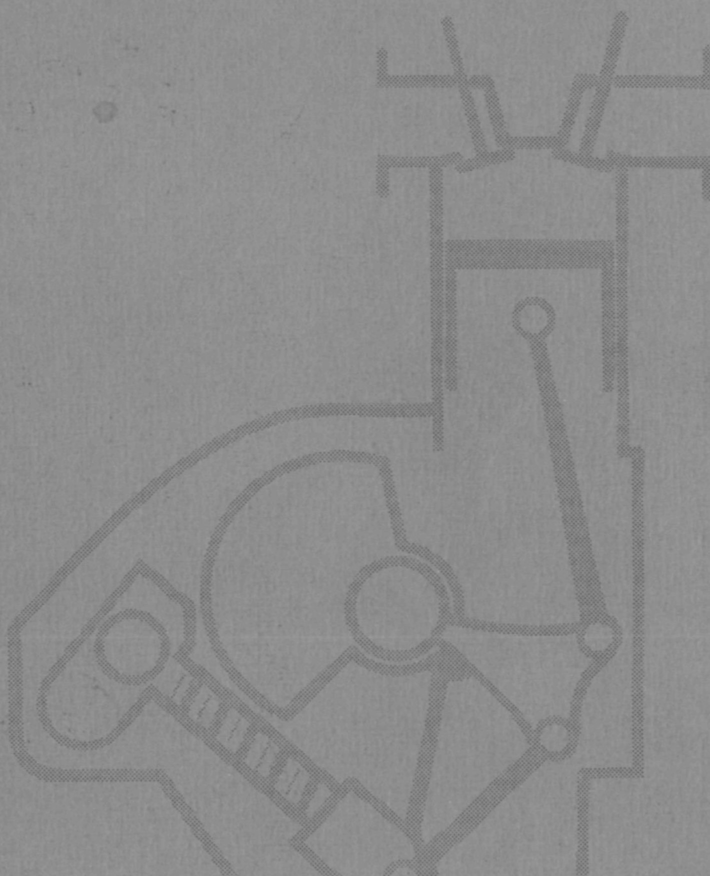
Kinematic and dynamic analysis of mechanisms, a finite element approach



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Kinematic and dynamic analysis of mechanisms, a finite element approach

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Abstract

The development of the finite element method for the numerical analysis of the mechanical behaviour of structures has been directed at the calculation of the state of deformation and stress of kinematically determinate structures. The discretized description of the kinematics of kinematically indeterminate structures as given in the finite element method is however also a good starting point for the numerical treatment of the analysis of mechanisms.

In the description of the kinematics of mechanisms the relations between deformations and displacements play a central role. For the calculation of the transfer functions of order one and two, being the basic information for the determination of velocity and acceleration, direct methods are presented, applicable to mechanisms consisting of undeformable links.

The description is completed with the formulation of dynamics, kinetostatics and vibrations. For mechanisms consisting of deformable links an approximate method is given.

The theory is applied to planar mechanisms. Examples demonstrate the use of the theory in kinematic, dynamic and kinetostatic problems.

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1

Introduction

Since many centuries mechanisms have found application in numerous fields of mechanical engineering. In many cases a bar linkage is used, especially when motion patterns with relatively large displacements must be realized. The design of such bar linkages is mainly based on kinematic considerations. The ever growing demands made on industrial machines have resulted in a growing interest, not only in mechanism synthesis, but also in analysis of the dynamic behaviour of mechanisms. An extra stimulus is provided by applications where mechanisms replace handwork that is either soulkilling or that must be done in an environment not particularly suited for human beings. These, mainly economic, motives in conjunction with the products of the modern technology, such as the computer and the methods of numerical analysis and data handling, lead to a progress in mechanism design. Characteristic for this progress is on the one hand a widening of the possibilities and a better insight in the behaviour of the mechanism. On the other hand the design process becomes faster and less costly.

In order to be able to contribute to these modern developments in 1972 a project group CADOM *) was started within the Laboratory for Mechanization of Production and Mechanisms of our Technical University. In the first instance this group has undertaken the task of the computer implementation of RANKERS' method of mechanism synthesis [1], [2]. As a member of this group the author was confronted with the specific problems in the field of mechanisms. The existing lack of appropriate methods

*) acronym for Computer Aided Design Of Mechanisms.

for the description of the motion of bar linkage mechanisms, especially for the more complex ones, was the direct inducement for the search for a numerical approach of the kinematics and the dynamic behaviour of bar mechanisms.

The starting point of the investigations was the theory for geometrically nonlinear structures of VISSER and BESSELING [3]. In the application of this theory to the problem of cross spring pivots the author used rigid elements to describe the kinematical relation between the blade spring ends A and B (fig. 1a). The idea of building models of mechanisms according this concept (fig. 1b) was left for a more fundamental approach.

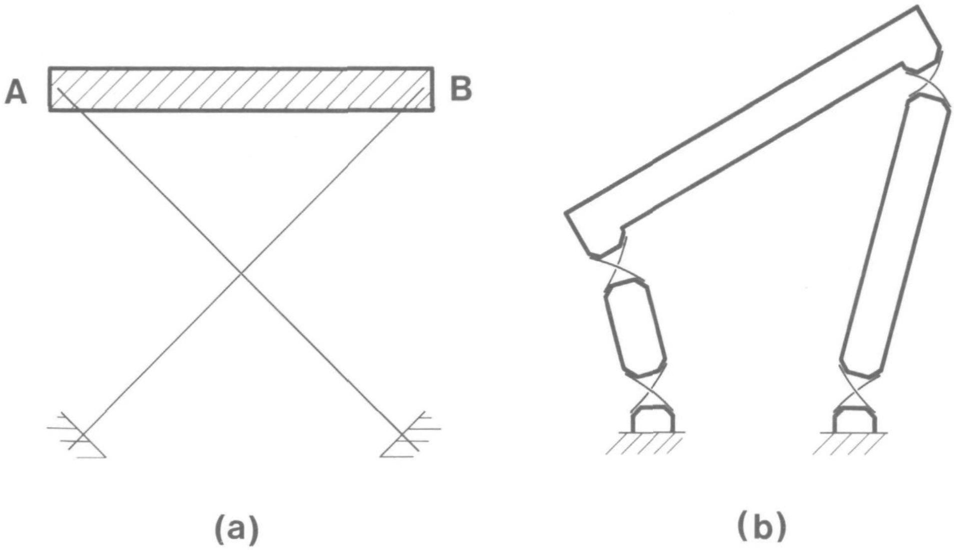

Fig. 1 Cross spring pivot and four-bar with cross spring pivots.

In a comparatively short time the description of the geometry of motion, i.e. the zero order kinematics of mechanisms was ready. When the theory in this embryonic form was presented at the Delft-Eindhoven-Twente colloquium on the Theory of Mechanisms, BOTTEMA drew attention to terms of higher order which upon revision appeared to be very useful for the description of the higher order transfer functions. This opened the way to dynamic problems, because now the calculation of the accelerations became possible where primary attempts of numerical differentiation had not led to promising results. Subsequent steps in the derivation of the equations of motion for the individual elements as well as for the mechanism as a whole did not give rise to serious problems.

The solution of the equations of motion is necessary in order to be able to determine the dynamic loading of the mechanism. In the present work we have concentrated our attention on the description of a mechanism-induction motor combination. For the induction motor use was made of a fourth order nonlinear model as presented by VAN DEN BURG [4].

The presentation of the kinetostatic and vibration problems contains nothing new in itself, but the uniformity of the description for a wide variety of mechanisms possibly stimulates further research.

Like in other modern developments we give a generalized approach for different types of mechanisms. These other developments are however mostly based on the Lagrangian equations of motion for systems of undeformable bodies [5]. In contrast to this our starting point is the description of kinematics as it is used in the finite element method for the description of the relation between displacements and deformation.

2

General theory

A general theory of kinematic and dynamic analysis is presented. There are several areas of application for this theory, but it has mainly been applied to planar mechanisms. However in order to emphasize the generality of the theory we have chosen for a derivation of the fundamental equations in a somewhat broader perspective.

In this chapter use is made of the index notation with the summation convention. The summation applies to repeated lower indices. The summation is then carried out over the full range of the indices concerned. If, for example, a and b are sets of n elements then

$$a_i\, b_i \equiv \sum_{i=1}^{n} a_i\, b_i\,.$$

2.1 SOME DEFINITIONS, PRELIMINARIES

In order to be able to speak about the definition of a mechanism or about the characteristics of the class of the mechanisms to be considered, it is necessary to recall shortly some notions from the finite element method.

In the finite element method a structure is divided into finite elements of a simple geometry by means of suitable sections. The description of position, deformation and state of stress of an infinite set of infinitesimal elements, as used in the continuum description, is replaced by the description of the position parameters and the generalized deformation and stress parameters of a finite set of elements, which have finite dimensions.

With respect to some reference state the momentaneous values of the position parameters or coordinates define a fixed number of deformation modes. The number of deformation modes is equal to the number of position parameters diminished by the number of degrees of freedom of the element as a rigid body. In this way it is possible to define for each element deformation quantities ε_i, which as functions of the coordinates x_j are invariant with respect to rigid body motions,

$$\varepsilon_i = D_i(x_j). \tag{1}$$

The mechanisms that we shall consider, can be characterized by the following definition:

"A structure is a mechanism iff there exists a nontrivial, kinematically admissible deformation-free velocity field".

Trivial, kinematically admissible deformation-free velocity fields are those motions in which the relative positions of the elements do not change. We shall concentrate upon the remaining motions without deformation. In accordance with the definition, the time derivative of (1),

$$\dot{\varepsilon}_i = D_{i,j}\, \dot{x}_j\, , \tag{2}$$

has solutions $\dot{x}_j \neq 0$ for $\dot{\varepsilon}_i = 0$ in the case of mechanisms. The comma preceding the index j denotes partial differentiation with respect to the j-th coordinate.

The coordinates fix the position of the elements in space. These coordinates can be separated into three groups:

- x^o: the coordinates with a fixed prescribed value, such as support coordinates,
- x^m: the coordinates which can be varied at will, or the so-called degrees of freedom of the mechanism,
- x^c: the remaining coordinates that are also referred to as the free coordinates.

In terms of these coordinates equation (2) becomes

$$\dot{\varepsilon}_i = D^c_{i,j}\,\dot{x}^c_j + D^m_{i,k}\,\dot{x}^m_k , \tag{3}$$

the superscript added to $D_{i,j}$ denoting the parts of $D_{i,j}$ associated with x^c and x^m.

The selection of the degrees of freedom of the mechanism makes it possible to consider the remaining coordinates x^c, which in general have to be determined as functions of time, as functions of these degrees of freedom x^m. Hence

$$x^c_i = x^c_i\,\{x^m_j(t)\}. \tag{4}$$

By means of the chain rule of differentiation the time derivatives of x^c can be expressed in the time derivatives of the degrees of freedom x^m,

$$\dot{x}^c_i = x^c_{i,j}\,\dot{x}^m_j , \tag{5}$$

$$\ddot{x}^c_i = x^c_{i,jk}\,\dot{x}^m_j\,\dot{x}^m_k + x^c_{i,j}\,\ddot{x}^m_j , \tag{6}$$

in which differentiation with respect to the j-th degree of freedom is denoted by an index j preceded by a comma. Also in this case the summation convention applies to repeated lower indices. The derivatives $x^c_{i,j}$ and $x^c_{i,jk}$ introduced here are functions of the degrees of freedom. They play an important role in kinematics, especially in the differential geometry of mechanisms.

In general we shall consider cases, in which only one degree of freedom is present or in which there exists a fixed relation between the degrees of freedom. In these cases the $x^c_{i,j}$ and $x^c_{i,jk}$ become functions of a single variable. We shall denote these functions as the transfer functions of the mechanisms of order one and two respectively. It is at hand to denote the coordinates x_i themselves as the transfer functions of order zero.

When differentiation with respect to the single degree of freedom is denoted by an accent ' we obtain

x^c_i : zero order transfer function,

$x_i^{c'}$: first order transfer function,

$x_i^{c''}$: second order transfer function.

2.2 KINEMATICS OF MECHANISMS CONSISTING OF UNDEFORMABLE LINKS

A widely used mathematical model of mechanism behaviour is the model in which the displacements due to the deformation of the links is neglected in comparison to the total motion. According to (1) the motion of the mechanism is then restricted to motions for which

$$D_i(x_k) = 0. \tag{7}$$

The problem of kinematics, as we prefer to pose it here, is the solution of the "free" coordinates in terms of the prescribed ones. Also, in order to be able to calculate velocities and accelerations later on, the first and second order transfer functions must be found. These and also the transfer functions of higher order are needed for the determination of several derived kinematical quantities of mechanisms, p.e. evolutes and polodes.

The zero order transfer function cannot be calculated directly from (7). For the first and higher order transfer functions however, equations will be derived which allow their direct calculation. Next an iterative calculation scheme for the zero order transfer function will be described.

Let us look at the motion of a mechanism which initially is in some reference position $x_k = x_k^r$ which satisfies (7). It is clear that, because the violation of the condition (7) due to even an infinitesimal displacement dx_k is prohibited, the following differential equation must hold

$$D_{i,j}\, dx_j = 0. \tag{8}$$

The introduction of the partitioning of $D_{i,j}$ into $D_{i,k}^c$ and $D_{i,l}^m$ makes a separation of knowns and unknowns possible

$$D_{i,k}^c\, dx_k^c + D_{i,l}^m\, dx_l^m = 0. \tag{9}$$

Now one particular element of x^m is selected to play the role of independent variable. When differentiation with respect to this independent variable is denoted by an accent ', equation (9) becomes a set of linear equations for the first order transfer function as defined in ch. 2.1:

$$D^c_{i,k}\, x'^c_k = -\, D^m_{i,l}\, x'^m_l . \tag{10}$$

By virtue of the definition of a mechanism, it is clear that the system allows nontrivial solutions. When we disregard for the moment the situation that there are more equations than unknowns, which may occur when the mechanism contains statically indeterminate parts, then the first order transfer function generally can be calculated as

$$x'^c_i = -\, D^{c^{-1}}_{i,j}\, D^m_{j,k}\, x'^m_k . \tag{11}$$

The derivation of expressions for the higher order transfer functions is highly facilitated by the introduction of the combined vectors x' and x'':

$$x' = \begin{vmatrix} x'^c \\ x'^m \end{vmatrix}, \quad x'' = \begin{vmatrix} x''^c \\ x''^m \end{vmatrix}, \quad \text{etc.} \tag{12}$$

Expressions for the transfer functions of order two and more can now be found by repeated differentiation of (9) with respect to the independent variable.

From

$$D_{i,j}\, x'_j = 0 \tag{13}$$

we obtain by differentiation:

$$D_{i,jk}\, x'_j\, x'_k + D_{i,j}\, x''_j = 0, \tag{14}$$

or, now partially separating x's in x^c and x^m:

$$D^c_{i,j}\, x''^c_j = -\, D_{i,jk}\, x'_j\, x'_k - D^m_{i,j}\, x''^m_j . \tag{15}$$

As x''^m is essentially zero, we obtain a set of linear equations very similar to (10), but now for the second order transfer function. Under the same conditions, which hold for (10), the solution of these equations reads:

$$x_i''^c = - D^{c^{-1}}_{i,j} D_{i,jk} x_j' x_k' . \quad (16)$$

Taking now the derivative of (14) we have

$$D_{i,jkl} x_j' x_k' x_l' + D_{i,jk} x_j'' x_k' + D_{i,jk} x_j' x_k'' +$$

$$+ D_{i,jk} x_j'' x_k' + D_{i,j} x_j''' = 0. \quad (17)$$

As the order of differentiation is immaterial it is obvious that the triple indexed quantity $D_{i,jk}$ is symmetrical in the indices j and k, so that we can take the middle terms together. Following the same procedure as in the derivation of (16) we arrive at

$$D^c_{i,j} x_j'''^c = - D_{i,jkl} x_j' x_k' x_l' - 3 D_{i,jk} x_j' x_k'' . \quad (18)$$

From these equations we can calculate $x_i'''^c$ and it will be understood that formulae for still higher orders may be derived in the same way.

It is observed that the matrix $D^c_{i,j}$ plays a dominant role in the theory, both in the kinematic part and the description of the force transmittance. Because of the resemblance of this matrix and the notion of the "fundamental relation" in classical kinematics, see p.e. [6], it is proposed to refer to this matrix with the name: *fundamental matrix*.

A second order approximation for the movement of the mechanism from the reference position to an adjacent position is given with the aid of the first and second order transfer functions in the reference position:

$$x = x^r + x'^r \Delta x^m + \tfrac{1}{2} x''^r (\Delta x^m)^2 . \quad (19)$$

Δx^m stands for the change of the independent variable, which is intentionally varied. The errors of higher order involved in this approximation show up when the residual deformations in the new position are calculated. Let these deformations, calculated according the nonlinear expression (1) have values ζ_i. These residuals should vanish in the case of a mechanism consisting of undeformable links. Hence the newly found position has to be corrected by additional displacements u_i which are approximated by the linear expressions:

$$u_i = - D^{c^{-1}}_{i,j} \zeta_j . \qquad (20)$$

This correction procedure must be repeated until the calculated deformations are vanishingly small. Of course the matrix $D^c_{i,j}$ must be updated with the changing geometry during this iterative process. As the equations, which are used, are correct for the first and second order terms, the iterative process converges very rapidly.

