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On the implied approximations appearing in simplified Emissivity for the investigation of the Thermal Emission from Dense Media

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***Abstract*—According to our recent modal representation, the origin of thermal radiation can be associated to the distribution of a finite number of current sources, independent one from the other (the Degrees of Freedom). In a companion paper we have also shown that, if one is only interested in the energy radiated, the current sources can be mathematically replaced by a much larger number of equivalent sources, that are unphysical, as they are imposed to be uncorrelated at any distance and are distributed all over the volume. A classic procedure for estimating the electromagnetic energy emitted by such ensemble of energetically equivalent currents is presented. The derivation presented here can just as well be applied to another set of energetically equivalent currents, the Quantum born Rytov currents. To arrive to a final analytical expression for the radiation, multiple simplifying approximations are used. Treating the problem in transmission rather than in reception provides useful insights on their limits and applicability.**

***Index Terms*—Thermal noise, radiometry, electromagnetics, emissivity, thermal radiation**

I. INTRODUCTION

The field of radiometry comprises the evaluation of the thermally induced radiation generated by a generic material. The peculiarity of this radiation is that its sources are typically assumed to be temporally and spatially incoherent [1] and thus the corresponding generated Poynting vectors are indicated as brightness expressed in $Wm^{-2}sr^{-1}$. The expression for the brightness of a black body (BB) was postulated by Planck's radiation law [2], which is not a classic electromagnetic (EM) description of radiation. Rytov [1] in the 60's, devised the amplitude of the equivalent currents that, distributed incoherently over the warm body, would radiate the same energy as implied by the BB brightness. The actual current values where described in terms of their autocorrelation. Since BBs do not exist, Rytov accounted for the real material properties by introducing in the currents amplitude the effective losses in the medium.

In [3], we presented an alternative derivation for the thermally induced fields, in material bodies, resorting to the concept of Degrees of Freedom (DoF). In [4] we have shown that the average radiation from the DoF can be calculated resorting to equivalent average currents, whose spatial distribution can be expressed in the same format as Rytov's currents. Using either procedure, Rytov or the DoF, the equivalent currents oriented along any of the $\hat{p}$ polarizations can be expressed as the superposition of incoherent currents distributed over small cubic volumes of side Δ centered in different positions $\vec{r}_n$:

$$\vec{J}_p^{\infty}(\vec{r}) = \hat{p}\,\frac{I^{\infty}}{\Delta^2}\,\mathrm{rect}(\vec{r}-\vec{r}_n, \Delta^3) \tag{1}$$

Since the currents were assumed incoherent, only their modulus plays a role in evaluating the radiated energy and this amplitude is expressed as

$$|I^{\infty}| = \sqrt{\Delta A_{\infty}^{\Delta^3}\left(\vec{J}_p^{\infty}\right)} \tag{2}$$

where $A_{\infty}^{\Delta^3}$ is the autocorrelation on the domain Δ^3, assumed to be derived in an infinite medium. The value of the autocorrelation depends on the procedure used to estimate the autocorrelation: either the DoF procedure or the Rytov procedure

$$\begin{aligned} A_{\infty}^{\Delta^3}\left(\vec{J}_p^{DoF}\right) &= k_B T\,\lambda_0^2\,\frac{3\alpha}{\zeta_0^2\pi}\,Re\left\{\frac{1}{(\sigma^*-j\omega\varepsilon_0\varepsilon_r)}\right\}\frac{1}{d^3} \\ A_{\infty}^{\Delta^3}\left(\vec{J}_p^{Rytov}\right) &= -\omega Im\{\varepsilon\}\,\frac{4hf}{e^{\frac{hf}{k_B T}}-1} \end{aligned} \tag{3}$$

In (3) the material was assumed to be characterized by an effective dielectric constant ε which can be expressed as

$$\varepsilon = \varepsilon_0\varepsilon_{r,eff} = \varepsilon_0(\varepsilon_{r,r} + j\varepsilon_{r,i}) \tag{4}$$

where ε_0 is the free space dielectric constant, $\varepsilon_{r,r}$ and $\varepsilon_{r,i}$ are the real and imaginary part of the relative effective dielectric constant with the imaginary part accounting for the Joule heating losses. In almost all literature, [5,6] the brightness is derived from the absorptivity resorting to reciprocity [7]. In modern literature even the impact of multiple reflections in stratified media have been addressed in many good papers such as [8-10].

