The order structure and topology on the space of measurable functions

by

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1. Introduction

In probability theory, L^p spaces for $p > 0$ together with the topology of convergence in probability have been widely applied. However, in that case we restrict ourselves to only a part of all the measurable functions and to an underlying probability space. One of the main aims of this thesis is to generalize this concept to the set of all measurable functions with the usual a.e. equivalence classes (which we call L^0) and (possibly) non-finite measure spaces. The other main aim is to establish an ordered structure on this L^0 space. We will start our discussion by first defining L^0 and establishing that L^0 is an ordered vector space. After this we will define the generalisation of the topology of convergence in probability, which is called the topology of convergence in measure and we will explore properties of this topology relating to the convergence in measure, continuity of maps, metrizability (i.e. under what circumstances can this topology be generated by a single metric?), completeness (i.e. convergence of Cauchy sequences) and local convexity (when can the topology be generated by a set of semi-norms?). After this we will generalise the notion of essential suprema from sets of random variables to sets of measurable functions. It turns out that for most properties considered in this text, such as metrizability, completeness and the existence of an essential supremum, requiring the underlying measure space to be σ-finite (i.e. the underlying space can be covered by countably many measurable sets of finite measure) suffices. Discussions on weaker conditions in these instances are also included in this text.

The last section of this thesis contains an overview of a widely used application of our L^0 space: namely Stochastic Differential Equations (SDEs). The main focus of this section is obtaining a maximal solution to such equation under local Lipschitz conditions.

The following Appendix contains an overview of commonly used definitions and results from the theory on general topological spaces, Grönwall's inequality for measurable functions (used in the section on SDEs) and different characterisations of the Radon-Nikodym theorem relating to the corresponding derivative possibly taking non-finite values (used in the section on the essential supremum).

For the properties of the topology of convergence in measure on L^0 a commonly returning source of the results will be "Measure Theory Volume 2", written by David Fremlin. This thesis will, in part, display his vision as described in the book in a comprehensive and orderly way together with other results from different sources. Furthermore, not mentioned details, generalizations and further results will be displayed as well.

2. The structure on L^0

We start our discussion with the definition of the space of measurable functions up to a.e. equivalence and establish that it is a linear space.

2.1. Definition of L^0.

Assume (S, Σ, μ) is a measurable space. Unless otherwise specified, we associate $\mathbb{R}$ with the Borel sigma algebra.

Definition 2.1. $\mathscr{L}^0(S, \Sigma, \mu) = \Big\{ f : S \to \mathbb{R} : f \textit{ is measurable} \Big\}$.

In order to be able to define the L^0 space, we first need to define the a.e. (almost everywhere) equal equivalence classes.

Definition 2.2. *Let $f \in \mathscr{L}^0(S, \Sigma, \mu)$. Then $g \in \dot{f} \iff g =_{a.e.} f$.*

Next, we will to provide that this indeed defines an equivalence class on $\mathscr{L}^0(S, \Sigma, \mu)$.

Theorem 2.3. *For $f \in L^0(S, \Sigma, \mu)$, $\dot{f}$ as defined above is an equivalence class.*

Proof.
(1) $\mu(\{f \neq f\}) = \mu(\varnothing) = 0$, and so $f =_{a.e.} f$.
(2) $f =_{a.e} g \iff \mu(\{f \neq g\}) = 0 \iff \mu(\{g \neq f\}) = 0 \iff g =_{a.e.} f$.
(3) Suppose $f =_{a.e.} g$ and $g =_{a.e} h$. Since: $\{g = f \neq h\} \subseteq \{g \neq h\}$ and $\{g \neq f \neq h\} \subseteq \{g \neq f\}$ we have:

$$\begin{aligned} 0 \leq \mu(\{f \neq h\}) &= \mu(\{g = f \neq h\} \cup \{g \neq f \neq h\}) \\ &\leq \mu(\{g \neq h\}) + \mu(\{g \neq f\}) = 0. \end{aligned}$$

It follows that $\mu(\{f \neq h\} = 0$.

□

Now we can finally define $L^0(S, \Sigma, \mu)$, which is the space of such equivalence classes.

Definition 2.4. $L^0(S, \Sigma, \mu) = \Big\{ \dot{f} : f \in \mathscr{L}^0(S, \Sigma, \mu) \Big\}$.

2.2. Linear structure on L^0.

Now that we have defined the L^0 space, we can consider properties of this space. We know that $\mathscr{L}^0(S, \Sigma, \mu)$ has linear structure (is a vector space). On this space, the addition of two measurable functions is defined pointwise. We can define addition on the $L^0(S, \Sigma, \mu)$ space in such a way that a lot of properties from $\mathscr{L}^0(S, \Sigma, \mu)$ are inherited, in specific the linear structure.

Definition 2.5. *For $\dot{f}, \dot{g} \in L^0(S, \Sigma, \mu), c \in \mathbb{R}$, we can define addition and scalar multiplication on $L^0(S, \Sigma, \mu)$ as follows:*

$$\dot{f} \oplus \dot{g} = (f \dot{+} g)$$
$$c\dot{f} = (\dot{cf}).$$

Remark 2.6.
(1) This definition is well defined:
- The sum of two measurable functions is measurable, so $\dot{f} \oplus \dot{g} \in L^0(S, \Sigma, \mu)$
- Different representation will not change the resulting equivalence class: Let $f_1, f_2, g_1, g_2 \in \mathscr{L}^0(S, \Sigma, \mu)$ such that $f_1 =_{a.e.} f_2$ and $g_1 =_{a.e.} g_2$. Then since $\{f_1 + g_1 \neq f_2 + g_2\} \subseteq \{f_1 \neq f_2\} \cup \{g_1 \neq g_2\}$:
$0 \leq \mu(\{f_1 + g_1 \neq f_2 + g_2\}) \leq \mu(\{f_1 \neq f_2\}) + \mu(\{g_1 + g_2\}) = 0 + 0 = 0.$
Hence: $(f_1 \dot{+} g_1) = (f_2 \dot{+} g_2)$.

- In the same fashion as above, we can conclude that $c\dot{f} \in L^0(S, \Sigma, \mu)$ and if $f_1 =_{a.e.} f_2$, then $cf_1 =_{a.e.} cf_2$.

(2) For convenience, we will write $\dot{f} + \dot{g}$ instead of $\dot{f} \oplus \dot{g}$.

Using this definition for addition and scalar multiplication, it follows (almost directly) from the fact that $\mathscr{L}^0(S, \Sigma, \mu)$ has a linear structure, that $L^0(S, \Sigma, \mu)$ has a linear structure.

Corollary 2.7. *$L^0(S, \Sigma, \mu)$ is a vector space over $\mathbb{R}$, where addition and scalar multiplication are defined as in definition 2.5.*

Proof. Let $\dot{f}, \dot{g}, \dot{h} \in L^0(S, \Sigma, \mu)$ and $a, b \in \mathbb{R}$.

(1) $\dot{f} + \dot{g} \overset{\text{def2.5}}{=} (f \dot{+} g) \in L^0(S, \Sigma, \mu)$.
(2) $c\dot{f} \overset{\text{def2.5}}{=} (\dot{cf}) \in L^0(S, \Sigma, \mu)$.
(3) $\dot{f}+(\dot{g}+\dot{h}) \overset{\text{def2.5}}{=} \dot{f}+(g \dot{+} h) \overset{\text{def2.5}}{=} (f + \dot{g} + h) = ((f + \dot{g}) + h) \overset{\text{def2.5}}{=} (f \dot{+} g)+ \dot{h} \overset{\text{def2.5}}{=} (\dot{f} + \dot{g}) + \dot{h}$.
(4) $\dot{f} + \dot{g} \overset{\text{def2.5}}{=} (f \dot{+} g) = (g \dot{+} f) \overset{\text{def2.5}}{=} \dot{g} + \dot{f}$.
(5) Consider $0(x) = 0 \quad \forall x \in S$. Since 0 is measurable: $0 \in \mathscr{L}^0$ and so $\dot{f} + \dot{0} \overset{\text{def2.5}}{=} (f \dot{+} 0) = \dot{f}$.
(6) $f \in \mathscr{L}^0 \implies (-f) \in \mathscr{L}^0$, so $\dot{f} + (\dot{-f}) \overset{\text{def2.5}}{=} (f + \dot{(}-f)) = \dot{0}$.
(7) $a(b\dot{f}) \overset{\text{def2.5}}{=} a(\dot{bf}) \overset{\text{def2.5}}{=} (a\dot{bf}) \overset{\text{def2.5}}{=} (ab)\dot{f}$.
(8) $1\dot{f} = \dot{1f}$.
(9) $a(\dot{f} + \dot{g}) \overset{\text{def2.5}}{=} a(f \dot{+} g) \overset{\text{def2.5}}{=} (a(f \dot{+} g)) = (af \dot{+} ag) \overset{\text{def2.5}}{=} a\dot{f} + a\dot{g}$.
(10) $(a + b)\dot{f} \overset{\text{def2.5}}{=} ((a \dot{+} b)f) = (af \dot{+} bf) \overset{\text{def2.5}}{=} (\dot{af}) + (\dot{bf}) \overset{\text{def2.5}}{=} a\dot{f} + b\dot{f}$.

□

2.3. **Multiplicative structure on L^0.** We know that for $f, g \in \mathscr{L}^0$, since the (pointwise) multiplication of two measurable functions is again measurable, we have that $f \cdot g \in \mathscr{L}^0$. Let $f_1, f_2, g_1, g_2 \in \mathscr{L}^0$. Then in case $f_1 =_{a.e.} f_2$ and $g_1 =_{a.e.} g_2$ we have $f_1 \cdot g_1 =_{a.e.} f_2 \cdot g_2$ (the proof is similar to the one in remark 2.6).
Hence the following definition is well-defined on $L^0(S, \Sigma, \mu)$:
$\dot{f} \otimes \dot{g} = (f \dot{\cdot} g)$, and for convenience, we will again use the notation: $\dot{f} \cdot \dot{g}$.

2.4. **Order structure on L^0.** A straightforward property of $\mathbb{R}$ is that it has an ordered structure: indeed, if we take two elements from $\mathbb{R}$, we can always say that one is smaller/greater than the other or equal by subtracting the two numbers. It is not as easily seen as on $\mathbb{R}$, but on $L^0(S, \Sigma, \mu)$ we can also compare two elements and say how these two elements relate to each other with respect to their 'size'. This notion is defined below.

Definition 2.8. *For $\dot{f}, \dot{g} \in L^0(S, \Sigma, \mu)$: $\dot{f} \leq \dot{g} \iff f \leq_{a.e.} g$.*

Remark 2.9. As it holds that for $f_1, f_2, g_1, g_2 \in \mathscr{L}^0(S, \Sigma, \mu)$, in case $f_1 =_{a.e.} f_2$ and $g_1 =_{a.e.} g_2$, then: $f_1 \leq_{a.e.} g_1 \implies f_2 \leq_{a.e.} g_2$ (Proof similar to the one seen in remark 2.6): this definition is well defined.

It can be shown that this notion of ordering in $L^0(S, \Sigma, \mu)$ behaves like an ordering is expected to behave when having the example of $\mathbb{R}$ in mind. The next lemma will help us prove an intuitive property we expect from an ordering.

Lemma 2.10. *For f,g,h* $\in \mathscr{L}^0(S,\Sigma,\mu)$ $f \leq_{a.e.} g \leq_{a.e.} h \implies f \leq_{a.e.} h$.

Proof. Let$s \in \{f > h\}$, then s must be in one of the following sets:

(1) $s \in \{g \leq h \text{ and } g < f\}$ or
(2) $s \in \{h \leq g \text{ and } g < f\}$ or
(3) s $\in \{h < g \text{ and } f \leq g\}$.

Note that $\{g \leq h \text{ and } g < f\} \subseteq \{g < f\}, \{h \leq g \text{ and } g < f\} \subseteq \{g < f\}$ and $\{h < g \text{ and } f \leq g\} \subseteq \{h < g\}$. It follows that:

$$
\begin{aligned}
0 &\leq \mu(\{f > h\}) \\
&\leq \mu(\{g \leq h \text{ and } g < f\} \cup \{h \leq g \text{ and } g < f\} \cup \{h < g \text{ and } f \leq g\}) \\
&\leq \mu(\{g \leq h \text{ and } g < f\} + \mu(\{h \leq g \text{ and } g < f\}) + \mu(\{h < g \text{ and } f \leq g\}) \\
&\leq \mu(\{g < f\}) + \mu(\{g < f\}) + \mu(\{h < g\}) = 0 + 0 + 0 = 0.
\end{aligned}
$$

□

Theorem 2.11. *Let* $u, v, w \in L^0(S, \Sigma, \mu)$*. Then* $u \leq v \leq w \implies u \leq w$.

Proof. From theorem 2.10 and definition 2.8 it follows that if: $u = \dot{f}, v = \dot{g}, w = \dot{h}$, then: $u \leq v \leq w \iff f \leq_{a.e.} g \leq_{a.e.} h \implies f \leq_{a.e.} h \iff u \leq w$. □

Remark 2.12. Let $u, v, w \in L^0(S, \Sigma, \mu)$

- Using similar proving techniques as in theorem 2.10 one can prove that:
  $u \leq u$
  $u \leq v \leq u \implies u = v$
  $u \leq v \implies u + w \leq v + w$
  $0 \leq u \implies 0 \leq cu, c$ non-negative scalar
  Because of the last two properties above, we can call $L^0(S, \Sigma, \mu)$ (per definition) a **partially ordered linear space**.

Having established that L^0 is an partially ordered linear space and that it, until now, verifies our intuitive feeling of how an ordering behaves, one might also hope to find that we can define a minimum/maximum of two elements in our ordered space. It turns out that since we can do this on $\mathscr{L}^0(S, \Sigma, \mu)$, we also have this property on $L^0(S, \Sigma, \mu)$. Such ordered spaces where we can define minima and maxima are called Riesz spaces.

Definition 2.13. *A set X is called a* Riesz space *if it is a partially ordered linear space, where* $\forall x, y \in X : \exists z, w \in X$ *so that* $\forall a \in X$*:*

$$
\begin{aligned}
z \leq a &\iff x \leq a \text{ and } y \leq a \\
a \leq w &\iff a \leq w \text{ and } a \leq w.
\end{aligned}
$$

We denote : $z = \sup\{x, y\}$ *and* $w = \inf\{x, y\}$.

Before we can prove that $L^0(S, \Sigma, \mu)$ is a Riesz space, we will prove a well-known equivalent characterisation of a maximum of two elements in an a.e. context.

Lemma 2.14. *For* $f_1, f_2, g \in \mathscr{L}^0(S,\Sigma,\mu)$*:*

$$f_1 \leq_{a.e.} g \text{ and } f_2 \leq_{a.e.} g \iff \max(f_1, f_2) \leq_{a.e.} g.$$

Proof. $\implies$
First note, that if f_1, f_2 are measurable, $\max(f_1, f_2)$ is measurable as well and so we can consider $s \in \{\max(f_1, f_2) > g\} \in \Sigma$. Then:

(1) $s \in \{f_1 > g \text{ and } f_2 > g\}$ or
(2) $s \in \{f_1 > g \text{ and } f_2 \leq g\}$ or
(3) $s \in \{f_1 \leq g \text{ and } f_2 > g\}$.

Hence:

$$\begin{aligned}0 &\leq \mu(\{\max(f_1, f_2) > g\}) \\ &\leq \mu(\{f_1 > g \text{ and } f_2 > g\} \cup \{f_1 > g \text{ and } f_2 \leq g\} \cup \{f_1 \leq g \text{ and } f_2 > g\}) \\ &\leq 2\mu(\{f_1 > g\}) + \mu(\{f_2 > g\}) = 0.\end{aligned}$$

$\Longleftarrow$

Given is that $\mu\{\max(f_1, f_2) > g\}$=0. We know $\{f_1 > g\} \subseteq \{\max(f_1, f_2) > g\}$, so $\mu(\{f_1 > g\}) = 0$. Analogous argument for f_2. $\square$

Given lemma 2.14, we can now swiftly prove that L^0 indeed is a Riesz space.

Theorem 2.15. *$L^0(S, \Sigma, \mu)$ is a Riesz space.*

Proof. Let $u = \dot{f}, v = \dot{g} \in L^0(S, \Sigma, \mu)$. I claim that $\sup\{u, v\} = \dot{\max(f, g)}$.
Indeed: Let $w = \dot{h} \in L^0(S, \Sigma, \mu)$, then:
$\dot{\max(f, g)} \leq w \iff \max(f, g) \leq_{a.e.} h \iff f \leq_{a.e.} h$ and $g \leq_{a.e.} h \iff u \leq w$ and $v \leq w$.
Analogously, it can be proven that $\inf\{u, v\} = \dot{\min(f, g)}$. $\square$

Now that we have established that L^0 is a Riesz space, we can now define identities which follow (almost) directly from $\mathscr{L}^0$ which involve absolute values. The absolute value is formally defined below.

Definition 2.16. *For $u \in L^0(S, \Sigma, \mu)$: $|u| = \sup\{u, -u\}$.*

Remark 2.17. Note that this definition is analogous to the definition for $f \in \mathscr{L}^0$: $|f(x)| = \max(f(x), -f(x))$.

Below we will prove that if $u = \dot{f}$, then $|u|$ is just the equivalence class containing $|f|$.

Theorem 2.18. *Let $u = \dot{f}, v = \dot{g} \in L^0(S, \Sigma, \mu), c \in \mathbb{R}$. Then the following identities hold:*
(1) $|(\dot{f})| = (\dot{|f|})$ *(We can take the dot out of the absolute value)*
(2) $|cu|$=$|c||u|$
(3) $\sup\{u, v\} = \frac{1}{2}(u + v + |u - v|)$
(4) $\inf\{u, v\} = \frac{1}{2}(u + v - |u - v|)$
(5) $|u + v| \leq |u| + |v|$.

Proof. We can prove that first identity as follows:
$|(\dot{f})| \overset{\text{def } 2.16}{=} \sup\{\dot{f}, \dot{-f}\} \leq \dot{g} \overset{\text{def } 2.13}{\iff} \dot{f} \leq \dot{g}$ and $\dot{-f} \leq \dot{g} \overset{\text{def } 2.8}{\iff} f \leq_{a.e.} g$ and $-f \leq_{a.e.} g \overset{\text{theorem } 2.14}{\iff} \max(f, -f) \leq_{a.e.} g \overset{\text{def } 2.8}{\iff} (\dot{|f|}) \leq \dot{g}$.
By using that: $|cf| = c|f|, \max(f, g) = \frac{1}{2}(f + g + |f - g|), \min(f, g) = \frac{1}{2}(f + g - |f - g|), |f + g| \leq |f| + |g|$ for $f, g \in \mathscr{L}^0(S, \Sigma, \mu), c \in \mathbb{R}$, the proofs of the other identities are analogous to that of the first identity. $\square$

We know that for $f \in \mathscr{L}^0 : f^+(x) \overset{\text{def}}{=} \max(f(x), 0), f^-(x) \overset{\text{def}}{=} \max(-f(x), 0)$ so that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. From the previous theorems one might expect that if we, for $u \in L^0$,define u^+ analogous to it's counterpart in $\mathscr{L}^0$, that these identities also hold in L^0. This is indeed the case and is stated in the next definition and theorem. The proof of the theorem is similar to that of theorem 2.18.

Definition 2.19. *For $u \in L^0(S, \Sigma, \mu)$:*

$$u^+ = \sup\{u, \dot{0}\} \text{ and } u^- = \sup(-u, \dot{0}).$$

Theorem 2.20. *For $u \in L^0(S, \Sigma, \mu)$:*

(1) $u = u^+ - u^-$
(2) $|u| = u^+ + u^-$.

3. Topology of convergence in measure

In the previous section we established that $L^0(S, \Sigma, \mu)$ has a linear structure and is ordered. The natural next questions that arise are questions regarding defining (non-trivial) topologies on this space. In this section we will define the topology of convergence in measure on $L^0(S, \Sigma, \mu)$. This topology will generalise the well known notion of convergence in probability to more general measure spaces.

We will define a topology on $\mathscr{L}^0$, which we can then extend to L^0. The topology is defined via a set of functions on $\mathscr{L}^0$ whose output is on the non-negative real numbers line.

Definition 3.1. *Let $F \in \Sigma$ such that $\mu(F) < \infty$, then $\tau_F : \mathscr{L}^0(S, \Sigma, \mu) \to \mathbb{R}_{\geq 0}$:*

$$\tau_F(f) = \int_S \min(|f|, 1_F) d\mu.$$

We set: $P = \{\tau_F : \mu(F) < \infty\}$ as the set of all such functions for (S, Σ, μ).

Remark 3.2. This definition is well defined:

- Since $|f|$ and 1_F are two measurable functions, $\min(|f|, 1_F)$ is measurable as well.
- Since $\min(|f|, 1_F) \leq 1_F$ on S:
  $\int_S \min(|f|, 1_F) d\mu \leq \int_S 1_F d\mu = \mu(F) < \infty$.

For a given function as defined in definition 3.1, it turns out that the triangle inequality holds.

Theorem 3.3. *For f,g $\in \mathscr{L}^0(S, \Sigma, \mu)$:*

$$\tau_F(f + g) \leq \tau_F(f) + \tau_F(g).$$

Proof. Note that:
$0 \leq \min(|f + g|, 1_F) \leq \min(|f| + |g|, 1_F) \leq \min(|f|, 1_F) + \min(|g|, 1_F)$.
This last inequality can be verified by checking every possible case individually. Since the integral is an increasing function and is linear, we obtain the identity by taking the integral on the left side and right side of the above inequalities. □

From the definition of τ_F (definition 3.1), one might wonder whether we can define a metric on $\mathscr{L}^0$ using this function. After all, when we set:

$$\phi_F(f, g) = \tau_F(f - g)$$

we then have the following properties:

(1) $\phi_F(f, h) = \tau_F(f - g + g - h) \overset{\text{thm } 3.3}{\leq} \tau_F(f - g) + \tau_F(g - h) = \phi_F(f, g) + \phi_F(g, h)$
(2) $\phi_F(f, g) = \tau_F(f - g) \geq 0$
(3) $\phi_F(f, g) = \phi_F(g, f)$.

But now consider the case that $S = \mathbb{R}$, $\Sigma = \mathscr{B}(\mathbb{R})$, $\mu = \lambda$ (Lebesgue measure), $F = (0, 1]$ and take $f(x) = 0$ and $g(x) = 1_{\{0\}}(x)$. Then since $f(x) =_{a.e.} g(x)$: $|f - g| =_{a.e.} 0$. And so: $\phi_F(f, g) = 0$ while f $\neq$ g in $\mathscr{L}^0$.
Another problem that arises is that $\min(|f|, 1_F)$ sets f to 0 outside of F, so any difference between two functions outside of F will not be considered. This can be seen, for example, in case we again have: $S = \mathbb{R}$, $\Sigma = \mathscr{B}(\mathbb{R})$, $\mu = \lambda$ (Lebesgue measure), $F = [0, 1)$ but now set $f(x) = 1_{[1,\infty)}$ and $g(x) = 0$. Then $f \neq g$, but since

$\min(|f-g|, 1_F)(x) = 0 : \phi_F(f,g) = 0$ for this case. This shows that ϕ_F (in its current form) is not a metric in every case. But since it does have the three listed properties above, we can call ϕ_F (per definition) a **pseudometric** on $\mathscr{L}^0$.

Using all of the above, we can now define the topology of convergence in measure on $\mathscr{L}^0(S, \Sigma, \mu)$.

Definition 3.4. *The topology of convergence on $\mathscr{L}^0$ is defined as follows:*
$G \subseteq \mathscr{L}^0$ is called open $\iff$
$\forall f \in G \quad \exists F_f \in \Sigma \quad (\mu(F_f) < \infty)$ *and* $\exists \delta_f > 0 : \phi_{F_f}(f,g) < \delta_f \implies g \in G$.

Remark 3.5. This definition is reminiscent of the definition of a set being open in $\mathbb{R}$. Here, δ_f serves the same purpose as ϵ and $\phi_{F_f}(f,g) < \delta_f \implies g \in G$ the same purpose as a open ball with radius ϵ around a point x in an open subset of $\mathbb{R}$ having to be fully contained in that open set.

