

CHAPTER 13

A HIGHER ORDER THEORY FOR DEEP WATER WAVES

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ABSTRACT

The classical problem of describing the characteristics of deep water waves of finite amplitude is considered. The method of analysis initially follows that of Nekrasov, but differs in that a non-linear algebraic equation is derived. This equation is solved to the third, fifth and fifteenth order by means of a digital computer and the data is presented in tabular form. Expressions for the wave speed and wave shape, predicted by the analysis, are compared with the results obtained by Stokes. The highest wave in water is also discussed.

INTRODUCTION

Recent advances in the development of gravity wave theories have focused increasing attention on the fact that linear theories are inadequate to describe certain free surface phenomena (Wiegel, 1964, pp. 2-3). One such phenomenon is the deep water wave of finite height, which has been studied by many researchers over the course of the past century. Among the more important papers which have appeared on this subject are those of Stokes (1847, 1880), Rayleigh (1876), Wilton (1914) and Levi-Civita (1925). A review of some of these works, as well as others which include the effect of finite depth, may be found in Oceanographical Engineering (Wiegel, 1964, Chapter 2).

An analysis of deep water waves by Nekrasov (1951), which differs from that of Stokes, recently appeared in the Russian literature. A relatively intensive study of this paper was subsequently conducted by Milne-Thomson (1960), who included a condensation of some parts of Nekrasov's paper in the fourth edition of his Theoretical Hydrodynamics. Moreover, an English translation of the entire work has been prepared by the Mathematics Research Center at the University of Wisconsin.

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Nekrasov's analysis leads to a non-linear integral equation. However, he did not obtain a general solution to the equation, but rather a solution limited to waves of very small amplitude. Moreover, this solution is in marked disagreement with that obtained by Stokes.

In an attempt to coordinate the work of Nekrasov and Stokes into one analysis, the present study will begin with the fundamental concepts of the former. The advantage of Nekrasov's development is that it presents and initiates the problem in a relatively lucid manner. However, rather than develop the non-linear integral equation of Nekrasov, the present study will center on an algebraic equation. After this equation has been derived, it will be shown that it is essentially identical to that obtained by Stokes (1880), thereby establishing the connection between the two theories.

With the aid of a digital computer, third, fifth, and fifteenth order solutions of the algebraic equation have been obtained, and will be presented in tabular form. The analysis will then be compared with those obtained by others, including Stokes. In particular the dependence of wave speed and wave shape on wave height will be examined. The discussion will be concluded by some observations on the highest wave in water.

SOLUTION OF THE WAVE PROBLEM

DEVELOPMENT OF THE THEORY

Let us consider a train of oscillatory waves of finite height traveling from right to left along the free surface of an infinitely deep body of water. The waves have a constant profile, which is as yet unknown. They travel with a speed, c , which is likewise unknown.

We superimpose over the entire field a uniform flow moving from left to right with a magnitude, c . As a result, the wave form is brought to rest, and by choosing a coordinate system which is likewise at rest, the flow beneath the free surface is seen to be steady. This steady wave will turn out to be more amenable to analysis than the progressive wave, whose properties are under consideration.

Fig. 1 shows a segment of the wave train, as seen in the complex z -plane. The y -axis is chosen to pass through the crest of the wave, C . The wave length is L ; the wave height, measured from trough to crest, is H ; and the flow field, extending over one wave length, is bounded by $A_\infty BCDA_\infty$, where BCD , the unknown free surface, is given by $y_0 = y_0(x_0)$. The still water level is located at $y = y_s$, a distance which remains to be determined.

The flow is assumed to be two-dimensional and irrotational

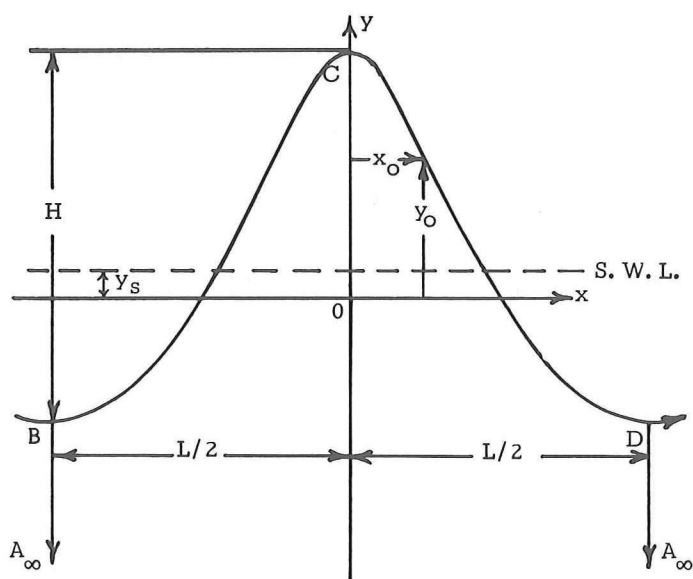


Figure 1. - Definition sketch of the physical z-plane.

(the latter assumption is discussed in detail by Stokes (1847), in the Appendix of his classical work). Therefore, we may define a complex potential

$$w \equiv \phi + i\psi \quad (1)$$

where ϕ is the potential function and ψ the streamfunction. As a consequence of the fact that the complex potential is analytic we obtain the Cauchy-Riemann equations, which may be related to the velocity components,

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad (2)$$

From these expressions it may be shown that both the potential function and the stream function satisfy Laplace's equation

$$\nabla^2 \phi = 0 \quad \text{and} \quad \nabla^2 \psi = 0 \quad (3)$$

On the free surface, BCD, the kinematic condition that the free surface is a streamline is given by

$$\psi = 0 \quad \text{at} \quad y = y_0 \quad (4)$$

and the dynamic condition that the pressure is a constant may be applied to the Bernoulli equation, to give

$$q_0^2 + 2gy_0 = K \quad \text{at} \quad y = y_0 \quad (5)$$

where q_0 is the particle speed at the free surface and K is the Bernoulli constant.