In this contribution, from the expressions (1)-(2) of the distributed currents we, similarly to [11] derive the radiated energy using a direct radiation procedure to derive the brightness. The advantages of the direct radiation method are multiple:

1) The average radiated energy can be evaluated also in the near field.

2) The radiated energy can be calculated using different levels of approximations: a) rigorous Green's functions calculations, b) approximate Physical Optics (PO) integrations or even c) geometrical optics (GO) ray tracing, for low dispersivity materials. The GO procedure leads to the

expression of the brightness in terms of analytical emissivity formulas.

3) These latter analytical formulas are equivalent to the expressions that are most often used in often in text-books, while being derived in a reception formalism. We believe that the derivation from direct radiation provides useful boundaries for their applicability.

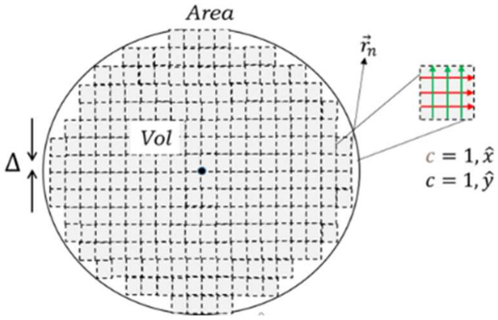

Fig. 1 Discretized representation of the currents postulated used in this paper on a finite spherical domain. The elementary blocks presents incoherent phases, but the same current amplitude.

II. FLUX OF POYNTING VECTOR DUE TO UNCORRELATED CURRENTS ON FINITE DOMAIN

A. PO Approximation

The first approximation that allows a simplified analysis of the thermally induced radiation from a finite body is to assume that the amplitude of the currents is the same in every cell and equal to that of the currents in an infinite body, as in Fig. 1. This approximation can be indicated as the Physical Optics (PO) one, in analogy of what is done in the field of quasi optics, where the perturbation of the currents due to finite edges are neglected. These currents were indicated as as $\vec{J}^{\infty}$ in eq. (1). When these currents are distributed on the finite domain, their collection can be expressed as

$$\vec{J}_{PO,p}(\vec{r}) = \sum_{n=1}^{N} \hat{p}\frac{I^{\infty}}{\Delta^2}\mathrm{rect}(\vec{r}-\vec{r}_n, \Delta^3) \quad (5)$$

which is valid for every $p = x, y, z$. In (5) $N = \frac{Vol}{\Delta^3}$, and n is a formal index providing a summation over a finite number of points in space:

$$\vec{r}_n = \Delta[n_x\hat{x} + n_y\hat{y} + n_z\hat{z}] \quad (6)$$

B. Poynting vector

The energy flux due to the finite body can be obtained by superimposing the Poynting vector fluxes emerging from all the cubic cells that are identified by the sub-gridding for the PO currents, Fig. 1. This is because all the currents in the different cells are incoherent. These incoherent currents together generate the brightness, which in our treatment will only emerge at the end of this paper, when we simplify the expressions for the energy radiated. Also each of the three current components, $p = x, y, z$ in each cell produce independent Poynting vectors. The flux of each Poynting vector contribution outside the volume, Vol containing a specific PO current cell centered in $\vec{r}_n$, and oriented along any of the possible directions $\hat{p} = \hat{x}, \hat{y}, \hat{z}$, can be indicated as Φ_n^p. It can be expressed as the integral over the surface, S, surrounding the volume, Vol, as

$$\Phi_n^p = \iint_S \vec{s}^p(\vec{r}, \vec{r}_n) \cdot \hat{n}(\vec{r}) d\vec{r} \quad (7)$$

In (7) $\vec{s}^p(\vec{r}, \vec{r}_n)$ indicates the Poynting vector contribution transmitted at the interface from the medium to free space, at the surface point $\vec{r}$, and $\vec{r}_n$ indicates the specific contribution from a sub-cell of volume Δ^3.