Theorem 3.6. *The topology of convergence of measure on $\mathscr{L}^0$ is a topology on $\mathscr{L}^0$.*

Proof.
(1) That $\varnothing$ and $\mathscr{L}^0$ are open in $\mathscr{L}^0$ follow directly from the definition.
(2) Let $G_i \subseteq \mathscr{L}^0$ be open $\forall\ i \in I$. Let $f \in \bigcup\limits_{i \in I} G_i$, then per definition $f \in G_j$ for some $j \in I$, so $\exists F_f, \delta_f > 0$ such that $\phi_{F_f}(f,g) < \delta_f \to g \in G_j \subseteq \bigcup\limits_{i \in I} G_i$
(3) Let $G_i \subseteq \mathscr{L}^0$ be open for $i \in \{1,..,n\}$. Let $f \in \bigcap\limits_{i=1}^{n} G_i$, then $f \in G_i$ for all $i = 1,..,n$ and correspondingly (by the definition of openness) we have F_f^i and δ_f^i for $i = 1,..,n$.
Note that $\mu(\bigcup\limits_{i=1}^{n} F_f^i) \leq \sum\limits_{i=1}^{n} \mu(F_f^i) < \infty$.
Choose $F_f = \bigcup\limits_{i=1}^{n} F_f^i$ and $\delta_f = \min(\delta_f^1, ..., \delta_f^n)$.
Since $\min(|f-g|, 1_{F_f^i}) \leq \min(|f-g|, 1_{\bigcup_{i=1}^{n} F_f^i})$ and the integral is increasing:
$\phi_{F_f^i}(f,g) \leq \phi_{\bigcup_{i=1}^{n} F_f^i}(f,g)$. Hence for any $i = 1,..,n$:
$\phi_{F_f^i}(f,g) \leq \phi_{F_f}(f,g) < \delta_f < \delta_f^i$ gives $g \in G_i$ (by assumed openness of G_i) for $i = 1,..,n$.
In conclusion: $\bigcap\limits_{i=1}^{n} G_i$ is open.

□

Now that we have established this topology on $\mathscr{L}^0$, we can now extend this to L^0. Note that for $f, g \in \mathscr{L}^0$: if $f =_{a.e.} g$, then $\min(|f|, 1_F) =_{a.e.} \min(|g|, 1_F)$, hence $\tau_F(f) = \tau_F(g)$ (see definition 3.1). It follows that the following definitions are well defined:

Definition 3.7. *Let $F \in \Sigma$ such that $\mu(F) < \infty$, then: $\bar{\tau}_F : L^0(S, \Sigma, \mu) \to \mathbb{R}_{\geq 0}$:*
$$\bar{\tau}_F(\dot{f}) = \tau_F(f) \text{ and } \bar{\phi}_F(\dot{f}, \dot{g}) = \phi_F(f,g).$$

Because of these definitions, we can now easily extend the topology of convergence in measure structure on $\mathscr{L}^0$ to L^0 without losing any properties that were proven before. The proofs of the theorems below are analogous to the $\mathscr{L}^0$ cases.

Theorem 3.8. *$\bar{\phi}_F$ is a pseudometric on L^0*

Definition 3.9. *The topology of convergence on L^0 is defined as follows:*
$G \subseteq L^0$ is called open $\iff$
$\forall u \in G \quad \exists F_u \in \Sigma \quad (\mu(F_u) < \infty)$ *and* $\exists \delta_u > 0 : \bar{\phi}_{Fu}(v, u) < \delta_u \implies v \in G.$

Theorem 3.10. *The topology of convergence in measure on L^0 is a topology on L^0.*

4. Convergence in measure

In the previous section, we have defined a topology on $\mathscr{L}^0(S,\Sigma,\mu)$ called the topology of convergence in measure. In this section, we will work out the notion of sequential convergence with respect to this topology.

Definition 4.1. *Let $f_n, f \in \mathscr{L}^0(S,\Sigma,\mu)$ for n=1,2,... . Then $f_n \to f$ in measure $\iff \forall F \in \Sigma$ such that $\mu(F) < \infty : \phi_F(f_n, f) \to 0 \quad (n \to \infty)$.*

Proposition 4.2. *Definition 4.1 is equivalent to the definition of convergence in general topological spaces (see Appendix A.8).*

Proof. $\implies$
Assume definition 4.1 and let $G \subseteq \mathscr{L}^0(S,\Sigma,\mu)$ be an open neighbourhood of $f \in \mathscr{L}^0(S,\Sigma,\mu)$. Then by definition 3.4, for $f \in G \quad \exists F_f \in \Sigma \quad (\mu(F_f) < \infty)$ and $\exists \delta_f > 0$ such that: $v \in \mathscr{L}^0(S,\Sigma,\mu)$ and $\phi_{F_f}(v,f) < \delta_f \implies v \in G$. By our assumption it holds that $\phi_{F_f}(f_n, f) \to 0$. It holds that $\exists N : \forall n \geq N \ \phi_{F_f}(f_n, f) < \delta_f$ which implies that $f_n \in G$ for $n \geq N$.
$\impliedby$
Suppose that for any open neighbourhood G of $f \in \mathscr{L}^0(S,\Sigma,\mu) \quad \exists N : \forall n \geq N : f_n \in G$. Let $\epsilon > 0$ and $F \in \Sigma$ (with finite measure). Define:

$$U(f;\phi_F;\epsilon) = \{g \in \mathscr{L}^0(S,\Sigma,\mu) : \phi_F(f,g) < \epsilon\}.$$

Then $U(f;\phi_F;\epsilon)$ is open and so for by our assumption, for $n \geq N : \phi_F(f_n, f) < \epsilon$. As $\epsilon > 0$ and $F \in \Sigma$ (with finite measure) were arbitrary, the proof is now complete. □

Remark 4.3. Convergence as defined in definition 4.1, does not have unique limits in general. Take, for example, $([0,1), \mathscr{B}([0,1)), \lambda)$ with $f_n(x) = \frac{1}{n}, f(x) = 0, g(x) = 1_{\{0\}}$. Then $f_n \to f$ in measure as well as $f_n \to g$ in measure, while $f \neq g$.

It turns out that this type of convergence is weaker than convergence a.e. and this is given in the following theorem and example.

Theorem 4.4. *If $(f_n)_{n\geq 1} \subset \mathscr{L}^0$, then $f_n \to_{a.e.} f \implies f_n \to f$ in measure .*

Proof. Note that $f_n \to_{a.e.} f \iff \mu(\{\lim_{n\to\infty} f_n \neq f\}) = 0 \iff \mu(\{\lim_{n\to\infty} |f_n - f| \neq 0\}) = 0$. And so since $\{|f_n - f| > 0\} \supseteq \{\min(|f_n - f|, 1_F) > 0\}$ we have that $\mu(\{|f_n - f| > 0\}) \geq \mu(\{\min(|f_n - f|, 1_F) > 0\}) \geq 0$ and as this inequality holds for all n, we have hence that $\min(|f_n - f|, 1_F) \to_{a.e.} 0$. Since $\min(|f_n - f|, 1_F)$ is dominated by 1_F, by the DCT (Dominated Convergence Theorem), we obtain:

$$\lim_{n\to\infty} \phi_F(f_n, f) = \int 0 d\mu = 0.$$

□

The following counterexample to the converse of theorem 4.4 is a worked out version of the one found at [1, p. 174 245Cc].

Example 4.5. Let μ be the Lebesgue measure on [0,1] and define:

$$f_n(x) = 2^m 1_{[2^{-m}k, 2^{-m}(k+1)]}(x)$$

where $n+1 = 2^m + k$ (n, m, k integers) and $0 \leq k < 2^m$. Note that $2^m \geq 1$.
This results into that for $F \in \Sigma$ with $\mu(F) < \infty$:
If $x \in [2^{-m}k, 2^{-m}(k+1)]$, then $\min(2^m 1_{[2^{-m}k, 2^{-m}(k+1)]}, 1_F)(x) \leq 1$
If $x \notin [2^{-m}k, 2^{-m}(k+1)]$, then $\min(2^m 1_{[2^{-m}k, 2^{-m}(k+1)]}, 1_F)(x) = 0$.
Hence we obtain:

$$\begin{aligned} 0 \leq \phi_F(f_n, 0) &= \int_{[0,1]} \min(2^m 1_{[2^{-m}k, 2^{-m}(k+1)]}, 1_F) d\mu \\ &\leq \int_{[0,1]} 1_{[2^{-m}k, 2^{-m}(k+1)]} d\mu \\ &= \mu([2^{-m}k, 2^{-m}(k+1)]) = 2^{-m}. \end{aligned}$$

Since m goes to infinity when n goes to infinity, we see that $\phi_F(f_n, 0) \to 0$. But also $\limsup_{n\to\infty} f_n = \infty$, hence no a.e. convergence.

In order to extend these results to L^0, we first need to make sure that a.e. convergence does not depend on the represented element in the equivalence class.

Theorem 4.6. *Let $(f_n)_{n\geq 1}, (g_n)_{n\geq 1} \subseteq \mathscr{L}^0$: If $f_n =_{a.e.} g_n \quad \forall n \geq 1, f =_{a.e.} g$ and $f_n \to_{a.e.} f$ then $g_n \to_{a.e.} g$*

Proof.

$$\begin{aligned} \mu(\{\lim_{n\to\infty} g_n \neq g\}) &= \mu(\{\lim_{n\to\infty} g_n \neq f\}) && (g =_{a.e.} f) \\ &= \mu(\{\lim_{n\to\infty} f_n \neq f\}) = 0 && (g_n =_{a.e.} f_n) \end{aligned}$$

□

Now we give a definition of a.e. convergence in L^0 which is well defined (we call this order*-convergent).

Definition 4.7. *Let $f, f_n \in \mathscr{L}^0$, then $(\dot{f}_n)_{n\geq 1}$ is order*-convergent to $\dot{f} \iff \lim_{n\to\infty} f_n =_{a.e.} f$*

Now that we have extended the concept of almost everywhere convergence in $\mathscr{L}^0(S, \Sigma, \mu)$ to $L^0(S, \Sigma, \mu)$, we can now extend most of the previous results in this section to $L^0(S, \Sigma, \mu)$, where the proofs are analogous to the $\mathscr{L}^0$ case.

Definition 4.8. *Let $\dot{f}_n, \dot{f} \in L^0(S, \Sigma, \mu)$ for n=1,2,... . Then $\dot{f}_n \to \dot{f}$ in measure $\iff \forall F \in \Sigma$ such that $\mu(F) < \infty : \bar{\phi}_F(\dot{f}_n, \dot{f}) \to 0 \quad (n \to \infty)$.*

Proposition 4.9. *Definition 4.8 is equivalent to the definition of convergence in general topological spaces (see Appendix A.8).*

Theorem 4.10. *If $(\dot{f}_n)_{n\geq 1} \subseteq L^0$, then $\dot{f}_n$ is order*-convergent to $\dot{f}$ implies that $\dot{f}_n \to \dot{f}$ in measure but the converse of generally false.*

Remark 4.11. One of the only results that does not extend to L^0 is the one stated in remark 4.3. But still, in general, $L^0(S, \Sigma, \mu)$ does not have unique limits. Consider the case: $L^0(S, \{S, \varnothing\}, \mu)$, where $\mu(S) = \infty$. Then any sequence $(\dot{f}_n)_{n\geq 1}$ converges in measure to any $\dot{f} \in L^0(S, \{S, \varnothing\}, \mu)$.

The most obvious solution to not obtaining unique limits is requiring that the topology of convergence in measure is Hausdorff. Define $\mathscr{T}$ as the set containing all open sets with respect to the topology of convergence in measure on $L^0(S, \Sigma, \mu)$. It turns out that that requiring $\mathscr{T}$ to be Hausdorff is equivalent to requiring (S, Σ, μ) to be semi-finite which is formally defined below.

Definition 4.12. *A measure space (S, Σ, μ) is defined to be semi-finite if $\forall A \in \Sigma$ with $\mu(A) = \infty \quad \exists B \in \Sigma$ with $B \subseteq A$ and $\mu(B) \in (0, \infty)$.*

Example 4.13. The measure space as described at remark 4.11 (namely $(S, \{S, \varnothing\}, \mu)$, where $\mu(S) = \infty$) is not semi-finite.

The proof below of theorem 4.14, is the proof found at [1, p. 175 245Ea] with some details worked out.

Theorem 4.14. *(S, Σ, μ) is semi-finite $\iff$ $\mathscr{T}$ is Hausdorff.*

Proof. $\implies$
Suppose (S, Σ, μ) is semi-finite and that $u = \dot{f}, v = \dot{g}$ are distinct members of L^0. Then it holds that $\mu(\{x : f(x) \neq g(x)\}) > 0 \quad$ (if not: $f =_{a.e.} g \implies \dot{f} = \dot{g}$). As (S, Σ, μ) is semi-finite: $\exists F \in \Sigma : \mu(F) < \infty$ with $F \subseteq \{x : f(x) \neq g(x)\}$. We know that $\bar{\phi}_F(u, v) \geq 0$, where $\bar{\phi}_F(u, v) = 0 \iff f =_{a.e.} g$ on F which is not possible in this case, so $\bar{\phi}_F(u, v) > 0$. As u, v were arbitrary, $\mathscr{T}$ is Hausdorff by [1, p. 506 2A3L].
$\impliedby$
Suppose that $\mathscr{T}$ is Hausdorff and that $E \in \Sigma$ with $\mu(E) > 0$. Then $u = \dot{1}_E \neq \dot{0}$ and so there are open neighbourhoods G_u of u and G_0 of 0 such that $G_u \cap G_0 = \varnothing$. Since G_u is open: $\exists F_u \in \Sigma$ (finite measure), and $\delta_u > 0$ such that: $\bar{\phi}_{F_u}(u, v) < \delta_u \implies v \in G_u$. As $\dot{0} \notin G_u : \bar{\phi}_{F_u}(u, \dot{0}) > \delta_u > 0$. Since $u = \dot{1}_E$ we hence see that: $\mu(E \cap F_u) = \bar{\phi}_{F_u}(u, \dot{0}) > 0$. And so: as $F_u \cap E$ has finite measure and E was arbitrary, our proof is now complete. $\square$

5. Continuity on L^0

Now that we have established a topology on L^0 (topology of convergence in measure), a natural question that arises is: When are maps from L^0 (or product space of L^0) to L^0 continuous? We know for the usual topology on $\mathbb{R}^2$ for instance that the maps $(x, y) \mapsto x + y$ and $(x, y) \mapsto xy$ are continuous, but are these maps also continuous when we have L^0 as our (co-)domain and the topology of convergence in measure? To answer these questions, we first need to get familiar with the concept of a topology being linear.

That the map $(x, y) \mapsto x + y$ is continuous on a topological space might seem natural, but this is actually a special property and is entailed in the following definition.

Definition 5.1. *A linear topological space is a linear space U over $\mathbb{R}$ (or $\mathbb{C}$) together with a topology $\mathscr{T}$, where the maps:*

$$U \times U \to U \text{ given by } (u, v) \mapsto u + v$$
$$\mathbb{R} \times U \to U \text{ given by } (\alpha, u) \mapsto \alpha u$$

are both continuous. $U \times U$ and $\mathbb{R} \times U$ are given with their product topologies.

Remark 5.2. Since the composition of continuous functions is continuous, the map $(u, v) \to u - \alpha v$ is also continuous

Thus far we have only worked with L^0 and the topology of convergence is measure, which is defined via a set of pseudo-metrics $P = \{\phi_F : F \in \Sigma, \mu(F) < \infty\}$ and these pseudo-metrics were on their part defined via the functionals, τ_F, from definition 3.1. This topology is actually a special case of a class of topologies on a linear space U over $\mathbb{R}$ defined via a family of functionals $\tau : U \to [0, \infty]$ and where $\phi(u, v) = \tau(u - v)$ defines a pseudo-metric. Given a family of functionals T and family of corresponding pseudo-metrics P, one can define openness analogous to definition 3.9, namely:
On a general space S together with the topology generated the set of pseudo-metrics $P = \{\phi_i : i \in I\}$ (I is some index set):

$$G \subseteq S \text{ is called open}$$
$$\Longleftrightarrow$$
$$\forall u \in S \quad \exists \phi_i \in P \text{ and } \exists \delta_u > 0 : v \in S \text{ and } \phi_i(u, v) < \delta_u \implies v \in G.$$

This class of topologies has special properties which makes (dis)proving it is a linear topology easier.
The proof of the following theorem can be found at [1, p. 514 2A5B].

Theorem 5.3. *Let U be a linear space over $\mathbb{R}$, and T a family of functionals $\tau : U \to [0, \infty]$ and suppose all τ have the following properties:*

(1) $\tau(u + v) \leq \tau(u) + \tau(v)$
(2) $\tau(\alpha u) \leq \tau(u)$,*where* $|\alpha| \leq 1$
(3) $\lim_{\alpha \to 0} \tau(\alpha u) = 0$

For $\tau \in T$, we define $\phi_\tau : U \times U \to [0, \infty]$ by $\phi_\tau(u, v) = \tau(u - v)$. Then each ϕ_τ is a pseudometric on U and the topology defined by P=$\{\phi_\tau : \tau \in T\}$ gives a linear topological space

With the theorem above, we can now more easily prove that L^0 indeed is a linear topology.

The proof below of theorem 5.4 is a worked out version of the one found at [1, p. 174 245Da].

Theorem 5.4. *Let (X, Σ, μ) be a measure space. Then the topology of convergence in measure is a linear topology on $L^0(\mu)$.*

Proof. By theorem 5.3, we only have to check that $\bar{\tau}_F$ as defined in theorem 3.7 satisfies the 3 conditions.

(1) By the definition of $\bar{\tau}F$ and theorem 3.3, it immediately follows that: $\bar{\tau}_F(\dot{f}+\dot{g}) \leq \bar{\tau}_F(\dot{f}) + \bar{\tau}_F(\dot{g})$.
(2) Let $u = \dot{f} \in L^0$ and $c \in \mathbb{R} : |c| \leq 1$. Then since:
$|cf| \leq |f| \implies \min(|cf|, 1_F) \leq_{a.e.} \min(|f|, 1_F)$, we get that (since the integral is increasing) $\tau_F(cf) \leq \tau_F(f)$ and so by definition 3.7: $\bar{\tau}_F(cu) \leq \bar{\tau}_F(u)$
(3) Given $u = \dot{f} \in L^0$ and $\epsilon > 0$. Since $\lim\limits_{n\to\infty} 2^{-n}f =_{a.e.} 0$ (pointwise). We have that $\lim\limits_{n\to\infty} \min(|2^{-n}f|, 1_F) =_{a.e.} 0$ and so by the DCT:
$$\lim_{n\to\infty} \bar{\tau}_F(2^{-n}u) = \int 0 d\mu = 0$$
hence per definition of a converging sequence in $\mathbb{R}$": for any $\epsilon > 0$ there exists a n such that $\bar{\tau}_F(2^{-n}u) < \epsilon$. Now take $c \in \mathbb{R}$ such that $|c| \leq 2^{-n}$. Then since the integral is increasing:
$$\bar{\tau}_F(cu) \overset{\text{def3.1}}{=} \bar{\tau}_F(|c|u) \leq \bar{\tau}_F(2^{-n}u) < \epsilon$$
As ϵ was arbitrary we have now proven per definition that $\lim\limits_{c\to 0} \bar{\tau}_F(cu) = 0$.

□

In order to prove that more maps are continuous (such as the absolute value function) it will be more convenient for us to once again prove some results related to topologies based on pseudometrics, as introduced below remark 5.2, which will make our lives easier. The theorems we will prove are reminiscent of different characterisations a function $f : \mathbb{R} \to \mathbb{R}$ of being continuous (we can say that the inverse image of an open set is open or we could equivalently use the epsilon/delta definition). First we will, as done in the cases of $f : \mathbb{R} \to \mathbb{R}$, introduce the notion of an "open ball" in the setting of topologies based on pseudo-metrics.

Definition 5.5. *Let X be a topology defined on P, a non-empty set of pseudometrics. Then for $\phi_0, .., \phi_n \in P$, $x \in X$ and $\epsilon > 0$:*
$U(x; \phi_0, ..., \phi_n; \epsilon) = \{y \in X : \max\limits_{i\leq n} \phi_i(y, x) < \epsilon\}$.

As done in the case of $\mathbb{R}$, one can define openness of a set via "open balls":

Definition 5.6. *Under the same conditions as definition 5.5: $G \subseteq X$ is open $\iff \forall x \in G \quad \exists \phi_0, ..., \phi_n \in P$ and $\delta > 0 : U(x; \phi_0, ..., \phi_n; \delta) \subseteq G$*

One might readily notice that this definition is not exactly on par with the characterisation of a set being open in L^0 given earlier. It can be shown that these definitions are equivalent and below it will be shown that they are equivalent in the case of L^0

Proposition 5.7. *The definition used in definition 3.9 is equivalent with definition 5.6 in the L^0 case.*

Proof. $\Longrightarrow$
Assume definition 3.9 and let $G \subseteq L^0$ be open. Then $\forall f \in G \quad \exists \phi_{F_f}$ and $\delta_f > 0$: $U(f; \phi_{F_f}; \delta_f) \subseteq G$ and this satisfies definition 5.6.
$\Longleftarrow$
Now assume $G \subseteq L^0$ is open in the sense of definition 5.6. Then:

$$\forall f \in G \quad \exists \phi_{F_0}, ..., \phi_{F_n} \in P \text{ and } \delta > 0 : U(f; \phi_{F_0}, ..., \phi_{F_n}; \delta) \subseteq G.$$

Take $F_f = \bigcup_{i=0}^{n} F_i$. Then since $\phi_{F_f} \geq \phi_{F_i} \quad \forall i \geq 0$:

$$U(f; \phi_{F_f}; \delta) \subseteq U(f; \phi_{F_0}, ..., \phi_{F_n}; \delta) \subseteq G.$$

□

Now that we have established the notion of being open via "open balls", it should not come as a surprise that these "open balls" themselves are open.
The proof below of theorem 5.8 can also be found at [1, p. 505 2A3G].

Theorem 5.8. *Under the same conditions as definition 5.5, $U(x; \phi_0, .., \phi_n; \epsilon)$ is open*

Proof. Take $y \in U(x; \phi_0, .., \phi_n; \epsilon)$. Set $\nu = \max_{i \leq n} \phi_i(y, x)$ and $\delta = \epsilon - \nu$.
In case $z \in U(y; \phi_0, .., \phi_n; \delta)$, then

$$\phi_i(z, x) \leq \phi_i(z, y) + \phi_i(y, x) < \delta + \nu = \epsilon \ \forall i = 1, .., n$$

as $\phi_i \leq \max_{i \leq n} \phi_i$ pointwise. Since i was arbitrary:

$$U(y; \phi_0, .., \phi_n; \delta) \subseteq U(x; \phi_0, .., \phi_n; \epsilon).$$

□

With the theorems above, we can now prove a characterisation of continuity in this context which is reminiscent of the ϵ, δ definition in $\mathbb{R}$.
The proof is a worked out version of [1, p. 505 2A3H].