On the lower boundary, the complex velocity at A_∞ is equivalent to the superimposed speed, c , and this condition is therefore given by

$$u - iv = c \quad \text{at} \quad y = -\infty \quad (6)$$

Also, a direct consequence of this condition and Eqs. 2 and 4, is that

$$\psi = -\infty \quad \text{at} \quad y = -\infty \quad (7)$$

The problem, as developed to this point, is to obtain the solutions of Eqs. 3 subject to the boundary conditions, Eqs. 4, 5, 6, and 7. The fundamental difficulty which immediately presents itself is that the location of the free surface, or upper boundary, is unknown. In order to eliminate this difficulty in the z -plane, the technique of conformal

mapping will be employed, so that the location of the free surface may be identified in the auxiliary ζ -plane.

The conformal transformation to be used was first suggested by Nekrasov (1951), and has also been discussed by Milne-Thomson (1960). It is given by

$$z = \frac{iL}{2\pi} (\ln \zeta + a_1 \zeta + \frac{1}{2} a_2 \zeta^2 + \frac{1}{3} a_3 \zeta^3 + \dots) \quad (8)$$

where

$$\zeta \equiv r \exp(i\chi)$$

and where the constant coefficients, a_j , are real, but as yet unknown.*

The ζ -plane is shown in Fig. 2. As one can verify by application of the mapping function, Eq. 8, the region bounded by $A_\infty BCDA_\infty$ in the z -plane is mapped inside the unit circle in the ζ -plane subject only to the proper evaluation of the constant coefficients, a_j . Point A_∞ appears at the origin, point C on the unit circle at $\chi = 0$ and points B and D on the unit circle at $\chi = \pi$ and $\chi = -\pi$ respectively, separated by a slit. The problem then, is to obtain values of a_j such that the free surface is mapped onto the unit circle of the ζ -plane. The transformation may subsequently be used to locate the free surface in the z -plane, and incidentally provide the wave speed and the other characteristics of the wave.

Before proceeding to an analysis of the boundary conditions in the ζ -plane, it is convenient to examine some additional properties of the transformation. For example, by separating real and imaginary parts of the mapping function and evaluating them on the free surface, that is, on the unit circle of the ζ -plane, we obtain the parametric equations that describe the free surface, $y_0 = y_0(x_0)$,

$$x_0 = -\frac{L}{2\pi} (\chi + a_1 \sin \chi + \frac{1}{2} a_2 \sin 2\chi + \frac{1}{3} a_3 \sin 3\chi + \dots) \quad (9)$$

$$y_0 = +\frac{L}{2\pi} (a_1 \cos \chi + \frac{1}{2} a_2 \cos 2\chi + \frac{1}{3} a_3 \cos 3\chi + \dots) \quad (10)$$

Also by differentiating Eq. 8 we obtain the complex operator,

$$\frac{dz}{d\zeta} = \frac{iL}{2\pi} \left(\frac{1}{\zeta} + a_1 + a_2 \zeta + a_3 \zeta^2 + \dots \right) = \frac{iL}{2\pi} \frac{f(\zeta)}{\zeta} \quad (11)$$

* Since $z = z(\zeta)$, Eq. 8 is, in fact, an inverse transformation.

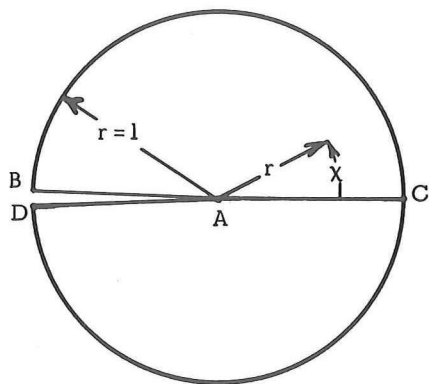


Figure 2. - Definition sketch of the auxiliary ζ -plane.

where

$$f(\zeta) \equiv \sum_{j=0}^{\infty} a_j \zeta^j \quad (12)$$

and

$$a_0 = 1.0 \quad (13)$$

Moreover, on the free surface

$$\zeta_0 = \exp(i\chi) \quad (14)$$

so that

$$f(\zeta_0) = \sum_{j=0}^{\infty} a_j \exp(ij\chi) = R_0 \exp(i\theta_0) \quad (15)$$

where the modulus of $f(\zeta_0)$ is given by

$$R_0 \equiv \left[\left(\sum_{j=0}^{\infty} a_j \cos j\chi \right)^2 + \left(\sum_{j=0}^{\infty} a_j \sin j\chi \right)^2 \right]^{1/2} \quad (16)$$

and the argument of $f(\zeta_0)$ is given by

$$\theta_0 \equiv \cos^{-1} \left[\frac{\sum_{j=0}^{\infty} a_j \cos j\chi}{R_0} \right] \quad (17)$$

In view of the mapping function given by Eq. 8, the boundary conditions, Eqs. 4, 5, 6, and 7, become

$$\psi = 0 \quad \text{at } r = 1 \quad (18)$$

$$q_0^2 + 2gy_0 = K \quad \text{at } r = 1 \quad (19)$$

$$u - iv = c \quad \text{at } \zeta = 0 \quad (20)$$

$$\psi = -\infty \quad \text{at } r = 0 \quad (21)$$

respectively, in the ζ -plane.