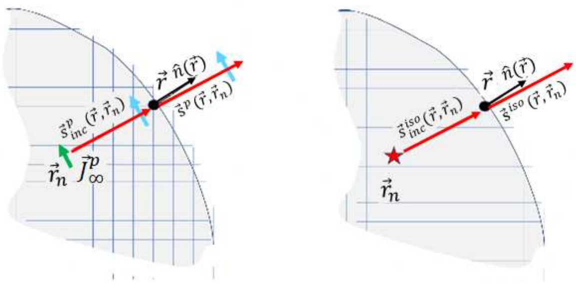

Fig. 2 a) Poynting vector, incident and transmitted through a specific point $\vec{r}$ on the surface emerging from a current component, $\vec{J}_{\infty}^{p}$ located in $\vec{r}_n$. b) Isotropic equivalent contribution due to three orthogonal current components in $\vec{r}_n$.

$\vec{s}^p(\vec{r}, \vec{r}_n)$ is proportional to $|I^{\infty}|^2$ from (2). There can be a difficulty in evaluating and interpreting $\vec{s}^p(\vec{r}, \vec{r}_n)$ if the losses in the medium are important. Unfortunately the media which are most often encountered in radiometric problems are typically dispersive and lossy especially at low frequencies. Accordingly the evaluation of $\vec{s}^p(\vec{r}, \vec{r}_n)$ in (7) is not straight forward. The analysis can be much simpler at frequencies at which the dispersivity is low, which fortunately coincides with those frequencies at which the radiation is stronger.

C. Low dispersivity material

In the following we will assume that the dispersivity in the lossy medium is moderate: i.e. $-Im\{k\} < 0.5\, Re\{k\}$, with $k = \omega\sqrt{\varepsilon_0\mu_0}\sqrt{\varepsilon_{r,eff}}$. In this situation we assume that the direction of propagation of the waves launched by each source is well defined with Poynting vector in $\vec{r}$, associated to a wave emerging from $\vec{r}_n$, essentially parallel to the unit vector $\frac{(\vec{r}-\vec{r}_n)}{|\vec{r}-\vec{r}_n|}$. With reference to Fig. 2, $\hat{n}(\vec{r})$ in (7) indicates the external normal to the surface in $\vec{r}$. In turn, the transmitted Poynting vector is the product of the incident one $\vec{s}_{inc}^p(\vec{r}, \vec{r}_n)$ and the corresponding power transmission coefficient from the inside of the warm body to free space: this latter transmission coefficient can be associated either TE or TM ($\bar{\bar{T}}^{TE}_{TM}$) polarization depending of the polarization of the incident wave defined by the current considered and the specific ray linking the considered point on the surface, $\vec{r}$ and the source current location, $\vec{r}_n$,

$$\vec{s}^p(\vec{r}, \vec{r}_n, \hat{n}) = \bar{\bar{T}}^{TE}_{TM}\left[\frac{\vec{r}-\vec{r}_n}{|\vec{r}-\vec{r}_n|}, \hat{n}\right] * \vec{s}_{inc}^p(\vec{r}, \vec{r}_n) \quad (8)$$

The flux from the $n-th$ cell due to all the three orthogonal currents associated to the $n-th$ cubic cell in Fig. 1 can be expressed as

$$\Phi_n = \sum_{p=1}^{3} \Phi_n^p \quad (9)$$

Accordingly the total flux due to all the current sources distributed in the volume can be expressed as

$$\Phi = \sum_{n=1}^{N} \Phi_n \tag{10}$$

In principle (10) can be calculated fairly simply, numerically.

III. Approximate Expression for the Radiated Energy from Low Dispersivity Materials

To facilitate the calculation of the average (in time) flux one can introduce a *number of approximations* that can even lead to analytical expressions for the energy radiated by finite bodies in some particular situations.

A. Unpolarized fields

Since the thermally excited equivalent dipoles in an infinite medium will present varying polarizations the average energy that they radiate can be assumed to be emerging from current sources that generate unpolarized fields. This allows to superpose the Poynting vectors of the eigen modes represented in each cubic cell of side Δ. From the DoF procedure the number of eigen modes should have been 6, 3 due to the electric currents and 3 due to the magnetic currents, while the Rytov procedure only considers 3 orthogonal electric currents. In order to unify the treatment for the two typology of representations we will assume that we only have electric currents but at the end we will multiply the energy radiated by the DoF procedure by a factor 2.

