, it follows that the Gauss map G is the *jump transformation* of F associated to the interval $(1/2, 1]$, meaning that for x as above with $x \neq 0$,

$$\min\{j \geq 0 \mid F^j(x) \in (1/2, 1]\} = a_1 - 1, \quad \text{and} \quad G(x) = F^{a_1}(x);$$

see, e.g., §11.4 of [11].

4.2. Farey expansions and Farey convergents

4.2.1. The Farey tent map and RCF-convergents and mediants

In [19], Ito studied the ergodic properties of the dynamical system $([0,1], \mathcal{B}, \mu, F)$ and showed via matrix relations that F generates all convergents and mediant convergents of the RCF-expansion of any irrational $x \in (0,1)$. We reproduce this fact here, fixing notation[8] along the way.

Recall from (15) that $F(x) = A_{\varepsilon(x)}^{-1} \cdot x$, or $x = A_{\varepsilon(x)} \cdot F(x)$. Setting

$$x_n := F^n(x) \qquad \text{and} \qquad \varepsilon_{n+1} = \varepsilon_{n+1}(x) := \varepsilon(x_n), \qquad n \geq 0, \tag{17}$$

we find for each $n \geq 0$ that $x_n = A_{\varepsilon(x_n)} \cdot F(x_n) = A_{\varepsilon_{n+1}} \cdot x_{n+1}$. Repeatedly applying this beginning from $x = x_0$, we have

$$x = (A_{\varepsilon_1} A_{\varepsilon_2} \cdots A_{\varepsilon_n}) \cdot x_n.$$

Let $A_{[0,0]} := I_2$ be the identity matrix and for $n > 0$,

$$A_{[0,n]} = A_{[0,n]}(x) := A_{\varepsilon_1} A_{\varepsilon_2} \cdots A_{\varepsilon_n}. \tag{18}$$

With this notation,

$$x_n = F^n(x) = A_{[0,n]}^{-1} \cdot x, \quad n \geq 0.$$

We wish to determine the entries of $A_{[0,n]}$. For $x = [0; a_1, a_2, \dots]$ irrational and $n \geq 0$, let $j_n = j_n(x)$ and $\lambda_n = \lambda_n(x)$ be the unique integers[9] satisfying

[7] If the expansion of x is finite, we set the remaining digits equal to ∞, e.g., $x = [0; a_1, \dots, a_n, \infty, \infty, \dots]$. This also holds for x equal to $0 = [0; \infty, \infty, \dots]$ and $1 = [0; 1, \infty, \infty, \dots]$, interpreting $\infty - 1 = \infty$.

[8] Notation is largely recycled from [19] but with matrix entries permuted.

[9] These should be thought of in light of Euclid's division lemma: for integers $n, a \geq 0$, there exist unique integers j and λ such that $n = ja + \lambda$ with $0 \leq \lambda < a$. Instead of summing a fixed integer a with itself j times, we sum the first j RCF-digits $a_1, a_2, \dots, a_j$ of x.

$$n = a_1 + a_2 + \cdots + a_{j_n} + \lambda_n, \qquad j_n \geq 0, \quad 0 \leq \lambda_n < a_{j_n+1}. \tag{19}$$

From (16), we have

$$\varepsilon_1\varepsilon_2\varepsilon_3 \cdots = 0^{a_1-1}10^{a_2-1}10^{a_3-1}1\cdots,$$

so

$$\varepsilon_1\varepsilon_2\cdots\varepsilon_n = 0^{a_1-1}10^{a_2-1}1\cdots 0^{a_{j_n}-1}10^{\lambda_n}. \tag{20}$$

Denote the entries of $A_{[0,n]}$, $n \geq 0$, by

$$\begin{pmatrix} u_n & t_n \\ s_n & r_n \end{pmatrix} = \begin{pmatrix} u_n(x) & t_n(x) \\ s_n(x) & r_n(x) \end{pmatrix} := A_{[0,n]},$$

and observe that for any $k \in \mathbb{Z}$,

$$A_0^k A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^k \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k+1 \end{pmatrix}. \tag{21}$$

From (20) and (21), it follows for $n > 0$ that

$$\begin{aligned} \begin{pmatrix} u_n & t_n \\ s_n & r_n \end{pmatrix} = A_{[0,n]} &= A_{\varepsilon_1}\cdots A_{\varepsilon_n} \\ &= A_0^{a_1-1}A_1\cdots A_0^{a_{j_n}-1}A_1A_0^{\lambda_n} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}\cdots\begin{pmatrix} 0 & 1 \\ 1 & a_{j_n} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda_n & 1 \end{pmatrix} \\ &= \begin{pmatrix} p_{j_n-1} & p_{j_n} \\ q_{j_n-1} & q_{j_n} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda_n & 1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_n p_{j_n} + p_{j_n-1} & p_{j_n} \\ \lambda_n q_{j_n} + q_{j_n-1} & q_{j_n} \end{pmatrix}, \end{aligned} \tag{22}$$

where p_j/q_j is the j^{th} RCF-convergent of x (see also Lemma 1.1 of [19]). Equality of the first and final expressions also holds for $n = j_n = \lambda_n = 0$ since, in this case, both matrices are the identity I_2. As a sequence, the quotients of the left-hand columns of the matrices in (22) are

$$\begin{aligned} \left(\frac{u_n}{s_n}\right)_{n\geq 0} = \left(\frac{\lambda_n p_{j_n} + p_{j_n-1}}{\lambda_n q_{j_n} + q_{j_n-1}}\right)_{n\geq 0} = \Bigg(&\frac{p_{-1}}{q_{-1}}, \frac{p_0 + p_{-1}}{q_0 + q_{-1}}, \dots, \frac{(a_1-1)p_0 + p_{-1}}{(a_1-1)q_0 + q_{-1}}, \\ &\frac{p_0}{q_0}, \frac{p_1 + p_0}{q_1 + q_0}, \dots, \frac{(a_2-1)p_1 + p_0}{(a_2-1)q_1 + q_0}, \dots, \\ &\frac{p_{j-1}}{q_{j-1}}, \frac{p_j + p_{j-1}}{q_j + q_{j-1}}, \dots, \frac{(a_{j+1}-1)p_j + p_{j-1}}{(a_{j+1}-1)q_j + q_{j-1}}, \dots \Bigg), \end{aligned} \tag{23}$$

i.e., the map F generates all RCF-convergents and mediants. Notice that the denominators $(s_n)_{n\geq 0}$ do not form an increasing sequence. Supposedly to 'remedy' this, in [19] Ito instead considers the sequence $((u_n+t_n)/(s_n+r_n))_{n\geq 0}$ with increasing denominators. However, in light of Proposition 4.1 below, we find it more natural to study $(u_n/s_n)_{n\geq 0}$.

4.2.2. Lehner and Farey expansions

Originally, there was no continued fraction expansion associated to the Farey tent map F. Such expansions do exist and can be obtained from a map introduced by Lehner in 1994; see [26]. The *Lehner map* (also referred to as 'the mother of all continued fractions' in [12]) is the map $L:[1,2]\to[1,2]$ defined by

$$L(x) := \begin{cases} 1/(2-x), & x\leq 3/2,\\ 1/(x-1), & x>3/2. \end{cases}$$

For $x\in[1,2]$ and each $n\geq 0$, set

$$(b_n,e_{n+1}) = (b_n(x),e_{n+1}(x)) := \begin{cases} (2,-1), & L^n(x)\leq 3/2,\\ (1,1), & L^n(x)>3/2. \end{cases}$$

The digits (b_n,e_{n+1}) generate the so-called *Lehner expansion* of $x\in[1,2]$,

$$x=[b_0/1;e_1/b_1,e_2/b_2,\dots], \tag{24}$$

which is a SRCF-expansion (see [12,26]).

Lehner studied expansions of the form (24) generated by L but no dynamical properties of this map. In [12] it is observed that the dynamical systems $([0,1],\mathcal{B},\mu,F)$ and $([1,2],\mathcal{B},\rho,L)$ are isomorphic via the translation $x\mapsto x+1$, where ρ is the absolutely continuous, L-invariant measure with density $1/(x-1)$. Through this isomorphism, the Farey tent map F can be used to generate a *Farey expansion* for each $x\in[0,1]$ (see also [18]). Indeed, for $x\in[0,1]$, let $\varepsilon_{n+1}=\varepsilon_{n+1}(x)$, $n\geq 0$, be as in (17), and let $[b_0/1;e_1/b_1,e_2/b_2,\dots]$ be the Lehner expansion of $x+1$. Then $x=[b_0-1/1;e_1/b_1,e_2/b_2,\dots]$, and we have

$$\begin{aligned}(b_n,e_{n+1}) &= \left\{\begin{matrix} (2,-1), & L^n(x+1)\leq 3/2\\ (1,1), & L^n(x+1)>3/2\end{matrix}\right\} = \left\{\begin{matrix} (2,-1), & F^n(x)\leq 1/2\\ (1,1), & F^n(x)>1/2\end{matrix}\right\}\\ &= \left\{\begin{matrix} (2,-1), & \varepsilon_{n+1}=0\\ (1,1), & \varepsilon_{n+1}=1\end{matrix}\right\} = (2-\varepsilon_{n+1},2\varepsilon_{n+1}-1).\end{aligned}$$

Hence F generates SRCF-expansions, called *Farey expansions*:

$$x=[(1-\varepsilon_1)/1;(2\varepsilon_1-1)/(2-\varepsilon_2),(2\varepsilon_2-1)/(2-\varepsilon_3),\dots]. \tag{25}$$

The convergents

$$P_n/Q_n = [(1-\varepsilon_1)/1; (2\varepsilon_1 - 1)/(2-\varepsilon_2), (2\varepsilon_2 - 1)/(2-\varepsilon_3), \dots, (2\varepsilon_n - 1)/(2-\varepsilon_{n+1})]$$

of the Farey expansion of x are called the *Farey convergents* of x. In [13] (Proposition 3.1), it is observed that the sequence $(P_n/Q_n)_{n\geq -1}$ of Farey convergents is precisely the sequence $(u_n/s_n)_{n\geq 0}$ from (23) of RCF-convergents and mediants generated by F:

Proposition 4.1. *For each $n \geq 0$,*

$$\begin{pmatrix} u_n \\ s_n \end{pmatrix} = \begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix},$$

where P_n/Q_n is the n^{th} Farey convergent of x.

5. Ito's natural extension of the Farey tent map

We now come to one of the central tools of this article: the natural extension of the Farey tent map, originally introduced by Ito in 1989 ([19]). In §5.1, we recall the definition of Ito's natural extension and discuss a one-to-one correspondence between orbits under the natural extension map and Farey convergents. Via this correspondence, we find that certain subregions of the domain of Ito's natural extension give rise to certain types of Farey convergents. In §5.2, we discuss induced transformations of Ito's natural extension and their connection with subsequences of Farey convergents. Moreover, we revisit a theorem of Brown and Yin from [6] stating that the natural extension of the Gauss map is isomorphic to a certain induced transformation of Ito's natural extension, and we recall from [13] that the entropy of our induced systems may be computed in terms of the measures of their domains (Theorem 5.6 below). As in §4, much of the material of this section can be found in [13], but we include it here for completeness and for some minor notational and definitional changes.

5.1. The natural extension of the Farey tent map

In [19], Ito determined a planar natural extension $(\Omega, \mathcal{B}, \bar{\mu}, \mathcal{F})$ of the dynamical system $([0,1], \mathcal{B}, \mu, F)$ associated to the Farey tent map. The map $\mathcal{F} : \Omega \to \Omega$ is defined for each $z = (x, y) \in \Omega$ by

$$\mathcal{F}(z) := \left(A_{\varepsilon(x)}^{-1} \cdot x, A_{\varepsilon(x)} \cdot y\right) = \begin{cases} \left(\frac{x}{1-x}, \frac{y}{1+y}\right), & x \leq 1/2, \\ \left(\frac{1-x}{x}, \frac{1}{1+y}\right), & x > 1/2, \end{cases} \tag{26}$$

where again $\Omega = [0,1]^2$, and $\bar{\mu}$ is the infinite, σ-finite, absolutely continuous $\mathcal{F}$-invariant measure with density $1/(x+y-xy)^2$. Since $([0,1], \mathcal{B}, \mu, F)$ is ergodic, so is its natural extension $(\Omega, \mathcal{B}, \bar{\mu}, \mathcal{F})$.

Notice that $\mathcal{F}$ is simply the Farey tent map F in the first coordinate. Setting $\varepsilon_{n+1} = \varepsilon_{n+1}(x) = \varepsilon(x_n)$ as in (17), we find that

$$z_n = (x_n, y_n) := \mathcal{F}^n(z) = \left(A_{[0,n]}^{-1} \cdot x, A_{[n,0]} \cdot y\right), \quad n \geq 0, \tag{27}$$

where $A_{[0,n]}$ is defined as in (18), and

$$A_{[n,0]} = A_{[n,0]}(x) := A_{\varepsilon_n} A_{\varepsilon_{n-1}} \cdots A_{\varepsilon_1}, \quad n \geq 1.$$

The entries of $A_{[n,0]}$ may be computed explicitly in terms of those of $A_{[0,n]}$ (recall (22)). Indeed, if $x = [0; a_1, a_2, \dots]$, we have for $n > 0$ that

$$\begin{aligned}
A_{[n,0]} &= A_{\varepsilon_n} \cdots A_{\varepsilon_1} I_2 \\
&= A_0^{\lambda_n} A_1 A_0^{a_{j_n}-1} \cdots A_1 A_0^{a_1-1} A_1 A_1^{-1} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & \lambda_n + 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_{j_n} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & \lambda_n + 1 \end{pmatrix} \left(\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{j_n} \end{pmatrix} \right)^T \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & \lambda_n + 1 \end{pmatrix} \begin{pmatrix} p_{j_n-1} & q_{j_n-1} \\ p_{j_n} & q_{j_n} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} q_{j_n} - p_{j_n} & p_{j_n} \\ (\lambda_n+1) q_{j_n} + q_{j_n-1} - ((\lambda_n+1) p_{j_n} + p_{j_n-1}) & (\lambda_n+1) p_{j_n} + p_{j_n-1} \end{pmatrix} \\
&= \begin{pmatrix} r_n - t_n & t_n \\ s_n + r_n - (u_n + t_n) & u_n + t_n \end{pmatrix},
\end{aligned}$$

and the first and final expressions are also equal to I_2 for $n = 0$. Notice, furthermore, that $A_{[0,n]}^T$ and $A_{[n,0]}$ are conjugate under A_1:

$$A_1 A_{[0,n]}^T = \begin{pmatrix} t_n & r_n \\ u_n + t_n & s_n + r_n \end{pmatrix} = A_{[n,0]} A_1. \tag{28}$$

5.1.1. $\mathcal{F}$-orbits and Farey convergents

Interpreting the map $\mathcal{F}$ symbolically and geometrically leads to a natural correspondence between $\mathcal{F}$-orbits and Farey convergents. Fix $z = (x, y) \in \Omega$ with (finite[10] or infinite) RCF-expansions

$$(x, y) = ([0; a_1, a_2, \dots], [0; b_1, b_2, \dots]). \tag{29}$$

One verifies using (16) and (26) that

[10] As in (16), if the expansion of x or y is finite, we set the remaining digits equal to ∞. If $x = 1/2$, we take the shorter of its two RCF-expansions, namely $x = [0; 2]$.

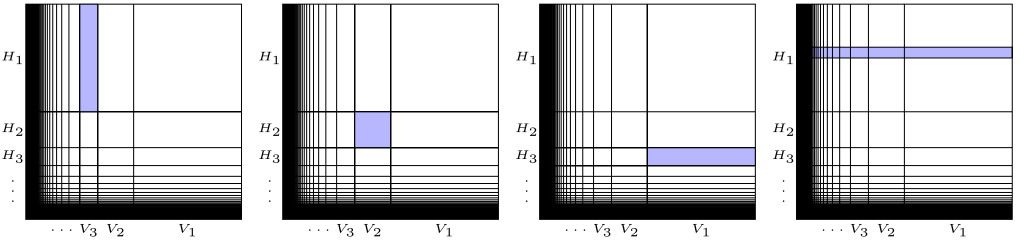


Fig. 3. From left to right: The sets $V_3 \cap H_1$, $\mathcal{F}(V_3 \cap H_1)$, $\mathcal{F}^2(V_3 \cap H_1)$ and $\mathcal{F}^3(V_3 \cap H_1)$, respectively.

$$\mathcal{F}(z) = \begin{cases} ([0; a_1 - 1, a_2, \dots], [0; 1 + b_1, b_2, \dots]), & a_1 > 1, \\ ([0; a_2, a_3, \dots], [0; 1, b_1, b_2, \dots]), & a_1 = 1. \end{cases} \tag{30}$$

Recalling the vertical and horizontal regions from (13), the image of the rectangle $V_a \cap H_b$ for $a > 1$ is thus the rectangle $\mathcal{F}(V_a \cap H_b) = V_{a-1} \cap H_{b+1}$ immediately below and to the right of the original rectangle, and the image of the right-half V_1 of Ω is the top half $\mathcal{F}(V_1) = H_1$, up to a Lebesgue-null set. In particular, the iterates $\mathcal{F}^\lambda$, $0 \leq \lambda < a$, 'slide' the rectangle $V_a \cap H_1$ 'down-and-right' through a rectangles, and the next image $\mathcal{F}^a(V_a \cap H_1)$ is mapped back as a subset of H_1 (see Fig. 3).

Now let $z = (x, y)$ be as in (29) with x irrational, and fix some $n \geq 0$. Recall from (19) that n may be written uniquely as

$$n = a_1 + a_2 + \cdots + a_{j_n} + \lambda_n,$$

where $0 \leq \lambda_n < a_{j_n+1}$. Repeatedly applying (30), one finds

$$\begin{aligned} z_n &= \mathcal{F}^n(z) \\ &= \begin{cases} ([0; a_1 - \lambda_n, a_2, \dots], [0; \lambda_n + b_1, b_2, \dots, \dots]), & n < a_1, \\ ([0; a_{j_n+1} - \lambda_n, a_{j_n+2}, \dots], [0; \lambda_n + 1, a_{j_n}, \dots, a_2, a_1 - 1 + b_1, b_2, \dots]), & n \geq a_1. \end{cases} \end{aligned} \tag{31}$$

In particular, if $z \in H_1$ so that $b_1 = 1$, then z_n belongs to (the closure of) $V_{a_{j_n+1} - \lambda_n} \cap H_{\lambda_n+1}$ for all $n \geq 0$.

Remark 5.1. The closure is needed in the previous statement if and only if $z = (x, 1)$ with $a_1 = 1$ and $a_1 \leq n < a_1 + a_2$. Indeed, in this case

$$z_n = (x_n, y_n) = ([0; a_2 - \lambda_n, a_3, \dots], [0; \lambda_n + 1, 1]).$$

Hence $y_n = 1/(\lambda_n + 2)$ which implies $z_n \notin H_{\lambda_n+1}$ lies on the lower boundary of H_{λ_n+1}. In all other cases for which $z \in H_1$, one has in fact $z_n \in V_{a_{j_n+1} - \lambda_n} \cap H_{\lambda_n+1}$ for all $n \geq 0$. In particular, this annoyance for $z = (x, 1)$ and $a_1 = 1$ is 'corrected' for $n \geq a_1 + a_2$, and the closures are no longer needed. We shall frequently overlook this subtlety and make no mention of the special case $a_1 = 1$, and thus some claims should be understood up to this minor technicality. See also Remark 5.4 below.