Eqs. 18 and 21 are precisely the boundary conditions for a clockwise irrotational vortex in the ζ -plane, so that the complex potential for this flow is

$$w = \frac{icL}{2\pi} \ln \zeta \quad (22)$$

By separating the real and imaginary parts of this expression and rearranging terms, we obtain

$$\chi = - \frac{2\pi\phi}{cL} \quad (23)$$

and

$$r = \exp \left(\frac{2\pi\psi}{cL} \right) \quad (24)$$

It is therefore apparent that χ is the normalized potential function, while r is functionally related to the stream function.

In order to obtain the velocity components we may use Eq. 11 in addition to Eq. 22 to compute the complex velocity,

$$u - iv = \frac{dw}{dz} = \frac{dw}{d\zeta} \frac{d\zeta}{dz} = \frac{c}{f(\zeta)} \quad (25)$$

It is seen that this expression satisfies the third boundary condition, Eq. 20, since Eq. 12 shows that $f(\zeta) = 1$ at $\zeta = 0$.

At the free surface Eq. 25 becomes, in view of Eq. 15,

$$u_0 - iv_0 = \frac{c}{f(\zeta_0)} = \frac{c}{R_0} \exp(-i\theta_0) \quad (26)$$

and therefore the velocity components are given by

$$\begin{aligned} u_0 &= \frac{c}{R_0} \cos \theta_0 \\ v_0 &= \frac{c}{R_0} \sin \theta_0 \end{aligned} \quad (27)$$

By taking the ratio of the velocity components, it may also be seen that θ_0 is the local slope angle which the free surface makes with a horizontal, in the physical z -plane.

It remains then to deal with the dynamic boundary condition, Eq. 19,

all other conditions having been satisfied by the complex potential described in Eq. 22. We therefore substitute Eqs. 27 into Eq. 19, so as to obtain

$$\frac{c^2}{R_0^2} + 2gy_0 = K \quad (28)$$

Eq. 28 was developed by Nekrasov, who proceeded to relate this equation to Eq. 15 in order to derive a non-linear integral equation, for which a general solution has not as yet been found.

In the present case we will also consider Eq. 28. However, the equation will be solved directly, though approximately, by algebraic means. We may define the following dimensionless terms:

$$\begin{aligned} x'_0 &\equiv \frac{gx_0}{c_L^2} = \frac{2\pi x_0}{L}, & y'_0 &\equiv \frac{gy_0}{c_L^2} = \frac{2\pi y_0}{L} \\ c' &\equiv \frac{c}{c_L} & K' &\equiv \frac{K}{\frac{c_L^2}{2}} \end{aligned} \quad (29)$$

where the wave speed from the linear theory is given by

$$c_L = \sqrt{\frac{gL}{2\pi}} \quad (30)$$

Using these dimensionless terms, we may convert Eq. 28 to the normalized form,

$$c'^2 + 2y'_0 R_0^2 = K' R_0^2 \quad (31)$$

where,

$$y'_0 = \sum_{j=1}^{\infty} \frac{a_j}{j} \cos j\chi \quad (32)$$

Furthermore, by expanding terms in Eq. 16, and applying trigonometric identities, it may be shown that

$$R_0^2 = A_0 + 2 \sum_{j=1}^{\infty} A_j \cos j\chi \quad (33)$$

where

$$A_0 \equiv \sum_{k=0}^{\infty} a_k^2 \quad (34)$$

$$A_j \equiv \sum_{k=j}^{\infty} a_{k-j} a_k \quad (35)$$

By substituting Eqs. 32 to 35 into Eq. 31, the problem of the deep water wave of finite amplitude is reduced to one of finding the solution of the algebraic equation,

$$c'^2 + 2 \left[\sum_{j=1}^{\infty} \frac{a_j}{j} \cos j\chi \right] - K' \left[\left(\sum_{k=0}^{\infty} a_k^2 \right) + \left(2 \sum_{j=1}^{\infty} \left(\sum_{k=j}^{\infty} a_{k-j} a_k \right) \cos j\chi \right) \right] = 0 \quad (36)$$

It should be noted that to this point in the analysis no approximations have been made. Therefore Eq. 36 is an exact representation of the problem.

In order to solve the problem for a finite number of coefficients, it will be necessary to truncate the infinite trigonometric series which appear in Eq. 36. As a consequence,

$$R_0^2 \approx A_0 + 2 \sum_{j=1}^n A_j \cos j\chi \quad (37)$$

where

$$A_0 \approx \sum_{k=0}^n a_k^2 \quad (38)$$

$$A_j \approx \sum_{k=j}^n a_{k-j} a_k \quad (39)$$

and

$$y_0' \approx \sum_{j=1}^n \frac{a_j}{j} \cos j\chi \quad (40)$$

By combining and expanding Eqs. 37 and 40 to the n th order we obtain

$$y_0' R_0^2 \approx \left(\sum_{k=1}^n \frac{a_k}{k} A_k \right) + \sum_{j=1}^n \sum_{k=1}^n \frac{a_k}{k} (A_{k-j} + A_{k+j}) \cos j\chi \quad (41)$$

where it is understood that absolute value signs are omitted on the subscripts of the terms on the right-hand side, and furthermore

$$A_\ell = 0 \quad \text{if } |\ell| > n \quad (42)$$

It should be noted that in deriving Eq. 41 harmonics higher than the n th have been omitted so that the expression is not an exact representation of $y_0' R_0^2$.

The expressions for R_0^2 , Eq. 37, and $y_0' R_0^2$, Eq. 41, may now be substituted into Eq. 31. By equating the coefficients of the harmonics, we obtain

$$c'^2 + 2 \sum_{k=1}^n \frac{a_k}{k} A_k = K' A_0 \quad (\text{0th harmonic}) \quad (43)$$

$$\sum_{k=1}^n \frac{a_k}{k} (A_{k-j} + A_{k+j}) = K' A_j \quad (\text{jth harmonic - } j=1, 2, \dots, n) \quad (44)$$

where absolute value signs are omitted on the subscripts, and A_ℓ vanishes if $|\ell| > n$.