Theorem 5.9. *Let X, Y be sets, where P and Θ denote the (non-empty) sets of pseudometrics on X and Y respectively and $\mathscr{X}, \mathscr{Y}$ the corresponding topologies. Then: $\phi : X \to Y$ is continuous $\iff$ $\forall x \in X, \theta \in \Theta, \epsilon > 0$ there exist $\phi_0, ..., \phi_n \in P, \delta > 0$ such that if $y \in X$ and $\max_{i \leq n} \phi_i(y, x) < \delta$, then $\theta(\phi(x), \phi(y)) < \epsilon$*

Proof. $\Longrightarrow$
Suppose ϕ is continuous and take $x \in X, \theta \in \Theta, \epsilon > 0$. By theorem 5.8, $U(\phi(x); \theta, \epsilon) \in \mathscr{Y}$. So per definition of continuity $G = \phi^{-1}(U(\phi(x); \theta, \epsilon) \in \mathscr{X}$. Now, let $x \in G$, so (since G is open) $\exists \phi_0, ..., \phi_n \in P, \delta > 0 : U(x; \phi_0, .., \phi_n; \delta) \subseteq G$. So whenever $y \in X$ and $\max_{i \leq n} \phi_i(y, x) < \delta : \theta(\phi(x), \phi(y)) < \epsilon$ holds.
$\Longleftarrow$
Take $H \in \mathscr{Y}$ and consider $G = \phi^{-1}(H)$. If $x \in G$, then $\phi(x) \in H$ and so, since H is open, $\exists \theta_0, .., \theta_n \in \Theta$ and $\epsilon > 0$ such that $U(\phi(x); \theta_0, .., \theta_n; \epsilon) \subseteq H$. By our assumption, it holds that for each $i \leq n \quad \exists \phi_{i0}, .., \phi_{im_i} \in P$ and $\delta_i > 0$ such that:

$$y \in X \text{ and } \max_{j \leq m_i} \phi_{ij}(y, x) < \delta_i \implies \theta_i(\phi(x), \phi(y)) < \epsilon.$$

Set $\delta = \min_{i \leq n} \delta_i$, then $U(x; \phi_{00}, .., \phi_{0m_0}, .., \phi_{nm_n}; \delta) \subseteq G$.
Indeed for $i \leq n$ and $y \in U(x; \phi_{00}, .., \phi_{0m_0}, .., \phi_{nm_n}; \delta)$:

$$\max_{j \leq m_i} \phi_{ij}(x, y) \leq \max_{j \leq m_i, i=1,..,n} \phi_{ij}(x, y) < \delta \leq \delta_i \implies \theta_i(\phi(x), \phi(y)) < \epsilon.$$

As i was arbitrary, this gives that $\phi(y) \in U(\phi(x); \theta_0, .., \theta_n; \epsilon) \subseteq H$, which gives: $y \in G$. Hence as $U(x; \phi_{00}, .., \phi_{0m_0}, .., \phi_{nm_n}; \delta) \subseteq G$, G is open by definition 5.6. □

Remark 5.10. Under the same circumstances as theorem 5.9, when we now have a map $\psi : X^k \to Y$ (where X^k denotes the k times product topology of X, where X is as defined in the theorem) it can be proven that the statements following are equivalent [1, p. 510 2A3T]:

$$\psi : X^k \to Y \text{ continuous}$$
$$\iff$$
$$\forall (x_1, ..., x_k) \in X^k, \theta \in \Theta, \epsilon > 0 : \exists \phi_{0j}, .., \phi_{nj} \in P \quad \exists \delta > 0 :$$
$$\max_{i \leq n} \phi_{ij}(y_j, x_j) < \delta \quad \forall j = 1, ..., k \implies \theta(\psi(x_1, .., x_k), \psi(y_1, .., y_k)) < \epsilon.$$

Now we can finally prove that more maps are continuous

Theorem 5.11. *The following maps are all continuous under the topology of convergence in measure of L^0 and the usual product topology on $(L^0)^2$:*

(1) $(u, v) \mapsto \sup\{u, v\}$
(2) $(u, v) \mapsto \inf\{u, v\}$
(3) $u \mapsto |u|$
(4) $u \mapsto u^+$
(5) $u \mapsto u^-$

Proof. We start by proving that $u \mapsto |u|$ is continuous.
For any $u, v \in L^0$, it holds that $||u| - |v|| \leq |u - v|$. Indeed: using theorem 2.18: $|u| = |(u - v) + v| \leq |u - v| + |v| \implies |u| - |v| \leq |u - v|$. By symmetry it also holds that $|v| - |u| \leq |u - v|$, hence proven that $||u| - |v|| \leq |u - v|$. And so (since the integral is increasing) $\bar{\phi}_F(|u|, |v|) \leq \bar{\phi}_F(u, v)$ (where F has finite measure). By continuity of the map $u \mapsto u$ (theorem 5.3), theorem 5.9 states that for any $\phi_F \in P, u \in L^0, \epsilon > 0$ we can find $\bar{\phi}_{F_0}, .., \bar{\phi}_{F_n} \in P$ and $\delta > 0$ such that: $\max_{i \leq n} \bar{\phi}_{F_i}(u, v) < \delta \implies \bar{\phi}_F(u, v) < \epsilon$. Since $\bar{\phi}(|u|, |v|) \leq \bar{\phi}_F(u, v)$ holds for any $u, v \in L^0$, we have that: $\max_{i \leq n} \bar{\phi}_{F_i}(u, v) < \delta \implies \bar{\phi}_F(|u|, |v|) \leq \bar{\phi}_F(u, v) < \epsilon$.
Hence proven by theorem 5.9, that $u \mapsto |u|$ is continuous.
Since sup and inf can be written in the form of theorem 2.18 and as addition and subtraction are continuous (theorem 5.4) as well as compositions of continuous functions being continuous: so are $\sup, \inf, ^+, ^-$. □

For maps from $\mathbb{R}^k$ to $\mathbb{R}$ such as $(x_1, .., x_k) \mapsto x_1$ and $(x_1, .., x_k) \mapsto x_1 \cdot x_k^5$ it is well established that they are continuous. One might ask themselves whether similar maps from $(L^0)^k \to L^0$ are continuous as well. It turns out that continuity of maps $\mathbb{R}^k \to \mathbb{R}$ implies continuity of analogous maps $(L^0)^k \to L^0$.
In order to make this result precise and then prove it, we first want to link measurable functions $f : X \to \mathbb{R}$ to measurable functions $h : \mathbb{R} \to \mathbb{R}$. This can be done by means of composing f with h. It turns out that this composition is again a measurable function.

Lemma 5.12. *Let (X, Σ, μ) be a measure space. Let $h : \mathbb{R} \to \mathbb{R}$ be measurable and $f : X \to \mathbb{R}$ also be measurable. Then $h \circ f$ is measurable as well.*

Proof. Let $H \in \mathscr{B}(\mathbb{R})$. Then since: $(h \circ f)^{-1}(H) = f^{-1}(h^{-1}(H))$, we have that since h is measurable $h^{-1}(H) \in \mathscr{B}(\mathbb{R})$ and so by using this fact and that f is also measurable, we get that $f^{-1}(h^{-1}(H)) \in \Sigma$. □

Remark 5.13.

- Using this result, it can be proven that if $h : \mathbb{R}^k \to \mathbb{R}$ is measurable and $f_i : X \to \mathbb{R}$ is measurable $\forall i = 1, .., k$, then $h \circ (f_1, .., f_k) : \mathbb{R} \to \mathbb{R}$ is measurable as well [2].
- As done in previous sections, as $h \circ f$ is in the space of measurable functions, we can translate it to L^0. We define the translation by $\bar{h}$:
  $h : \mathbb{R}^k \to \mathbb{R}$, then $\bar{h} : (L^0)^k \to L^0$:
  $$\bar{h}(\dot{f_1}, .., \dot{f_k})) = (h \circ (f_1, \dot{..}, f_k))$$
  This definition is well defined: Suppose $f_1 =_{a.e.} f'_1, ..., f_k =_{a.e.} f'_k$. Then:
  $$\begin{aligned} 0 &\leq \mu(\{h \circ (f_1, .., f_k) \neq h \circ (f'_1, .., f'_k\}) \\ &\leq \mu(\{f_1 \neq f'_1\} \cup ... \cup \{f_k \neq f'_k\}) \leq 0. \end{aligned}$$

It turns out that if h is continuous, then $\bar{h}$ is continuous as well.
The proof below is a generalisation of the proof seen at [1, p. 174 245Dc].

Theorem 5.14. *For any continuous function $h : \mathbb{R}^k \to \mathbb{R}$, $\bar{h}$ (as defined in remark 5.13) is continuous.*

Proof. Take $u_i = \dot{f_i} \in L^0$(for $i = 1, .., k$), $F \in \Sigma \quad (\mu(F) < \infty)$ and $\epsilon > 0$. Then by remark 5.10, it suffices to prove that $\exists \delta > 0 : \bar{\phi}_F(u_i, v_i) < \delta \quad \forall i = 1, .., k \implies \bar{\phi}_F(\bar{h}(v_1, .., v_k), \bar{h}(u_1, .., u_k)) < \epsilon$. We will do a proof by contradiction.
Suppose $\exists \delta > 0 : \bar{\phi}_F(u_i, v_i) < \delta \quad \forall i = 1, .., k \implies \bar{\phi}_F(\bar{h}(v_1, .., v_k), \bar{h}(u_1, .., u_k)) < \epsilon$ does not hold. Then for some $\epsilon > 0$ we can find $\forall n \in \mathbb{N}$ a $v_{in} = \dot{g_{in}}$ such that $\bar{\phi}_F(v_{in}, u_i) \leq 4^{-n} \quad \forall i = 1, .., k$ but $\bar{\phi}_F(\bar{h}(v_{1n}, .., v_{kn}), \bar{h}(u_1, .., u_k)) \geq \epsilon$. Set

$$E_{in} = \{x \in F : |g_{in}(x) - f_i(x)| \geq 2^{-n}\} \text{ where } i \in \{1, .., k\}.$$

Then we see that:

$$\begin{aligned} \bar{\phi}_F(v_{in}, u_i) &= \int_F \min(|g_{in} - f_i|, 1) d\mu \\ &\geq \int_F \min(|g_{in} - f_i|, 1) 1_{E_{in}} d\mu \\ &\geq 2^{-n} \int_F 1_{E_{in}} d\mu && (|g_{in} - f_i| \geq 2^{-n} \text{ on } E_{in} \text{ and } 1 \geq 2^{-n}) \\ &= 2^{-n} \mu(E_{in}) && (E_{in} \subseteq F) \end{aligned}$$

and so the following holds:

$$\mu(E_{in}) \leq 2^n \bar{\phi}_F(v_{in}, u_i) \leq 2^n \cdot 4^{-n} = 2^{-n} \qquad (\bar{\phi}_F(v_{in}, u_i) \leq 4^{-n})$$

It also holds that since E_{im} is measurable, so is $E_i = \bigcap_{n \geq 1} \bigcup_{m \geq n} E_{im}$ and that:

$$\mu(E_i) = \lim_{n \to \infty} \mu(\bigcup_{m \geq n} E_{im})$$

$$\leq \lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(E_{im})$$

$$\leq \lim_{n\to\infty} \sum_{m=n}^{\infty} 2^{-m} = 0 \qquad \text{(tail of a convergent series, hence 0).}$$

So E_i has 0 measure. And this implies that $E = \bigcup_{i=1}^{k} E_i$ also has 0 measure.
But $\lim_{n\to\infty} g_{in}(x) = f_i(x) \quad \forall x \in F \setminus E, i = 1, .., k$ since

$$F \setminus E = \{x \in F : \forall i = 1, .., k \quad \exists n \geq 1 \quad : \forall m \geq n \quad |g_{im}(x) - f_i(x)| \leq 2^{-m}\}$$

and since we assumed h is a continuous function from $\mathbb{R}^k$ to $\mathbb{R}$

$$\lim_{n\to\infty} h(g_{1n}(x), .., g_{kn}(x)) = h(f_1(x), .., f_k(x)) \quad \forall x \in F \setminus E.$$

$\bar{\phi}_F(\bar{h}(v_{1n}, .., v_{kn}), \bar{h}(u_1, .., u_k)) =$
$\int_{F\setminus E} \min(1, |h(g_{1n}, .., g_{kn}) - h(f_1, .., f_k)|) d\mu + \int_E \min(1, |h(g_{1n}, .., g_{kn}) - h(f_1, .., f_k)|) d\mu$
and that:

$$\int_E \min(1, |h(g_{1n}, ..g_{kn}) - h(f_1, .., f_k)|) d\mu \leq \int_E 1 d\mu = \mu(E) = 0$$

and also:

$$0 \leq \min(1, |h(g_{1n}, .., g_{kn}) - h(f_1, .., f_k)|) \leq |h(g_{1n}, .., g_{kn}) - h(f_1, .., f_k)|$$

so that by the squeeze theorem: $\lim_{n\to\infty} \min(1, |h(g_{1n}, .., g_{kn}) - h(f_1, .., f_k)|) = 0$. Hence by the DCT $(\min(1, |h(g_{1n}, .., g_{kn}) - h(f_1, .., f_k)| \leq 1)$:

$$\lim_{n\to\infty} \int_{F\setminus E} \min(1, |h(g_{1n}, .., g_{kn}) - h(f_1, .., f_k)|) d\mu = 0$$

so overall: $\lim_{n\to\infty} \bar{\phi}_F(\bar{h}(v_{1n}, .., v_{kn}), \bar{h}(u_1, .., u_k)) = 0$ and we have hence reached our contradiction. □

Remark 5.15. With theorem 5.14, we know have that continuity of maps $h : \mathbb{R}^k \to \mathbb{R}$ one to one translates to continuity in analogous cases for maps $(L^0)^k \to L^0$. Examples of now established continuous maps are $(L^0)^k \to L^0$ are:

(1) $(u, v) \mapsto u \cdot v$
(2) $(u, v) \mapsto u$
(3) $(u, v) \mapsto \cos(u)$.

6. L^0 METRIC SPACE

In previous sections we have defined pseudometrics on L^0, which on their part defined the topology of convergence in measure. We have also seen that these pseudometrics are not metrics in general, but we have thus far not explored whether we can use the defined pseudo-metrics to define a metric on L^0. It turns out that, under a certain assumption on our measure space (X, Σ, μ), we can indeed do this where the defined metric itself generates the same topology as the topology of convergence in measure.

Definition 6.1. *We call a topology* (X, T) metrizable*, if we can define a metric on* X *whose standard topology on this metric space generates* T.

Remark 6.2. Consider a metric space (M, d). We know that since d is a metric, it is also a pseudometric. Now consider the topology on M generated by the set of pseudo-metrics $P = \{d\}$ (as defined in text below 5.2). Then one can easily observe that the topology generated by P is equivalent to the metric space topology (see Appendix A.4) on (M, d).

As we want to prove that under some assumption, we can define a metric which generates the same topology in measure, by remark 6.2 we have to hence prove that two topologies generated by pseudo-metrics are equivalent. Remark 6.3 will provide a method to show two such topologies are equivalent.

Remark 6.3. Suppose we have a set X and two non-empty families P, Θ of pseudo-metrics on X, generating the topologies $\mathcal{P}, \mathcal{G}$ on X and suppose we want to check whether the two topologies are equivalent. We can do this by firstly considering the identity map $\phi : (X, \mathcal{P}) \to (X, \mathcal{G})$ and check whether this map is continuous. As both topologies are generated by pseudo-metrics, this is equivalent as saying that: $\forall \theta \in \Theta, x \in X \quad \exists \phi_0, .., \phi_n \in P$ and $\delta > 0 : y \in X$ and $\max\limits_{i \leq n} \phi_i(y, x) < \delta \implies \theta(y, x) < \epsilon$. When we now reverse the roles of $\mathcal{P}$ and $\mathcal{G}$, we obtain a method to determine when $\mathcal{P} = \mathcal{G}$.

Definition 6.4. *A measure space* (X, Σ, μ) *is called* σ*-finite* $\iff$ X *can be covered by countably many non-decreasing sets of measurable sets* E_n *of finite measure.*

Example 6.5. Consider $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$. Then this space is σ-finite as $\mathbb{R} = \bigcup\limits_{n=1}^{\infty} [-n, n]$ and $\forall n \in \mathbb{N} : \lambda([-n, n]) = 2n < \infty$.

The following proof is an adjusted version of the one found at [1, p. 175 245Eb].

Theorem 6.6. (X, Σ, μ) *is a* σ*-finite measure space* $\iff$ *the topology of convergence in measure on* $L^0(X, \Sigma, \mu)$ *is metrizable.*

Proof. $\implies$
Let $X = \bigcup\limits_{n=0}^{\infty} E_n$, where $\mu(E_n) < \infty \; \forall n \geq 0$. Then we claim that

$$\bar{\phi}(u, v) = \sum_{n=0}^{\infty} \frac{\bar{\phi}_{E_n}(u, v)}{1 + 2^n \mu(E_n)} \text{ defines a metric.}$$

First we will prove that $\bar{\phi}(u, v)$ is a pseudo-metric.

(1) (Triangle inequality). Let $u = \dot{f}, v = \dot{g}, w = \dot{h} \in L^0$. We have proven earlier that: $\bar{\phi}_{E_n}(\dot{f}, \dot{h}) \leq \bar{\phi}_{E_n}(\dot{f}, \dot{g}) + \bar{\phi}_{E_n}(\dot{g}, \dot{h})$ (See theorem 3.8).
Hence $\forall n \geq 0$:
$$\frac{\bar{\phi}_{E_n}(u,v)}{1+2^n\mu(E_n)} \leq \frac{\bar{\phi}_{E_n}(u,w)+\bar{\phi}_{E_n}(w,v)}{1+2^n\mu(E_n)}.$$
We have also seen earlier that: $\bar{\phi}_{E_n}(u,v) \leq \mu(E_n) < \infty$ and so it holds that:
$$\frac{\bar{\phi}_{E_n}(u,v)}{1+2^n\mu(E_n)} \leq \frac{\mu(E_n)}{1+2^n\mu(E_n)} \leq \frac{1}{2^n}$$
and this proves that $\bar{\phi}(u,v) < \infty$ as
$$\bar{\phi}(u,v) = \sum_{n=0}^{\infty} \frac{\bar{\phi}_{E_n}(u,v)}{1+2^n\mu(E_n} \leq \sum_{n=0}^{\infty} \frac{1}{2^n} < \infty.$$
Combining everything gives:
$$\bar{\phi}(u,v) = \sum_{n\geq 0} \frac{\bar{\phi}_{E_n}(u,v)}{1+2^n\mu(E_n)} \leq \sum_{n\geq 0} \frac{\bar{\phi}_{E_n}(u,w)+\bar{\phi}_{E_n}(w,v)}{1+2^n\mu(E_n)} = \bar{\phi}(u,w) + \bar{\phi}(w,v)$$
(2) (Non-negativity) $\bar{\phi}(\dot{f}, \dot{g}) \geq 0$, since $\phi_{E_n}(f,g) \geq 0 \quad \forall n$.
(3) (Symmetry) $\bar{\phi}(\dot{f}, \dot{g}) = \bar{\phi}(\dot{g}, \dot{f})$, since $\phi_{E_n}(f,g) = \phi_{E_n}(g,f) \quad \forall n$.

Now consider $\bar{\phi}(u,v) = 0$ for $u = \dot{f}, v = \dot{g}$. Then since $\bar{\phi}(u,v)$ is a sum over non-negative numbers, all summands must be equal to 0. And so we get that $\int \min(|f-g|, 1_{E_n})d\mu = 0 \quad \forall n$. Since $\min(|f-g|, 1_{E_n})$ is non-negative, we hence obtain that $\min(|f-g|, 1_{E_n}) =_{a.e.} 0$, so $\mu(\{\min(|f-g|, 1_{E_n}) \neq 0\}) = 0 \quad \forall n$. Note that:

$$\min(|f-g|, 1_{E_n})(s) \neq 0 \iff 1_{E_n}(s) \neq 0 \text{ and } |f-g|(s) \neq 0 \iff 1_{E_n}(s)|f-g|(s) \neq 0.$$

Hence we obtain that

$$0 = \mu(\{\min(|f-g|, 1_{E_n}) \neq 0\}) = \mu(\{|f-g|1_{E_n} \neq 0\})$$

which means per definition that $f =_{a.e.} g$ on E_n.
Now, since $X = \bigcup_{n\geq 0} E_n$ it can be seen that

$$\bigcup_{n\geq 0} \{|f-g|1_{E_n} \neq 0\} = \{|f-g| \neq 0\}.$$

Indeed, if $s \in \{|f-g| \neq 0\}$, then since $s \in X \quad \exists n \in \mathbb{N} : s \in E_n$, hence $s \in \{|f-g|1_{E_n} \neq 0\}$. The other way around holds trivially.
Since $(E_n)_{n\geq 0}$ is non-decreasing, we have that $(\{|f-g|1_{E_n} \neq 0\})_{n\geq 0}$ is non-decreasing, hence obtain: $\mu(|f-g| \neq 0) = \lim_{n\to\infty} \mu(|f-g|1_{E_n} \neq 0) = 0$. And so we have that $f =_{a.e.} g$, so per definition: $u = v$.
We will now prove that this metric generates the topology of convergence in measure, using remark 6.2 and 6.3. Let $F \in \Sigma$, $\mu(F) < \infty$ and $\epsilon > 0$. Then since $\bigcup_{n\geq 1} E_n = X$, we have that $\bigcap_{n\geq 1} F \setminus E_n = \varnothing$, so it holds that $\lim_{n\to\infty} \mu(F \setminus E_n) = 0$, meaning that $\exists n : \mu(F \setminus E_n) \leq \frac{1}{2}\epsilon$. Using this, we can now establish that if $u, v \in L^0$ and $\bar{\phi}(u,v) \leq \frac{\epsilon}{2(1+2^n\mu(E_n))}$, then $\bar{\phi}_F(u,v) < \epsilon$.
Indeed, let $u = \dot{f}, v = \dot{g}$, then since $\forall m \geq 0$ it holds that:

$$\frac{\bar{\phi}_{Em}(u,v)}{1+2^m\mu(E_m)} \leq \sum_{m=0}^{\infty} \frac{\phi_{E_m}(u,v)}{1+2^m\mu(E_m)} = \bar{\phi}(u,v).$$

Hence it holds that for n :
$\bar{\phi}_{E_n}(u,v) \le (1+2^n\mu(E_n))\bar{\phi}(u,v) \le \frac{\epsilon}{2}$ and also $\bar{\phi}_{F\setminus E_n}(u,v) \le \mu(F\setminus E_n) \le \frac{\epsilon}{2}$.
So overall we obtain:

$$\begin{aligned}\bar{\phi}_F(u,v) &= \int \min(|f-g|, 1_F)d\mu \\ &\le \int \min(|f-g|, 1_{E_n})d\mu + \int \min(|f-g|, 1_{F\setminus E_n})d\mu < \epsilon.\end{aligned}$$

In the other direction, first write:

$$\bar{\phi}(u,v) = \sum_{n=0}^{\infty} \tfrac{\bar{\phi}_{E_n}(u,v)}{1+2^n\mu(E_n)} = \sum_{n=0}^{m} \tfrac{\bar{\phi}_{E_n}(u,v)}{1+2^n\mu(E_n)} + \sum_{n=m}^{\infty} \tfrac{\bar{\phi}_{E_n}(u,v)}{1+2^n\mu(E_n)}.$$

As we have earlier established that $\bar{\phi}(u,v)$ is a convergent series, we can choose $m : \sum_{n=m}^{\infty} \frac{\bar{\phi}_{E_n}(u,v)}{1+2^n\mu(E_n)} \le \frac{\epsilon}{2}$. Then when we have: $\bar{\phi}_{E_m}(u,v) \le \frac{\epsilon\mu(E_0)}{4}$, we see that as:

$$\sum_{n=0}^{m} \tfrac{\mu(E_0)}{1+2^n\mu(E_n)} \le \sum_{n=0}^{m} \tfrac{\mu(E_0)}{1+2^n\mu(E_0)} \le \sum_{n=0}^{m} \tfrac{1}{2^n} \le 2.$$

And so the following also holds:

$$\begin{aligned}\bar{\phi}(u,v) = \sum_{n=0}^{\infty} \frac{\bar{\phi}_{E_n}(u,v)}{1+2^n\mu(E_n)} &= \sum_{n=0}^{m} \frac{\bar{\phi}_{E_n}(u,v)}{1+2^n\mu(E_n)} + \sum_{n=m}^{\infty} \frac{\bar{\phi}_{E_n}(u,v)}{1+2^n\mu(E_n)} \\ &\le \frac{\epsilon}{4}\sum_{n=0}^{m} \frac{\mu(E_0)}{1+2^n\mu(E_n)} + \frac{\epsilon}{2} \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

These show that $\{\bar{\phi}\}$ defines the same topology as $\{\bar{\phi}_F : \mu(F) < \infty\}$.
$\Longleftarrow$
Suppose the topology of convergence in measure is metrizable by a metric $\bar{\phi}$. Then since the set $\{u \in L^0 : \bar{\phi}(u,0) < 2^{-n}\}$ is open by the topology on the metric space, we know by the topology of convergence in measure that:
$\forall n \in \mathbb{N} \quad \exists$ measurable set F_n of finite measure and a $\delta_n > 0$ (see definition 3.9):

$$\bar{\phi}_{F_n}(u,0) < \delta_n \implies \bar{\phi}(u,0) < 2^{-n}.$$

Set $E = X \setminus \bigcup_{n\in\mathbb{N}} F_n$. If E does not have zero measure, then $u = \dot{1}_E \neq 0$, hence since $\bar{\phi}$ is a metric $\exists n \in \mathbb{N}$: $\bar{\phi}(u,0) > 2^{-n}$ (else $\bar{\phi}(u,0) = 0$, contradiction as $u \neq 0$). Now:

$$\mu(E\cap F_n) = \bar{\phi}_{F_n}(u,0) \ge \delta_n \text{ while } E \cap F_n = \varnothing, \text{ hence a contradiction.}$$

So $\mu(E) = 0$ and $X = \bigcup_{m\ge 1} F_m \cup E$ is a countable union of sets of finite measure and is non decreasing: hence X is σ-finite. $\square$

7. Completeness of L^0

In the previous sections we have defined convergence on $L^0(S,\Sigma,\mu)$ with respect to the topology of convergence in measure. In the case of sequences in $\mathbb{R}$, we know that every Cauchy sequence converges with respect to the standard topology. In this section we define and will consider restrictions on our measure space (S,Σ,μ) such that every Cauchy sequence in $L^0(S,\Sigma,\mu)$ converges. Before we do this, we will make a slight detour and build up general theory surrounding defining completeness using filters. This generalises the notion of a sequence being Cauchy with respect to a metric or norm to non-metrizable and not normable spaces.