Recall from Proposition 4.1 and (23) that the $(n-1)^{\text{st}}$ Farey convergent of x is

$$\frac{u_n}{s_n} = \frac{\lambda_n p_{j_n} + p_{j_n-1}}{\lambda_n q_{j_n} + q_{j_n-1}}. \tag{32}$$

Identifying the n^{th} point $z_n \in V_{a_{j_n+1}-\lambda_n} \cap H_{\lambda_n+1}$ of the $\mathcal{F}$-orbit of $z \in H_1$ with the $(n-1)^{\text{st}}$ Farey convergent u_n/s_n, we find that certain subregions $R \subset \Omega$ correspond to certain types of RCF-convergents or mediants. For instance, if $R = H_1$, then $z_n \in R$ if and only if $\lambda_n = 0$ in (32), i.e., u_n/s_n is a RCF-convergent. More generally, if $R = H_{\lambda+1}$, then $z_n \in R$ if and only if $\lambda_n = \lambda$, i.e., u_n/s_n is a RCF-convergent $(\lambda = 0)$ or a 'λ^{th} mediant' convergent $(\lambda > 0)$.

Moreover, setting $R = V_a$, we have $z_n \in R$ if and only if $a = a_{j_n+1} - \lambda_n$, or $\lambda_n = a_{j_n+1} - a$. Hence $z_n \in R$ if and only if u_n/s_n is a RCF-convergent $(a_{j_n+1} = a)$ or '$(a-1)^{\text{st}}$-from-final' mediant convergent $(a_{j_n+1} > a)$. Lastly, setting $R = V_{a-\lambda} \cap H_{\lambda+1}$, we have $z_n \in R$ if and only if $\lambda_n = \lambda$ and $a_{j_n+1} = a$, i.e., u_n/s_n is a RCF-convergent $(\lambda = 0)$ or 'λ^{th} mediant' convergent $(\lambda > 0)$ with partial denominator $a_{j_n+1} = a$.

These observations naturally lead us to consider the dynamics of Ito's natural extension $(\Omega, \mathcal{B}, \bar{\mu}, \mathcal{F})$ restricted to certain subregions of the domain in order to 'pick out' desired subsequences of Farey convergents.

5.2. Inducing Ito's natural extension

A $\bar{\mu}$-measurable set $R \subset \Omega$ is called *inducible*[11] if either $0 < \bar{\mu}(R) < \infty$ or $R = \Omega$. In the former case (i.e., $\bar{\mu}(R) < \infty$), we call R *proper*. For R inducible, define the *hitting time to* R, denoted $N_R : \Omega \to \mathbb{N} \cup \{\infty\}$, by

$$N_R(z) := \inf\{n \geq 1 \mid \mathcal{F}^n(z) \in R\}. \tag{33}$$

Since $(\Omega, \mathcal{B}, \bar{\mu}, \mathcal{F})$ is conservative and ergodic ([6]), $\bar{\mu}$–a.e. $z \in \Omega$ enters R infinitely often under iterates of $\mathcal{F}$ (see Remark 2.2.1 of [11]). Unless otherwise stated, we assume throughout that the null set of points from any $S \subset \Omega$ whose $\mathcal{F}$-orbits enter R at most finitely many times are removed from S, and—abusing notation—denote this new set again by S. Define $\mathcal{F}_R : \Omega \to R$ by

$$\mathcal{F}_R(z) := \mathcal{F}^{N_R(z)}(z).$$

The *induced map of* $\mathcal{F}$ *on* R is the map $\mathcal{F}_R$ restricted to R, and the *induced measure* $\bar{\mu}_R$ is defined by

[11] We remark that the definition of *inducible* given here is broader than that in [13], where it is also required that $\bar{\mu}(\partial R) = 0$. This latter condition (called *$\bar{\mu}$-continuity*) is not needed for our present purposes.

$$\bar{\mu}_R(S) := \begin{cases} \frac{\bar{\mu}(S)}{\bar{\mu}(R)}, & R \neq \Omega, \\ \bar{\mu}(S), & R = \Omega, \end{cases} \qquad \text{for all} \quad S \in \mathcal{B},\ S \subset R.$$

Ergodicity of the *induced system* $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ follows from that of $(\Omega, \mathcal{B}, \bar{\mu}, \mathcal{F})$. Notice that $\bar{\mu}_R$ is a probability measure if and only if R is proper.

Writing $z = (x, y)$ and setting $A_R(z) := A_{[0,N_R(z)]}(x)$, $A_R^{-1}(z) := (A_R(z))^{-1}$ and $A_R^T(z) := (A_R(z))^T$, Equations (27) and (28) give

$$\mathcal{F}_R(z) = \left(A_R^{-1}(z) \cdot x, A_1 A_R^T(z) A_1^{-1} \cdot y\right). \tag{34}$$

We denote the entries of $A_R(z)$ by

$$\begin{pmatrix} u_R(z) & t_R(z) \\ s_R(z) & r_R(z) \end{pmatrix} := A_R(z) = A_{\varepsilon_1} \cdots A_{\varepsilon_{N_R(z)}}. \tag{35}$$

For $n \geq 0$, set $z_n^R = (x_n^R, y_n^R) := \mathcal{F}_R^n(z)$ and define $N_n^R(z)$ by $N_0^R(z) := 0$ and

$$N_n^R(z) := \sum_{\ell=0}^{n-1} N_R(z_\ell^R), \quad n \geq 1. \tag{36}$$

When the point z is understood, we use the suppressed notation $N_n^R := N_n^R(z),\ n \geq 0$. We remark that when $R = \Omega$, $N_n^R = n$ for all $n \geq 0$. In general, the sequence $(N_n^R)_{n\geq 1}$ gives the indices $N \geq 1$ for which the forward orbit $(\mathcal{F}^N(z))_{N\geq 0}$ of z enters the region R, so $\mathcal{F}_R^n(z) = \mathcal{F}^{N_n^R}(z),\ n \geq 0$.

Let $A_0^R(z) := I_2$ be the identity matrix, and for $n \geq 1$ set

$$A_n^R(z) := A_R(z_{n-1}^R) = A_{\varepsilon_{N_{n-1}^R+1}} \cdots A_{\varepsilon_{N_n^R}}. \tag{37}$$

Notice that for $m \geq 1$ and $n \geq 0$,

$$A_m^R(z_n^R) = A_R(z_{n+m-1}^R) = A_{n+m}^R(z). \tag{38}$$

Moreover, set

$$A_{[m,n]}^R(z) := A_m^R(z) A_{m+1}^R(z) \cdots A_n^R(z), \qquad 0 \leq m \leq n, \tag{39}$$

and denote the entries of $A_{[0,n]}^R(z)$ by

$$\begin{pmatrix} u_n^R(z) & t_n^R(z) \\ s_n^R(z) & r_n^R(z) \end{pmatrix} := A_{[0,n]}^R(z) = A_{\varepsilon_1} \cdots A_{\varepsilon_{N_n^R}}. \tag{40}$$

When the point z is understood, we use the suppressed notation

$$A_{[0,n]}^R = \begin{pmatrix} u_n^R & t_n^R \\ s_n^R & r_n^R \end{pmatrix} := A_{[0,n]}^R(z), \quad n \geq 0. \tag{41}$$

From (40) and (18), we have $A^R_{[0,n]} = A_{[0,N^R_n]}$ and thus, by (22),

$$\begin{pmatrix} u^R_n & t^R_n \\ s^R_n & r^R_n \end{pmatrix} = \begin{pmatrix} u_{N^R_n} & t_{N^R_n} \\ s_{N^R_n} & r_{N^R_n} \end{pmatrix} = \begin{pmatrix} \lambda_{N^R_n} p_{j_{N^R_n}} + p_{j_{N^R_n}-1} & p_{j_{N^R_n}} \\ \lambda_{N^R_n} q_{j_{N^R_n}} + q_{j_{N^R_n}-1} & q_{j_{N^R_n}} \end{pmatrix}, \quad n \geq 0. \tag{42}$$

The following lemma will be useful in §6.3 below.

Lemma 5.2. *For any $z = (x, y) \in \Omega$, $u_R(z), s_R(z) \in \mathbb{Z}$ satisfy $s_R(z) > 0$ and $0 \leq u_R(z) \leq s_R(z)$.*

Proof. It is clear from (35) that $u_R(z), s_R(z) \in \mathbb{Z}$. Moreover, $A_R(z) = A^R_{[0,1]}$, so setting $N = N^R_1$, (42) gives

$$\begin{pmatrix} u_R(z) \\ s_R(z) \end{pmatrix} = \begin{pmatrix} u_N \\ s_N \end{pmatrix} = \begin{pmatrix} \lambda_N p_{j_N} + p_{j_N-1} \\ \lambda_N q_{j_N} + q_{j_N-1} \end{pmatrix}.$$

Since $N > 0$, (19) implies either $j_N > 0$ or $\lambda_N > 0$. If $j_N = 0$, then $u_R(z) = \lambda_N p_0 + p_{-1} = p_{-1} = 1$ and $s_R(z) = \lambda_N q_0 + q_{-1} = \lambda_N > 0$, so the claim holds. If $j_N > 0$, then $s_R(z) \geq q_{j_N-1} \geq q_0 = 1$, and $u_R(z)/s_R(z)$ lies between p_{j_N-1}/q_{j_N-1} and p_{j_N+1}/q_{j_N+1}, which are fractions between 0 and 1. Thus $0 \leq u_R(z) \leq s_R(z)$. □

Notice from (42) that $(u^R_n/s^R_n)_{n\geq 0}$ is a subsequence of the Farey convergents $(u_n/s_n)_{n\geq 0}$. In particular, the correspondence $z_n \leftrightarrow u_n/s_n$ between the $\mathcal{F}$-orbit of $z \in H_1$ and the Farey convergents of x gives a correspondence $z^R_n \leftrightarrow u^R_n/s^R_n$ between the subsequence $(z^R_n)_{n\geq 0} = (z_{N^R_n})_{n\geq 0}$ of points in the $\mathcal{F}$-orbit of z entering R and the subsequence $(u^R_n/s^R_n)_{n\geq 0} = (u_{N^R_n}/s_{N^R_n})_{n\geq 0}$ of Farey convergents. Hence a subregion R naturally determines a subsequence of Farey convergents. We illustrate with the examples discussed at the end of §5.1:

Example 5.3. The region $R = H_{\lambda+1}$ corresponds to RCF-convergents ($\lambda = 0$) or λ^{th} mediants ($\lambda > 0$):

$$\left\{ \frac{u^R_n}{s^R_n} \right\}_{n\geq 0} = \left\{ \frac{\lambda_n p_{j_n} + p_{j_n-1}}{\lambda_n q_{j_n} + q_{j_n-1}} \;\middle|\; \lambda_n = \lambda \right\}_{n\geq 0}.$$

The vertical regions $R = V_a$ give—in addition to the RCF-convergents p_{j-1}/q_{j-1} for which $a_{j+1} = a$—final mediants, next-to-final mediants, and so on for $a = 1, 2, \ldots$, respectively:

$$\left\{ \frac{u^R_n}{s^R_n} \right\}_{n\geq 0} = \left\{ \frac{\lambda_n p_{j_n} + p_{j_n-1}}{\lambda_n q_{j_n} + q_{j_n-1}} \;\middle|\; \lambda_n = a_{j_n+1} - a \right\}_{n\geq 0}.$$

The regions $R = V_{a-\lambda} \cap H_{\lambda+1}$ pick out RCF-convergents ($\lambda = 0$) or λ^{th}-mediants ($\lambda > 1$) corresponding to partial denominators a in the RCF-expansion of x:

$$\left\{\frac{u_n^R}{s_n^R}\right\}_{n\geq 0} = \left\{\frac{\lambda_n p_{j_n} + p_{j_n-1}}{\lambda_n q_{j_n} + q_{j_n-1}} \;\middle|\; \lambda_n = \lambda,\ a_{j_n+1} = a\right\}_{n\geq 0}.$$

Remark 5.4. For the reasons explained in Remark 5.1, some of the statements in Example 5.3 are false for $z = (x, 1)$, where $x = [0; a_1, a_2, \dots]$ with $a_1 = a(x) = 1$. For some of the examples in §7, it will be advantageous to 'fix' this. In such cases, we may 'adjust' the map $\mathcal{F}_R$ so that the corresponding statements on the subsets of Farey convergents are true for all $z = (x, 1)$. For instance, when $R = H_1$ and $z = (x, 1)$ with $x > 1/2$, the sequence $(u_n^R/s_n^R)_{n\geq 0}$ skips the RCF-convergent p_0/q_0 of x. To 'catch' this convergent, we instead consider the map $\overline{\mathcal{F}}_{H_1} : \overline{H}_1 \to \overline{H}_1$ where for $z = (x, y) \in \overline{H}_1$, $\overline{\mathcal{F}}_{H_1}(z) := \mathcal{F}^{a(x)}(z)$ if $x \neq 0$ and $\overline{\mathcal{F}}_{H_1}(z) := z$ if $x = 0$. The maps $\overline{\mathcal{F}}_{H_1}$ and $\mathcal{F}_{H_1}$ agree on $H_1\backslash(A \cup B)$, where $A = (1/2, 1] \times \{1\}$ and $B = \{0\} \times (1/2, 1]$. The dynamical systems $(\overline{H_1}, \mathcal{B}, \bar{\mu}_{\overline{H}_1}, \overline{\mathcal{F}}_{H_1})$ and $(H_1, \mathcal{B}, \bar{\mu}_{H_1}, \mathcal{F}_{H_1})$ are isomorphic under the identity map and thus—from an ergodic-theoretic point of view—indistinguishable. Moreover, the map $\overline{\mathcal{F}}_{H_1}$ 'fixes' the subtlety in Remark 5.1 since, for $z \in A$, $\overline{\mathcal{F}}_{H_1}(z) = ((1 - x)/x, 1/2) \in \overline{H}_1$. Thus we *do* include the convergent p_0/q_0 in $(u_n^R/s_n^R)_{n\geq 0}$. Throughout, we shall often consider such altered systems without mention, denoting them again by $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$.

Consider again $R = H_1$, which gives as a subsequence $(u_n^R/s_n^R)_{n\geq 0}$ of Farey convergents the RCF-convergents of x. Now R consists of all points $z = (x, y)$ as in (29) with $b_1 = 1$, so—after the alteration of Remark 5.4—we find from (31) that for $x \neq 0$, $N_R(z) = a_1 = a(x)$ and

$$\mathcal{F}_R(z) = \mathcal{F}^{a_1}([0; a_1, a_2, \dots], [0; 1, b_2, b_3, \dots]) = ([0; a_2, a_3, \dots], [0; 1, a_1, b_2, b_3, \dots]). \quad (43)$$

Notice the similarity between this induced map and the map $\mathcal{G}$ from (12); they both act essentially as a two-sided shift on RCF-expansions. In fact, Brown and Yin proved in 1996 that a copy of the Gauss natural extension is found sitting (inverted, scaled and 'suspended' from $y = 1$) within $(\Omega, \mathcal{B}, \bar{\mu}, \mathcal{F})$ as the induced system $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ with $R = H_1$ (Theorem 1 of [6]):

Theorem 5.5 *(Brown–Yin, 1996 [6]). The induced system $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ with $R = H_1$ is isomorphic to the Gauss natural extension $(\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$.*

Using Theorem 5.5, one can exploit knowledge about the Gauss natural extension $(\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$ to infer properties of other induced systems $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$. This is used, for instance, in the proof[12] of Theorem 4.6 of [13], which states that the measure-theoretic entropy $h(\mathcal{F}_R) = h_{\bar{\mu}_R}(\mathcal{F}_R)$ of the induced system $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ is inversely proportional to the $\bar{\mu}$-measure of R:

[12] The aforementioned $\bar{\mu}$-continuity condition assumed on inducible regions R in [13] is *not* needed in the proof.

Theorem 5.6. *For any proper, inducible subregion $R \subset \Omega$,*

$$h(\mathcal{F}_R) = \frac{\pi^2}{6\bar{\mu}(R)}.$$

Remark 5.7. We remark here the striking resemblance between Theorem 5.6, Remark 5.10 of [23] on the entropy of S-expansions and Theorem 2 of [24] (conjectured in Remark 2 of [28]) on the entropy of α-CFs. From the results of §7.2 and §7.3 below, Theorem 5.6 may be viewed as simultaneously extending these results from [23,24].

6. Inducing contractions of the mother of all continued fractions

We have seen in §5.2 that inducible subregions $R \subset \Omega$ naturally determine subsequences $(u_n^R/s_n^R)_{n\geq 0} = (u_{N_n^R}/s_{N_n^R})_{n\geq 0}$ of the convergents of Farey expansions. In this section, we construct new GCF-expansions whose convergents are precisely the subsequences $(u_n^R/s_n^R)_{n\geq 0}$ (Corollary 6.12). These GCFs arise from a general procedure described in §6.1 called *contraction*, which—under very mild assumptions—allows one to produce from a given GCF a new GCF whose convergents are any desired subsequence of the original convergents. In §6.2, we use induced transformations of Ito's natural extension of the Farey tent map ('the mother of all continued fractions') to govern contractions of Farey expansions. In §6.3 we introduce a dynamical system—isomorphic to the induced system $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$—which acts essentially as a two-sided shift on contracted Farey expansions.

6.1. Contraction

Recall the singularisation procedure discussed in §3.3: beginning with a SRCF-expansion, one may (simultaneously) singularise at possibly countably many positions to produce a new SRCF-expansion whose convergents are a subsequence of the original convergents. However, singularisation at position n is subject to the condition that the partial numerator α_{n+2} and partial denominator β_{n+1} are both equal to 1. Moreover, beginning from a RCF-expansion, this constraint on partial denominators implies that consecutive convergents cannot be removed via singularisation.

In this subsection, we recall an old acceleration technique of Seidel from 1855 ([40]; see also [37]), called *contraction*, which overcomes these obstacles. Although our main interest is in producing GCF-expansions whose convergents are subsequences of Farey convergents, we present contraction in the general, abstract setting of GCFs discussed in §2.1, as we feel this technique has been largely overlooked[13] and can be fruitfully applied to other continued fraction algorithms. For the same reason, we include a proof of Seidel's theorem (Theorem 6.6) below.

[13] As mentioned in the introduction (§1), contraction is used in the analytic theory of continued fractions, but usually only for subsequences of odd or even integers ([27]). See also [5].

Definition 6.1. A GCF $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \dots]$ is called *contractable* if $Q_{[m+1,n]} \neq 0$ for all $0 \leq m \leq n$. The *contracted continued fraction* (CCF) of a contractable GCF $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \dots]$ with respect to a strictly increasing sequence of non-negative integers $(n_k)_{k\geq 0}$ is the GCF $[\beta_0'/\alpha_0'; \alpha_1'/\beta_1', \alpha_2'/\beta_2', \dots]$, where

$$\begin{pmatrix} \alpha_{k+1}' \\ \beta_{k+1}' \end{pmatrix} := \begin{pmatrix} -\det(B_{[n_{k-1}+2,n_k+1]})Q_{[n_{k-2}+2,n_{k-1}]}Q_{[n_k+2,n_{k+1}]} \\ Q_{[n_{k-1}+2,n_{k+1}]} \end{pmatrix}, \qquad k \geq -1,$$

with $n_k := k$ for $k < 0$.

Remark 6.2. The requirement that a contractable GCF satisfies $Q_{[m+1,n]} \neq 0$ for all $0 \leq m \leq n$ guarantees that the partial numerators α_{k+1}' are nonzero for all $k \geq -1$, i.e., that a CCF is in fact a GCF as defined in §2.1. Indeed, we have $\det(B_{[n_{k-1}+2,n_k+1]}) \neq 0$ for all $k \geq -1$ as noted in §2.1, and both $Q_{[n_{k-2}+2,n_{k-1}]} \neq 0,\ k \geq 1$, and $Q_{[n_k+2,n_{k+1}]} \neq 0,\ k \geq -1$, by assumption. Moreover, for $k = -1$ and $k = 0$, $Q_{[n_{k-2}+2,n_{k-1}]} = 1 \neq 0$; see (2).

This requirement also guarantees that the scalars c_k in Theorem 6.6 below are nonzero.

Remark 6.3. Notice that the partial numerators α_n' of a CCF in general *do not* satisfy $|\alpha_n'| = 1$, even if this is true of the original GCF. Hence contraction does not necessarily send SRCFs to SRCFs.