The simplification arises from the fact that the TE/TM transmission coefficients in (8) can be replaced with the average of the TE and TM transmission coefficients. Moreover, the amplitudes of all the incident Poynting vectors are the same for all electric dipoles and for every polarization since they all generate the same energy over the 4π solid angle. The approximate expression for the flux from the n_{th} cube is:

$$\Phi_n \approx 3 \times \iint_S \overline{\overline{\mathrm{T}}}_{\mathrm{ave}}(\vec{r}) * \vec{s}_{iso}^{\,inc}(\vec{r}, \vec{r}_n) \cdot \hat{n}(\vec{r}) d\vec{r} \tag{11}$$

where $\vec{s}_{iso}^{\,inc}(\vec{r}, \vec{r}_n)$ represents the isotropic Poynting vector due to the $n-th$ cell. In turn, this latter can also be expressed as a function of only the equivalent generating electric current concentrated in the cube of side Δ also centered at $\vec{r}_n$, see Fig. 2b. The equivalent modulus square of electric current distributed on the isotropic dipole is $|I_{iso}^{\infty}|^2 = |I^{\infty}|^2 \frac{2}{3}$. Accordingly the term $\vec{s}_{iso}^{\,inc}(\vec{r}, \vec{r}_n)$ from the integrand in (11) represents the isotropic Poynting vector due to an infinitesimal current distribution:

$$\vec{\mathrm{s}}_{\mathrm{iso}}^{\,inc}(\vec{r}, \vec{r}_n) = \frac{2}{3} |I^{\infty}|^2 \zeta_0 \frac{1}{\lambda_0^2} \sqrt{\varepsilon_{r,eff}}^{\,*} \Delta^2 \frac{e^{-2\alpha|\vec{r}-\vec{r}_n|}}{4|\vec{r}-\vec{r}_n|^2} \frac{(\vec{r}-\vec{r}_n)}{|\vec{r}-\vec{r}_n|} \tag{12}$$

where $\alpha = -Im\{k\}$. Superimposing the contributions of all the cells to the flux outside the considered volume leads to

$$\Phi_{tot} = 3 \iint_S \sum_{n=1}^{N} \overline{\overline{\mathrm{T}}}_{\mathrm{ave}}(\vec{r}, \vec{r}_n) * \vec{\mathrm{s}}_{\mathrm{iso}}^{\,inc}(\vec{r}, \vec{r}_n) \cdot \hat{n}(\vec{r}) d\vec{r} \tag{13}$$

The expression in (13) can be further simplified by replacing the summation in n with a integration over the volume of the considered body. To obtain this goal, the Poynting vector, $\vec{\mathrm{s}}_{\mathrm{iso}}^{\,inc}(\vec{r}, \vec{r}_n)$ at observation points, $\vec{r}$, due to a source in $\vec{r}_n$ can be replaced by an integration over the cubic cell volume Δ_n^3, surrounding $\vec{r}_n$ as follows:

$$\vec{\mathrm{s}}_{\mathrm{iso}}^{\,inc}(\vec{r}, \vec{r}_n) \approx \frac{1}{\Delta^3} \iiint_{\Delta_n^3} \vec{\mathrm{s}}_{\mathrm{iso}}^{\,inc}(\vec{r}, \vec{r}') d\vec{r}' \tag{14}$$

If one then recognizes that thanks to the chosen sub-discretization

$$\sum_{n=1}^{N} \iiint_{\Delta_n^3} \dots\dots d\vec{r}' = \iiint_{Vol} \dots\dots d\vec{r}' \tag{15}$$

The average flux due to all the degrees of freedom can be expressed as a double integral through the entire volume as

$$\Phi_{tot} = \frac{3}{\Delta^3} \iint_S \iiint_{Vol} \overline{\overline{\mathrm{T}}}_{\mathrm{ave}}(\vec{r}, \vec{r}') * \vec{\mathrm{s}}_{\mathrm{iso}}^{\,inc}(\vec{r}, \vec{r}') \cdot \hat{n}(\vec{r})\, d\vec{r}'\, d\vec{r} \tag{16}$$

