DELFT UNIVERSITY OF TECHNOLOGY

DEPARTMENT OF AERONAUTICAL ENGINEERING

Report VTH - 158

A UNIFIED THEORY FOR LINEARIZED SHOCK-ON-SHOCK INTERACTION PROBLEMS

by

N. L. ARORA

DELFT - THE NETHERLANDS

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ERRATA II

Page	Line	Correction
12	2nd of Eqs.(2.2.8)	replace $p_1(x, u, z, t)$ by $p_1(x, y, z, t)$
35	Eq.(3.3.4) R.H.S.	should read: $(K/\bar{b})\ s\ \exp\{-\bar{b}sy\}\ G(s)$
51	Eq.(3.6.12) R.H.S.	last term should read: $\frac{\varepsilon}{\beta}(k_1\Lambda_{21} - k_2\Lambda_{22})(1 - \frac{\beta}{m + m_1}\frac{y}{\tau})$
103	5	delete the sentence: "Due to the constant.....discontinuity at I."
138	Fig. 15b	intersection of the sonic wavelet BCD with the wing should be denoted by E
144	Fig. 25	shape of the disturbed shock is wrong; it should be recalculated by using the corrected Eq.(3.6.12)
52	3	replace voordien by voordoen
	4 from below	replace aaneengestroomde by aangestroomde

ABSTRACT

The non-uniform flow field produced, as a result of the interaction of a plane shock of arbitrary strength with supersonically moving aerodynamic obstacles (like two-dimensional thin aerofoils, slender bodies of revolution and three-dimensional thin wings), behind the shock is analyzed. The problem is posed generally in terms of initial and boundary values for the perturbation pressure. The solution is constructed by a systematic use of integral transforms. The density field and the shock shape are also deduced.

As detailed examples the results of the theory are applied to the interaction with a thin wedge, a slender conical projectile, a flat plate delta wing with supersonic leading edges and a thin yawed wedge. Various numerical results are presented for the examples considered.

SAMENVATTING

In dit proefschrift wordt een gelinearizeerde theorie ontwikkeld die in staat is op uniforme wijze de interactie problemen te behandelen, die zich voordien indien een vlakke schokgolf van willekeurige sterkte getroffen wordt door, met supersone snelheid, bewegende lichamen als twee-dimensionale dunne vleugels, axisymmetrische slanke lichamen en drie-dimensionale dunne vleugels.

Het probleem wordt geformuleerd in de vorm van een begin- en randwaarde-probleem voor de verstoringsdruk van het niet-uniforme stromingsveld dat zich als resultaat van de interactie achter de schok voordoet. De analytische uitdrukkingen worden verkregen door de methode van de integraal-transformaties toe te passen. Tevens worden het dichtheidsveld en de vorm van de schok afgeleid.

In hoofdstuk I van de dissertatie wordt een overzicht gegeven van de litteratuur en worden de basis-hypothesen besproken.

In hoofdstuk 2 worden ten eerste de bewegings-vergelijkingen van de niet-stationnaire stroming gelinearizeerd. Ten tweede worden de relaties opgesteld die de onbekende storingen aan de stroomafwaartse zijde van de schokgolf uitdrukken in de bekende stroomopwaartse verstoringen. Tenslotte worden de uitdrukkingen voor het stroomopwaartse storingsveld gegeven.

In hoofdstukken 3, 4, 5 en 6 wordt de theorie behandeld voor de schok-schok interactie van dunne twee-dimensionale vleugels, axisymmetrische slanke lichamen, drie-dimensionale dunne vleugels, en scheef aangestroomde twee-dimensionale vleugels. De theorie wordt toegepast op de interactie met een dunne wig, een slank conisch projectiel, een symmetrische delta vleugel bij invalshoek nul, een vlakke delta vleugel met supersone voorranden en een dunne scheef-aaneengestroomde wig.

In het laatste hoofdstuk 7 worden de verschillende numerieke resultaten gegeven, die verkregen werden voor de in de voorafgaande hoofdstukken behandelde voorbeelden en een algemene bespreking volgt.

ERRATA

"A unified theory for linearized shock-on-shock interaction problems"

by N.L. Arora

Page	Line	For	Read
V	13	Specifications	Specification
XI	4	derivatives	derivative
1	3†	1965	1957
4	2	Blanksnship	Blankenship
12	11	shcock	shock
	last of Eqs.(2.2.8)	v_1	$\underline{v}_1$
13	first	The	the
14	7†	peerturbation	perturbation
16	end of Eqs.(2.3.5)	,	.
20	6	(2.3.4)	(2.3.3)
21	Eq.(2.4.6)	$\partial^2\phi/\partial y^2$	$\partial^2\phi/\partial r^2$
24	Eq.(2.4.13) L.H.S.	$\partial\phi(x_1, y, z)/\partial z$	$\partial\phi(x_1, y, z)/\partial y$
25	Eq.(2.4.15a) R.H.S.	$F_1(\xi, \varsigma)$	$F_1(\xi, \zeta)$
27	7	(2.3.4)	(2.3.3)
31	5	$v_1 = v_o = - U\psi_y$,	$v_1 - v_o = - U\psi_y$
	5†	$x \leqslant m\tau$	$x < m\tau$
32	14	Eq. (2.1.24)	Eq. (2.2.24)
	under Eq.(3.3.4)	$f\{\bar{a}(\bar{\tau} + \lambda_o x)\}$	$f\{\bar{a}(\bar{\tau} + \lambda_o \bar{x})\}$
	3†	(3.1.5b)	(3.2.5b)
	last	(2.1.16); (2.1.18)	(2.2.16); (2.2.18)
33	first	Eqs. (2.1.17)	Eqs. (2.2.17)
34	first	(k/β)	$-(k/\beta)$
39	end of Eq.(3.4.16)	$\Big)d\xi$	$d\xi\Big)$,
41	10	(2.1.9)	(2.2.19)
42	2†	condtion	condition
44	10	Fig. 11b	Fig. 11a
47	under Eq.(3.5.6)	ϕ_o should read: $\phi_o = \arctan\{1/(M^{-2} - 1)^{\frac{1}{2}}\}$	
50	under Eq.(3.6.7)	$\bar{N}$ should read: $\bar{N} = \bar{b}(\bar{\tau}^2 - \bar{x}^2 - y^2)^{\frac{1}{2}}$	

†count from below

Page	Line	For	Read
53	10	§2.4.3	§2.4
57	4	§3.3	§3.4
	Eq.(4.4.8) L.H.S.	$\{\exp s\lambda_o\bar{x}\}$	$\exp\{s\lambda_o\bar{x}\}$
61	4	$\bar{\tau} - (R \pm \lambda_i\xi)$	$\bar{\tau} - (\bar{R} \pm \lambda_i\xi)$
64	10	(4.4.15)	(4.4.19)
70	4	$\bar{x} > 0$	$\bar{x} < 0$
71	end of Eq.(5.4.10)	$G_1(s, \zeta)$	$G_1(s, z)$
	Eq.(5.4.11)	$z \to \infty$	$z \to \pm\infty$
72	under Eq.(5.4.14)	$G_1(s, \nu)$	$G_1(s, z)$
73	first	after L^{-1} insert parentheses $\Big(\ \Big)$	
	$7^{\dagger}$	reorganized	recognized
75	5	Eq.	Eqs.
77	under Eq.(5.5.7)	$f_1(\bar{x}, \bar{\tau}, z)$	$f_1(\bar{x}, z, \bar{\tau})$
		$F_2(\bar{x}, \bar{\tau}, z)$	$F_2(\bar{a\tau}, z)$
78	first	should read: This can further be expressed as	
80	first	$\bar{x} > 0$	$\bar{x} < 0$
81	$4^{\dagger}$	after L^{-1} insert parentheses $\Big(\ \Big)$	
90	last of Eqs.(6.2.1)	should read: $\underline{V} = \underline{V}_1 + \varepsilon\, \underline{v}_1(x, y, z) + O(\varepsilon^2)$	
91	3	$P = p_1/\gamma p_1$	$p = p_1/\gamma P_1$
92	end of Eqs.(6.2.8)	,	),
93	$9^{\dagger}$	delete $\rho = \rho_o/R_o$	
		$p = p_o/\gamma P_o$	$\bar{p} = p_o/\gamma P_o$
95	$5^{\dagger}$ & $4^{\dagger}$	6.2.5	6.2.15
96	$6^{\dagger}$	(6.2.6b)	(6.2.16b)
97	$7^{\dagger}$	(6.2.6)	(6.2.16)
	$2^{\dagger}$	(6.4.8)	(6.4.7)
99	first	Figure 20	Figure 21
111	16	Snedden	Sneddon
126	8	on the R.H.S. integral, insert $d\zeta$	
130	Eq.(A 4.25) R.H.S.	second term second integral, insert $d\mu$	
	end of Eq.(A 4.26)	$d\xi$	$d\zeta$
137	Fig. 13	ordinate represents M'	
138	Fig. 15b	below the wing, I should read: I'	

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CONTENTS

NOMENCLATURE†

a_o, a_1	velocity of sound in region (0) and (1).
$\bar{a}$	defined in (3.3.4); also in (6.2.14).
a, b	defined in (6.4.7).
A, B, C, D	constants defined in (2.3.11).
A_o	$= - m_1^2$, in (3.3.4); $= - \bar{m}_1^2$, in Chapter 6.
A_i	constants defined in (3.4.14); also in (6.4.8).
$\bar{b}$	$= \beta/\bar{a}$.
B_o	constant defined in (3.3.11); also in (6.3.5).
B_{o1}, B_{o2}	constants defined in (5.6.2) & (5.6.4).
$\bar{c}$	defined in (3.4.21).
c_p, c_v	specific heats of medium at constant pressure and constant volume.
B_i, C_i, D_i, E_i	constants defined in (5.5.11) & (5.6.8).
E, F, G	constants defined in (2.3.12).
$f(x_1)$	function defining the shape of an aerofoil or a slender body of revolution.
$F(\xi)$	function defined in (2.4.11).
$F_1(\xi, \zeta)$, $F_2(\xi, \zeta)$	functions defined in (2.4.15).
$F(\ )$, $F^{-1}(\ )$	Fourier exponential transform and its inverse.
$F_c(\ )$, $F_c^{-1}(\ )$	Fourier cosine transform and its inverse.
$F_s(\ )$, $F_s^{-1}(\ )$	Fourier sine transform and its inverse.
H	unit step function.
J	$= c_p - c_v$, specific gas constant.
J_o	Bessel function of the first kind.
k_1, k_2	constants defined in (2.4.17).