Definition 7.1. *Let U be a set. Then $\mathcal{F} \subseteq \mathcal{P}(U)$ is called a* filter *on U if:*

(1) $A_1, ..., A_n \in \mathcal{F}$, then $\bigcap\limits_{i=1}^{n} A_i \in \mathcal{F}$
(2) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq U$, then $B \in \mathcal{F}$
(3) $\varnothing \notin \mathcal{F}$ and $U \in \mathscr{F}$.

Example 7.2. Let U be a non-empty set and let $(u_n)_{n\geq 1}$ be a sequence in U. Define $\mathcal{F} = \{F \subseteq U : \{n : u_n \notin F\}$ is finite $\}$. Then $\mathcal{F}$ is a filter on U. Indeed:

(1) Let $A, B \in \mathcal{F}$. Then since $\{n : u_n \notin A\}$ and $\{n : u_n \notin B\}$ are finite sets, so is $\{n : u_n \notin A \cap B\}$ as $\{u_n \notin A \cap B\} = \{u_n \notin A$ or $u_n \notin B\}$, finite set. Hence: $A \cap B \in \mathcal{F}$
(2) Let $A \in \mathcal{F}$ and suppose $A \subseteq B \subseteq U$. Then since $u_n \notin B \implies u_n \notin A$: $\{n : u_n \notin B\} \subseteq \{n : u_n \notin A\}$, finite. Hence: $B \in \mathcal{F}$.
(3) Since $\{n : u_n \notin \varnothing\} = \mathbb{N}$: $\varnothing \notin \mathcal{F}$. It also holds that $U \in \mathscr{F}$.

Definition 7.3. *Let U be a linear space (vector space) over $\mathbb{R}$ (or $\mathbb{C}$) and $\mathcal{T}$ a linear space topology on U (see def 5.1). A filter $\mathcal{F}$ (as defined in definition 7.1) is called* Cauchy *if for every open set in G in U containing 0 $\exists F \in \mathcal{F}$: F-F=$\{u - v : u, v \in F\} \subseteq G$.*

Definition 7.4. *Let $\mathcal{F}$ be a Cauchy filter on U. Then $\mathcal{F}$ is said to converge to $x \in U$ (Notation: $\mathcal{F} \to x$) if every open set containing x belongs to $\mathcal{F}$.*

Under the same circumstances as definition 7.4, if we now assume that $(U, \mathcal{T})$ is Hausdorff any converging Cauchy filter $\mathcal{F} \to x \in U$ has a unique limit.
Indeed: Suppose $\mathcal{F} \to x_1 \in U$ and $\mathcal{F} \to x_2 \in U$, where $x_1 \neq x_2$, then $\mathcal{F}$ contains all open sets containing x_1 as well as all open sets containing x_2. Then, since U is Hausdorff, there is an open neighbourhood U_{x_1} of x_1 and U_{x_2} of x_2 such that: $U_{x_1} \cap U_{x_2} = \varnothing$. Since $U_{x_1}, U_{x_2} \in \mathcal{F}$ we have that $U_{x_1} \cap U_{x_2} = \varnothing \in \mathcal{F}$(definition 7.1), contradiction.
In general, Cauchy filters do not have unique limits. For instance, consider: $U = \mathbb{R}$ and $\mathcal{T} = \{\varnothing, \mathbb{R}\}$. It is an easy check that this is a linear topology. It holds that $\mathcal{F} = \{\mathbb{R}\}$ is a filter on U as:

(1) $\mathbb{R} \in \mathcal{F}$
(2) $\varnothing \notin \mathcal{F}$.

$\mathcal{F}$ is even Cauchy as $\mathbb{R}$ is the only open set containing 0, hence: $\mathbb{R} - \mathbb{R} \subseteq \mathbb{R}$. But $\mathcal{F} \to x$ for any $x \in \mathbb{R}$.

Definition 7.5. *Under the same conditions as definition 7.3, U is said to be* complete *if every Cauchy filter on U is convergent (convergent as in definition 7.4).*

In case we are in the situation as defined under remark 5.2, it turns out that $\mathcal{F}$ being Cauchy is equivalent with being able to find an $F \in \mathcal{F}$ such that the difference between all elements in F is less or equal to ϵ.
The proof below is an adjusted version of the one found at [1, p. 516 2A3G].

Lemma 7.6. *Let U be linear space over $\mathbb{R}$ (or $\mathbb{C}$) and let T be a family of functionals from U to $\mathbb{R}$ (or $\mathbb{C}$) (see text below remark 5.2) defining a linear space topology. Then a filter $\mathcal{F}$ is Cauchy $\iff$ $\forall \tau \in T$ and $\epsilon > 0 \quad \exists F \in \mathcal{F} : \tau(u-v) < \epsilon \quad \forall u, v \in F$*

Proof. $\implies$
Suppose $\mathcal{F}$ is Cauchy, $\tau \in T$ and $\epsilon > 0$. Then by theorem 5.8, $U(0, \phi_\tau; \epsilon)$ is open, hence by definition 7.3 $\exists F \in \mathcal{F} : F - F \subseteq G$, so $\tau(u-v) < \epsilon \quad \forall u, v \in F$.
$\impliedby$
Let G be an open set containing 0, then by definition 5.6 $\exists \tau_0, ..., \tau_n \in T$ and $\epsilon > 0$ such that $U(0, \phi_{\tau_0}, ..., \phi_{\tau_n}; \epsilon) \subseteq G$. For each $i \leq n$, $\exists F_i \in \mathcal{F}$ such that $\tau_i(u-v) < \epsilon \ \forall u, v \in F_i$. Now, per definition of a filter, $F = \bigcap\limits_{i \leq n} F_i \in \mathcal{F}$ and $\forall u, v \in F : \tau_i((u-v)-0) = \phi_{\tau_i}(u-v, 0) < \epsilon \quad \forall i \leq n$. And so $F - F \subseteq G$. $\square$

Now that we have established the definition of being complete with the help of filters, one might wonder if this is equivalent to our usual notion of every Cauchy sequence converging in a normed space or metric space. It is indeed equivalent and we will prove this for the case that U is a normed space. The proof of the case that U is a metric space is analogous.
This proof is a worked out version of the one found at [1, p. 516 2A5G].

Proposition 7.7. *Let $(U, \|.\|)$ be a normed space over $\mathbb{R}$ (or $\mathbb{C}$) and let $\mathcal{T}$ be the linear topology defined on U from the set of functionals $T = \{\|.\|\}$.
Then U is complete in the sense of 7.5 $\iff$ every Cauchy sequence (wrt $\|.\|$) in U converges*

Proof. $\implies$
Let $(u_n)_{n \geq 1}$ be a Cauchy sequence in U. Define $\mathcal{F}$=$\{F \subseteq U : \{n : u_n \notin F\}$ is finite $\}$. Then $\mathcal{F}$ is a filter on U by example 7.2. Let $\epsilon > 0$ and take $m \in \mathbb{N}$ such that $\|u_j - u_k\| < \epsilon$ whenever $j, k \geq m$ (this m exists per definition of u_n being Cauchy), then $F = \{u_j : j \geq m\} \in \mathcal{F}$ (since $\{n : u_n \notin F\} = \{1, .., m-1\}$ is a finite set) and $\|u - v\| < \epsilon \quad \forall u, v \in F$. Now, let $G \subseteq U$ be open such that $0 \in G$. Then $\exists \epsilon > 0$ such that $U(0; \|.\|; \epsilon) \subseteq G$ (by def 5.6). Hence we have (per definition of F): $F - F \subseteq U(0, \|.\|, \epsilon) \subseteq G$, where $F \in \mathcal{F}$, hence proven that $\mathcal{F}$ is Cauchy and since U is assumed to be complete in the sense of definition 7.5, it has a limit u. Now for any $\epsilon > 0$, the set $\{v \in L^0 : \|v - u\| < \epsilon\} = U(u; \|.\|; \epsilon)$ is an open set containing u (by theorem 5.8), hence it belongs to $\mathcal{F}$ (by definition 7.4) and so $\{n : u_n \notin U(u; \|.\|; \epsilon)\} = \{n : \|u_n - u\| \geq \epsilon\}$ is finite. But this means that $\exists m : \|u_n - u\| < \epsilon$ whenever $n \geq m$. As ϵ was arbitrary: $\lim\limits_{n \to \infty} u_n = u$.
$\impliedby$
Let $\mathcal{F}$ be a Cauchy filter on U. Then for each $n \geq 1$, we can choose a $F_n \in \mathcal{F}$ such that $\|u - v\| < 2^{-n} \quad \forall u, v \in F_n$ (since $U(0, \|.\|, 2^{-n})$ is open in U). For each $n \geq 1$, $F'_n = \bigcap\limits_{i \leq n} F_i$ belongs again to $\mathcal{F}$ hence can't be empty by definition 7.1. Choose $u_n \in F'_n$. If $m \in \mathbb{N}$ and $k, j \geq m$, then $u_j, u_k \in F_m$ (as $F'_j \subseteq F'_m$ so $u_j \in F'_m$, hence $u_j \in F_i \quad \forall i \leq m$). But this means $\|u_j - u_k\| < 2^{-m}$, so u_n is a

Cauchy sequence and has hence a limit (by assumption), say u.
Now take $\epsilon > 0$ and $m \in \mathbb{N}$: $2^{-m+1} < \epsilon$. As u_n converges to u, $\exists k \geq m$: $\|u_k - u\| < 2^{-m}$. Since we have earlier established that $u_k \in F_m$ we see that (as $\forall u, v \in F_m : \|u - v\| < 2^{-m}$ and $\|v - u\| \leq \|v - u_k\| + \|u_k - u\| < 2 \cdot 2^{-m}$):

$$\begin{aligned} F_m &\subseteq \{v \in L^0 : \|v - u_k\| < 2^{-m}\} \\ &\subseteq \{v \in L^0 : \|v - u\| < 2^{-m+1}\} \\ &\subseteq \{v \in L^0 : \|v - u\| < \epsilon\} \end{aligned}$$

and this implies that (per def of filter) $\{v \in L^0 : \|v - u\| < \epsilon\} \in \mathcal{F}$. Since ϵ arbitrary, $\mathcal{F} \to u$ as we know by def 5.6 that for an open set containing u, say A, $\exists \epsilon > 0$ such that $\{v : \|v - u\| < \epsilon\} \subseteq A$, meaning that per definition of a filter $A \in \mathcal{F}$. As $\mathcal{F}$ was arbitrary, U is complete. $\square$

Now that we have established general theory about complete linear topologies, we can finally prove that under a certain assumption on (X, Σ, μ), L^0 is complete under the topology of convergence in measure. This assumption roughly states that every set in Σ with ∞ measure can be approximated by a set with finite measure and that every subset of Σ can be approximated by a single set $H \in \Sigma$ which acts as the 'least upper bound' of this subset of Σ. This notion is made precise below.

Definition 7.8. *A measure space (X, Σ, μ) is called* localizable *if:*

(1) It is semi-finite (see definition 4.12)

(2) $\forall \mathcal{E} \subseteq \Sigma \quad \exists H \in \Sigma$ such that:

- *$\forall E \in \mathcal{E} : E \setminus H$ is negligible (meaning: $\mu(E \setminus H) = 0$))*
- *If $G \in \Sigma$ and $\forall E \in \mathcal{E}$, $E \setminus G$ negligible, then $H \setminus G$ also negligible.*

We call this H the essential supremum *of $\mathcal{E}$.*

It turns out that being localizable is weaker than being σ-finite. This we will prove via the notion of strictly localizable.

Definition 7.9. *Let (X, Σ, μ) be a measure space. Then μ is called strictly localizable if there exists a partition $(X_i)_{i \in I}$ of X into measurable sets of finite measure such that:*

$$\Sigma = \{E : E \subseteq X, E \cap X_i \in \Sigma \quad \forall i \in I\}$$
$$\mu(E) = \sum_{i \in I} \mu(E \cap X_i).$$

Remark 7.10. The index set I in definition 7.9 may be uncountable.

The proofs of both proposition 7.11 and proposition 7.12 show and elaborate on the proofs found at [1, p. 13 211L].

Proposition 7.11. *A σ-finite measurable space is strictly localizable.*

Proof. Let (X, Σ, μ) be a σ-finite measure space ($X = \bigcup_{n=1}^{\infty} E_n$) and let F_n be a disjoint sequence of measurable sets of finite measure covering X defined as follows: since $X = \bigcup_{n=1}^{\infty} E_n$, define $F_1 = E_1, F_2 = E_2 \setminus E_1, F_3 = E_3 \setminus E_1 \cup E_2$ etc.

If $E \in \Sigma$, then $E \cap F_n \in \Sigma$ for all n and $\mu(E) = \sum_{n=1}^{\infty} \mu(E \cap F_n)$. In case $E \subseteq X$ and $E \cap F_n \in \Sigma \quad \forall n$, then $E = \bigcup_{n \in \mathbb{N}} E \cap F_n \in \Sigma$. $\square$

Proposition 7.12. *A strictly localizable measure space is localizable.*

Proof. Let (X, Σ, μ) be a strictly localizable measure space and let $(X_i)_{i\in I}$ be the corresponding decomposition of X.
Let $\mathcal{E}$ be family of measurable subsets of X. Let $\mathcal{F}$ be the family of measurable subsets $F \subseteq X$ such that $\mu(F \cap E) = 0 \; \forall E \in \mathcal{E}$.
Note that $\varnothing \in \mathcal{F}$ and if $(F_n)_{n\geq 1}$ is a sequence in $\mathcal{F}$, then $\bigcup\limits_{n\in\mathbb{N}} F_n \in \mathcal{F}$.
For each $i \in I$, set $\gamma_i = \sup\{\mu(F \cap X_i) : F \in \mathcal{F}\}$ and choose a sequence $(F_{in})_{n\in\mathbb{N}}$ in $\mathcal{F}$: $\lim\limits_{n\to\infty} \mu(F_{in} \cap X_i) = \gamma_i$.
Set

$$\begin{aligned} F_i &= \bigcup_{n\in\mathbb{N}} F_{in} \in \mathcal{F} \\ F &= \bigcup_{i\in I} F_i \cap X_i \subseteq X \text{ and} \\ H &= X \setminus F. \end{aligned}$$

We see that $F \cap X_i = F_i \cap X_i \in \Sigma \; \forall i \in I$, and by the definition strictly localizable: $F \in \Sigma$ which gives $H \in \Sigma$.
For any $E \in \mathcal{E}$ it holds:

$$\begin{aligned} \mu(E \setminus H) &= \mu(E \cap F) \\ &= \sum_{i\in I} \mu(E \cap F \cap X_i) \\ &= \sum_{i\in I} \mu(E \cap F_i \cap X_i) = 0 \end{aligned}$$

Thus $F \in \mathcal{F}$. If $G \in \Sigma$ and $\mu(E \setminus G) = 0 \; \forall E \in \mathcal{E}$, then $X \setminus G$, $F' = F \cup (X \setminus G) \in \mathcal{F}$ and so it holds that $\gamma_i \geq \mu(F' \cap X_i) \; \forall i \in I$.
And so we obtain:

$$\begin{aligned} \mu(F \cap X_i) &= \mu(\bigcup_{n\in\mathbb{N}} F_{in} \cap X_i) \qquad (F \cap X_i = F_i \cap X_i = \bigcup_{n\in\mathbb{N}} F_{in} \cap X_i) \\ &\geq \mu(F_{in} \cap X_i) \end{aligned}$$

which gives: $\mu(F \cap X_i) \geq \sup\limits_n \{\mu(F_{in} \cap X_i)\} = \gamma_i$.
And so, as also $F' \cap X_i \supseteq F \cap X$, we obtain:

$$\gamma_i \geq \mu(F' \cap X_i) \geq \mu(F \cap X_i) \geq \gamma_i.$$

So, $\mu(F \cap X_i) = \mu(F' \cap X_i) \; \forall i \in I$.
Because $\mu(X_i) < \infty$ it follows that $\mu((F' \setminus F) \cap X_i) = 0 \; \forall i \in I$. Summing over i yields: $\mu(F' \setminus F) = 0 \implies \mu(H \setminus G) = 0$. □

In the following example, we will show that being localizable is weaker than being σ-finite.

Example 7.13. [1, p. 14 211N] Take $X = \mathbb{R}$ and $\Sigma = \mathscr{B}(\mathbb{R})$ together with counting measure μ:

$$\mu(A) = \begin{cases} |A| & |A| < \infty \\ \infty & |A| = \infty. \end{cases}$$

Then μ is not σ-finite, because if it were $\mathbb{R}$ would be a countable union of countable sets, hence countable which is not true.
μ is strictly localizable: Set $X_x = \{x\}\ \forall x \in X$. Then $(X_x)_{x\in X}$ is a partition of X and for any $E \subseteq X$, $\mu(E \cap X_x) = 1$ if $x \in X$ else 0. And so we have that $\mu(E) = \sum_{x\in X} \mu(E \cap X_x)$, which implies that μ is localizable by proposition 7.12.

Under the assumption that (X, Σ, μ) is localizable, it turns out that that if we define a function f on a measurable set of X, under certain conditions of f, we can approximate f by a function defined on the whole of X (in an almost everywhere sense). This approximation is entailed in the next theorem.
The proof below represents the proof found at [1, p. 28 213N] together with some details worked out.

Theorem 7.14. *Let (X, Σ, μ) be a localizable measure space. Suppose Φ is a family of measurable real valued functions defined on measurable subsets of X such that : $f, g \in \Phi \implies f =_{a.e.} g$ on $dom(f) \cap dom(g)$. Then $\exists\ h : X \to \mathbb{R}$ such for every $f \in \Phi$: $h =_{a.e.} f$ on $dom(f)$.*

Proof. For $q \in \mathbb{Q}$ and $f \in \Phi$, set

$$E_{fq} = \{x \in dom(f) : f(x) \geq q\} \in \Sigma.$$

For each $q \in \mathbb{Q}$, let E_q be an essential supremum of $\{E_{fq} : f \in \Phi\}$ (see definition 7.8).
Define:

$$h^*(x) = \sup\{q : q \in \mathbb{Q}, x \in E_q\} \in [-\infty, \infty]$$

for $x \in X$ and set $\sup \varnothing = -\infty$.
If $f, g \in \Phi$ and $q \in \mathbb{Q}$, then:

$$\begin{aligned} E_{fq} \setminus (X \setminus (dom(g) \setminus E_{gq})) &= E_{fq} \cap dom(g) \setminus E_{gq} \\ &\subseteq \{x : x \in dom(f) \cap dom(g), f(x) \neq g(x)\} \end{aligned}$$

is negligible (by the definition of Φ) and so as f was arbitrary:

$$E_q \setminus (X \setminus (dom(g) \setminus E_{gq})) = E_q \cap dom(g) \setminus E_{gq}$$

is negligible as well (by definition of essential supremum).
As $E_{gq} \setminus E_q$ is negligible as well (definition essential supremum), we have that:

$$E_{gq} \triangle (E_q \cap dom(g))$$

is negligible as well and so

$$H_g = \bigcup_{q\in\mathbb{Q}} E_{gq} \triangle (E_q \cap dom(g))$$

is also negligible. But if $x \in dom(g) \setminus H_g$, then $\forall q \in \mathbb{Q} : x \in E_q \iff x \in E_{gq}$(this immediately follows from the definition of $dom(g) \setminus H_g$ when written out). It follows that for such $x : h^*(x) = g(x)$ and so $h^* =_{a.e.} g$ on $dom(g)$ and this holds $\forall g \in \Phi$.
The function h^* is not necessarily real valued but is measurable as it holds that:

$$\{x : h^*(x) > a\} = \bigcup_{q\in\mathbb{Q}} \{E_q : q > a\}$$

Indeed: Let $x \in \{x : h^*(x) > a\}$, then $h^*(x) > a$, so $\sup\{q \in \mathbb{Q} : x \in E_q\} > a$, so $x \in E_q$ for some $q > a$.
Let $x \in \bigcup_{q\in\mathbb{Q}} \{E_q : q > a\}$. Then $x \in E_q$ for some $q > a$. Which means $\sup\{q \in \mathbb{Q} : x \in E_q\} > a$.
Hence, we can modify h^* by setting

$$h(x) = \begin{cases} h^*(x) & h(x) \in \mathbb{R} \\ 0 & h^*(x) \in \{-\infty, \infty\}. \end{cases}$$

Then we have obtained a measurable real valued function $h : X \to \mathbb{R}$, where for any function $g \in \Phi : h(x) =_{a.e.} g(x)\ \forall x \in dom(g)$. □

The proof of completeness under when we assume localizability will make use of the following lemma below.

Lemma 7.15. *Suppose $(f_n)_{n\geq 1}$ is a sequence in $L^0(S, \Sigma, \mu)$ for which it holds that for $F \in \Sigma$:*

$$\mu(\{x \in F : |f_{n+1}(x) - f_n(x)| \geq 2^{-n}\} \leq 2^{-n}.$$

Then $(f_n)_{n\geq 1}$ converges a.e. on F.

Proof. Define the following set:

$$H_n = \{x \in F : |f_{n+1}(x) - f_n(x)| \geq 2^{-n}\}.$$

Then $\mu(H_n) \leq 2^{-n}$ (as established earlier). Now define $B_n = \bigcup_{m\geq n} H_m$. As H_m is measurable, so is B_n. We also see that B_n is non-increasing, hence we have that:

$$\begin{aligned} \mu(\bigcap_{n=1}^{\infty} \bigcup_{m\geq n} H_m) &= \lim_{n\to\infty} \mu(\bigcup_{m\geq n} H_m) \\ &\leq \lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(H_m) \\ &\leq \lim_{n\to\infty} \sum_{m=n}^{\infty} 2^{-m} = 0 \qquad \text{(tail convergent series).} \end{aligned}$$

We have hence proven that $\bigcap_{n\geq 1} \bigcup_{m\geq n} H_m$ has 0 measure.
If $x \in F \setminus \bigcap_{n\geq 1} \bigcup_{m\geq n} H_m$, then per definition $\exists k : x \in F \setminus \bigcup_{m\geq k} H_m$, so that $|f_{m+1}(x) - f_m(x)| \leq 2^{-m}\ \ \forall m \geq k$, hence we see that $(f_n(x))_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, hence is convergent (we define this limit as $f(x)$). Since $\bigcap_{n=1}^{\infty} \bigcup_{m\geq n} H_m$ had zero measure, we can hence say:

$$\lim_{n\to\infty} f_n(x) =_{a.e.} f(x) \text{ on } F.$$

□

Now we can finally prove that under the condition of (X, Σ, μ) being localizable, L^0 is complete under the topology of convergence in measure.
The proof below is an adapted version of the one found at [1, p. 175 245E].