Substituting, the expression for the isotropic Poynting vectors from (12) in (16) the total flux can be expressed as

$$\Phi_{tot} = \frac{1}{\Delta} \frac{1}{2} |I^{\infty}|^2 \zeta_0 \frac{1}{\lambda_0^2} \sqrt{\varepsilon_{r,eff}}^{\,*} \times I_{int} \tag{17}$$

where the compact notation I_{int} indicates the integrations over the volume and the surface

$$I_{int} = \iint_S \iiint_{Vol} \frac{e^{-2\alpha|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|^2} \overline{\overline{\mathrm{T}}}_{\mathrm{ave}}(\vec{r}, \vec{r}') * \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|} \cdot \hat{n}(\vec{r}) d\vec{r}' d\vec{r} \tag{18}$$

B. Large surface curvatures and no multiple reflections

The expression in (17) requires the evaluation of the integration in (18) which in general needs to be performed numerically. However, as it is customary in radiometric problems, a few further simplifications allow to express the integrals in simpler form. The main approximation is the assumption that the radius of curvature, r_{curv}, of the radiating warm body is large in terms of the wavelength, λ, and that the attenuation within the body at the considered frequency, is high enough that multiple reflections within the body can be neglected. To introduce these simplifications one can first identify for each point, $\vec{r}$, in the surface identified as S in (18), that the entire volume contributes to the integrand incident Poynting vector.

$$I_{int} = \iint_S \vec{I}_V(\vec{r}) \cdot \hat{n} d\vec{r} \tag{19}$$

where $\vec{I}_V(\vec{r})$ indicates the volumetric integral

$$\vec{I}_V(\vec{r}) = \iiint_{Vol} \frac{e^{-2\alpha|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|^2} \overline{\overline{\mathrm{T}}}_{\mathrm{ave}}(\vec{r}, \vec{r}') * \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}\, d\vec{r}' \tag{20}$$

In correspondence of each $\vec{r}$, one can identify an auxiliary flat surface with the same normal $\hat{n}(\vec{r})$ as the real one. If the radius of curvature of the actual surface is large with respect to the wavelength in the medium and with respect to the penetration depth, δ, the volumetric integration leading to $\vec{I}_V(\vec{r})$ can be approximated as the integration over a semi-sphere centred in $\vec{r}$. Accordingly eq. (19) can be approximated as

$$\vec{I}_V(\vec{r}) \cdot \hat{n} d\vec{r} \approx I_{\mathrm{HS}}(\vec{r}) d\vec{r} \tag{21}$$

where

$$I_{\mathrm{HS}}(\vec{r}) = \iiint_{HS} \frac{e^{-2\alpha r''}}{r''^2} \overline{\overline{\mathrm{T}}}_{\mathrm{ave}}(\vec{r}, \vec{r}'') * \hat{r}_{in} \cdot (-\hat{z})\, d\vec{r}'' \qquad (22)$$

In the above a local reference system was introduced with origin in $\vec{r}$ and with $\hat{n} = (-\hat{z})$. To perform the integration over the hemisphere a spherical coordinate parametrization can be used which suggests a change of variable $\vec{r}' \rightarrow \vec{r}''$used in (22) can be defined so that $\hat{r}_{in} = \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}$ is expressed as $-\hat{r}''$. See Fig.3 for a visualization of the approximations and refence systems applied in this section to the surface geometry. Introducing explicitly this spherical representation and also that the transmission power coefficient is now expressed only in terms of local coordinates $\overline{\overline{\mathrm{T}}}_{\mathrm{ave}}(\vec{r}, \vec{r}'') = \overline{\overline{\mathrm{T}}}_{\mathrm{ave}}(\phi'', \theta'')$, it results that I_{HS} can be expressed as

$$I_{\mathrm{HS}}(\vec{r}) = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \overline{\overline{\mathrm{T}}}_{\mathrm{ave}}(\phi'', \theta'') * \hat{r}_{in} \cdot (-\hat{z}) \sin\theta''\, d\theta'' d\phi''$$
$$\times \int_0^l \frac{e^{-2\alpha r''}}{r''^2} r''^2\, dr'' \qquad (23)$$