For a mechanism with given kinematical dimensions, starting from a given position we are able to calculate subsequent positions for a sequence of values of the independent variable, thus constructing the zero order transfer function of the mechanism. During this process the first and second order transfer functions are calculated also for each position of the mechanism.

The computation scheme is visualized in fig. 2.

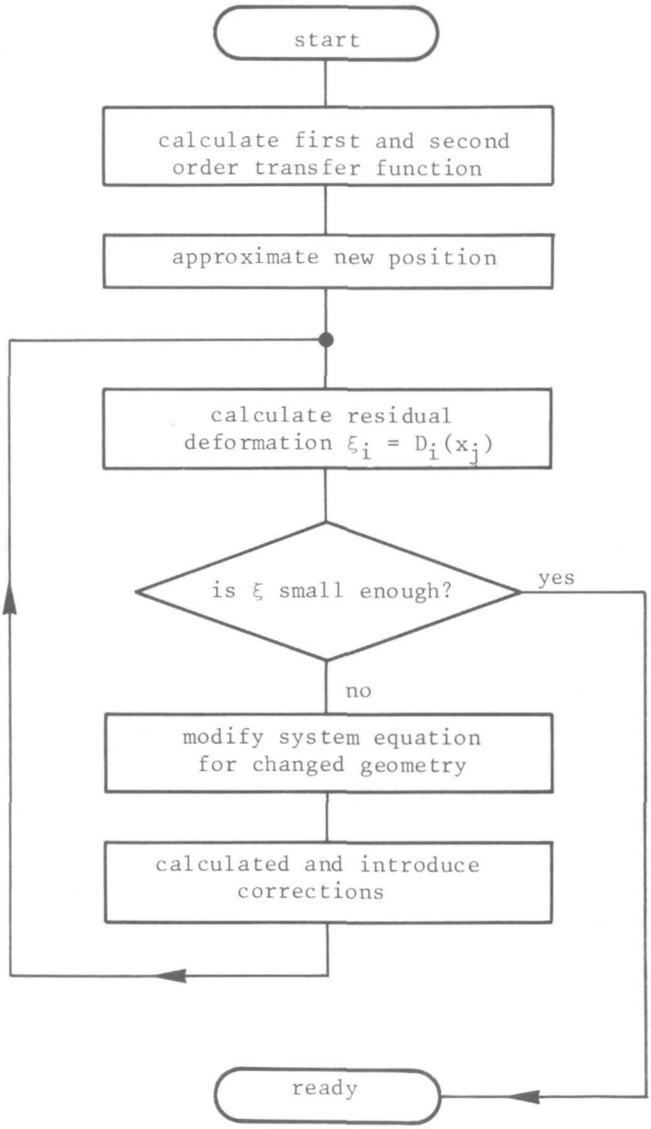

Fig. 2 Computation scheme.

2.3 EQUATIONS OF MOTION, EQUILIBRIUM EQUATIONS AND STRESS STRAIN RELATIONS

Virtual deformation modes can be represented by:

$$\delta\varepsilon_j = D^c_{i,j}\,\delta x^c_j + D^m_{i,k}\,\delta x^m_k \,. \tag{21}$$

With the aid of the associated vector of generalized stresses σ_i and a vector of nodal loads f_i the equation of virtual work can be written in the form

$$f_l\,\delta x_l = \sigma_i\,\delta\varepsilon_i = \sigma_i D^c_{i,j}\,\delta x^c_j + \sigma_i D^m_{i,k}\,\delta x^m_k \,. \tag{22}$$

According to the principle of virtual work this equation must hold for all kinematically admissible displacements δx^c_j and δx^m_k. This condition yields the equilibrium equations:

$$\sigma_i\, D^c_{i,j} = f_j\,; \quad \sigma_i\, D^m_{i,k} = r_k \,. \tag{23}$$

The vector r_k represents the driving forces. The nodal load vector f_j contains the externally applied forces f^e_k.

The validity of the principle of virtual work is extended to dynamics by adding the negative flux of the momentum to the external forces, provided we work in an inertial coordinate system. For these so-called inertial forces we shall assume that they can be written in the usual form with the aid of a mass matrix M multiplied by the vector of accelerations, $\ddot{x}_j$:

$$f_k = f^e_k - M_{kj}\,\ddot{x}_j \,. \tag{24}$$

Even for the specific applications we have in mind, namely mechanisms, it cannot in general be assumed that there is a simple relation between stresses and strains, because it may happen that a mechanism must be described with the aid of damping, nonlinear springs etcetera. Nevertheless we shall focus our attention here on the most frequently encountered type of stress strain law

$$\sigma_i = S_{ij}\,\varepsilon_j. \tag{25}$$

Other types of stress-strain relations have to be treated as special cases.

2.4 MECHANISMS CONTAINING DEFORMABLE LINKS

The influence of the flexibility of the links on the behaviour of a mechanism can be an important aspect, especially in heavy (dynamically) loaded mechanisms. A general approach for this problem leads to a very complicated mathematical model. The governing equations are the expressions for the strains, the equations of motion and the stress-strain law:

$$\varepsilon_i = D_i(x_j),$$

$$\sigma_i\,D_{i,k} = f_k^e - M_{kj}\,\ddot{x}_j,$$

$$\sigma_i = S_{ij}\,\varepsilon_j. \tag{26}$$

By elimination of the generalized stresses σ_i and deformations ε_i it is possible to arrive at a system of simultaneous second order differential equations:

$$D_{i,j}\,S_{ik}\,D_k(x_l) = f_j^e - M_{jk}\,\ddot{x}_k. \tag{27}$$

This system is highly nonlinear and apart from numerical integration, there is in general no possibility to obtain solutions.

It may be expected that real mechanisms have been dimensioned such that the deformation of the links during operation is of minor importance. In that practical case the general description is abandoned for a more manageable formulation based upon the premise that the difference between the rigid link model solution and the flexible link model solution is small. It must then, of course be realized that the predictive value of calculations according this new formulation may become little when the deformation is no longer small.

When the assumption of small influence of deformation is worked out, one obtains a solution consisting of two parts. One is the solution for the unloaded rigid link mechanism. The other is a correction of this solution, completely determined by the displacements resulting from the inertia forces and the externally applied forces, calculated according the positions and accelerations of the rigid link model. In fact the equations can be found by linearization of the equations from the general formulation. Elsewhere [7] it has been shown that an asymptotic theory for small flexibility leads to the same results. We confine ourselves here to the presentation of the equations for the corrections x^{corr}:

$$D^{c}_{j,i}\, S_{jk}\, D^{c}_{k,l}\, x^{corr}_{l} = f^{e}_{i} - M_{ij}\, \ddot{x}_{j}. \tag{28}$$

In par. 5.1 the approach outlined above will be applied to flexible links in planar mechanisms.

3

Kinematical analysis

The kinematical analysis of mechanisms is of great importance. First of all the primary function of a mechanism is very often formulated in terms of kinematical quantities. In addition various geometrical relations resulting from the kinematical analysis of the rigid link mechanism are essential for the dynamic analysis. From these we mention the differential relations between the link positions.

The geometrical properties of the motion of a mechanism are characterized by transfer functions. These transfer functions form the basic information for the description of the differential geometry and the instantaneous kinematics.

In this chapter we shall start with the presentation of a number of finite elements and we shall propose expressions for the deformations in terms of element coordinates. Next it will be described how a kinematical model of a mechanism is build up with these elements. Aside from the topological and geometrical description of the mechanism, also the definition of the kinematic boundary conditions are discussed. After having made a few remarks about the system equations and their solution, a computer programme, which will be referred to as PLANAR, will be discussed. The chapter ends with an example of application.

3.1 DESCRIPTION OF THE SINGLE ELEMENTS

All finite element methods start with a definition of the properties of the finite elements. In this paragraph a number of finite elements are presented, especially fit for the description of kinematics.

In principle all finite elements, for which a large displacement description is available, can be used for this purpose. We shall confine ourselves at first to the presentation of those elements used in the description of planar mechanisms. In addition we shall describe two elements which might be used for the analysis of spatial mechanisms. The description of the latter is not exhaustive and must be seen as an illustration of the method in three dimensional problems.

In the finite element method it is known that the following properties of the elements are sufficient for convergence with respect to uniform mesh refinement:

- the elements must describe all possible rigid body motions exactly,
- the mutual compatibility of the displacements on the element boundaries must be ensured,
- all homogeneous states of deformation can be described.

Particularly in view of the calculation of the forces on mechanisms later on, we should obey these rules already when constructing elements for the kinematical description. However for the specific kinematic applications to linkages in the three-dimensional case we shall introduce a rather artificial deformation mode for the spatial quadrilateral element.

The planar truss element

Describing the planar truss element we are in the convenient situation that this element is eminently suitable to demonstrate the basic approach. The element shown in fig. 3 is well known as the basic element of planar trusses. A truss element is only able to carry tensile and compressive loads and it has one deformation parameter, namely the elongation.

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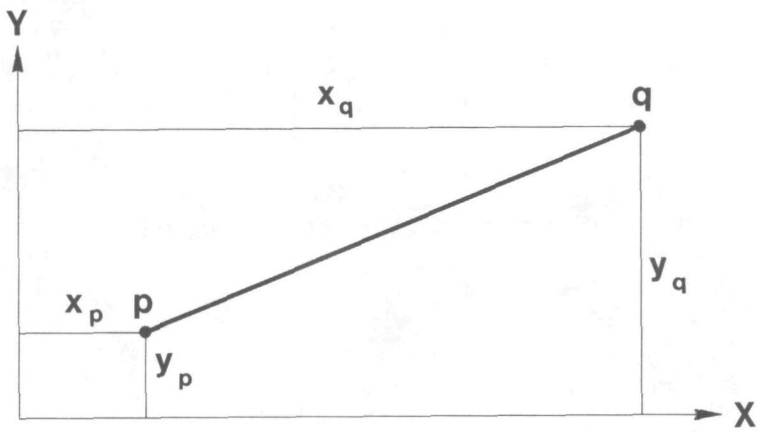

Fig. 3 Truss element.

As indicated in the picture, the position of the element is determined by the position of the endpoints p and q. The position of these nodal points is characterized by the four coordinates

$$x_i = \{x_p,\ y_p,\ x_q,\ y_q\}. \tag{29}$$

As a rigid body this element has three degrees of freedom for planar motions. In addition to these rigid body modes, the four coordinates also describe a single mode of deformation: the elongation of the element. Therefore the expression (1) of the generalized deformation in terms of the coordinates can be formulated as:

$$\begin{aligned} \varepsilon_1 &= D_1(x_i) = \\ &= \left[(x_q-x_p)^2 + (y_q-y_p)^2\right]^{\frac{1}{2}} - \left[(x_q^r-x_p^r)^2 + (y_q^r-y_p^r)^2\right]^{\frac{1}{2}} = \\ &= \ell - \ell^r. \end{aligned} \tag{30}$$

The superscript r indicates that the coordinates and the length must be taken in some reference position of the element. The derivatives $D_{1,j}$ and $D_{1,jk}$, evaluated in the reference position, are easily found. For example we have:

$$D_{1,1} = \frac{\partial D_1}{\partial x_p}\bigg|_{x_j \to x_j^r} =$$

$$= -(x_q - x_p)\left[(x_q-x_p)^2 + (y_q-y_p)^2\right]^{-\frac{1}{2}}\bigg|_{x_j \to x_j^r} =$$

$$= \cos\beta^r, \tag{31}$$

β^r denoting the angle between the element axis and the positive x-axis in the reference position. Likewise we find:

$$D_{1,12} = \frac{\partial^2 D_1}{\partial x_p \partial y_p}\bigg|_{x_i \to x_i^r} =$$

$$= \frac{\partial}{\partial y_p}\left\{-(x_q-x_p)\left[(x_q-x_p)^2 + (y_q-y_p)^2\right]^{-\frac{1}{2}}\right\}\bigg|_{x_i \to x_i^r} =$$

$$= -(x_q-x_p)(y_q-y_p)\left[(x_q-x_p)^2 + (y_q-y_p)^2\right]^{-3/2}\bigg|_{x_i \to x_i^r} =$$

$$= -\frac{1}{\ell^r}\cos\beta^r \sin\beta^r. \tag{32}$$

Proceeding in this way we obtain the following results:

$$x_i = \{x_p,\ y_p,\ x_q,\ y_q\},$$

$$D_1 = \left[(x_q-x_p)^2 + (y_q-y_p)^2\right]^{\frac{1}{2}} - \left[(x_q^r-x_p^r)^2 + (y_q^r-y_p^r)^2\right]^{\frac{1}{2}},$$

$$D_{1,j} = \left[-\cos\beta^r,\ -\sin\beta^r,\ \cos\beta^r,\ \sin\beta^r\right],$$

$$D_{1,jk} = \frac{1}{\ell^r}\begin{bmatrix} \sin^2\beta & -\sin\beta\cos\beta & -\sin^2\beta & \sin\beta\cos\beta \\ -\sin\beta\cos\beta & \cos^2\beta & \sin\beta\cos\beta & -\cos^2\beta \\ -\sin^2\beta & \sin\beta\cos\beta & \sin^2\beta & -\sin\beta\cos\beta \\ \sin\beta\cos\beta & -\cos^2\beta & -\sin\beta\cos\beta & \cos^2\beta \end{bmatrix}\Bigg|_{\beta \to \beta^r}. \tag{33a,b,c,d}$$

Higher derivatives can be found in the same way.

The beam element

Although the truss element is a powerful element in the kinematical analysis, an element with rotational position parameters can hardly be missed in the description of planar mechanisms. Not only the rotative input motion of many mechanisms has been the reason for including such an element, but in particular the extra possibilities for the coupling of elements together. The common beam element, with two translations and one rotation defined in its nodes, has proved to be a handsome tool in the description of mechanisms.

The element coordinates are:

$$x_i = \{x_p, y_p, \beta_p, x_q, y_q, \beta_q\}. \tag{34}$$

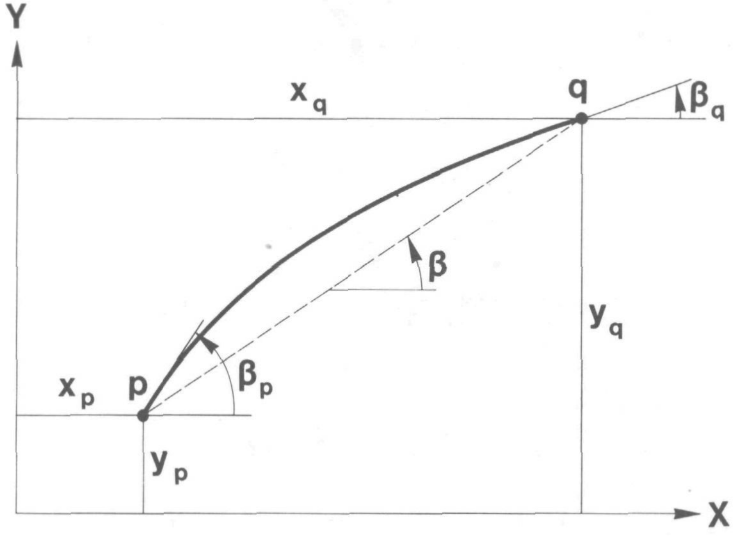

Fig. 4 The beam element: coordinates.

In addition to the deformation parameter ε_1 described before two bending deformations can be defined, for which the relative rotations of the beam in p and q are taken. If we assume that in the reference position the angles β_p and β_q are both equal to β^r, then we have:

$$\varepsilon_2 = (\beta_p - \beta)\ell^r ,$$
$$\varepsilon_3 = (\beta - \beta_q)\ell^r . \tag{35}$$

It must be remarked that the choice of the deformation parameters is not unambigeous. Other choices are possible and have their own merits. For example BESSELING in his description of spatial, geometrically nonlinear problems had to introduce a definition different from ours [8].

With the same procedure as used for the truss element, again the derivatives $D_{i,j}$ and $D_{i,jk}$ can be derived. In carrying out this differentiation we must be aware of the fact that β is a function of the xy-coordinates:

$$\beta = \arccos \{(x_q - x_p)[(x_q - x_p)^2 + (y_q - y_p)^2]^{-\frac{1}{2}}\}. \qquad (36)$$

The results obtained in this way are:

$$x_i = \{x_p,\ y_p,\ \beta_p,\ x_q,\ y_q,\ \beta_q\},$$

$$D_1 = [(x_q - x_p)^2 + (y_q - y_p)^2]^{\frac{1}{2}} - [(x_q^r - x_p^r)^2 + (y_q^r - y_p^r)^2]^{\frac{1}{2}},$$

$$D_2 = [\beta_p - \arccos \{(x_q - x_p)[(x_q - x_p)^2 + (y_q - y_p)^2]^{-\frac{1}{2}}\}]\cdot \ell^r,$$

$$D_3 = [\beta_q + \arccos \{(x_q - x_p)[(x_q - x_p)^2 + (y_q - y_p)^2]^{-\frac{1}{2}}\}]\cdot \ell^r,$$

$$D_{i,j} = \left.\begin{bmatrix} -\cos\beta & -\sin\beta & 0 & +\cos\beta & +\sin\beta & 0 \\ -\sin\beta & +\cos\beta & \ell & +\sin\beta & -\cos\beta & 0 \\ +\sin\beta & -\cos\beta & 0 & -\sin\beta & +\cos\beta & -\ell \end{bmatrix}\right|_{\beta \to \beta^r},$$

$$D_{1,ij} = \frac{1}{\ell^r} \left.\begin{bmatrix} +\sin^2\beta & -\sin\beta\cos\beta & 0 & -\sin^2\beta & +\sin\beta\cos\beta & 0 \\ -\sin\beta\cos\beta & +\cos^2\beta & 0 & +\sin\beta\cos\beta & -\cos^2\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin^2\beta & +\sin\beta\cos\beta & 0 & +\sin^2\beta & -\sin\beta\cos\beta & 0 \\ +\sin\beta\cos\beta & -\cos^2\beta & 0 & -\sin\beta\cos\beta & +\cos^2\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right|_{\beta \to \beta^r},$$

$$D_{2,ij} = \frac{1}{\ell^r} \left[\begin{matrix} -\sin 2\beta & +\cos 2\beta & 0 & +\sin 2\beta & -\cos 2\beta & 0 \\ +\cos 2\beta & +\sin 2\beta & 0 & -\cos 2\beta & -\sin 2\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ +\sin 2\beta & -\cos 2\beta & 0 & -\sin 2\beta & +\cos 2\beta & 0 \\ -\cos 2\beta & -\sin 2\beta & 0 & +\cos 2\beta & +\sin 2\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right] \Bigg|_{\beta \to \beta^r},$$

$$D_{3,ij} = - D_{2,ij} . \tag{37}$$

The out of plane torsion element

The out of plane torsion element is a linear element with its axis perpendicular to the plane in which planar motions take place (fig. 5).