In view of the fact that the unknown terms in Eqs. 43 and 44 are all functions of the height of the wave, it is convenient to include an equation for wave height. The wave height is seen to be equal to the sum of the displacements of the crest and trough from the x -axis. Therefore, using Eq. 10,

$$H = (y_0)_{\chi=0} + (-y_0)_{\chi=\pi} = \frac{L}{\pi} \sum_{j=1}^n \frac{a_j}{j} \quad (j=1, 3, 5, \dots, n) \quad (45)$$

Normalizing with respect to wave length, the dimensionless wave height is

$$H' = \frac{1}{\pi} \sum_{j=1}^n \frac{a_j}{j} \quad (j=1, 3, 5, \dots, n) \quad (46)$$

Eqs. 43, 44 and 46 are therefore seen to constitute a set of $(n+2)$ equations

in $(n+2)$ unknowns $(c', K', a_1, a_2, \dots, a_n)$ for any desired value of dimensionless wave height, H' .

COMPUTER SOLUTION

In setting up the equations for computer solution, the coefficient, K' was eliminated between the first of Eqs. 44 ($j = 1$) and each successive equation, ($j = 2, 3, \dots, n$), thereby reducing Eqs. 44 to $(n-1)$ equations in $(n-1)$ unknowns $(a_1, a_2, \dots, a_{n-1})$, for a fixed value of a_n . The simultaneous solution of these $(n-1)$ non-linear algebraic equations was accomplished with the aid of the Newton-Raphson iteration (Hildebrand, 1956). By this technique the problem was reduced to one of obtaining the solution of a set of $(n-1)$ linear algebraic equations at each iteration. The Jordan reduction method was then used to solve the set of linear equations for $a_1, a_2, \dots, a_{n-2}$ and a_{n-1} . Substitution of these terms into Eq. 46, Eq. 44 (first harmonic), and Eq. 43, with the aid of Eqs. 38 and 39, yielded values for H' , K' and c' respectively. The entire procedure was repeated for various values of a_n so that all of the terms could be tabulated for uniform intervals of the dimensionless wave height. Intervals of 0.01 were chosen for the tabulation.

The results, as obtained from a CDC 1604 computer are given for the third order ($n = 3$), fifth order ($n = 5$) and fifteenth order ($n = 15$) in Tables 1, 2 and 3. In view of the rapid convergence of the a_j series, for lower values of H' , the computations in these cases were not carried to the fifteenth order, as is indicated in Table 3.

Values of H/L were obtained correct to ± 0.00001 , while all other terms appearing in the tables were computed correct to the last place shown. It should be noted that this precision was obtained in satisfying Eqs. 43, 44 and 46, the equations which were programmed on the computer. However, in looking back to Eq. 36, which represents an exact expression of the problem, it is seen that the validity of the various approximations employed in the subsequent development is reflected in the degree to which the tabulated values satisfy this general expression of the problem. For the lower values of H/L where convergence of the a_j series is rapid, and lower order solutions are sufficient, Eq. 36 is satisfied very precisely. However, as H/L increases, the truncation of the trigonometric series and, also the omission of higher harmonics in the development of $(y'_0 R_0^2)$ result in a less precise satisfaction of Eq. 36.

No attempt will be made here to examine the convergence of the trigonometric series (as given in Eq. 8) formed by the coefficients. For a discussion of this question, reference may be made to Levi-Civita (1925), who presented a convergence proof.

Table 1. - Deep water wave coefficients - Third order.

H/L	c'	K'	a ₁	a ₂	a ₃
.142	1.0979	1.3751	.37171	.28318	.22320
.140	1.0951	1.3653	.36778	.27696	.21614
.130	1.0817	1.3179	.34755	.24651	.18256
.120	1.0696	1.2732	.32641	.21703	.15175
.110	1.0587	1.2314	.30436	.18853	.12366
.100	1.0487	1.1925	.28139	.16106	.09830
.090	1.0396	1.1568	.25749	.13478	.07576
.080	1.0314	1.1245	.23261	.10988	.05615
.070	1.0241	1.0957	.20672	.08665	.03957
.060	1.0178	1.0705	.17979	.06541	.02611
.050	1.0124	1.0491	.15183	.04653	.01575
.040	1.0079	1.0315	.12288	.03040	.00836
.030	1.0044	1.0177	.09304	.01738	.00363
.020	1.0020	1.0079	.06247	.00782	.00110
.010	1.0005	1.0020	.03137	.00197	.00014
.000	1.0000	1.0000	.00000	.00000	.00000

Table 2. - Deep water wave coefficients - Fifth order.

H/L	c'	K'	a ₁	a ₂	a ₃	a ₄	a ₅
.142	1.1099	1.3933	.34066	.25987	.21964	.18964	.16118
.140	1.1063	1.3825	.33811	.25489	.21307	.18215	.15342
.130	1.0900	1.3304	.32413	.23004	.18173	.14753	.11848
.120	1.0756	1.2816	.30832	.20519	.15253	.11702	.08910
.110	1.0628	1.2366	.29086	.18038	.12534	.09023	.06467
.100	1.0513	1.1954	.27176	.15575	.10026	.06712	.04487
.090	1.0412	1.1582	.25100	.13154	.07756	.04772	.02943
.080	1.0323	1.1250	.22855	.10806	.05754	.03207	.01799
.070	1.0246	1.0958	.20440	.08573	.04051	.02009	.01006
.060	1.0180	1.0705	.17861	.06500	.02665	.01149	.00502
.050	1.0124	1.0491	.15131	.04640	.01601	.00582	.00215
.040	1.0079	1.0315	.12270	.03035	.00845	.00248	.00074
.030	1.0045	1.0177	.09299	.01737	.00365	.00081	.00018
.020	1.0020	1.0079	.06246	.00782	.00110	.00016	.00002
.010	1.0005	1.0020	.03137	.00197	.00014	.00001	.00000
.000	1.0000	1.0000	.00000	.00000	.00000	.00000	.00000

DISCUSSION

COMPARISON WITH STOKES'S THEORY

As has already been pointed out, Eq. 36 is an exact representation of the problem of the deep water wave of finite amplitude. A comparison of this equation with Eq. 13 in the second paper by Stokes (1880) reveals that they are identical.* This is, in fact, not too surprising since Stokes's development of Eq. 13 is just as free of approximation as is the development of Eq. 36.

