ON THE THEORY OF 3-PHASE SQUIRREL-CAGE INDUCTION MOTORS INCLUDING SPACE HARMONICS AND MUTUAL SLOTTING

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ABSTRACT - In this paper general equations for the asynchronous squirrel-cage motor which contain the influence of space harmonics and the mutual slotting are derived by using among others the power-invariant symmetrical component transformation and a time-dependent transformation with which, under certain circumstances, the rotor-position angle can be removed from the coefficient matrix. The developed models implemented in a machine-independent computer program form powerful tools, with which the influence of space harmonics in relation to the geometric data of specific motors can be analyzed for steady-state and transient performances. Simulations and measurements are presented in a companion paper.

Keywords - asynchronous machines, general theory, space harmonics, mutual slotting, transients.

1. INTRODUCTION

In general the most difficult problems in the theory and modeling of induction motors are saturation of the magnetic circuit and all "parasitic effects" caused by higher harmonics in the magnetic field in the air gap. Both phenomena become even more complicated if the influence of the slotting of stator and rotor surfaces are considered.

Gradually, mathematical models of induction machines have been worked out which take more details of the real geometrical construction into account and enable increasingly closer insight into their influence. At first models were developed which included space harmonics but which ignored those combinations of harmonics which could cause the multiple armature reaction [1,2,3]. These models include only the asynchronous torques and do not include the synchronous and pulsating torques. In [5] the general equations for the squirrel-cage induction motor are derived by means of harmonic analysis. Although these equations describe the dynamic behavior, for numerical calculations, a relatively large computation time is required because of the problem of the rotor position angle. In [6] this model is extended by the addition of the influence of stator slotting. In [7] and [8] dynamic induction motor models have been developed where the squirrel cage and the 3-phase stator winding are represented by equivalent polyphase windings. In the model presented in [7] it is possible to simplify the set of equations when no more than two harmonics per phase group are taken into account, but it is not general for all squirrel-cage motors. In [9] a transformation has been developed which simplifies the general set of equations and under certain circumstances transforms the set of equations in such a way that the ever-present influence of the rotor position angle, when considering the multiple armature reaction, can be removed from the parameters and only appears in the source voltage. This enables a particular solution of the differential equations. In [9], however, a smooth air gap which could have a significant influence on the results was assumed. Further, the zero-sequence component was not included in the equations.

In this paper the model presented in [9] is extended by the addition of the mutual slotting and the zero-sequence component. The influence of saturation due to the main field can, as a linear magnetic circuit is presupposed, be taken into account by making an increase in the air gap. The influence of the saturation due to the slot-leakage fluxes can be dealt with by additional widening of the slot openings.

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2. MATHEMATICAL MODEL

After applying the group transformation to the rotor equations and the symmetrical component transformation to both the stator and the rotor equations, the following set of matrix equations arises [6,10]:

$$U'_s = R'_s\, I'_s + d/dt(\, L'_{ss}\, I'_s + L'_{sr}\, I'_r\,)$$
$$0 = R'_r\, I'_r + d/dt(\, L'_{rs}\, I'_s + L'_{rr}\, I'_r\,) \quad (1)$$

$$T_e = \frac{1}{2}\left(I'^{*T}_r \frac{\partial L'_{rs}}{\partial\theta} I'_s + I'^{*T}_s \frac{\partial L'_{sr}}{\partial\theta} I'_r + I'^{*T}_r \frac{\partial \Delta L'_{rr}}{\partial\theta} I'_r \right) \quad (2)$$

where θ is the rotor-position angle. A list of symbols is provided in section 6.

The elements of the voltage and current matrices are:

$$U'_s = [u_s^{\,0}, u_s^{\,+}, u_s^{\,-}]^T \;;\quad I'_s = [i_s^{\,0}, i_s^{\,+}, i_s^{\,-}]^T$$

$$I'_r = [i_{r0}, i'_{r1}, \ldots, i_{rK}, \ldots, i'^{\,*}_{r1}]^T \text{ , subscript } K=N/2z$$

where N is the number of rotor slots and z the highest common factor of N and the number of pole pairs p.

The influence of the rotor slotting on the self-inductance of the stator is only taken into account by the Carter factor [10] and therefore no ΔL_{ss}-matrix appears in expression (2). In the rotor self-inductance and the mutual inductance the slotting is taken into account completely.

The transformed stator parameters are:

$$R'_s = R_s = \text{diag}[R_s, R_s, R_s] \;;\; L'_{ss} = \text{diag}[\tilde{L}_{s0}, \tilde{L}_s, \tilde{L}_s]$$

where the zero-sequence inductance $\tilde{L}_{s0}$ contains the harmonics $\nu=3,9,15,\ldots$ and the positive sequence inductance $\tilde{L}_s$ contains the harmonics $\nu=1,-5,7,-11,\ldots$

For the rotor the transformed parameters are:

$$R'_r = \text{diag}[\tilde{R}_{r0}, \tilde{R}_{r1}, \ldots, \tilde{R}_{rK}, \ldots \tilde{R}_{r1}],$$

$$L'_{rr} = \text{diag}[\tilde{L}_{r0}, \tilde{L}_{r1}, \ldots, \tilde{L}_{rK}, \ldots, \tilde{L}_{r1}] + \Delta L'_{rr}$$

where $\Delta L'_{rr} = \sum_n \Delta\tilde{L}_{rn}(e^{jnN\theta}\, Q + e^{-jnN\theta}\, Q^{*T})$; $n=1,2,3,\ldots$

After these transformations the complex mutual inductance matrix becomes as given in equation (3) where $g=0,\pm1,\pm2,\ldots$ $\nu = 1,-5,7,-11,13,\ldots$ $\mu'=\pm\mu$, $\mu=3,9,15,\ldots$ In each element of (3) only those harmonics appear that fulfill all the constraints that are given in this matrix. In [6,10] a comprehensive determination of the

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separate parameters is provided. Appendix I gives a short overview of the resultant parameters.

$$L'_{sr}=\begin{bmatrix} \sum\limits_{\mu'=gN/z} m_{\mu'} & \sum\limits_{\mu'=gN/z+1} m_{\mu'} & \cdots & \sum\limits_{\mu'=gN/z+(N/z-1)} m_{\mu'} \\ \sum\limits_{\nu=gN/z} m_{\nu} & \sum\limits_{\nu=gN/z+1} m_{\nu} & \cdots & \sum\limits_{\nu=[gN/z+(N/z-1)]} m_{\nu} \\ \sum\limits_{\nu=-gN/z} m^*_{\nu} & \sum\limits_{\nu=-(gN/z+1)} m^*_{\nu} & \cdots & \sum\limits_{\nu=-[gN/z+(N/z-1)]} m^*_{\nu} \end{bmatrix} \qquad (3)$$

The structure of the L'_{sr}-matrix

The behavior of the squirrel-cage induction motor is strongly determined by the magnetic coupling between stator and rotor [9]. The presence of a harmonic in an element of the mutual inductance matrix is directly connected to the number of rotor bars N and by z. On the grounds of the present symmetry it is possible to write the L'_{sr}-matrix as follows:

$$L'_{sr}=\begin{bmatrix} M_{10} & M_{11}\cdots M_{1(K-1)} & M_{1K} & M^*_{1(K-1)}\cdots & M^*_{11} \\ M_{20} & M_{21}\cdots M_{2(K-1)} & M_{2K} & M_{3(K-1)}\cdots & M_{31} \\ M^*_{20} & M^*_{31}\cdots M^*_{3(K-1)} & M^*_{2K} & M^*_{2(K-1)}\cdots & M^*_{21} \end{bmatrix} \qquad (4)$$

where

$M_{1i}=2\hat{M}_{1i}\cos\epsilon_i$ (i=0 or K) ; $M_{1i}=\hat{M}_{1i}e^{j\epsilon_i}$ (i=1,....K-1)

$M_{2i}=\hat{M}_{2i}e^{j\alpha_i}$ (i=0,...K) ; $M_{3i}=\hat{M}_{3i}e^{j\beta_i}$ (i=1,....K-1)

and in general $\hat{M}_{ki}e^{j\eta}=\sum\limits_{\nu} m_\nu=\sum\limits_{\nu}^{n}\hat{m}_\nu e^{j\nu p\theta}$,k=1,2,3 (5)

Time-dependent transformations

The rotor and stator variables are transformed with complex, power-invariant, time-dependent transformations, as presented in [9]. These transformations change the L'_{sr}-matrix in such a way that the mutual connections between the positive and negative sequence components of the stator currents disappear. The transformation matrices for the rotor voltage and current are:

$$A=\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & a_1 & \cdot & \cdot & \cdot & b_1 \\ \cdot & \cdot & a_2 & \cdot & b_2 & \cdot \\ & & & \ddots & & \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ & & & \ddots & & \\ \cdot & \cdot & b^*_2 & \cdot & a^*_2 & \cdot \\ 0 & b^*_1 & \cdot & \cdot & \cdot & a^*_1 \end{bmatrix} ;\; B=\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & c_1 & \cdot & \cdot & \cdot & d_1 \\ \cdot & \cdot & c_2 & \cdot & d_2 & \cdot \\ & & & \ddots & & \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ & & & \ddots & & \\ \cdot & \cdot & d^*_2 & \cdot & c^*_2 & \cdot \\ 0 & d^*_1 & \cdot & \cdot & \cdot & c^*_1 \end{bmatrix}$$

which satisfy the condition: $A^T.B^*=U$ or $A^{-1}=B^{*T}$,where U is the unity matrix, and where

$a_i=D_i\hat{M}_{2i}/E_i^2\, e^{j(\gamma_i-\alpha_i)}$ $\quad b_i=D_i\hat{M}_{3i}/E_i^2\, e^{-j(\gamma_i-\beta_i)}$

$c_i=\hat{M}_{2i}/D_i\, e^{j(\gamma_i-\alpha_i)}$ $\quad d_i=-\hat{M}_{3i}/D_i\, e^{-j(\gamma_i-\beta_i)}$

$D_i=\surd(\hat{M}^2_{2i}+\hat{M}^2_{3i})$ $\quad E_i=\surd(\hat{M}^2_{2i}-\hat{M}^2_{3i})$ if $(\hat{M}_{2i}\neq\hat{M}_{3i})$

for i = 1,2,..,K-1.