Theorem 7.16. *Let (X,Σ,μ) be a localizable measure space. Then $L^0(X,\Sigma,\mu)$ is complete under the topology of convergence in measure.*

Proof. Suppose (X,Σ,μ) is localizable and let $\mathcal{F}$ be a Cauchy filter on L^0. Per definition of completeness (definition 7.5), we have to prove the Cauchy filter converges (see definition 7.4). Let F be a set of finite measure. Then since $\{x \in L^0 : \bar{\phi}_F(x,0) \leq 4^{-n}\}$ is open in L^0 (see theorem 5.8) and contains 0, per definition of a Cauchy filter $\exists A_n(F) \in \mathcal{F}$ such that $A_n(F) - A_n(F) \subseteq \{\bar{\phi}_F(x,0) < 4^{-n}\}$, hence meaning that $\phi_F(u,v) < 4^{-n} \quad \forall u,v \in A_n(F)$. Per definition of $\mathcal{F}$ being a filter (definition 7.1): $\bigcap_{k\leq n} A_k(F) \in \mathcal{F}$ and is non-empty, so we can choose $u_n^F \in \bigcap_{k\leq n} A_k(F)$. Then as

$$u_{n+1}^F \in \bigcap_{k\leq n+1} A_k(F) \subseteq A_n(F) \text{ and } u_n^F \in A_n(F)$$

we can say:

$$\bar{\phi}_F(u_{n+1}^F, u_n^F) \leq 4^{-n} \leq 2^{-n}.$$

As $u_n^F \in L^0$, we can write $u_{F_n} = \dot{f}_n^F$, where $f_n^F : X \to \mathbb{R}$ measurable.
Consider the following set: $\{x \in F : |f_{n+1}^F(x) - f_n^F(x)| \geq 2^{-n}\}$.
It holds that $2^{-n} 1_{\{x\in F:|f_{n+1}^F(x)-f_n^F(x)|\geq 2^{-n}\}} \leq \min(|f_{n+1}^F - f_n^F|, 1_F)$.
Indeed:

(1) Case 1: $s \in \{x \in F : |f_{n+1}^F(x) - f_n^F(x)| \geq 2^{-n}\}$. Then the left-hand side of the proven inequality gives 2^{-n}. In case $|f_{n+1}^F - f_n^F|(s) \geq 1$, the right-hand side is 1 and $2^{-n} \leq 1$ holds. In case $|f_{n+1}^F - f_n^F|(s) < 1$, we have that the inequality that we are proving gives: $|f_{n+1}^F - f_n^F|(s) \geq 2^{-n}$ and this holds.
(2) Case 2: $s \in \{x \in F : |f_{n+1}^F(x) - f_n^F(x)| < 2^{-n}\}$. As then the left-hand side is 0, the inequality is automatically satisfied.

And so, since the integral is an increasing function, we can take the integral on both sides of the inequality above and preserve the inequality. Doing so gives:

$$2^{-n}\mu(\{x \in F : |f_{n+1}^F(x) - f_n^F(x)| \geq 2^{-n}\}) \leq \bar{\phi}_F(u_{n+1}^F, u_n^F) \leq 4^{-n}.$$

Hence obtaining

$$\mu(\{x \in F : |f_{n+1}^F(x) - f_n^F(x)| \geq 2^{-n}\}) \leq 2^n \bar{\phi}_F(u_{n+1}^F, u_n^F) \leq 2^n 4^{-n} = 2^{-n}.$$

Hence by lemma 7.15, we obtain that

$$\lim_{n\to\infty} f_n^F(x) =_{a.e.} f^F(x) \text{ on } F.$$

Claim: If E, G are two sets of finite measure such that $E \subseteq G$, then $\bar{\phi}_E(u_n^E, u_n^G) \leq 2 \cdot 4^{-n}$.
Proof claim: $A_n(E), A_n(G) \in \mathcal{F}$ implies that also $A_n(E) \cap A_n(G) \in \mathcal{F}$ and is non-empty, hence they must have some point w in common.
This gives:

$$\begin{aligned}
\bar{\phi}_E(u_n^E, u_n^G) &\leq \bar{\phi}_E(u_n^E, w) + \bar{\phi}_E(w, u_n^G) \\
&\leq \bar{\phi}_E(u_n^E, w) + \bar{\phi}_G(w, u_n^G) && (E \subseteq G) \\
&\leq 2 \cdot 4^{-n}.
\end{aligned}$$

Consequently (in the same fashion as before)

$$\mu(\{x \in E : |f_n^G(x) - f_n^E(x)| \geq 2^{-n}\}) \leq 2^n \bar{\phi}_E(u_n^G, u_n^E) \leq 2^{-n+1}$$

and also

$$\lim_{n\to\infty} f_n^G - f_n^E =_{a.e.} 0 \text{ on } E \implies f^E =_{a.e.} f^G \text{ on } E.$$

Consequently, since we took $E \subseteq G$ as arbitrary sets of finite measure, it follows that for $E \subseteq F$, since $E \cup F$ has finite measure and $E \subseteq E \cup F$: $f^E =_{a.e.} f^{E\cup F}$ on E and $f^F =_{a.e.} f^{E\cup F}$ on F, implying that $f^E =_{a.e.} f^F$ on $E \cap F$.
Because μ is localizable, it follows by theorem 7.14, that $\exists f$ measurable function on X such that $f =_{a.e.} f^E$ on E for any measurable set E with finite measure. Now consider $u = \dot{f} \in L^0$ (f as defined above). Then for any set E of finite measure:

$$\begin{aligned}\bar{\phi}_E(u, u_n^E) &= \int_E \min(1, |f - f_n^E|) d\mu \\ &= \int_E \min(1, |f^E - f_n^E| d\mu \to 0 \quad (f =_{a.e.} f^E \text{ on } E \text{ and } f_n^E \to_{a.e.} f^E \text{ on } E).\end{aligned}$$

Now:

$$\begin{aligned}\inf_{A\in\mathcal{F}} \sup_{v\in A} \bar{\phi}_E(v, u) &\leq \inf_{n\in\mathbb{N}} \sup_{v\in A_n(E)} \bar{\phi}_E(v, u) \\ &\leq \inf_{n\in\mathbb{N}} \sup_{v\in A_n(E)} (\bar{\phi}_E(v, u_n^E) + \bar{\phi}_E(u, u_n^E)) \\ &\leq \inf_{n\in\mathbb{N}} \sup_{v\in A_n(E)} 2 \cdot 4^{-n} + \bar{\phi}_E(u, u_n^E) = 0.\end{aligned}$$

As E was arbitrary, it follows that $\mathcal{F} \to u$.
Indeed, since $\inf_{A\in\mathcal{F}} \sup_{v\in A} \bar{\phi}_E(v, u) = 0$, we know $\sup_{v\in A} \bar{\phi}_E(v, u)$ gets arbitrarily close to 0, as an identity dependent on $A \in \mathcal{F}$. Now, let G be an open set containing v. Then by definition 5.6, $\exists \delta > 0$ and $\bar{\phi}_E : U(v; \bar{\phi}_E, \delta) \subseteq G$. As $\sup_{v\in A} \bar{\phi}_E(v, u)$ gets arbitrarily close to 0, we can choose a $A \in \mathcal{F}$ and $\delta' < \delta$ such that:
$A \subseteq \{v \in L^0 : \bar{\phi}_E(v, u) < \delta'\} \subseteq \{v \in L^0 : \bar{\phi}_E(v, u) < \delta\} \subseteq G$. Hence since $A \in \mathcal{F}$, per definition of a filter (definition 7.1): $G \in \mathcal{F}$. Since $\mathcal{F}$ was arbitrary: L^0 is complete. □

Remark 7.17. In case we assume that L^0 is σ-finite (which is stronger than being localizable by theorem 7.11), we have that that the topology of convergence in measure is metrizable by theorem 6.6 (namely by the metric $\bar{\phi}$ as defined in theorem 6.6). For a Cauchy sequence $(\dot{f}_n)_{n\geq 1}$, it holds that:

$$0 = \lim_{n\to\infty} \bar{\phi}(\dot{f}_n, \dot{f}_{n+1}) = \lim_{n\to\infty} \sum_{m=1}^{\infty} \tfrac{\phi_{E_m}(f_n, f_{n+1})}{1+2^m\mu(E_m)}.$$

Since it holds for all n that:

$$\tfrac{\phi_{E_m}(f_n, f_{n+1})}{1+2^m\mu(E_m)} \leq \tfrac{\mu(E_m)}{1+2^m\mu(E_m)} \leq \tfrac{1}{2^m}$$

By the DCT (with respect to the counting measure):

$$\lim_{n\to\infty} \bar{\phi}(\dot{f}_n, \dot{f}_{n+1}) = \sum_{m=1}^{\infty} \lim_{n\to\infty} \tfrac{\phi_{E_m}(f_n, f_{n+1})}{1+2^m\mu(E_m)} = 0.$$

Hence it holds that $\lim_{n\to\infty} \phi_{E_m}(f_n, f_{n+1}) = 0$ $\forall m \geq 1$. And so, for each m, we can find a subsequence of f for which the following holds:

$$\phi_{E_m}(f_{n_k}^{E_m}, f_{n_{k+1}}^{E_m}) \leq 2^{-k}$$

Using the same method as in the proof of theorem 7.16:

$$\mu(\{x \in E_m : |f_{n_{k+1}}^{E_m}(x) - f_{n_k}^{E_m}(x)| \geq 2^{-k}\}) \leq 2^{-k}$$

and so by, by lemma 7.15, $(f_{n_k}^{E_m})_{k\geq 1}$ converges a.e. on E_m (to f^{E_m}). Then it holds that $f^{E_m} =_{a.e.} f^{E_n}$ on $E_n \cap E_m$, hence using that σ-finite implies localizability: there exists a $h : X \to \mathbb{R}$ such that $h =_{a.e.} f^{E_n}$ on E_n $\forall n \geq 1$. And so we obtain:

$$\begin{aligned}\lim_{n\to\infty} \bar{\phi}(\dot{f}_n, \dot{h}) &= \lim_{n\to\infty} \sum_{m=1}^{\infty} \frac{\phi_{E_m}(f_n, h)}{1 + 2^m \mu(E_m)} \\ &\leq \sum_{m=1}^{\infty} \lim_{n\to\infty} \frac{\phi_{E_m}(f_{n_k}^{E_m}, f_n) + \phi_{E_m}(f_{n_k}^{E_m}, h)}{1 + 2^m \mu(E_m)}\end{aligned}$$

Choosing n_k sufficiently large, we obtain that the sum above will tend to 0 as $(\dot{f}_n)_{n\geq 1}$ is Cauchy and $h =_{a.e.} f^{E_m}$ on E_m.

8. Local convexity of L^0

The intuition behind a topological vector space being locally convex is that if we have an open set containing 0 in our space, then we can always find a convex set fully contained in the open set which also contains 0. As this is a strong property to have, we will in this section consider whether local convexity applies in general to L^0 and the topology of convergence in measure. In order to define local convexity, we will first introduce semi-norms. With these semi-norms we will define local convexity in a manner that is equivalent to the intuition above.

Definition 8.1. *A* semi-norm *on a vector space X is a map $p : X \to \mathbb{R}$ such that $\forall x, y \in X, s \in \mathbb{R}$:*

(1) $p(x) \geq 0$
(2) $p(sx) = |s|p(x)$
(3) $p(x+y) \leq p(x) + p(y)$

Example 8.2. Consider $C(\mathbb{R})$ (the space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$) and define $p_n : C(\mathbb{R}) \to \mathbb{R}$

$$p_n(f) = \max_{x \in [-n,n]} |f(x)|.$$

This definition is well-defined as f being continuous implies $|f|$ being continuous, which means $|f(x)|$ achieves its maximum on the compact interval $[-n, n]$.
p_n is a semi-norm:

(1) $p_n(f) \geq 0$
(2) Let $s \in \mathbb{R}$. Then $p_n(sf) = \max_{x \in [-n,n]} |sf(x)| = |sf(a)|$ for some $a \in [-n, n]$.
Hence we have that:
$$|s||f(x)| \leq |s||f(a)| \; \forall x \in [-n, n].$$
This means that in case $s \neq 0$, we have that
$$|f(x)| \leq |f(a)| \; \forall x \in [-n, n].$$
But as $a \in [-n, n]$ this means:
$$|f(a)| = p_n(f)$$
And so:
$$p_n(sf) = |s||f(a)| = |s|p_n(f).$$
In case $s = 0$, the equality above also holds.
(3)
$$\begin{aligned} p_n(f+g) &= \max_{x \in [-n,n]} |f(x) + g(x)| \\ &\leq \max_{x \in [-n,n]} (|f(x)| + |g(x)|) \\ &\leq p_n(f) + p_n(g). \end{aligned}$$

Definition 8.3. *A* locally convex topological vector space *is a vector space X along a family $\mathcal{P}$ of semi-norms on X, which define a topology on X. The topology on X under $\mathcal{P}$ is generated by sets of the form $p^{-1}(U)$, where $p \in \mathcal{P}$ and $U \subseteq \mathbb{R}$ open.*

Remark 8.4. There is an equivalent formulation of a topological vector space X being locally convex which states that every neighbourhood around the 0 element contains a convex set from a fixed basis. This basis, which only consists of convex sets, is also a set of neighbourhoods of 0.

Proving local convexity by using the definition can sometimes be tedious. It turns out that in case a specific set of functions from our topological X to $\mathbb{R}$ is known, we have a relatively easy to check requirement for local convexity to hold. This specific set is called the dual of X and is defined below.

Definition 8.5. *Let* (X, T) *be a topological space. Then the dual of* X *is defined as:*

$$X^* = \{\phi : X \to \mathbb{R} : \phi \textit{ is linear and continuous } \}$$

Example 8.6. Consider $\mathbb{R}$ equipped with the standard topology. Then $\mathbb{R}^* = \{\phi(x) = ax \| a \in \mathbb{R}\}$. Indeed, let ϕ be a linear function on $\mathbb{R}$ and set $a = \phi(1)$. Then we see: $\phi(x) = \phi(x \cdot 1) = x\phi(1) = ax$. Since all functions of these form are continuous we have hence proven our assertion.

Example 8.7. [4, p. 5,6 Prop. 7.13, Thm. 7.14] Let $1 < p < \infty$ and choose p' such that:

$$\tfrac{1}{p} + \tfrac{1}{p'} = 1.$$

For a measure space $(X, \mathcal{A}, \mu)$, if $f \in L^{p'}(S)$, then

$$F(g) = \textstyle\int fgd\mu.$$

defines a functional from $L^p(S)$ to $\mathbb{R}$. Assuming the standard norms on L^p and $L^{p'}$, the fact that $F(g)$ only takes values in $\mathbb{R}$ follows directly from Hölder's inequality:

$$\textstyle\int |fg|d\mu \leq \|f\|_{p'} \|g\|_p < \infty.$$

As we have now shown that $|fg|$ is integrable, fg is also integrable and so:

$$|F(g)| = |\textstyle\int fgd\mu| \leq \int |fg|d\mu < \infty.$$

One can also easily see that F is linear. It also holds that F is Lipschitz as the $f \in L^{p'}(S)$ is fixed:

$$\begin{aligned} |F(g) - F(h)| &= |\int f(g-h)d\mu| \\ &\leq \int |f(g-h)|d\mu \\ &\leq \|f\|_{p'} \|g-h\|_p && \text{(Hölder's inequality)} \end{aligned}$$

which implies F is continuous. Thus we have now established that:

$$(L^p(S, \mathcal{A}, \mu))^* \supseteq \{\textstyle\int fgd\mu : g \in L^p(S)\}.$$

In this case $(1 < p < \infty)$ the left inclusion also holds.

A property of a locally convex topological space X is that any linear and continuous function whose domain is a vector subspace of X can be extended to a function in X^*.

Theorem 8.8. [5]*(Hahn-Banach theorem) Let* X *be a locally convex topological vector space over* $\mathbb{R}$*,* M *a vector subspace of* X *and* $f : M \to \mathbb{R}$ *linear and continuous. Then* f *has a linear and continuous extension to all of* X*.*

Using Hahn-Banach, we can finally prove a requirement of X being locally convex with respect to the dual.
The proof below is inspired by the approach seen at [7] (at this source, the theorem is proved for the case X being a normed vector space.)

Corollary 8.9. *If X is a locally convex topological vector space, where $X \neq \{0\}$ and is not just generated the trivial semi-norm (the trivial semi-norm is the map from X that maps every element to 0) then $X^* \neq \{0\}$.*

Proof. As the topology on X is generated by non-trivial semi-norms, we have that $\exists x \in X$ and $p \in \mathcal{P}$ such that $p(x) \neq 0$. Then it holds that $p(\frac{x}{p(x)}) = 1$. Define $y = \frac{x}{p(x)}$ and the functional $\phi : \text{span}\{y\} \to \mathbb{R}$ by $\phi(\alpha y) = \alpha$. Then it holds that ϕ is linear:

(1) $\phi(\alpha y + \beta y) = \phi((\alpha + \beta)y) = \alpha + \beta = \phi(\alpha y) + \phi(\beta y)$
(2) $\phi(c(\alpha y)) = \phi((c\alpha)y) = c(\alpha) = c\phi(\alpha y)$

It also holds that ϕ is bounded by the semi-norm:

$$|\phi(\alpha y)| = |\alpha| \leq 2|\alpha| \cdot 1 = 2|\alpha| \cdot p(y) = 2p(\alpha y).$$

which implies that ϕ is continuous [6]. Hence, by theorem 8.8, ϕ has a continuous linear extension to all of X, (say $\bar{\phi}$). Then $\bar{\phi} \in X^*$ and $\bar{\phi} \neq 0$, hence $X^* \neq \{0\}$. □

Remark 8.10. In case X is a locally convex topological vector space generated by just the trivial semi-norm (say p), then $X^* = \{0\}$.
Indeed: Note that $\forall A \in \mathscr{B}(\mathbb{R})$

$$p^{-1}(A) = \begin{cases} X & ,0 \in A \\ \varnothing & ,0 \notin A. \end{cases}$$

Hence, X has the trivial topology. Now let $\phi \in X^*$. Then ϕ is continuous and so $\forall y \in \mathbb{R}$ $\phi^{-1}(\{y\})$ is either X or $\varnothing$. This means that ϕ has the form:

$$\phi(x) = C, \text{ where } C \in \mathbb{R} \text{ is a constant.}$$

But then by the linearity of ϕ :

$$C = \phi(-x) = -\phi(x) = -C$$

which implies that $C = 0$. Hence proven that $X^* = \{0\}$.

Now that we have established this result, we can finally prove that $L^0(X, \Sigma, \mu)$ with the topology of convergence in measure is not locally convex (in general). The corollary below is a worked out version of the one seen at [8].

Corollary 8.11. *$L^0([0,1], \mathcal{B}([0,1]), \lambda)$ together with the topology of convergence in measure is not locally convex*

Proof. Consider $L^0([0,1]) = \{\dot{f} : f : [0,1] \to \mathbb{R} \text{ measurable } \}$. Then we have earlier defined the pseudometric: $\bar{\phi}_F(\dot{f}, \dot{g}) = \int_F \min(|f-g|, 1) d\lambda$ and that

$$\dot{f}_n \to \dot{f} \text{ in measure } \iff \phi_F(f_n, f) \to 0, \text{ where } F \in \mathcal{B}([0,1]).$$

Assume $L^0([0,1])$ is locally convex. Then by corollary 8.9: $\exists \phi \in (L^0)^* : \phi(\dot{f}) \neq 0$ for some $\dot{f} \in L^0$. Let

$$g_{2^k+m} = 1_{[\frac{m}{2^k}, \frac{m+1}{2^k}]} \text{ for } k \in \mathbb{N} \text{ and } m \in \{0, ..., 2^k - 1\}.$$

Then note that: $f = \sum_{n_k=2^k}^{2^{k+1}-1} g_{n_k} f$. Hence:

$$0 < |\phi(f)| = |\phi(\sum_{n_k=2^k}^{2^{k+1}-1} g_{n_k} f)|$$

$$= |(\sum_{n_k=2^k}^{2^{k+1}-1} \phi(g_{n_k} f))| \qquad \text{(linearity)}.$$

$$\stackrel{\triangle}{\leq} \sum_{n=2^k}^{2^{k+1}-1} |\phi(g_{n_k} f)| \quad \forall k \in \mathbb{N}.$$

Then it holds that

$$\forall k \in \mathbb{N} \quad \exists n_k \in \{2^k, ..., 2^{k+1}-1\} : |\phi(g_{n_k} f)| \geq \tfrac{1}{2^k}|\phi(f)|.$$

Indeed, suppose $\exists k : \forall n_k \quad 2^k|\phi(g_{n_k} f)| < |\phi(f)|$. Then

$$2^k|\phi(g_{n_k} f)| < |\phi(f)| \leq \sum_{n_k=2^k}^{2^{k+1}-1} |\phi(g_{n_k} f)| \leq 2^k \max_{n_k} |\phi(g_{n_k} f)|.$$

Since the left-hand side is supposed to hold $\forall n_k$:

$$2^k \max_{n_k} |\phi(g_{n_k} f)| < 2^k \max_{n_k} |\phi(g_{n_k} f)|.$$

Hence we have reached a contradiction.
Let $h_k = 2^k g_{n_k} f$, then since

$$\mu(|h_k| > 0) = \mu(2^k g_{n_k} |f| > 0) \leq \mu(g_{n_k} > 0) = \mu([\tfrac{m}{2^k}, \tfrac{m+1}{2^k}]) = \tfrac{1}{2^k}$$

it can be proven that $h_k \to 0$ in measure. To see this, first note that

$$\min(|h_k|, 1) \leq 1_{\{|h_k|>0\}}.$$

This holds as:

(1) Case 1: $x \in \{|h_k| > 0\}$. Then $\min(|h_k|(x), 1) \leq 1 = 1_{\{|h_k|>0\}}(x)$
(2) Case 2: $x \in \{|h_k| = 0\}$. Then $\min(|h_k|(x), 1) = 0 = 1_{\{|h_k|>0\}}(x)$.

And so we get:

$$0 \leq \bar{\phi}_F(h_k, 0) = \textstyle\int_F \min(|h_k|, 1) d\lambda \leq \int_F 1_{|h_k|>0} d\lambda \leq \tfrac{1}{2^k},\ F \in \mathcal{B}([0,1])$$

so it converges in measure.
Now consider the sequence $(n_k)_{k\geq 1}$ such that $|\phi(g_{n_k} f)| \geq \frac{1}{2^k}|\phi(f)|$ (as established before). We see that:

$$\phi(g_{n_k} f) \geq \tfrac{1}{2^k}|\phi(f)| \iff |2^k \phi(g_{n_k} f)| \geq |\phi(f)| \stackrel{\text{linearity}}{\iff} |\phi(h_k)| \geq |\phi(f)|.$$

Since $h_k \to 0$ in measure and ϕ continuous, we hence see that $0 \geq |\phi(f)| > 0$, contradiction. $\square$

Remark 8.12.

(1) The result of corollary 8.11 can be extended to a more general setting. It turns out that for $L^0(S, \Sigma, \mu)$ with a finite non-atomic (see [9, p. 395 Def 10.51] for definition non-atomic. For example $\mathbb{N}$ together with the counting measure is atomic and $\mathbb{R}$ with Lebesgue measure is non-atomic) the dual of $L^0(S, \Sigma, \mu)$ is trivial [9, p. 481 Thm 13.41] and so is not locally convex by corollary 8.9 for the topology of convergence in measure. This result also holds for L^p spaces together with the topology of convergence in measure (where $0 < p \leq \infty$)([9, p. 482 Thm. 13.43]). Showing, for instance, that $L^p([0,1], \mathcal{B}([0,1]), \lambda)$ is not locally convex as well.

(2) For $0 < p < 1$, $L^p(S, \Sigma, \mu)$ together with the topology induced by the metric:

$$d(f, g) = \int |f - g|^p d\mu$$

is locally convex if and only if the range of μ consists of only finitely many values [17, p. 5 Thm 3.2].