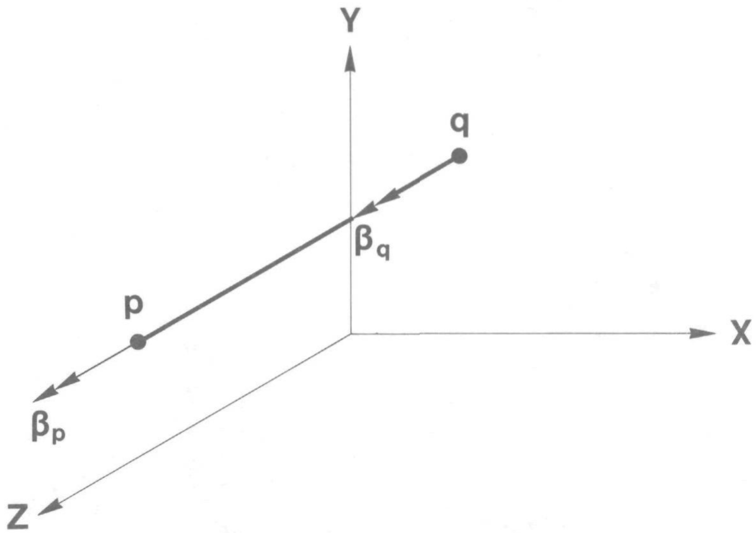

Fig. 5 The out of plane torsion element.

The need for this element became manifest when the kinematic description as presented in this thesis was used for the optimalisation of mechanism parameters by KLEIN BRETELER and VAN WOERKOM [9, 10]. The element can be used to vary the angle between two beam elements.

The single deformation parameter is the angle of twist. Without further comments the definitions and derivatives are summarized as:

$$x_i \quad = \{\beta_p,\ \beta_q\}, \tag{38a}$$

$$D_1 \quad = \beta_q - \beta_p, \tag{38b}$$

$$D_{1,i} = [-1,\ +1], \tag{38c}$$

$$D_{1,ij} = 0. \tag{38d}$$

The spatial truss element

This element is the simplest spatial element. For a global description we refer to the planar version of this element. The element position is characterized by the six end point coordinates as given in fig. 6.

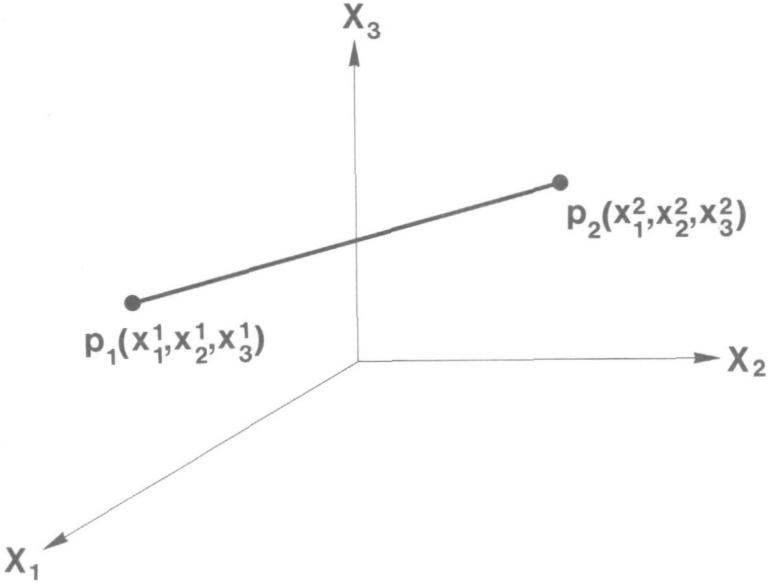

Fig. 6 The spatial truss element.

$$x_i = \{x_1^1,\ x_2^1,\ x_3^1,\ x_1^2,\ x_2^2,\ x_3^2\} = \{x_i^p\} \quad \begin{matrix} p = 1,2 \\ i = 1,2,3 \end{matrix} \tag{39a}$$

The single deformation parameter is given by:

$$D_1 = \ell - \ell^r ; \; \ell = \left[(x_1^2 - x_1^1)^2 + (x_2^2 - x_2^1)^2 + (x_3^2 - x_3^1)^2\right]^{\frac{1}{2}}. \qquad (39b)$$

For the derivatives of the deformation parameter we then have:

$$\frac{\partial D_1}{\partial x_i^p} = - \cos \alpha_i \text{ if } p = 1,$$
$$+ \cos \alpha_i \text{ if } p = 2, \qquad (39c)$$

$$\frac{\partial^2 D_1}{\partial x_i^p \partial x_j^q} = \frac{1}{\ell}\left[\delta_{ij} - \cos \alpha_i \cos \alpha_j\right] \text{ if } p = q,$$
$$- \frac{1}{\ell}\left[\delta_{ij} - \cos \alpha_i \cos \alpha_j\right] \text{ if } p \neq q, \qquad (39d)$$

in which $\cos \alpha_i$ denotes the direction cosines of the directed line segment p_1p_2.

The tetrahedron and the quadrilateral space element

With six spatial truss elements a tetrahedron element can be build (fig. 7).

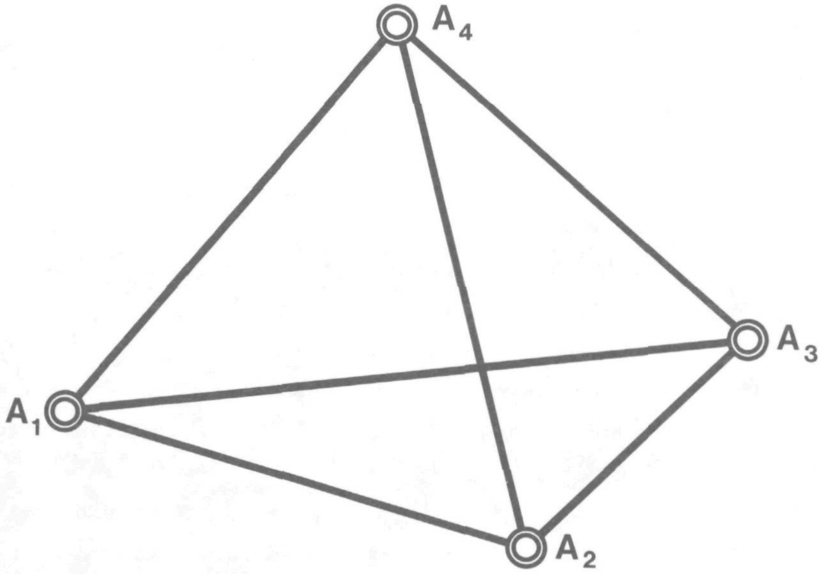

Fig. 7 The tetrahedron and quadrilateral space element.

The deformation parameters then consist of the deformation parameters of the truss elements forming this new element. In the special case that the four vertices A_1, A_2, A_3 and A_4 are coplanar the element can be regarded as a quadrilateral space element. But now the six deformation parameters of the six line elements cannot adequately describe the deformation of the element, because they cannot guarantee that the four points remain coplanar. Therefore we must replace one of the deformation parameters by a new one. For kinematic applications the volume of the tetrahedron, which must vanish in the case of coplanar vertices, can be taken as the new deformation parameter:

$$\varepsilon_1 = \left\| \begin{matrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{matrix} \right\| \qquad (40)$$

The situation described here is identical to the planar case when a coupler triangle is defined by the length of its sides. Then also an indeterminacy occurs when the three points become colinear. Instead of using the area of the triangle as a deformation parameter, there also the out of plane torsion element can be used to avoid this difficulty.

3.2 THE TOPOLOGICAL DESCRIPTION

In the topological description of the mechanism we define the number and the type of the constituting elements and the way they are interconnected, without regarding the specific dimensions of the elements and their absolute or relative position.

It is noted that the term "link", which is commonly used in theoretical kinematics, is not used here. In many cases the links coincide with the elements, but in general they don't. This is illustrated in fig. 8, where we have a four-bar linkage consisting of 7 elements.

Apart from the description of the kinematic functioning of the mechanism the choice of the elements must be determined such that it fits the description of force transmittance and vibration in further calculation steps. If, for example, the bending stress in the link BB_0

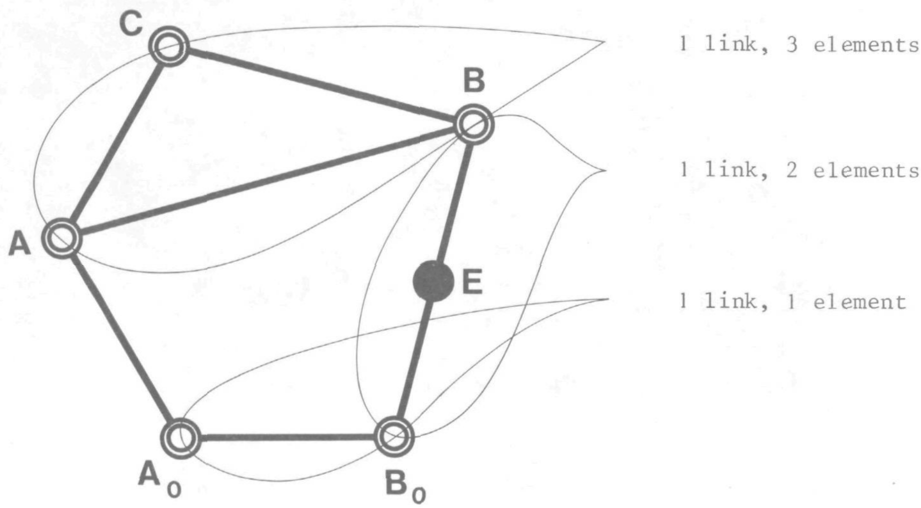

Fig. 8 Links and elements.

(see fig. 8) is to be calculated later on, or if the transverse vibration of this link must be described in a later stage of the study of this mechanism, it is necessary to subdivide this link into two or more beam elements which are rigidly connected to each other.

An essential point of the topological description is the specification of the element interconnections. This can be accomplished by indicating which element coordinates are shared and which are not. With a pin joint connection between two beam elements, the translational coordinates are shared, while the rotations are not. When on the contrary such elements are rigidly connected also the rotations are shared (see fig. 9).

Slide connections ask for a special treatment. Mechanisms in which one or more links are coupled by slide connections instead of pin joints, can be described by simple manipulations of the equations. The method will be explained with the aid of the example depicted in fig. 10.

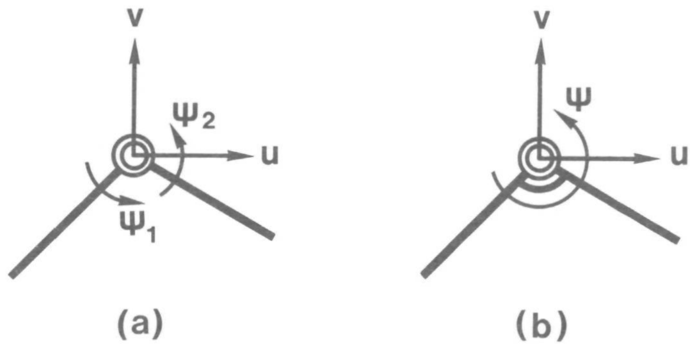

Fig. 9 Pin joint (a) and rigid connection (b).

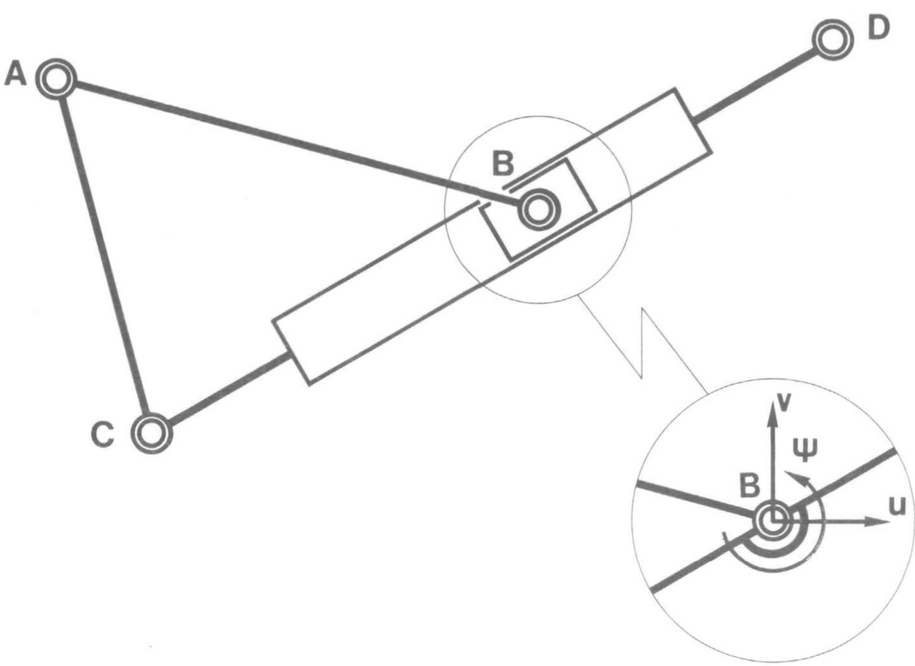

Fig. 10 Slide connection.

In this mechanism the link AB can slide along the link CD, but its relative rotation with respect to this link is unconstrained. When modelling this mechanism we subdivide the link CD into two beam elements, rigidly connected in B according to fig. 9b. The link AB, which may be a truss or a beam element, is then pin joint connected in B (see enlargement). The specification of the sliding connection can now be completed by expressing that the position of the node B on CD is not fixed. This can be done by releasing the deformation condition of the beam elements CB and BD. The beams are allowed to change their length. In the case that the length of CD itself is important we must add the condition that the longitudinal deformation in this link is found by adding the deformations of the elements CB and BD in which the link was subdivided. So instead of using two deformation parameters ε_1^{CB} and ε_1^{BD}, only their sum $\varepsilon_1^{CB} + \varepsilon_1^{BD}$ appears in the description. Although this procedure follows naturally from the kinematic approach, we admit that it is far from trivial. It is possible, may be not simple, to translate the procedures for the specification of the connections into a language more familiar to the designer. Attempts in this direction have been made by several authors [11].

3.3 THE DIMENSIONS OF THE MECHANISM

For numerical calculations the kinematical dimensions of the mechanism must be known. For a finite element description it is sufficient to define one specific position of the mechanism by the coordinates of the nodes. From the viewpoint of theoretical kinematics however it is usual to give kinematical dimensions of the links, such as their length and relative orientation, rather than a set of coordinates. Both methods can be used, provided that in the second method provisions are made to exclude a possible unwanted configuration of the instantaneous centres of rotation, which is accomplished by the addition of a set of estimated coordinate values.

3.4 KINEMATIC BOUNDARY CONDITIONS, THE INPUT MOTION

Depending on the specific problem at hand a number of coordinates have prescribed values. In particular the coordinates of the links, which are chosen to constitute the frame, have prescribed constant values. One or more coordinates may have non-constant prescribed values. These coordinates define the input motion of the mechanism. A frequently used coordinate for this aim is the input crank angle, but in some cases a slider position may be more convenient.

The input motion is not always defined by prescribing one of the coordinates. It is also possible to have a motion of a mechanism controlled by the action of a hydraulic or pneumatic actuator. This type of input motion can be looked upon as the result of a prescribed deformation of a line element representing the actuator. In the next chapter more attention is given to this type of input motion.

3.5 SYSTEM EQUATIONS AND THEIR SOLUTION

Once the mechanism is defined we can proceed and set up the system equations according the general theory presented in chapter 2. The first and higher order transfer functions $x_k^{(i)c}$ all follow from a set of linear equations of the type:

$$D^c_{j,k} x_k^{(i)c} = b_j, \qquad (41)$$

D being the nonsingular fundamental matrix of the mechanism. The transfer functions are then calculated in succession of their order, starting with the first order transfer function. As the matrix D is the same for all these transfer functions, the solutions are obtained by inversion of the fundamental matrix and subsequent multiplication with the various right hand terms. The availability of the inverse of D proved to be of great value in some other applications, of which we mention the calculation of partial derivatives needed in the optimalisation of mechanisms [9, 10]. The matrix D is a non-symmetric matrix. In the implementation of the theory in a computer programme this led to a Gausz inversion procedure in which in each elimination step the largest element of a row was used as pivot.