Example 6.4. We compute the CCF with respect to $(n_k)_{k\geq 0} = (2k)_{k\geq 0}$ of $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \dots] = [1/1; 1/2, 2/3, 3/4, \dots]$, i.e., $\alpha_0 = 1$ and $\alpha_n = \beta_{n-1} = n$ for all $n > 0$. We first note that the recurrence relations (2) and the fact that all of the partial numerators and partial denominators are positive imply that this GCF is in fact contractable. From Definition 6.1,

$$\begin{pmatrix} \alpha_0' \\ \beta_0' \end{pmatrix} = \begin{pmatrix} -\det(B_{[0,0]})Q_{[-1,-2]}Q_{[1,0]} \\ Q_{[0,0]} \end{pmatrix} = \begin{pmatrix} \alpha_0 \cdot 1 \cdot 1 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1' \\ \beta_1' \end{pmatrix} = \begin{pmatrix} -\det(B_{[1,1]})Q_{[0,-1]}Q_{[2,2]} \\ Q_{[1,2]} \end{pmatrix} = \begin{pmatrix} \alpha_1 \cdot 1 \cdot \beta_2 \\ \beta_2\beta_1 + \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 \cdot 3 \\ 3 \cdot 2 + 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix},$$

$$\begin{aligned} \begin{pmatrix} \alpha_2' \\ \beta_2' \end{pmatrix} &= \begin{pmatrix} -\det(B_{[2,3]})Q_{[1,0]}Q_{[4,4]} \\ Q_{[2,4]} \end{pmatrix} = \begin{pmatrix} -\alpha_2\alpha_3 \cdot 1 \cdot \beta_4 \\ \beta_4(\beta_3\beta_2 + \alpha_3) + \alpha_4\beta_2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \cdot 3 \cdot 1 \cdot 5 \\ 5(4 \cdot 3 + 3) + 4 \cdot 3 \end{pmatrix} = \begin{pmatrix} -30 \\ 87 \end{pmatrix}, \end{aligned}$$

and for $k > 1$,

$$\begin{pmatrix} \alpha_{k+1}' \\ \beta_{k+1}' \end{pmatrix} = \begin{pmatrix} -\det(B_{[2k,2k+1]})Q_{[2k-2,2k-2]}Q_{[2k+2,2k+2]} \\ Q_{[2k,2k+2]} \end{pmatrix}$$

$$= \begin{pmatrix} -\alpha_{2k}\alpha_{2k+1} \cdot \beta_{2k-2} \cdot \beta_{2k+2} \\ \beta_{2k+2}(\beta_{2k+1}\beta_{2k} + \alpha_{2k+1}) + \alpha_{2k+2}\beta_{2k} \end{pmatrix}$$

$$= \begin{pmatrix} -((2k)(2k+1)) \cdot (2k-1) \cdot (2k+3) \\ (2k+3)((2k+2)(2k+1) + (2k+1)) + (2k+2)(2k+1) \end{pmatrix}$$

$$= \begin{pmatrix} -(2k-1)(2k)(2k+1)(2k+3) \\ (2k+1)((2k+3)^2 + (2k+2)) \end{pmatrix}.$$

The first few terms of the CCF are thus

$$[1/1; 3/8, -30/87, -420/275, -1890/623, -5544/1179, -12870/1991, -25740/3107, \ldots].$$

Before proving that the convergents of a CCF are a subsequence of the original GCF-convergents, we need the following:

Lemma 6.5. *For any GCF,*

$$Q_{[m+1,n]} = \frac{Q_n P_{m-1} - P_n Q_{m-1}}{\det B_{[-1,m]}} \quad \textit{for all integers } -1 \le m \le n.$$

Proof. If $m = n$, then both the left- and right-hand sides of the expression equal 1. For $m < n$, $Q_{[m+1,n]}$ is the bottom-right entry of

$$B_{[m+1,n]} = B^{-1}_{[-1,m]} B_{[-1,n]} = \frac{1}{\det B_{[-1,m]}} \begin{pmatrix} Q_m & -P_m \\ -Q_{m-1} & P_{m-1} \end{pmatrix} \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix},$$

which is evidently $\frac{Q_n P_{m-1} - P_n Q_{m-1}}{\det B_{[-1,m]}}$. □

Let $[\beta'_0/\alpha'_0; \alpha'_1/\beta'_1, \alpha'_2/\beta'_2, \ldots]$ be the CCF of a contractable GCF $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \ldots]$ with respect to $(n_k)_{k\ge0}$. For $-1 \le m \le n$, let

$$\begin{pmatrix} P'_{[m,n-1]} & P'_{[m,n]} \\ Q'_{[m,n-1]} & Q'_{[m,n]} \end{pmatrix} = B'_{[m,n]} := B_{[m,n]}([\beta'_0/\alpha'_0; \alpha'_1/\beta'_1, \alpha'_2/\beta'_2, \ldots]),$$

and when $m = -1$, set $P'_n := P'_{[-1,n]}$ and $Q'_n := Q'_{[-1,n]}$.

Theorem 6.6 *(Seidel, 1855 [40]). With notation as above,*

$$\begin{pmatrix} P'_k \\ Q'_k \end{pmatrix} = c_k \begin{pmatrix} P_{n_k} \\ Q_{n_k} \end{pmatrix}, \quad k \ge -2, \qquad \textit{where} \quad c_k = \prod_{j=0}^{k-1} Q_{[n_{j-1}+2, n_j]},$$

with $n_k := k$, $k < 0$, and the product defining c_k set equal to 1 for $k < 1$. In particular, the CCF with respect to $(n_k)_{k\ge0}$ of a contractable GCF-expansion $x = [\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \ldots] \in \mathbb{C}$ with convergents $(P_k/Q_k)_{k\ge-1}$ is a GCF-expansion of x with convergents $(P_{n_k}/Q_{n_k})_{k\ge-1}$.

Proof. The proof of the first statement is by induction on k. The statement trivially holds for $k < 0$. Now let $k+1 \geq 0$ and suppose the statement is true for k and $k-1$. By the recurrence relations (3) and Definition 6.1—letting U represent either P or Q—we compute

$$\begin{aligned} U'_{k+1} =& \beta'_{k+1}U'_k + \alpha'_{k+1}U'_{k-1} \\ =& Q_{[n_{k-1}+2,n_{k+1}]}c_kU_{n_k} - \det(B_{[n_{k-1}+2,n_k+1]})Q_{[n_{k-2}+2,n_{k-1}]}Q_{[n_k+2,n_{k+1}]}c_{k-1}U_{n_{k-1}} \\ =& c_k\left(Q_{[n_{k-1}+2,n_{k+1}]}U_{n_k} - \det(B_{[n_{k-1}+2,n_k+1]})Q_{[n_k+2,n_{k+1}]}U_{n_{k-1}}\right). \end{aligned}$$

By Lemma 6.5,

$$Q_{[n_{k-1}+2,n_{k+1}]} = \frac{Q_{n_{k+1}}P_{n_{k-1}} - P_{n_{k+1}}Q_{n_{k-1}}}{\det B_{[-1,n_{k-1}+1]}} \qquad \text{and}$$
$$Q_{[n_k+2,n_{k+1}]} = \frac{Q_{n_{k+1}}P_{n_k} - P_{n_{k+1}}Q_{n_k}}{\det B_{[-1,n_k+1]}},$$

so the above computation gives

$$U'_{k+1} = \frac{c_k\left((Q_{n_{k+1}}P_{n_{k-1}} - P_{n_{k+1}}Q_{n_{k-1}})U_{n_k} - (Q_{n_{k+1}}P_{n_k} - P_{n_{k+1}}Q_{n_k})U_{n_{k-1}}\right)}{\det B_{[-1,n_{k-1}+1]}}.$$

For both $U = P$ and $U = Q$, the numerator of the previous expression simplifies to

$$c_k(Q_{n_k}P_{n_{k-1}} - P_{n_k}Q_{n_{k-1}})U_{n_{k+1}}.$$

Using Lemma 6.5 once more, we have

$$U'_{k+1} = c_k\frac{Q_{n_k}P_{n_{k-1}} - P_{n_k}Q_{n_{k-1}}}{\det B_{[-1,n_{k-1}+1]}}U_{n_{k+1}} = c_kQ_{[n_{k-1}+2,n_k]}U_{n_{k+1}} = c_{k+1}U_{n_{k+1}}.$$

This proves the first statement. The second statement follows from the first, since for a contractable GCF, $Q_{[n_{j-1}+2,n_j]} \neq 0$ for all $j \geq 0$ implies $c_k \neq 0$ for all $k \geq -1$, and $x = \lim_{k\to\infty}\frac{P_k}{Q_k} = \lim_{k\to\infty}\frac{P_{n_k}}{Q_{n_k}}$. □

Example 6.7. Continuing with Example 6.4, for $[\beta_0/\alpha_0;\alpha_1/\beta_1,\alpha_2/\beta_2,\ldots] = [1/1;1/2, 2/3,3/4,\ldots]$ one computes

$$\left(\begin{pmatrix}P_n\\Q_n\end{pmatrix}\right)_{n\geq 0} = \left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix},\begin{pmatrix}11\\8\end{pmatrix},\begin{pmatrix}53\\38\end{pmatrix},\begin{pmatrix}309\\222\end{pmatrix},\begin{pmatrix}2119\\1522\end{pmatrix},\begin{pmatrix}16687\\11986\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}148329\\106542\end{pmatrix},\begin{pmatrix}1468457\\1054766\end{pmatrix},\begin{pmatrix}16019531\\11506538\end{pmatrix},\ldots\right).$$

For the CCF

$$[\beta'_0/\alpha'_0; \alpha'_1/\beta'_1, \alpha'_2/\beta'_2, \dots] = [1/1; 3/8, -30/87, -420/275, -1890/623, -5544/1179, \\ -12870/1991, -25740/3107, \dots]$$

of $[1/1; 1/2, 2/3, 3/4, \dots]$ with respect to $(2k)_{k\geq 0}$, we find

$$\left(\begin{pmatrix} P'_k \\ Q'_k \end{pmatrix}\right)_{k\geq 0} = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 11 \\ 8 \end{pmatrix}, 3\begin{pmatrix} 309 \\ 222 \end{pmatrix}, 3\cdot 5\begin{pmatrix} 16687 \\ 11986 \end{pmatrix}, 3\cdot 5\cdot 7\begin{pmatrix} 1468457 \\ 1054766 \end{pmatrix}, \\ 3\cdot 5\cdot 7\cdot 9\begin{pmatrix} 190899411 \\ 137119578 \end{pmatrix}, \dots\right).$$

As fractions, $P'_k/Q'_k = P_{2k}/Q_{2k}$ for $k \geq 0$.

6.2. Contracted Farey expansions

Throughout this subsection, $R \subset \Omega$ is an inducible subregion and $z = (x, y) \in \Omega$ with x irrational. Using notation from §2.1 and §4.2, let $[\beta_0/\alpha_0; \alpha_1/\beta_1, \alpha_2/\beta_2, \dots]$ denote the Farey expansion of x, i.e., $\alpha_0 = 1$, $\beta_0 = 1 - \varepsilon_1$, and for $n > 0$, $\alpha_n = 2\varepsilon_n - 1$ and $\beta_n = 2 - \varepsilon_{n+1}$; see (25). Below, we shall perform contraction on Farey expansions, but first we must show that Farey expansions are in fact contractable.

Proposition 6.8. *The Farey expansion of an irrational $x \in (0, 1)$ is contractable.*

Proof. We must show that $Q_{[m+1,n]} \neq 0$ for any $0 \leq m \leq n$. By Lemma 6.5, this is equivalent to $Q_n P_{m-1} \neq P_n Q_{m-1}$ for all $0 \leq m \leq n$. By Proposition 4.1, the Farey convergents $(P_j/Q_j)_{j\geq -1} = (u_j/s_j)_{j\geq 0}$ are all RCF-convergents and mediants, which are distinct by (9) and (10). □

Recall the definition of $N_n^R = N_n^R(z)$ from (36).

Definition 6.9. The *contracted Farey expansion* of x with respect to R and $z = (x, y)$, denoted

$$[\beta_0^R/\alpha_0^R; \alpha_1^R/\beta_1^R, \alpha_2^R/\beta_2^R, \dots] = [\beta_0^R(z)/\alpha_0^R(z); \alpha_1^R(z)/\beta_1^R(z), \alpha_2^R(z)/\beta_2^R(z), \dots],$$

is the CCF of the Farey expansion of x with respect to $(n_k)_{k\geq 0}$, where $n_k := N_{k+1}^R - 1$, $k \geq 0$. If $z = (x, 1)$, we call this the *contracted Farey expansion* of x with respect to R.

Remark 6.10. Using (31), one can show that for any $z = (x, y)$ and $z' = (x', y')$ in Ω with $x = x'$, $|\mathcal{F}^n(z) - \mathcal{F}^n(z')| \to 0$. Using this and the fact that $(\Omega, \mathcal{B}, \bar{\mu}, \mathcal{F})$ is conservative and ergodic ([6]), one finds that for any inducible $R \subset \Omega$ with $\bar{\mu}(\mathrm{int}(R)) > 0$, the forward $\mathcal{F}$-orbit of $(x, 1)$ enters R infinitely often for Lebesgue–a.e. $x \in (0, 1)$; see Remark 4.3 of [13]. Hence, for such R, the contracted Farey expansion of x with respect to R exists for Lebesgue–a.e. $x \in (0, 1)$.

In order to study contracted Farey expansions using the dynamics of $\mathcal{F}_R$, we wish to understand these expansions and their convergents in terms of entries from the matrices $A_R(z)$ and $A^R_{[m,n]}(z)$ from (35) and (39) rather than $B_{[m,n]}$. To this end, we begin with a lemma. Let $z^R_n = \mathcal{F}^n_R(z)$ and recall that $s_R(z)$ and $s^R_n(z)$ denote the bottom-left entries of the matrices $A_R(z)$ and $A^R_{[0,n]}(z)$, respectively; see (35) and (40).

Lemma 6.11. *For any $z \in \Omega$ and $0 \le j < k$, one has $Q_{[N^R_j+1, N^R_k-1]} = s^R_{k-j}(z^R_j)$. In particular, if $k = j+1$, then $Q_{[N^R_j+1, N^R_{j+1}-1]} = s_R(z^R_j)$.*

Proof. First, notice that for any $n > 0$,

$$\det A_{[0,n]} = \det\left(A_{\varepsilon_1} \cdots A_{\varepsilon_n}\right) = \prod_{j=1}^{n}(1-2\varepsilon_j) = \det(B_{-1}B_0 \cdots B_n) = \det B_{[-1,n]}, \quad (44)$$

and equality of the left- and right-hand sides also holds for $n = 0$ since both sides equal 1. Then by Lemma 6.5, Proposition 4.1, and Equations (41) and (42),

$$\begin{aligned} Q_{[N^R_j+1, N^R_k-1]} &= \frac{Q_{N^R_k-1}P_{N^R_j-1} - P_{N^R_k-1}Q_{N^R_j-1}}{\det B_{[-1,N^R_j]}} = \frac{s_{N^R_k}u_{N^R_j} - u_{N^R_k}s_{N^R_j}}{\det A_{[0,N^R_j]}} \\ &= \frac{s^R_k u^R_j - u^R_k s^R_j}{\det A^R_{[0,j]}}, \end{aligned}$$

where $A^R_{[0,j]} = A^R_{[0,j]}(z)$. For the first statement, it suffices to show that the right-hand side of the previous line equals the bottom-left entry of $A^R_{[0,k-j]}(z^R_j)$. From (38), (39) and (41), we have

$$\begin{aligned} A^R_{[0,k-j]}(z^R_j) &= A^R_1(z^R_j)A^R_2(z^R_j)\cdots A^R_{k-j}(z^R_j) \\ &= A^R_{j+1}(z)A^R_{j+2}(z)\cdots A^R_k(z) \\ &= \left(A^R_{[0,j]}\right)^{-1} A^R_{[0,k]} = \frac{1}{\det A^R_{[0,j]}}\begin{pmatrix} r^R_j & -t^R_j \\ -s^R_j & u^R_j \end{pmatrix}\begin{pmatrix} u^R_k & t^R_k \\ s^R_k & r^R_k \end{pmatrix}, \end{aligned}$$

and the first statement follows. The second statement follows immediately from the first and the fact that $A^R_{[0,1]}(z^R_j) = A^R_1(z^R_j) = A_R(z^R_j)$; see (38) and (39). □

Now let $[\beta^R_0/\alpha^R_0; \alpha^R_1/\beta^R_1, \alpha^R_2/\beta^R_2, \ldots]$ be the contracted Farey expansion of x with respect to R and z, and for $-1 \le m \le n$, let

$$\begin{pmatrix} P^R_{[m,n-1]} & P^R_{[m,n]} \\ Q^R_{[m,n-1]} & Q^R_{[m,n]} \end{pmatrix} = B^R_{[m,n]} := B_{[m,n]}([\beta^R_0/\alpha^R_0; \alpha^R_1/\beta^R_1, \alpha^R_2/\beta^R_2, \ldots]) \quad (45)$$

and

$$P_n^R := P_{[-1,n]}^R, \qquad Q_n^R := Q_{[-1,n]}^R. \tag{46}$$

Then Theorem 6.6, Proposition 4.1 and Lemma 6.11 imply:

Corollary 6.12. *With notation as above,*

$$\begin{pmatrix} P_k^R \\ Q_k^R \end{pmatrix} = c_k^R \begin{pmatrix} u_{k+1}^R \\ s_{k+1}^R \end{pmatrix}, \quad k \geq -1, \qquad \textit{where} \quad c_k^R = \prod_{j=0}^{k-1} s_R(z_j^R),$$

with $c_k^R = 1$ *for* $k < 1$. *In particular, the contracted Farey expansion of* x *with respect to* R *and* $z = (x, y)$ *has convergents* $(u_k^R / s_k^R)_{k \geq 0}$.

Corollary 6.12 describes the convergents of a contracted Farey expansion in terms of the entries of $A_{[0,n]}^R$ (see (41)). Proposition 6.14 below gives an alternative description of the partial numerators and partial denominators α_k^R, β_k^R in terms of entries of the matrices $A_R(z)$ (see (35)). For this, we introduce three integer-valued maps on Ω: let $d_R, \alpha_R, \beta_R : \Omega \to \mathbb{Z}$ be defined for $z \in \Omega$ by

$$d_R(z) := \begin{cases} s_R(\mathcal{F}_R^{-1}(z)), & \text{if } \mathcal{F}_R^{-1}(z) \text{ is defined,} \\ 1, & \text{otherwise,} \end{cases}$$
$$\alpha_R(z) := -\det(A_R(z)) d_R(z) s_R(\mathcal{F}_R(z)) \tag{47}$$

and

$$\beta_R(z) := s_R(z) u_R(\mathcal{F}_R(z)) + r_R(z) s_R(\mathcal{F}_R(z)). \tag{48}$$

Remark 6.13. Notice that Lemma 5.2 implies $d_R(z)$ is a positive integer for any z. We claim that $d_R(z) = 1$ whenever $z = (x, 1)$. By definition of $\mathcal{F}$ (Equation (26)), $\mathcal{F}^{-1}(z) \in (1/2, 1] \times \{0\}$, and $\mathcal{F}^{-n}(z) \in [0, 1/2] \times \{0\}$ for $n > 1$. If $\mathcal{F}_R^{-1}(z)$ is not defined (e.g., if R does not intersect the line $[0, 1] \times \{0\}$), then $d_R(z) = 1$ by definition. Otherwise, $\mathcal{F}_R^{-1}(z) = \mathcal{F}^{-n}(z)$ for some $n \geq 1$, and

$$A_R(\mathcal{F}_R^{-1}(z)) = A_0^{n-1} A_1 = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$$

implies $d_R(z) = s_R(\mathcal{F}_R^{-1}(z)) = 1$. In either case, $d_R(z) = 1$ for $z = (x, 1)$ as claimed.