(3) For $L^0(\mathbb{N})$ together with the product topology, it holds that the product topology is generated by the family of semi-norms:

$$p_n(x) = |x(n)|.$$

Hence by definition 8.3, $L^0(\mathbb{N})$ is locally convex in this instance. What can also be observed is that convergence in measure is equivalent with convergence with the product topology.
Indeed, suppose that $(\dot{x}_n)_{n\geq 1}, \dot{x} \in L^0(\mathbb{N})$ and $E \in \mathcal{P}(\mathbb{N})$. Note that the equivalence class of any element in $L^0(\mathbb{N})$ only consists of that one element under the counting measure. Then as

$$\sum_{j=1}^{k} \min(|x_n(j) - x(j)|, 1) 1_{\{j\}} \uparrow \min(|x_n - x|, 1)$$

by the Monotone Convergence Theorem we obtain:

$$\bar{\phi}_E(\dot{x}_n, \dot{x}) = \sum_{j \in E} \min(|x_n(j) - x(j)|, 1)$$

Hence $\bar{\phi}_E(\dot{x}_n, \dot{x}) \to 0 \iff \forall j \in E : x_n(j) \to x(j)$. And so we obtain convergence under the product topology and convergence of topology in measure are equivalent. By theorem 6.6, the topology of convergence in measure on $L^0(\mathbb{N})$ is metrizable (as $\mathbb{N}$ together with the counting measure is σ-finite) and $L^0(\mathbb{N})$ under the product topology is metrizable as well metrizable as $\mathbb{N}$ is countable ([9, p. 206 Lemma 5.74]). This implies that both topologies are equivalent ([13]). Hence we can conclude that $L^0(\mathbb{N})$ under the topology of convergence in measure is locally convex.
$L^p(\mathbb{N})$ for $0 < p < 1$ under the topology of convergence in measure is not locally convex by the second remark (as the counting measure does not take finitely many values in this case).

9. Essential supremum

In applications of measure theory to financial markets, it is common to let the probability distribution of the market depend on time. This means that each second corresponds to a potentially different random variable and so it is valuable to us to consider these different distributions in a set. With sets in $\mathbb{R}$ one is able to define a supremum and one might want to be able to have this concept defined for a set of random variables to be able to predict behavior of the random variables over time. Hence, in this section we will define a supremum over a set of measurable functions. The supremum over a set of non-negative measurable functions is called an essential supremum and is defined below.

Definition 9.1. *Let $(\Omega, \mathcal{F}, Q)$ be a measure space and let $\mathcal{X}$ be a non-empty set of non-negative measurable functions defined on $(\Omega, \mathcal{F}, Q)$. The essential supremum of $\mathcal{X}$, denoted by $\operatorname{ess\,sup} \mathcal{X}$ is a measurable function X^* satisfying:*

(1) $\forall X \in \mathcal{X} : X \leq_{a.e.} X^*$
(2) If Y is a measurable satisfying $X \leq_{a.e.} Y \quad \forall X \in \mathcal{X}$, then $X^* \leq_{a.e.} Y$

Remark 9.2. This definition of an essential supremum differs from the one used for localizability (see definition 7.8).

Corollary 9.3. *Under the same conditions as definition 9.1, if the essential supremum exists then it is unique (in an a.e. sense).*

Proof. Suppose X_1^* and X_2^* are both the essential supremum of $\mathcal{X}$. Then since $X \leq_{a.e.} X_1^*$ and $X \leq_{a.e.} X_2^* \quad \forall X \in \mathcal{X}$ we have by part two of the definition that: $X_1^* \leq_{a.e.} X_2^*$ and $X_2^* \leq_{a.e.} X_1^*$, hence obtaining that $X_1^* =_{a.e.} X_2^*$ □

In order to prove existence of X^*, we will first define a measure which is absolutely continuous with respect to Q. Then it turns out that X^* is actually the Radon-Nikodym derivative of Q with that said measure.

Definition 9.4. *Given $\mathcal{X}$ as defined in definition 9.1 and given $A \in \mathcal{F}$, then we call $\pi = (K; A_1, .., A_K; X_1, .., X_K)$ an $\mathcal{X}-$partition of A if:*

(1) $K \in \mathbb{N}$
(2) $A_i \cap A_j = \varnothing \quad \forall i \neq j$, *where* $A_i \in \mathcal{F}$ *and* $\bigcup_{i=1}^{K} A_i = A$
(3) $X_1, ..., X_K \in \mathcal{X}$

For $\lambda \in (0, \infty]$, we define:
$\mu_\pi^\lambda(A) = \int_\Omega \sum_{k=1}^{K} \min(X_k, \lambda) 1_{A_k} dQ$ and $\mu^\lambda(A) = \sup\{\mu_\pi^\lambda(A) : \pi\ \mathcal{X}\text{-partition of } A\}$.

Theorem 9.5.

(1) μ^λ *is a non-negative function on* $\mathcal{F}$.
(2) μ^λ *is finitely additive*

Proof.

(1) Per definition, $\lambda > 0$, $X_k \geq 0$ (as $\mathcal{X}$ consists of non-negative measurable functions) and $1_{A_k} \geq 0$, hence $\mu_\pi^\lambda(A) \geq 0$ for any partition π, and so $\mu^\lambda(A) \geq 0$.

(2) Let $A, B \in \mathcal{F}$: $A \cap B = \varnothing$.
Then I will first show that:
$\{\mu_\pi^\lambda(A \cup B) : \pi \text{ partition}\} = \{\mu_{\pi_A}^\lambda(A) + \mu_{\pi_B}^\lambda(B) : \pi_A, \pi_B \text{ partitions of } A,\ B\}$.
First I will show that set on the right is contained in set on the left.
Since $A \cap B = \varnothing$, the following is a partition of $A \cup B$:

$$(k_A + k_B; A_1, .., A_{k_A}, B_{k_A+1}, ..., B_{k_A+k_B}, X_1, ..., X_{k_A}, X_{k_A+1}, ..., X_{k_A+k_B})$$

where

$$\pi_A = (k_A, A_1, .., A_k, X_1, .., X_{k_A}) \quad \text{partition of A}$$
$$\pi_B = (k_B, B_{k_A+1}, .., B_{k_A+k_B}, X_{k_A+1}, .., X_{k_A+k_B}) \quad \text{partition of B.}$$

we have: $\mu_\pi^\lambda(A \cup B) = \int \sum_{k=1}^{k_A+k_B} \min(X_k, \lambda) 1_{A_k} dQ = \mu_{\pi_A}^\lambda(A) + \mu_{\pi_B}^\lambda(B)$.
Next, I will show that the left set is contained in the right set.
Let $\pi = (K; A_1, ..., A_K; X_1, .., X_K)$ be a partition of $A \cup B$.
Then

$$\pi_A = (K; A_1 \setminus B, ..., A_K \setminus B; X_1, ..., X_K)$$
$$\text{and}$$
$$\pi_B = (k; A_1 \setminus A, ..., A_K \setminus A; X_1, ..., X_K)$$

are $\mathcal{X}-$ partitions of A and B respectively and since

$$1_{A_i \setminus A} + 1_{A_i \setminus B} = 1_{A_i} \ \forall i = 1, .., K$$

we obtain that $\mu_\pi^\lambda(A \cup B) = \mu_{\pi_A}^\lambda(A) + \mu_{\pi_B}^\lambda(B)$.
Hence we obtain our desired result:

$$\begin{aligned}
\mu^\lambda(A \cup B) &= \sup\{\mu_\pi^\lambda(A \cup B) : \pi\} \\
&= \sup\{\mu_{\pi_A}^\lambda(A) + \mu_{\pi_B}^\lambda(B) : \pi_A, \pi_B\} \\
&= \sup\{\mu_{\pi_A}^\lambda(A) : \pi_A\} + \sup\{\mu_{\pi_B}^\lambda(B) : \pi_B\} \qquad (\mu_{\pi_A}^\lambda, \mu_{\pi_B}^\lambda \geq 0) \\
&= \mu^\lambda(A) + \mu^\lambda(B).
\end{aligned}$$

.

□

Remark 9.6. Per definition, we have that $u_\pi^\infty(A) = \int \sum_{k=1}^{K} X_k 1_{A_k} dQ$ for some partition π of A. Note that it also holds that $\lim_{\lambda \to \infty} \min(X_k, \lambda) = X_k$. Since this limit is increasing in λ and $\min(X_k, \lambda)$ is measurable, we obtain that by the Monotone Convergence Theorem:

$$\begin{aligned}
\mu_\pi^\infty(A) &= \lim_{\lambda \to \infty} \int \sum_{k=1}^{K} \min(X_k, \lambda) 1_{A_k} dQ \\
&= \sup_{\lambda \in (0,\infty)} \int \sum_{k=1}^{K} \min(X_k, \lambda) 1_{A_k} dQ \qquad (\text{ integral increasing }) \\
&= \sup_{\lambda \in (0,\infty)} \mu_\pi^\lambda(A).
\end{aligned}$$

Hence the following holds:
$\mu^\infty(A) = \sup_\pi \sup_{\lambda \in (0,\infty)} \mu_\pi^\lambda(A) = \sup_{\lambda \in (0,\infty)} \sup_\pi \mu_\pi^\lambda(A) = \sup_{\lambda \in (0,\infty)} \mu^\lambda(A)$.

The proof of the following lemma for the case that $\lambda = \infty$ is partly inspired from the proof seen at [10, p. 324 Lemma A.2].

Lemma 9.7. *Suppose* $(\Omega, \mathcal{F}, \mu)$ *is a* $\sigma-$*finite space. Then* $\lambda \in (0, \infty]$, μ^λ *is* σ-*additive.*

Proof. Case 1: $\lambda < \infty$.
Assume $(\Omega, \mathcal{F}, Q)$ is a finite space. Since $\mu^\lambda(\varnothing) = 0$ and μ^λ is finitely additive, it suffices to show that for any sequence $(A_n)_{n\geq 1}$ with $A_n \downarrow \varnothing$ that $\mu^\lambda(A_n) \to 0$ $(n \to \infty)$ (see [16]). Note that for any partition π of A_n, it holds that:

$$\begin{aligned}\mu^\lambda_\pi(A_n) &= \int \sum_{k=1}^{K} \min(X_k, \lambda) 1_{A_{k,n}} dQ \\ &\leq \lambda Q(A_n).\end{aligned}$$

Since Q is measure, we have that $\lim_{n\to\infty} Q(A_n) = 0$, so we obtain that:

$$\lim_{n\to\infty} \mu^\lambda(A_n) = \lim_{n\to\infty} \sup\{\mu^\lambda_\pi(A_n) : \pi\} \leq \lambda \lim_{n\to\infty} Q(A_n) = 0.$$

Hence by the lemma, μ^λ is σ-additive. As μ^λ is $\sigma-$additive for finite spaces, it is also $\sigma-$additive for σ-finite spaces by theorem 9.8.
Case 2: $\lambda = \infty$.
Let $(A_j)_{j\geq 1}$ be an increasing sequence of sets in $\mathcal{F}$ with $A = \bigcup_{j=1}^{\infty} A_j$. By remark 9.6 it holds:

$$\begin{aligned}\lim_{j\to\infty} \mu^\infty(A_j) &= \sup_{j\in(0,\infty)} \sup_{\lambda\in(0,\infty)} \mu^\lambda(A_j) \\ &= \sup_{\lambda\in(0,\infty)} \sup_{j\in(0,\infty)} \mu^\lambda(A_j) \\ &= \sup_{\lambda\in(0,\infty)} \mu^\lambda(A) = \mu^\infty(A).\end{aligned}$$

Now let $(B_i)_{i\geq 1}$ be a sequence of disjoint sets in $\mathcal{F}$ with $\bigcup_{i=1}^{\infty} B_i = B$. Define $B'_j = \bigcup_{i=1}^{j} B_i$, then $(B'_j)_{j\geq 1}$ is increasing and $\bigcup_{j=1}^{\infty} B'_j = B$, hence by the result above

$$\begin{aligned}\lim_{j\to\infty} \mu^\lambda(B'_j) = \mu^\lambda(B) &\iff \lim_{j\to\infty} \mu^\lambda\Big(\bigcup_{i=1}^{j} B_i\Big) = \mu^\lambda(B) \\ &\overset{\text{thm9.5}}{\iff} \lim_{j\to\infty} \sum_{i=1}^{j} \mu^\lambda(B_i) = \mu^\lambda(B).\end{aligned}$$

□

Before I prove the theorem on the existence of the essential supremum of a set of non-negative measurable functions on a σ-finite space, I will first state and prove a lemma that shows that it suffices to show the existence for functions on finite spaces.

Lemma 9.8. *Suppose (S, Σ, Q) is a σ-finite space, where $S = \bigcup_{n=1}^{\infty} S_n$ and $Q(S_n) < \infty\ \forall n \geq 1$. Then for $\nu : \Sigma \to [0, \infty) : \nu(A) = \sum_{n=1}^{\infty} \frac{Q(S_n \cap A)}{1+2^n Q(S_n)}$ it holds that (S, Σ, ν) is a finite measure space.*
Furthermore, for measurable functions $X, Y : (S, \Sigma) \to \mathbb{R}$ it holds that:

$$X =_{a.e.} Y \text{ on } (S, \Sigma, \mu) \iff X =_{a.e.} Y \text{ on } (S, \Sigma, \nu).$$

Proof. I will first show that (S, Σ, ν) is a measure space:

(1) $\nu(\varnothing) = \sum_{n=1}^{\infty} \frac{Q(S_n \cap \varnothing)}{1+2^n Q(S_n)} = 0$.
(2) Let $A = \bigcup_{m \geq 1} A_m$ where $A_i \cap A_j = \varnothing\ \forall i \neq j$ and $A_i \in \Sigma\ \forall i \geq 1$. Then first it should be noted that:

$$\nu(A) = \sum_{n=1}^{\infty} \frac{Q(S_n \cap A)}{1 + 2^n Q(S_n)} \leq \sum_{n=1}^{\infty} \frac{Q(S_n)}{1 + 2^n Q(S_n)} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

Hence we can conclude the following:

$$\begin{aligned}
\nu(\bigcup_{m \geq 1} A_m) &= \sum_{n=1}^{\infty} \frac{Q(\bigcup_{m \geq 1} (S_n \cap A_m))}{1 + 2^n Q(S_n)} \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{Q((S_n \cap A_m))}{1 + 2^n Q(S_n)} && (Q \text{ is } \sigma\text{-finite}) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{Q((S_n \cap A_m))}{1 + 2^n Q(S_n)} && (\text{Sum is finite}) \\
&= \sum_{m=1}^{\infty} \nu(A_m).
\end{aligned}$$

It also holds that for a set $A \in \Sigma$:

$$Q(A) = 0 \iff \nu(A) = 0.$$

Indeed, in case $Q(A) = 0$ it follows trivially that $Q(S_n \cap A) = 0$ for any n, hence proving $\nu(A) = 0$.
In case $\nu(A) = 0$, we have that all summands must be equal to 0. This means that for any $n \geq 1$: $Q(S_n \cap A) = 0$. Hence: $0 \leq Q(A) = Q(\bigcup_{n=1}^{\infty} (S_n \cap A)) \leq \sum_{n=1}^{\infty} Q(S_n \cap A) = 0$
In case $X =_{a.e.} Y$ on (S, Σ, μ), we have that $\mu(\{\omega \in S : X(\omega) \neq Y(\omega)\}) = 0$, hence by our previous proof $X =_{a.e.} Y$ on (S, Σ, ν) and by symmetry, this also holds the other way around. $\square$

Remark 9.9. From the last part of lemma 9.8, it follows that if for a set of non-negative measurable functions $\mathcal{X}$ has an essential supremum on the finite space

(S, Σ, ν), then this essential supremum is equal (a.e.) to the essential supremum of $\mathcal{X}$ on the σ-finite space (S, Σ, Q).

Now we can finally prove existence of the essential supremum. The proof below is a generalisation of the proof seen at [10, p. 324 Lemma A.3] (the proof at this source only proves existence for the case that $(\Omega, \mathcal{F}, Q)$ is a probability space).

Theorem 9.10. *Suppose $(\Omega, \mathcal{F}, Q)$ is σ-finite. Let $\mathcal{X}$ be a non empty family of non-negative measurable functions. Then $X^* = \operatorname{ess\,sup} \mathcal{X}$ exists.*
Furthermore, if $\mathcal{X}$ is closed under pairwise maximization (meaning: $X, Y \in \mathcal{X} \implies \max(X, Y) \in \mathcal{X}$), then there exists non-decreasing sequence $(Z_n)_{n \geq 1}$, of measurable functions in $\mathcal{X}$ such that $X^* =_{a.e.} \lim_{z \to \infty} Z_n$.

Proof. First note that μ^∞ is absolutely continuous with respect to Q. Indeed: let $A \in \mathcal{F}$ such that $Q(A) = 0$. Then we see that:

$$0 \leq \mu_\pi^\lambda(A) = \int \sum_{k=1}^{K} \min(X_k, \lambda) 1_{A_k} dQ \leq \lambda \int 1_A dQ = \lambda Q(A) = 0.$$

Since π was taken arbitrary here: $\mu^\lambda(A) = 0$ and this gives

$$\mu^\infty(A) = \sup_{\lambda \in (0,\infty)} \mu^\lambda(A) = 0.$$

Hence by Radon-Nikodym, $\frac{d\mu^\infty}{dQ}$ exists (and can possibly take the value ∞, see Appendix C), and we define $X^* = \frac{d\mu^\infty}{dQ}$. Now, X^* has the following properties:

(1) $X^* \geq_{a.e.} 0$
(2) $\forall A \in \mathcal{F} : \mu^\infty(A) = \int 1_A X^* dQ$

Let $X \in \mathcal{X}$ and $A \in \mathcal{F}$ and consider the partition $\pi_1 = (1; A; X)$.
We have that for this π_1:

$$\int 1_A X dQ = \mu_{\pi_1}^\infty(A) \leq \sup_\pi \mu_\pi^\infty(A) = \mu^\infty(A) = \int 1_A X^* dQ.$$

Since A was arbitrary, we see that for $A = \{X > X^*\}$:

$$\int 1_{X > X^*} X dQ \leq \int 1_{X > X^*} X^* dQ \implies \int (X - X^*) 1_{X > X^*} dQ \leq 0.$$

Since the measurable function inside the integral is greater or equal to 0, we see that that integral is greater or equal to 0, hence: $(X - X^*) 1_{X > X^*} =_{a.e.} 0$, so $X \leq_{a.e.} X^*$. So condition 1 of definition 9.1 is satisfied.
Let Y be as in condition 2 of definition 9.1. Note that for any partition π of A:

$$\int 1_A X^* dQ = \mu^\infty(A) = \sup_\pi (\int \sum_{k=1}^{K} 1_{A_k} X_k dQ) \leq \int 1_A Y dQ$$

hence $\int 1_A X^* dQ \leq \int 1_A Y dQ \quad \forall A \in \mathcal{F}$. So we obtain $X^* \leq_{a.e.} Y$ (using the same argument as before).
Now assume $\mathcal{X}$ is closed under pairwise maximization and let ν be the measure on $(\Omega, \mathcal{F})$ as defined in lemma 9.8. By the above argument, $\mathcal{X}$ has an essential supremum, X_ν^*, with respect to the finite measure space $(\Omega, \mathcal{F}, \nu)$. Given $n \in \mathbb{N}$, we know that $\mu^n(\Omega) < \infty$ (with respect to measure ν). Per definition of a supremum, we can define a partition $\pi^{(n)} = (k^{(n)}, A_1^{(n)}, .., A_{k^{(n)}}^{(n)}, X_1^{(n)}, ..., X_{k^{(n)}}^{(n)})$ of Ω satisfying:

$$\mu^n(\Omega) \leq \mu^n_{\pi^{(n)}}(\Omega) + \tfrac{1}{n}.$$

Then $Y_n = \max(X_1^{(n)}, ..., X_{k^{(n)}}^{(n)})$ is also in $\mathcal{X}$ (by our assumption) and we see that since $\min(X_k^{(n)}, n) \leq X_k^{(n)} \leq \max(X_1^{(n)}, ..., X_{k^{(n)}}^{(n)}) = Y_n$, we acquire:

$$\mu^n_{\pi^{(n)}}(\Omega) = \int \sum_{k=1}^{K} \min(X_k^{(n)}, n) 1_{A_{k^{(n)}}} d\nu \leq \int \sum_{k=1}^{K} Y_n 1_{A_{k^{(n)}}} d\nu = \int Y_n d\nu$$

hence obtaining:

$$\mu^n(\Omega) \leq \int Y_n d\nu + \tfrac{1}{n}.$$

Likewise $Z_n = \max(Y_1, ..., Y_n)$ is in $\mathcal{X}$ and $\mu^n(\Omega) \leq \int Z_n d\nu + \frac{1}{n}$. Then letting $n \to \infty$ gives:
$\int X_\nu^* 1_\Omega d\nu = \mu^\infty(\Omega) \leq \lim_{n\to\infty} \int Z_n d\nu \overset{\text{MCT}}{=} \int (\lim_{n\to\infty} Z_n) d\nu.$
Also as

$$\lim_{n\to\infty} Z_n \leq_{a.e.} X_\nu^* \qquad (Z_n \in \mathcal{X} \quad \forall n \implies Z_n \leq_{a.e.} X_\nu^* \quad \forall n)$$

We see that $X_\nu^* =_{a.e.} \lim_{n\to\infty} Z_n$. Hence, by lemma 9.8 we have that $X_\nu^* =_{a.e.} \lim_{n\to\infty} Z_n$ with respect to Q as well. Since, by remark 9.9, X_ν^* is also the essential supremum of $\mathcal{X}$ with respect to Q, this finishes the proof. □

Under the restriction that $(\Omega, \mathcal{F}, Q)$ is σ-finite, we found that X^* exists and is the Radon-Nikodym derivative $\frac{d\mu^\infty}{dQ}$. In this case it can possibly take the value ∞. For the case of an interval (a, b) in $\mathbb{R}$, we know that $\sup(a, b) = b$. So requiring that the interval is bounded above, suffices to ensure that the supremum is finite in this case. We can impose a similar requirement on $\mathcal{X}$ for it to have an essential supremum that only takes finite values.

Theorem 9.11. *Let $(\Omega, \mathcal{F}, Q)$ be σ-finite and let $\mathcal{X}$ be a set of non-negative measurable functions for which it holds that it is almost everywhere bounded by a measurable function f who is almost everywhere bounded by a constant M i.e.:*

$$X \leq_{a.e.} f \leq_{a.e.} M < \infty \;\forall X \in \mathcal{X}$$

Then essential supremum of $\mathcal{X}$ only takes values in $\mathbb{R}$.

Proof. By Radon-Nikodym, it suffices to show that μ^∞ is σ-finite (as then $X^* = \frac{d\mu^\infty}{dQ}$ will take finite values, see Appendix C). We know $X = \bigcup_{n\geq 1} B_n$, where $Q(B_n) < \infty \;\forall n$. For an arbitrary $\mathcal{X}-$partition π of B_n it holds that:

$$\begin{aligned}\mu^\lambda_\pi(B_n) &= \sum_{k=1}^{K} \int \min(\lambda, X_k) 1_{A_k} dQ \\ &\leq MQ(B_n) < \infty\end{aligned}$$

Hence μ^λ is σ-finite (as established upper-bound does not depend on π) $\forall \lambda \in (0, \infty)$. And so as

$$\mu^\infty(B_n) = \sup_{\lambda \in (0,\infty)} \mu^\lambda(B_n)$$

μ^∞ is σ-finite as well (as established upper-bound also does not depend on λ). □

Example 9.12. Let $(\Omega, \mathcal{F}, Q)$ be a σ-finite space and let $\mathcal{X}$ be a set of non-negative measurable functions, closed under pairwise maximization. Define the following set:

$$\Psi = \{\tfrac{X}{1+X} : X \in \mathcal{X}\}.$$

Then as $\frac{X}{1+X} \leq_{a.e.} 1$ $\forall X \in \mathcal{X}$, we have that by theorem 9.11 that Ψ has an essential supremum which takes values in $\mathbb{R}$. Suppose that $\operatorname{ess\,sup} \Psi =_{a.e} 1$. Then by theorem 9.10, we have that $\exists$ non-decreasing sequence such that

$$\tfrac{X_n}{1+X_n} \in \Psi \text{ and } \lim_{n\to\infty} \tfrac{X_n}{1+X_n} =_{a.e.} 1.\ \forall n \geq 1.$$

For this sequence to exist it must hold that $X_n \to_{a.e.} \infty$ $(n \to \infty)$, which implies that $\mathcal{X}$ has no finite essential supremum.

Theorem 9.11 states that if $\mathcal{X}$ is bounded above by a function f which itself bounded above by a constant (a.e.), then the essential supremum of $\mathcal{X}$ only takes values in $\mathbb{R}$. It turns out that the requirement that f should be bounded above by a constant (a.e.) is not needed in order for the theorem to still hold. For this I will first extend the definition of the supremum of two elements in a Riesz space (as seen at definition 2.13) to the supremum of $|I|$ elements (where I is some index set) as follows:
$\sup(u_\tau : \tau \in I)$ is the element in a Riesz space X for which holds:

$$\forall a \in X\text{: } \sup(u_\tau : \tau \in I) \leq a \iff u_\tau \leq a\ \forall \tau \in I$$

Note that this element does not necessarily exist in X.