If $\hat{M}_{2i}=\hat{M}_{3i}=0$ then $a_i=c_i=1$, $b_i=d_i=0$.

The relations between the old and the new variables are given by

$$U'_r=A.U''_r\ ;\ I'_r=B.I''_r \qquad (6)$$

Because of the structure of the transformation matrix B the transformed currents can be written as follows:

$$I''_r=[\,i_{r0},i''_{r1},i''_{r2},\ldots,i_{rK},\ldots,i''^*_{r2},i''^*_{r1}]$$

The equations are further developed by transforming the stator variables, using the following relations between the old and new quantities:

$$U'_s=C\,U''_s \text{ and } I'_s=C\,I''_s \text{ where } C=\mathrm{diag}[\,1,\ e^{j\gamma},\ e^{-j\gamma}] \qquad (7)$$

The new stator variables are defined as:

$U''_s=[\,u_{s0},u_{sf},u_{sb}]^T$ and $I''_s=[\,i_{s0},i_{sf},i_{sb}]^T$

where $u_{sb}=u^*_{sf}$ and $i_{sb}=i^*_{sf}$.

The introduced angles γ, and $\gamma_1,\ldots\ldots,\gamma_{K-1}$ will later be determined in relation to specific motor data.

The transformation of equations (1) and (2) using relations (6) and (7) yields:

$$U''_s=R''_sI''_s+C^{-1}\frac{dC}{dt}(L''_{ss}I''_s+L''_{sr}I''_r)+\frac{d}{dt}(L''_{ss}I''_s+L''_{sr}I''_r) \qquad (8)$$

$$0=R''_rI''_r+A^{-1}\frac{dA}{dt}(L''_{rs}I''_s+L''_{rr}I''_r)+\frac{d}{dt}(L''_{rs}I''_s+L''_{rr}I''_r)$$

$$T_e=\mathrm{Re}[I''^{*T}_r\frac{\partial L''_{rs}}{\partial\theta}I''_s+I''^{*T}_rA^{-1}\frac{\partial A}{\partial\theta}L''_{rs}I''_s+I''^{*T}_rL''_{rs}\frac{\partial C^{-1}}{\partial\theta}CI''_s$$
$$+I''^{*T}_rA^{-1}\frac{\partial A}{\partial\theta}\Delta L''_{rr}I''_r+\frac{1}{2}I''^{*T}_r\frac{\partial\Delta L''_{rr}}{\partial\theta}I''_r] \qquad (9)$$

where

$L''_{ss}=C^{-1}L'_{ss}C=L'_{ss}$, $R''_s=C^{-1}R'_sC=R'_s$, $L''_{sr}=C^{-1}L'_{sr}B=L''^{*T}_{rs}$

and $L''_{rr}=A\,L'_{rr}B$ and $R''_{rr}=A\,R'_{rr}B$.

In the derivation of the electromagnetic torque equation the fact that $\Delta L''_{rr}=\Delta L''^{*T}_{rr}$ was used.

The mutual inductance matrix becomes:

$$L''_{sr}=\begin{bmatrix} M_{10} & M_{01}\cdots M_{0(K-1)} & M_{1K} & M^*_{0(K-1)}\cdots & M^*_{01} \\ M_{20} & M_1\ \cdots M_{(K-1)} & M_{2K} & 0\ \ \cdots & 0 \\ M^*_{20} & 0\ \ \cdots 0 & M^*_{2K} & M^*_{(K-1)}\cdots & M^*_1 \end{bmatrix} \qquad (10)$$

where $M_{1i}=2\hat{M}_{1i}\cos\epsilon_i$ and $M_{2i}=\hat{M}_{2i}e^{j(\alpha_i-\gamma)}$ for i=0 or K

$M_i=\hat{M}_ie^{j(\gamma_i-\gamma)}$ for i=1,....K-1

$\hat{M}_i=E^2/D_i$ (if $\hat{M}_{2i}=\hat{M}_{3i}=0$ then $\hat{M}_i=0$).

$M_{0i}=\frac{1}{D_i}[\hat{M}_{1i}\hat{M}_{2i}e^{j(\gamma_i-\alpha_i+\epsilon_i)}-\hat{M}_{1i}\hat{M}_{3i}e^{j(\gamma_i-\beta_i-\epsilon_i)}]$

for i=1,....K-1

if $\hat{M}_{2i}=\hat{M}_{3i}=0$, $M_{0i}=\hat{M}_{1i}e^{j\epsilon_i}$

In the second and third row, the mutual inductance matrix now contains inductances which consist of a real part $\hat{M}$ and a complex exponential function. $\hat{M}$ depends, in general, on the rotor position angle θ . The angles γ_i and γ in the complex functions are still to be determined. The free angles γ_i also appear in the zero sequence components in the first row.

The new rotor matrices become:

$$L''_{rr}=\begin{bmatrix}\tilde{L}_{r0} & \cdot & \cdot & 0\\ \cdot & \tilde{L}_{r1} & \cdot & k_i\tilde{L}_{r1}\\ & \cdot & \cdot & \cdot\\ \cdot & \cdots & \tilde{L}_{rK} & \cdots\\ & \cdot & & \cdot\\ 0 & k_i^*\tilde{L}_{r1} & & \tilde{L}_{r1}\end{bmatrix}+\Delta L''_{rr};\quad R''_{r}=\begin{bmatrix}\tilde{R}_{r0} & \cdot & \cdot & 0\\ \cdot & \tilde{R}_{r1} & \cdot & k_i\tilde{R}_{r1}\\ & \cdot & \cdot & \cdot\\ \cdot & \cdots & \tilde{R}_{rK} & \cdots\\ & \cdot & & \cdot\\ 0 & k_i^*\tilde{R}_{r1} & & \tilde{R}_{r1}\end{bmatrix} \quad (11)$$

where

$$k_i=\frac{-2\hat{M}_{2i}\hat{M}_{3i}}{D_i^2}e^{j(\alpha_i+\beta_i-2\gamma_i)},\ \text{if } \hat{M}_{2i}=\hat{M}_{3i}=0:\ k_i=0 \quad (12)$$

Matrix $\Delta L''_{rr}$ can be written as: $\Delta L''_{rr}(i,k)=\sum_n \Delta\tilde{L}_{rn}\Delta_n(i,k)$

where

$$\Delta_n(i,k)=(c_kc_i^*e^{j\kappa}+d_k^*d_ie^{-j\kappa})q_{ik}+(c_kc_i^*e^{-j\kappa}+d_k^*d_ie^{j\kappa})q_{ki}+$$
$$+(c_kd_ie^{j\kappa}+d_k^*c_i^*e^{-j\kappa})q_{N/z-k,i}+(c_kd_ie^{-j\kappa}+d_k^*c_i^*e^{j\kappa})q_{i,N/z-k}$$

$\kappa=nN_s\theta$.

For $\Delta L''_{rr}$ it holds that:

$\Delta L''_{rr}(i,k)=\Delta L''^*_{rr}(k,i)$ and $\Delta L''_{rr}(i,k)=\Delta L''_{rr}(N/z-k,N/z-i)$

The factors $A^{-1}\frac{dA}{dt}$ and $A^{-1}\frac{\partial A}{\partial\theta}$ in equations (8) and (9) can be written as:

$$A^{-1}\frac{dA}{dt}=\begin{bmatrix}0 & \cdot & \cdot & \cdot & \cdot & 0\\ \cdot & e_1 & \cdot & \cdot & \cdot & f_1\\ \cdot & \cdot & e_2 & \cdot & f_2 & \cdot\\ \cdot & \cdot & \cdots & 0 & \cdots & \cdot\\ \cdot & \cdot & f_2^* & \cdot & e_2^* & \cdot\\ 0 & f_1^* & \cdot & \cdot & \cdot & e_1^*\end{bmatrix};\quad A^{-1}\frac{\partial A}{\partial\theta}=\begin{bmatrix}0 & \cdot & \cdot & \cdot & \cdot & 0\\ \cdot & g_1 & \cdot & \cdot & \cdot & h_1\\ \cdot & \cdot & g_2 & \cdot & h_2 & \cdot\\ \cdot & \cdot & \cdots & 0 & \cdots & \cdot\\ \cdot & \cdot & h_2^* & \cdot & g_2^* & \cdot\\ 0 & h_1^* & \cdot & \cdot & \cdot & g_1^*\end{bmatrix}$$

where

$$g_i=j\left[\frac{\partial\gamma_i}{\partial\theta}-\frac{\frac{\partial\alpha_i}{\partial\theta}\hat{M}_{2i}^2-\frac{\partial\beta_i}{\partial\theta}\hat{M}_{3i}^2}{E_i^2}\right]+\left[\frac{1}{D_i}\frac{\partial D_i}{\partial\theta}-\frac{1}{E_i}\frac{\partial E_i}{\partial\theta}\right],$$

$$e_i=j\left[\frac{d\gamma_i}{dt}-\frac{\frac{d\alpha_i}{dt}\hat{M}_{2i}^2-\frac{d\beta_i}{dt}\hat{M}_{3i}^2}{E_i^2}\right]+\left[\frac{1}{D_i}\frac{dD_i}{dt}-\frac{1}{E_i}\frac{dE_i}{dt}\right], \quad (13)$$

$$h_i=\left[-j\left(\frac{\partial\alpha_i}{\partial\theta}-\frac{\partial\beta_i}{\partial\theta}\right)\frac{\hat{M}_{2i}\hat{M}_{3i}}{E_i^2}+\frac{1}{E_i^2}\left(\hat{M}_{2i}\frac{\partial\hat{M}_{3i}}{\partial\theta}-\hat{M}_{3i}\frac{\partial\hat{M}_{2i}}{\partial\theta}\right)\right]e^{j\theta_i}$$

$$f_i=\left[-j\left(\frac{d\alpha_i}{dt}-\frac{d\beta_i}{dt}\right)\frac{\hat{M}_{2i}\hat{M}_{3i}}{E_i^2}+\frac{1}{E_i^2}\left(\hat{M}_{2i}\frac{d\hat{M}_{3i}}{dt}-\hat{M}_{3i}\frac{d\hat{M}_{2i}}{dt}\right)\right]e^{j\theta_i}$$

$\theta_i=\alpha_i+\beta_i-2\gamma_i$.