A second connection between Stokes's development and that presented here may be constructed from the mapping function, Eq. 8. By separating real and imaginary parts, and introducing Eqs. 23 and 24, we obtain

$$x = \frac{\phi}{c} + \frac{L}{2\pi} \sum_{k=1}^{\infty} \frac{a_k}{k} \exp\left(\frac{2\pi k \psi}{cL}\right) \sin\left(\frac{2\pi k \phi}{cL}\right) \quad (47)$$

and

$$y = \frac{\psi}{c} + \frac{L}{2\pi} \sum_{k=1}^{\infty} \frac{a_k}{k} \exp\left(\frac{2\pi k \psi}{cL}\right) \cos\left(\frac{2\pi k \phi}{cL}\right) \quad (48)$$

which, incidentally, may be used to plot the potential- and stream-lines in the z -plane. A comparison of these expressions with Stokes's Eqs. 9 and 10 reveals that they are also identical.†

If we now turn our attention to the problem of solving Eq. 36, an examination of his analysis reveals that Stokes was forced to make another assumption, in addition to truncating the infinite trigonometric series and omitting higher harmonics. This assumption consisted of omitting those products of the coefficients a_j , the sum of whose subscripts exceeded the order of the solution. The basis for this assumption was that a_j/j is of order j . There is little question that this is true for small values of H/L . However, for larger values of the dimensionless wave height the coefficients do not vary greatly as j increases, and the assumption appears less justified.

It is of course well-known that Stokes did not carry his analysis beyond the fifth order. However, in a subsequent paper, Wilton (1914)

* Expand both equations to any arbitrary order and in Eq. 36 replace a_k by Stokes's term $(-kA_k)$, K' by C , c'^2 by $(1/g)$.

† Replace x by Stokes's $(-x)$, y by $(-y)$, a_k by $(-kA_k)$, $L/2\pi$ by 1, and c by 1.

developed a twelfth order solution, also employing approximations to simplify his calculations. No attempt will be made here to compare the tabulated values of the coefficients obtained in this study with the expressions for the coefficients derived by Stokes and Wilton, since all three studies are based on the solution of Eq. 36 and the results for like orders of solution, where they are available, are, understandably, quite similar. Rather, a comparison of the predicted wave speeds and wave shapes will be made in the subsequent sections, with primary emphasis on the significance of higher order solutions.

WAVE SPEED

The equation for computing the wave speed may be obtained by combining Eqs. 43, 38 and 39 to give

$$c = \sqrt{\frac{gL}{2\pi}} \left[K' \sum_{k=0}^n a_k^2 - 2 \sum_{k=1}^n \frac{a_k}{k} \left(\sum_{j=k}^n a_{j-k} a_j \right) \right]^{1/2} \quad (49)$$

which may be evaluated to any order. The results for the third, fifth, and fifteenth orders, are given in Tables 1, 2 and 3, and have also been plotted against wave height in Fig. 3. In order to compare Eq. 49 with the expressions obtained by Stokes (1847), for the third order,

$$c = \sqrt{\frac{gL}{2\pi}} \left[1 + \frac{\pi^2 H^2}{2L^2} \right] \quad (50)$$

and from Stokes's analysis (Levi-Civita, 1925 and Beach Erosion Board, 1942), for the fifth order

$$c = \sqrt{\frac{gL}{2\pi}} \left[1 + \frac{\pi^2 H^2}{L^2} + \frac{\pi^4 H^4}{2L^4} \right]^{1/2} \quad (51)$$

these two equations are also represented in Fig. 3.

An examination of Fig. 3 shows that in predicting wave speed as a function of wave height, there is, in general, good agreement between the results obtained by Stokes and those of the present study. In fact, for values of the dimensionless wave height ranging from 0.00 to 0.06 the curves are virtually identical, suggesting that the third order developments of both studies (see Table 1) are entirely adequate for determining the wave speed in this range of wave height.

For values of the dimensionless wave height exceeding 0.06, the curves diverge to more than 1% of the dimensionless wave speed. In view