† The symbols which are used locally are not listed here.

K	constant defined in (3.3.10); also in (6.4.7).
K_o, K_1	modified Bessel functions of the second kind.
$\underline{i}$, $\underline{j}$, $\underline{k}$	unit vectors in the x, y and z directions.
$L(\ )$, $L^{-1}(\ )$	Laplace transform and its inverse.
m	$= (V - U)/a_1$, (2.3.13).
$\bar{m}$	defined in (6.2.5).
m_1	$= (W + U)/a_1$, obstacle Mach number behind the shock, (2.5.1).
$\bar{m}_1$	$= (W + U \cos\chi)/a_1$, (6.3.4).
M	$= V/a_o$, Mach number of the plane shock.
M'	$= W/a_o$, Mach number of the obstacle.
M_1	$= U/a_1$, Mach number of the undisturbed main flow behind the shock.
$\bar{M}$	defined in (6.1.3).
p_o, p_1	perturbed values of pressure ahead of and behind the shock.
$\bar{p}$, p	$\bar{p} = p_o/\gamma P_o$, $p = p_1/\gamma P_1$, dimensionless values.
P	pressure of the flow field.
P_o, P_1	pressure ahead of and behind the undisturbed shock.
q_o, q_1	perturbed values of radial velocities ahead of and behind the shock.
$\bar{q}$, q	$\bar{q} = q_o/V$, $q = q_1/a_1$, dimensionless values.
Q_1, Q_2	constants defined in (3.4.17).
r	radial direction.
r_c	$= \{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}$, (3.4.16).
$\bar{R}$	$= \{(\bar{x} - \xi)^2 + r^2\}^{\frac{1}{2}}$, used in Chapter 4.
	$= \{(\bar{x} - \xi)^2 + (z - \zeta)^2 + y^2\}^{\frac{1}{2}}$, used in Chapter 5.
R	density of the flow field.
R_o, R_1	density ahead of and behind the undisturbed shock.
s_1	perturbed value of specific entropy behind the shock.
s	$= s_1/c_p$, dimensionless value; also used as transformed parameter, corresponding to $\bar{\tau}$.

S	specific entropy of the flow field.
S_1	specific entropy of the fluid in region (1).
t	time variable.
$\underline{v}_o(u_o, v_o, w_o)$, $\underline{v}_1(u_1, v_1, w_1)$	perturbed values of velocity components in the x, y and z directions, ahead of and behind the shock.
$\underline{\bar{v}}(\bar{u}, \bar{v}, \bar{w})$	$= \underline{v}_o/V$, dimensionless values.
$\underline{v}(u, v, w)$	$= \underline{v}_1/a_1$, dimensionless values.
U	velocity of the undisturbed flow behind the undisturbed plane shock.
V	speed of the plane shock in medium at rest.
$\underline{V}$	velocity vector of the flow field.
V_o	defined in (6.1.1).
V_1	defined in (6.1.2).
W	speed of the obstacle in medium at rest.
x, y, z	axial, vertical and spanwise directions.
(x_1, y), (x_1, r), (x_1, y, z)	co-ordinates fixed in the obstacle.
(x, y), (x, r), (x, y, z)	co-ordinates fixed in the undisturbed flow behind the shock.
$\bar{x}$	Lorentz variable, (3.3.1).
z'	variable defined by (6.2.3).
α	transform parameter corresponding to y (or r).
β	$= (M'^2 - 1)^{\frac{1}{2}}$.
$\bar{\beta}$	$= (\bar{M}^2 - 1)^{\frac{1}{2}}$.
γ	$= c_p/c_v$, the ratio of specific heats, taken as 1.4 for perfect gas.
δ	delta function.
ε	perturbation parameter; also used as semi-vertex angle for a wedge, half angle for a cone, angle of attack for a flat plate wing.
$\eta(x_1, z)$	function defining the shape of a wing.
θ	$= \arctan (1/n)$, semi-apex angle of a delta wing.
Θ	temperature of the flow field.

λ	$= (s^2 + \alpha^2)^{\frac{1}{2}}$, used in Chapters 3 & 4;
	$= (s^2 + \alpha^2 + \nu^2)^{\frac{1}{2}}$, used in Chapter 5.
λ_o	defined in Eq. (3.3.4); also in §6.4.
λ_1	$= \lambda_o$.
λ_2, λ_3	roots of Eq. (3.4.13); also in §6.4.
λ_4	$= (1 - \bar{b}^2)^{\frac{1}{2}}$.
λ_5	$= - \lambda_4$.
$\Lambda s'$, $\Pi s'$	constants defined in (2.3.10); also in (6.2.15).
μ	angle defined by (6.1.2).
ν	transform parameter, corresponding to z.
ρ_o, ρ_1	density perturbations ahead of and behind the shock.
$\bar{\rho}$, ρ	$\bar{\rho} = \rho_o/R_o$, $\rho = \rho_1/R_1$, dimensionless values.
Σ	projection of the wing surface on x-z plane.
τ	$= a_1 t$, reduced space variable, corresponding to time t.
$\bar{\tau}$	Lorentz variable, (3.3.1).
ϕ	perturbation velocity potential.
ϕ_o, ϕ_1, ϕ_2	angles defined in (3.5.6).
χ	angle of yaw of the aerofoil with respect to the shock plane.
χ'	angle defined by (6.1.1).
ψ	shock displacement.
ψ_t	local shock oscillating velocity.
ψ_r, ψ_y, ψ_z	local shock deflections.
ξ, ζ, μ	variables of integration.

Subscripts

0, 1	conditions ahead of and behind the plane shock, in regions (0) and (1), respectively.
i	= 0, 1, ...5, indices.
t, r, y, z	partial differentiation with respect to t, r, y and z.

$(\underline{variable})$	represents the vector.

Superscripts

$*, **, ***$	denote the transformed function.
$(function)'$	derivatives of a function with respect to its argument.

Chapter 1

INTRODUCTION

1.1. Preliminary remarks

In the present work an attempt is made to deduce under certain conditions a systematic and unified theory for the non-uniform flow field produced behind a plane shock of arbitrary strength when it encounters an aerodynamic obstacle moving at supersonic speed. The supersonic obstacle is supposed to have a weak attached shock and a collision between the two shocks is involved. This has been termed a shock-on-shock or simply shock-shock interaction. The aerodynamic obstacles to be considered are aerofoils, axisymmetric slender bodies and three-dimensional wings. The problem is of some practical importance in connection with blast effects on supersonic aircrafts, from the viewpoint of weapon analysis and the vulnerability of either a missile or a re-entry vehicle to blast.

A theoretical investigation of the flow field produced by the interaction of a plane shock with an obstacle, stationary or moving, is difficult. This is not only because of the non-linear nature of the problem, but also because of the occurence of non-uniform shock waves which imply variations in the entropy of the fluid and a loss of the irrotational nature of the motion. To simplify the problem sufficiently for a theoretical attack to be successful, two possible courses are open. The first is to linearize the basic equations of motion on the assumption that the incident shock is weak. Since the entropy changes across a weak shock are of the third order in the shock strength (cf. Liepmann & Roshko 1965), the entropy variations are also effectively eliminated and, in fact, the problem belongs to the theory of acoustics.

The second possibility is to consider an incident plane shock of arbitrary strength and linearize the basic equations on the assumption that the obstacle produces only small perturbations in the uniform flow behind the shock. Although the entropy variations are no longer negligible, it appears that they can be suppressed from the theoretical investigation by basing the analysis on the pressure variations in the fluid. The present investigation belongs to this category.

1.2. General survey

The earliest work on the diffraction of a plane shock wave was done by Bargmann (1945). He used the pseudo-stationary property to develop a first order solution for a weak shock reflected at a concave corner of small angle. This work assumed the irrotational flow behind the shock. It was followed up theoretically by Lighthill (1949a, 1950) who took into account the vorticity behind the curved shock wave. Lighthill's derivation, free from the restriction to weak shocks, was nevertheless restricted to wedges because an essential element of his method is the cone-field transformation. He reduced the problem to a boundary value problem of Riemann-Hilbert type and solved it by the method of complex variables. A more general approach was developed by Ting & Ludloff (1952), in which hyperbolic equations are used throughout and an arbitrary shape of the aerofoil can be assumed. They obtain the pressure and density field in the entire domain behind the advancing plane shock in explicit analytic form. Fletcher, Taub & Bleakney (1951) and White (1951) presented the results of interferometric experiments on shock wave diffraction and these results compared favourably with the theories of Lighthill and Ting & Ludloff. The experimental data were recorded by Bleakney, White & Griffith (1950).

The considerations of Ting & Ludloff (1952) for the diffraction of a plane shock by a symmetrical aerofoil were later modified by Ludloff & Friedmann (1952) to be applicable to the diffraction of plane shocks by axisymmetric bodies of arbitrary profiles. Chester (1954) extended the problem of Lighthill (1949a) to the case of thin infinite wedges at yaw with respect to the incident plane shock.