Remark 9.13. Note that if we take $X = L^0$ as our Riesz space, the above definition generalises the concept of an essential supremum to L^0. Indeed, for a set $\dot{\mathcal{X}} = \{\dot{u}_\tau : \tau \in I\}$ (where u_τ is a non-negative measurable function), $\sup(\dot{u}_\tau : \tau \in I)$ must have the following two properties:

1. $\forall \dot{u}_\tau \in \dot{\mathcal{X}}$: $\dot{u}_\tau \leq \sup(\dot{u}_\tau : \tau \in I)$ $\forall \tau \in I$.
2. If $\dot{y} \in L^0$ such that $\dot{u}_\tau \leq \dot{y}$ $\forall \tau \in I$, then $\sup(\dot{u}_\tau) \leq \dot{y}$

By observing that in case of existence $\sup(\dot{u}_\tau : \tau \in I) = \dot{\overline{\sup(u_\tau : \tau \in I)}}$, we see that the above is equivalent with

1. $u_\tau \leq_{a.e.} \sup(u_\tau : \tau \in I)$ $\forall \tau \in I$.
2. If $u_\tau \leq_{a.e.} y$ $\forall \tau \in I$, then $\sup(u_\tau : \tau \in I) \leq_{a.e.} y$.

Hence obtaining that (in case of existence): $\sup(\dot{u}_\tau : \tau \in I) = \dot{\overline{\operatorname{ess\,sup}\{u_\tau : \tau \in I\}}}$.

Using this extension, we can now prove the assertion. The proof below is an adjusted and worked out version to the one found at [11, p. 126 Ex. 23.3iv] (adjusted to fit the terminology of this thesis).

Theorem 9.14. *Let $(\Omega, \mathcal{F}, Q)$ be a σ-finite measure space.*
Then any subset of $\mathcal{X}$ of $L^0(\Omega, \mathcal{F}, Q)$, where the equivalence classes contain non-negative functions and which is bounded above by a function $v \in L^0(\Omega, \mathcal{F}, Q)$ has a supremum (as defined above) which is in $L^0(\Omega, \mathcal{F}, Q)$.

Proof. Denote $\mathcal{X}$ by $\mathcal{X} = \{\dot{u}_\tau : \tau \in I\}$, where I is some index set and u_τ non-negative. Assume $\mathcal{X}$ is bounded above by $\dot{v} \in L^0(\Omega, \mathcal{F}, Q)$ (i.e. $u_\tau \leq_{a.e.} v$ $\forall \tau$). For $n \in \mathbb{N}$ define: $\dot{u}_{\tau,n} = \inf(\dot{u}_\tau, n)$ (in sense of definition 2.13), then $\dot{u}_n = \dot{\overline{\operatorname{ess\,sup}\{u_{\tau,n} : \tau \in I\}}}$ exists in L^0 for every fixed n by theorem 9.11. Now consider $\dot{u}_0 = \sup(\dot{u}_n : n \geq 1)$. Then since $\dot{u}_n \leq \dot{v}$ $\forall n$: $\sup(\dot{u}_n : n \geq 1) \leq \dot{v} < \infty$, so $\sup(u_n : n \geq 1) <_{a.e.} \infty$. Define $\sup'(u_n : n \geq 1) = \sup(u_n :$

$n \geq 1)1_{\{\sup(u_n : n \geq 1) \neq \infty\}}$. Then $\sup'(u_n : n \geq 1)$ is measurable, real valued and $\sup'(u_n : n \geq 1) =_{a.e.} \sup(u_n : n \geq 1)$ meaning that we can view $\dot{u}_0$ as an element of L^0. Writing out the extended definition of the supremum of $|I|$ elements and essential supremum gives that

$$\dot{u}_0 = \sup(\dot{u}_{\tau,n} : \tau \in I, n \in \mathbb{N}) = \sup(\dot{u}_\tau : \tau \in I).$$

The first equality can be seen as follows: Let $\dot{a} \in L^0$. Then:

$$\begin{aligned}
\dot{u}_0 \leq \dot{a} &\overset{\text{def}}{\iff} \dot{u}_n \leq \dot{a} \quad \forall n \geq 1 \\
&\overset{\text{def}}{\iff} \operatorname{ess\,sup}\{\dot{u_{\tau,n}} : \tau \in I\} \leq \dot{a} \quad \forall n \geq 1 \\
&\iff \operatorname{ess\,sup}\{u_{\tau,n} : \tau \in I\} \leq_{a.e.} a \quad \forall n \geq 1 \\
&\iff u_{\tau,n} \leq_{a.e.} a \quad \forall n \geq 1, \forall \tau \in I \\
&\iff \dot{u}_{\tau,n} \leq \dot{a} \quad \forall n \geq 1, \forall \tau \in I.
\end{aligned}$$

Hence we obtain that $\dot{u}_0 = \sup(\dot{u}_{\tau,n} : \tau \in I, n \in \mathbb{N})$.
Similarly, for the second equality:

$$\begin{aligned}
\sup(\dot{u}_{\tau,n} : \tau \in I, n \geq 1) \leq \dot{a} &\iff \dot{u}_{\tau,n} \leq \dot{a} \quad \forall \tau, n \\
&\iff u_{\tau,n} \leq_{a.e.} a \quad \forall \tau, n \\
&\iff u_\tau \leq_{a.e.} a \quad \forall \tau \\
&\iff \dot{u}_\tau \leq \dot{a} \quad \forall \tau.
\end{aligned}$$

Hence obtaining $\sup(\dot{u}_\tau : \tau \in I) = \sup(\dot{u}_{\tau,n} : \tau \in I, n \geq 1)$ and so $\sup(\dot{u}_\tau : \tau \in I) = \dot{u}_0$, where $\dot{u}_0$ can be viewed as an element of L^0. $\square$

Remark 9.15. The theorem even holds in case Q is semi-finite (in sense of definition 4.12) ([11, p. 127 Ex. 23.3iv]).

10. Application: Stochastic Differential Equations

Up until now, we have only considered theoretical aspects of $\mathscr{L}^0$ and L^0. One of the many applications of this theory are Stochastic Differential Equations (SDEs). We know that Ordinary Differential Equations (ODEs) are of the form:

$$\frac{d}{ds}x_s = b(s, x_s), x_0 = x(0) \tag{10.1}$$

where $x : \mathbb{R} \to \mathbb{R}$. In most physical applications however, ODEs will not suffice to model phenomena accurately. This can for example be seen in case we interpret (10.1) as a particle which has position x_t at time t and speed $b(t, x_t)$. The ODE does not take phenomena into account such as measurement errors. As measurement errors are probabilistic, the position of a random particle at time t can be more realistically described as a random variable rather than a fixed number. As such, we want to model differential equations as to obtain random variables for each t and on this we will give a broad introduction in this chapter. Such equations are of the form:

$$dX_t = b(t, X_t) + \sigma(t, X_t)dB_t \tag{10.2}$$

where X_t denotes a random variable at time t and $\sigma(t, X_t)dB_t$ a random component. The exact definition will be given in the next section.

10.1. The definition of a SDE.

We will start this section by formally defining equation 10.2. For this equation, we expected the solution to give a random variable X_t for each time t. The collection $(X_t)_{t\geq 0}$ of solutions for each time t is called a stochastic process and is defined below.

Definition 10.1. *Let $(\Sigma, \mathscr{A}, P)$ be a probability space. A d-dimensional* stochastic process *indexed by $I \subseteq [0, \infty)$ is a family of random variables $X_t : \Sigma \to \mathbb{R}^d$ $t \in I$ We write $X = (X_t)_{t\in I}$. I is called the index set and $\mathbb{R}^d$ the state space.*

As pointed out earlier, (10.2) has a random component $\sigma(t, X_t)dB_t$. In order to solve SDEs, we will have to make an assumption on the distribution of the random component. As one might expect, the random component is linked to the normal distribution via a Brownian Motion, which is defined below.

Definition 10.2. *A d-dimensional* Brownian Motion $(B_t)_{t\geq 0}$ *is a d-dimensional stochastic process on a probability space $(\Sigma, \mathscr{A}, P)$ indexed by $[0, \infty)$ which has the following properties:*

(1) $B_0(\omega) = 0 \quad a.s.$
(2) $B_{t_n} - B_{t_{n-1}}, ..., B_{t_1} - B_{t_0}$ *are independent* $\quad \forall 0 = t_0 < .. < t_n < \infty$.
(3) $B_t - B_s \sim B_{t+h} - B_{s+h} \quad \forall 0 \geq s \leq t, h \geq -s$
(4) $B_t - B_s \sim \mathscr{N}^d(0, t - s)$
(5) $t \mapsto B_t(\omega)$ *is a continuous function* $\forall \omega \in \Omega$.

Notation: A d-dimensional Brownian Motion will be denoted by BM^d.

Remark 10.3. For a d-dimensional Brownian Motion $B_t = (B_t^1, .., B_t^d)$ it holds that all components $B_t^1, .., B_t^d$ are independent and are one dimensional Brownian Motions. The converse holds as well [3, p. 17 Theorem 2.16].

For our SDEs, we will also make assumptions on the σ-algebras corresponding to the domain of X_t and B_t. The definitions below will be used for these assumptions.

Definition 10.4. *A* filtration *on a probability space* $(\Omega, \mathscr{F}, P)$ *is a nested family of* σ*-algebras* $\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F}, s \leq t$ *satisfying:*

(1) $\mathscr{F}_s = \bigcap\limits_{t>s} \mathscr{F}_t, s \geq 0$
(2) *All* $A \in \mathscr{F}$ *with* $P(A) = 0$ *are contained in* $\mathscr{F}_0$.

Remark 10.5. One interpretation of a filtration is that it is the σ-algebra that keeps all the information from the past preserved (as $\mathscr{F}_s \subset \mathscr{F}_t$ for $s \leq t$).

Definition 10.6. *Let* $(B_t)_{t\geq 0}$ *be a* BM^d *on* $(\Omega, \mathscr{F}, P)$. *A filtration* $(\mathscr{F}_t)_{t\geq 0}$ *is called* admissible *if:*

(1) $\mathscr{F}_t^B \subset \mathscr{F}_t \quad \forall t \geq 0$, *where* $\mathscr{F}_t^B = \sigma(B_s : s \leq t)$
(2) $B_t - B_s \perp\!\!\!\perp \mathscr{F}_s \quad \forall 0 \leq s \leq t$.

Before we can define a solution of (10.2), we first have to define what $\sigma(t, X_t)dB_t$ means. For this we have to build up some stochastic integration theory, where we define the stochastic integral as a limit over integrals of simple processes.

Definition 10.7. *A real valued stochastic process* $(f(t, \bullet))_{t\in[0,T]}$ *(in sense of definition 10.1) of the form*

$$f(t, \omega) = \sum_{j=1}^{n} \phi_{j-1}(\omega) 1_{[s_{j-1}, s_j)}(t)$$

where $n \geq 1$ *and* $0 = s_0 \leq s_1 \leq ... \leq s_n \leq T$ *and* $\phi_j \in L^\infty(\mathcal{F}_{s_j})$, *is called a* simple process.
We denote $\mathcal{E}_T$ *for the family of all simple processes on* $[0, T]$ *and* $\mathcal{L}_T^2$ *as the closure of* $\mathcal{E}_T$ *in* $L^2(P \otimes \lambda)$ *(with respect to the* L^2*-norm).*

Definition 10.8. *Let* $I \subseteq [0, \infty]$ *and* $(\mathcal{F}_t)_{t\in I}$ *be a filtration. Then a* martingale $(X_t, \mathcal{F}_t)_{t\in I}$ *is a real valued process* $X_t : \Omega \to \mathbb{R}^d$ *satisfying:*

(1) $\mathbb{E}(|X_t|) < \infty$.
(2) X_t *is* $\mathcal{F}_t$ *measurable* $\forall t \in I$.
(3) $\mathbb{E}(X_t|\mathcal{F}_s) = X_s \; \forall s, t \in I \; s \leq t$.

Example 10.9. The BM^d as defined at definition 10.2 is a martingale with respect to any admissible filtration as defined at 10.6 ([3, p. 49 Ex. 5.2]).

Using the two definitions above, we can define the stochastic integral for simple processes.

Definition 10.10. *Let* M *be a continuous* L^2 *martingale and* $f \in \mathcal{E}_T$. *Then we define the* stochastic integral *as follows:*

$$\int_0^T f(s) dM_s = \sum_{j=1}^{n} f(s_{j-1})(M(s_j) - M(s_{j-1}))$$

where $0 = s_0 \leq s_1 \leq ... \leq s_n = T$.

Similar to the definition the integral over a positive measurable function, the stochastic integral is defined as a limit of integrals of simple processes.

Definition 10.11. *Let* $(B_t)_{t\geq 0}$ *be a* BM^1 *and let* $f \in \mathcal{L}_T^2$. *Then the stochastic integral is defined as:*

$$\int_0^t f(s) dB_s = \lim_{n\to\infty} \int_0^t f_n(s) dB_s$$

where $(f_n)_{n\geq 1} \subseteq \mathcal{E}_T$ *such that* $\lim\limits_{n\to\infty} f_n = f$ *in* $L^2(\lambda_T \otimes P)$.

Remark 10.12. ([3, p. 216 Ex. 14.15]) Not all intuition from (Riemann) integrals in $\mathbb{R}$ carries over to stochastic integrals. For instance, in case that $(B_t)_{t\geq 0}$ is a BM^1 and $T > 0$, then:

$$\int_0^T B_t dB_t = \tfrac{1}{2}(B_T^2 - T).$$

Using all the tools defined above, we can finally formulate what it means to be solution of (10.2).

Definition 10.13. *Let* $(B_t, \mathscr{F}_t)$ *be a* BM^d *with admissible filtration* $\mathscr{F}_t$*, where for each t:* $B_t : (\Omega, \mathscr{F}_t, P) \to (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$.
Let $b : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $\sigma : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ *be measurable functions.*
A solution of

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

with initial condition $X_0 = \xi$ *is a d-dimensional stochastic process* $(X_t)_{t\geq 0}$ *where for each t it holds that the restricted map:*
$X : [0,t] \times \Omega \to \mathbb{R}^d$ *where* $(s,\omega) \mapsto X_s(\omega)$ *is* $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$ *measurable* $\forall t \geq 0$.
It furthermore holds that $X_t(\omega) = (X_t^1(\omega), .., X_t^d(\omega)) \in \mathbb{R}^d$ *has that:*

$$X_t^j = \xi^j + \int_0^t b_j(s, X_s)ds + \sum_{k=1}^d \int_0^t \sigma_{jk}(s, X_s)dB_s^k \quad \forall j = 1, .., d.$$

We usually use the following short-hand notation for the condition above:

$$X_t = \xi + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s.$$

Remark 10.14. The measurability condition on the restricted map on X_t is called progressive measurability.

In the remainder of this text, I will use the following notation for b and σ as defined in definition 10.13:

$$|b|^2 = \sum_{j=1}^n b_j^2$$

$$|\sigma|^2 = \sum_{j=1}^n \sum_{k=1}^d \sigma_{jk}^2.$$

It can also be shown, using Cauchy-Schwarz, that for a matrix-vector multiplication σx ($x \in \mathbb{R}^d$) we have the following relation:

$$\begin{aligned}
|\sigma x| &= \sum_{j=1}^n (\sum_{i=1}^d (x_i \sigma_{ji})^2) \\
&\leq \sum_{j=1}^n (\sum_{i=1}^d x_i^2)(\sum_{i=1}^d \sigma_{ji}^2) \\
&= \sum_{i=1}^d x_i^2 \sum_{j=1}^n \sum_{i=1}^d \sigma_{ji}^2 = |\sigma| \cdot |x|.
\end{aligned}$$

Now that we have formally defined SDEs, one might be interested in techniques on how to solve them. As the techniques are beyond the scope of this thesis, we will not treat them here. But to give one an intuition on what these solutions might look like, two SDEs and their solutions will be compared to their corresponding (non-random) ODE and their solutions.

Example 10.15. [3, p. 275 Ex. 18.3] Consider the following one dimensional SDE (where we assume $\beta(t)$ and $\delta(t)$ are integrable):

$$dX_t = \beta(t)X_t dt + \delta(t)X_t dB_t.$$

The solution to this SDE is:

$$X_t = X_0 \exp(\int\limits_0^t \beta(s) - \tfrac{1}{2}(\delta(s))^2 ds) + \int_0^t \delta(s) dB_s.$$

The non-random ODE corresponding to this SDE is:

$$x'(t) = \beta(t)x(t)$$

which has as the solution

$$x(t) = x_0 \exp(\int\limits_0^t \beta(s)ds).$$

Example 10.16. (Langevin equation) The following SDE:

$$dX_t = -aX_t dt + bdW_t \ (a, b \in \mathbb{R})$$

has the corresponding solution:

$$X_t = \exp(-at)X_0 + b\int\limits_0^t \exp(-a(t-s))dW_s$$

The corresponding ODE:

$$x'(t) = -ax(t)$$

has as the solution:

$$x(t) = x_0 \exp(-at).$$

What one might readily notice in the examples above, is that the solutions to the corresponding ODE is always contained in solution to the SDE. Hence, one can interpret the solutions to SDEs as the solutions to the ODE ”plus a random component”.

10.2. **Existence and uniqueness SDE.** Main questions surrounding differential equation are always related to the existence and uniqueness of solutions. For the case of SDEs, for the case that $\sigma \equiv 0$ we have an ODE about which we have a lot of information regarding existence and uniqueness available to us in case we assume Lipschitz conditions on the coefficients.
Therefore, it seems natural to extend these Lipschitz conditions to general cases of SDEs. When we require the coefficients of our SDEs to be Lipschitz we will mean the following:

$$\sum_{j=1}^{n}|b_j(t,x)-b_j(t,y)|^2+\sum_{j=1}^{n}\sum_{k=1}^{d}|\sigma_{jk}(t,x)-\sigma_{jk}(t,y)|^2 \leq L^2|x-y|^2 \tag{10.3}$$

$\forall x, y \in \mathbb{R}^n, t \in [0,T], L = L_T\ \forall T > 0$.
When we assume a linear growth on the coefficients, it is meant that:

$$\sum_{j=1}^{n}|b_j(t,x)|^2+\sum_{j=1}^{n}\sum_{k=1}^{d}|\sigma_{jk}(t,x)|^2 \leq M^2(1+|x|)^2 \tag{10.4}$$

for $x \in \mathbb{R}^n, t \in [0,T]$ and $M = M_T\ \forall T > 0$.
Before we can state and prove the theorems regarding existence and uniqueness, we will first establish a bound the expected difference between two processes satisfying (10.2). For the proof of the bound on the processes (and other proofs in this section) we will need to establish a few inequalities from number theory, which are given in the remark below.

Remark 10.17. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|^k$, $k \geq 1$. Then as f is convex, the following holds:

$$|\frac{1}{2}a+\frac{1}{2}b|^k \leq \frac{1}{2}|a|^k+\frac{1}{2}|b|^k \iff$$
$$2^{-k}|a+b|^k \leq 2^{-1}|a|^k+2^{-1}|b|^k \iff$$
$$|a+b|^k \leq 2^{k-1}|a|^k+2^{k-1}|b|^k$$

We can extend this idea to the sum of three elements as follows. Define $b' = \frac{b}{2}, c' = \frac{c}{2}$, then:

$$\begin{aligned}|\frac{1}{3}a+\frac{1}{3}b+\frac{1}{3}c|^k &\leq \frac{1}{3}|a|^k+\frac{2}{3}|b'+c'|^k \\ &\leq \frac{1}{3}|a|^k+\frac{2}{3}(2^{k-1}|b'|^k+2^{k-1}|c'|^k)\end{aligned}$$

Hence obtaining the following inequality:

$$|a+b+c|^k \leq 3^{k-1}|a|^k+3^{k-1}|b|^k+3^{k-1}|c|^k$$

This argument can be extended to sums of an arbitrary amount of numbers (countable) using induction. In particular, we see that the following two inequalities hold ($k = 2$):

(1) $|a+b+c|^2 \leq 3|a|^2+3|b|^2+3|c|^2$
(2) $|a+b|^2 \leq 2|a|^2+2|b|^2$.

For the proof of the bound, we will also need an inequality from Stochastic Integral Theory. The more general statement and its proof can be found at [3, p. 213 Thm. 14.13d].

Corollary 10.18. *Assuming the Lipschitz condition 10.3, it holds that for σ as defined at definition 10.13:*

$$\mathbb{E}(\sup_{t\leq T}|\int_0^t \sigma(s,X_s)-\sigma(s,Y_s)dB_s|^2) \leq 4\int_0^T \mathbb{E}(|\sigma(s,X_s)-\sigma(s,Y_s)|^2)ds.$$

The proof of the corollary below is a worked out version of the one found at [3, p. 280 Thm. 18.9].

Corollary 10.19. *(Stability) Let $(B_t,\mathcal{F}_t)_{t\geq 0}$ be a BM^d and assume that the coefficients $b:[0,\infty)\times\mathbb{R}^n\to R^n$ and $\sigma:[0,\infty)\times R^n\to R^{n\times d}$ of (10.2) satisfy (10.3). If $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are any two solutions of (10.2) with $\mathcal{F}_0$ measurable initial condition*

$$X_0=\xi\in L^2(P) \text{ and } Y_0=\nu\in L^2(P)$$

then:

$$\mathbb{E}(\sup_{t\leq T}(|X_t-Y_t|^2)) \leq 3\exp\left(3L^2(T+4)T\right)\mathbb{E}(|\xi-\nu|^2).$$

Proof. It holds that

$$X_t-Y_t=(\xi-\nu)+\int_0^t(b(s,X_s)-b(s,Y_s)ds+\int_0^t\sigma(s,X_s)-\sigma(s,Y_s)dB_s$$

by the second to last given equation of remark 10.17:

$$\begin{aligned}
&\mathbb{E}(\sup_{t\leq T}|X_t-Y_t|^2)\\
&=\mathbb{E}(\sup_{t\leq T}|(\xi-\nu)+\int_0^t b(s,X_s)-b(s,Y_s)ds+\int_0^t\sigma(s,X_s)-\sigma(s,Y_s)dB_s|^2)\\
&\leq\mathbb{E}(\sup_{t\leq T}(3(\xi-\nu)^2+3(\int_0^t b(s,X_s)-b(s,Y_s)ds)^2+3(\int_0^t\sigma(s,X_s)-\sigma(s,Y_s)dB_s)^2)\\
&=3\,\mathbb{E}((\xi-\nu)^2)+3\,\mathbb{E}((\int_0^T b(s,X_s)-b(s,Y_s)ds)^2)+3\,\mathbb{E}((\int_0^T\sigma(s,X_s)-\sigma(s,Y_s)dB_s)^2)
\end{aligned}$$

As it holds that:

$$\begin{aligned}
(\int_0^T|b(s,X_s)-b(s,Y_s)\cdot 1|ds)^2 &\overset{\text{C.S.}}{\leq}\int_0^T|b(s,X_s)-b(s,Y_s)|^2ds\int_0^T 1^2ds\\
&=T\int_0^T|b(s,X_s)-b(s,Y_s)|^2ds.
\end{aligned}$$

we obtain:

$$\begin{aligned}
&\mathbb{E}(\sup_{t\leq T}|\int_0^t b(s,X_s)-b(s,Y_s)ds|^2)\\
&\leq\mathbb{E}(\sup_{t\leq T}(\int_0^t|b(s,X_s-b(s,Y_s)|ds)^2)
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}((\int_0^T |b(s,X_s) - b(s,Y_s) \cdot 1| ds)^2) \\
&\leq \mathbb{E}(\int_0^T |b(s,X_s) - b(s,Y_s)|^2 ds) T && \text{(Cauchy-Schwarz)} \\
&\leq L^2 T (\int_0^T \mathbb{E}(|X_s - Y_s|^2) ds && \text{(Lipschitz)}
\end{aligned}
$$

For σ it holds:

$$
\begin{aligned}
&\mathbb{E}(\sup_{t \leq T} |\int_0^t \sigma(s,X_s) - \sigma(s,Y_s) dB_s|^2) \\
&\leq 4 \int_0^T \mathbb{E}(|\sigma(s,X_s) - \sigma(s,Y_s)|^2) ds && \text{(Cor. 10.18)} \\
&\leq 4L^2 \int_0^T \mathbb{E}(|X_s - Y_s|^2) ds.
\end{aligned}
$$

Hence we obtain that:

$$
\begin{aligned}
\mathbb{E}(\sup_{t \leq T} |X_t - Y_t|^2) &\leq 3\,\mathbb{E}(|\xi - \nu|^2) + 3L^2(T+4) \int_0^T \mathbb{E}(|X_s - Y_s|^2) ds \\
&\leq 3\,\mathbb{E}(|\xi - \nu|^2) + 3L^2(T+4) \int_0^T \mathbb{E}(\sup_{r \leq s} |X_r - Y_r|^2) ds.
\end{aligned}
$$

Now Grönwall's inequality (see Appendix B) with $u(T) = \mathbb{E}(\sup_{t \leq T} |X_t - Y_t|^2)$, $a(s) = 3\,\mathbb{E}(|\xi - \nu|^2), b(s) = 3L^2(T+4)$ yields:

$$
\mathbb{E}(\sup_{t \leq T} |X_t - Y_t|^2) \leq 3 \exp(3L^2(T+4)T)\,\mathbb{E}(|\xi - \nu|^2).
$$

□

Using the established bound on two solutions, we can now prove uniqueness. The proof below is the one found at [3, p. 281 Cor. 18.10] with an extra detail added.