It is apparent from (23) that the radial spreading in the denominator simplifies with the differential in spherical coordinates leading to

$$I_{\mathrm{HS}}(\vec{r}) = \frac{1 - e^{-2\alpha l}}{2\alpha} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \overline{\overline{\mathrm{T}}}_{\mathrm{ave}} * \hat{r}_{in} \cdot (-\hat{z}\,) \sin\theta''\, d\theta'' d\phi'' \qquad (24)$$

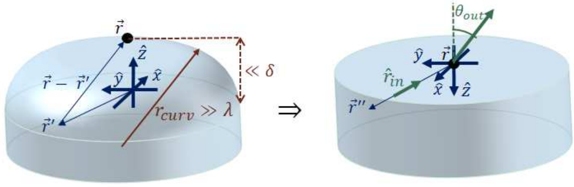

Fig. 3 Visualization of the approximations and the coordinate reference systems adopted to evaluate the outward flux corresponding to a point $\vec{r}$ on the surface of the body.

To perform the integration in θ'' one can first recognize that

$$\frac{\overline{\overline{\mathrm{T}}}_{\mathrm{ave}}(\phi'', \theta'') * \hat{r}_{in}}{|\overline{\overline{\mathrm{T}}}_{\mathrm{ave}}(\phi'', \theta'') * \hat{r}_{in}|} \cdot (-\hat{z}) = \cos\theta_{out} \qquad (25)$$

Using then Snell's law $\sin\theta'' = \frac{\sin\theta_{out}}{Re\{\sqrt{\varepsilon_{r,eff}}\}}$ the integration in (24) becomes

$$I_{\mathrm{HS}}(\vec{r}) = \frac{1-e^{-2\alpha l}}{2\alpha} \frac{1}{Re\{\sqrt{\varepsilon_{r,eff}}\}} \times$$
$$\int_0^{2\pi} \int_0^{\theta''_{\mathrm{crit}}} |\overline{\overline{\mathrm{T}}}_{ave}(\phi'', \theta'') * \hat{r}_{in}| \cos\theta_{out} \sin\theta_{out}\, d\theta'' d\phi'' \qquad (26)$$

where it has been recognized that the modulus of the transmission coefficient is equal to zero for elevation angles larger than the critical angle defined at the dielectric air interface, $\theta'' > \theta''_{crit}$. To perform this integration it is best to introduce the change of variable $d\phi'' = d\phi_{out}$; $d\theta'' = \frac{d\theta''}{d\theta_{out}} d\theta_{out}$ which leads to

$$I_{\mathrm{HS}}(\vec{r}) = \frac{1-e^{-2\alpha l}}{2\alpha} \frac{1}{Re\{\sqrt{\varepsilon_{r,eff}}\}} \int_0^{2\pi} \int_0^{\pi/2} |\overline{\overline{\mathrm{T}}}_{ave}(\phi_{out}, \theta_{out}) * \hat{r}_{in}|$$
$$\cos\theta_{out} \sin\theta_{out} \frac{d\theta''}{d\theta_{out}} d\theta_{out} d\phi_{out} \qquad (27)$$

C. Analytical approximations

The integration in θ_{out} is dominated by the points where $\theta_{out} \approx 0$ for which $\theta'' \approx \frac{\theta_{out}}{Re\{\sqrt{\varepsilon_{r1}}\}}$ so that $\frac{d\theta''}{d\theta_{out}} = \frac{1}{Re\{\sqrt{\varepsilon_{r,eff}}\}}$. In this case, where we can approximate $|\overline{\overline{\mathrm{T}}}_{ave}(\phi'', \theta'') \cdot \hat{r}_{in}| = T_{ave}(0,0)$ it results that

$$I_{\mathrm{HS}}(\vec{r}) = \frac{1-e^{-2\alpha l}}{2\alpha} \frac{2\pi T_{ave}(0,0)}{Re^2\{\sqrt{\varepsilon_{r,eff}}\}} \int_0^{\pi/2} \cos\theta_{out} \sin\theta_{out}\, d\theta_{out} \qquad (28)$$