If $\hat{M}_{2i}=\hat{M}_{3i}=0$ then $e_i=f_i=g_i=h_i=0$.

The matrices $C^{-1}\frac{dC}{dt}$ and $\frac{\partial C^{-1}}{\partial\theta}C$ are:

$C^{-1}\frac{dC}{dt}=j\frac{d\gamma}{dt}\text{diag}[0,1,-1]$ and $\frac{\partial C^{-1}}{\partial\theta}C=-j\frac{\partial\gamma}{\partial\theta}\text{diag}[0,1,-1]$

As follows from equations (10) and (11), mutual connections between positive and negative sequence stator currents are replaced by additional mutual connections between positive and negative sequence rotor currents.

From the general equations in matrix form it appears that in general:

L''_{sr}, L''_{rr}, $\Delta L''_{rr}$ and R''_{rr} depend on θ.

L''_{rr} and R''_{rr} are functions of $\hat{M}_{2i}.\hat{M}_{3i}.e^{j(\alpha_i+\beta_i-2\gamma_i)}$.

$A^{-1}dA/dt$ depends on $\hat{M}_{2i}.\hat{M}_{3i}.e^{j(\alpha_i+\beta_i-2\gamma_i)}$, $d\gamma_i/dt$, $d\alpha_i/dt$ and $d\beta_i/dt$.

$A^{-1}\partial A/\partial\theta$ depends on $\hat{M}_{2i}.\hat{M}_{3i}.e^{j(\alpha_i+\beta_i-2\gamma_i)}$, $\partial\gamma_i/\partial\theta$, $\partial\alpha_i/\partial r$ and $\partial\beta_i/\partial\theta$.

$\Delta L''_{rr}$ contains exponential functions with $N_s\theta$, α_i, α_k, β_i, β_k, γ_i and γ_k.

$C^{-1}dC/dt$ and $C^{-1}\partial C/\partial\theta$ depend on $d\gamma/dt$ and $\partial\gamma/\partial\theta$ respectively.

In order to see the advantages of these transformations the structures of the mutual inductance matrix and the rotor self-inductance matrix have to be examined in order to develop simpler equations. In [9] it is shown that 3-phase induction motors can be divided into two types, each with a specific kind of multiple armature reaction. Both types can be recognized by the structure of their mutual-inductance matrix. This also holds when the mutual slotting is taken into account. In Table 1. representatives of these two types are given.

3. THE FIRST TYPE OF ARMATURE REACTION

The first type of armature reaction occurs in 3-phase induction motors where $N/z=k.3$ and this can be seen in the structure of the L'_{sr}-matrix.

An example of this type is given in Table 1.a from which the following properties can be derived:

either $M_{2i}=0$ or $M_{3i}=0$ or $M_{2i}=M_{3i}=0$ and $M_{20}=M_{2K}=0$.

From this it follows that $A=B$ and that A itself becomes unitary. The transformation is then only an angle transformation and does not influence the amplitudes. The elements in the matrices containing $M_{2i}M_{3i}$ disappear. Further, it is easy to see in equations (12) and (13) that for this type $k_i=f_i=h_i=0$. This means that all exponential terms in the general equations (8) and (9) disappear, except in the mutual inductances M_i in expression (10). The exponential functions in the second and third row of the mutual inductance matrix can now be removed by introducing the following constraint:

$$\gamma_i=\gamma.$$

column number

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
m_{15} m^*_{15}			m_3			m^*_9			m_9			m^*_3		
	m_1 m_{31} m_{-29}			m_{19} m_{-11}			m_7 m_{-23}			m_{25} m_{-5}			m_{13} m_{-17}	
		m^*_{-17} m^*_{13}			m^*_{-5} m^*_{25}			m^*_{-23} m^*_7			m^*_{-11} m^*_{19}			m^*_{-29} m^*_1 m^*_{31}

a) First type: N = 30, z = 2, N/z = 15

0	1	2	3	4	5	6	7	8	9	10	11	12	13
	m_{15} m^*_{27}		m_3 m^*_{39}		m_{33} m^*_9		m_{21} m^*_{21}		m_9 m^*_{33}		m_{39} m^*_3		m_{27} m^*_{15}
	m_1		m_{31} m_{-11}		m_{19} m_{-23}		m_7 m_{-35}		m_{37} m_{-5}		m_{25} m_{-17}		m_{13} m_{-29}
	m^*_{-29} m^*_{13}		m^*_{-17} m^*_{25}		m^*_{-5} m^*_{37}		m^*_{-35} m^*_7		m^*_{-23} m^*_{19}		m^*_{-11} m^*_{31}		m^*_1

b) Second type: N = 28, z = 2, N/z = 14

0	1	2	3	4	5	6	7	8	9	10
m_{33} m^*_{33}	m_{45} m^*_{21}	m^*_9	m_3	m_{15} m^*_{51}	m_{27} m^*_{39}	m_{39} m^*_{27}	m_{51} m^*_{15}	m^*_3	m_9	m_{21} m^*_{45}
m_{-11} m_{55}	m_1	m_{13}	m_{25} m_{-41}	m_{37} m_{-29}	m_{49} m_{-17}	m_{-5}	m_7	m_{19} m_{-47}	m_{31} m_{-35}	m_{43} m_{-23}
m^*_{-11} m^*_{55}	m^*_{-23} m^*_{43}	m^*_{31} m^*_{-35}	m^*_{19} m^*_{-47}	m^*_7	m^*_{-5}	m^*_{49} m^*_{-17}	m^*_{37} m^*_{-29}	m^*_{25} m^*_{-41}	m^*_{13}	m^*_1

c) Second type: N = 22, z = 2, N/z = 11

Table 1. Distribution of harmonics in the L'_{sr}-matrix.

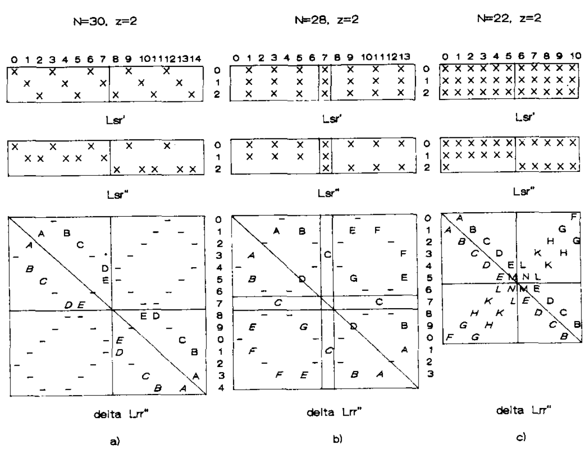

Table 2. The structure of L'_{sr}, L''_{sr} and $\Delta L''_{rr}$ for both types of armature reaction. The italic symbols mean complex conjugate values.

From Table 2.a, which gives a general impression of the structure of the L'_{sr}-, L''_{sr}- and $\Delta L''_{rr}$-matrices for the first type of armature reaction, it can be concluded that for this type the sequence components of the stator are linked neither by the L''_{sr}- nor by the $\Delta L''_{rr}$-matrix. This means that the zero-sequence component does not have any influence on the electro-mechanical behavior of the motor and therefore yields the conclusion:

> Motors with the first type of armature reaction do not display any difference in electro-mechanical behavior for a star-connected or delta-connected stator winding.