Ehler & Shoemaker (1959) solved the problem of the linearized interaction between a weak shock wave meeting a half plane moving subsonically or supersonically at any angle of incidence. In contrast Smyrl (1963) considers the impact of a plane shock of arbitrary strength which encounters a thin two-dimensional wedge and a wedge yawed with respect to the shock plane, moving at supersonic speed. The problem is linearized and the methods of solution are based on those of Lighthill (1949a) and Chester (1954). Smyrl extended the solution to include the case of aerofoils of arbitrary shapes by a superposition of wedges or cone fields. This work was followed up by Blankenship (1965) for shock-shock interaction on slender supersonic cones. Blankenship proceeded on the same lines as Smyrl to formulate the linearized problem, and furnished a numerical solution.

Whitham (1957, 1958, 1959) has developed an approximate theory for the prediction of shock patterns associated with the interaction between a blast wave and two- or three-dimensional bodies. The diffraction pattern predicted by Whitham's theory concerns the shape and the location of the diffracted shock (or Mach shock) at any time. Bryson & Gross (1961) experimentally investigated the diffraction of a plane strong shock by several cones, a cylinder and a sphere. The diffraction pattern, in particular, the shape of the diffracted shock and the loci of the Mach triple points, compared favourably with the theoretical results based on Whitham's theory. Whitham's technique is based on kinematical considerations and does not analyse the pressure distribution. However, Miles (1965) has extended Whitham's ideas to predict the pressure field due to diffraction of a plane shock by a supersonically moving thin wedge, but his results, when compared with those of Smyrl turned out to be qualitatively satisfactory. An excellent review of the related problems of reflection and diffraction of shock wave upto 1963 is contained in a survey article by Pack (1964).

Recently Inger (1966a,b) has studied the blast wave impingement on a slender wedge at hypersonic speed. The analysis proceeds by taking the blast shock to be very weak relative to the original wedge shock, thereby allowing formulation of a small perturbation theory for the transient disturbance field.

In the theoretical analyses of Smyrl and Blanksnship concerned with wedges and cones, advantage is taken of the fact that the flow configuration behind the disturbed shock grows in size proportional with respect to time. Hence no fundamental length or time scale exists. Thus a conical transformation can be employed thereby suppressing one independent variable. Furthermore in both investigations, as a first step to proceed with the analysis, assumptions are introduced for the prediction of the basic flow pattern developed behind the shock. The postulated models, however, are to be substantiated by the visual observations. For example, Smyrl based his prediction of the flow model for two-dimensional wedges on shallow water experiments by Klein (1966). For interaction with the slender supersonic cones, Blankenship adopted the flow model of Smyrl. Later Blankenship & Merrit (1966) supported the hypothesis, made earlier by Blankenship (1965), based on the experimental results (shadowgraphs) obtained by Merrit & Aronson (1966) and Brown & Mullaney (1965). Predicting the flow pattern is thus the first critical step of their analyses.

Another approach requiring little detailed information concerning the flow pattern behind the disturbed shock has been proposed by the author (1968). He obtained a closed form analytic solution for the flow field produced as a result of interaction of a plane shock of arbitrary strength with a supersonically moving axisymmetric slender body, as contrasted with the numerical solution of Blankenship (1965) for a supersonic slender cone. The same considerations were also used to treat the interaction with two-dimensional aerofoils of arbitrary shape, free from the restriction to cone fields. Later the author (1969) extended his theory to cover the shock-shock interaction with three-dimensional planar wings, and also the case of interaction with two-dimensional aerofoils at yaw with respect to the shock plane. It may be pointed out that the author was first to report on the three-dimensional interaction problems. Recently Ting & Gunzburger (1969) also presented the solution of the diffraction of a shock wave by moving thin symmetric wings, using an extension of the approach of Ting & Ludloff

(1952). Their method turns out to be tedious as compared to the approach of the present author. It is the purpose of this dissertation to present the author's investigations in a coherent fashion.

1.3. Statement of the problem and assumptions

We consider a plane shock of arbitrary strength moving freely at supersonic speed V into a gas at rest imparting a uniform velocity U to the fluid behind it, Fig.1a. The density, pressure and sonic velocity ahead of and behind the shock are denoted by R_o, P_o, a_o, and R_1, P_1, a_1. An aerodynamic obstacle of infinite length, having a pointed nose, is moving in the gas at rest in the direction opposite to the shock with a supersonic speed $W > a_o$. At the instant t = 0, the shock front is assumed to coincide with the nose of the body, Fig.1b.

For time $t \leq 0$, there are three flow regimes (0), (1) and (2), Fig.1. In region (0) the gas is at rest, region (1) is that of uniform flow behind the plane shock and region (2) is in general a spatially non-uniform region. The regions (0) and (2) are separated by a weak shock or Mach wave emanating from the nose of the body.

For time t > 0, the body penetrates the plane shock, Fig.2. We intend to obtain the solution for the non-uniform flow field produced as a result of the interaction behind the shock. The only physical parameters which define the problem are V, W, P_o, R_o, and the function defining the shape of the body.

We choose a co-ordinate system (x, y, z) with origin 0 fixed relative to the undisturbed flow behind the plane shock such that at t = 0 the shock front just coincides with the nose of the body. The x-axis is taken along the streamwise direction, the y-axis in the vertical direction and the z-axis in the spanwise direction. The corresponding co-ordinates for the axisymmetric flow will be (x, r), where r denotes the radial direction.

The incident plane shock may be regarded as a moving surface of zero thickness, which is then simply a mathematical discontinuity (cf. Courant & Friedrich 1948). The gas on either side of the shock, being outside the shock, can be considered devoid of viscosity and heat conduction.

We also assume the perfect gas 'equation of state' to hold on either side of the shock.

The shape of the solid boundaries of the moving obstacle should be such that the inclination of the surface to the stream direction is everywhere small, so that the perturbations introduced into an otherwise uniform flow can be expected to be small.

The plane shock in traversing over the body and its associated field of region (2) gets diffracted, i.e. a non-uniform shock IBF (Fig.2) results due to small disturbances in the speed and the shape of the initially plane shock. The diffracted shock is assumed to meet the body surface normally to ensure that the flow remains tangential to the body surface across the shock.

The incident shock does not alter the disturbance field of region (2) ahead of it, since it is moving supersonically ($V > a_o$). The disturbances in region (2), assumed weak, are deduced on the assumption that the changes in the state of the gas are not only adiabatic but isentropic too.

The flow in the interaction region behind the non-uniform shock will be in general non-isentropic and rotational. The diffracted shock is assumed to be slightly disturbed from the undisturbed location of the plane shock. Hence the downstream perturbations in the interaction region will be small compared with the undisturbed flow of region (1). The linear treatment of the flow field based on region (1) is then permissible.

Consistent with the linearized theory, the conditions at the disturbed shock can be applied at the location of the undisturbed plane shock. For aerofoils and wings the boundary condition on the surface of the body may be applied at its projection on the x-z plane, i.e. at y = 0. For axisymmetric slender bodies the body axis lies along the x-axis and the boundary condition is specified in the vicinity of the body axis.

Flow picture due to interaction

In the following we shall discuss the salient features of the diffraction pattern developed behind the shock, say, for interaction with an axisymmetric slender body of infinite length, Fig.2.

For time t > 0, in the co-ordinate system chosen, the body is travelling in region (1) with, in general, supersonic speed $(W + U) > a_1$ in the direction of negative x-axis, starting from 0 at time t = 0. The flow pattern generated by the travelling body is in the present approximation a succession of spherical acoustic waves, giving rise to the spherical wavelet BCDE and the envelop AC of the spherical wavelets. Thus AC forms an attached Mach wave emanating from the nose of the body.

The plane shock travels to the right, with speed (V - U) for the observer fixed in 0, and enters the region (0) and (2). The gas in region (0) then has velocity U to the left, while in region (2) this speed is superimposed upon the perturbation field. In the disturbed region behind the shock the air enters from the right across the diffracted shock IBF.

Due to the disturbances in the shock by the presence of the body, the flow behind the shock will be non-isentropic, rotational flow giving rise to pressure-density and entropy-density perturbations. In addition the perturbations in region (2) which are essentially pressure-density perturbations, on passing through the shock will also give rise to pressure-density and entropy-density perturbations. The pressure perturbations give rise to sound waves while the entropy perturbations are attached to the fluid elements. The entropy perturbations due to the body effect generated at the shock are to be found roughly in the region BOF, while the fluid elements of region (2) which have passed through the shock are found roughly in the region IOF. In the region IOF there are then entropy-density variations in addition to pressure-density variations. The pressure perturbations emerging from the shock IF give rise to spherical acoustic waves. The envelop of these spherical wavelets is another Mach wave ID from the shock intersection I and tangent to the wavelet BCDE.

A contact discontinuity appears in the approximate position IO, which essentially divides the air into two non-mixing regions (3) and (4), thus separating a flow which comes from region (2) from that which initially comes from region (0). The type of three shock intersection that occurs at B is similar to the well known experimental phenomenon of Mach-reflection with the presence of three shock configuration (Mach shock BF, deflected shock BI, and the reflected shock BC) accompanied by a contact discontinuity surface (also called a slip stream) along the approximate position BO. Across a contact discontinuity there is no flow of fluid; the pressure is continuous there, though the tangential component of velocity, as well as the density, temperature and entropy, or their derivatives may in general be discontinuous.

The features of the flow field discussed above remain essentially the same for interaction with two-dimensional aerofoils. For three-dimensional wings they have to be visualized in three-dimensions. However, it may be pointed out here that the approach developed in this work does not depend much on this flow picture. It is far more that by the method employed the flow pattern can be seen to emerge from the solution of the problem.