One motivation for defining $d_R(z)$ as above lies in the second statement of the following proposition: when $d_R(z) = 1$, the notation simplifies and we need not consider the index 1 partial numerator α_1^R as a separate case.

Proposition 6.14. *The digits of the contracted Farey expansion of* x *with respect to* R *and* $z = (x, y)$ *are given by*

$$\begin{pmatrix}\alpha_0^R\\ \beta_0^R\end{pmatrix}=\begin{pmatrix}s_R(z_0^R)\\ u_R(z_0^R)\end{pmatrix},\qquad \begin{pmatrix}\alpha_1^R\\ \beta_1^R\end{pmatrix}=\begin{pmatrix}\alpha_R(z_0^R)/d_R(z_0^R)\\ \beta_R(z_0^R)\end{pmatrix}\qquad \textit{and}$$
$$\begin{pmatrix}\alpha_{k+1}^R\\ \beta_{k+1}^R\end{pmatrix}=\begin{pmatrix}\alpha_R(z_k^R)\\ \beta_R(z_k^R)\end{pmatrix},\quad k>0.$$

When $d_R(z)=1$ (e.g., when $z=(x,1)$ by Remark 6.13), this becomes

$$\begin{pmatrix}\alpha_0^R\\ \beta_0^R\end{pmatrix}=\begin{pmatrix}s_R(z_0^R)\\ u_R(z_0^R)\end{pmatrix}\qquad \textit{and}\qquad \begin{pmatrix}\alpha_{k+1}^R\\ \beta_{k+1}^R\end{pmatrix}=\begin{pmatrix}\alpha_R(z_k^R)\\ \beta_R(z_k^R)\end{pmatrix},\quad k\geq 0.$$

Proof. From Definitions 6.1 and 6.9, we have

$$\begin{pmatrix}\alpha_{k+1}^R\\ \beta_{k+1}^R\end{pmatrix}=\begin{pmatrix}-\det(B_{[N_k^R+1,N_{k+1}^R]})Q_{[N_{k-1}^R+1,N_k^R-1]}Q_{[N_{k+1}^R+1,N_{k+2}^R-1]}\\ Q_{[N_k^R+1,N_{k+2}^R-1]}\end{pmatrix},\quad k\geq -1,\quad (49)$$

where $N_k^R:=k$ for $k<0$. For $k=-1$,

$$\begin{pmatrix}\alpha_0^R\\ \beta_0^R\end{pmatrix}=\begin{pmatrix}-\det(B_{[0,0]})Q_{[-1,-2]}Q_{[N_0^R+1,N_1^R-1]}\\ Q_{[0,N_1^R-1]}\end{pmatrix}=\begin{pmatrix}Q_{[N_0^R+1,N_1^R-1]}\\ Q_{[0,N_1^R-1]}\end{pmatrix}.$$

Lemma 6.11 gives $Q_{[N_0^R+1,N_1^R-1]}=s_R(z_0^R)$, and Lemma 6.5 and Proposition 4.1 give

$$Q_{[0,N_1^R-1]}=\frac{Q_{N_1^R-1}P_{-2}-P_{N_1^R-1}Q_{-2}}{\det B_{[-1,-1]}}=P_{N_1^R-1}=u_1^R=u_R(z_0^R).$$

Thus the claim holds for $\alpha_0^R,\ \beta_0^R$. Next, notice from Equation (44) that for any $0\leq m<n$,

$$\det B_{[m+1,n]}=\frac{\det B_{[-1,n]}}{\det B_{[-1,m]}}=\frac{\det A_{[0,n]}}{\det A_{[0,m]}}=\det(A_{\varepsilon_{m+1}}\cdots A_{\varepsilon_n}),$$

so by (37), we have for $k\geq 0$ that

$$\det(B_{[N_k^R+1,N_{k+1}^R]})=\det(A_{\varepsilon_{N_k^R+1}}\cdots A_{\varepsilon_{N_{k+1}^R}})=\det(A_R(z_k^R)).$$

This, Equation (49) and Lemma 6.11 give for $k\geq 0$

$$\begin{aligned}\begin{pmatrix}\alpha_{k+1}^R\\ \beta_{k+1}^R\end{pmatrix}&=\begin{pmatrix}-\det(B_{[N_k^R+1,N_{k+1}^R]})Q_{[N_{k-1}^R+1,N_k^R-1]}Q_{[N_{k+1}^R+1,N_{k+2}^R-1]}\\ Q_{[N_k^R+1,N_{k+2}^R-1]}\end{pmatrix}\\ &=\begin{cases}\begin{pmatrix}-\det(A_R(z_0^R))s_R(z_1^R)\\ s_2^R(z_0^R)\end{pmatrix}, & k=0,\\ \begin{pmatrix}-\det(A_R(z_k^R))s_R(z_{k-1}^R)s_R(z_{k+1}^R)\\ s_2^R(z_k^R)\end{pmatrix}, & k>0.\end{cases}\end{aligned}$$

When $k=0$, this gives $\alpha_1^R = \alpha_R(z_0^R)/d_R(z_0^R)$. When $k>0$, $s_R(z_{k-1}^R) = s_R(\mathcal{F}_R^{-1}(z_k^R)) = d_R(z_k)$, so $\alpha_{k+1}^R = \alpha_R(z_k^R)$. Moreover, for $k \geq 0$, $s_2^R(z_k^R)$ is the bottom-left entry of

$$\begin{aligned} A_{[0,2]}^R(z_k^R) &= A_1^R(z_k^R)A_2^R(z_k^R) = A_R(z_k^R)A_R(z_{k+1}^R) \\ &= \begin{pmatrix} u_R(z_k^R) & t_R(z_k^R) \\ s_R(z_k^R) & r_R(z_k^R) \end{pmatrix} \begin{pmatrix} u_R(z_{k+1}^R) & t_R(z_{k+1}^R) \\ s_R(z_{k+1}^R) & r_R(z_{k+1}^R) \end{pmatrix}; \end{aligned}$$

see (40), (39), (38) and (35). Thus $s_2^R(z_k^R) = s_R(z_k^R)u_R(z_{k+1}^R) + r_R(z_k^R)s_R(z_{k+1}^R) = \beta_R(z_k^R)$. This proves the first statement. The latter statement follows immediately from the first. □

We refer the reader to §7 below for examples using Proposition 6.14.

6.3. A two-sided shift for contracted Farey expansions

In this subsection, we associate to the induced system $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ an isomorphic dynamical system $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$ acting essentially as a two-sided shift for contracted Farey expansions. This new system will serve several purposes in §7 below: we will see that $(\Omega_{H_1}, \mathcal{B}, \bar{\nu}_{H_1}, \tau_{H_1}) = (\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$ is the natural extension of the Gauss map; for certain subregions $R \subset H_1$, $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$ will coincide with a two-sided shift system associated to S-expansions in [23]; and in §7.3, we describe the natural extension of each of Nakada's α-CFs, $0 < \alpha \leq 1$, as an induced system $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ by using the isomorphic system $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$.

To ease exposition, we impose some restrictions on our inducible subregion $R \subset \Omega$ throughout this subsection. First, we assume that R is bounded away from the origin and that for any $z=(x,y) \in R$, $y>0$. Furthermore, we assume that $s_R(z)=1$ for all $z \in R$, and hence—by Lemma 5.2—that $u_R(z) \in \{0,1\}$. The regions R considered in §7 below shall satisfy these assumptions.

Remark 6.15. A two-sided shift space may be constructed without the restriction $s_R(z) = 1$, but in general the domain Ω_R consists of several planar 'sheets,' and the invariant measure $\bar{\nu}_R$ is a sum of measures which—restricted to each of these sheets—has density of the form in Theorem 6.16 below. However, this more general system is not needed for our purposes.

Define $\varphi_R : R \to \mathbb{R}^2$, where for $z=(x,y) \in R$,

$$\begin{aligned} \varphi_R(z) = (X(z), Y(z)) &:= \left(\begin{pmatrix} 1 & -u_R(z) \\ 0 & 1 \end{pmatrix} \cdot x,\ \begin{pmatrix} -1 & 1 \\ 1-u_R(z) & u_R(z) \end{pmatrix} \cdot y \right) \qquad (50) \\ &= \begin{cases} \left(x, \frac{1-y}{y}\right), & u_R(z)=0, \\ (x-1, 1-y), & u_R(z)=1. \end{cases} \end{aligned}$$

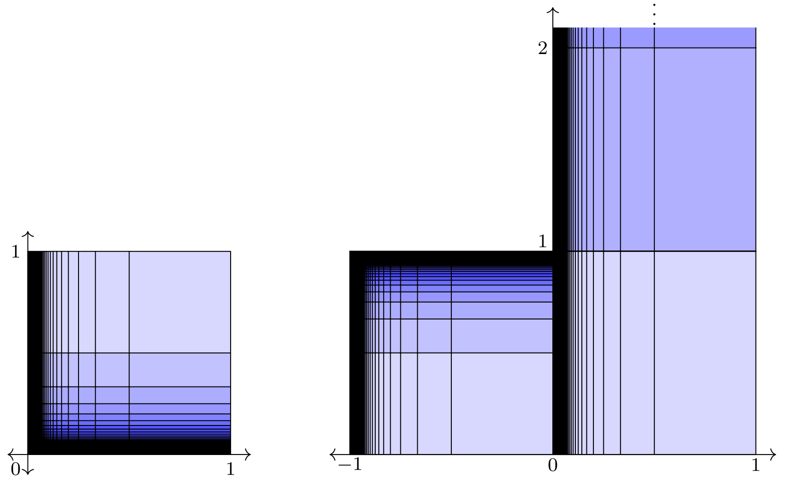


Fig. 4. Left: The domain $\Omega = [0,1]^2$. Right: The first quadrant shows part of the image of $\Omega \backslash ([0,1] \times \{0\})$ under the map $(x,y) \mapsto (x,(1-y)/y)$; the second quadrant shows the image of Ω under the map $(x,y) \mapsto (x-1,1-y)$.

The map φ_R is injective, except possibly on the null-set of points $\{(x,y) \in R \mid x \in \{0,1\}\}$; see Fig. 4. Setting $\Omega_R := \varphi_R(R)$, its inverse (off of the image of the aforementioned null-set) $\varphi_R^{-1} : \Omega_R \setminus (\{0\} \times [0,\infty)) \to R$ is given by

$$\varphi_R^{-1}(X,Y) = \begin{cases} \left(X, \frac{1}{Y+1}\right), & X > 0, \\ (X+1, 1-Y), & X < 0. \end{cases}$$

If $z = \varphi_R^{-1}(X,Y)$, this may also be written

$$\begin{aligned} \varphi_R^{-1}(X,Y) &= \left(\begin{pmatrix} 1 & u_R(z) \\ 0 & 1 \end{pmatrix} \cdot X, \begin{pmatrix} u_R(z) & -1 \\ u_R(z)-1 & -1 \end{pmatrix} \cdot Y \right) \\ &= \left(X + u_R(z), \frac{u_R(z)Y - 1}{(u_R(z)-1)Y - 1} \right). \end{aligned} \tag{51}$$

Define $\tau_R : \Omega_R \to \Omega_R$ by

$$\tau_R(X,Y) := \begin{cases} (X,Y), & X = 0, \\ \varphi_R \circ \mathcal{F}_R \circ \varphi_R^{-1}, & X \neq 0. \end{cases}$$

We obtain a dynamical system $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$, where $\bar{\nu}_R := \bar{\mu}_R \circ \varphi_R^{-1}$ denotes the push-forward measure of $\bar{\mu}_R$ under φ_R. By construction, $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ and $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$ are isomorphic. Recall the definitions of α_R and β_R from (47) and (48).

Theorem 6.16. *The map* $\tau_R : \Omega_R \to \Omega_R$ *is given by* $\tau_R(0, Y) = (0, Y)$ *and for* $X \neq 0$,

$$\tau_R(X, Y) = \left(\frac{\alpha_R(z)}{X} - \beta_R(z), \frac{1}{\beta_R(z) + \alpha_R(z)Y} \right),$$

where $z = \varphi_R^{-1}(X, Y)$, *and the measure* $\bar{\nu}_R$ *has density*

$$\frac{1}{\bar{\mu}(R)(1 + XY)^2}.$$

Remark 6.17. We remark here the resemblance between the measures and maps from $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$ and the natural extension $(\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$ of the Gauss map from §3.1. We shall return to this point in §7.1 below.

Proof of Theorem 6.16. We begin with the statement about the map τ_R. By definition, $\tau_R(0, Y) = (0, Y)$, so let $(X, Y) \in \Omega_R$ with $X \neq 0$. Set $(X', Y') := \tau_R(X, Y)$, $z = (x, y) := \varphi_R^{-1}(X, Y)$ and $z' = (x', y') := \mathcal{F}_R(z)$, and note that $(X', Y') = \varphi_R(z')$. Set $u = u_R(z)$ and $u' = u_R(z')$. Using Equations (50), (34), (51), and symmetry of A_1, respectively,

$$\begin{aligned}
(X', Y') &= \left(\begin{pmatrix} 1 & -u' \\ 0 & 1 \end{pmatrix} \cdot x', \begin{pmatrix} -1 & 1 \\ 1 - u' & u' \end{pmatrix} \cdot y' \right) \\
&= \left(\begin{pmatrix} 1 & -u' \\ 0 & 1 \end{pmatrix} A_R^{-1}(z) \cdot x, \begin{pmatrix} -1 & 1 \\ 1 - u' & u' \end{pmatrix} A_1 A_R^T(z) A_1^{-1} \cdot y \right) \\
&= \left(\begin{pmatrix} 1 & -u' \\ 0 & 1 \end{pmatrix} A_R^{-1}(z) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot X, \right. \\
&\qquad \left. \begin{pmatrix} -1 & 1 \\ 1 - u' & u' \end{pmatrix} A_1 A_R^T(z) A_1^{-1} \begin{pmatrix} u & -1 \\ u - 1 & -1 \end{pmatrix} \cdot Y \right) \\
&= \left(\begin{pmatrix} 1 & -u' \\ 0 & 1 \end{pmatrix} A_R^{-1}(z) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot X, \right. \\
&\qquad \left. \left(\begin{pmatrix} -1 & 1 \\ 1 - u' & u' \end{pmatrix}^{-T} A_1^{-1} A_R^{-1}(z) A_1 \begin{pmatrix} u & -1 \\ u - 1 & -1 \end{pmatrix}^{-T} \right)^{-T} \cdot Y \right).
\end{aligned}$$

One easily computes

$$\begin{pmatrix} -1 & 1 \\ 1 - u' & u' \end{pmatrix}^{-T} A_1^{-1} = \begin{pmatrix} 1 & -u' \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_1 \begin{pmatrix} u & -1 \\ u - 1 & -1 \end{pmatrix}^{-T} = -\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},$$

so $(X', Y') = (M \cdot X, M^{-T} \cdot Y)$, where (recall $u = u_R(z)$, $u' = u_R(z')$ and $s_R(z) = s_R(z') = 1$)

$$
\begin{aligned}
M &= \begin{pmatrix} 1 & -u' \\ 0 & 1 \end{pmatrix} A_R^{-1}(z) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{\det A_R(z)} \begin{pmatrix} 1 & -u' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_R(z) & -t_R(z) \\ -s_R(z) & u_R(z) \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \\
&= \frac{1}{\det A_R(z)} \begin{pmatrix} r_R(z) + s_R(z)u' & (r_R(z)u - t_R(z)) - u'(u_R(z) - s_R(z)u) \\ -s_R(z) & u_R(z) - s_R(z)u \end{pmatrix} \\
&= \frac{1}{\det A_R(z)} \begin{pmatrix} r_R(z) + s_R(z)u' & r_R(z)u - t_R(z) \\ -s_R(z) & 0 \end{pmatrix} \\
&= \frac{-1}{\det A_R(z)} \begin{pmatrix} -\left(r_R(z)s_R(z') + s_R(z)u_R(z')\right) & -\det(A_R(z)) \\ 1 & 0 \end{pmatrix} \\
&= \frac{-1}{\det A_R(z)} \begin{pmatrix} -\beta_R(z) & \alpha_R(z) \\ 1 & 0 \end{pmatrix}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\tau_R(X,Y) &= (X', Y') = (M \cdot X, M^{-T} \cdot Y) \\
&= \left(\begin{pmatrix} -\beta_R(z) & \alpha_R(z) \\ 1 & 0 \end{pmatrix} \cdot X, \begin{pmatrix} 0 & 1 \\ \alpha_R(z) & \beta_R(z) \end{pmatrix} \cdot Y \right),
\end{aligned}
$$

proving the claim about τ_R.

Next we prove the statement about the density of $\bar{\nu}_R$. Let S be a measurable subset of Ω_R. Using a change of variables,

$$
\begin{aligned}
\bar{\nu}_R(S) =& \bar{\mu}_R \circ \varphi_R^{-1}(S) = \int\limits_{\varphi_R^{-1}(S)} d\bar{\mu}_R = \frac{1}{\bar{\mu}(R)} \iint\limits_{\varphi_R^{-1}(S)} \rho(x,y)\,dx\,dy \\
=& \frac{1}{\bar{\mu}(R)} \iint\limits_S \rho(\varphi_R^{-1}(X,Y)) |\det J|\,dX\,dY,
\end{aligned}
$$

where

$$
\rho(x,y) := \frac{1}{(x+y-xy)^2}
$$

is the density of $\bar{\mu}$ and J is the Jacobian of φ_R^{-1} at $(X,Y) \in S$. Let $u = u_R(z) \in \{0,1\}$, where $z = \varphi_R^{-1}(X,Y)$. By Equation (51), the Jacobian of φ_R^{-1} at (X,Y) is

$$
J = \begin{pmatrix} 1 & 0 \\ 0 & \frac{u((u-1)Y-1)-(u-1)(uY-1)}{((u-1)Y-1)^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-1}{((u-1)Y-1)^2} \end{pmatrix}.
$$

Moreover,

$$\begin{aligned}\rho(\varphi_R^{-1}(X,Y)) &= \left((X+u) + \frac{uY-1}{(u-1)Y-1} - (X+u)\frac{uY-1}{(u-1)Y-1}\right)^{-2} \\ &= \left(\frac{(X+u)((u-1)Y-1)+(uY-1)-(X+u)(uY-1)}{((u-1)Y-1)}\right)^{-2} \\ &= \left(\frac{1+XY}{(u-1)Y-1}\right)^{-2}\end{aligned}$$

so that

$$\rho(\varphi_R^{-1}(X,Y))|\det J| = \left(\frac{(u-1)Y-1}{1+XY}\right)^2 \frac{1}{((u-1)Y-1)^2} = \frac{1}{(1+XY)^2}. \quad \square$$

For given $(X,Y) = (X(z),Y(z)) \in \Omega_R$, set

$$(X_n^R, Y_n^R) = (X_n^R(z), Y_n^R(z)) := \tau_R^n(X,Y), \quad n \geq 0. \tag{52}$$

Then, for $X_n^R \neq 0$,

$$z_n^R = \mathcal{F}_R^n(z) = \mathcal{F}_R^n \circ \varphi_R^{-1}(X,Y) = \varphi_R^{-1} \circ \tau_R^n(X,Y) = \varphi_R^{-1}(X_n^R, Y_n^R), \quad n \geq 0. \tag{53}$$

The next result states that the map τ_R acts essentially as a two-sided shift operator on contracted Farey expansions.