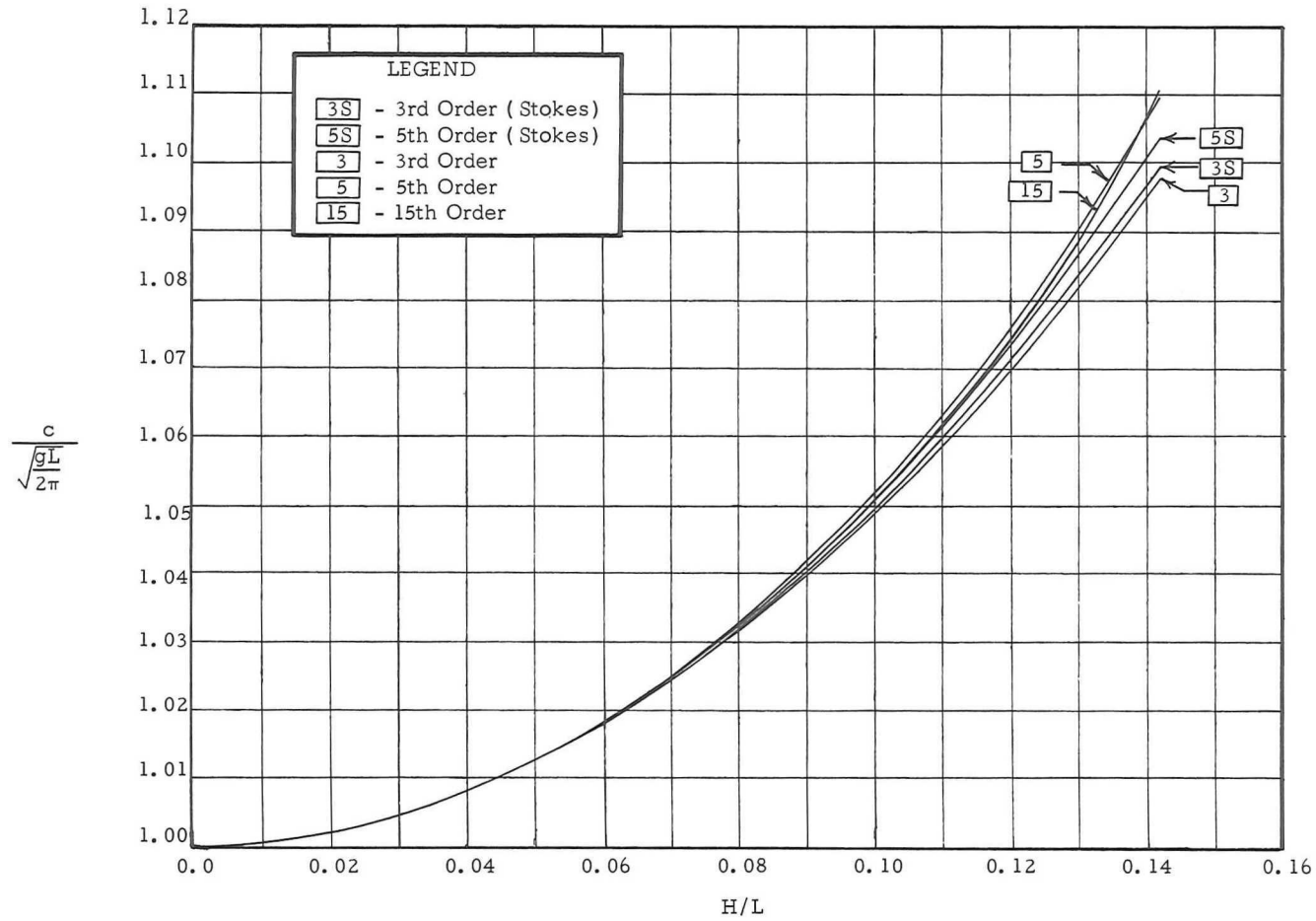


Figure 3. - The effect of wave height on wave speed.

of the fact that very few waves whose wave height exceeded $H/L = 0.1$ have been observed, either in the sea or in the laboratory, this divergence is perhaps of only limited physical significance. Nevertheless, in the continuing search for an understanding of the mechanisms which limit the height of a deep water wave, a precise evaluation of the wave speed, which takes the higher harmonics into account, may prove to be of assistance. Under these circumstances the use of the fifteenth order development (Table 3), should be considered.

WAVE SHAPE

In order to describe the wave shape or profile of a deep water wave, we seek a relationship between the free surface coordinates, x_0 and y_0 . The parametric set of equations,

$$x_0 = -\frac{L}{2\pi} \left(\chi + a_1 \sin \chi + \frac{1}{2} a_2 \sin 2\chi + \frac{1}{3} a_3 \sin 3\chi \dots \right) \quad (9)$$

$$y_0 = +\frac{L}{2\pi} \left(a_1 \cos \chi + \frac{1}{2} a_2 \cos 2\chi + \frac{1}{3} a_3 \cos 3\chi \dots \right) \quad (10)$$

represents such a relationship. Numerical values relating x_0 and y_0 may be obtained by substituting arbitrary values of the normalized potential function, χ . This procedure is adequate for a graphical representation of wave shape. Stokes (1880), however, suggests that the two equations may be reduced to one expression, with the aid of Lagrange's theorem (see Whittaker and Watson, 1963). It should be noted that the reduction becomes quite laborious for higher order solutions.

Fig. 4 shows the shape of a deep water wave, whose height is given by $H/L = 0.1$, with computations carried out to the fifteenth order. In order to determine the location of the profile, it is first necessary to determine the elevation of the still water level. This may be accomplished by noting that the still water level, y_s , is located such that the net area bounded by the free surface and the still water level vanishes. Therefore we have

$$0 = \int_0^{L/2} (y_0 - y_s) dx_0 \quad (52)$$

By substituting the parametric profile expressions, Eqs. 9 and 10, this equation may be solved for the still water elevation, to give