1.4. Some remarks on formulation and solution

The general shock-shock interaction problem is posed in terms of initial- and boundary-values for the time-dependent pressure perturbations. The boundary conditions on the velocity components or pressure are required at the solid boundaries of the downstream flow regions. Additional boundary conditions are formulated at the approximate location of the shock. The formulation is essentially similar to that of Ting & Ludloff (1952) and Ludloff & Friedmann (1952) for the diffraction of plane shocks by stationary bodies. The solution is sought by the application of integral transforms and is shown to lead to the various field representations valid for the different regions. The solution is also used to describe the entropy and density field, and the shape of the shock front.

The same considerations hold for studying the shock-shock interaction with two-dimensional aerofoils, axisymetric slender bodies and three-dimensional planar wings. For the interaction with thin yawed aerofoils a slightly different point of view is employed, since in such a case we cannot indicate a moment when the interaction begins; the incident shock is being disturbed at all times by the aerofoil and its associated field. By considering the flow in a suitable reference frame the time may be eliminated entirely from the problem, while the method of solution is the same as before.

The literature of relevance to the application of integral transforms to the boundary value problem is rather extensive, and space will not permit even a short review of it. It might be recorded, however, that the integral transforms have been used by Gunn (1947) and Miles (1948) for the solution of steady supersonic flows over planar wings, and by Stewartson (1950) and Temple (1953) for unsteady supersonic motion, among others. Werner (1961) has treated the interaction of a plane shock with a cellular vortex field by the use of Laplace transforms.

1.5. Outline of the thesis

We may close this chapter with some remarks on the order of material presented in the subsequent chapters.

In Chapter 2, firstly the linearization of the equation of motion of unsteady non-isentropic flow is carried out. Secondly, the shock relations are deduced connecting the unknown downstream perturbations with the known upstream perturbations at the disturbed shock. Finally, the expressions for the upstream disturbance are presented.

In Chapters 3, 4, 5, and 6 the general theory of shock-shock interaction is deduced for thin two-dimensional aerofoils, axisymmetric slender bodies, three-dimensional thin wings, and aerofoils at yaw with respect to the incident shock. As detailed examples the results of the theory are applied to the interaction with a thin wedge, a slender cone, a flat plate delta wing with supersonic leading edges, and a thin wedge at yaw.

In the final Chapter 7 the various numerical results are presented for the examples considered in the preceding chapters, and a discussion follows.

Chapter 2

FUNDAMENTAL EQUATIONS AND RELATIONS

2.1. Introductory remarks

In this chapter the fundamental equations and other relations are presented which form the basis of the analysis to be presented in the subsequent chapters.

In §2.2 the general equations of unsteady inviscid motion are given, which hold in the flow field behind the disturbed shock. They are then linearized assuming the perturbations to be small compared to the parameters which determine the uniform state of flow behind the plane unperturbed shock. These equations of motion are linear equations in the perturbation pressure, velocities and density. Elimination of all variables except pressure results in a wave equation. Physically this equation implies that all pressure variations propagate with the mean speed of sound. In addition they satisfy the conditions at the shock front and on the body downstream.

In §2.3 the shock relations across a plane shock, propagating without any disturbances, are given first. Then the Rankine-Hugoniot equations are considered to derive the conditions on the disturbed shock front, which relate the unknown downstream perturbations to the known upstream perturbations and the shock displacement. The procedure has been carried out by Moore (1953) for two-dimensional interaction with a plane shock and by Chang (1957) for the case of interaction with an oblique shock.

In §2.4 the results of the usual linearized theory are presented for the supersonic motion of two-dimensional aerofoils, axisymmetric slender bodies and three-dimensional planar wings in the gas at rest. These perturbations are thus considered known and are prescribed ahead of the shock front in its motion on the obstacle.

Finally in §2.5 the Mach number of the obstacle behind the shock is discussed.

2.2. Governing equations

2.2.1. Equations of motion

The unsteady flow of an ideal gas is governed by the equations

Cons. of mass $$\partial R/\partial t + \nabla.(R\underline{V}) = 0, \tag{2.2.1}$$

Cons. of momentum $$\partial \underline{V}/\partial t + (\underline{V}.\nabla)\underline{V} = -(1/R)\nabla P, \tag{2.2.2}$$

Cons. of energy $$c_v R\{\partial \Theta/\partial t + (\underline{V}.\nabla)\Theta\} = -P(\nabla.\underline{V}), \tag{2.2.3}$$

Eq. of state $$P = JR\Theta, \tag{2.2.4}$$

where R, P, Θ, $\underline{V}$ represent the density, pressure, temperature and the velocity vector of the flow, and J the specific gas constant. Denoting the ratio of specific heats by $\gamma(= c_p/c_v)$, Eqs. (2.2.1), (2.2.3) and (2.2.4) can be combined to yield the adiabatic relation

$$(\partial/\partial t + \underline{V}.\nabla)(P/R^\gamma) = 0. \tag{2.2.5}$$

This equation states essentially that (P/R^γ) is constant for a fluid element and can serve to replace Eqs. (2.2.3) and (2.2.4) from which it is derived. For processes where the entropy is not same throughout, it is convenient to relate Eq. (2.2.5) to the entropy of an element by the second law of thermodynamics, which can be expressed as

$$(\partial/\partial t + \underline{V}.\nabla)S = 0, \tag{2.2.6}$$

with S denoting the specific entropy. It is then easily found by integrating (2.2.5) that for a fluid element

$$P/R^\gamma = \exp\{(S - S_1)/c_v\}, \tag{2.2.7}$$

where the subscript 1 represents some reference state which can be chosen later.

2.2.2. Disturbance field behind the shock

The flow field behind the disturbed shock is non-isentropic in general. This field may be treated as a time-dependent small disturbance from the state of relative rest of region (1), Fig. 2. The basic system is then the pressure field due to the undisturbed plane shock. Upon this field a perturbation produced by the presence of the obstacle and by the small changing diffraction is superposed. The equations of motion of §2.2.1 can be linearized based upon an expansion in terms of a small parameter ε, which can be interpreted as the maximum slope (small compared with unity) of the wing surface. We assume an expansion in ε for the non-uniform flow region behind the shcock as

$$\left.\begin{aligned} R &= R_1 + \varepsilon\rho_1(x, y, z, t) + O(\varepsilon^2) + \ldots, \\ P &= P_1 + \varepsilon p_1(x, u, z, t) + O(\varepsilon^2) + \ldots, \\ S &= S_1 + \varepsilon s_1(x, y, z, t) + O(\varepsilon^2) + \ldots, \\ \underline{V} &= \varepsilon \underline{v}_1(x, y, z, t) + O(\varepsilon^2) + \ldots, \end{aligned}\right\} \quad (2.2.8)$$

noting that the velocity of the undisturbed flow in region (1) is zero for the chosen co-ordinate system. Here S_1 is the specific entropy of the gas in region (1); $\underline{v}_1 = \underline{i}\, u_1 + \underline{j}\, v_1 + \underline{k}\, w_1$, the vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ are the unit vectors in the x, y and z directions; ρ_1, p_1, s_1, (u_1, v_1, w_1) are the perturbation values of the density, pressure, entropy and velocities for the disturbed flow. The expansions (2.2.8) are inserted in the Eqs. (2.2.1), (2.2.2), (2.2.5), (2.2.6) and (2.2.7). Equating the coefficients of like powers in ε we obtain in first approximation

$$\partial\rho_1/\partial t + R_1(\partial u_1/\partial x + \partial v_1/\partial y + \partial w_1/\partial z) = 0, \quad (2.2.9)$$

$$\begin{gathered} \partial u_1/\partial t = -(1/R_1)\,\partial p_1/\partial x, \quad \partial v_1/\partial t = -(1/R_1)\,\partial p_1/\partial y, \\ \partial w_1/\partial t = -(1/R_1)\,\partial p_1/\partial z, \end{gathered} \quad (2.2.10)$$

$$\partial p_1/\partial t = a_1^2\,\partial\rho_1/\partial t, \quad (2.2.11)$$

$$\partial s_1/\partial t = 0, \quad (2.2.12)$$

and

$$s_1 = c_v(p_1/P_1 - \gamma\rho_1/R_1), \quad (2.2.13)$$

where $a_1 = (\gamma P_1/R_1)^{\frac{1}{2}}$, The velocity of sound in region (1). From Eqs. (2.2.10) we can also obtain

$$\partial\underline{\omega}/\partial t = 0, \qquad (2.2.14)$$

where $\underline{\omega}$ is the perturbed vorticity vector whose components are $(\partial w_1/\partial y - \partial v_1/\partial z)$, $(\partial u_1/\partial z - \partial w_1/\partial x)$ and $(\partial v_1/\partial x - \partial u_1/\partial y)$.

From Eqs. (2.2.12) and (2.2.14) it is clear that any variation in entropy or vorticity at a point will remain constant in time.