As the positive-sequence and negative-sequence rotor currents are not linked, the final equations only contain the forward-sequence component of the stator variables and the positive-sequence components of the rotor variables. This leads to the following voltage equations:

$$u_{sf} = R_s i_{sf} + j\frac{d\gamma}{dt}\Psi_{sf} + \frac{d}{dt}\Psi_{sf}$$

$$0 = \tilde{R}_{rk} i''_{rk} + e_k \Psi''_{rk} + \frac{d}{dt}\Psi''_{rk} \qquad ,k = 1,2,\dots K-1$$

the flux-current relations:

$$\Psi_{sf} = \tilde{L}_s i_{sf} + \sum_{i=1}^{K-1} \hat{M}_i i''_{ri}$$

$$\Psi''_{rk} = \hat{M}_k i_{sf} + \tilde{L}_{rk} i''_{rk} + \sum_{i=1}^{K-1} \Delta L''_{rr}(k,i) i''_{ri}$$

and the electromagnetic torque equation:

$$T_e = 2\,\mathrm{Re}\left[\sum_{i=1}^{K-1}[i''^*_{ri}(\frac{\partial \hat{M}_i}{\partial\theta} + g_i \hat{M}_i) i_{sf}] + \sum_{i=1}^{K-1}\sum_{k=1}^{K-1}[\frac{1}{2}\frac{\partial \Delta L''_{rr}(i,k)}{\partial\theta} + g_i \Delta L''_{rr}(i,k)] i''_{rk} i''^*_{ri}\right]$$

where

$\hat{M}_i = \hat{M}_{2i}$, $e_i = j(\omega - d\alpha_i/dt)$, $g_i = -j\,\partial\alpha_i/\partial\theta$ when $\hat{M}_{3i} = 0$

$\hat{M}_i = -\hat{M}_{3i}$, $e_i = j(\omega - d\beta_i/dt)$, $g_i = -j\,\partial\beta_i/\partial\theta$ when $\hat{M}_{2i} = 0$

For this type it holds that no equation for a 0-column or a middle-column K, as far as it exists, can be present. As the maximum number of voltage equations can be counted:
For the stator: 1
For the rotor : K-1, where K = N/2z when N/z is even
K = N/2z+1/2 when N/z is odd.

A further reduction in the number of equations is achieved when the mutual inductance matrix contains columns which are not filled with inductances. These potential equations yield trivial solutions and can therefore be omitted. If the motor is connected to a symmetrical power source then the transformed voltage becomes

for the star connection: $u_{sf} = \frac{1}{\sqrt{2}} U_N e^{j(\omega t + \phi' - \gamma)}$

and

for the delta connection: $u_{sf} = \frac{1}{2}\sqrt{6}\, U_N e^{j(\omega t + \phi - \gamma)}$

where $\phi' = \phi - \pi/6$ and U_N is the terminal voltage (rms).
Upon taking the remaining free angle

$$\gamma = \omega t + \phi' \qquad (\text{or } \omega t + \phi) ,$$

$\frac{d\gamma}{dt} = \omega$ and u_{sf} becomes a DC voltage.
This possibility to transform u_{sf} into a DC voltage here enables the formulation of the conclusions [9]:

* The starting characteristics of this type of motor do not depend on the moment of switching, but only on the rotor position angle at s=1 as $\hat{M}_i$ and $\Delta L''_{rr}(i,k)$ still remain functions of θ.

* This type has only one synchronous torque which appears at s=1. When the rotor moves, only pulsating torques exist.

4. THE SECOND TYPE OF ARMATURE REACTION

Three-phase squirrel-cage motors for which N/z is not a multiple of three display the multiple-armature reaction of the second type. This type possesses the following characteristic geometrical properties (see figures 1.b and c, 2.b and c):
In the mutual inductance matrix both M_{2i} and M_{3i} can have values not equal to zero. Further, M_{20} and M_{2K} never occur together. From Table 2.b and c, which gives a general impression of the structures of the L'_{sr}-, L''_{sr}- and $\Delta L''_{rr}$-matrices for the second type of armature reaction, it can be concluded that for this type the sequence components of the stator and the rotor are linked by the L''_{sr}- and by the $\Delta L''_{rr}$-matrix respectively.
Therefore, for the second type, the equations in their most general form have to be used when all harmonics have to be taken into account. These equations (8) and (9) can be written as:

$$u_{s0} = R_s i_{s0} + \frac{d}{dt}(\psi_{s0})$$

$$u_{sf} = R_s i_{sf} + \frac{d}{dt}(\psi_{sf}) + j\frac{d\gamma}{dt}\psi_{sf}$$

$$0 = R_{rm} i_{rm} + \frac{d}{dt}(\psi''_{rm}) \quad , \quad m = 0 \text{ or } K$$

$$0 = \tilde{R}_{rk}(i''_{rk} + k_k i''^{*}_{rk}) + e_k \psi''_{rk} + f_k \psi''^{*}_{rk} + \frac{d}{dt}(\psi''_{rk}), \quad k=1,2,\ldots,K-1$$

with the flux-current relations:

$$\psi_{s0} = \tilde{L}_{s0} i_{s0} + M_{1m} i_{rm} + \sum_{i=1}^{K-1} [M_{0i} i''_{ri} + M^{*}_{0i} i''^{*}_{ri}]$$

$$\psi_{sf} = \tilde{L}_s i_{sf} + M_{2m} i_{rm} + \sum_{i=1}^{K-1} M_i i''_{ri}$$

$$\psi''_{rm} = M_{1m} i_{s0} + M^{*}_{2m} i_{sf} + M_{2m} i^{*}_{sf} + (\tilde{L}_{rm} + \Delta_{mm}) i_{rm} + \sum_{i=1}^{K-1} [\Delta_{mi} i''_{ri} + \Delta_{m-i} i''^{*}_{ri}]$$

$$\psi''_{rk} = M^{*}_{0k} i_{s0} + M^{*}_{k} i_{sf} + \tilde{L}_{rk}(i''_{rk} + k_k i''^{*}_{rk}) + \Delta_{km} i_{rm} + \sum_{i=1}^{K-1} [\Delta_{ki} i''_{ri} + \Delta_{k-i} i''^{*}_{ri}]$$

and the electromagnetic torque equation:

$$T_e(t) = \mathrm{Re}\Bigg[\Big[2.\sum_{i=1}^{K-1} [(\frac{\partial M^{*}_{0i}}{\partial\theta} + g_i M^{*}_{0i} + h_i M_{0i}) i''^{*}_{ri}] + i_{rm}\frac{\partial M_{1m}}{\partial\theta}\Big] i_{s0} + 2\Big[\sum_{i=1}^{K-1}\{[\frac{\partial M^{*}_{i}}{\partial\theta} + (g_i - j\frac{\partial\gamma}{\partial\theta})M^{*}_{i}] i''^{*}_{ri} + h^{*}_{i} M^{*}_{i} i''_{ri}\} + i_{rm}(\frac{\partial M^{*}_{2m}}{\partial\theta} - j\frac{\partial\gamma}{\partial\theta}M^{*}_{2m})\Big] i_{sf}$$
$$+ \sum_{k=1}^{K-1}\Big[(\frac{\partial\Delta_{mk}}{\partial\theta} i''_{rk} + \frac{\partial\Delta_{m-k}}{\partial\theta} i''^{*}_{rk}) i^{*}_{rm} + \sum_{i=1}^{K-1}(\frac{\partial\Delta_{ik}}{\partial\theta} i''_{rk} + \frac{\partial\Delta_{i-k}}{\partial\theta} i''^{*}_{rk}) i''^{*}_{ri}\Big]$$
$$+ 2\sum_{i=1}^{K-1}\Big[(g_i\Delta_{im} + h_i\Delta^{*}_{m-i}) i_{rm} + \sum_{k=1}^{K-1}[(g_i\Delta_{ik} + h_i\Delta^{*}_{k-i}) i''_{rk} + (g_i\Delta_{i-k} + h_i\Delta^{*}_{ik}) i''^{*}_{rk}]\Big] i''^{*}_{ri}\Bigg]$$

In these equations the following short notations are used:

$$\Delta_{ki} = \Delta L''_{rr}(k,i) \quad ; \quad \Delta_{k-i} = \Delta L''_{rr}(k, N/z - i)$$

However for normally constructed machines, usually a characteristic set of harmonics can be selected which represents the most significant influence on the behavior of the machine.
When in every column in the mutual inductance matrix this set of harmonics fulfills the following constraint:

$$\gamma = \gamma_i = [\alpha_i + \beta_i]/2 \quad (= \alpha_K \text{ or } \alpha_0) \qquad (14)$$

the exponential functions in the second and third row of the L'_{sr}-matrix, as well as in the factors k_i, h_i and f_i disappear. Constraint (14) can only be fulfilled if every element of the L'_{sr}-matrix contains at the most only one inductance.
The inductances in the first row remain functions of θ, however, in consequence of the constraint (14) the exponential functions in M_{0i} in expression (10) become each other's complex conjugate:

$$M_{0i} = \frac{1}{D_i}\hat{M}_{1i}(\hat{M}_{2i}e^{j\epsilon} - \hat{M}_{3i}e^{-j\epsilon}) \text{ where } \epsilon = \gamma_i - \alpha_i + \epsilon_i = -\gamma_i + \beta_i + \epsilon_i$$

Normally this results in one single angle for all columns ($\pm 3\gamma$ for combinations of the lower zero-sequence harmonics and $\pm g.3\gamma$ for the higher, where g=0,1,2,...).
After having analyzed the behavior of the squirrel-cage motor with the second type of armature reaction, from the general model some simplified models can be derived. In this paper only the equations for a first-order approximation for the star connection with ungrounded neutral will be given which provides a system of differential equations with constant coefficients. The θ-dependent elements in the first row of the L'_{sr}-matrix disappear from the equations. ($i_{s0}=0$). The selected set of harmonics has to fulfill constraint (14). The $\Delta L''_{rr}$-matrix contains, in principle, inductances which depend on θ, however, in this case, most elements become constant and, in analogy with the elements in the first row of the L'_{sr}-matrix, the few remaining θ-dependent elements in the $\Delta L''_{rr}$-matrix acquire the same angle dependence: $\pm g.6\gamma$.
The contribution of these θ-dependent terms is most often a second-order effect and will be ignored in this model. This results in the following system of differential equations with constant coefficients.