It is apparent that the approximate expression in (28) can be integrated analytically, using

$$\int_0^{\pi/2} \cos\theta_{out} \sin\theta_{out}\, d\theta_{out} = \frac{1}{2} \qquad (29)$$

Recognizing that $\alpha = \frac{1}{\delta}$, eq. (28) becomes

$$I_{\mathrm{HS}}(\vec{r}) = (1 - e^{-2\alpha l}) \frac{\delta\pi}{Re^2\{\sqrt{\varepsilon_{r,eff}}\}} \frac{1}{2} T_{ave}(0,0) \qquad (30)$$

A comparison between the numerical evaluation of Eq. (27) and Eq. (30) for a material with $\varepsilon_{r,eff} = 80 - j$ at a frequency of 10GHz is shown in Fig. 4.

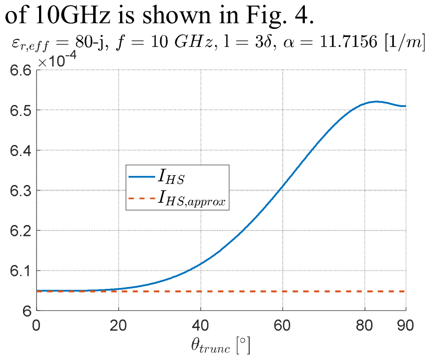

Fig. 4 Numerical evaluation of Eq. (27), denoted I_{HS}, together with the analytical expression from Eq. (30), which is indicated as $I_{HS,approx}$. The quantities are being plotted versus the truncation angle, θ_{trunc} ,where the variability of the functions $|\overline{\overline{\mathrm{T}}}_{ave}(\phi_{out}, \theta_{out}) * \hat{r}_{in}|$ and $\frac{d\theta''}{d\theta_{out}}$ is cut off. The material has a relative effective permittivity of $80 - j$, length $l = 3\delta$ and the frequency is set to 10 GHz.

When evaluating Eq. (27) and Eq. (30), $|\overline{\overline{\mathrm{T}}}_{ave}(\phi_{out}, \theta_{out}) * \hat{r}_{in}|$ is taken to be: $\left|\frac{\eta_1}{\eta_2}\right| \left|\frac{|\tau_\perp|^2 + |\tau_{||}|^2}{2}\right|$, in the notation of [12]. This fraction represents the average of the power transmission coefficients (as a ratio of Poynting vectors) associated to orthogonal and parallel polarization, between two semi-infinite, homogenous, dielectric media, see [12]. The term $\frac{d\theta''}{d\theta_{out}}$, using Snell's Law and the assumption that $Re\{k\} \gg Im\{k\}$ can be written as

$$\frac{d\theta''}{d\theta_{out}} \approx \frac{1}{Re\{\varepsilon_{r,eff}\}} \frac{\cos(\theta_{out})}{\sqrt{1 - \frac{\sin^2(\theta_{out})}{Re^2\{\varepsilon_{r,eff}\}}}} \qquad (31)$$

For the numerical results in Fig. 4, within the range $0 \le \theta_{out} \le \theta_{trunc}$, the expressions provided for $|\overline{\overline{\mathrm{T}}}_{ave}\,(\phi_{out}, \theta_{out}) * \hat{r}_{in}|$ and $\frac{d\theta''}{d\theta_{out}}$ have been implemented. When $\theta_{out} \ge \theta_{trunc}$, the values at $\theta_{out} = \theta_{trunc}$ were used. Overall, the analytical expression from Eq. (30) seems to approximate the integral in Eq. (27) well within a 10% deviation even when θ_{trunc} is taken all the way to 90°.