Theorem 10.20. *(Uniqueness) Let $(B_t, \mathcal{F}_t)$ be a BM^d and assume $b : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$ as defined at definition 10.13 satisfies the Lipschitz condition 10.3. Then for any two solutions $(X_t)_{t \geq 0}$,$(Y_t)_{t \geq 0}$ with $\mathcal{F}_0$ measurable condition $X_0 = Y_0 = \xi \in L^2(P)$ it holds that:*

$$
P(\forall t \geq 0 : X_t = Y_t) = 1.
$$

Proof. The bound at corollary 10.19 shows that:

$$
\mathbb{E}(\sup_{t \leq n} |X_t - Y_t|^2) \leq 0 \; \forall n \geq 0.
$$

Hence we have that:

$$P(\forall t \in [0,n] : X_t = Y_t) = 1 \ \forall n \geq 1.$$

Define $A_n = \{\forall t \in [0,n] : X_t = Y_t\}$. Then as $(A_n)_{n\geq 1}$ is non-increasing:

$$\begin{aligned} P(\forall t \geq 0 : X_t = Y_t) &= P(\bigcap_{n=1}^{\infty} A_n) \\ &= \lim_{n\to\infty} P(A_n) = 1. \end{aligned}$$

□

The proof of the following theorem, which shows existence of an unique solution to our SDE under the Lipschitz and linear growth condition is based on the proof seen at [3, p. 282 Thm. 18.11].

Theorem 10.21. *(Existence) Let $(B_t, \mathcal{F}_t)$ be a BM^d and assume coefficients $b : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,\infty) \times \mathbb{R}^n \to R^n$ as defined at definition 10.13 satisfy Lipschitz condition (10.3) and linear growth condition (10.4). Then for every $\mathcal{F}_0$ measurable initial condition $X_0 = \xi \in L^2(P)$ there exists a unique solution $(X_t)_{t\geq 0}$ of equation (10.2) that satisfies:*

$$\mathbb{E}(\sup_{t\leq T} |X_s|^2) \leq k_T \, \mathbb{E}((1+|\xi|)^2) \ \forall T > 0.$$

Proof. Just as for ODEs, we will use a Picard iteration scheme. Define

$$\begin{aligned} X_0(t) &= \xi \\ X_{n+1}(t) &= \xi + \int_0^t \sigma(s, X_n(s)) dB_s + \int_0^t b(s, X_n(s)) ds. \end{aligned}$$

(1) By the last inequality given at remark 10.17, we see that:

$$\begin{aligned} |X_{n+1}(t) - \xi| &= |\int_0^t \sigma(s, X_n(s)) dB_s + \int_0^t b(s, X_n(s)) ds|^2 \\ &\leq 2|\int_0^t \sigma(s, X_n(s)|^2 + 2|\int_0^t b(s, X_n(s) ds|^2. \end{aligned}$$

Fully analogous to the computations in the proof of corollary 10.19, where we now set $X_s = X_n(s)$, omit the Y terms and use the linear growth condition (10.4) instead of Lipschitz, we obtain:

$$\begin{aligned} \mathbb{E}(\sup_{t\leq T}(|X_{n+1}(t) - \xi|)^2) &\leq 2M^2(T+4) \int_0^T \mathbb{E}((1+|X_n(s)|)^2) ds \\ &\leq 2M^2(T+4) \int_0^T \mathbb{E}(\sup_{s\leq T}(1+|X_n(s)|)^2) ds \\ &= 2M^2(T+4) \int_0^T 1 ds \, \mathbb{E}(\sup_{s\leq T}(1+|X_n(s)|)^2) \end{aligned}$$

$$= 2M^2(T+4)T\,\mathbb{E}(\sup_{s\le T}(1+|X_n(s)|)^2).$$

(2) Similar to (1), we have:

$$|X_{n+1}(t)-X_n(t)| \le 2|\int_0^t b(s,X_n(s))-b(s,X_{n-1}(s))ds|^2 + \\ 2|\int_0^t \sigma(s,X_n(s))-\sigma(s,X_{n-1}(s)dB_s|^2$$

Analogous to the computations of the proof of corollary 10.19 using $X = X_n$ and $Y = X_{n-1}$ we see that:

$$\mathbb{E}(\sup_{t\le T}|X_{n+1}(t)-X_n(t)|^2) \le 2L^2(T+4)\int_0^T \mathbb{E}(|X_n(s)-X_{n-1}(s)|^2)ds. \tag{10.5}$$

Define:

$$\phi_n(T) = \mathbb{E}(\sup_{t\le T}|X_{n+1}(t)-X_n(t)|^2) \text{ and } c_T = 2L^2(T+4).$$

Then we can establish using induction that $\forall n \ge 0$:

$$\phi_n(T) \le c_T^n \tfrac{T^n}{n!}\phi_0(T)$$

Base case: For $n = 0$, result follows immediately.

Induction step: Suppose $\phi_{k-1}(s) \le c_s^{k-1}\frac{s^{k-1}}{(k-1)!}\phi_0(s)$ $(s \in [0,T])$ holds. Then:

$$\begin{aligned}
\phi_k(T) &\le 2L^2(T+4)\int_0^T \mathbb{E}(|X_k(s)-X_{k-1}(s)|^2)ds && (10.5)\\
&\le 2L^2(T+4)\int_0^T \mathbb{E}(\sup_{w\le s}(|X_k(s)-X_{k-1}(s)|^2))ds\\
&\le 2L^2(T+4)\int_0^T \phi_{k-1}(s)ds\\
&\le c_T\int_0^T c_s^{k-1}\frac{s^{k-1}}{(k-1)!}\phi_0(s)ds && \text{(Induction assumption)}\\
&\le c_T^k\int_0^T \frac{s^{k-1}}{(k-1)!}ds\phi_0(T) && (\phi_0, c_s^{k-1} \text{ increase in } s)\\
&= c_T^k\frac{T^k}{(k)!}\phi_0(T).
\end{aligned}$$

As the final estimate from the first part of the proof (1), for $n = 0$, is equivalent to:

$$\phi_0(t) = \mathbb{E}(\sup_{s\le T}|X_1(t)-\xi|^2) \le 2M^2(T+4)T\,\mathbb{E}((1+|\xi|)^2)$$

we obtain:

$$\sum_{n=0}^{\infty}(\mathbb{E}(\sup_{s\leq T}|X_{n+1}-X_n|^2))^{\frac{1}{2}} \leq \sum_{n=0}^{\infty}(c_T^n\frac{T^n}{n!}\phi_0(T))^{\frac{1}{2}}$$
$$\leq \sum_{n=0}^{\infty}(c_T^n\frac{T^n}{n!}2M^2(T+4)T\,\mathbb{E}((1+|\xi|)^2))^{\frac{1}{2}}$$
$$= \mathbb{E}((1+|\xi|)^2)^{\frac{1}{2}}\sum_{n=0}^{\infty}(c_T^n\frac{T^n}{n!}2M^2(T+4)T)^{\frac{1}{2}}$$
$$= \mathbb{E}((1+|\xi|)^2)^{\frac{1}{2}}C(L,M,T)$$

where $C_T = C(L,M,T) = \sum_{n=0}^{\infty}(c_T^n\frac{T^n}{n!}2M^2(T+4)T)^{\frac{1}{2}}$ and this sum converges for any $T>0$.

(3) Let $n \geq m$. By the triangle-inequality for the sup-norm it holds that:

$$\sup_{s\leq T}|X_n(s)-X_m(s)| = \sup_{s\leq T}|\sum_{j=m+1}^{n}X_j - X_{j-1}| \leq \sum_{j=m+1}^{n}\sup_{s\leq T}|X_j-X_{j-1}|.$$

Now applying the $L^2(P)$-norm to the last inequality above gives:

$$\mathbb{E}(\sup_{s\leq T}|X_n-X_m|^2)^{\frac{1}{2}} = \mathbb{E}((\sup_{s\leq T}|\sum_{j=m+1}^{n}X_j-X_{j-1}|)^2)^{\frac{1}{2}}$$
$$\leq \sum_{j=m+1}^{n}\mathbb{E}((\sup_{s\leq T}|X_j-X_{j-1}|)^2)^{\frac{1}{2}}$$
$$\leq \sum_{j=m+1}^{\infty}\mathbb{E}((\sup_{s\leq T}|X_j-X_{j-1}|)^2)^{\frac{1}{2}}.$$

Therefore, as the right-hand side is the tail of a convergent series, $(X_n)_{n\geq 0}$ is a Cauchy sequence in L^2 and we denote its limit by X. The above inequality shows that there exists a sequence $m(k)$ such that:

$$\lim_{k\to\infty}|X(s)-X_{m(k)}(s)| =_{a.e.} 0.$$

Since $X_{m(k)}$ is continuous and adapted, so is X. Moreover if we set $m=0$ and use that $X_{m(k)} \to_{.a.e} X$ we have that:

$$\mathbb{E}(\sup_{s\leq T}|X(s)-\xi|^2)^{\frac{1}{2}} \leq \sum_{n=0}^{\infty}\mathbb{E}(\sup_{t\leq T}|X_{n+1}-X_n|^2)^{\frac{1}{2}}$$
$$\leq C_T(\mathbb{E}((1+|\xi|)^2))^{\frac{1}{2}}.$$

by the inequalities obtained during this proof. This inequality gives us the desired inequality from the theorem with $k_T = (C_T+1)^2$. Indeed, first note that (using reverse triangle inequality for the sup-norm):

$$\sup_{s\leq T}|X(s)-\xi| \geq |\sup_{s\leq T}|X(s)|-|\xi||.$$

Taking the L^2 norm on both sides of the inequality yields:

$$\mathbb{E}((\sup_{s\leq T}|X_s|-|\xi|)^2) \leq C_T^2\,\mathbb{E}((1+|\xi|)^2)$$

Now using the reverse triangle inequality for the L^2 gives us:

$$\begin{aligned}(\mathbb{E}((\sup_{s\leq T}|X_s|-|\xi|)^2)^{\frac{1}{2}} &\geq |(\mathbb{E}(\sup_{t\leq T}|X_s|^2))^{\frac{1}{2}}-(\mathbb{E}(|\xi|^2))^{\frac{1}{2}}| \\ &\geq (\mathbb{E}(\sup_{s\leq T}|X_s|^2))^{\frac{1}{2}}-(\mathbb{E}(|\xi|^2))^{\frac{1}{2}}.\end{aligned}$$

Hence we obtain:

$$\begin{aligned}(\mathbb{E}(\sup_{s\leq T}|X_s|^2))^{\frac{1}{2}} &\leq (C_T^2(\mathbb{E}((1+|\xi|)^2)))^{\frac{1}{2}}+(\mathbb{E}(|\xi|^2))^{\frac{1}{2}} \\ &\leq ((C_T+1)\,\mathbb{E}((1+|\xi|))^2)^{\frac{1}{2}}.\end{aligned}$$

□

10.3. **Localization.** In the previous section we have shown that under a global Lipschitz (with respect to $x, y \in \mathbb{R}^n$) and linear growth condition, we have existence and uniqueness of solutions. In this section, we will establish that we can weaken conditions and still obtain uniqueness and existence. We will also show that we can establish uniqueness of solutions until a certain stopping time which will be formally defined below.

Definition 10.22. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration. A random time $\tau:\Omega\to[0,\infty]$ is called a stopping time if*

$$\{\tau\leq t\}\in\mathcal{F}_t\ \forall t\geq 0.$$

Stopping times are useful if we want to know when a process (X_t):

- Leaves or enters a set for the first time.
- Hit its maximum.
- Returns to 0.

Example 10.23. ([3, p. 53 Section 5.2]) Examples of stopping times with respect to process a (X_t) are entry and hitting times into a set $A\in\mathcal{B}(\mathbb{R}^d)$:

- Entry time: $\mathring{\tau}_A=\inf\{t\geq 0: X_t\in A\}$
- Hitting time: $\tau_A=\inf\{t>0: X_t\in A\}$

The first lemma shows that under certain conditions we obtain that the coordinates of X_t are equal a.s. until a certain stopping time.

Lemma 10.24. [3, p. 286 Lemma 18.15] *Let $(B_t,\mathcal{F})_{t\geq 0}$ be a BM^d. Consider the SDEs*

$$dX_t^j=b_j(t,X_t^j)dt+\sigma_j(t,X_t^j)dB_t,\ j=1,2$$

with initial condition: $X_0^1=X_0^2=\xi\in L^2(P)$. Assume ξ is $\mathcal{F}_0$ measurable and that the coefficients $b_j:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ and $\sigma_j:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ satisfy (10.3) $\forall x,y\in\mathbb{R}^n$ with global Lipschitz constant L. If for some $T>0, R>0$:

$$b_1|_{[0,T]\times B(0,R)}=b_2|_{[0,T]\times B(0,R)}\ \text{and}$$
$$\sigma_1|_{[0,T]\times B(0,R)}=\sigma_2|_{[0,T]\times B(0,R)}$$

(where $B(0,R)$ is a ball with around 0 with radius R) we have for stopping times $\tau_j=\inf\{t\geq 0:|X_t^j-\xi|\geq R\}\wedge T$ that it holds that:

$$P(\tau_1=\tau_2)=1\ \text{and}\ P(\sup_{s\leq\tau_1}|X_s^1-X_s^2|=0)=1.$$

Proof. Observe that:

$$\begin{aligned} X^1_{t\wedge\tau_1} - X^1_{t\wedge\tau_2} &= \int\limits_0^{t\wedge\tau_1} b_1(s,X^1_s) - b_2(s,X^2_s)ds + \int\limits_0^{t\wedge\tau_1} (\sigma_1(s,X^1_s) - \sigma_2(s,X^2_s)dB_s \\ &= \int\limits_0^{t\wedge\tau_1} b_2(s,X^1_s) - b_2(s,X^2_s)ds + \int\limits_0^{t\wedge\tau_1} \sigma_2(s,X^1_s) - \sigma_2(s,X^2_s)dB_s \end{aligned}$$

where it was used that by our assumption (same holds for σ):

$$\begin{aligned} b_1(s,X^1_s) - b_2(s,X^2_s) &= b_1(s,X^1_s) - b_2(s,X^1_s) + b_2(s,X^1_s) - b_2(s,X^2_s) \\ &= b_2(s,X^1_s) - b_2(s,X^2_s). \end{aligned}$$

Analogous to the steps seen in the proof of corollary 10.19 where we in this case use $X_t = X^1_{t\wedge\tau_1}, Y_t = X^2_{t\wedge\tau_1}$ together with remark 10.17 to obtain:

$$\mathbb{E}(\sup_{s\leq T}|X^1_{t\wedge\tau_1} - X^2_{t\wedge\tau_1}|^2) \leq 2L^2(T+4)\int\limits_0^T \mathbb{E}(\sup_{r\leq s}|X^1_{r\wedge\tau_1} - X^2_{r\wedge\tau_1}|^2)ds$$

Grönwall's inequality with $a(s) = 0, b(s) = 2L^2(T+4)$ yields:

$$\mathbb{E}(\sup_{r\leq s}|X^1_{r\wedge\tau_1} - X^2_{r\wedge\tau_1}|^2) = 0 \ \forall s \leq T.$$

From this it follows that:

$$X^1_{\bullet\wedge\tau_1} =_{a.s.} X^2_{\bullet\wedge\tau_1}$$

in particular $\tau_1 \leq_{a.s.} \tau_2$. Reversing the roll of X_1 and X_2 finishes the proof. □

The following theorem shows that we can actually weaken the Lipschitz condition (10.3) by substituting the restriction $\forall x, y \in \mathbb{R}$ to $\forall x, y \in K$ where K is a compact subset of $\mathbb{R}^n$. The proof of this theorem can be found at [3, p. 287 Thm 18.17].

Theorem 10.25. *Let $(B_t, \mathcal{F}_t)_{t\geq 0}$ be a BM^d and assume that the coefficients $b : [0,\infty)\times\mathbb{R}^n$ and $\sigma : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^{n\times d}$ of (10.2) satisfy linear growth (10.4) $\forall x, y \in \mathbb{R}^n$ and the Lipschitz condition 10.3 $\forall x, y \in K$ for every compact set $K \subset \mathbb{R}^n$ with local Lipschitz constants $L_{T,K}$. For every $\mathcal{F}_0$ measurable initial condition $X_0 = \xi \in L^2(p)$ there exists an unique solution $(X_t)_{t\geq 0}$ which satisfies:*

$$\mathbb{E}(\sup_{s\leq T}|X_s|^2) \leq k_T\,\mathbb{E}((1+|\xi|)^2)\ \forall T > 0.$$

Thus far we have only talked about existence and uniqueness of solutions for all $t \geq 0$. It turns out that if we assume a local Lipschitz condition [12, p. 245 (34.5.2)] (instead of a Lipschitz condition as defined at (10.3)) we can still obtain uniqueness and existence of solutions but until a certain maximal stopping time. The following theorem makes the claim about existence of solutions until a maximal stopping time more precise:

Theorem 10.26. [12, p. 246 Thm. 34.7] *For a SDE as defined at definition 10.13 together with the local Lipschitz condition* [12, p. 245 (34.5.2)]*, there exists a P a.s. uniquely defined stopping time τ and an unique P a.s. process X on $[0,\tau]$ (i.e. $X_{t\wedge\tau}(\omega)$ solves the SDE $\forall t > 0$) for which hold:*

(1) X is a solution of our SDE on $[0,\tau]$.

(2) *For all stopping times τ' for which there exists a process X' which is a solution on $[0, \tau']$ we have that $[0, \tau'] \subseteq [0, \tau]$ and $X = X'$ on $[0, \tau']$.*

The proof of this theorem makes use of the following two lemma's.

Lemma 10.27. [12, p. 247 Lemma 34.8] *Let X, Y be two solutions of (10.2) on the stochastic interval $[0, \tau']$. Then $X =_{a.s.} Y$ on $[0, \tau']$.*

Remark 10.28. The proof of this lemma 10.27 can be constructed in a similar way as the proof of lemma 10.24.

Lemma 10.29. [12, p. 248 Lemma 34.8] *Let X be a solution of (10.2) on $[0, \tau]$ and X' a solution on $[0, \tau']$. Then there exists a solution on $[0, \max(\tau, \tau')]$.*

Proof. Because of lemma 10.27, we have that X and X' coincide on $[0, \min(\tau, \tau')]$. Therefore it holds that the process Y defined on $[0, \max(\tau, \tau')]$ by

$$Y(\omega) = \begin{cases} X(\omega) & \omega \in [0, \tau] \\ X'(\omega) & \omega \in [0, \tau'] \end{cases}$$

is a solution of (10.2). ☐

Now consider the following set of stopping times:

$$\Psi = \{\tau : \exists \text{ process } X \text{ such that } (X, \tau) \text{ is a solution of } (10.2)\ \}.$$

As τ is a random variable on a probability space, is non-negative and Ψ is non-empty: we have that $\operatorname{ess\,sup} \Psi$ exists by theorem 9.10. As Ψ is also closed under pairwise maximization by lemma 10.29 (and the fact that the maximum of two stopping times is again a stopping time ([3, p. 341 A.12b])) there exists a non-decreasing sequence τ_n such that $\tau_n \to_{a.s.} \tau$ $(n \to \infty)$ and to each such τ_n corresponds a solution X_n. It is clear that τ corresponds to the maximum stopping time solution and that X_n converges a.s. to the X as described in theorem 10.26.

Appendix A. Topological spaces

As the basics of general topology are not taught in any of the non-elective bachelor of science courses at the TU Delft, I will briefly state some concepts from topology that I frequently used in the thesis.

Definition A.1. *(Topology) A topological space is a set M with a collection $\mathcal{T}$ of subsets $U \subseteq M$. We call a subset $U \subseteq M$ open if it is in $\mathcal{T}$. The open sets are required to satisfy the following rules:*

(1) $\varnothing, M \in \mathcal{T}$.
(2) Unions of open sets are open (possibly uncountable).
(3) Finite intersections of open sets are open.

Definition A.2. *Let M be a topological space. An open neighbourhood of $x \in M$ is an open set $U \subseteq M$ that contains x.*

Definition A.3. *(Continuity) Let M, N be topological spaces. Then the map $\phi : M \to N$ is called continuous if $\phi^{-1}(U)$ is open in M for any open set U of N.*

Definition A.4. *(Topology on metric spaces) For a metric space (M, d) we define the topology $\mathcal{T}_d$ by declaring that $U \subseteq M$ is open if $\forall x \in U$ $\exists \epsilon > 0$:*

$$B_\epsilon(x) = \{y \in M : d(x, y) < \epsilon\} \subseteq U.$$

Definition A.5. *(Subspace topology) Let M be a topological space. Let $\Sigma \subseteq M$. Then the subspace topology of Σ is the collection of sets $U \cap \Sigma$, where $U \subseteq M$ is open in M.*

Definition A.6. *(Product topology) If M, N are topological spaces we define a subset of $M \times N$ to be open if it is the union of sets of the form $U \times V$ where U is open in M and V open in N.*

Remark A.7. Needless to say that the topology on metric spaces, subspace topology and product topology are indeed topologies by definition A.1.

Definition A.8. *(Convergence) A sequence $(x_n)_{n\geq 1}$ of points in a topological space M converges to $x \in M$ if for every open neighbourhood U of x there exists a $N > 0$ such that: $x_n \in U$ $\forall n \geq N$.*

Definition A.9. *(Hausdorff spaces) A topological space M is called Hausdorff if for any two distinct points $x, y \in M$ there exists an open neighbourhood U_x of x and U_y of y such that: $U_x \cap U_y = \varnothing$.*

Appendix B. Grönwall's inequality

In the section on Stochastic Differential Equations, a frequently used result is Grönwall's inequality for measurable functions. As this form of Grönwall's inequality has not been stated in any of the non-elective courses, I will state it here for the sake of completeness. The proof can be found at [3, p. 360 Thm. A.43].

Theorem B.1. *Let $u, a, b : [0,\infty) \to [0,\infty)$ be measurable functions satisfying:*

$$u(t) \leq a(t) + \int\limits_0^t b(s)u(s)ds \ \forall t \geq 0.$$

Then the following holds:

$$u(t) \leq a(t) + \int\limits_0^t a(s)b(s)\exp(\textstyle\int_s^t b(r)dr)ds \ \forall t \geq 0$$

Appendix C. Radon-Nikodym

The Radon-Nikodym theorem relating to the existence of the Radon-Nikodym derivative has a few different formulations which lead in the end to the derivative taking only finite or also infinite values. In my thesis I used two different formulations of Radon-Nikodym which I will state here.

Theorem C.1. [14] *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space (and μ is a positive measure). Let ν be another measure on $(X, \mathcal{A})$ where ν is absolutely continuous with μ. Then there exists a measurable function $f : (X, \mathcal{A}) \to [0, \infty]$ such that:*

$$\forall A \in \mathcal{A} : \mu(A) = \int 1_A f d\nu.$$

Remark C.2. In case we assume ν is σ-finite as well, the measurable function f will only take finite values. ([15]).

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