$$u_{sf} = R_s i_{sf} + \frac{d}{dt}(\Psi_{sf}) + j\frac{d\gamma}{dt}\Psi_{sf}$$

$$0 = R_{rm} i_{rm} + \frac{d}{dt}(\Psi''_{rm}) \quad , \quad m = 0 \text{ or } K$$

$$0 = \tilde{R}_{rk}(i''_{rk} + k_k i''^{*}_{rk}) + e_k\Psi''_{rk} + f_k\Psi''^{*}_{rk} + \frac{d}{dt}\Psi''_{rk}, \quad k=1,2,\ldots,K-1$$

and the flux-current relations:

$$\Psi_{sf} = \tilde{L}_s i_{sf} + \hat{M}_{2m} i_{rm} + \sum_{i=1}^{K-1}\hat{M}_i i''_{ri}$$

$$\Psi''_{rm} = \hat{M}_{2m}(i_{sf} + i^{*}_{sf}) + (\tilde{L}_{rm} + \Delta_{mm}) i_{rm} + \sum_{i=1}^{K-1}[\Delta_{mi} i''_{ri} + \Delta_{m-i} i''^{*}_{ri}]$$

$$\Psi''_{rk} = \hat{M}_k i_{sf} + \tilde{L}_{rk}(i''_{rk} + k_k i''^{*}_{rk}) + \Delta_{km} i_{rm} + \sum_{i=1}^{K-1}[\Delta_{ki} i''_{ri} + \Delta_{k-i} i''^{*}_{ri}]$$

The complete L-matrix is real and constant. In calculations it has to be inverted only once, which saves much computer application time.

The torque equation becomes:

$$T_e = 2.\mathrm{Re}\Big[\sum_{i=1}^{K-1}[g_i i''^{*}_{ri} + h^{*}_i i''_{ri}]\hat{M}_i i_{sf} - j\frac{\partial\gamma}{\partial\theta}[\hat{M}_{2m} i_{rm} + \sum_{i=1}^{K-1}\hat{M}_i i''^{*}_{ri}] i_{sf} +$$

$$+\sum_{i=1}^{K-1}[(g_i\Delta_{im} + h_i\Delta_{m-i})i_{rm} + \sum_{k=1}^{K-1}\{(g_i\Delta_{ik} + h_i\Delta_{k-i})i''_{rk} +$$

$$+(g_i\Delta_{i-k} + h_i\Delta_{ki})i''^{*}_{rk}\}]i''_{ri}\Big]$$

where the factors e_i, g_i, f_i and h_i become:

$$e_i = g_i\frac{d\theta}{dt} \quad ; \quad g_i = j\left[\frac{\partial\gamma_i}{\partial\theta} - \frac{\frac{\partial\alpha_i}{\partial\theta}\hat{M}^2_{2i} - \frac{\partial\beta_i}{\partial\theta}\hat{M}^2_{3i}}{E^2_i}\right]$$

$$f_i = h_i\frac{d\theta}{dt} \quad ; \quad h_i = -j\left(\frac{\partial\alpha_i}{\partial\theta} - \frac{\partial\beta_i}{\partial\theta}\right)\frac{\hat{M}_{2i}\hat{M}_{3i}}{E^2_i}$$

Note that when not more then one harmonic inductance is present in every element of the mutual inductance matrix L'_{sr}, see expression (4), the factors g_i and h_i are constant too.

As this system has constant coefficients a particular solution is possible when the slip s is constant [10]. Upon assuming a symmetric voltage system between the terminals of the star-connected stator winding the transformed stator voltage can be written as:

$$u_{sf} = \frac{1}{\sqrt{2}} U_N\, e^{j(\omega t + \phi' - \gamma)}$$

For a constant value of the slip s, γ can be written as: $\gamma = \gamma' p\theta = \gamma'(1-s)\omega t$ and after introducing $s_\nu = 1 - \gamma'(1-s)$, the transformed voltage becomes:

$$u_{sf} = \underline{U}_{sf}\, e^{j(s_\nu \omega t + \phi')} \quad , \text{ where } \underline{U}_{sf} = \frac{1}{\sqrt{2}} U_N \quad \text{and}$$

U_N is the rms-value of the terminal voltage.

From here it is clear that, because of the presence of groups of harmonics that result in different γ's, the synchronous torques which change into pulsating torques with a frequency of $2s_\nu\omega t$ outside their synchronous speeds always cause synchronous torques to appear simultaneously with pulsating torques. This contrasts with the first type where only one synchronous torque exists at stand-still.
For the delta connection similar models can be derived. In these models the L''_{sr}- matrix remains θ dependent because of the elements in row 1.

One difficulty in this development of a system with a maximum of constant elements is constituted by the practical fact that for column 1 it may be necessary to consider more inductances in element 3 in the interests of obtaining more accurate results [9].

The complete set of equations, which describes the transient state includes the mechanical equation of motion:

$$T_e - T_{me} = J\frac{d^2\theta}{dt^2} + d_{me}\frac{d\theta}{dt}$$

where T_{me} is the load torque, J the inertia of the complete rotor mass and d_{me} the damping coefficient.

5. CONCLUSIONS

In this paper general equations for squirrel-cage induction motors have been derived based on the real geometry of the motor. The squirrel cage has been described by its meshes; no equivalent windings have been used. By means of complex time-dependent transformations free angles are introduced which are very helpfully simplifying the set of equations when the specific geometrical properties of the both types, in which the asynchronous machines fundamentally can be divided, are taken into account. The equations derived are general in the sense that all space harmonics are taken into account, due to the MMF as well as to the double slotting. This provides a better calculation of the synchronous, pulsating and asynchronous torques. The final equations enable the formulation of some specific properties of both types in connection to their electromechanical behavior. Further they are valid for star and delta connection and for any arbitrary source voltage.

6. ABBREVIATIONS AND SYMBOLS

i(t), u(t)	instantaneous values (complex or real)
$i^*(t)$, $u^*(t)$	conjugate complex values
I,*U*,*L*	matrices
I^T,U^T,etc	transposed matrices
I,U	rms values (complex or real)
$\hat{L}$, $\hat{M}$, etc	magnitudes (real)
$\tilde{L}$, $\tilde{M}$, etc	inductances after group- and symmetrical-component transformation.
i',*I*',L'_{sr},etc	quantities after group- and symmetrical-component transformation.
i",*I*",L''_{sr},etc	quantities after the time-dependent transformation.

Symbols (in SI-units)

d_{me}	coefficient of mechanical damping
i,I,*I*	current
j	complex number j=(0,1)
J	inertia of rotor mass
K	integer number K=N/2z or N/2z+1/2
L,*L*	inductance
m_k	mutual inductance
M	mutual inductance
N	number of rotor slots
N_s	number of stator slots
p	number of pole pairs
R,*R*	resistance
s	slip
T_e	electromagnetic torque
T_{me}	load torque
u,U,*U*	voltage
U	unit matrix
z	greatest common divisor of N and p
α, β, ϵ	angle in composed inductance of one element in the mutual inductance matrix.
γ	angle introduced by transformation
θ	rotor position angle
ν,μ	integer numbers
ϕ	phase angle of stator phase voltage
ϕ'	phase angle of stator terminal voltage
Ψ,Ψ,ψ	flux linkage
ω	angular frequency of the source voltage
Δ_{ki}	short notation for $\Delta L_{rr}(k,i)$
ΔL_{rr}	rotor inductance matrix due to stator slotting

Subscripts

i_{sf}	forward component of stator current
i_{sb}	backward component of stator current
i_r	rotor mesh current

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BIOGRAPHY

Gerardus Chr. Paap was born in Rotterdam, the Netherlands on February 2, 1946. He received his M.SC. from Delft University of Technology in 1972 and his Ph.D. from the Technical University of Lodz in 1988. From 1973 he has been with the Faculty of Electrical Engineering of the Delft University of Technology. First, from 1973 to 1985, with the Electrical Machines and Drives Division where he lectured on the fundamentals of electrical machines and from 1985 with the Power Systems Laboratory as Assistant Professor. His research interests include power systems transients and the dynamics of electrical machines.

APPENDIX I

The resultant parameters which appear in the equations after group and symmetrical component transformation are:
the zero sequence inductance:

$$\tilde{L}_{s0} = \mu_0 \frac{6\, w_s^2\, D\, l}{\pi\, p^2\, \delta} \sum_\nu \frac{\xi_{s\nu}^2}{\nu^2} \zeta_{s\nu} + \tilde{L}_{s0\sigma} \quad \text{with } \nu = 3,9,15,\ldots.$$

the positive sequence inductance :

$$\tilde{L}_{s} = \mu_0 \frac{3\, w_s^2\, D\, l}{\pi\, p^2\, \delta} \sum_\nu \frac{\xi_{s\nu}^2}{\nu^2} \zeta_{s\nu} + \tilde{L}_{s\sigma} \quad \text{with } \nu = 1,-5,7,-11,13,.$$

where δ the effective air gap, w_s the number of turns per stator phase in series, D diameter of stator bore and l the effective iron length. $\tilde{L}_{s0\sigma}$ and $\tilde{L}_{s\sigma}$ are the zero-sequence and positive-sequence leakage inductances.
For the rotor the transformed parameters are:

$$\tilde{R}_{rk} = 2R_R + R_{rb}\left(2.\sin \frac{k\pi p}{N} \right)^2 , \quad k = 0,1,\ldots,(N/z-1)$$

$$\tilde{L}_{r0} = \frac{\beta}{N+\beta} \tilde{L}_N + \tilde{L}_{r0\sigma};\; \tilde{L}_{rk} = \tilde{L}_N + \tilde{L}_{rk\sigma}, \quad k = 1,2,\ldots,(N/z-1)$$

$$\tilde{L}_{rk\sigma} = 2L_{R\sigma} + L_{rb\sigma}\left(2.\sin \frac{k\pi p}{N} \right)^2 \quad k = 0,1,\ldots,(N/z-1)$$

where R_R the resistance of a rotor-ring segment and R_{rb} the resistance of a bar. $L_{R\sigma}$ and $L_{rb\sigma}$ are the leakage inductances of a ring segment and a bar respectively.
β is the rate between the conductance of the unipolar flux path with cross-section A_0 and the effective air gap Δ and one rotor tooth with cross-section A_t.