The perturbation quantities can be expressed in dimensionless form as follows

$$\rho_1/R_1 = \rho, \quad p_1/\gamma P_1 = p, \quad s_1/c_p = s, \quad \underline{v}_1/a_1 = \underline{v}, \qquad (2.2.15)$$

where $\underline{v} = \underline{i}\, u + \underline{j}\, v + \underline{k}\, w$. We can also replace the time variable t by a reduced space variable $\tau = a_1 t$. Using the dimensionless parameters, the Eqs. (2.2.9)-(2.2.11) and (2.2.13) can be written as

$$\partial\rho/\partial\tau + \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0, \qquad (2.2.16)$$

$$\partial u/\partial\tau = -\,\partial p/\partial x, \quad \partial v/\partial\tau = -\,\partial p/\partial y, \quad \partial w/\partial\tau = -\,\partial p/\partial z, \qquad (2.2.17)$$

$$\partial p/\partial\tau = \partial\rho/\partial\tau, \qquad (2.2.18)$$

and

$$s = p - \rho. \qquad (2.2.19)$$

Axisymmetric flow: For interaction with a slender body of revolution, the disturbed field behind the shock can be regarded axially symmetric. The flow variables are then functions of (x, r, t), r being the radial direction. For the disturbed field we can assume that there is an expansion beginning with a term of order zero followed by a term proportional to ε^2, where ε can be interpreted as fineness ratio (small compared with unity) of the body. Thus we write

$$\left.\begin{aligned} P &= P_1 + \varepsilon^2\, p_1(x, r, t) + \ldots, \\ R &= R_1 + \varepsilon^2\, \rho_1(x, r, t) + \ldots, \\ \underline{V} &= \varepsilon^2\, \underline{q}_1(x, r, t) + \ldots, \end{aligned}\right\} \qquad (2.2.20)$$

in a co-ordinate system at rest relative to the uniform flow of region (1). Here $\underline{q}_1 = \underline{i}\, u_1 + \underline{j}\, q_1$, u_1 and q_1 being the axial and radial perturbed velocities. It is far from obvious which power of ε correponds to the next term. References to such terms of higher order can be found in the literature (cf. e.g. Lighthill 1954). Based on expansion (2.2.20) we linearize the equations of motion (2.2.1), (2.2.2) and (2.2.5). Expressing the disturbance parameters in dimensionless form we obtain

$$\partial\rho/\partial\tau + \partial u/\partial x + (\partial q/\partial r + q/r) = 0, \tag{2.2.21}$$

$$\partial u/\partial\tau = - \partial p/\partial x, \quad \partial q/\partial\tau = - \partial p/\partial r, \tag{2.2.22}$$

together with

$$\partial p/\partial\tau = \partial\rho/\partial\tau, \tag{2.2.23}$$

where $q = q_1/a_1$, $u = u_1/a_1$, and ρ, p are defined in (2.2.15).

It is noted that the Eqs. (2.2.16) through (2.2.18) can be combined as to eliminate ρ, u, v and w, and thus obtain the following equation for p alone

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - \frac{\partial^2 p}{\partial\tau^2} = 0, \tag{2.2.24}$$

a wave equation. Similarly for axisymmetric case, the Eqs. (2.2.21)-(2.2.23) can be combined to yield the wave equation for p

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} - \frac{\partial^2 p}{\partial\tau^2} = 0. \tag{2.2.25}$$

The Eq. (2.2.24) or (2.2.25) is not affected by the fact that the entropy perturbations are present due to the disturbed shock, as already pointed out in chapter 1.

The Eqs. (2.2.24) and (2.2.25) for the peerturbation pressure p will be solved for shock-shock interaction with thin aerofoils and wings, and with slender bodies of revolution, subject to two initial conditions and the appropriate boundary conditions. The boundary conditions to be considered are on the body surface and on the disturbed shock front. In what follows we shall discuss the conditions on the shock.

2.3. Shock relations

2.3.1. Undisturbed shock propagation

First we consider that the plane shock propagates without encountering any disturbances. The Rankine-Hugoniot shock wave equations, which are the statement of conservation of mass, momentum and energy applied across the plane shock together with the equation of state, give (cf. Fig. 1a)

$$\left.\begin{aligned} R_1(V - U) &= R_o V, \\ P_1 - P_o &= R_o VU, \\ \tfrac{1}{2}(V - U)^2 + \frac{\gamma}{\gamma - 1}(P_1/R_1) &= \tfrac{1}{2}V^2 + \frac{\gamma}{\gamma - 1}(P_o/R_o), \end{aligned}\right\} \qquad (2.3.1)$$

for an observer fixed in the shock plane. The subscripts 0 and 1 pertain to the flow values in region (0) and region (1). Eqs. (2.3.1) solved for U, P_1 and R_1 give

$$\left.\begin{aligned} U &= \frac{2}{\gamma + 1}(1 - a_o^2/V^2), \\ P_1 &= \frac{2}{\gamma + 1}R_o(V^2 - \frac{\gamma - 1}{2\gamma}a_o^2), \\ R_1 &= \frac{\gamma - 1}{\gamma + 1}R_o/(1 + \frac{2}{\gamma - 1}a_o^2/V^2), \end{aligned}\right\} \qquad (2.3.2)$$

where $a_o = (\gamma P_o/R_o)^{\frac{1}{2}}$ is the velocity of sound in region (0). Writing $M = V/a_o > 1$ for the shock Mach number and $M_1 = U/a_1$ for the Mach number of the uniform flow behind the shock, we can obtain from (2.3.2)

$$\left.\begin{aligned} P_1/P_o &= \{2\gamma M^2 - (\gamma - 1)\}/(\gamma + 1), \\ R_1/R_o &= (\gamma + 1)M^2/\{(\gamma - 1)M^2 + 2\}, \\ a_1/a_o &= [\{2\gamma M^2 - (\gamma - 1)\}\{(\gamma - 1)M^2 + 2\}]^{\frac{1}{2}}/\{(\gamma + 1)M\}, \\ M_1 &= 2(M^2 - 1)/[\{2\gamma M^2 - (\gamma - 1)\}\{(\gamma - 1)M^2 + 2\}]^{\frac{1}{2}}. \end{aligned}\right\} \qquad (2.3.3)$$

2.3.2. Disturbed shock front

The plane shock in traversing over the obstacle and its associated disturbance field (due to the obstacle velocity $W > a_o$) denoted by the perturbation parameters ρ_o, p_o, (u_o, v_o, w_o), undergoes a small unsteady displacement. Let this displacement in x-direction be denoted by $\psi(y, z, t)$, and is assumed having the same order of magnitude as the displacement of the obstacle from the mean plane and also the disturbances ahead of the shock, i.e. of $O(\varepsilon)$ for thin wings. Hence the position of the disturbed shock at any instant may be expressed as

$$x = (V - U)t + \psi(y, z, t) + O(\varepsilon^2) + \dots, \qquad (2.3.4)$$

in the co-ordinate system chosen behind the shock. Due to the unsteady displacement the incident shock moves no longer in a uniform fashion. It is well known that the Rankine-Hugoniot conditions, nevertheless, apply provided the velocity components are taken relative to the unsteady shock. A second point to notice is that the dependence of ψ on y and z causes the shock to deviate from its position in a plane perpendicular to the x-axis. The shock in fact becomes locally oblique shock, and in addition to the relations for the normal shock the continuity of the velocity components tangential to the shock has tobe invoked. Since the deviations from the normal shock are only small we can write for the shock relations of the perturbed shock (cf. Fig. 3)

$$\left.\begin{aligned}
&(R_1 + \rho_1)(V + \psi_t - U - u_1) = (R_o + \rho_o)(V + \psi_t - u_o),\\
&(P_1 + p_1) - (P_o + p_o) = (R_o + \rho_o)(V + \psi_t - u_o)(U + u_1 - u_o),\\
&\tfrac{1}{2}\{(V + \psi_t - U - u_1)^2 + v_1^2 + w_1^2\} + \frac{\gamma}{\gamma - 1}(P_1 + p_1)/(R_1 + \rho_1)\\
&\qquad = \tfrac{1}{2}\{(V + \psi_t - u_o)^2 + v_o^2 + w_o^2\} + \frac{\gamma}{\gamma - 1}(P_o + p_o)/(R_o + \rho_o),
\end{aligned}\right\} \qquad (2.3.5)$$

The normal to the shock wave is no longer the x-axis, but is determined by its direction cosines which are proportional to $(-1, \psi_y, \psi_z)$. If $\underline{n}$ is a unit vector normal to the shock and $\underline{v}_L$ and $\underline{v}_R$ denote the velocity vectors on the left and right side of the shock, the continuity of the tangential components of velocity across the shock can be expressed as

$$\underline{n} \times \underline{v}_L = \underline{n} \times \underline{v}_R.$$

This leads to

$$\begin{vmatrix} \underline{i} & -1 & -(V + \psi_t - U - u_1) \\ \underline{j} & \psi_y & v_1 \\ \underline{k} & \psi_z & w_1 \end{vmatrix} = \begin{vmatrix} \underline{i} & -1 & -(V + \psi_t - u_o) \\ \underline{j} & \psi_y & v_o \\ \underline{k} & \psi_z & w_o \end{vmatrix},$$

which on expansion gives the relations

$$\left.\begin{aligned} w_1\psi_y - v_1\psi_z &= w_o\psi_y - v_o\psi_z, \qquad O(\varepsilon^2) \\ v_1 - (V + \psi_t - U - u_1)\psi_y &= v_o - (V + \psi_t - u_o)\psi_y, \\ w_1 - (V + \psi_t - U - u_1)\psi_z &= w_o - (V + \psi_t - u_o)\psi_z. \end{aligned}\right\} \qquad (2.3.6)$$

In the Eqs. (2.3.5) and (2.3.6), the terms of order ε^o are the customary Rankine-Hugoniot relations given by (2.3.1). Retaining quantities of first order, we obtain the conditions which the disturbance field must satisfy at the disturbed shock.

$$\left.\begin{aligned} &\rho_1(V - U) + R_1(\psi_t - u_1) = \rho_o V + R_o(\psi_t - u_o), \\ &p_1 - p_2 = \rho_o VU + R_o(\psi_t - u_o)U + R_o V(u_1 - u_o), \\ &(V - U)(\psi_t - u_1) + \frac{\gamma}{\gamma - 1}(P_1/R_1)(1 + p_1/P_1 - \rho_1/R_1), \\ &\quad = V(\psi_t - u_o) + \frac{\gamma}{\gamma - 1}(P_o/R_o)(1 + p_o/P_o - \rho_o/R_o), \\ &\qquad v_1 - v_o = - U\psi_y, \\ &\qquad w_1 - w_o = - U\psi_z. \end{aligned}\right\} \qquad (2.3.7)$$