3.6 THE COMPUTER PROGRAMME "PLANAR"

On the basis of the underlying theory a computer programme was written which contains a subset of the elements described in this chapter, namely beams and truss elements. The basic programme part handles the kinematic part of the theory and calculates the transfer functions up to order two and some important matrices. These then can be transferred to other attached subprogrammes which use this basic information for further calculations. The basic programme part, due to the simple formulation, is relatively small. It contains about 250 FORTRAN statements. The programme input is very flexible and allows the definition of different types of mechanisms. An example of the possibilities is given in fig. 11. The computer programme is organized such that the user is able to insert one or more subprogrammes which he may take from an existing library or which must be programmed by himself. It is clear that in the latter case the user must have a fair knowledge of the underlying theory.

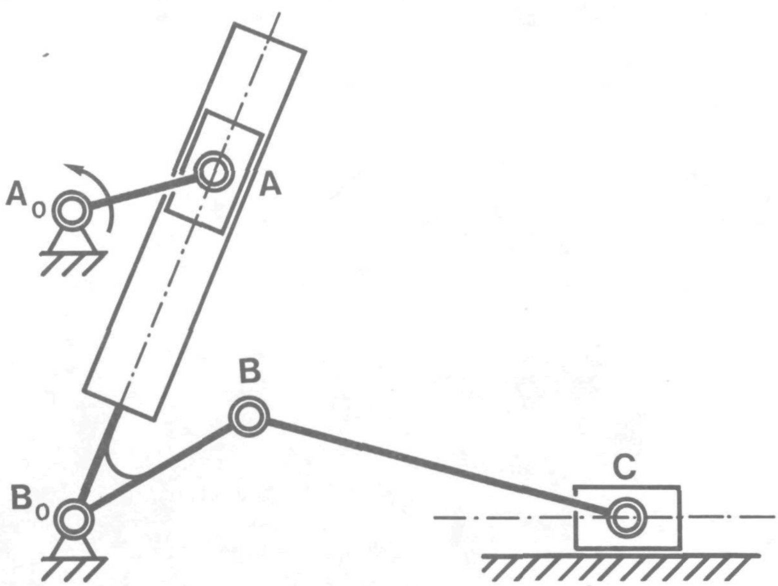

Fig. 11 Sample mechanism.

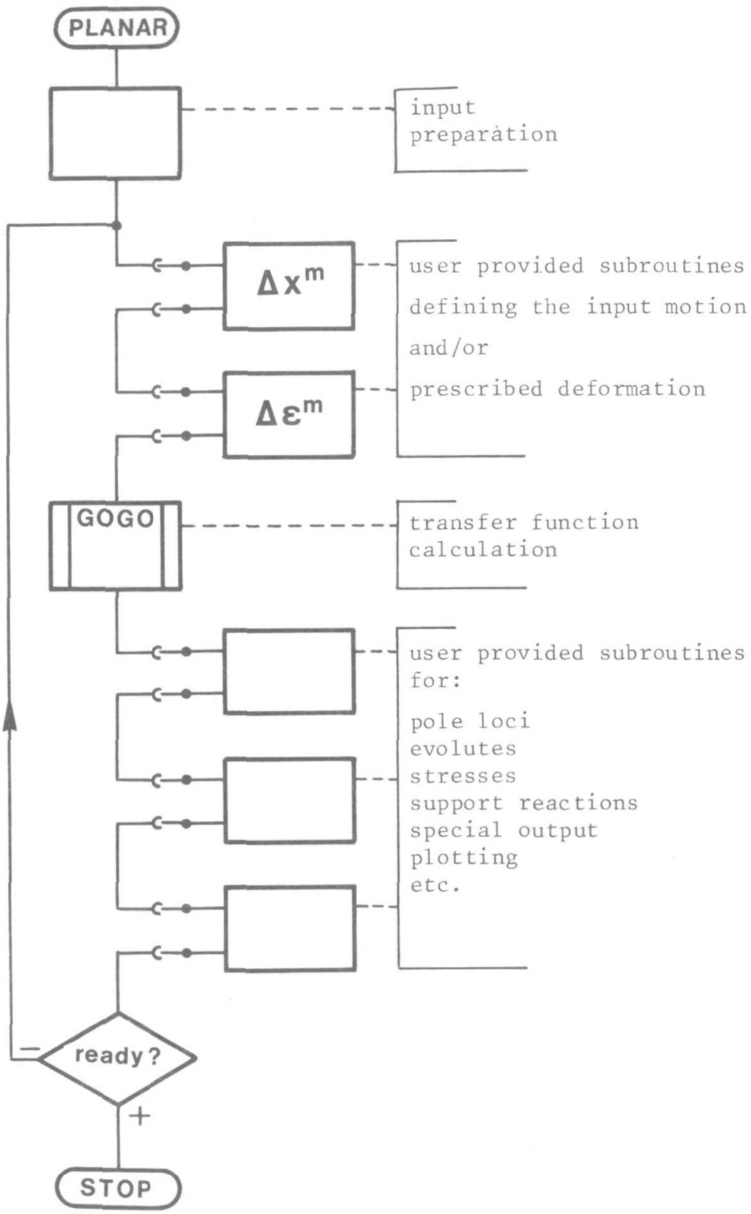

Fig. 12 Computer programme set up.

The programme is organized as a loop which is executed once for each increment of the input variable (fig. 12). Results can be given as output after every cycle or must be stored in order to be processed after a complete revolution of the input crank.

3.7 EXAMPLE [12]

As an example we shall present an analysis of a Stephenson - 2 mechanism [13], depicted in fig. 13. The indicated link is taken as the input link.

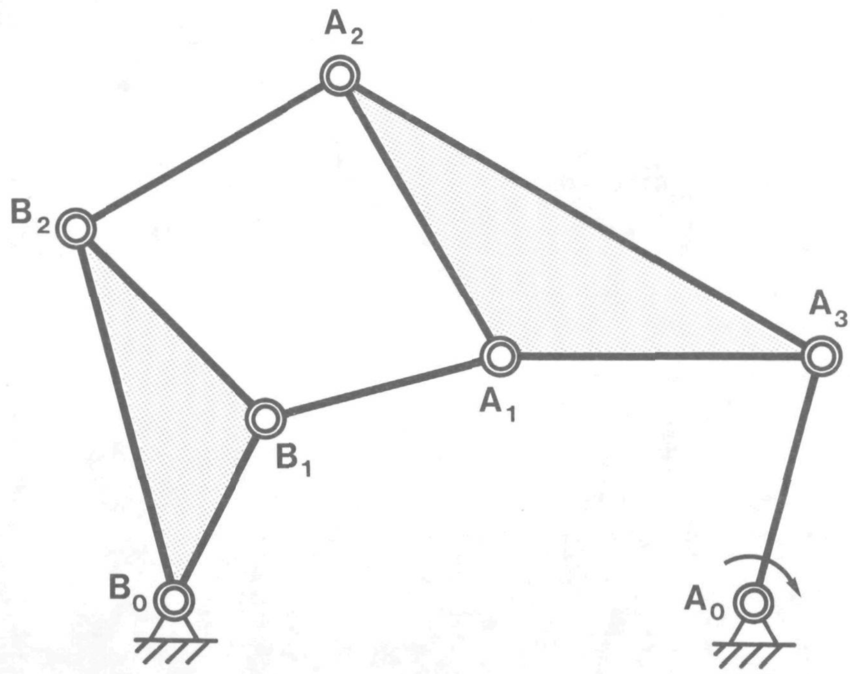

Fig. 13 Stephenson-2 mechanism.

As a special case we shall choose the kinematic dimensions such that the mechanism is a replacement mechanism for an inverted slider-crank mechanism (fig. 14a).

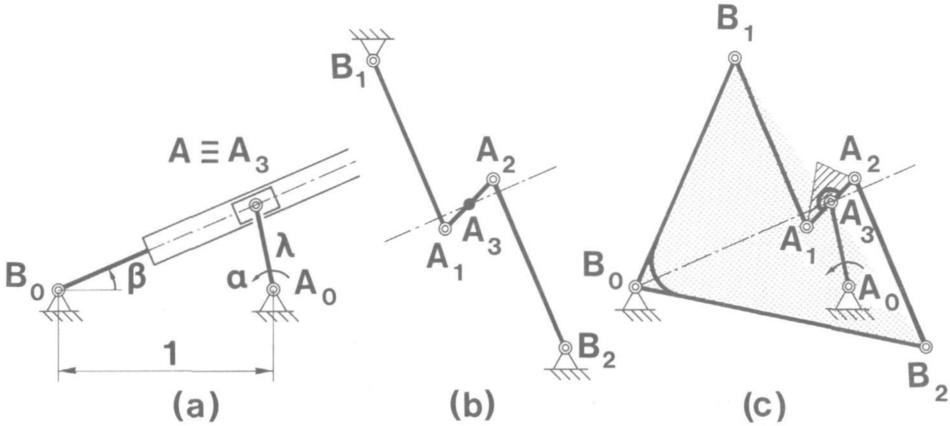

Fig. 14 Slider-crank (a), Watt's (b) and replacement mechanism (c).

The slide connection of the inverted slider-crank can be replaced by an approximate straight-line mechanism. Here we shall use the straight-line mechanism according Watt (fig. 14b). The replacement mechanism found in this way is indeed a Stephenson-2 mechanism (fig. 14c).

In accordance to WEISBACH [14] the dimensions of the Watt's linkage are chosen such that a reasonable rectilinear guidance is obtained. This results in a mechanism that is antimetric in the half-stroke position and that shows zero deviation in the extreme positions (fig. 15).

The required stroke of the Watt's linkage can be determined from the original mechanism (fig. 14a). The remaining free parameter is the extreme value of the angle α, which for practical purposes should not exceed a value of $\alpha_{max} = 2\pi/18$. According fig. 15 the quantities

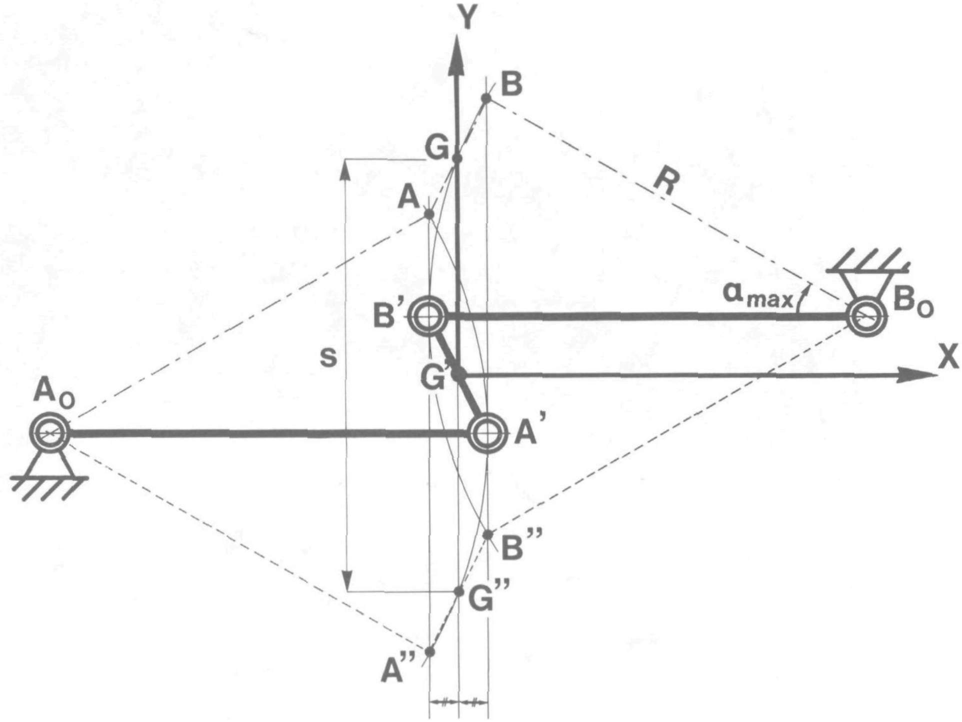

Fig. 15 Watt's straight-line mechanism.

$$s = GG'' = 2R\sin\alpha_{max},$$

$$x_{B_0} = \frac{1}{2} R\ (1 + \cos\alpha_{max})\ ,$$

$$y_{B_0} = R\sin\alpha_{max} - R\sqrt{1 - \frac{1}{4}\ (1 + \cos\ \alpha_{max})^2},$$

determine the geometry of the mechanism.

In fig. 16 the replacement mechanism is given. The free coordinates x_i^c and the mechanism degree of freedom $x_1^m = \psi_7^o$ are indicated. The ternary links 134 and 256 are realized according the method indicated in fig. 9a. In order to be able to compare the motion of the replacement mechanism with that of the original inverted slider-crank mechan-

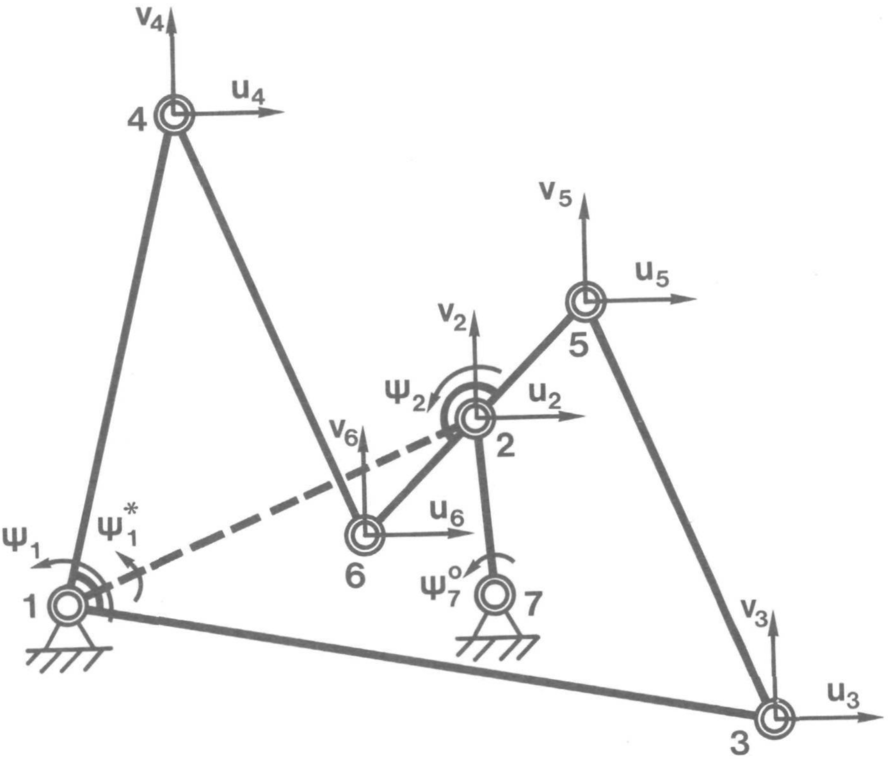

Fig. 16 Replacement mechanism.

ism the dashed element is added. This element is pin connected in node 1 and 2. The slide connection in 2 is realized by removing the condition of zero elongation of this element. The slider rotation ψ_1^* can now immediately be compared to the rotation ψ_1 of the replacement mechanism.

In the calculation with the computer programme the nodal coordinates of the mechanism in the starting position must be calculated instead of being read from cards. Due to the modularity of the programme this is easily accomplished by replacing the subroutine, in which usually the coordinates are read, by a subroutine in which now the coordinates are calculated.

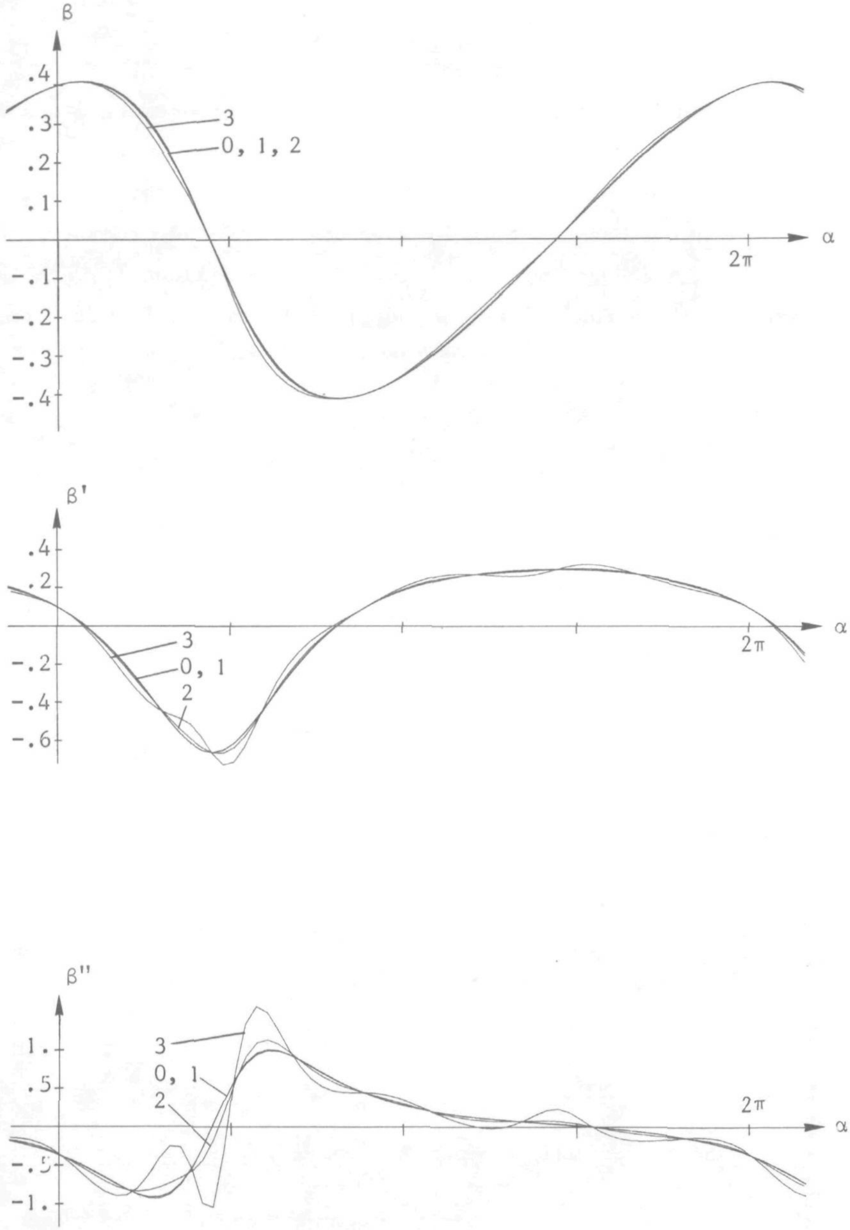

Fig. 17 Transfer functions of replacement mechanisms.