IV. Brightness and Emissivity

Resorting to the approximations of section III we could express the surface-volumetric integral representation of the PO Poynting vector in terms of a surface-solid angle integration. This allows us to recognize the standard brightness as a simplified representation of the Poynting vector flux due to incoherent sources. This is achieved replacing Eq. (27) in (17) and introducing the notation

$$\Phi_{tot} = \iint_S \iint_{2\pi} B(\Omega)\, d\Omega d\vec{r} \tag{32}$$

where

$$B(\Omega) = \frac{1}{\Delta}\frac{1}{2}|I^{\infty}|^2 \zeta_0 \frac{1}{\lambda_0^2}\sqrt{\varepsilon_{r,eff}}^{*}\frac{1-e^{-2\alpha l}}{2\alpha}\frac{1}{Re\{\sqrt{\varepsilon_{r,eff}}\}} |\overline{\overline{\mathrm{T}}}_{ave}\,(\phi_{out}, \theta_{out}) \cdot \hat{r}_{in}| \cos\theta_{out} \frac{d\theta''}{d\theta_{out}}$$

This expression highlights the origin of the $\cos\theta_{out}$ dependence of the brightness at times indicated as the Lambertian, which is typically assumed in radiometry. In (32) one can recognize the term $\frac{|I^{\infty}|^2}{\Delta}$ which was previously expressed, eq. (2), as the autocorrelation of the radiating currents over the cubic cell Δ. The analytical approximations introduced in eq. (28)-(30) are typically also adopted in radiometric contexts and lead to

$$B(\Omega) \approx B(\Omega = 0) = A^{\Delta^3}\,\frac{1}{4}\zeta_0 \frac{1}{\lambda_0^2}(1 - e^{-2\alpha l})\frac{\delta T_{ave}(0,0)}{Re\{\sqrt{\varepsilon_{r,eff}}\}} \tag{33}$$

Where we have assumed $\sqrt{\varepsilon_{r,eff}}^{*} \approx Re\{\sqrt{\varepsilon_{r,eff}}\}$, which again requires the recognition that these expressions should only be used for weakly dispersive media.

Accordingly substituting (33) in (32) and focusing only on Rytov currents

$$\Phi_{tot}^{Rytov} \approx S\frac{\pi}{2}A_{Rytov}^{\Delta^3}\,\zeta_0 \frac{1}{\lambda_0^2}(1 - e^{-2\alpha l})\frac{\delta T_{ave}(0,0)}{Re\{\sqrt{\varepsilon_{r,eff}}\}} \tag{34}$$

In the case that the radiometric fields were derived from the DoF procedure the flux would have to be complemented by the contributions associated to the magnetic currents and thus

$$\Phi_{tot}^{DoF} \approx S\frac{\pi}{2}[2A_{DoF}^{\Delta^3}]\,\zeta_0 \frac{1}{\lambda_0^2}(1 - e^{-2\alpha l})\frac{\delta T_{ave}(0,0)}{Re\{\sqrt{\varepsilon_{r,eff}}\}} \tag{35}$$

These fluxes can be expressed as

$$\Phi_{tot} = \begin{bmatrix} A_{Rytov}^{\Delta^3} \\ 2A_{DoF}^{\Delta^3} \end{bmatrix} F(S, l; \varepsilon_{r,eff}, \lambda_0) \tag{36}$$

Where

$$F(S, l; \epsilon_r, \lambda_0) = S\left[\frac{\delta(1-e^{-2\alpha l})}{\lambda_0^2}\, T_{ave}(0,0)\frac{\pi}{2}\frac{\zeta_0}{Re\left\{\sqrt{\varepsilon_{r,eff}}\right\}}\right] \tag{37}$$

is a factor that depends from the geometry the material and the frequency and remains the same independently from the methodology used to approximate the autocorrelation of currents.

V. Conclusions

The EM and quantum procedures to derive the currents in the infinite warm body differ in the value for the currents I^{∞}. However in both cases the spectral energy radiated can be expressed resorting to exactly the same methodology, except for the magnitude of the currents. Here we have presented an approximate procedure to evaluate the radiated energy using the approximation that the current in each element of the warm volume is the same as in the corresponding infinite material. Moreover we have shown that the direct calculation of the radiated energy can lead to an analytical expression which is essentially equivalent to the ones that are found in standard radiometry textbooks. However in obtaining this result we have had to assume that the frequency dispersion is limited, that the radius of curvature of the warm body is large with respect to the wavelength, that the currents radiate in average as isotropic sources, and finally that the brightness is characterized by its value at broadside. With this direct radiation we can quantify the loss of accuracy linked to each of these approximations.

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