$$\Delta\tilde{L}_{rn} = \frac{1}{2} \frac{\hat{\Lambda}_{sn}}{\Lambda_{s0}} \zeta_{srn}\, \xi_{\Delta skn}\, \tilde{L}_N, \quad \text{where } \tilde{L}_N = \mu_0 \frac{\pi D l}{N\delta}$$

For the n-th harmonic of the stator-conductance wave it holds for the elements in the matrix $Q[q(i,k)]$:
$n.N_s/N \neq 1,2,3,\ldots.$

$q(i,k) = 0$	if $i = k$
$q(i,k) \neq 0$	if $n.N_s + (i-k)p = gN$; $g=0,1,2,..$
$q(i,k) = \beta/(N+\beta)$	if $i = 1$ or $k = 1$
$q(i,k) = 1$	if $i \neq k$ and $k \neq 1$

The elements in the resultant mutual inductance in expression (5) are:

$$\hat{M}_{ij} = \sqrt{\left[\sum_{k=1}^{n} \hat{m}_k^2 + 2 \sum_{l=1}^{n-1} \sum_{k=l+1}^{n} \hat{m}_l\, \hat{m}_k \cos\{(\nu_l - \nu_k)p\theta\} \right]}$$

$$\eta = \arctan\left[\sum_{k=1}^{n} m_k \sin(\nu_k p\theta) / \sum_{k=1}^{n} \hat{m}_k \cos(\nu_k p\theta) \right]$$

and the original inductances in expression (3):

$$m_\nu = \hat{m}_\nu\, e^{j\nu p\theta} \;;\; \hat{m}_\nu = \sqrt{(3N)}.\tilde{L}_N \frac{w_s}{\pi p} \frac{1}{\nu} \xi_{s\nu} \xi_{r\nu} \xi_{sk\nu} \zeta_{s\nu}$$

where $\nu = \nu, \mu$

$\xi_{s\nu}$	is the winding factor of the stator.
$\zeta_{s\nu}$	is the stator slot factor.
$\xi_{sk\nu}$	is the skew factor
$\xi_{r\nu}$	is the rotor slot factor
ζ_{srn}	is the factor of mutual slotting
$\xi_{\Delta skn}$	is the skew factor in ΔL_{rr}

PERFORMANCE ANALYSIS OF PERMANENT MAGNET SYNCHRONOUS MOTORS PART:II OPERATION FROM VARIABLE SOURCE AND TRANSIENT CHARACTERISTICS

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Abstract

A comprehensive analysis of transient performance of a permanent magnet (PM) synchronous motor, operating from a variable source is described. Internal damping is modeled and optimum values of design parameters which improve transient performance are obtained. The effects of varying the supply voltage and frequency on the optimum values of these parameters are demonstrated. Simulated nonlinear response to a load change and starting performance are discussed. The results illustrate that significant improvement in transient performance can be achieved, over a wide range of voltages and frequencies, when some parameters are properly chosen.

Keywords: permanent-magnet synchronous motors

INTRODUCTION

Modeling of permanent magnet and steady state performance of PM motors have been a subject of growing interest since the early 80's [1-3]. The effects of varying supply frequency on motor parameters and steady state performance have also been studied [4]. Moreover, the characteristics of this motor, when operated from an inverter with phase controller, has been considered [5]. Starting performance was firstly discussed by Honsinger [6]. He investigated the effects of the rotor cage parameters on the run-up characteristics and valuable conclusions were reached. This study has been followed up by Rahman et al who examined the effect of saturation on starting performance [7]. Miller [8] developed a pull-in criterion and studied the run-up characteristics of line start PM motors.

Synchronization of PM motors results from the action of synchronizing torque. This torque is attributable to both the magnet and the difference between the axes reactances. Also, the hunting oscillations, following a load change, are damped by the action of damping torque. This torque component is significantly affected by the motor design parameters. The effects of all design parameters on dynamic performance are extensively examined using the damping and synchronizing torque technique [9]. The effective parameters which increase motor damping are identified and their optimum values, that improve dynamic performance, are obtained.

The object of this paper is to examine the effects of supply voltage and frequency on the optimum values of the effective design parameters and also on the starting performance. The results give physical interpretation to transient performance when the motor operates from a variable source. They also show that with a proper choice of some parameters, the transient performance can be significantly improved over a wide range of operating voltages and frequencies.

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MODELING OF TORQUE COMPONENTS

Algorithm

Modeling and accurate prediction of damping and synchronizing torques provide a quantitative assessment and physical realization of motor performance. Damping torque is defined as the torque component in-phase with the rotor speed, while the synchronizing torque is that in-phase component with rotor angle. This indicates that, following a small load change, the deviation in motor torque can be written as [9] :

$$\Delta T_M(t) = K_D \cdot \Delta\omega(t) + K_S \cdot \Delta\delta(t) \tag{1}$$

Where K_D and K_S are damping and synchronizing torque coefficients, respectively. These coefficients must be positive for the stable motor. Using minimization procedures to minimize squared errors, the following algorithm is obtained:

$$\sum^{N} \Delta T_M(t) \cdot \Delta\delta(t) = K_S \sum^{N} [\Delta\delta(t)]^2 + K_D \sum^{N} \Delta\omega(t) \cdot \Delta\delta(t) \tag{2}$$

$$\sum^{N} \Delta T_M(t) \cdot \Delta\omega(t) = K_D \sum^{N} [\Delta\omega(t)]^2 + K_S \sum^{N} \Delta\omega(t) \cdot \Delta\delta(t) \tag{3}$$

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The above equations give the values of the torque coefficients K_D and K_S which can be solved numerically. This solution can be obtained either on-line, in experimental applications or off-line with digital simulation, over a period of time t= N.T, where T is the integration step and N is the number of iterations. The advantage of using this technique is that it gives the torque components directly from the motor nonlinear response without approximations.

Optimum Parameters

The PM motor used in this investigation, and also in Part I [9], is a 4 h.p., 2 pole, 1-phase, 230 Volt. The per-unit values of motor parameters' are given in the Appendix. The damping and synchronizing torque algorithm has been applied to assess the effects of all design parameters on the dynamic performance of PM motors[9]. To examine the effect of a specific parameter, a 50% load disturbance was applied for 20 ms. The corresponding deviations in rotor angle, rotor speed and electrical torque were obtained over a period of 3 seconds. This is required to solve the decomposition algorithm, Eqns.2-3, and compute the time invariant torque coefficients, K_D and K_S. It has been concluded that a proper choice of the cage resistances, R_D and R_Q, increases damping and improves dynamic performance. Each of these parameters has an optimum value at which maximum damping occurs. These values are designated as R_QOpt and R_DOpt. The question now is to what extent the supply voltage and frequency affects both R_DOpt and R_QOpt and how these parameters affect starting performance. This is described subsequently.

OPERATION FROM VARIABLE SOURCE

Sensitivity Of Optimum Parameters

At each operating voltage and frequency, a 50% load disturbance was applied for 20 ms and the corresponding deviations in rotor angle, rotor speed and electrical torque were obtained. While varying either R_D or R_Q the torque decomposition algorithm has been applied to define their optimum values at this voltage and frequency. This has been repeated to define both R_QOpt and R_DOpt versus supply voltage at different frequencies as shown in Figs. 1-2. These results illustrate that both R_QOpt and R_DOpt vary within a limited region over the whole operating range of voltages and frequencies. Moreover, a study was made concerning the internal damping, over a wide range of voltages and frequencies, using:

(i) one set of optimum R_D and R_Q values, determined at the nominal voltage and frequency, and

(ii) the optimum values of R_D and R_Q, corresponding to each operating voltage and frequency.

Figs.3 & 4. show the results of this study. This demonstrates clearly that the optimum R_Q and R_D may be chosen at the nominal voltage and frequency with only a slight reduction in the internal motor damping. Therefore, these optimum values of parameters are used in computing the results presented subsequently.