Here we shall confine ourselves to the presentation of the results of a parameter study with $\lambda = 0.4$ and $\alpha_{max} = \frac{k\pi}{12}$, $k = 0,1,2,3$.

The diagrams in fig. 17 show the transfer functions β, β' and β'' of the original mechanism ($k = 0$) and those of the replacement mechanism for $k = 1,2,3$. Evaluating these results it is obvious that the output motion of the replacement mechanism for $k = 1$ is in excellent agreement with the original mechanism ($k = 0$) even in the second order transfer function. The replacement mechanism for $k = 2$ is reasonable. For the $k = 3$ replacement mechanism the second order behaviour is bad, due to the relatively large amplitudes of the Watt mechanism.

4

Dynamic analysis

In the kinematic description of a mechanism the time independent kinematic properties are determined by means of the transfer functions. In many applications this information will be sufficient because the designer expects, justifiably or not, that the mechanism will behave nicely and that the motion can be derived from the transfer functions in a simple manner (for example by using the assumption of constant speed). In a number of cases however the time dependence of the motion cannot be obtained by means of such simple methods. In these situations, the forces that cause the motion must be included in the analysis. Roughly the forces can be separated into two groups. First of all there are the functional forces, originating from the specific process for which the mechanism is made. Secondly we have parasitic forces, which are not necessary for this process but which are inherent to the realization of the process by means of the chosen mechanism. Inertia forces and friction forces belong to this category.

The subject of this chapter, the dynamic analysis, is defined as the determination of the motion of the mechanism as a function of time. This function must be determined from the action of the forces to which the mechanism is subjected. In addition to the average speed as a global result more detailed information on the motion may be required. The knowledge of the variations inside a cycle of the motion is a prerequisite for the determination of the internal stress resultants, which will be described in the next chapter.

In this stage of the analysis of the mechanism the choice of the mechanical model will not affect the result very much. As the analysis is intended for the description of one degree of freedom mechanisms, a single second order differential equation of motion for the whole mechanism is derived by reduction of the various forces to the link associated with this single degree of freedom. It will be shown that with the aid of the principle of virtual work this reduction can be carried out by means of the transfer functions in a convenient way.

When the driving forces are constant or when they are known functions of the crank angle or the time, then the motion can be found by integration of the equation of motion. The integration constants are chosen on the basis of considerations like constant energy in the system or average speed. In the most frequently occurring case the driving forces, e.g. the input torque, are not given functions, but they are determined by a set of differential equations. In that situation the system consisting of prime mover and mechanism must be analysed as a whole. In particular the case in which the prime mover is an asynchronous squirrel cage electric motor, will be treated in this chapter.

4.1 THE EQUATION OF MOTION

The equation of motion for a mechanism can be derived by several methods. First of all we mention the method in which the generalized stresses are eliminated from the equations of motion of the individual elements as indicated in chapter 2. However in the present chapter the notion of generalized stresses will not be used in the derivation of the equation of motion.

The starting point of the derivation is the principle of virtual work in which it is stated that the motion of a mechanical system with rigid links is such that for kinematically admissible virtual displacements δx_k the total virtual work done by the associated forces $\bar{f}_k$ vanishes identically. The forces consist of a part f_k^c acting in the sense of the velocities $\dot{x}_k^c$ (see 2.1 for definition) and a part f_k^m acting in the sense of the prescribed velocities $\dot{x}_k^m$. Possible distributed loads, like gravity forces, are replaced by statically equivalent forces in the nodal points. The inertia forces resulting from the motion of the concentrated and distributed mass of the links, can adequately be described with the aid of the mechanism mass matrix in which the contributions of the links are combined. The inertia forces must in fact be included in $\bar{f}_k$, but as we like to trace these terms in the derivation, they will be treated separately, hence

$$\bar{f}^c_k = f^c_k - M^c_{kj}\,\ddot{x}_j\,,$$

$$\bar{f}^m_k = f^m_k - M^m_{kj}\,\ddot{x}_j\,. \tag{42}$$

The virtual work equation now reads

$$\delta W = \delta x^c_k\,[f^c_k - M^c_{kj}\,\ddot{x}_j] + \delta x^m_k\,[f^m_k - M^m_{kj}\,\ddot{x}_j]. \tag{43}$$

Kinematically admissible displacements are further subject to the kinematic condition

$$\delta x^c_k = x'^c_k\,\delta\alpha, \quad \delta x^m_k = x'^m_k\,\delta\alpha, \tag{44}$$

in which α stands for the kinematic degree of freedom of the mechanism viz. the input crank angle, while the x'^c_k and x'^m_k are the transfer functions of the first order as defined before. Substitution of (44) into the expression for the virtual work yields

$$\delta W = \{x'^c_k\,[f^c_k - M^c_{kj}\,\ddot{x}_j] + x'^m_k\,[f^m_k - M^m_{kj}\,\ddot{x}_j]\}\,\delta\alpha. \tag{45}$$

As this expression must vanish identically for all possible $\delta\alpha$ the following equation of motion is obtained:

$$x'^c_k\,[f^c_k - M^c_{kj}\,\ddot{x}_j] + x'^m_k\,[f^m_k - M^m_{kj}\,\ddot{x}_j] = 0. \tag{46}$$

In order to bring this equation in a more convenient and more recognizable form, it is observed that in normal cases of single degree of freedom mechanisms, the elements of x'^m_k are all zero except the one related to the degree of freedom of the mechanism, which has the value one. The only contribution to the second term of (46) is then of the form

$$T_{in} - J\ddot{\alpha}\,,$$

where T_{in} has the meaning of the input driving force and J stands for the mass, directly coupled to the degree of freedom α. In the case of a mechanism with a rotating input T_{in} and J are the input torque and the moment of inertia (e.g. a flywheel coupled to the input crank). Accordingly (46) is rewritten as

$$T_{in} = J\ddot{\alpha} - x_k^{'c}\,[f_k^c - M_{kj}^c\ddot{x}_j]. \tag{47}$$

The acceleration vector $\ddot{x}$ can be expressed in terms of the transfer functions x' and x'' according to (6)

$$\ddot{x}_i = x_i''\dot{\alpha}^2 + x_i'\ddot{\alpha}\,.$$

Introduction into (47) leads to

$$T_{in} = [J + x_k^{'c}M_{kj}^c x_j']\,\ddot{\alpha} + x_k^{'c}M_{kj}^c x_j''\dot{\alpha}^2 - x_k^{'c}f_k^c\,. \tag{48}$$

When furthermore use is made of the fact that $x_i^{''m} = 0$, and if we assume that J includes all inertia terms related with the input crank, then (48) becomes

$$T_{in} = [J + x_k^{'c}M_{kj}^{cc}x_j^{'c}]\,\ddot{\alpha} + x_k^{'c}M_{kj}^{cc}x_j^{''c}\dot{\alpha}^2 - x_k^{'c}f_k^c\,. \tag{49}$$

Here M_{kj}^{cc} stands for that part of the mass matrix M^c that corresponds to the free coordinates x^c.

The factor of $\ddot{\alpha}$ is better known as the reduced moment of inertia J_{red}. In the same way we may call $x_k^{'c}f_k^c$ the reduced force f_{red}. If M is a constant matrix, then (49) reduces to the well known form

$$T_{in} = J_{red}\ddot{\alpha} + \frac{1}{2}\,\frac{dJ_{red}}{d\alpha}\,\dot{\alpha}^2 - f_{red}. \tag{50}$$

This equation is frequently encountered in the literature. Although the formulation of the coefficients J_{red} and f_{red} in terms of the transfer functions is characteristic for the present method, it will be clear that the equation of motion and its solution has been the subject of many investigations. The general solution of the equation of motion has not been the subject of the present investigation. Only some ad hoc cases will be considered.

Mass matrix for elements used as a link in a mechanism

In the equation of motion (46) and also in the kinetostatics the inertia forces are taken into account by means of mass matrices. In the description with mass matrices the distributed inertia forces are replaced by equivalent concentrated forces in the nodes. In the case in which the links are not allowed to deform, the determination of these equivalent nodal forces is not unique because we have 6 nodal displacement parameters per element available and there are only three rigid body modes with the corresponding inertia forces (two translational and one rotational). The final choice is based upon various considerations, one of which is that we want to treat both ends in the same way. We have chosen for an approach in which the inertia forces are associated with the end point translations of each element. In this case the mass matrix becomes

$$\bar{M}_r = \frac{ml}{6} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} . \qquad (51)$$

This mass matrix is found by setting the virtual work of the inertia forces equal to the virtual work of the equivalent forces in the nodes.

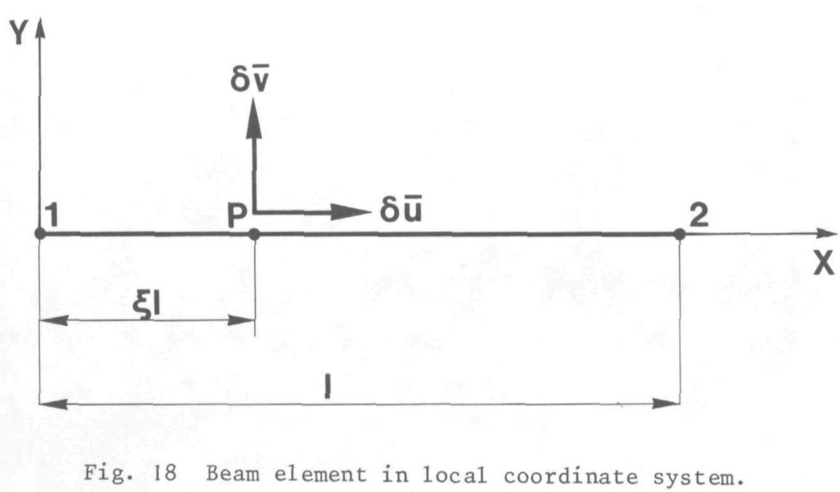

Fig. 18 Beam element in local coordinate system.

The acceleration components $\ddot{x}_p$ and $\ddot{y}_p$ of a point $P = P(\xi)$ of the link as shown in fig. 18 are given by a linear interpolation:

$$\begin{vmatrix} \ddot{x}_p \\ \ddot{y}_p \end{vmatrix} = \begin{pmatrix} 1-\xi & 0 & \xi & 0 \\ 0 & 1-\xi & 0 & \xi \end{pmatrix} \begin{vmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \end{vmatrix} . \tag{52}$$

For the virtual displacements we take a linear function:

$$|\delta\bar{u} \;\; \delta\bar{v}| = |\delta x_1 \;\; \delta y_1 \;\; \delta x_2 \;\; \delta y_2| \begin{bmatrix} 1-\xi & 0 \\ 0 & 1-\xi \\ \xi & 0 \\ 0 & \xi \end{bmatrix} \tag{53}$$

For the virtual work of the inertia forces then follows:

$$-\int_0^{\ell} |\delta\bar{u} \;\; \delta\bar{y}| \, m \begin{vmatrix} \ddot{x}_p \\ \ddot{y}_p \end{vmatrix} dx = -|\delta x_1 \;\; \delta y_1 \;\; \delta x_2 \;\; \delta y_2| \; \bar{M}_r \begin{vmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \end{vmatrix} , \tag{54}$$

from which the mass matrix (51) is obtained. As the mass matrix has been derived in the local element coordinate system, which in general does not coincide with the global coordinate system, it must be transformed by:

$$M_r = T^t \bar{M}_r T. \tag{55}$$

This orthogonal coordinate transformation leaves the mass matrix unchanged. The mass matrix is independent of the momentaneous orientation of the link.

4.2 THE MECHANISM AND ITS PRIME MOVER

As it was pointed out in the introduction to chapter 4 it is in many circumstances necessary to include the dynamic characteristics of the prime mover in the dynamic analysis of mechanisms. In order to elucidate the use of the equation of motion and in order to emphasize the need of general formulations, two practical examples will be presented. In the first example we shall be concerned with the description of the behaviour of a system formed by a mechanism driven by an asynchronous squirrel cage induction motor. In the second example it will be shown how mechanisms actuated by hydraulic or pneumatic actuators can be described.

Mechanisms driven by induction motors

Judging from the literature the problem of a mechanism driven by an induction motor is an important problem. When studying the contributions of various authors one discovers that there is little unanimity about the model which is used for the induction motor. It is therefore with some hesitation that we present the results of our theoretical and experimental investigations of the last few years. Our investigations were prompted by questions such as: "How can a mechanical engineer choose a suitable induction motor for a given mechanism and which mathematical model should he use in order to obtain a reliable description of the behaviour". It is impossible to give a definite answer to these questions without more specifications of the problem at hand. We shall however furnish the tools necessary for an understanding and an evaluation of the various models.

Most models which are presented in the literature are simplifications of the more general induction motor models, well known in the field of electrical engineering. The present state of computing facilities does not prevent the application of these complicated models. A serious drawback is formed by the lack of information about the important parameters of these models, as they are usually not included in the normal specification of an electric motor. In this paragraph such a general model is presented in the form of the well accepted model described by VAN DEN BURG [4]. It is a fourth order, doubly non-linear model of a squirrel cage induction motor with rotational symmetry. The parameters of this model can be derived from the information normally available in the product specification of the manufacturer. This is what actually will be done in an example. In a number of cases the various models proposed in the literature may be sufficient to describe the behaviour of the system mechanism and induction motor. The model presented here can serve as a reference in the evaluation of these models.

The 3-phase induction motor model

The model for the description of the behaviour of the 3-phase induction motor is taken from the thesis of VAN DEN BURG [4], dealing with the stability of an electrical transmission shaft. The model is founded on the general theory of electrical machines and presumes the following characteristics:

a) linear reciprocal properties of the ferromagnetics,
b) rotational symmetric rotor and stator,
c) constant coupling coefficient κ,
d) supply voltage system with neglible internal impedance.

For the precise details and the backgrounds of the model we must refer to the original work. The model is formulated in terms of the dimensionless components of the electro-magnetic flux vector quantities x_{ds} and x_{qs} for the stator (s) and x_{dr} and x_{qr} for the rotor (r). Furthermore we have the slip s, the dimensionless angular velocity $\dot{\theta}$ and the dimensionless output torque μ. The descriptive equations of the electrical-mechanical behaviour of the induction motor can then be given as follows:

$$\begin{vmatrix} \dot{x}_{ds} \\ \dot{x}_{qs} \\ \dot{x}_{dr} \\ \dot{x}_{qr} \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} - \begin{bmatrix} \gamma_s & -1 & -\kappa\gamma_s & 0 \\ +1 & \gamma_s & 0 & -\kappa\gamma_s \\ -\kappa\gamma_r & 0 & \gamma_r & -s \\ 0 & -\kappa\gamma_r & +s & \gamma_r \end{bmatrix} \begin{vmatrix} x_{ds} \\ x_{qs} \\ x_{dr} \\ x_{qr} \end{vmatrix} \tag{56}$$

$$s = 1 - \dot{\theta}, \quad \mu = x_{dr}x_{qs} - x_{ds}x_{qr}, \quad \dot{} = \frac{d}{d\tau} \, . \tag{57}$$

The model is essentially nonlinear due to the relation for μ and the appearance of s in the differential equation for the fluxes.

In addition to the coupling coefficient κ the model contains two machine time constants γ_r and γ_s. For asynchronous machines of normal design, the range of values for these parameters is given by $0.1 < (\gamma_s, \gamma_r) < 0.5$ and $0.9 < \kappa < 0.98$. A dimensionless time parameter $\tau = \omega_o t$, ω_o being the supply frequency, is used. The output torque is made dimensionless by means of a reference torque M^*.

The determination of the parameter values

Some of the parameters of the model are closely related to the specifications normally given by the manufacturer. Especially M^* and γ_r can be determined from these to accuracies within 15%, which is sufficient for the applications we have in mind.