Proposition 6.18. *Let $[\beta_0^R/\alpha_0^R; \alpha_1^R/\beta_1^R, \alpha_2^R/\beta_2^R, \ldots]$ denote the contracted Farey expansion of $x \in (0,1)\backslash\mathbb{Q}$ with respect to R and $z = (x,y) \in R$. Then for $n \geq 0$,*

$$\begin{aligned}(X_n^R, Y_n^R) = ([0/1; \alpha_{n+1}^R/\beta_{n+1}^R, \alpha_{n+2}^R/\beta_{n+2}^R, \ldots], \\ [0/1; 1/\beta_n^R, \alpha_n^R/\beta_{n-1}^R, \ldots, \alpha_1^R/\beta_0^R, \alpha_0^R/(1/y-1)]).\end{aligned}$$

Proof. For each $n \geq 0$, set

$$\begin{aligned}(T_n, V_n) := ([0/1; \alpha_{n+1}^R/\beta_{n+1}^R, \alpha_{n+2}^R/\beta_{n+2}^R, \ldots], \\ [0/1; 1/\beta_n^R, \alpha_n^R/\beta_{n-1}^R, \ldots, \alpha_1^R/\beta_0^R, \alpha_0^R/(1/y-1)]).\end{aligned}$$

Using (4) and (5), one finds that for each $n \geq 0$,

$$\begin{aligned}(T_{n+1}, V_{n+1}) &= \left(\begin{pmatrix} 0 & \alpha_{n+1}^R \\ 1 & \beta_{n+1}^R \end{pmatrix}^{-1} \cdot T_n, \begin{pmatrix} 0 & 1 \\ \alpha_{n+1}^R & \beta_{n+1}^R \end{pmatrix} \cdot V_n\right) \\ &= \left(\frac{\alpha_{n+1}^R}{T_n} - \beta_{n+1}^R, \frac{1}{\beta_{n+1}^R + \alpha_{n+1}^R V_n}\right).\end{aligned} \tag{54}$$

We will show by induction that $(X_n^R, Y_n^R) = (T_n, V_n)$ for all $n \geq 0$. By (50), Proposition 6.14 and the fact that $s_R(z) = 1$ for all z,

$$X_0^R = \begin{pmatrix} 1 & -u_R(z) \\ 0 & 1 \end{pmatrix} \cdot x = \begin{pmatrix} \alpha_0^R & -\beta_0^R \\ 0 & 1 \end{pmatrix} \cdot x = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha_0^R \\ 1 & \beta_0^R \end{pmatrix} \right)^{-1} \cdot x.$$

Setting $n = 0$ in (4) and multiplying both sides by $B_{[-1,0]}^{-1}$ reveals that $X_0^R = T_0$. Similarly,

$$Y_0^R = \begin{pmatrix} -1 & 1 \\ 1-u_R(z) & u_R(z) \end{pmatrix} \cdot y = \begin{pmatrix} -1 & 1 \\ \alpha_0^R - \beta_0^R & \beta_0^R \end{pmatrix} \cdot y$$
$$= \begin{pmatrix} 0 & 1 \\ \alpha_0^R & \beta_0^R \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \cdot y = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha_0^R \\ 1 & \beta_0^R \end{pmatrix} \right)^T \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \cdot y.$$

Since $\left(\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot y = 1/y - 1$, Equation (6) gives $Y_0^R = V_0$. Now suppose that $(X_n^R, Y_n^R) = (T_n, V_n)$ for some $n \geq 0$. By Theorem 6.16, Proposition 6.14, our inductive hypothesis and (54),

$$\begin{aligned}(X_{n+1}^R, Y_{n+1}^R) &= \left(\frac{\alpha_R(z_n^R)}{X_n^R} - \beta_R(z_n^R), \frac{1}{\beta_R(z_n^R) + \alpha_R(z_n^R) Y_n^R} \right) \\ &= \left(\frac{\alpha_{n+1}^R}{T_n} - \beta_{n+1}^R, \frac{1}{\beta_{n+1}^R + \alpha_{n+1}^R V_n} \right) \\ &= (T_{n+1}, V_{n+1}). \quad \square \end{aligned}$$

For the remainder of this subsection, we restrict our attention to the full-measure subset of points $z \in R$ for which $z_n^R = (x_n^R, y_n^R) := \mathcal{F}_R^n(z) \in R$ is defined and $x_n^R \neq 0$ for all $n \in \mathbb{Z}$ (in particular, $x \notin \mathbb{Q}$). We remark that as $\mathcal{F}_R$ is totally invariant on this subset, the induced system $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ and its restriction to this full-measure subset are isomorphic. The same is true of the system $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$ and its restriction to the image under φ_R of our full-measure, totally $\mathcal{F}_R$-invariant subset of R. Abusing notation, we denote these restricted, isomorphic systems again by $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ and $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$.

Now, let

$$\Delta(0/1; \alpha_1/\beta_1, \alpha_2/\beta_2, \ldots, \alpha_n/\beta_n) \times \Delta(0/1; 1/\beta_0, \alpha_0/\beta_{-1}, \alpha_{-1}/\beta_{-2}, \ldots, \alpha_{-(m-1)}/\beta_{-m}) \subset \Omega_R$$

be the (possibly empty) set of points $(X(z), Y(z)) \in \Omega_R$ satisfying

$$\alpha_R(z_j^R) = \alpha_{j+1} \quad \text{and} \quad \beta_R(z_k^R) = \beta_{k+1}$$

for all $-m \leq j \leq n-1$ and $-m-1 \leq k \leq n-1$. The following result is needed in §7.3 below when we realise the natural extensions of the α-CFs as induced systems $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$.

Proposition 6.19. *The Borel σ-algebra $\mathcal{B}$ restricted to Ω_R is generated by the sets*

$$\Delta(0/1;\alpha_1/\beta_1,\alpha_2/\beta_2,\dots,\alpha_n/\beta_n)\times\Delta(0/1;1/\beta_0,\alpha_0/\beta_{-1},\alpha_{-1}/\beta_{-2},\dots,\alpha_{-(m-1)}/\beta_{-m}).$$

Proof. We first remark that each of these sets belongs to $\mathcal{B}$ since each may be written as an intersection of preimages of integers under compositions of the measurable functions α_R, β_R, $\mathcal{F}_R^{\pm1}$ and φ_R^{-1}. Next, notice that there are only countably many such sets, so it suffices to show that any open set $U\in\Omega_R$ can be written as *some* union of these. It thus suffices to show that for any $(X,Y)=(X(z),Y(z))\in U$, there exists some set

$$\begin{aligned}D_n=&\,\Delta(0/1;\alpha_1/\beta_1,\alpha_2/\beta_2,\dots,\alpha_n/\beta_n)\\&\times\Delta(0/1;1/\beta_0,\alpha_0/\beta_{-1},\alpha_{-1}/\beta_{-2},\dots,\alpha_{-(n-2)}/\beta_{-(n-1)})\end{aligned}\tag{55}$$

such that $(X,Y)\in D_n\subset U$. By definition, (X,Y) belongs to each $D_n,\ n\geq1$, for which

$$\alpha_{j+1}=\alpha_R(z_j^R)\quad\text{and}\quad\beta_{k+1}=\beta_R(z_k^R)\tag{56}$$

for all $-(n-1)\leq j\leq n-1$ and $-n\leq k\leq n-1$. Thus, to prove that there is some n for which $D_n\subset U$, it suffices to show that the Euclidean diameters of the sets D_n tend to 0 uniformly in n. For this, it suffices to show that

$$|X-c_n|\to0\qquad\text{and}\qquad|Y-d_n|\to0$$

uniformly in n, where—recycling notation—(X,Y) is an arbitrary point in D_n and

$$\begin{aligned}(c_n,d_n):=&([0/1;\alpha_1/\beta_1,\alpha_2/\beta_2,\dots,\alpha_n/\beta_n],\\&[0/1;1/\beta_0,\alpha_0/\beta_{-1},\alpha_{-1}/\beta_{-2},\dots,\alpha_{-(n-2)}/\beta_{-(n-1)}]).\end{aligned}$$

Fix D_n as in (55) and assume $(X,Y)=(X(z),Y(z))\in D_n$ so that (56) holds. Proposition 6.14 and the fact that $s_R(z)=1$ (and hence $d_R(z)=1$) imply that the digits of the contracted Farey expansion of x with respect to R and $z=(x,y)$ are given by

$$\begin{pmatrix}\alpha_0^R\\\beta_0^R\end{pmatrix}=\begin{pmatrix}1\\u_R(z)\end{pmatrix}\qquad\text{and}\qquad\begin{pmatrix}\alpha_{j+1}^R\\\beta_{j+1}^R\end{pmatrix}=\begin{pmatrix}\alpha_R(z_j^R)\\\beta_R(z_j^R)\end{pmatrix},\quad j\geq0.$$

The previous line and (50) give $X=x-\beta_0^R$. Letting $P_n^R,\ Q_n^R$ be as in (46), the previous line and (56) give

$$\frac{P_n^R}{Q_n^R}=[\beta_0^R/\alpha_0^R;\alpha_1^R/\beta_1^R,\dots,\alpha_n^R/\beta_n^R]=[\beta_0^R/1;\alpha_1/\beta_1,\dots,\alpha_n/\beta_n],$$

so also $c_n=\frac{P_n^R}{Q_n^R}-\beta_0^R$. By Corollary 6.12 and (42),

$$\frac{P_n^R}{Q_n^R} = \frac{u_{n+1}^R}{s_{n+1}^R} = \frac{\lambda_N p_{j_N} + p_{j_N-1}}{\lambda_N q_{j_N} + q_{j_N-1}},$$

where $N = N_{n+1}^R(z)$. By (9) and (10),

$$|X - c_n| = \left| x - \frac{P_n^R}{Q_n^R} \right| = \left| x - \frac{\lambda_N p_{j_N} + p_{j_N-1}}{\lambda_N q_{j_N} + q_{j_N-1}} \right| \leq \left| x - \frac{p_{j_N-1}}{q_{j_N-1}} \right| \leq \frac{1}{q_{j_N-1}^2},$$

where the final inequality follows classical arguments in the theory of RCFs. Since R is bounded away from the origin, there is some integer $M > 0$ such that for any integer $a \geq 1$, the number of rectangles $V_{a-\lambda} \cap H_{\lambda+1}$, $0 \leq \lambda < a$, intersecting R is no greater than M. By (19) and the fact that $z_{n+1}^R \in V_{a_{j_N+1}-\lambda_N} \cap H_{\lambda_N+1}$, this implies that $j_N = j_{N_{n+1}^R(z)}$ grows uniformly in n. Since the denominators q_j of RCF-convergents are strictly increasing, we have that $|X - c_n| \to 0$ uniformly in n.

It remains to show that $|Y - d_n| \to 0$ uniformly in n. Let $n \geq 1$, and consider $z_{-n}^R = (x_{-n}^R, y_{-n}^R) = \mathcal{F}_R^{-n}(z)$. By Proposition 6.14, the digits of the contracted Farey expansion of x_{-n}^R with respect to R and z_{-n}^R are given by

$$\begin{pmatrix} \alpha_0^R(z_{-n}^R) \\ \beta_0^R(z_{-n}^R) \end{pmatrix} = \begin{pmatrix} 1 \\ u_R(z_{-n}^R) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \alpha_{k+1}^R(z_{-n}^R) \\ \beta_{k+1}^R(z_{-n}^R) \end{pmatrix} = \begin{pmatrix} \alpha_R(z_{-n+k}^R) \\ \beta_R(z_{-n+k}^R) \end{pmatrix}, \quad k \geq 0.$$

Since $(X_n^R(z_{-n}^R), Y_n^R(z_{-n}^R)) = (X, Y) \in D_n$ (see (52)), we have by (56)

$$\alpha_{j+1} = \alpha_R(z_{-n+(n+j)}^R) = \alpha_{n+j+1}^R(z_{-n}^R) \quad \text{and} \quad \beta_{k+1} = \beta_R(z_{-n+(n+k)}^R) = \beta_{n+k+1}^R(z_{-n}^R)$$

for all $-(n-1) \leq j \leq n-1$ and $-n \leq k \leq n-1$. In particular, applying Proposition 6.18 to z_{-n}^R, we find

$$Y = Y_n^R(z_{-n}^R) = [0/1; 1/\beta_n^R(z_{-n}^R), \alpha_n^R(z_{-n}^R)/\beta_{n-1}^R(z_{-n}^R), \ldots,$$
$$\alpha_1^R(z_{-n}^R)/\beta_0^R(z_{-n}^R), \alpha_0^R(z_{-n}^R)/(1/y_{-n}^R - 1)]),$$

while

$$\begin{aligned} d_n &= [0/1; 1/\beta_0, \alpha_0/\beta_{-1}, \alpha_{-1}/\beta_{-2}, \ldots, \alpha_{-(n-2)}/\beta_{-(n-1)}] \\ &= [0/1; 1/\beta_n^R(z_{-n}^R), \alpha_n^R(z_{-n}^R)/\beta_{n-1}^R(z_{-n}^R), \ldots, \alpha_2^R(z_{-n}^R)/\beta_1^R(z_{-n}^R)])]. \end{aligned}$$

Set

$$\begin{aligned} B_{[-1,n]}^R(z_{-n}^R) :=& B_{[-1,n]}([\beta_0^R(z_{-n}^R)/\alpha_0^R(z_{-n}^R); \alpha_1^R(z_{-n}^R)/\beta_1^R(z_{-n}^R), \ldots, \alpha_n^R(z_{-n}^R)/\beta_n^R(z_{-n}^R)]) \\ =& \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha_0^R(z_{-n}^R) \\ 1 & \beta_0^R(z_{-n}^R) \end{pmatrix} \begin{pmatrix} 0 & \alpha_1^R(z_{-n}^R) \\ 1 & \beta_1^R(z_{-n}^R) \end{pmatrix} \cdots \begin{pmatrix} 0 & \alpha_n^R(z_{-n}^R) \\ 1 & \beta_n^R(z_{-n}^R) \end{pmatrix}, \end{aligned}$$

and denote the entries by

$$B^R_{[-1,n]}(z^R_{-n}) = \begin{pmatrix} P^R_{n-1}(z^R_{-n}) & P^R_n(z^R_{-n}) \\ Q^R_{n-1}(z^R_{-n}) & Q^R_n(z^R_{-n}) \end{pmatrix}.$$

Then Equation (6) gives

$$Y = (B^R_{[-1,n]}(z^R_{-n}))^T \cdot \begin{pmatrix} \frac{1}{y^R_{-n}} - 1 \end{pmatrix} = \frac{P^R_{n-1}(z^R_{-n})(1 - y^R_{-n}) + Q^R_{n-1}(z^R_{-n})y^R_{-n}}{P^R_n(z^R_{-n})(1 - y^R_{-n}) + Q^R_n(z^R_{-n})y^R_{-n}},$$

while

$$d_n = (B^R_{[-1,n]}(z^R_{-n}))^T \cdot 0 = \frac{Q^R_{n-1}(z^R_{-n})}{Q^R_n(z^R_{-n})}.$$

Notice by Proposition 6.14, Equation (47) and the fact that $s_R(z) = 1$ for all z, that

$$|\det(B^R_{[-1,n]}(z^R_{-n})| = |\alpha^R_0(z^R_{-n})\alpha^R_1(z^R_{-n}) \cdots \alpha^R_n(z^R_{-n})| = 1.$$

Moreover, recall that $y^R_{-n} \in [0, 1]$. We thus compute

$$\begin{aligned} |Y - d_n| &= \left| \frac{P^R_{n-1}(z^R_{-n})(1 - y^R_{-n}) + Q^R_{n-1}(z^R_{-n})y^R_{-n}}{P^R_n(z^R_{-n})(1 - y^R_{-n}) + Q^R_n(z^R_{-n})y^R_{-n}} - \frac{Q^R_{n-1}(z^R_{-n})}{Q^R_n(z^R_{-n})} \right| \\ &= \frac{|P^R_{n-1}(z^R_{-n})Q^R_n(z^R_{-n}) - P^R_n(z^R_{-n})Q^R_{n-1}(z^R_{-n})||1 - y^R_{-n}|}{|P^R_n(z^R_{-n})(1 - y^R_{-n}) + Q^R_n(z^R_{-n})y^R_{-n}||Q^R_n(z^R_{-n})|} \\ &\le \frac{1}{|P^R_n(z^R_{-n}) + (Q^R_n(z^R_{-n}) - P^R_n(z^R_{-n}))y^R_{-n}|Q^R_n(z^R_{-n})} \\ &\le \frac{1}{\min\left\{P^R_n(z^R_{-n}), Q^R_n(z^R_{-n})\right\} Q^R_n(z^R_{-n})}. \end{aligned}$$

By Corollary 6.12,

$$\min\left\{P^R_n(z^R_{-n}), Q^R_n(z^R_{-n})\right\} Q^R_n(z^R_{-n}) = \min\left\{u^R_{n+1}(z^R_{-n}), s^R_{n+1}(z^R_{-n})\right\} s^R_{n+1}(z^R_{-n}),$$

so it suffices to show that

$$\min\left\{u^R_n(z), s^R_n(z)\right\} s^R_n(z) \to \infty$$

uniformly in n. Write $u^R_n(z) = \lambda_N p_{j_N} + p_{j_N-1} \ge p_{j_N-1}$ and $s^R_n(z) = \lambda_N q_{j_N} + q_{j_N-1} \ge q_{j_N-1}$ where $N = N^R_n(z)$. As before, since R is bounded away from the origin, $j_N = j_{N^R_n(z)}$ grows uniformly in n. Since $x \notin \mathbb{Q}$, there is some n large enough (independent of z) for which $u^R_n(z) \ge p_{j_N-1} \ge 1$. Since the RCF-convergent denominators q_j are strictly increasing for $j > 0$,

$$\min\left\{u^R_n(z), s^R_n(z)\right\} s^R_n(z) \ge \min\left\{p_{j_N-1}, q_{j_N-1}\right\} q_{j_N-1} \to \infty$$

uniformly in n. □

Notice from Propositions 6.14 and 6.18 that

$$X(z) = [0/1; \alpha_R(z_0^R)/\beta_R(z_0^R), \alpha_R(z_1^R)/\beta_R(z_1^R), \dots].$$

From the proof of Proposition 6.19, it is evident that the convergents

$$d_n = [0/1; 1/\beta_0, \alpha_0/\beta_{-1}, \alpha_{-1}/\beta_{-2}, \dots, \alpha_{-(n-2)}/\beta_{-(n-1)}]$$

of the GCF

$$[0/1; 1/\beta_0, \alpha_0/\beta_{-1}, \alpha_{-1}/\beta_{-2}, \dots]$$

with

$$\alpha_{j+1} = \alpha_R(z_j^R) \quad \text{and} \quad \beta_{j+1} = \beta_R(z_j^R), \quad j < 0,$$

also converge to $Y(z)$. We thus obtain GCF-expansions of both $X(z)$ and $Y(z)$ on which τ_R acts as a two-sided shift:

Corollary 6.20. *For $z \in R$ for which z_n^R is defined for all $n \in \mathbb{Z}$,*

$$\begin{aligned}(X(z), Y(z)) = \big(&[0/1; \alpha_R(z_0^R)/\beta_R(z_0^R), \alpha_R(z_1^R)/\beta_R(z_1^R), \dots],\\ &[0/1; 1/\beta_R(z_{-1}^R), \alpha_R(z_{-1}^R)/\beta_R(z_{-2}^R), \dots]\big),\end{aligned}$$

and for any $n \in \mathbb{Z}$, $\tau_R^n(X(z), Y(z))$ equals

$$\begin{aligned}\big(&[0/1; \alpha_R(z_n^R)/\beta_R(z_n^R), \alpha_R(z_{n+1}^R)/\beta_R(z_{n+1}^R), \dots],\\ &[0/1; 1/\beta_R(z_{n-1}^R), \alpha_R(z_{n-1}^R)/\beta_R(z_{n-2}^R), \dots]\big).\end{aligned}$$

7. Examples of contracted Farey expansions

In this section we consider several examples of explicit, inducible regions R and the contracted Farey expansions they produce. We shall find in §7.1 RCFs, in §7.2 the second-named author's S-expansions, and in §7.3 Nakada's α-CFs for $\alpha \in (0, 1]$. Throughout this section, any reference to the induced system $(H_1, \mathcal{B}, \bar{\mu}_{H_1}, \mathcal{F}_{H_1})$ is to the 'altered' system from Remark 5.4.