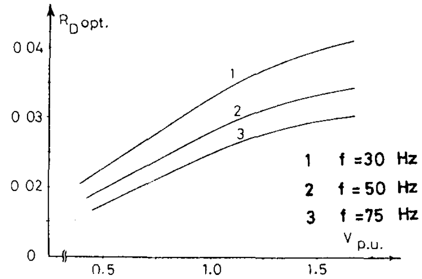


Fig. 1 Effects of voltage and frequency on optimum value of R_D

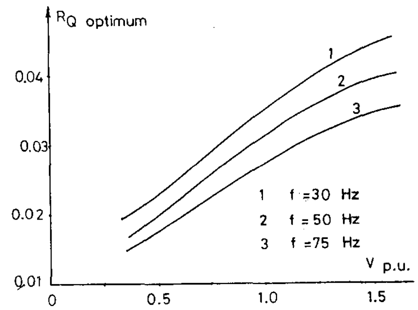


Fig. 2 Effects of voltage and frequency on optimum value of R_Q

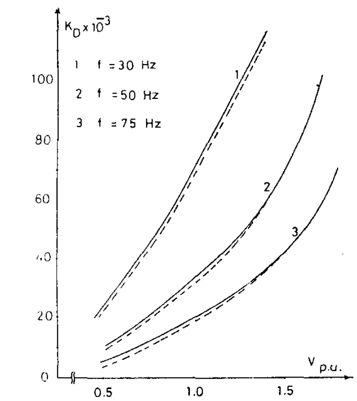


Figure 3 Torque coefficients versus V
——— R_Dopt at nominal v & f
– – – – R_Dopt at every V & f

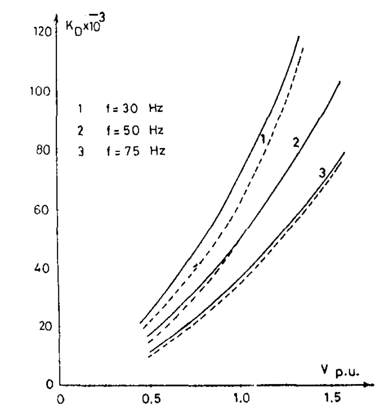


Figure 4 Torque coefficients versus V
——— R_Qopt at nominal v & f
– – – – R_Qopt at every V & f

Damping Characteristics

The objects of this section are to:

1. examine the effects of operating voltage and frequency on damping and synchronizing torques.
2. study the effects of using the optimum parameters on torque coefficients in comparison with the nominal parameters.

The results of this investigation are concluded in Fig.5. Concerning the first object, the results illustrate that motor damping increases with either increasing the supply voltage or decreasing the frequency. Deterioration in dynamic performance at low voltage may be attributed to lack of damping torque. Moreover, there is an optimum value of supply voltage at which maximum synchronous torque occurs. This optimum voltage is related directly to supply frequency. Failure in synchronization at low-frequencies and high voltages may be due to lack of synchronizing torque. Therefore, at each synchronous speed, the optimum operating voltage is that value which provides maximum synchronizing torque.

A comparison between damping and synchronizing torque coefficients obtained using either the nominal or the optimum parameters, over a wide range of voltage and frequencies, is demonstrated in Fig.5. It is clear that the use of the optimum parameters increases motor damping at all voltages and frequencies. As an example, the damping torque is increased by about 100% at V=0.50 p.u. and f= 30 Hz which represents a significant improvement in motor dynamics. However, the obtained synchronizing torques, computed using both nominal and optimum parameters, are similar. Therefore, the solid lines in Fig. 5 represent the synchronizing torques for both cases.

The synchronizing torque is significantly affected by field strength over the whole range of operating voltages and frequencies. To demonstrate this, the optimum parameters are considered and the torque components are obtained versus I_F as shown in Fig.6. These results illustrate that increasing the magnet strength increases the synchronizing torque without significant variations in damping coefficient.

Time Response

To further elucidate the above important remarks, the motor is subjected to a pulse load increase, and the time response is shown in Fig.7. This represents a comparison between motor responses at different voltages and frequencies, using either the nominal or the optimum parameters. The results illustrate well-damped oscillations and a reduction in the rotor first swing, at all operating voltages and frequencies, when the optimum parameters are used. Comparing Figs.7.a and 7.b shows that at a given supply frequency, the increase in supply voltage increases motor damping and reduces rotor first swing which indicates an increase in stability reserve. These results confirm those previously predicted from Fig.5.

Fig.8 shows a comparison between motor

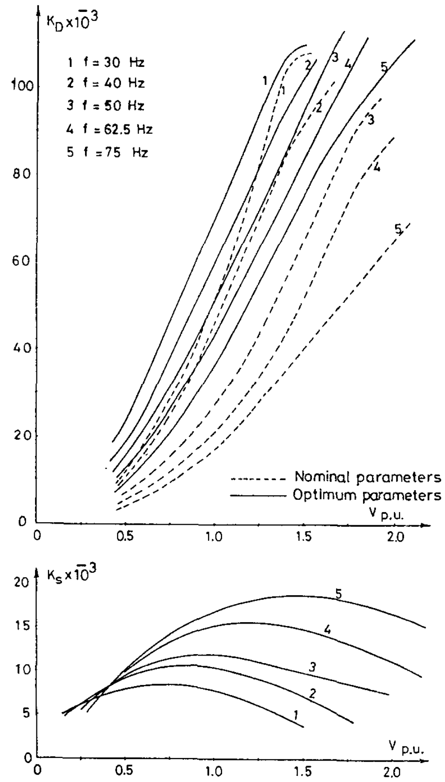


Figure 5 Torque coefficients versus V

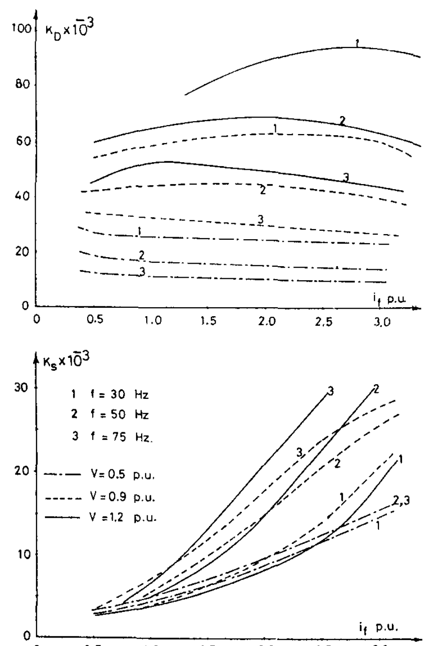


Figure 6 Torque coefficients versus I_F

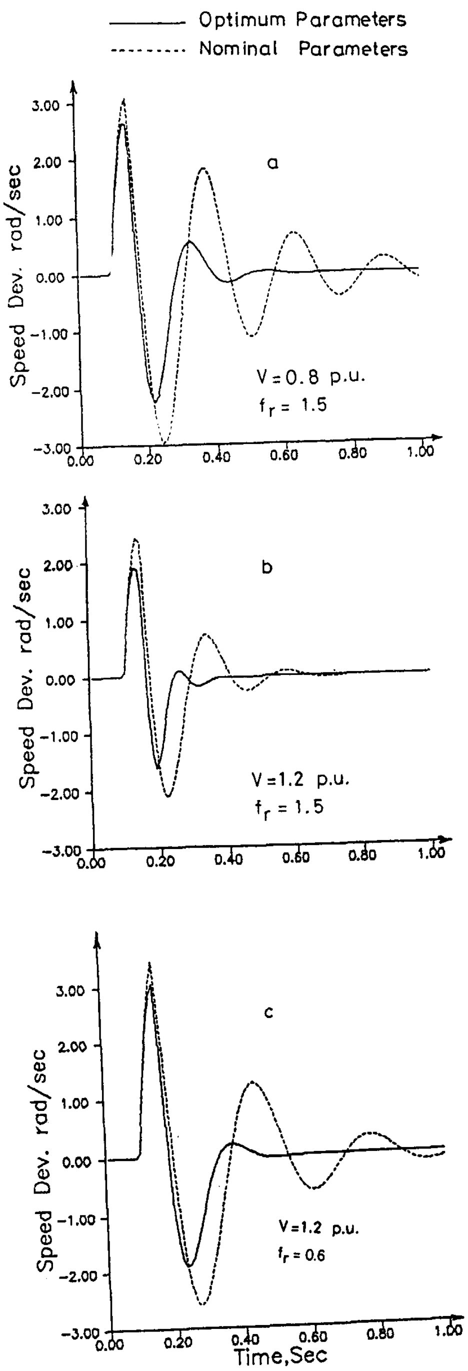


Figure 7 Comparison of time response

response at a constant frequency and variable voltage. These remarks confirm those extracted from Fig.5, indicating that either increasing supply voltage or reducing frequency, improves damping and increase motor stability reserve. This is indicated by the reduction in the first rotor swing in both cases.

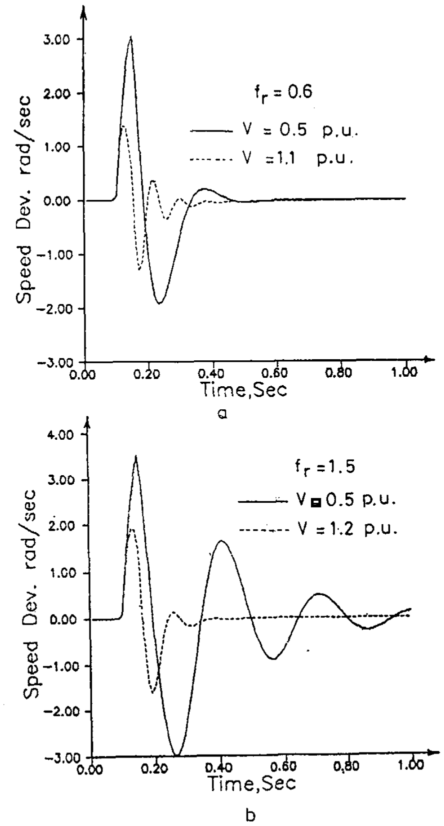


Figure 8 Response at constant frequency

An important test has been carried out to obtain the motor response at low and high voltage levels using three different values of supply frequency. These results are shown in Fig.9, which illustrates:

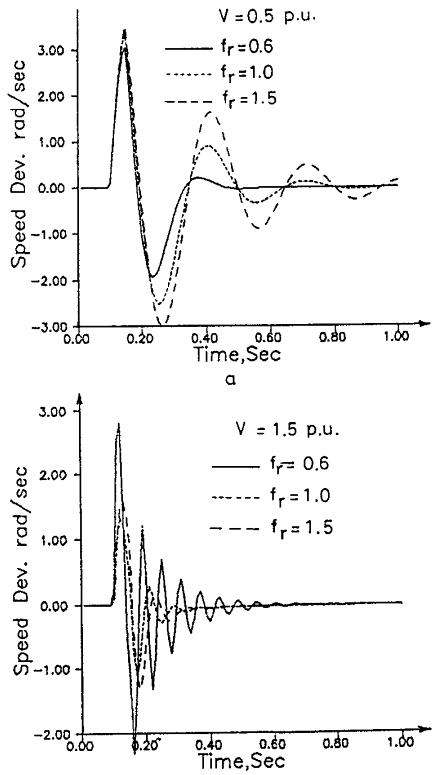
1. At low frequency (curve 1, in Fig.9.a - 9.b), the performance is improved at low voltage due to high damping torque and deteriorates at high voltage due to lack of the synchronizing torque.
2. There is an optimum operating voltage corresponding to each synchronous speed. This voltage can be chosen to yields the maximum synchronizing torque.