According to the so-called KLOSZ-formula [15] for the steady state torque

$$\mu_c = \frac{\kappa}{s^c(1+\gamma_s^2)/\gamma_r + 2\kappa^2\gamma_s + \gamma_r(1+\sigma^2\gamma_s^2)/s^c},$$

$$\sigma = 1 - \kappa^2, \tag{58}$$

the maximum torque has a value of

$$\mu_{max} \approx \frac{\kappa}{2(1+\kappa^2\gamma_s)}. \tag{59}$$

Since in most specifications the overload capacity M_{max}/M_{nom} is given, the reference torque M^* becomes

$$M^* = \frac{2(1+\kappa^2\gamma_s)}{\kappa} \cdot \frac{M_{max}}{M_{nom}} \cdot M_{nom}. \tag{60}$$

For normal machines in a first approximation the parameter γ_r is equal to the slip at breakdown torque, s_k. Again s_k can be estimated from the overload capacity:

$$s_k/s_{nom} = 2(1+\kappa^2\gamma_s)\ M_{max}/M_{nom}. \tag{61}$$

In these calculations we take normal values for κ and γ_s e.g. $\kappa = 0.95$ and $\gamma_s = 0.3$.

The value of γ_s cannot be derived from information normally available in the manufacturer's catalogue. Generally γ_r and γ_s do not differ very

Motor type: 3-phase squirrel cage induction motor

Manufacturer: Ateliers de Constructions Electriques de Charleroi

Type number: AK 160 M44 N

Quantity	Symbol	Value	Units	Source
Power	W	11000	W	given
Synchronous speed	n_s	1500	rpm	given
Nominal speed	n	1450	rpm	given
Nominal slip	s_{nom}	0.033	[1]	$(n_s-n)/n_s$
Nominal angular speed	ω_{nom}	152	rad/s	$2\pi n/60$
Synchronous ang. speed	ω_s	157	rad/s	$2\pi n_s/60$
Overload capacity	M_{max}/M_{nom}	2.4	[1]	given
Relative slip at max torque	s_k/s_{nom}	5.67	[1]	$2(1+\kappa^2\gamma_s)M_{max}/M_{nom}$
Slip at max torque	s_k	0.19	[1]	$(s_k/s_{nom})s_{nom}$
	γ_r	0.19	[1]	s_k
	γ_s	0.19	[1]	γ_r
Relative starting current	I_s/I_n	6.2	[1]	given
Power factor	$\cos\phi$	0.82	[1]	given
Coupling coefficient	κ	0.95	[1]	$I_n\sqrt{1-\cos^2\phi}/I_s$
Supply frequency	ω_o	314	rad/s	given
Nominal torque	$M_{nom\ 1500}$	72.4	Nm	W/ω_{nom}
Reference torque	M^*_{1500}	431	Nm	$\frac{1}{\kappa}\frac{s_k}{s_{nom}}M_{nom}$
Moment of inertia	P_{1500}	1	kgm^2	given
Number of poles	$2i$	4	[1]	ω_o/ω_s
Reference moment of inertia	P_{1500}	.0087	kgm^2	$i\ M_{1500}/\omega_o^2$
Dimensionless moment of inertia	Π	114	[1]	P_{1500}/P^*_{1500}

Table I Calculation of motor constants.

much so that in cases where information is lacking, we shall assume γ_s to be equal to γ_r.

If the starting current is given, a rough estimate for the coupling coefficient κ can be obtained using the properties of the HEILAND-diagram. From the general appearance of this diagram the stray coefficient $\sigma = 1-\kappa^2$ can be estimated as the ratio of the nominal idle current and the starting current I_s

$$\sigma = I_{nom} \sqrt{1-\cos^2\phi}/I_s. \tag{62}$$

When the information necessary for this estimate is lacking or unreliable, it is advised to take a standard value $\kappa = 0.95$. An example of the determination of the motor constants is given in Table I.

Calculations with the induction motor model

Instead of using standard methods for the integration of the equations describing the motor model and the mechanism we have developed a method which after all appears to be very much alike the well known predictor corrector method. The mechanism dependent terms in the equation of motion are calculated beforehand and transferred to the integration procedure in the form of a fourier series. As in our approach a description of the starting behaviour is possible, the initial conditions describe the system at rest. A typical integration step is depicted in fig. 19.

For the dimensionless time increment Δt a value of 0.1 was a good value. This means that a complete cycle of the supply voltage is subdivided into about 60 intervals. A check on the correctness of the integration process was done by comparing the results with VAN DEN BURG's results which were obtained by analog computation. The computerprogramme was made to stop when a steady state solution was achieved. To accomplish this, after every say 1000 steps it was checked whether the solution had a repetitive character or not.

A typical result is shown in fig. 20. For comparison a result of a linearized first order model according to HABIGER [16] is given in fig. 21. The latter result was obtained by WIJN [17] who made extensive theoretical and experimental investigations of a transfer mechanism coupled with an induction motor.

initial (rest) conditions:

$$\theta = \theta_0,\ \dot{\theta}_0 = 0,\ \mathbf{x} = 0,\ \dot{\mathbf{x}} = 0 \Longrightarrow T_0 = 0$$

induction motor prediction:

$$s_0 = 1 - \dot{\theta}_0$$
$$\mathbf{A}_0 = \mathbf{A}(s_0)$$
$$\Delta\mathbf{x}_0 = [\mathbf{u} + \mathbf{A}_0\mathbf{x}_0]\,\Delta t$$
$$\mathbf{x}_1 = \mathbf{x}_0 + \Delta\mathbf{x}_0$$

mechanism prediction:

$$\ddot{\theta}_0 = \frac{1}{J}\,(T_0 - \tfrac{1}{2}J'\dot{\theta}_0^2 - f_{red})$$
$$\dot{\theta}_1 = \dot{\theta}_0 + \ddot{\theta}_0\Delta t$$
$$\theta_1 = \theta_0 + \dot{\theta}_0\Delta t + \tfrac{1}{2}\ddot{\theta}\Delta t^2$$

induction motor correction:

$$s_1 = 1 - \dot{\theta}_1$$
$$\mathbf{A}_1 = \mathbf{A}_1(s_1)$$
$$\Delta\mathbf{x}_1 = [\mathbf{u} + \mathbf{A}_1\mathbf{x}_1]\,\Delta t$$
$$\mathbf{x}_2 = \mathbf{x}_0 + \tfrac{1}{2}[\Delta\mathbf{x}_0 + \Delta\mathbf{x}_1]$$
$$T_2 = \mathrm{Prod}\,(\mathbf{x}_2)$$

mechanism correction:

$$\ddot{\theta}_1 = \frac{1}{J}\,(T_2 - \tfrac{1}{2}J'\dot{\theta}_0^2 - f_{red})$$
$$\dot{\theta}_2 = \dot{\theta}_0 + \tfrac{1}{2}(\ddot{\theta}_0 + \ddot{\theta}_1)\,\Delta t$$
$$\theta_2 = \theta_0 + \tfrac{1}{2}(\dot{\theta}_2 + \dot{\theta}_1)\,\Delta t$$

result:

$$\theta_2,\ \dot{\theta}_2,\ \mathbf{x}_2,\ T_2$$

Fig. 19 Integration of induction motor-mechanism system.

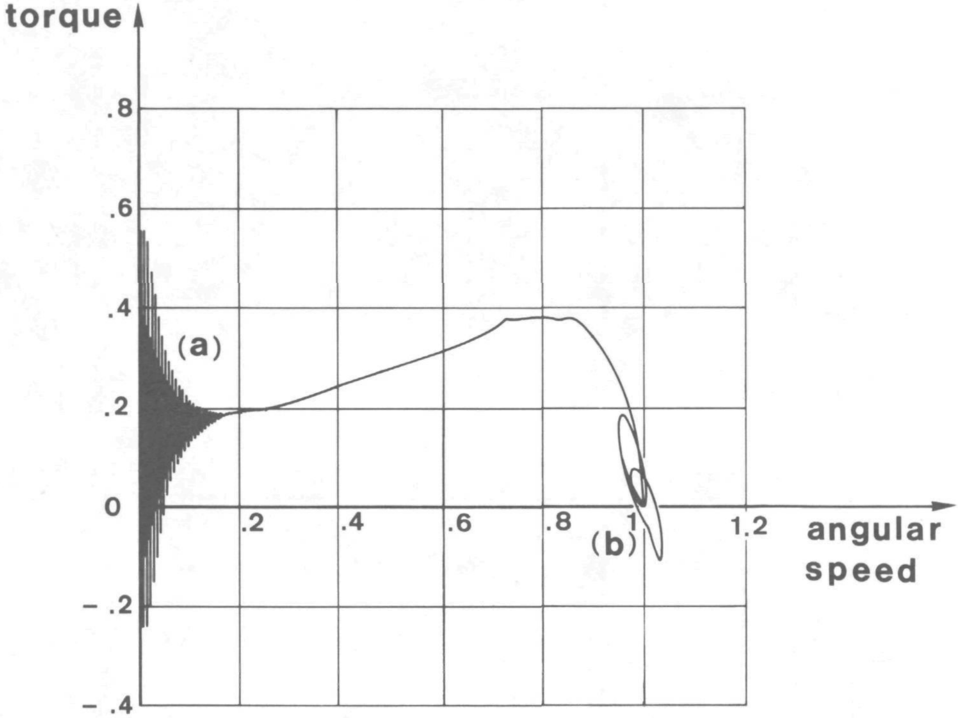

Fig. 20 Starting behaviour (a) and steady state solution (b).

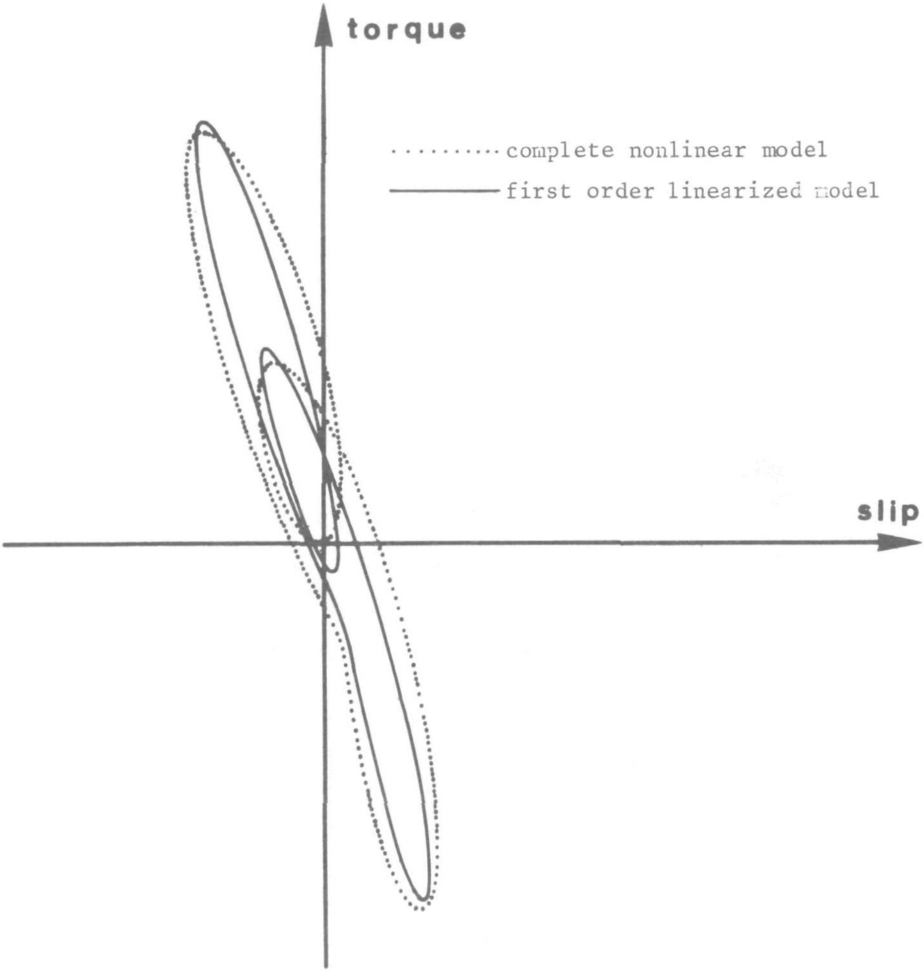

Fig. 21 Steady state slip torque diagram.

Mechanisms driven by a linear actuator

In a large number of applications mechanisms are driven by linear actuators such as hydraulic or pneumatic cylinders. Especially when the actuator is not attached to the machine frame, as in fig. 22, it is useful to incorporate the actuator in the kinematic and dynamic description.

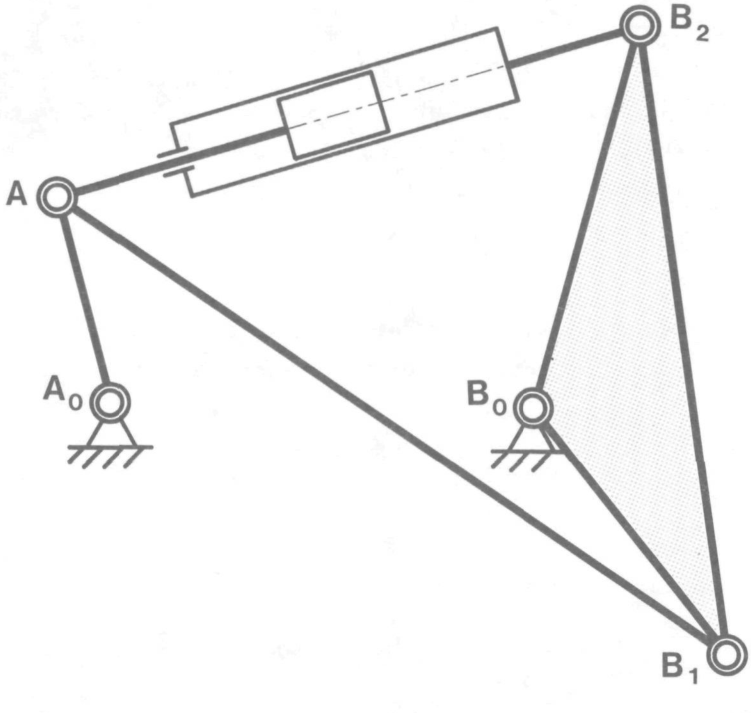

Fig. 22 Example of application of a linear actuator.

In the preceding sections the input motion of the mechanism has been defined by prescribing the value of one or more displacement parameters. It is obvious that in systems like fig. 22 no equivalent set of prescribed displacement parameter values can be found, unless the solution is known on forehand. If however the actuator is considered as a special line element of which the elongation ε is prescribed, it is possible to formulate the problem without much difficulties.

Taking this elongation as the new independent variable, ε_a^o, it will be shown how the equations must be modified to describe this type of mechanisms.

The transfer functions can be found with the same equations as used in section 2.2, but the right hand terms are slightly modified. Instead of for example a right hand term $-D^m_{i,j}.x'^m_j$ we now must add a vector ε'^o. This is a vector containing only zeros except for the entry associated with the actuator.

The equations of motion also remain the same. As usual the vector of accelerations is expressed in terms of the modified transfer functions and the time derivatives of the actuator stroke:

$$\ddot{x}_i = x'_i \ddot{\varepsilon}^o_a + x''_i \dot{\varepsilon}^{o2}_a ; \quad x'_i = \frac{dx_i}{d\varepsilon^o_a}, \quad x''_i = \frac{d^2 x_i}{d\varepsilon^{o2}_a}. \tag{63}$$

The force associated with the actuator stroke is the normal force component of the line element used for the description of the actuator. From the general expression for the stresses (23)

$$\sigma_k = D^{-1}_{k,i} f_i - D^{-1}_{k,j} M_{ji} \ddot{x}_i , \tag{64}$$

the actuator force σ_a can be separated. In this way we arrive at the following relation between the actuator force and the elongation ε^o_a

$$\sigma_a = D^{-1}_{a,i}.f_i - D^{-1}_{a,j}.M_{ji}.\ddot{x}_i , \tag{65}$$

or

$$\sigma_a = D^{-1}_{a,i}.f_i - D^{-1}_{a,j}.M_{ji}.(x'_i \ddot{\varepsilon}^o_a + x''_i \dot{\varepsilon}^{o2}_a). \tag{66}$$

In most cases neither ε^o_a, nor σ_a will be a given function of the time. Generally ε^o_a or σ_a is related to a control variable p(t), e.g.

$$\sigma_a = \sigma_a\{p(t),\ \varepsilon_a,\ \dot{\varepsilon}_a\} . \tag{67}$$

When this relation is known the system is completely defined. Given a set of initial values the equations can be solved by numerical integration. The stresses in the remaining links can then be determined with the unused equations (64). For a recent publication about models of pneumatic actuators we refer to the work of BIALAS [18].

5

Kinetostatics and Vibrations

Generally the aim of the kinetostatic analysis is to provide information about the strength and the stiffness of a mechanism. How much and in which form this information is necessary depends fully on the functional design specifications, such as lifetime and precision. It may be clear that no general paradigm covering the general problem of strength and stiffness can be presented. Following the lines of the finite element method we shall confine ourselves here to the presentation of the general description of strength and stiffness and to the related problems such as vibrations.

The term kinetostatics refers to those theories and computations that provide us with knowledge about the internal stress distribution of the mechanism, including bearing and reaction forces. The motion as a function of time, i.e. the solution of the equation of motion, is a prerequisite and must be determined beforehand. Furthermore the mass distribution and the stiffness of the components of the mechanism must be known in this stage. When the flexibility of some components of the mechanism is not negligible the kinetostatic analysis can easily be extended to the calculation of the associated extra displacements. If these displacements turn out to be comparatively large, then structural vibrations are likely to occur.

This chapter is mainly devoted to the presentation of the equations to be used in kinetostatics and vibration analysis. The discussion of the kinetostatics focusses on the application to planar mechanisms. Finally some guidelines for the formulation and the solution of vibration problems are indicated.