7.1. Regular continued fractions, revisited

Set $R = H_1$, and recall from Theorem 5.5 above that the induced system $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ is isomorphic to the Gauss natural extension $(\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$. We re-obtain this fact here through the use of contracted Farey expansions and the two-sided shift of §6.3.

Proof of Theorem 5.5. Let $z = (x, y) \in R$ with $x \neq 0$ be as in (29). Using (30), we find that $N_R(z) = a_1 = a(x)$, and by (35),

$$\begin{pmatrix} u_R(z) & t_R(z) \\ s_R(z) & r_R(z) \end{pmatrix} = A_R(z) = A_{\varepsilon_1} \cdots A_{\varepsilon_{a_1}} = A_0^{a_1-1} A_1 = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a(x) \end{pmatrix}. \tag{57}$$

In particular, $s_R(z) = 1$ for all z, so we are in the setting of §6.3. We know that $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ is isomorphic to $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$; we shall show that this latter system is precisely $(\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$. Since $u_R(z) = 0$ for all z, the map $\varphi_R : R \to \mathbb{R}^2$ from (50) is

$$\varphi_R(z) = \left(x, \frac{1-y}{y} \right) \qquad \text{for all } z = (x, y) \in R,\ x \neq 0, \tag{58}$$

and thus $\Omega_R = \varphi_R(R) = \Omega$, up to a null set. Since $\bar{\mu}(R) = \log 2$, Theorem 6.16 gives that $\bar{\nu}_R = \bar{\nu}_G$. Moreover, from Equations (47), (48), and (57) we find

$$\begin{pmatrix} \alpha_R(z) \\ \beta_R(z) \end{pmatrix} = \begin{pmatrix} 1 \\ a(x) \end{pmatrix}. \tag{59}$$

But if $(X, Y) = \varphi_R(z)$, Equation (58) gives $X = x$, so by Theorem 6.16 and Equation (11),

$$\tau_R(X, Y) = \left(\frac{1}{X} - a(X), \frac{1}{a(X) + Y} \right) = \mathcal{G}(X, Y).$$

Thus $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R) = (\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$. □

Let $z = (x, y) \in R$ as in (29) (so $b_1 = 1$) with $x \notin \mathbb{Q}$, and notice that repeated use of (43) gives

$$z_k^R = (x_k^R, y_k^R) = \mathcal{F}_R^k(x, y) = ([0; a_{k+1}, a_{k+2}, \dots], [0; 1, a_k, \dots, a_1, b_2, b_3, \dots]).$$

Thus, by Proposition 6.14 and Equation (59), the digits of the contracted Farey expansion of x with respect to $R = H_1$ and $z = (x, y) \in R$ are

$$\begin{pmatrix} \alpha_0^R \\ \beta_0^R \end{pmatrix} = \begin{pmatrix} s_R(z) \\ u_R(z) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \alpha_{k+1}^R \\ \beta_{k+1}^R \end{pmatrix} = \begin{pmatrix} 1 \\ a(x_k^R) \end{pmatrix} = \begin{pmatrix} 1 \\ a_{k+1} \end{pmatrix}.$$

That is, the contracted Farey expansion of x with respect to $R = H_1$ and $z = (x, y)$ recovers the RCF-expansion $[0/1; 1/a_1, 1/a_2, \dots] = [0; a_1, a_2, \dots]$ of x.

7.2. S-expansions, revisited

We also find S-expansions (and thus Minkowski's diagonal CFs, Bosma's optimal CFs, and Nakada's α-CFs for $\alpha \geq 1/2$; see [23] and §3.3 above) as special instances of contracted Farey expansions. Indeed, let $S \subset \Omega$ be a singularisation area, i.e., S is $\bar{\nu}_G$-measurable set with $\bar{\nu}_G(\partial S) = 0$ satisfying both

(a) $S \subset V_1$ and
(b) $S \cap \mathcal{G}(S) = \varnothing$,

and let $[\beta_0^S/\alpha_0^S; \alpha_1^S/\beta_1^S, \alpha_2^S/\beta_2^S, \dots]$ be the S-expansion of $x = [0; a_1, a_2, \dots] \in (0,1) \setminus \mathbb{Q}$ obtained by simultaneously singularising at all positions n for which $\mathcal{G}^n(x,0) \in S$ (see Definitions 4.4, 4.5 of [23] and §3.3 above). For $n \geq -1$ let

$$B^S_{[-1,n]} = \begin{pmatrix} P^S_{n-1} & P^S_n \\ Q^S_{n-1} & Q^S_n \end{pmatrix} := B_{[-1,n]}([\beta_0^S/\alpha_0^S; \alpha_1^S/\beta_1^S, \alpha_2^S/\beta_2^S, \dots]).$$

From remarks preceding Theorem 4.13 and Theorem 5.3.i of [23], it follows that $P^S_{-2} = Q^S_{-1} = 0$, $P^S_{-1} = Q^S_{-2} = 1$, and for $k \geq 0$,

$$\begin{pmatrix} P^S_k \\ Q^S_k \end{pmatrix} = \begin{pmatrix} p_{j_k^S} \\ q_{j_k^S} \end{pmatrix},$$

where p_j/q_j is the j^{th} RCF-convergent of x and $(j_k^S)_{k \geq 0}$ is the subsequence of powers $j \geq 0$ for which $\mathcal{G}^j(x,0) \in \Delta := \Omega \backslash S$.

We wish to determine a proper, inducible subregion $R \subset \Omega$ for which the contracted Farey expansion $[\beta_0^R/\alpha_0^R; \alpha_1^R/\beta_1^R, \alpha_2^R/\beta_2^R, \dots]$ of x with respect to R coincides with the S-expansion of x. By Remark 2.2, it suffices to find R such that $P^R_k = P^S_k$ and $Q^R_k = Q^S_k$ for all $k \geq 0$, with

$$B^R_{[-1,k]} = \begin{pmatrix} P^R_{k-1} & P^R_k \\ Q^R_{k-1} & Q^R_k \end{pmatrix}$$

as in (45) and (46).

It seems natural to set $R := \varphi^{-1}_{H_1}(\Delta) \subset H_1$, where $\varphi_{H_1} : H_1 \to \Omega$ is the isomorphism map between $(H_1, \mathcal{B}, \bar{\mu}_{H_1}, \mathcal{F}_{H_1})$ and $(\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$ from (58) above satisfying $\varphi_{H_1} \circ \mathcal{F}_{H_1}(z) = \mathcal{G} \circ \varphi_{H_1}(z)$ for all $z \in H_1$. However, in the classical setting of RCFs and, in particular, S-expansions, one uses the one-to-one correspondence between points in the $\mathcal{G}$-orbit of $(x,0)$ and RCF-convergents p_n/q_n, which come from the *right-hand column* of the matrix $\left(\begin{smallmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{smallmatrix}\right)$. On the other hand, for contracted Farey expansions we use the one-to-one correspondence between points in the $\mathcal{F}_R$-orbit of $(x,1)$ and contracted Farey convergents u_n^R/s_n^R coming from the *left-hand column* of the matrix $A^R_{[0,n]} = A_{[0,N_n^R]}$ from (41). When $R = \varphi^{-1}_{H_1}(\Omega) = H_1$, the matrix $A^{H_1}_{[0,n]}$ is of the form $\left(\begin{smallmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{smallmatrix}\right)$, so the

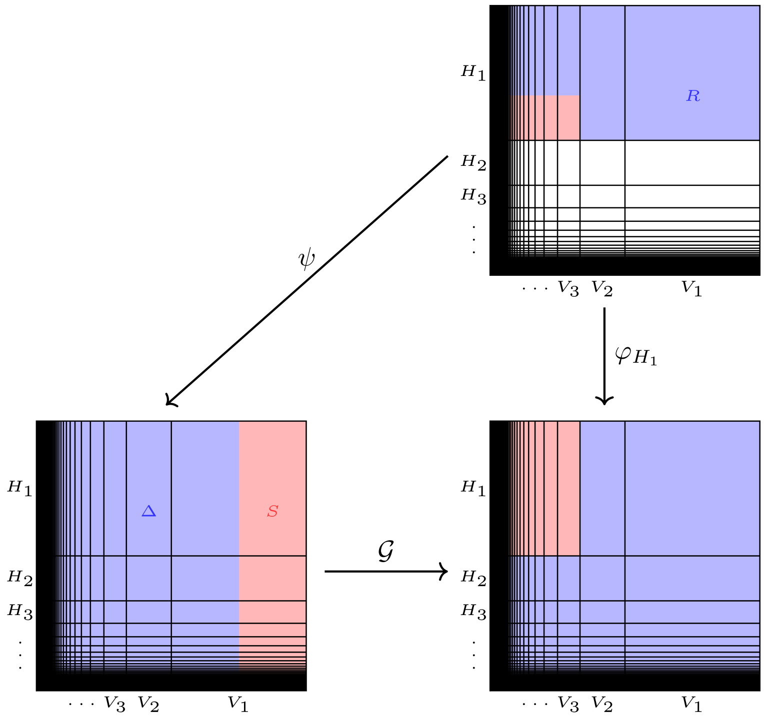
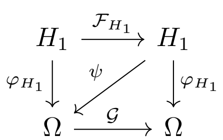


Fig. 5. Bottom-left: A singularisation area S and its complement Δ in Ω. Bottom-right: The images of S and Δ under $\mathcal{G}$. Top-right: The region $R = \psi^{-1}(\Delta)$ and its complement in H_1.

one-to-one correspondence in this setting is between $\mathcal{F}^n_{H_1}(x,1) = \varphi^{-1}_{H_1} \circ \mathcal{G}^n(x,0)$ and p_{n-1}/q_{n-1}. This indexing discrepancy is fixed by instead considering the isomorphism map $\psi := \mathcal{G}^{-1} \circ \varphi_{H_1}$ between $(H_1, \mathcal{B}, \bar{\mu}_{H_1}, \mathcal{F}_{H_1})$ and $(\Omega, \mathcal{B}, \bar{\nu}_G, \mathcal{G})$:



Set

$$R := \psi^{-1}(\Delta) = H_1 \backslash \psi^{-1}(S);$$

see Fig. 5. Notice that for any $z \in H_1$, either $\varphi_{H_1}(z) \in \Delta$ or $\mathcal{G} \circ \varphi_{H_1}(z) \in \Delta$; otherwise, both $\varphi_{H_1}(z)$ and $\mathcal{G} \circ \varphi_{H_1}(z)$ belong to S, contrary to condition (ii) of a singularisation area. Thus, either $\psi^{-1} \circ \varphi_{H_1}(z) \in R$ or $\psi^{-1} \circ \mathcal{G} \circ \varphi_{H_1}(z) \in R$. But $\psi^{-1} \circ \varphi_{H_1} = \mathcal{F}_{H_1}$ and $\psi^{-1} \circ \mathcal{G} \circ \varphi_{H_1} = \mathcal{F}^2_{H_1}$, so for any $z \in H_1$, either $\mathcal{F}_{H_1}(z) \in R$ or $\mathcal{F}^2_{H_1}(z) \in R$. The entries of the matrices $A_R(z)$ depend on whether $\mathcal{F}_{H_1}(z) \in R$:

Lemma 7.1. *For any $z = (x, y) \in H_1$ with $x = [0; a_1, a_2, \dots]$,*

$$\begin{pmatrix} u_R(z) & t_R(z) \\ s_R(z) & r_R(z) \end{pmatrix} = A_R(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} & \text{if } \mathcal{F}_{H_1}(z) \in R, \\ \begin{pmatrix} 1 & a_2 \\ 1 & a_2+1 \end{pmatrix} & \text{if } \mathcal{F}_{H_1}(z) \notin R. \end{cases}$$

Proof. First suppose that $\mathcal{F}_{H_1}(z) \in R$. Now $\mathcal{F}_{H_1}(z) = \mathcal{F}^{a_1}(z)$, and for all $1 \le j < a_1$, $\mathcal{F}^j(z) \notin H_1$ implies $\mathcal{F}^j(z) \notin R \subset H_1$. Thus $N_R(z) = a_1$, and by (35) we have

$$A_R(z) = A_0^{a_1-1} A_1 = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}.$$

If $\mathcal{F}_{H_1}(z) = \mathcal{F}^{a_1}(z) \notin R$, then $\mathcal{F}^2_{H_1}(z) = \mathcal{F}^{a_1+a_2}(z) \in R$. Since $\mathcal{F}^j(z) \notin H_1$ for $1 \le j < a_1 + a_2$ with $j \ne a_1$, we have $N_R(z) = a_1 + a_2$ and—by (35)—

$$A_R(z) = A_0^{a_1-1} A_1 A_0^{a_2-1} A_1 = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} = \begin{pmatrix} 1 & a_2 \\ a_1 & a_2 a_1 + 1 \end{pmatrix}.$$

But $\mathcal{F}_{H_1}(z) \notin R$ is equivalent to $\varphi_{H_1}(z) \notin \Delta$, or $\varphi_{H_1}(z) \in S$. Since φ_{H_1} acts as the identity on the first coordinate and $S \subset V_1$ by condition (a) of a singularisation area, this implies $a_1 = 1$. □

By Lemma 7.1, $s_R(z) = 1$ for all $z \in H_1$ and, in particular, for all $z = (x, 1)$. By Corollary 6.12, Equation (42), and the fact that $R \subset H_1$ (so $\lambda_{N^R_{k+1}} = 0$),

$$\begin{pmatrix} P^R_k \\ Q^R_k \end{pmatrix} = \begin{pmatrix} u^R_{k+1} \\ s^R_{k+1} \end{pmatrix} = \begin{pmatrix} p_{j_{N^R_{k+1}}-1} \\ q_{j_{N^R_{k+1}}-1} \end{pmatrix}, \qquad k \ge 0.$$

Writing $N^R_{k+1} = a_1 + \dots + a_{j_{N^R_{k+1}}}$ (see (19)), we see that the indices $j_{N^R_{k+1}}$, $k \ge 0$, are precisely the powers $j > 0$ for which

$$\mathcal{F}^{a_1+\dots+a_j}(x, 1) = \mathcal{F}^j_{H_1}(x, 1) \in R.$$

Equivalently, these are the powers $j > 0$ for which

$$\varphi_{H_1}^{-1} \circ \mathcal{G}^j \circ \varphi_{H_1}(x, 1) \in R = \psi^{-1}(\Delta) = \varphi_{H_1}^{-1} \circ \mathcal{G}(\Delta),$$

or $\mathcal{G}^{j-1}(x, 0) \in \Delta$. Thus $j_{N^R_{k+1}} - 1 = j^S_k$, and

$$\begin{pmatrix} P^R_k \\ Q^R_k \end{pmatrix} = \begin{pmatrix} p_{j_{N^R_{k+1}}-1} \\ q_{j_{N^R_{k+1}}-1} \end{pmatrix} = \begin{pmatrix} p_{j^S_k} \\ q_{j^S_k} \end{pmatrix} = \begin{pmatrix} P^S_k \\ Q^S_k \end{pmatrix}, \qquad k \ge 0.$$

By Remark 2.2, this proves:

Proposition 7.2. *The contracted Farey expansion of x with respect to $R = \psi^{-1}(\Delta)$ coincides with the S-expansion of x.*

In §5 of [23], a two-dimensional ergodic system[14] $(\Gamma_S, \mathcal{B}, \rho, \tau)$ is constructed corresponding to the two-sided shift operator for S-expansions. We briefly recall this system here and show that it coincides with $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$ as defined in §6.3. (Note by Lemma 7.1 that $s_R(z) = 1$ for all $z \in R$, so we are in the setting of §6.3.) Set

$$\Delta^- := \mathcal{G}(S) \qquad \text{and} \qquad \Delta^+ := \Delta \backslash \Delta^-.$$

Define $M : \Delta \to \mathbb{R}^2$ for $z = (x, y)$ by

$$M(z) := \begin{cases} (x, y), & z \in \Delta^+, \\ \left(\frac{-x}{1+x}, 1 - y\right), & z \in \Delta^-, \end{cases}$$

and let $\Gamma_S := M(\Delta)$. The map $\tau : \Gamma_S \to \Gamma_S$ is defined by $\tau := M \circ \mathcal{G}_\Delta \circ M^{-1}$, where $\mathcal{G}_\Delta : \Delta \to \Delta$ is the map $\mathcal{G}$ induced on Δ, i.e., $\mathcal{G}_\Delta(z) = \mathcal{G}(z)$ if $\mathcal{G}(z) \in \Delta$ and $\mathcal{G}_\Delta(z) = \mathcal{G}^2(z)$ otherwise. The measure ρ is the probability measure on $(\Gamma_S, \mathcal{B})$ with density $1/((\log 2)\bar{\nu}_G(\Delta)(1 + XY)^2)$ (see Theorem 5.9 of [23]). Setting

$$X_k^S := [0/1; \alpha_{k+1}^S/\beta_{k+1}^S, \alpha_{k+2}^S/\beta_{k+2}^S, \ldots], \qquad k \geq 0,$$

$Y_0^S := 0$ and

$$Y_k^S := [0/1; 1/\beta_k^S, \alpha_k^S/\beta_{k-1}^S, \ldots, \alpha_2^S/\beta_1^S], \qquad k \geq 1,$$

where $x = [\beta_0^S/\alpha_0^S; \alpha_1^S/\beta_1^S, \alpha_2^S/\beta_2^S, \ldots]$ is the S-expansion of x, it is observed following Definition 5.8 of [23] that

$$(X_k^S, Y_k^S) = \tau^k(X_0^S, Y_0^S), \qquad k \geq 0.$$

Note that by Propositions 6.18 and 7.2, τ^n and τ_R^n agree for all $n \geq 0$ when evaluated at $(X_0^S, Y_0^S) = (X_0^S, 0)$. We claim that in fact $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R) = (\Gamma_S, \mathcal{B}, \rho, \tau)$. By (50) and Lemma 7.1,

$$\varphi_R(z) = \begin{cases} \left(x, \frac{1-y}{y}\right), & \mathcal{F}_{H_1}(z) \in R, \\ (x - 1, 1 - y), & \mathcal{F}_{H_1}(z) \notin R. \end{cases} \tag{60}$$

Lemma 7.3. *For any $z \in \Delta$, $M(z) = \varphi_R \circ \psi^{-1} \circ \mathcal{G}_\Delta^{-1}(z)$.*

---

14 We replace the original notation Ω_S from [23] by Γ_S to avoid confusion with Ω_R defined §6.3. However, we shall see in Proposition 7.4 below that, in fact, $\Gamma_S = \Omega_R$.

Proof. Suppose first that $z = (x, y) \in \Delta^+$. Now, since $z \notin \Delta^- = \mathcal{G}(S)$, we have $\mathcal{G}^{-1}(z) \in \Omega \setminus S = \Delta$. Hence $\mathcal{G}_\Delta^{-1}(z) = \mathcal{G}^{-1}(z)$. Then

$$\varphi_R \circ \psi^{-1} \circ \mathcal{G}_\Delta^{-1}(z) = \varphi_R \circ \varphi_{H_1}^{-1}(z) = \varphi_R \left(x, \frac{1}{1+y}\right).$$

Notice that

$$\mathcal{F}_{H_1}\left(x, \frac{1}{1+y}\right) = \psi^{-1} \circ \varphi_{H_1}\left(x, \frac{1}{1+y}\right) = \psi^{-1}(z) \in \psi^{-1}(\Delta) = R,$$

so by (60), $\varphi_R\,(x, 1/(1+y)) = z$. Thus, for $z \in \Delta^+$, $\varphi_R \circ \psi^{-1} \circ \mathcal{G}_\Delta^{-1}(z) = z = M(z)$.