Here we may note that to the first approximation the shock displacement derivatives ψ_y and ψ_z remain uncoupled and so also the velocities v_1 and w_1. Simplifying the set (2.3.7) on the assumption that the flow ahead of the shock in region (2) is isentropic, i.e. $p_o/P_o = \gamma\rho_o/R_o$, one obtains for the disturbed shock relations

$$\left.\begin{aligned} (\psi_t - u_1)/V &= B_1(\psi_t - u_o)/V + B_2\, p_o/P_o, \\ \rho_1/R_1 &= C_1(\psi_t - u_o)/V + C_2\, p_o/P_o, \\ p_1/P_1 &= D_1(\psi_t - u_o)/V + D_2\, p_o/P_o, \\ v_1/V &= v_o/V - (U/V)\psi_y, \\ w_1/V &= w_o/V - (U/V)\psi_z, \end{aligned}\right\} \qquad (2.3.8)$$

where the coefficients are constants depending upon the Mach number of the undisturbed shock, M:

$$B_1 = \frac{(\gamma - 1) - 2/M^2}{(\gamma + 1)}, \qquad B_2 = \frac{2(\gamma - 1)}{\gamma(\gamma + 1)M^2},$$

$$C_1 = \frac{4}{(\gamma - 1)M^2 + 2}, \qquad C_2 = \frac{1}{\gamma}\left(1 - \frac{2(\gamma - 1)}{(\gamma - 1)M^2 + 2}\right),$$

$$D_1 = \frac{4\gamma M^2}{2\gamma M^2 - (\gamma - 1)}, \qquad D_2 = \frac{2M^2 - (\gamma - 1)}{2\gamma M^2 - (\gamma - 1)}.$$

The perturbation parameters behind the shock may be made dimenionless as in (2.2.15). For the parameters ahead of the shock we write

$$p_o/\gamma P_o = \bar{p}, \quad u_o/V = \bar{u}, \quad v_o/V = \bar{v}, \quad w_o/V = \bar{w}. \tag{2.3.9}$$

The shock relations (2.3.8) may then be put in the form

$$\rho = \Lambda_{11}\,\bar{u} + \Lambda_{12}\,\bar{p} + \Pi_{11}\,\psi_\tau, \tag{2.3.10a}$$

$$p = \Lambda_{21}\,\bar{u} + \Lambda_{22}\,\bar{p} + \Pi_{21}\,\psi_\tau, \tag{2.3.10b}$$

$$u = \Lambda_{31}\,\bar{u} + \Lambda_{32}\,\bar{p} + \Pi_{31}\,\psi_\tau, \tag{2.3.10c}$$

$$v = \Lambda_{41}\,\bar{v} + \Pi_{41}\,\psi_y, \tag{2.3.10d}$$

$$w = \Lambda_{41}\,\bar{w} + \Pi_{41}\,\psi_z, \tag{2.3.10e}$$

where

$$\Lambda_{11} = \frac{-4}{2 + (\gamma - 1)M^2}, \qquad \Lambda_{12} = 1 - \frac{2(\gamma - 1)}{2 + (\gamma - 1)M^2},$$

$$\Lambda_{21} = \frac{-4M^2}{2\gamma M^2 - (\gamma - 1)}, \qquad \Lambda_{22} = \frac{2M^2 - (\gamma - 1)}{2\gamma M^2 - (\gamma - 1)},$$

$$\Lambda_{31} = \frac{(\gamma - 1)M^2 - 2}{(\gamma + 1)M}(a_o/a_1), \qquad \Lambda_{32} = \frac{-2(\gamma - 1)}{(\gamma + 1)M}(a_o/a_1),$$

$$\Lambda_{41} = M\,a_o/a_1,$$

$$\Pi_{11} = \frac{4\,a_o/a_1}{M\{2 + (\gamma - 1)M^2\}}, \qquad \Pi_{31} = \frac{2(1 - 1/M^2)}{(\gamma + 1)},$$

$$\Pi_{21} = \frac{4M\,a_o/a_1}{2\gamma M^2 - (\gamma - 1)}, \qquad \Pi_{41} = -\,M_1.$$

For interaction with slender bodies of revolution, we denote the unsteady displacement in the incident shock by $\psi(r, t)$. It will be assumed of the same order of magnitude as the axisymmetric perturbations which are now functions of (x, r, t). By following the similar procedure as in the above discussion, we arrive at the same relations as obtained in (2.3.10) except that the last two of relations (2.3.10) are replaced by a single relation.

$$q = \Lambda_{41}\,\bar{q} + \Pi_{41}\,\psi_r. \tag{2.3.10f}$$

where $q = q_1/a_1$, $\bar{q} = q_o/V$, q_o and q_1 being the radial components of the disturbance velocities ahead of and behind the shock.

Along with the upstream perturbations (to be specified), the shock relations (2.3.10) also include the derivatives of the shock displacement. The displacement is determined only after the solution of the flow problem. However we may notice that ψ_τ can be eliminated from (2.3.10b) and (2.3.10c), while ψ_τ, ψ_y and ψ_z can be eliminated from (2.3.10b), (2.3.10d) and (2.3.10e) alternatively by cross-differentiation. Thus at the disturbed shock we obtain

$$u = \frac{1}{A}\,(p - B\,\bar{u} - C\,\bar{p}), \tag{2.3.11a}$$

$$\frac{\partial v}{\partial \tau} = \frac{1}{D}\,\left(\frac{\partial p}{\partial y} - \Lambda_{21}\,\frac{\partial \bar{u}}{\partial y} - \Lambda_{22}\,\frac{\partial \bar{p}}{\partial y}\right) + \Lambda_{41}\,\frac{\partial \bar{v}}{\partial \tau}, \tag{2.3.11b}$$

$$\frac{\partial w}{\partial \tau} = \frac{1}{D}\,\left(\frac{\partial p}{\partial z} - \Lambda_{21}\,\frac{\partial \bar{u}}{\partial z} - \Lambda_{22}\,\frac{\partial \bar{p}}{\partial z}\right) + \Lambda_{41}\,\frac{\partial \bar{w}}{\partial \tau}, \tag{2.3.11c}$$

where $A = \Pi_{21}/\Pi_{31}$, $B = \Lambda_{21} - A\,\Lambda_{31}$, $C = \Lambda_{22} - A\,\Lambda_{32}$, $D = \Pi_{21}/\Pi_{41}$.

For interaction with the axisymmetric slender bodies, by virtue of relations (2.3.10b) and (2.3.10f), we obtain

$$\frac{\partial q}{\partial \tau} = \frac{1}{D}\,\left(\frac{\partial p}{\partial r} - \Lambda_{21}\,\frac{\partial \bar{u}}{\partial r} - \Lambda_{22}\,\frac{\partial \bar{p}}{\partial r}\right) + \Lambda_{41}\,\frac{\partial \bar{q}}{\partial \tau}, \tag{2.3.11d}$$

which will replace the conditions (2.3.11b) and (2.3.11c). It must be noted that in Eqs. (2.3.11) the derivatives with τ are taken while travelling with the shock.

Also from the shock relations (2.3.10a) and (2.3.10b), we can eliminate ψ_τ and thus obtain at the disturbed shock

$$\rho = E\,p + F\,\bar{u} + G\,\bar{p}, \tag{2.3.12}$$

where $E = \Pi_{11}/\Pi_{21}$, $F = \Lambda_{11} - E\,\Lambda_{21}$ and $G = \Lambda_{12} - E\,\Lambda_{22}$.

It may be emphasised here that consistent with the linearization the conditions at the shock will be applied at its undisturbed location, i.e. at $x = m\tau$, where

$$m = (V - U)/a_1 = M(a_o/a_1) - M_1.$$

Making use of the plane shock relations (2.3.4) we obtain

$$m = \{(\gamma - 1)M^2 + 2\}^{\frac{1}{2}}/\{2\gamma M^2 - (\gamma - 1)\}^{\frac{1}{2}}, \qquad (2.3.13)$$

a function of M.

The conditions (2.3.11) will be used later in conjunction with the equations of motion to obtain a single boundary condition at the shock $(x = m\tau)$, for p. The condition (2.3.12) will be used for determining the density field, after the solution of the pressure field is completed.

2.4. Specification of upstream disturbance

The perturbations in region (2) are due to the obstacle moving with constant supersonic speed W in the medium at rest, i.e. region (0) denoted by the state R_o, P_o and a_o (cf. Fig. 1). Hence the time-independent solution can be obtained for region (2) by the usual linearized potential theory. Since the incident shock does not affect the flow ahead of it, the region (2) can be considered merely truncated (cf. Fig. 2).