5.1 KINETOSTATIC ANALYSIS

Real mechanisms, as they are used in a wide variety of technical applications, are not coplanar, but the links that constitute the mechanism move in parallel planes (fig. 23).

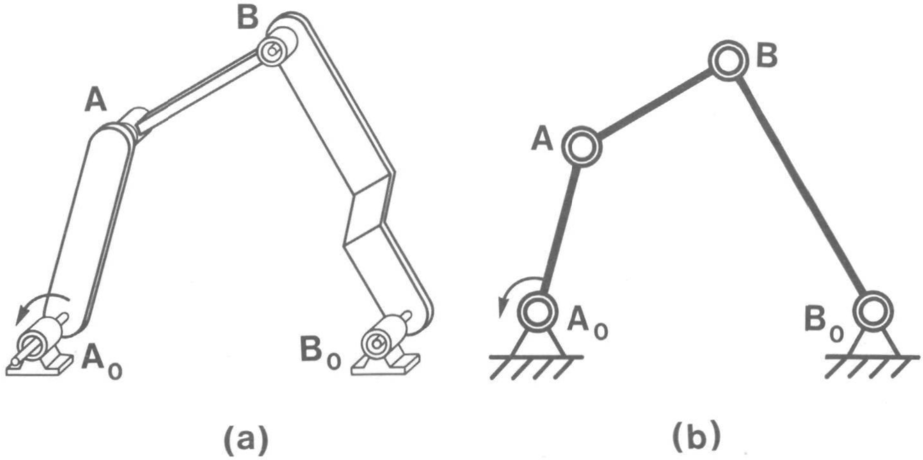

Fig. 23 Planar mechanism (a) and coplanar model (b).

In the first design stages a coplanar model of the mechanism, in which the links are considered to move all in the same plane instead of in different parallel planes, is very useful. From the kinematical point of view such a model is identical to the real mechanism and it serves the study of the primary function of the mechanism; that is how it generates a special motion pattern or force distribution. With the coplanar model of the mechanism the description of deformation and load transmittance is not correct. Nevertheless the results obtained from the coplanar model are valuable to the designer, because the calculated forces in the links of the coplanar model have a strong indicative value and are of great help to the engineer when he is materializing the various links and interconnections.

The equations for the internal stress resultants are (23):

$$D_{j,i}\,\sigma_j = f_i^e - M_{ij}\ddot{x}_j \,. \qquad (68)$$

Since in coplanar models the matrix $D_{i,j}$ is nonsingular, the stresses can be obtained from

$$\sigma_k = D_{k,i}^{-1}\left[f_i^e - M_{ij}\ddot{x}_j\right]. \qquad (69)$$

Here the accelerations are taken from the result of the dynamic analysis of the mechanical system, comprising prime mover and mechanism.

The forces, calculated in this manner, can never be more than a very rough estimate. Due to the non-coplanarity of real mechanisms extra bending and torsion will occur, which leads to considerable loss of strength and stiffness. Therefore a mechanical model is introduced which allows the links to move in separate parallel planes. In fig. 24 an example of a finite element model of a sample mechanism is sketched.

Generally such a model of the mechanism is a statically indeterminate structure. Then the displacements and the stresses for a sequence of positions priorly found with the coplanar model can be determined in a straightforward manner by the so-called displacement method. The number of displacement parameters, about 10 in the sample coplanar model of fig. 23, has increased to about 80 in the non-coplanar model of fig. 24. Some calculation time can perhaps be gained by separating the constant and the variable parts in the equations, this however at the price of loss of transparancy of the computer programme.

In principle this type of calculations can be done with the aid of available finite element programme systems such as ASKA, etc. However these programmes are complicated due to their generality and the adaptation of these programmes to our problem will be time consuming.

Not without purpose we didn't start to point out the possibility of calculating the displacements of a coplanar model, because in that case we are forced to estimate values for the in plane flexibilities due to out of plane deformation, which affects the stiffness to a great extent. In our opinion the designer should rather be relieved from this task

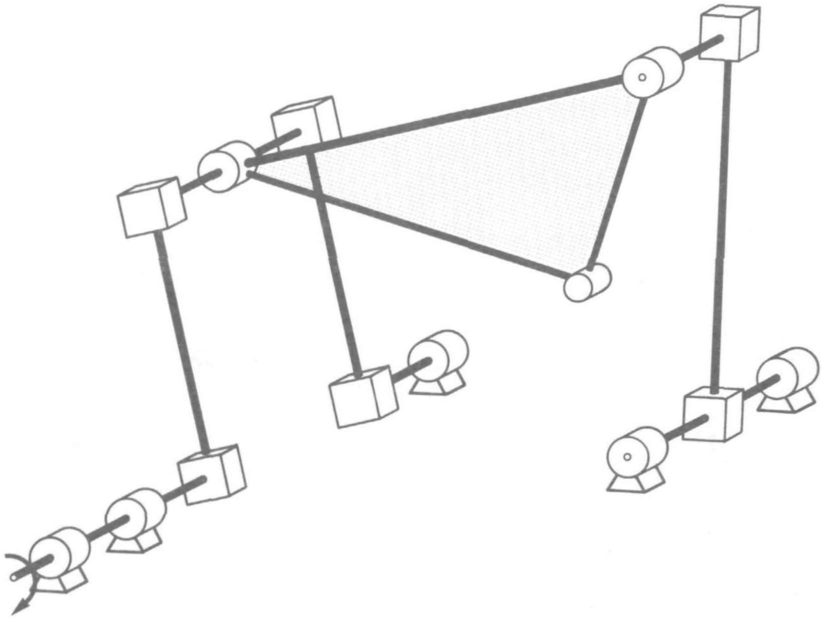

Fig. 24 Non-coplanar model of mechanism.

where a finite element calculation can do this job better and cheaper.

Finally we want to consider what can be done with the results of the kinetostatic analysis. Normally the mechanism is to be designed such that it is capable to perform its function during a given lifetime according to a given specification. Apart from the written specification we must be aware of the existence of the unwritten specifications concerning noise production and allowable vibration levels, to say nothing of safety aspects.

A number of the various application possibilities are visualized in fig. 25.

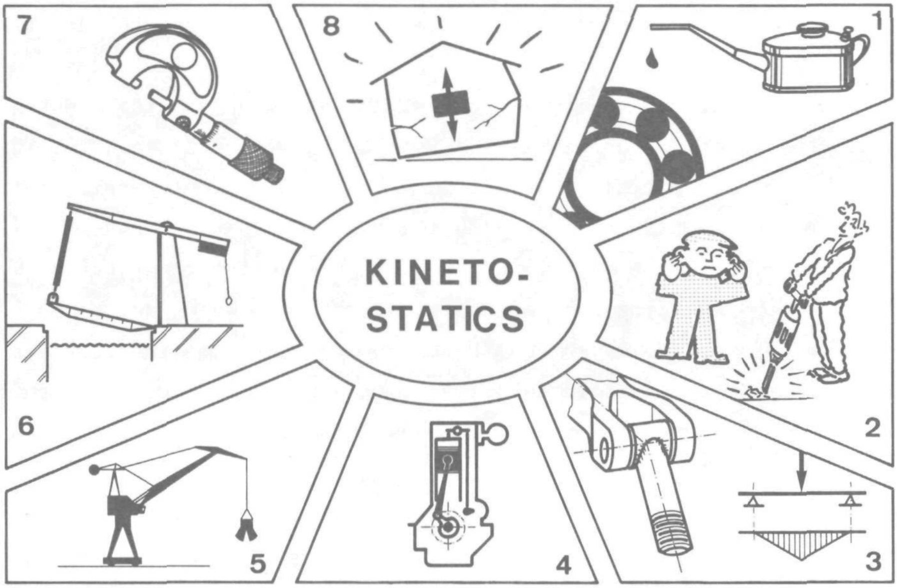

Fig. 25 Keyrole of kinetostatic analysis: bearing forces for the design (3), lifetime and service (1) of the connections, reaction forces for vibration nuisance (8), static and dynamic loads for balancing (6) and fatigue load (4,5), deflections for position accuracy (7).

From the calculated forces the varying state of stress in the various components can be determined. Depending on the specific situation this calculated state of stress is to be compared with the allowable states of stress representative for the situation. The knowledge of the loading situation of the connections between the links of the mechanism is indispensable for a sound design. The influence of the finite stiffness on the performance of the mechanism can be characterized by the deflections under operating conditions, especially in the case of mechanisms meant to generate loci with a high positional accuracy.

5.2 VIBRATION ANALYSIS

Apart from some very special applications, mechanisms are not meant to produce a certain vibration level. On the contrary, mechanism design is mostly directed to the materialisation of a kinematic chain, avoiding vibrations as much as possible. As in many cases a suitable description of the dynamical behaviour is lacking, the designer adheres a design philosophy which is best expressed as: "design for stiffness". Doing so it is often not realized that not only the stiffness properties but also the mass distribution is of prime influence on the dynamical behaviour. The reason of this attitude of the designer can partly be traced to the existing lack of practical relevant knowledge in an accessible form. Too often the treatment of mechanism vibrations is nothing more than the solution of a difficult mathematical problem of which the relevance for practical cases is left undiscussed. And only in a few cases serious attempts have been made to make the results readily accessible to the designer. Thus the possibility of feedback from the applications scene is cut off. Without further proof it can be stated that the development in this field depends highly on the attention given to these aspects.

In all fairness it must be remarked that the problem at hand is very complicated indeed. The main problem is the difficult choice of the model. The question which type of model is most suitable has certainly not yet been settled. Furthermore we have the problem of the determination and evaluation of the solutions. In many cases the actual dynamical behaviour of a mechanism is strongly influenced by friction and clearance. These effects will have to be determined by means of measurements on authentic mechanisms. The discrepancy between the measured values and the values calculated with the ideal mechanism without friction and clearance may be eliminated by an approximate description of these troublesome effects.

For the moment our contribution is restricted to the development of methods that make the classical methods better accessible for a wide variety of mechanisms. With the observations made above in mind it will be demonstrated how the problem of vibrations can be formulated in a general sense. This will be done for the problem of the vibrations of a rigid link mechanism driven by an induction motor and for the vibrations of a flexible link mechanism with constant speed of the input crank.

Vibrations of a rigid link mechanism driven by an induction motor

When the influence of the finite stiffness of the mechanical elements is neglected there is still the possibility of vibrations. Let us consider the system of fig. 26, consisting of a mechanism rigidly connected to an induction motor.

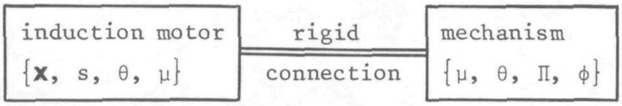

Fig. 26 Rigid link mechanism driven by induction motor.

The motion of the mechanism is governed by its equation of motion (50), presented here in an appropriate dimensionless form.

$$\mu = \Pi\ddot{\theta} + \tfrac{1}{2}\Pi'\dot{\theta}^2 - \phi. \qquad (70)$$

For the induction motor we have the system (57), which is written as:

$$\dot{\mathbf{x}} = \mathbf{b} - \mathbf{A}(s)\mathbf{x},$$
$$s = 1 - \dot{\theta},$$
$$\mu = \mathrm{Prod}(\mathbf{x}). \qquad (71)$$

In a previous chapter it has been shown how a stationary periodic solution of this system of equations (70) and (71) can be obtained. Generally this solution is not the only possible solution. The question of the possibility of vibrations or the related problem of the stability of a stationary solution is an interesting problem of practical importance. Let the total "motion" $\{\theta, \mu, \mathbf{x}, s\}$ consist of the solution of the stationary periodic motion already found, $\{\theta_0, \mu_0, \mathbf{x}_0, s_0\}$, and an additional small deviation, $\{\theta_1, \mu_1, \mathbf{x}_1, s_1\}$, such that

$$\theta = \theta_0 + \theta_1, \quad \theta_1 << \theta_0,$$
$$\mu = \mu_0 + \mu_1, \quad \mu_1 << \mu_0,$$
$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1, \quad \mathbf{x}_1 << \mathbf{x}_0,$$
$$s = s_0 + s_1, \quad s_1 << s_0. \qquad (72)$$

When the quantities Π, ϕ and $\mathbf{A}$ are developed in Taylor series in the neighbourhood of the given stationary solution, then after introduction of the solution (72) a linear system of equations for $\{\theta_1, \mu_1, \mathbf{x}_1, s_1\}$ is obtained

$$\begin{aligned}
\mu_1 &= \Pi_0 \ddot{\theta} + \Pi_0' \dot{\theta}_0 \dot{\theta}_1 + \left[\Pi_0' \ddot{\theta}_0 + \tfrac{1}{2}\Pi'' \dot{\theta}_0^2 + \phi_0'\right] \theta_1 , \\
\dot{\mathbf{x}}_1 &= - \mathbf{A}_0 \mathbf{x}_1 - \frac{d\mathbf{A}}{ds} s_1 , \\
s_1 &= - \dot{\theta}_1 , \\
\mu_1 &= \text{Lin}\,(\mathbf{x}_0, \mathbf{x}_1).
\end{aligned} \tag{73}$$

The functions Π_0 and ϕ_0 are known periodic functions of θ_0. The matrix $d\mathbf{A}/ds$ is a constant matrix as $\mathbf{A}$ contains only the linear form of a.

The approach is quite similar to the description of torsional vibrations of crankshafts. It is expected that the method of the determination of the solutions of the system (73) will be very similar to the determination of solutions in the comparable cases treated by KOITER [19] and HOLZWEISSIG [20].

Vibrations of a flexible link mechanism operating at constant input speed

In addition to the vibrations caused by the mutual influence of the mechanism and its energy source, it is possible to study also deviations of the motion of the mechanism in the case of flexible links. For the sake of simplicity we shall assume a constant speed of the input crank of the mechanism. The starting point of this analysis are the exact equations of chapter 2.4 from which the stresses and the deformations have been eliminated

$$D_{j,i} S_{jk} D_k(x) = f_i - M_{ij} \ddot{x}_j . \tag{74}$$

It will be assumed that a (approximate) solution $x_i(t)$ of this system of equations is known. This solution may be a flexible link solution according chapter 2.4 (28), but also the solution of the rigid link model can be taken.

Consider adjacent solutions $x_i = \underline{x}_i + u_i$ of the system (74). Expanding all terms in the equations with respect to the small deviations u_i we obtain as a first order approximation:

$$\underline{D}_{j,i} S_{jk} \underline{D}_{k,\ell} u_\ell + D_{j,ik} u_k S_{j\ell} \underline{D}_\ell(\underline{x}_m) = f_{i,k} u_k - \underline{M}_{ij} \ddot{u}_j - \underline{M}_{ij,k} u_k \ddot{\underline{x}}_j . \tag{75}$$

The contributions on the left hand side are:

$\underline{D}_{j,i} S_{jk} \underline{D}_{k,\ell}$: the classical stiffness matrix of the system

$\underline{D}_\ell(\underline{x}) S_{j\ell} \underline{D}_{j,ik}$: the geometrical stiffness matrix.

The terms with $f_{j,k}$ and $M_{ij,k}$ originate from dependence on position of the loads and the mass matrix respectively.

The simplest form of (75) is obtained when for the fundamental state the stress free rigid link solution is taken and when the loads and the mass matrix are position independent. In that case we retain

$$D_{j,i} S_{jk} D_{k,\ell} u_\ell = - M_{ij} \ddot{u}_j \tag{76}$$

Both (75) and (76) are systems of linear differential equations with periodic coefficients of the form

$$\mathbf{A}\ddot{\mathbf{x}} + [\mathbf{B} + \mathbf{C}(t)]\mathbf{x} = 0; \quad \mathbf{C}(t+T) = \mathbf{C}(t). \tag{77}$$

Many investigators have been involved with this type of equations that always occurs when the problem of the infinitesimal stability of the periodic solutions of nonlinear systems is studied. General solution methods can be found in BOLOTIN [21]. It is interesting that apart from the main unstable regions also subharmonic, ultraharmonic and combination vibrations may occur.

The practical significance of these vibration possibilities is as yet not very clear.

5.3 EXAMPLE

In order to demonstrate the applicability of the kinetostatic analysis we shall consider a recent development of an automobile engine with low fuel consumption. The engine was invented by POULIOT in 1974 and it is reported recently in [22] as shown in fig. 27.

The major difference between the Pouliot engine and a conventional engine is that the conventional engine consists of a four-bar link mechanism with a fixed stroke, but the Pouliot engine has been designed with a six-bar mechanism with a variable stroke. The piston stroke depends on a coupler curve of a mechanism with changeable base $A_0B_0^*$ (fig. 28), controlled by a servo-motor which actuates a control screw.