$$y_s = \frac{L}{4\pi} \sum_{k=1}^n \frac{a_k^2}{k} \quad (53)$$

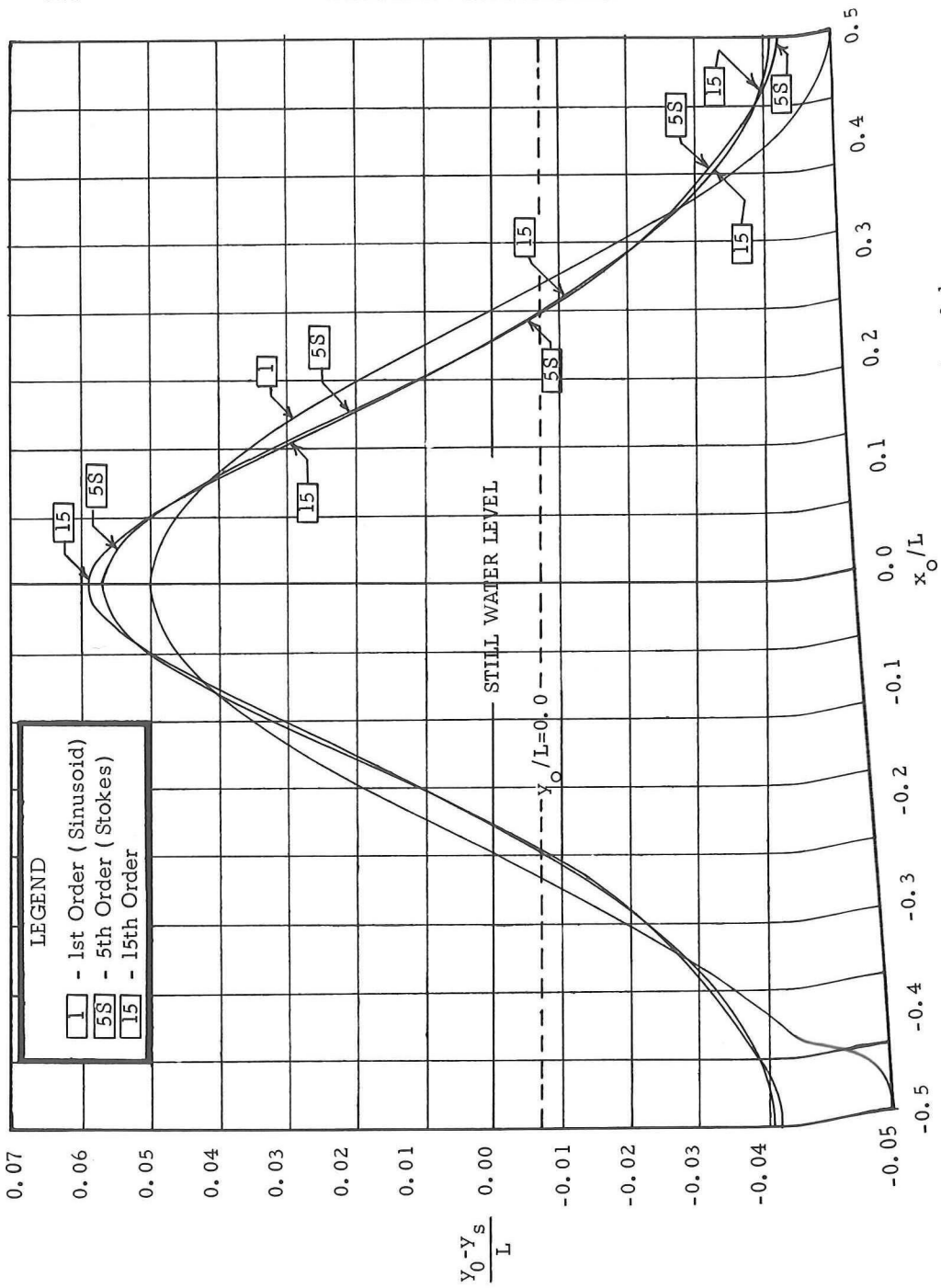


Figure 4. - Profile of a deep water wave - $H/L = 0.1$.

With the aid of this equation, the fifteenth order wave profile in Fig. 4 is oriented with respect to the still water elevation.

For purposes of comparison, Fig. 4 includes, in addition to the fifteenth order profile, the wave shapes which result from the first and fifth (Stokes) order developments. In the first order analysis the classical linear theory of Airy (1845) was employed to give the sinusoidal shape. The fifth order shape (Beach Erosion Board, 1942) was derived from the Stokes-Levi-Civita theory and is given by

$$y_0^I = \alpha \cos x_0^I + \left(\frac{1}{2}\alpha^2 + \frac{17}{24}\alpha^4\right) \cos 2x_0^I + \left(\frac{3}{8}\alpha^3 + \frac{153}{128}\alpha^5\right) \cos 3x_0^I - \frac{1}{3}\alpha^4 \cos 4x_0^I + \frac{125}{384}\alpha^5 \cos 5x_0^I \quad (54)$$

where

$$\alpha = \frac{1}{2}(2\pi H^I) - \frac{3}{64}(2\pi H^I)^3 - \frac{584}{12288}(2\pi H^I)^5 \quad (55)$$

The two higher order profiles shown in Fig. 4 exhibit the shallow, broad troughs and high, slim crests which are characteristic of finite-amplitude waves. This is seen in marked contrast to the first order sinusoidal profile which is more appropriately reserved for small amplitude waves.

A comparison of the fifth and fifteenth order profiles is also fruitful, although the differences are less significant than those involving the sinusoidal profile. The primary limitation of the fifth order expression, Eq. 54, is that the wave shape is described by the first five harmonics alone. In contrast the fifteenth order profile is described by the first fifteen harmonics. Thus the latter profile has a slimmer crest and a broader trough, as is clearly shown in Fig. 4.

A further result of including the higher harmonics is seen in a comparison of the smooth character of the fifteenth order profile with the somewhat wavy character of the fifth order profile. In particular the rather flat portion of the fifth order profile near $x_0/L = \pm 0.4$ shows the effects of using an inadequate number of harmonics. No such flat region is seen in the fifteenth order case. The higher harmonics, therefore, appear to describe the wave more precisely.

A further weakness of Eq. 54 is that, for the higher values of the dimensionless wave height, it is inexact. For example, for $H/L = 0.1$, the case shown in Fig. 4, the dimensionless wave height is shy of this value by 1%. For higher values the divergence from the proper value increases.

One may summarize then, that for many applications the fifth order profile is quite adequate, but for a more precise analysis of the problem the fifteenth order profile is a better representation.