The disturbance field ahead of the shock is, in fact, governed by the Eqs. (2.2.9)-(2.2.13), except that now the state of region (1) is replaced by the state of region (0) and the perturbation quantities are designated by the subscript 0. On the assumption that this flow is isentropic, the Eqs. (2.2.9)-(2.2.13) could be expressed as

$$\left.\begin{aligned} \partial \underline{v}_o/\partial t' &= -\nabla(p_o/R_o), \\ (1/a_o^2)\,\partial p_o/\partial t' + R_o\,\nabla.\underline{v}_o &= 0, \end{aligned}\right\} \qquad (2.4.1)$$

in a co-ordinate system (x', y, z, t') at rest relative to the fluid in region (0); $\underline{v}_o = \underline{i}\,u_o + \underline{j}\,v_o + \underline{k}\,w_o$ is the perturbation velocity in region (2). In view of the initial condition of uniform rest, the first of Eqs. (2.4.1) implies that $\underline{v}_o$ is the gradient of a scalar function, i.e. a potential ϕ exists. The Eqs. (2.4.1) can then be expressed as

$$\left.\begin{aligned} R_o\ \partial\phi/\partial t' &= -\ p_o, \\ \partial^2\phi/\partial t'^2 &= a_o^2\ \nabla^2\phi. \end{aligned}\right\} \tag{2.4.2}$$

Now it is convenient to use a system of axes fixed in the body. The fluid of region (0) then approaches the body from the left with a uniform speed W. The transformation to this system of axes, assuming that the origins coincide at time $t' = 0$, is formally achieved by writing

$$x' = x_1 - Wt',$$

if the new co-ordinates fixed in the body are (x_1, y, z). Thus the Eqs. (2.4.2) become

$$\left.\begin{aligned} & R_o(\partial\phi/\partial t' + W\ \partial\phi/\partial x_1) = -\ p_o, \\ & \frac{\partial^2\phi}{\partial t'^2} + 2W\frac{\partial^2\phi}{\partial x_1 \partial t'} + W^2\frac{\partial^2\phi}{\partial x_1^2} = a_o^2\ \nabla_1^2\phi, \end{aligned}\right\} \tag{2.4.3}$$

where ∇_1^2 involves derivatives with respect to x_1, y, z. In this system the flow pattern in region (2) is steady and the equations (2.4.3) reduce to

$$p_o = -\ R_o\ W\ \partial\phi/\partial x_1, \tag{2.4.4}$$

$$-(W^2/a_o^2 - 1)\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{2.4.5}$$

the well-known Prandtl-Glauert equations of linearized theory.

For axisymmetric flow around slender bodies of revolution, the perturbation velocity potential $\phi(x_1, r)$ in region (2) will be of $O(\varepsilon^2)$. The linearized equation which governs the flow is now

$$-(W^2/a_o^2 - 1)\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0, \tag{2.4.6}$$

The solution of the Eqs. (2.4.5) and (2.4.6) together with the appropriate boundary conditions at the surface of the obstacle forms the subject of the usual linearized theory. In the sequel we shall present the results of the linearized theory for the disturbances in region (2) for thin two-dimensional aerofoils, axisymmetric slender bodies and three-dimensional thin wings. These will be used for the known perturbations ahead of the shock, in the shock relations deduced in § 2.3.2.

Two-dimensional aerofoils

Let the upper surface of an aerofoil be

$$y = f(x_1), \quad \text{for } x_1 > 0; \quad \text{with } f(x_1) = 0, \quad \text{for } x_1 \leqslant 0,$$

in a co-ordinate system fixed at its leading edge. If the aerofoil is moving with supersonic speed $W > a_o$, the disturbances are described by two-dimensional potential in the upper half plane ($y > 0$)

$$\phi(x_1, y) = \begin{cases} -(W/\beta)\, f(x_1 - \beta y), & x_1 - \beta y > 0 \\ 0, & x_1 - \beta y \leqslant 0. \end{cases} \qquad (2.4.7)$$

where $\beta = (M'^2 - 1)^{\frac{1}{2}}$, and $M' = W/a_o$ the Mach number of the aerofoil. The potential in the lower half plane ($y < 0$) can also be determined for the given lower surface of the aerofoil, since the flow on the two sides of the aerofoil is independent of each other. In the above we may also assume that $\partial f(x_1)/\partial x_1$ is continuous and has continuous derivatives.

From (2.4.7) the perturbation velocity components can be written as

$$\left.\begin{aligned} u_o &= \partial\phi/\partial x_1 = -(W/\beta)\, f'(x_1 - \beta y), \\ v_o &= \partial\phi/\partial y = W\, f'(x_1 - \beta y), \end{aligned}\right\} \qquad (2.4.8)$$

where the prime on the function represents differentiation with respect to the argument. For the perturbation pressure we use the approximation

$$p_o = -R_o\, W\, u_o.$$

Axisymmetric slender bodies

We restrict the analysis to smooth slender bodies. Let the body of revolution be defined by

$$r = f(x_1), \qquad \text{for } x_1 > 0,$$

whose pointed nose is at $x_1 = 0$. We assume that $f'(x_1)$ is continuous and has continuous derivatives. Thus not only must the slope of the surface to the incident stream be small, but changes in it must be spread evenly over the length of the body, so that the curvature too is consistently small. Similarly the curvature must vary only gradually. These conditions ensure that the velocity due to disturbance caused in the uniform stream by the presence of the body is everywhere small compared with the velocity of the main stream. Under these rather stringent conditions of smoothness the boundary condition on the surface

$$q_o/(W + u_o) = f'(x_1),$$

which may be approximated as $q_o \doteqdot W f'(x_1)$ gives by a further approximation in the limit

$$\text{as } r \to 0, \quad r\, q_o \doteqdot W f(x_1)\, f'(x_1). \tag{2.4.9}$$

The disturbance potential $\phi(x_1, r)$ for the supersonic flow over an axisymmetric slender body is then given by

$$\phi(x_1, r) = -W \int_0^{x_1-\beta r} \frac{f(\xi)\, f'(\xi)}{\{(x_1 - \xi)^2 - \beta^2 r^2\}^{\frac{1}{2}}}\, d\xi. \tag{2.4.10}$$

The perturbation velocities are determined from

$$\left.\begin{aligned} u_o &= \partial\phi/\partial x_1 = -W \int_0^{x_1-\beta r} \frac{F(\xi)}{\{(x_1 - \xi)^2 - \beta^2 r^2\}^{\frac{1}{2}}}\, d\xi, \\ q_o &= \partial\phi/\partial r = \frac{W}{r} \int_0^{x_1-\beta r} \frac{(x_1 - \xi)\, F(\xi)}{\{(x_1 - \xi)^2 - \beta^2 r^2\}^{\frac{1}{2}}}\, d\xi, \end{aligned}\right\} \tag{2.4.11}$$

where $$F(\xi) = f'^2(\xi) + f(\xi)\, f''(\xi).$$

For the perturbation pressure we must use the quadratic approximation to Bernoulli's equation (cf. e.g. Lighthill 1954)

$$p_o/R_o = -W u_o - \tfrac{1}{2} q_o^2.$$

The second term is negligible except in the vicinity of the body surface where q_o^2 is comparable with u_o in magnitude. We shall, however, neglect the contribution of q_o^2 to p_o in the entire field, and consider the approximation

$$p_o \simeq - R_o W u_o .$$

This simplification will limit the analysis to very slender configurations, or infinitely weak conical shock waves.

Three-dimensional wings

We consider nearly plane wings which are pointed and whose leading edges are smooth functions of x. The mean surface of the wing lies in the plane $y = 0$, where the boundary condition on the wing surface can be applied. Following Ward (1955), the perturbation velocity potential for the supersonic flow $(W > a_o)$ past a wing symmetrical about its chord plane $y = 0$ may be given by

$$\phi(x_1, y, z) = -\frac{1}{\pi} \iint_{\Sigma'} \left\{\frac{\partial\phi(\xi, y, \zeta)}{\partial y}\right\}_{y=0} \frac{d\xi\, d\zeta}{\{(x_1 - \xi)^2 - \beta^2(z - \zeta)^2 - \beta^2 y^2\}^{\frac{1}{2}}}, \tag{2.4.12}$$

for $y > 0$, with respect to the co-ordinates fixed at the wing apex, x_1 being in the flow direction. Here Σ' is that part of the mean wing surface for which

$$\xi \leqslant x_1 - \beta\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}.$$

The potential in half space $y < 0$ is also determined, since ϕ is an even function of y.

If $y = \eta(x_1, z)$ represents the upper surface of a symmetric wing, we can have on the wing surface, at $y = 0$

$$\partial\phi(x_1, y, z)/\partial z = \begin{cases} W\, \partial\eta(x_1, z)/\partial x_1, & \text{on the wing projection } \Sigma \\ 0, & \text{elsewhere.} \end{cases} \tag{2.4.13}$$

We also assume that $\eta(x_1, z) = 0$ for $x_1 \leqslant 0$, and $\partial\eta/\partial x_1$ is continuous and has continuous derivatives.

For antisymmetrical wings, i.e. wings with zero thickness but at an incidence to the oncoming flow, the perturbation velocity potential can be expressed as

$$\phi(x_1, y, z) = -\frac{1}{\pi}\frac{\partial}{\partial y} \iint_{\Sigma'} \phi(\xi,+0,\zeta) \frac{d\xi\, d\zeta}{\{(x_1 - \xi)^2 - \beta^2(z - \zeta)^2 - \beta^2 y^2\}^{\frac{1}{2}}}, \tag{2.4.14}$$

for y > 0. Since ϕ in this case is an odd function of y, the potential can also be obtained in region y < 0.