Next, suppose that $z \in \Delta^-$. Then $z \in \mathcal{G}(S)$, so $\mathcal{G}^{-1}(z) \in S = \Omega \setminus \Delta$ and $\mathcal{G}_\Delta^{-1}(z) = \mathcal{G}^{-2}(z)$. Moreover, since $\mathcal{G}^{-1}(z) \in S \subset V_1$, we have $\mathcal{G}^{-1}(z) = (1/(x+1), 1/y - 1)$. With these observations, we find

$$\varphi_R \circ \psi^{-1} \circ \mathcal{G}_\Delta^{-1}(z) = \varphi_R \circ \varphi_{H_1}^{-1} \circ \mathcal{G}^{-1}(z) = \varphi_R \circ \varphi_{H_1}^{-1}\left(\frac{1}{x+1}, \frac{1}{y} - 1\right) = \varphi_R\left(\frac{1}{x+1}, y\right).$$

We claim that $\mathcal{F}_{H_1}(1/(x+1), y) \notin R$. This is equivalent to $\psi^{-1} \circ \varphi_{H_1}(1/(x+1), y) \notin \psi^{-1}(\Delta)$, or $\varphi_{H_1}(1/(x+1), y) \in S$. But $\varphi_{H_1}(1/(x+1), y) = (1/(x+1), 1/y-1) = \mathcal{G}^{-1}(z) \in S$ by assumption, so the claim holds. Thus, from (60), we have $\varphi_R(1/(x+1), y) = (-x/(x+1), 1-y)$ and $\varphi_R \circ \psi^{-1} \circ \mathcal{G}_\Delta^{-1}(z) = (-x/(x+1), 1-y) = M(z)$ for $z \in \Delta^-$. □

Proposition 7.4. *With* $R = \psi^{-1}(\Delta)$,

$$(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R) = (\Gamma_S, \mathcal{B}, \rho, \tau).$$

Proof. By Lemma 7.3,

$$\Omega_R = \varphi_R(R) = \varphi_R \circ \psi^{-1}(\Delta) = \varphi_R \circ \psi^{-1} \circ \mathcal{G}_\Delta^{-1}(\Delta) = M(\Delta) = \Gamma_S.$$

Moreover,

$$\begin{aligned}
\psi \circ \mathcal{F}_R(z) &= \begin{cases} \psi \circ \mathcal{F}_{H_1}(z), & \mathcal{F}_{H_1}(z) \in R, \\ \psi \circ \mathcal{F}_{H_1}^2(z), & \mathcal{F}_{H_1}(z) \notin R, \end{cases} \\
&= \begin{cases} \varphi_{H_1}(z), & \varphi_{H_1}(z) \in \Delta, \\ \mathcal{G} \circ \varphi_{H_1}(z), & \varphi_{H_1}(z) \notin \Delta, \end{cases} \\
&= \begin{cases} \mathcal{G} \circ \psi(z), & \mathcal{G} \circ \psi(z) \in \Delta, \\ \mathcal{G}^2 \circ \psi(z), & \mathcal{G} \circ \psi(z) \notin \Delta, \end{cases} \\
&= \mathcal{G}_\Delta \circ \psi(z),
\end{aligned}$$

so

$$\tau_R = \varphi_R \circ \mathcal{F}_R \circ \varphi_R^{-1} = \varphi_R \circ \psi^{-1} \circ \mathcal{G}_\Delta \circ \psi \circ \varphi_R^{-1} = M \circ \mathcal{G}_\Delta \circ M^{-1} = \tau.$$

Lastly, $\bar{\nu}_R = \rho$ since these are both probability measures on $\Omega_R = \Gamma_S$ with densities of the form $C(1+XY)^{-2}$, where C is a normalising constant. □

7.3. Nakada's α-continued fractions, revisited

Recall Nakada's parameterised family of α-CF maps from §3.2, which are defined for all $0 \le \alpha \le 1$. Moreover, recall from the end of §3.3 that the natural extensions of the α-CFs are realised as S-expansion systems, but only for $\alpha \ge 1/2$. Since, by §7.2, S-expansions are realised as contracted Farey expansions, so are Nakada's α-CFs for $\alpha \ge 1/2$. In this subsection we extend this fact to $\alpha > 0$, giving a new description of a planar natural extension of $([\alpha-1,\alpha], \mathcal{B}, \rho_\alpha, G_\alpha)$ as an explicit induced transformation $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ of Ito's natural extension of the Farey tent map (Theorem 7.11 below; cf. [24]).

Remark 7.5. One finds that $G_\alpha([\alpha-1,\alpha]) = [\alpha-1,\alpha)$, so $([\alpha-1,\alpha], \mathcal{B}, \rho_\alpha, G_\alpha)$ is isomorphic to the restriction of this system to $[\alpha-1,\alpha)$, which we denote by $([\alpha-1,\alpha), \mathcal{B}, \rho_\alpha, G_\alpha)$. The endpoint α was included in the domain in §3.2 so that we could speak of matching, which depends on the G_α-orbits of α and $\alpha-1$. However, it shall be more convenient in this subsection to consider the isomorphic system $([\alpha-1,\alpha), \mathcal{B}, \rho_\alpha, G_\alpha)$.

The domain R will be constructed in two steps: first, we define a subset $A \subset H_1$ via an integer-valued map k on H_1; second, R is defined by 'pushing' part of A down into $\Omega \setminus H_1$ with the map $\mathcal{F}$. Fix $\alpha \in (0,1]$ and define $k : H_1 \to \mathbb{N} \cup \{\infty\}$ by

$$k(z) := \inf\{j > 0 \mid \mathcal{F}_{H_1}^{-j}(z) \in [0,\alpha) \times [1/2,1]\}, \qquad z \in H_1,$$

and let

$$A := \{z \in H_1 \mid k(z) \text{ is odd}\};$$

see Fig. 6. Recall the definition of hitting times N_R from (33). The restriction of N_R to R—also denoted N_R—is called the *return time* to R. We wish to determine the return times N_A under $\mathcal{F}$. For this, we use the following:

Lemma 7.6. *For any $z = (x,y) \in H_1$,*

$$k(\mathcal{F}_{H_1}(z)) = \begin{cases} 1, & x < \alpha, \\ k(z)+1, & x \ge \alpha. \end{cases}$$

Proof. First, notice that $\mathcal{F}_{H_1}^{-1}(\mathcal{F}_{H_1}(z)) = z$ belongs to $[0,\alpha)\times[1/2,1]$ if and only if $x < \alpha$. Thus, if $x < \alpha$, then $k(\mathcal{F}_{H_1}(z)) = 1$. If $x \geq \alpha$, then $k(z)$ is the infimum of powers $j > 0$ for which

$$\mathcal{F}_{H_1}^{-(j+1)}(\mathcal{F}_{H_1}(z)) = \mathcal{F}_{H_1}^{-j}(z) \in [0,\alpha)\times[1/2,1].$$

Hence $k(\mathcal{F}_{H_1}(z)) = k(z) + 1$. □

Lemma 7.7. *The return times $N_A : A \to \mathbb{N}$ under $\mathcal{F}$ are*

$$N_A(z) = \begin{cases} a_1, & x < \alpha, \\ a_1 + a_2, & x \geq \alpha, \end{cases}$$

where $z = (x,y) \in A$ with $x = [0; a_1, a_2, \dots]$.

Proof. First, suppose that $x < \alpha$, and notice that for all $0 < j < a_1$, $\mathcal{F}^j(z) \in H_{j+1} \neq H_1$ implies $\mathcal{F}^j(z) \notin A \subset H_1$. On the other hand, by Lemma 7.6, $k(\mathcal{F}_{H_1}(z)) = 1$ is odd, so $\mathcal{F}_{H_1}(z) \in A$. Since $\mathcal{F}_{H_1}(z) = \mathcal{F}^{a_1}(z)$, we have $N_A(z) = a_1$.

Next, suppose that $x \geq \alpha$. As above, $\mathcal{F}^j(z) \notin H_1$ for all $0 < j < a_1+a_2$ with $j \neq a_1$, so $\mathcal{F}^j(z) \notin A$ for such j. Moreover, $z \in A$ implies that $k(z)$ is odd, and thus by Lemma 7.6, $k(\mathcal{F}_{H_1}(z)) = k(z)+1$ is even. Hence $\mathcal{F}^{a_1}(z) = \mathcal{F}_{H_1}(z) \notin A$. Write $z' = (x', y') := \mathcal{F}_{H_1}(z)$. Again by Lemma 7.6,

$$k(\mathcal{F}_{H_1}(z')) = \begin{cases} 1, & x' < \alpha, \\ k(z') + 1, & x' \geq \alpha. \end{cases}$$

But $k(z') = k(\mathcal{F}_{H_1}(z))$ is even, so in either case $k(\mathcal{F}_{H_1}(z'))$ is odd. Hence $\mathcal{F}^{a_1+a_2}(z) = \mathcal{F}_{H_1}^2(z) = \mathcal{F}_{H_1}(z') \in A$, and $N_A(z) = a_1 + a_2$. □

We now define the subregion $R \subset \Omega$ in terms of the set $A \subset H_1$. For each integer $a > 1$, let

$$A_a := A \cap V_a \cap ([\alpha, 1] \times [1/2, 1])$$

be the set of points $z = (x,y) \in A$ for which $x = [0; a_1, a_2, \dots]$ with $a_1 = a$ and $x \geq \alpha$. Next, define

$$R := A \cup \bigcup_{a>1} \bigcup_{\lambda=1}^{a-1} \mathcal{F}^\lambda(A_a) \tag{61}$$

as the region $A \subset H_1$ together with each A_a ‘pushed down’ into $\Omega\backslash H_1$ under $\mathcal{F}$ a maximal number of times; see Fig. 6. Notice that if $\alpha > 1/2$, then $A_a = \varnothing$ for $a > 1$ and hence $R = A$.

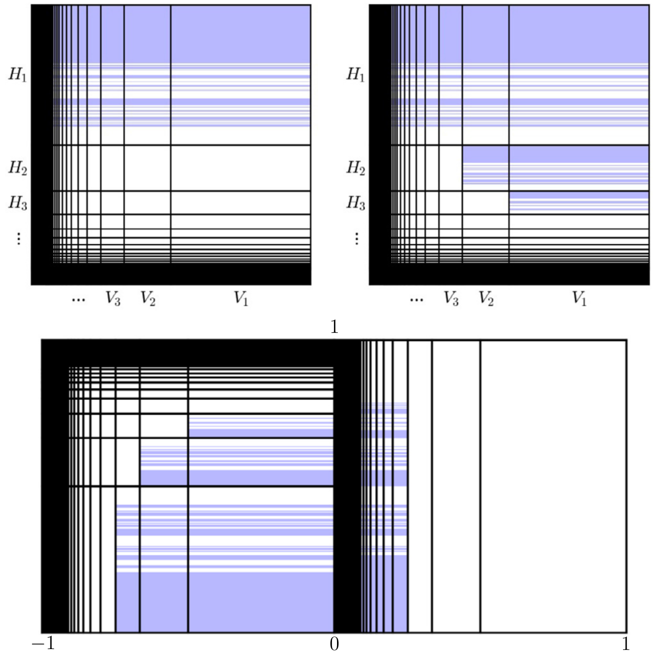


Fig. 6. Approximations of the regions A (top-left), R (top-right), and Ω_R (bottom) for $\alpha = 1/4$.

Remark 7.8. In Fig. 6, the region A consists of rectangles extending from $x = 0$ to $x = 1$, and the region R consists of A together with rectangles extending from $x = F(1/4) = 1/3$ to $x = 1$ and $x = F^2(1/4) = 1/2$ to $x = 1$. These 'full' rectangles are due to the fact that $\alpha = 1/4$ is of the form $\alpha = 1/n$ for some integer $n \geq 1$; see also [28], where the natural extensions of the α-CF maps are constructed for such α. For general $\alpha > 0$, one can show that A consists of rectangles extending from various $x = x_0 \in [0, 1)$ to $x = 1$.

Lemma 7.7 and the definition of R give the following:

Corollary 7.9. *The return times $N_R : R \to \mathbb{N}$ under $\mathcal{F}$ are given by $N_R = N_A$ if $\alpha > 1/2$ and*

$$N_R(z) = \begin{cases} a_1, & x < \alpha, \\ 1, & \alpha \leq x \leq 1/2, \\ a_2 + 1, & 1/2 < x, \end{cases}$$

if $\alpha \leq 1/2$, where $z = (x, y) \in A$ with $x = [0; a_1, a_2, \dots]$.

From this and Equation (35), we find that if $\alpha > 1/2$, then

$$A_R(z) = \begin{pmatrix} u_R(z) & t_R(z) \\ s_R(z) & r_R(z) \end{pmatrix} = \left\{ \begin{matrix} A_0^{a_1-1}A_1, & x < \alpha \\ A_1A_0^{a_2-1}A_1, & x \geq \alpha \end{matrix} \right\} = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}, & x < \alpha, \\ \begin{pmatrix} 1 & a_2 \\ 1 & a_2+1 \end{pmatrix}, & x \geq \alpha, \end{cases} \tag{62}$$

while if $\alpha \leq 1/2$,

$$A_R(z) = \begin{pmatrix} u_R(z) & t_R(z) \\ s_R(z) & r_R(z) \end{pmatrix} = \left\{ \begin{matrix} A_0^{a_1-1}A_1, & x < \alpha \\ A_0, & \alpha \leq x \leq 1/2 \\ A_1A_0^{a_2-1}A_1, & 1/2 < x \end{matrix} \right\}$$
$$= \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}, & x < \alpha, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \alpha \leq x \leq 1/2, \\ \begin{pmatrix} 1 & a_2 \\ 1 & a_2+1 \end{pmatrix}, & 1/2 < x. \end{cases} \tag{63}$$

Notice, in particular, that $s_R(z) = 1$ for all z, and R satisfies the assumptions of §6.3. Moreover,

$$u_R(z) = \begin{cases} 0, & x < \alpha, \\ 1 & x \geq \alpha, \end{cases} \tag{64}$$

so the map $\varphi_R : R \to \mathbb{R}^2$ from (50) is given by

$$\varphi_R(z) = \begin{cases} \left(x, \frac{1-y}{y}\right), & x < \alpha, \\ (x-1, 1-y), & x \geq \alpha. \end{cases} \tag{65}$$

The region $\Omega_R = \varphi_R(R)$ is shown in Fig. 6.

Before proving that $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ is the natural extension of $([\alpha-1, \alpha), \mathcal{B}, \rho_\alpha, G_\alpha)$, we determine the values of $\alpha_R(z)$ and $\beta_R(z)$ defined in (47) and (48).

Lemma 7.10. *Let $z = (x, y) \in R$ with $x \neq 0, 1$. Then*

$$\alpha_R(z) = \begin{cases} 1, & x < \alpha, \\ -1, & x \geq \alpha, \end{cases} \quad \textit{and} \quad \beta_R(z) = \begin{cases} \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor, & x < \alpha, \\ \left\lfloor \frac{1}{1-x} + 1 - \alpha \right\rfloor, & x \geq \alpha. \end{cases}$$

Proof. Let $z = (x, y) \in R$ with $x = [0; a_1, a_2, \ldots]$, and set $z' = (x', y') = \mathcal{F}_R(z)$. From (47), the fact that $s_R(z) = 1$ for all z, and (62) and (63), we have

$$\alpha_R(z) = -\det(A_R(z)) = \begin{cases} 1, & x < \alpha, \\ -1, & x \geq \alpha, \end{cases}$$

as claimed.

Next, from (48) and the fact that $s_R(z) = 1$ for all z, we have $\beta_R(z) = r_R(z) + u_R(z')$. If $\alpha > 1/2$, then from (62) and (64), we find that

$$\beta_R(z) = \left\{ \begin{matrix} a_1 + u_R(z'), & x < \alpha \\ a_2 + 1 + u_R(z'), & x \geq \alpha \end{matrix} \right\} = \begin{cases} a_1, & x < \alpha \text{ and } x' < \alpha, \\ a_1 + 1, & x < \alpha \text{ and } x' \geq \alpha, \\ a_2 + 1, & x \geq \alpha \text{ and } x' < \alpha, \\ a_2 + 2, & x \geq \alpha \text{ and } x' \geq \alpha. \end{cases} \tag{66}$$

Now suppose $\alpha \leq 1/2$. Notice that if $\alpha \leq x \leq 1/2$, then by (34) and (63),

$$x' = A_R^{-1}(z) \cdot x = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot x = \frac{x}{1-x} > x \geq \alpha \tag{67}$$

implies $u_R(z') = 1$. Hence, again from (63) and (64),

$$\beta_R(z) = \left\{ \begin{matrix} a_1 + u_R(z'), & x < \alpha \\ 1 + u_R(z'), & \alpha \leq x \leq 1/2 \\ a_2 + 1 + u_R(z'), & 1/2 < x \end{matrix} \right\} = \begin{cases} a_1, & x < \alpha \text{ and } x' < \alpha, \\ a_1 + 1, & x < \alpha \text{ and } x' \geq \alpha, \\ 2, & \alpha \leq x \leq 1/2, \\ a_2 + 1, & 1/2 < x \text{ and } x' < \alpha, \\ a_2 + 2, & 1/2 < x \text{ and } x' \geq \alpha. \end{cases} \tag{68}$$

The remainder of the proof consists of cases. Throughout, we repeatedly use the two inequalities $\alpha \leq 1 + x'$ and $x' < 1 + \alpha$, which follow from $\alpha \in (0, 1]$ and $x' \in [0, 1]$.

(i) Suppose that $x < \alpha$. We must show $\beta_R(z) = \lfloor \frac{1}{x} + 1 - \alpha \rfloor$. By (34), (62) and (63),

$$x' = A_R^{-1}(z) \cdot x = \begin{pmatrix} a_1 & -1 \\ -1 & 0 \end{pmatrix} \cdot x = \frac{1}{x} - a_1,$$

so

$$\frac{1}{x} + 1 - \alpha = x' + a_1 + 1 - \alpha.$$

(a) If $x' < \alpha$, then

$$a_1 \leq x' + a_1 + 1 - \alpha < a_1 + 1,$$

and by (66) and (68), $\beta_R(z) = a_1 = \lfloor \frac{1}{x} + 1 - \alpha \rfloor$.

(b) If $x' \geq \alpha$, then

$$a_1 + 1 \leq x' + a_1 + 1 - \alpha < a_1 + 2,$$

so by (66) and (68), $\beta_R(z) = a_1 + 1 = \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor$.

(ii) Now suppose that $x \geq \alpha$. We must show $\beta_R(z) = \left\lfloor \frac{1}{1-x} + 1 - \alpha \right\rfloor$.

(a) If $x \leq 1/2$, then from the computation in (67), $1 + x' = 1/(1-x)$. Hence

$$\frac{1}{1-x} + 1 - \alpha = 2 + x' - \alpha,$$

and (using $x' > x$)

$$2 \leq 2 + x - \alpha < 2 + x' - \alpha < 3.$$

By (68), $\beta_R(z) = 2 = \left\lfloor \frac{1}{1-x} + 1 - \alpha \right\rfloor$.

(b) Now suppose that $x > 1/2$. By (34), (62) and (63),

$$x' = A_R^{-1}(z) \cdot x = \begin{pmatrix} a_2 + 1 & -a_2 \\ -1 & 1 \end{pmatrix} \cdot x = \frac{x}{1-x} - a_2,$$

and

$$\frac{1}{1-x} + 1 - \alpha = \frac{x}{1-x} + 2 - \alpha = a_2 + 2 + x' - \alpha.$$

(1) If $x' < \alpha$, then

$$a_2 + 1 \leq a_2 + 2 + x' - \alpha < a_2 + 2,$$

and by (66) and (68), $\beta_R(z) = a_2 + 1 = \left\lfloor \frac{1}{1-x} + 1 - \alpha \right\rfloor$.