THE HIGHEST WAVE

In order to study the characteristics of the highest possible wave, it is necessary to add a further restriction to those imposed on the deep water wave by Eqs. 43, 44 and 46. This restriction was first suggested by Stokes in his discussion of the highest wave, when he postulated that a sharp peak would occur at the crest of this wave. Such a sharp peak must result in a stagnation point and therefore the velocity will vanish at the crest. From the dynamic boundary condition at the free surface, Eq. 19, we obtain, therefore

$$2gy_0 = K \quad \text{at } r = 1; \chi = 0 \quad (56)$$

or, by substituting the profile expression, Eq. 10, for y_0 ,

$$2 \sum_{k=1}^n \frac{a_k}{k} = K' \quad (57)$$

Since this equation introduces no new unknowns to those already appearing in Eqs. 43, 44 and 46, its inclusion results in a set of $n + 3$ equations with $n + 3$ unknowns, with H' now considered as unknown.

The computation was similar to that for waves of any height, trial values of a_n being chosen until Eq. 57 was satisfied, along with Eqs. 43, 44 and 46. For the fifteenth order the highest possible wave was computed to have a dimensionless wave height,

$$(H/L)_{\max} \approx 0.1442 \quad (58)$$

This compares well with values of

$$(H/L)_{\max} \approx 0.142 \quad (59)$$

obtained by Michell (1893) and

$$(H/L)_{\max} \approx 0.1418 \quad (60)$$

obtained by Havelock (1919). Michell and Havelock both base their analyses on series expansions about the peak, which converge relatively rapidly. In contrast the series obtained in the present analysis converges at an increasingly slow rate as the dimensionless wave height increases.

However, the computations suggest that, as the order of the analysis is increased, and more harmonics are included, the computed height for the highest wave is reduced. The method of the present analysis, if carried to sufficiently high orders, may therefore be expected to give results approaching those of Michell and Havelock. In the absence of such a computation, Eqs. 59 and 60 appear to be more reliable than Eq. 58. As a consequence, Tables 1, 2 and 3 include values of the coefficients for a maximum dimensionless wave height of 0.142, as suggested by Michell.

In concluding this discussion of maximum wave height, it should again be pointed out that very few observations of waves exceeding a dimensionless wave height of 0.1 have been reported. Whether or not the theoretical values for maximum wave height derived by Michell and Havelock are physically reliable, is, as yet, open to question.

CONCLUSION

A theoretical study of deep water waves having a finite height has been undertaken, based on the work of Nekrasov (1951). By means of a conformal transformation the problem is reduced to one of obtaining a solution to a non-linear algebraic equation, which is shown to be in agreement with an equation developed by Stokes (1880). The solution of this equation, for the coefficients of the harmonics that describe the shape of the free surface, is accomplished with the aid of a digital computer.

Machine computations have been carried out to the third, fifth and fifteenth order and the results are shown in Tables 1, 2 and 3. In each case the coefficients have been tabulated for various values of the dimensionless wave height. Values of the corresponding dimensionless wave speed are also included.

A comparison of the resulting wave speeds and wave shapes of deep water waves, with those obtained by Stokes, reveals generally good agreement. This is particularly true for the lower values of the dimensionless wave height, where excellent agreement is obtained. However, for larger wave heights, the inclusion of higher harmonics in the fifteenth order development results in higher values of the wave speed than were obtained from some of the lower order analyses, including those of Stokes. Similarly, for the larger wave heights the use of the fifteenth order development in predicting wave shape results in broader troughs and slimmer peaks than those predicted by the lower order analyses.

An application of the fifteenth order analysis to the determination of the highest wave in water shows good agreement with the results obtained by Michell (1893) and Havelock (1919).

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APPENDIX-NOTATION

A	point on the boundary of the wave field at $y = -\infty$
A_j	see Eq. 35
a_j	coefficient of the j th harmonic - see Eq. 8
B	point on the boundary of the wave field at the trough
C	point on the boundary of the wave field at the crest
c	wave speed
c'	$\equiv c/c_L$
c_L	first order wave speed = $\sqrt{gL/2\pi}$
D	point on the boundary of the wave field at the trough
$f(\zeta)$	see Eq. 12
g	acceleration due to gravity
H	wave height
H'	$\equiv H/L$
i	$\sqrt{-1}$
j	integer which identifies the j th harmonic - see Eq. 8
K	Bernoulli constant
K'	$\equiv K/c_L^2$
k	integer which identifies the k th harmonic
L	wave length
ℓ	integer which identifies the ℓ th harmonic
n	integer which identifies the highest harmonic and the order of the analysis
q_0	magnitude of the particle velocity at the free surface

R_0	modulus of $f(\zeta_0)$ - see Eq. 16; also $= c/q_0$
r	radial coordinate in the ζ -plane
u	x-component of the particle velocity
u_0	x-component of the particle velocity at the free surface
v	y-component of the particle velocity
v_0	y-component of the particle velocity at the free surface
w	$\equiv \phi + i\psi$
x	horizontal coordinate in the z -plane
x_0	horizontal free surface coordinate in the z -plane
x'_0	$\equiv gx_0/c_L^2 = 2\pi x_0/L$
y	vertical coordinate in the z -plane
y_0	vertical free surface coordinate in the z -plane
y'_0	$\equiv gy_0/c_L^2 = 2\pi y_0/L$
y_s	still water level in the z -plane
z	$\equiv x + iy$
α	see Eq. 55
ζ	$\equiv r \exp(i\chi)$
ζ_0	$= \exp(i\chi)$, or ζ at the free surface
θ_0	argument of $f(\zeta_0)$ - see Eq. 17; also local slope angle of the free surface in the z -plane
π	$\equiv 3.1415927$
ϕ	potential function
χ	tangential coordinate in the ζ -plane; also the normalized potential function
ψ	stream function