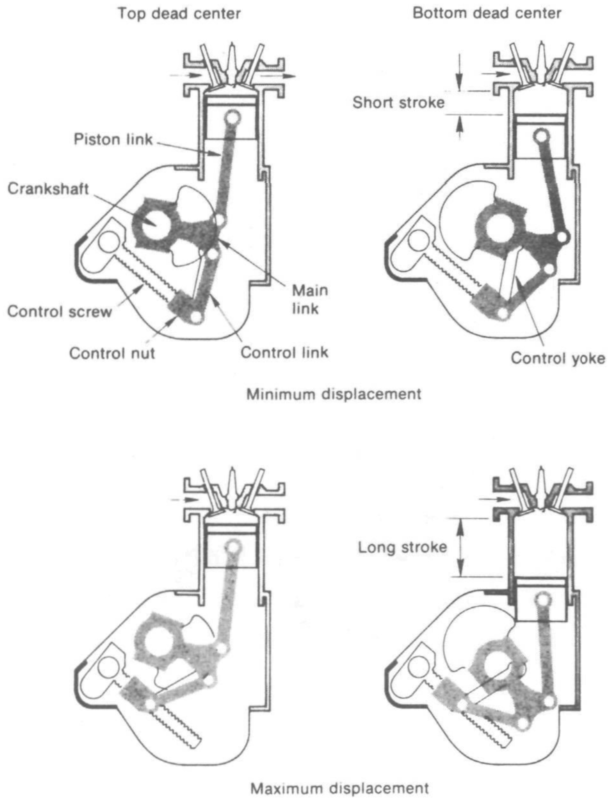

Fig. 27 The Pouliot engine [22].

From a kinematical point of view the principle of the engine, shown in fig. 28, consists of a four-bar mechanism $A_0ABB_0^*$. The piston link CD is connected to the coupler plane in C. The stroke is controlled by changing the position of the fixed pivot B_0^*. This is accomplished by shortening the link $B_0^*E_0$, representing the control screw.

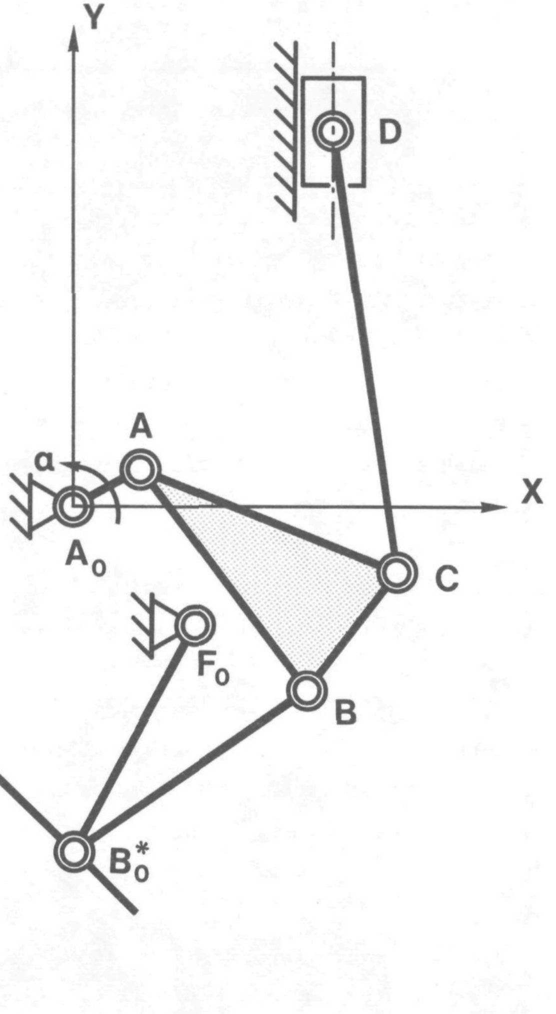

Fig. 28 Principle design of the Pouliot-engine.

The coordinates, which have been determined from the original pictures by measurement, are listed in Table II.

node	X	Y	node	X	Y
A_0	0.000	0.000	C	0.830	-0.100
A	0.180	-0.015	D	0.610	0.925
B	0.690	-0.420	E_0	-0.670	-0.160
B_0^*	0.210	-0.880	F_0	0.330	-0.200

Table II Nodal coordinates Pouliot-engine.

The input for the computer programme was prepared such that we have two modes of operation: rotation of the crank A_0A or stroke adjustment. The stroke is controlled by prescribing the axial deformation ε_1^0 of the control screw $E_0B_0^*$. So in the programme either $\alpha'^o = 1$ and $\varepsilon_1'^o = 0$, or $\alpha'^o = 0$ and $\varepsilon_1'^o = 1$.

Starting from the position according Table II four full rotations of the crank A_0A are made. After each full rotation the stroke is adjusted by shortening the control screw $E_0B_0^*$ by an amount of 0.2. The results of the calculations for the succesive full rotations are indicated as no. 1, 2, 3, 4. The first results are pure kinematical. In fig. 29 the piston displacement diagram for the four different stroke values is presented, together with the trajectories of the coupler point C.

In order to give an impression of what can be done we have made some kinetostatic calculations. In the case that the crankshaft is given a constant speed of 1 rad/s the contribution of the piston mass to the total forces is calculated. The piston mass was taken equal to the unity of mass. The forces have been calculated according to the general formulas (23). Some of the results are presented here in the form of diagrams.

Fig. 30 and 31 show the resulting force F_y and the resulting couple M about A_0, that the mechanism exerts on its surroundings. The resulting force F_x is zero.

In fig. 33 the force exerted on the crankshaft bearing has been plotted in a polar diagram. It is clear that plots of this type can be useful for the design of the bearing.

Finally in fig. 32 the axial force in the control screw is shown.

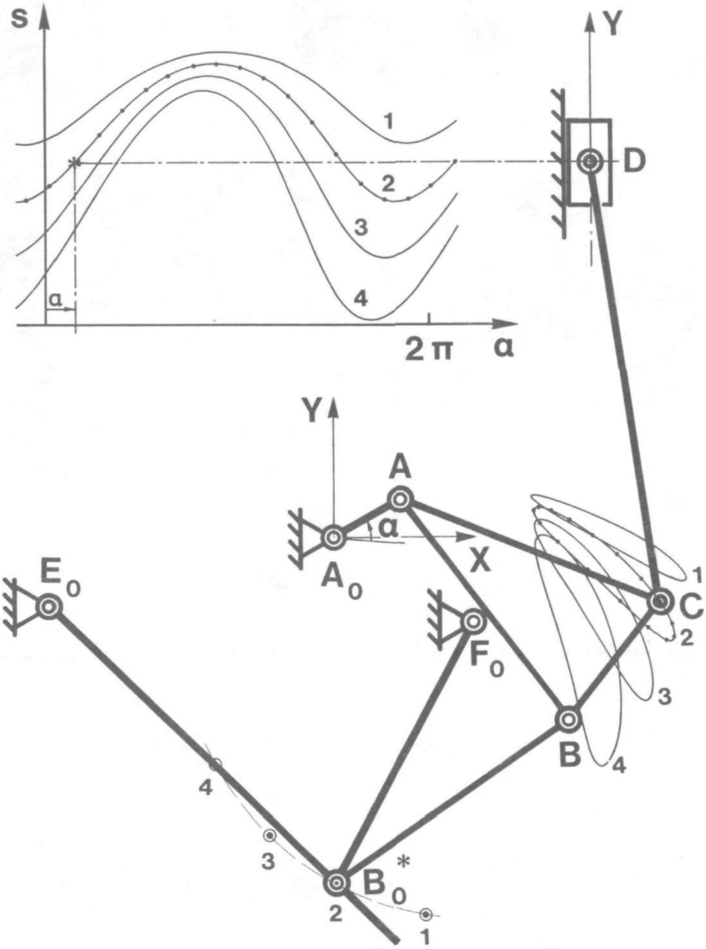

Fig. 29 Pouliot engine, kinematical results.

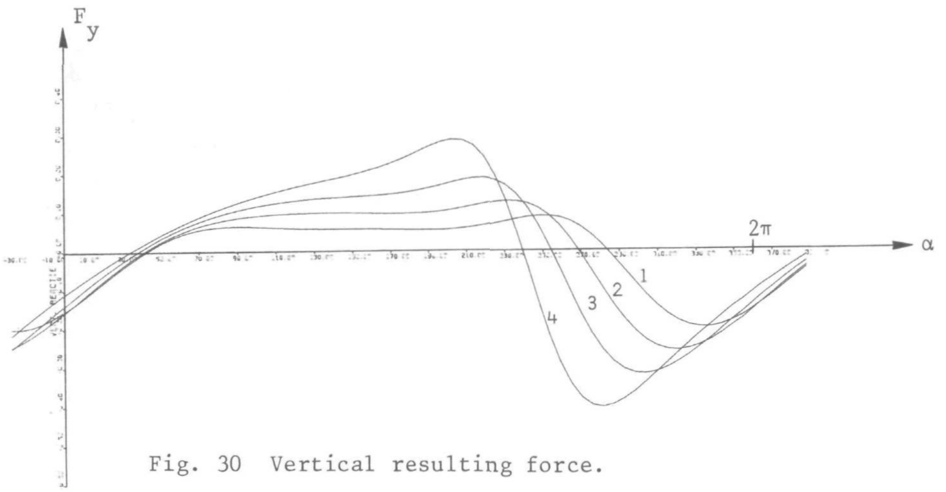

Fig. 30 Vertical resulting force.

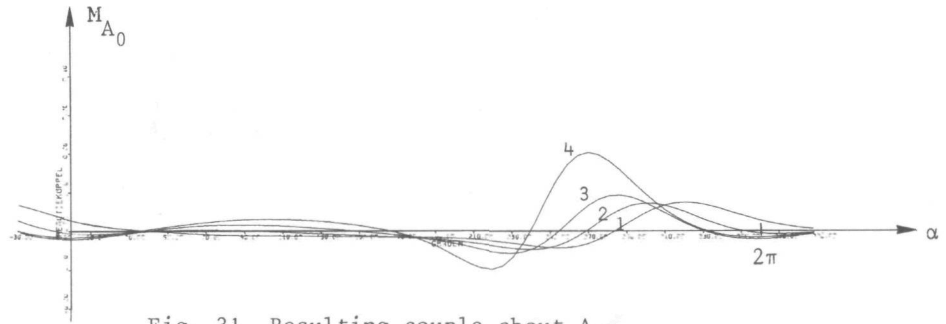

Fig. 31 Resulting couple about A_0.

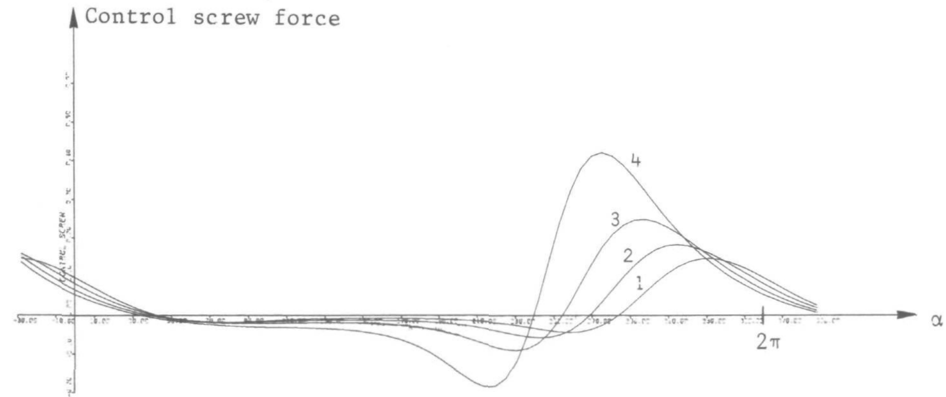

Fig. 32 Control screw force.

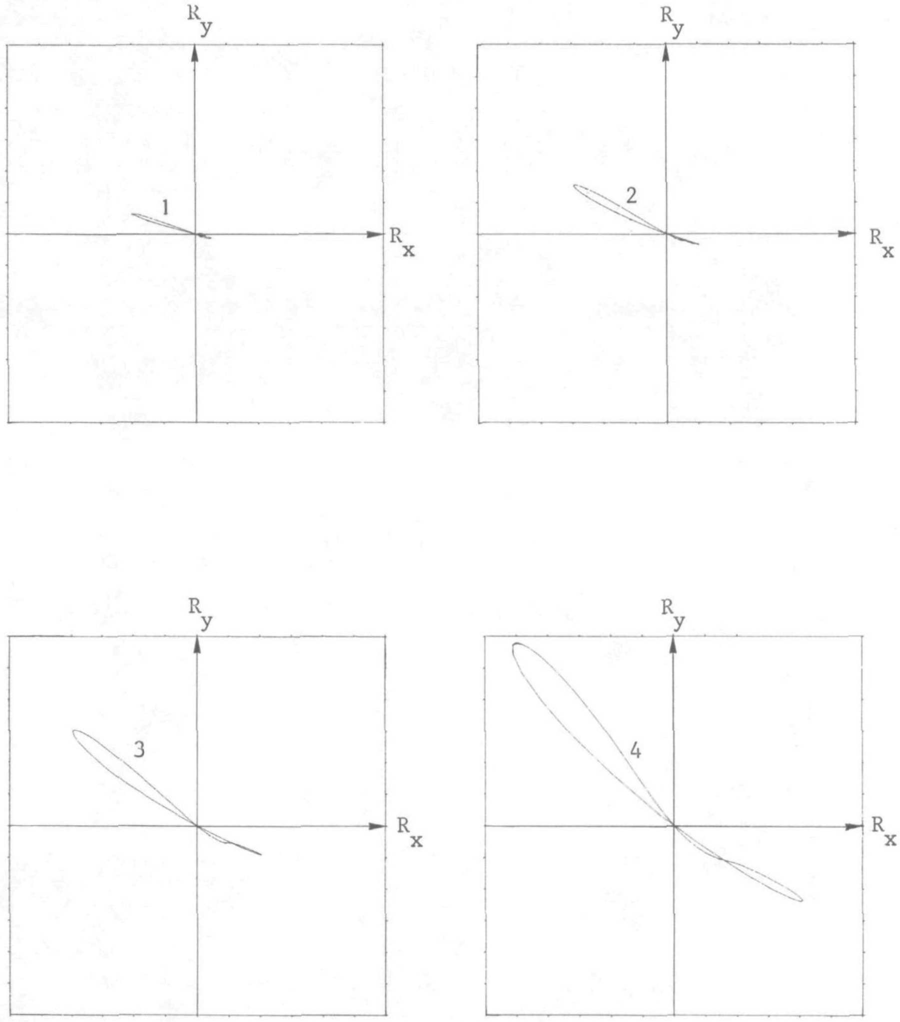

Fig. 33 Polar plot of crankshaft bearing force.

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Samenvatting

Bij de ontwikkeling van de eindige elementenmethode voor de numerieke analyse van het mechanisch gedrag van constructies heeft tot dusver de berekening van de vervormings- en spanningstoestand in vormvaste constructies centraal gestaan. De gediscretiseerde beschrijvingswijze van de kinematica van de constructie, zoals deze in de eindige elementenmethode wordt gegeven, vormt echter ook een goed uitgangspunt voor de numerieke analyse van mechanismen.

Na een inleidend hoofdstuk wordt in hoofdstuk 2 de algemene theorie beschreven. Karakteristiek voor de gevolgde werkwijze is de belangrijke rol die is toegekend aan de betrekkingen tussen vervormingen en verplaatsingen. Uitgaande van deze betrekkingen kunnen lineaire vergelijkingen worden opgesteld voor de z.g. eerste en tweede orde overdrachtsfuncties van mechanismen, opgebouwd uit onvervormbare schakels. Deze eerste en tweede orde overdrachtsfuncties zijn van essentieel belang voor de bepaling van snelheden en versnellingen. Vanuit een bekende momentane stand van het mechanisme kan met behulp van deze overdrachtsfuncties een benadering van een naburige stand worden verkregen. Met een iteratieve berekeningsmethode wordt deze geschatte positie gecorrigeerd zodat in de nieuwe stand de verschillende onderdelen zonder vervorming aan elkaar passen. Vervolgens worden de bewegingsvergelijkingen van een mechanisme afgeleid met behulp van het principe van de virtuele arbeid. Voor het in de praktijk belangrijke geval dat kleine vervormingen van de schakels in rekening moeten worden gebracht wordt tenslotte een bruikbare methode aangegeven.

In de volgende hoofdstukken wordt de algemene theorie nader uitgewerkt en toegepast op vlakke mechanismen.

Hoofdstuk 3 bevat daartoe allereerst een beschrijving van de voor deze aanpak geschikte elementen. Nadat is uiteengezet op welke wijze opbouw en geometrie van het mechanisme kan worden beschreven, wordt ingegaan op de wijze van opstellen en oplossen van de systeemvergelijkingen. Een computerprogramma "PLANAR", met als kern het berekenen van de overdrachtsfuncties, wordt besproken. De gevolgde werkwijze wordt toegelicht met een voorbeeld.

De dynamische analyse komt in hoofdstuk 4 aan de orde. Om de posities van de schakels als functie van de tijd te kunnen berekenen moeten oplossingen worden bepaald van de bewegingsvergelijking van het mechanisme. De wijze waarop het mechanisme wordt aangedreven is daarbij van grote betekenis. Ruime aandacht is dan ook besteed aan de beschrijving van een aandrijving met behulp van een asynchrone kooi-ankermotor.

In hoofdstuk 5 worden de kinetostatica en de trillingsvraagstukken behandeld. Het kunnen berekenen van het krachtenspel in een mechanisme is van onschatbare betekenis voor de constructeur. Aangegeven wordt op welke wijze het twee-dimensionale model kan worden vervangen door een meer realistisch model waarin parasitaire stijfheden beter tot hun recht kunnen komen. Bij de beschrijving van trillingsverschijnselen komt de betrekkelijke waarde van het bestuderen van geïdealiseerde systemen naar voren. Weliswaar kunnen vergelijkingen voor deze systemen worden opgesteld en opgelost, de betekenis voor werkelijke systemen is echter gering zolang toetsing aan experimentele gegevens achterwege blijft. Het hoofdstuk wordt besloten met een toepassingsvoorbeeld uit de kinetostatica.