(2) Lastly, if $x' \geq \alpha$, then

$$a_2 + 2 \leq a_2 + 2 + x' - \alpha < a_2 + 3,$$

so by (66) and (68), $\beta_R(z) = a_2 + 2 = \left\lfloor \frac{1}{1-x} + 1 - \alpha \right\rfloor$. $\square$

We are now in a position to prove:

Theorem 7.11. *The induced system $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ is the natural extension of $([\alpha - 1, \alpha), \mathcal{B}, \rho_\alpha, G_\alpha)$.*

Proof. When $\alpha = 1$, then $R = [0, 1) \times [1/2, 1)$, and $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ is isomorphic to $(H_1, \mathcal{B}, \bar{\mu}_{H_1}, \mathcal{F}_{H_1})$. The result follows from Theorem 5.5 and the fact that for $\alpha = 1$, $([\alpha - 1, \alpha), \mathcal{B}, \rho_\alpha, G_\alpha)$ is (isomorphic to) $([0, 1], \mathcal{B}, \nu_G, G)$.

Now suppose $\alpha \in (0,1)$. Since $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ and the system $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$ from §6.3 are isomorphic, it suffices to show that the latter system is the natural extension of $([\alpha-1,\alpha), \mathcal{B}, \rho_\alpha, G_\alpha)$. Throughout, we shall consider the restrictions of $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ and $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$ to the full-measure subsets on which $\mathcal{F}_R^n$ and τ_R^n are defined for all $n \in \mathbb{Z}$, and such that for any $(x,y) \in R$ and any $(X,Y) \in \Omega_R$, both x and X are irrational; see the discussion preceding Proposition 6.19. Since $G_\alpha([\alpha-1,\alpha)\backslash\mathbb{Q}) \subset [\alpha-1,\alpha)\backslash\mathbb{Q}$, we shall in fact show that $(\Omega_R, \mathcal{B}, \bar{\nu}_R, \tau_R)$ is the natural extension of $([\alpha-1,\alpha), \mathcal{B}, \rho_\alpha, G_\alpha)$ restricted to $[\alpha-1,\alpha)\backslash\mathbb{Q}$, which we denote $([\alpha-1,\alpha)\backslash\mathbb{Q}, \mathcal{B}, \rho_\alpha, G_\alpha)$

To distinguish the Borel σ-algebras restricted to Ω_R and $[\alpha-1,\alpha)\backslash\mathbb{Q}$, we shall denote these by $\mathcal{C}$ and $\mathcal{D}$, respectively. Notice that $([\alpha-1,\alpha)\backslash\mathbb{Q}, \mathcal{D}, \rho_\alpha, G_\alpha)$ is non-invertible and $(\Omega_R, \mathcal{C}, \bar{\nu}_R, \tau_R)$ is invertible. We will show (i) that $([\alpha-1,\alpha)\backslash\mathbb{Q}, \mathcal{D}, \rho_\alpha, G_\alpha)$ is a factor of $(\Omega_R, \mathcal{C}, \bar{\nu}_R, \tau_R)$ with factor map $\pi_X : \Omega_R \to [\alpha-1,\alpha)\backslash\mathbb{Q}$ being the projection onto the first coordinate, and (ii) that the factor map π_X satisfies

$$\bigvee_{n=0}^{\infty} \tau_R^n \circ \pi_X^{-1}(\mathcal{D}) = \mathcal{C},$$

where $\bigvee_{n=0}^{\infty} \tau_R^n \circ \pi_X^{-1}(\mathcal{D})$ is the smallest σ-algebra containing each σ-algebra $\tau_R^n \circ \pi_X^{-1}(\mathcal{D}),\ n \geq 0$.

(i) We must show that $\pi_X : \Omega_R \to [\alpha-1,\alpha)\backslash\mathbb{Q}$ is measurable, surjective, and satisfies $\pi_X \circ \tau_R = G_\alpha \circ \pi_X$ and $\bar{\nu}_R \circ \pi_X^{-1} = \rho_\alpha$. Certainly π_X is measurable, since for any Borel set $A \in \mathcal{D}$, $\pi_X^{-1}(A) = (A \times [0,1]) \cap \Omega_R \in \mathcal{C}$ is a Borel set in Ω_R. For surjectivity, suppose α has RCF-expansion $\alpha = [0; \alpha_1, \alpha_2, \ldots]$, and let $z = (x,y) \in H_1$ with $x = [0; a_1, a_2, \ldots] \notin \mathbb{Q}$ and $y = [0; 1, b, b, b, \ldots]$ for some $b > \alpha_1$. Then

$$\mathcal{F}_{H_1}^{-1}(z) = ([0; b, a_1, a_2, \ldots], [0; 1, b, b, \ldots]) \in [0,\alpha) \times [1/2, 1],$$

so $k(z) = 1$ is odd and $z \in A$. Similarly, $k(\mathcal{F}_{H_1}^{-n}(z)) = 1$ for all $n \geq 0$, so $\mathcal{F}_{H_1}^{-n}(z) \in A$ for all $n \geq 0$. This—together with Corollary 7.9—implies that $\mathcal{F}_R^n(z) \in R$ is defined for all $n \in \mathbb{Z}$. Since $x \in [0,1]\backslash\mathbb{Q}$ was arbitrary, (65) gives $\pi_X(\Omega_R) = \pi_X(\varphi_R(R)) = [\alpha-1,\alpha)\backslash\mathbb{Q}$, i.e., π_X is surjective.

Next, we show $\pi_X \circ \tau_R = G_\alpha \circ \pi_X$. Let $(X,Y) = (X(z), Y(z)) \in \Omega_R$, where $z = (x,y) \in R$, and notice from (65) that

$$X = \begin{cases} x, & x < \alpha, \\ x-1, & x \geq \alpha. \end{cases}$$

Moreover, $x < \alpha$ if and only if $X > 0$, and $x \geq \alpha$ if and only if $X < 0$. These observations, together with Theorem 6.16 and Lemma 7.10, give

$$\pi_X \circ \tau_R(X,Y) = \frac{\alpha_R(z)}{X} - \beta_R(z)$$

$$\begin{aligned}
&=\begin{cases} \frac{1}{X} - \left\lfloor \frac{1}{x} + 1 - \alpha \right\rfloor, & x < \alpha, \\ -\frac{1}{X} - \left\lfloor \frac{1}{1-x} + 1 - \alpha \right\rfloor, & x \geq \alpha, \end{cases} \\
&=\begin{cases} \frac{1}{X} - \left\lfloor \frac{1}{X} + 1 - \alpha \right\rfloor, & X > 0, \\ -\frac{1}{X} - \left\lfloor -\frac{1}{X} + 1 - \alpha \right\rfloor, & X < 0, \end{cases} \\
&=\frac{1}{|X|} - \left\lfloor \frac{1}{|X|} + 1 - \alpha \right\rfloor \\
&=G_\alpha \circ \pi_X(X,Y)
\end{aligned}$$

as desired. Lastly, notice that for any Borel set $A \in \mathcal{D}$, τ_R-invariance of $\bar{\nu}_R$ gives

$$\bar{\nu}_R \circ \pi_X^{-1}(G_\alpha^{-1}(A)) = \bar{\nu}_R \circ \tau_R^{-1}(\pi_X^{-1}(A)) = \bar{\nu}_R \circ \pi_X^{-1}(A),$$

so $\bar{\nu}_R \circ \pi_X^{-1}$ is an absolutely continuous, G_α-invariant probability measure. Uniqueness of ρ_α implies $\bar{\nu}_R \circ \pi_X^{-1} = \rho_\alpha$. Thus $([\alpha-1,\alpha), \mathcal{D}, \rho_\alpha, G_\alpha)$ is a factor of $(\Omega_R, \mathcal{C}, \bar{\nu}_R, \tau_R)$.

(ii) We now show that

$$\bigvee_{n=0}^{\infty} \tau_R^n \circ \pi_X^{-1}(\mathcal{D}) = \mathcal{C}.$$

The forward inclusion follows from measurability of π_X and τ_R^{-1}, so it suffices to show the backward inclusion. For this, it suffices to show that every element of a generating set of the Borel σ-algebra $\mathcal{C}$ on Ω_R can be written as $\tau_R^k \circ \pi_X^{-1}(D)$ for some $D \in \mathcal{D}$ and $k \geq 0$. By Proposition 6.19, $\mathcal{C}$ is generated by the sets

$$\begin{aligned}
C = &\Delta(0/1; \alpha_1/\beta_1, \alpha_2/\beta_2, \dots, \alpha_n/\beta_n) \\
&\times \Delta(0/1; 1/\beta_0, \alpha_0/\beta_{-1}, \alpha_{-1}/\beta_{-2}, \dots, \alpha_{-(m-1)}/\beta_{-m})
\end{aligned}$$

containing all points $(X(z), Y(z)) \in \Omega_R$ for which

$$\alpha_R(z_j^R) = \alpha_{j+1} \quad \text{and} \quad \beta_R(z_k^R) = \beta_{k+1}$$

for all $-m \leq j \leq n-1$ and $-m-1 \leq k \leq n-1$.
Let $D \in \mathcal{D}$ be the set of irrationals $X \in [\alpha-1, \alpha)$ for which

$$\operatorname{sgn}(G_\alpha^j(X)) = \alpha_{j-m} \quad \text{and} \quad \left\lfloor \frac{1}{|G_\alpha^k(X)|} + 1 - \alpha \right\rfloor = \beta_{k-m}$$

for all $1 \leq j \leq n+m$ and $0 \leq k \leq n+m$. Let $X \in [\alpha-1, \alpha) \setminus \mathbb{Q}$, $(X,Y) = (X(z), Y(z)) \in \pi_X^{-1}(\{X\})$, and $z_k^R = \mathcal{F}_R^k(z)$ for all $k \in \mathbb{Z}$. Using the fact that $G_\alpha \circ \pi_X = \pi_X \circ \tau_R$, Equations (52), (53) and Lemma 7.10 give

$$\operatorname{sgn}(G_\alpha^k(X)) = \alpha_R(z_k^R) \quad \text{and} \quad \left\lfloor \frac{1}{|G_\alpha^k(X)|} + 1 - \alpha \right\rfloor = \beta_R(z_k^R), \quad k \geq 0,$$

so $\pi_X^{-1}(D)$ is the set of points $(X(z), Y(z)) \in \Omega_R$ such that

$$\alpha_R(z_j^R) = \alpha_{j-m} \quad \text{and} \quad \beta_R(z_k^R) = \beta_{k-m}$$

for all $1 \leq j \leq n+m$ and $0 \leq k \leq n+m$. By Corollary 6.20, this is the set of points of the form

$$\begin{aligned} X = X(z) =& [0/1; \alpha_R(z_0^R)/\beta_R(z_0^R), \alpha_R(z_1^R)/\beta_R(z_1^R), \dots] \\ =& [0/1; \alpha_R(z_0^R)/\beta_{-m}, \alpha_{-(m-1)}/\beta_{-(m-1)}, \dots, \alpha_n/\beta_n, \\ & \alpha_R(z_{n+m+1}^R)/\beta_R(z_{n+m+1}^R), \dots] \end{aligned}$$

and

$$Y = Y(z) = [0/1; 1/\beta_R(z_{-1}^R), \alpha_R(z_{-1}^R)/\beta_R(z_{-2}^R), \dots].$$

Since $(X_{m+1}^R, Y_{m+1}^R) = \tau_R^{m+1}(X, Y)$ is of the form

$$X_{m+1}^R = [0/1; \alpha_1/\beta_1, \dots, \alpha_n/\beta_n, \alpha_R(z_{n+m+1}^R)/\beta_R(z_{n+m+1}^R), \dots]$$

and

$$Y_{m+1}^R = [0/1; 1/\beta_0, \alpha_0/\beta_{-1}, \dots, \alpha_{-(m-1)}/\beta_{-m}, \alpha_R(z_0^R)/\beta_R(z_{-1}^R), \dots],$$

we have $\tau_R^{m+1} \circ \pi_X^{-1}(D) = C$. $\square$

Remark 7.12. Recall from the end of §3.2 that there are several open questions about Nakada's α-CFs, including explicit descriptions of the values of the entropy $h(G_\alpha)$ for $\alpha < g^2$, $g = (\sqrt{5}-1)/2$, and of the densities of the invariant measures ρ_α ([24]). It is also open to explicitly compute the so-called *Legendre constant* for $\alpha < g^2$ ([15,35]).

Each of these questions may be answered with an understanding of the domain of the natural extension of $([\alpha-1, \alpha), \mathcal{B}, \rho_\alpha, G_\alpha)$; see, e.g., Theorem 5.6 for the entropy. To date, however, the description of this domain has proven to be unmanageable for these tasks. Our new description of the natural extension $(R, \mathcal{B}, \bar{\mu}_R, \mathcal{F}_R)$ could bring many of these questions within reach. Indeed, by (61), in order to understand R it suffices to understand the set $A \subset H_1$. We hope to return to these questions in subsequent work and suspect that matching (see §3.2 above) will play a crucial role in their resolution.

Data availability

No data was used for the research described in the article.

References

[1] C. Bonanno, C. Carminati, S. Isola, G. Tiozzo, Dynamics of continued fractions and kneading sequences of unimodal maps, Discrete Contin. Dyn. Syst. 33 (4) (2013) 1313–1332.
[2] W. Bosma, Optimal continued fractions, Ned. Akad. Wet. Indag. Math. 49 (4) (1987) 353–379.
[3] W. Bosma, Approximation by mediants, Math. Comput. 54 (189) (1990) 421–434.
[4] J. Bourdon, B. Daireaux, B. Vallée, Dynamical analysis of α-Euclidean algorithms, J. Algorithms 44 (1) (2002) 246–285.
[5] D. Bowman, Fibonacci contractions of continued fractions, Fibonacci Q. 52 (3) (2014) 206–214.
[6] G. Brown, Q. Yin, Metrical theory for Farey continued fractions, Osaka J. Math. 33 (4) (1996) 951–970.
[7] C. Carminati, S. Marmi, A. Profeti, G. Tiozzo, The entropy of α-continued fractions: numerical results, Nonlinearity 23 (10) (2010) 2429–2456.
[8] C. Carminati, G. Tiozzo, A canonical thickening of $\mathbb{Q}$ and the entropy of α-continued fraction transformations, Ergod. Theory Dyn. Syst. 32 (4) (2012) 1249–1269.
[9] C. Carminati, G. Tiozzo, Tuning and plateaux for the entropy of α-continued fractions, Nonlinearity 26 (4) (2013) 1049–1070.
[10] K. Dajani, C. Kalle, Invariant measures, matching and the frequency of 0 for signed binary expansions, Publ. Res. Inst. Math. Sci. 56 (4) (2020) 701–742.
[11] K. Dajani, C. Kalle, A First Course in Ergodic Theory, CRC Press, Boca Raton, FL, 2021.
[12] K. Dajani, C. Kraaikamp, The mother of all continued fractions, Colloq. Math. 84/85 (part 1) (2000) 109–123, Dedicated to the memory of Anzelm Iwanik.
[13] K. Dajani, C. Kraaikamp, S. Sanderson, A unifying theory for metrical results on regular continued fraction convergents and mediants, Math. Comp., Electronically published on January 22, 2025, https://doi.org/10.1090/mcom/4046, in press.
[14] H.E. Daniels, Processes generating permutation expansions, Biometrika 49 (1962) 139–149.
[15] J. de Jonge, C. Kraaikamp, Natural extensions for Nakada's α-expansions: descending from 1 to g^2, J. Number Theory 183 (2018) 172–212.
[16] A. Hurwitz, Über eine besondere Art der Kettenbruch-Entwicklung reeller Grössen, Acta Math. 12 (1) (1889) 367–405.
[17] M. Iosifescu, C. Kraaikamp, Metrical Theory of Continued Fractions, Math. Appl., vol. 547, Kluwer Academic Publishers, Dordrecht, 2002.
[18] M. Iosifescu, G.I. Sebe, On the metrical theory of a peculiar continued fraction expansion, Rev. Roum. Math. Pures Appl. 53 (5–6) (2008) 465–477.
[19] S. Ito, Algorithms with mediant convergents and their metrical theory, Osaka J. Math. 26 (3) (1989) 557–578.
[20] H. Jager, Some metrical observations on the approximation of an irrational number by its nearest mediants, Period. Math. Hung. 23 (1) (1991) 5–16.
[21] S. Kakutani, Induced measure preserving transformations, Proc. Imp. Acad. (Tokyo) 19 (1943) 635–641.
[22] A.Y. Khinchin, Continued Fractions, Dover Publications, Inc., Mineola, NY, 1997, With a preface by B.V. Gnedenko, Translated from the third (1961) Russian edition, Reprint of the 1964 translation.
[23] C. Kraaikamp, A new class of continued fraction expansions, Acta Arith. 57 (1) (1991) 1–39.
[24] C. Kraaikamp, T.A. Schmidt, W. Steiner, Natural extensions and entropy of α-continued fractions, Nonlinearity 25 (8) (2012) 2207–2243.
[25] J. Lagrange, Additions aux éléments d'algèbre d'Euler, Paris, Oeuvres VII (1798).
[26] J. Lehner, Semiregular continued fractions whose partial denominators are 1 or 2, in: The Mathematical Legacy of Wilhelm Magnus: Groups, Geometry and Special Functions, Brooklyn, NY, 1992, in: Contemp. Math., vol. 169, Amer. Math. Soc., Providence, RI, 1994, pp. 407–410.
[27] L. Lorentzen, H. Waadeland, Continued Fractions, vol. 1. Convergence Theory, second ed., Atlantis Studies in Mathematics for Engineering and Science, vol. 1, Atlantis Press, Publishing Co. Pte. Ltd., Paris, Hackensack, NJ, 2008.
[28] L. Luzzi, S. Marmi, On the entropy of Japanese continued fractions, Discrete Contin. Dyn. Syst. 20 (3) (2008) 673–711.
[29] H. Minkowski, Ueber die Annäherung an eine reelle Grösse durch rationale Zahlen, Math. Ann. 54 (1–2) (1900) 91–124.
[30] B. Minnigerode, Ueber eine neue methode, die pell'sche gleichung aufzulösen, Nachr. Gött. 23 (1873) 619–652.

[31] P. Moussa, A. Cassa, S. Marmi, Continued fractions and Brjuno functions, in: Continued Fractions and Geometric Function Theory (CONFUN), Trondheim, 1997 105 (1999) 403–415.
[32] H. Nakada, Metrical theory for a class of continued fraction transformations and their natural extensions, Tokyo J. Math. 4 (2) (1981) 399–426.
[33] H. Nakada, S. Ito, S. Tanaka, On the invariant measure for the transformations associated with some real continued-fractions, Keio Eng. Rep. 30 (13) (1977) 159–175.
[34] H. Nakada, R. Natsui, The non-monotonicity of the entropy of α-continued fraction transformations, Nonlinearity 21 (6) (2008) 1207–1225.
[35] R. Natsui, On the Legendre constant of α-continued fractions, J. Number Theory 131 (2011) 487–507.
[36] W. Parry, Ergodic properties of some permutation processes, Biometrika 49 (1962) 151–154.
[37] O. Perron, Die Lehre von den Kettenbrüchen, 2d ed., Chelsea Publishing Co., New York, 1950.
[38] A. Rényi, Valós számok elöállítására szolgáló algoritmusokról, Magy. Tud. Akad. Mat. Fiz. Oszt. Közl. 7 (1957) 265–293.
[39] S. Sanderson, Dynamics of number expansions and translation surfaces, Utrecht University, , Utrecht, 2025, https://doi.org/10.33540/2993.
[40] L. Seidel, Bemerkungen über den Zusammenhang zwischen dem Bildungsgesetze eines Kettenbruches und der Art des Fortgangs seiner Näherungsbrüche, Abh. der Kgl. Bayr. Akad. der Wiss., München, Zweite Klasse 7 (3) (1855) 559–602.
[41] G. Tiozzo, The entropy of α-continued fractions: analytical results, arXiv:0912.2379v1, 2009.
[42] G. Tiozzo, The entropy of Nakada's α-continued fractions: analytical results, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 13 (4) (2014) 1009–1037.