STARTING CHARACTERISTICS

Effects Of Optimum Parameters

During starting, the cage resistance must be large enough to compensate for the negative magnet torque. The object of this section is to demonstrate the effects of the optimum parameters on starting performance. It has been demonstrated[9], that damping is more sensitive to changes in R_Q rather than in R_D. Fig.10 shows the effects of varying R_D with different values of R_Q and loads. Examining these results indicates that using R_Q at its optimum value is necessary to :

1. achieve maximum improvements in dynamic performance;



b

Figure 9 Response at a constant voltage

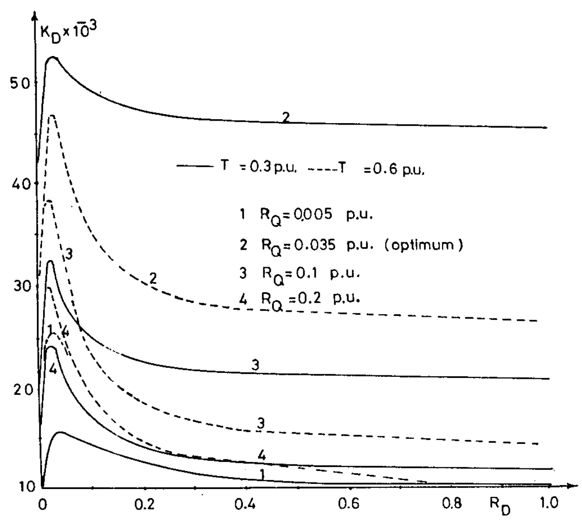


Fig. 10 Effect of cage resistances on internal damping

2. reduce any adverse effects on motor dynamics due to an increase in R_D more than its optimum value.

The above remarks are of significant importance to the starting performance. The value of R_D can be increased beyond its optimum value to substitute for the negative magnet torque. This will improve the starting performance without any adverse effects following changes in load torque. A comprehensive investigation has been undertaken to determine the optimum value of R_D which improves both starting performance and response following small load changes. The effects of these optimum parameters on starting are shown in Fig.11. The results illustrate poor starting, under different loads, when either R_Q or R_D deviated from its optimum value.

Starting From Variable Source

Fig.12 shows the effects of supply voltage and frequency on the starting performance. As explained in Fig.5, the loss of synchronism at low-frequency, high-voltage (Fig,12.a) is due to lack of synchronous torque. Also there is an optimum value of supply voltage which yields good starting performance at each synchronous speed. This optimum value may be determined as that value which increases the synchronizing torque coefficient

CONCLUSION

This paper defines the parameters which affect the transient performance of the PM motors over a wide range of operating voltages and frequencies. The results demonstrate that a proper choice of these effective parameters at the nominal voltage and frequency improve the motor performance over a wide range of supply voltages and frequencies. Increasing supply voltage and reducing frequency increase both motor damping and stability reserve. However, performance may deteriorate at low frequency with the increase of supply voltage due to lack of synchronous torque. The starting performance is significantly improved with the use of the optimum parameters. There is an optimum value of supply voltage corresponding to each synchronous speed at which maximum synchronizing torque occurs. The choice of operating voltage at this value has no adverse effects on the damping torque. Moreover, increasing the supply voltage above this optimum value causes a failure of the motor to reach synchronous speed due to lack of synchronizing torque. These results are of practical importance, giving some valuable insight in the PM motors dynamics when operated from a variable voltage and frequency source.

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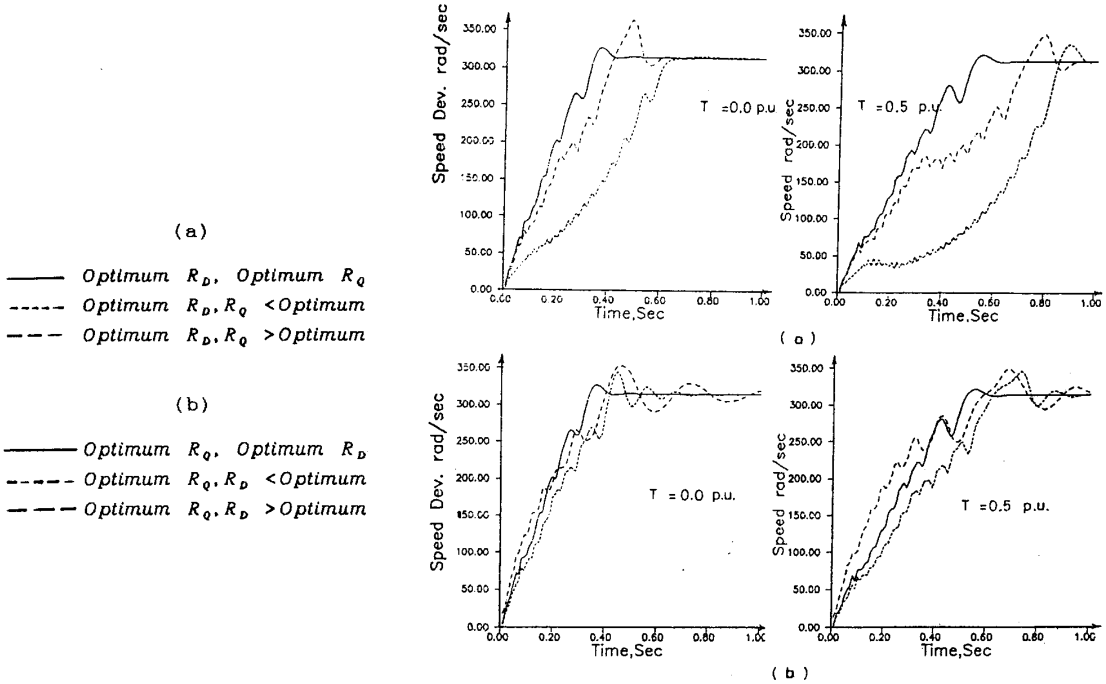

Figure 11 Effects of cage parameters on starting characteristics

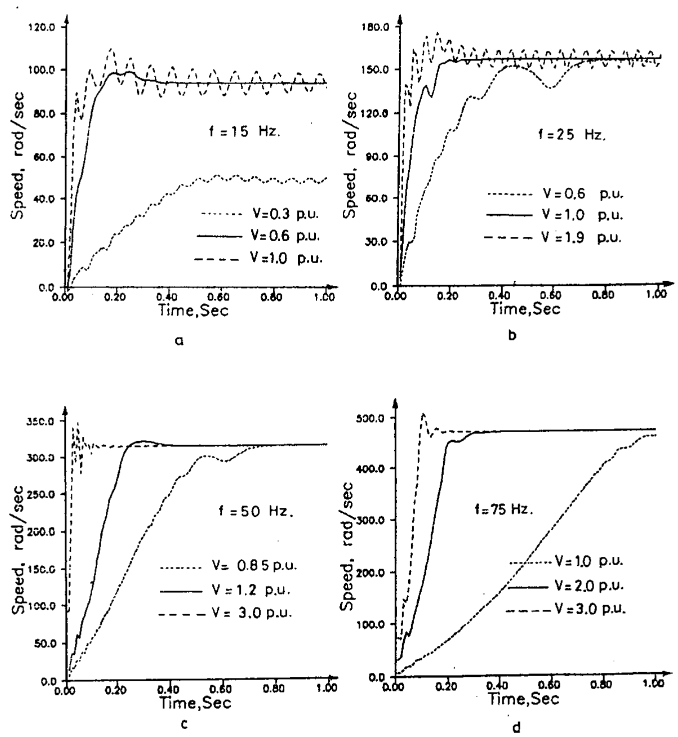

Figure 12 Effects of voltage and frequency on starting characteristics

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LIST OF SYMBOLS

f	Supply frequency Hz.
K_D	Damping torque coefficient, p.u/rad/sec.
K_S	Synchronizing torque coeff., p.u/rad.
f_r	Per unit frequency (f/f_o)
V	Supply voltage p.u.
R	Resistance p.u.
T	Torque p.u.
I_F	Equivalent field current
X	Reactance
δ	Load angle , rad.
ω	Angular velocity rad./sec.

SUBSCRIPTS

D	Direct axis damper circuit.
Q	Quadrature axis damper circuit.

APPENDIX

Motor Parameters in Per-Unit value[7]:

R_a = 0.0173 R_D = 0.054

R_Q = 0.108 i_f = 1.817

X_{do} = 0.543 X_{qo} = 1.086

X_{ado} = 0.478 X_{aqo} = 1.021

X_{Do} = 0.608 X_{Qo} = 1.151

H = 0.2510 sec.

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