The perturbation velocities and pressure can now be determined as

$$u_o = \partial\phi/\partial x_1, \quad v_o = \partial\phi/\partial y, \quad w_o = \partial\phi/\partial z \quad \text{and} \quad p_o = -R_o W u_o.$$

Thus using (2.4.12) together with (2.4.13), we are led to

$$u_o = -\frac{W}{\pi} \iint_{\Sigma'} \frac{F_1(\xi, \zeta)\, d\xi\, d\zeta}{\{(x_1 - \xi)^2 - \beta^2(z - \zeta)^2 - \beta^2 y^2\}^{\frac{1}{2}}}, \tag{2.4.15a}$$

for symmetrical wings, where at y = 0

$$F_1(\xi,\zeta) = \begin{cases} \partial^2\eta(\xi, \zeta)/\partial\xi^2, & \text{on } \Sigma \\ 0, & \text{elsewhere.} \end{cases}$$

From (2.4.14) we obtain

$$u_o = -\frac{W}{\pi}\frac{\partial}{\partial y} \iint_{\Sigma'} \frac{F_2(\xi, \zeta)\, d\xi\, d\zeta}{\{(x_1 - \xi)^2 - \beta^2(z - \zeta)^2 - \beta^2 y^2\}^{\frac{1}{2}}}, \tag{2.4.15b}$$

for the antisymmetrical wings, where at y = 0

$$F_2(\xi,\zeta) = \begin{cases} (1/W)\ \partial\phi(\xi, \zeta)/\partial\xi, & \text{on } \Sigma \\ 0, & \text{elsewhere.} \end{cases}$$

In the above we have given expressions for perturbation velocities and pressure in region (2), which are to be prescribed ahead of the shock in its passage on the obstacle. These perturbations are, however, with respect to the co-ordinates (x_1, y, z) fixed in the body. To relate these perturbations in the co-ordinate system (x, y, z, t) chosen behind the shock, we assume that the two co-ordinate systems coincide at time t = 0. At a later instant the two systems no longer coincide and we have

$$x_1 = x + (W + U)t = x + m_1\tau, \tag{2.4.16}$$

where $$m_1 = (W + U)/a_1 = M'(a_o/a_1) + M_1.$$

Also at the undisturbed location of the shock, $x = m\tau$. Hence to obtain the perturbations in region (2) at the shock, in the co-ordinate system (x, y, z, τ) we shall replace x_1 by $(m + m_1)\tau$ in the expressions (2.4.8), (2.4.11) and (2.4.15). The resulting expressions can further be expressed in dimensionless form by using (2.3.9). Thus we shall obtain for the upstream perturbation parameters at the shock $x = m\tau$:

For two-dimensional aerofoils,

$$\left.\begin{aligned} \bar{u} &= -\,(k_1/\beta)f'\{(m + m_1)\tau - \beta y\}, \\ \bar{v} &= -\,\beta\,\bar{u}, \quad \bar{p} = -\,(k_2/k_1)\,\bar{u}, \end{aligned}\right\} \tag{2.4.17}$$

where $k_1 = W/V = M'/M$ and $k_2 = (R_o/\gamma P_o)\,W^2 = M'^2$.

For axisymmetric slender bodies,

$$\left.\begin{aligned} \bar{u} &= -\,k_1 \int_0^{\xi_1} \frac{F(\xi)}{[\{(m + m_1)\tau - \xi\}^2 - \beta^2 r^2]^{\frac{1}{2}}}\,d\xi, \\ \bar{q} &= \frac{k_1}{r} \int_0^{\xi_1} \frac{\{(m + m_1)\tau - \xi\}F(\xi)}{[\{(m + m_1)\tau - \xi\}^2 - \beta^2 r^2]^{\frac{1}{2}}}\,d\xi, \\ \bar{p} &= -\,(k_2/k_1)\,\bar{u}, \end{aligned}\right\} \tag{2.4.18}$$

where $\xi_1 = (m + m_1)\tau - \beta r$.

For three-dimensional wings,

$$\bar{u} = -\frac{k_1}{\pi} \iint_{\Sigma'} \frac{F_1(\xi, \zeta)\,d\xi\,d\zeta}{[\{(m + m_1)\tau - \xi\}^2 - \beta^2\{(z - \zeta)^2 - y^2\}]^{\frac{1}{2}}}, \tag{2.4.19a}$$

(for symmetrical wings)

$$\bar{u} = -\frac{k_1}{\pi}\frac{\partial}{\partial y} \iint_{\Sigma'} \frac{F_2(\xi, \zeta)\,d\xi\,d\zeta}{[\{(m + m_1)\tau - \xi\}^2 - \beta^2\{(z - \zeta)^2 - y^2\}]^{\frac{1}{2}}}, \tag{2.4.19b}$$

(for antisymmetrical wings)

$$\bar{p} = -\,(k_2/k_1)\,\bar{u}.$$

The upstream perturbations defined by (2.4.17), (2.4.18) and (2.4.19) can now be used in the conditions at the disturbed shock, (2.3.10), or (2.3.11) and (2.3.12).

2.5. Obstacle Mach number behind the shock

For the obstacle moving in region (0), we have defined its Mach number as $M' = W/a_o$. When the obstacle penetrates the incident shock and is moving in region (1), its Mach number can be defined as

$$m_1 = (W + U)/a_1 = M'(a_o/a_1) + M_1,$$

for an observer fixed in the obstacle. Making use of the plane shock relations (2.3.4), we can write

$$m_1 = \frac{(\gamma + 1)M M' + 2(M^2 - 1)}{[\{2\gamma M^2 - (\gamma - 1)\}\{(\gamma - 1)M^2 + 2\}]^{\frac{1}{2}}}, \qquad (2.5.1)$$

a function of M and M'. Figure 4 shows the variation of m_1 with M' and M as a parameter, taking $\gamma = 1.4$. We notice that for real incident shocks ($M > 1$) and initially supersonic obstacles ($M' > 1$), the uniform flow over the obstacle behind the shock will always be supersonic.

Chapter 3

TWO-DIMENSIONAL AEROFOILS

3.1. Introductory remarks

This chapter is concerned with the study of shock-on-shock interaction with two-dimensional aerofoils of arbitrary shape. The flow pattern produced after the aerofoil has penetrated the shock is essentially the same as has been discussed in §1.3, except that now it has to be visualized in two dimensions. Thus in the interaction region we shall have Mach waves AC and AC′ (cf. Fig. 5) emanating from the leading edges of the aerofoil, the Mach waves ID and I′D′ from the shock intersections I and I′, together with the reflected waves BCDE and B′C′D′E′ propagating with sonic velocity a_1 from the center at O. The diffracted shock IF and I′F′ on the two sides of the aerofoil can have similar or opposite curvatures, depending upon the attitude of the aerofoil (cf. Fig. 5), though the conditions at the disturbed shock will be applied at its undisturbed location which lie in the same plane for y = 0. Since the uniform flow over the aerofoil behind the shock is supersonic, $(W + U) > a_1$, the flow pattern on the two sides will be independent of each other. Hence it is sufficient to consider the flow, say, in the upper half plane $(y > 0)$.

The flow parameters for the two-dimensional problem are functions of (x, y, τ) only, since such a flow can be considered to be independent of spanwise gradients. The interaction flow field behind the shock can then be described by the linearized equations of motion (2.2.16) through (2.2.18), or by the wave Eq. (2.2.24), with the z-derivatives omitted. Also the shock relations (2.3.11a) and (2.3.11b), while (2.3.11c) omitted, will be used together with the two-dimensional perturbation parameters (2.4.17). In the sequel we shall complete the formulation of the flow problem and present the solution.

In §3.2 the initial and boundary conditions are considered. In §3.3 the Lorentz transformation for x and τ is introduced and the complete formulation is written down in terms of new variables. In §3.4 the solution is sought by a systematic application of the integral transforms. The solution is used to describe the entropy and density field, and the shape of the shock front. In §3.5 the properties of the solution are discussed. Finally in §3.6 the results of §3.4 are applied to calculate the interaction field for a two-dimensional wedge.

3.2. Initial and boundary conditions

For the disturbed region behind the shock ($x \leqslant m\tau$), we can prescribe two initial conditions as follows

$$\text{for } \tau \leqslant 0, \qquad \left.\begin{aligned} p(x, y, \tau) &= 0, \\ \partial p(x, y, \tau)/\partial\tau &= 0. \end{aligned}\right\} \tag{3.2.1}$$

On the aerofoil

In the region behind the shock ($x \leqslant m\tau$) the tangency condition stating that the normal component of velocity on the surface of the aerofoil must vanish, can be expressed in the first approximation, at $y = 0$

$$v_1 = (W + U)\, f'\{x + (W + U)t\}, \tag{3.2.2}$$

where the function f describes the surface of the aerofoil and has been defined in §2.4,

$$f\{x + (W + U)t\} = 0, \text{ for } \{x + (W + U)t\} \leqslant 0.$$

Hence (3.2.2) gives in dimensionless form, for $-m_1\tau \leqslant x \leqslant m\tau$, $y = 0$

$$v = v_1/a_1 = m_1\, f'(x + m_1\tau). \tag{3.2.3}$$

Further for values $x < -m_1\tau$, $y = 0$ we have $v = 0$, since the air is undisturbed to the left of the aerofoil. The conditon (3.2.3) can then be used all along the axis $-\infty < x \leqslant m\tau$, since $f = 0$ for $x \leqslant -m_1\tau$.

Using the second momentum Eq. (2.2.17) and the tangency condition (3.2.3), we can derive a boundary condition for the y-derivative of p at y = 0. Thus at y = 0

$$\partial p/\partial y = -m_1^2 f''(x + m_1\tau), \qquad (3.2.4)$$

valid for $-\infty < x < m$. It may be noticed that the application of the second momentum Eq. (2.2.17) requires the calculation of $\partial v/\partial\tau$ at y = 0. At the shock location ($x = m\tau$), however, this calculation fails, since $\partial v/\partial\tau$ leads to infinite values there. Hence the condition (3.2.4) is applicable only for $x < m\tau$. At $x = m\tau$, $y = 0$, a different condition will be used to be deduced from the shock relations.

On the shock front

The boundary condition on the disturbed shock front can be furnished by the shock relations (2.3.11a) and (2.3.11b) which must be satisfied at $x = m\tau$, all along the half plane, $0 < y < \infty$. These relations are

$$u = \frac{1}{A}(p - B\,\bar{u} - C\,\bar{p}), \qquad (3.2.5a)$$

and

$$\frac{\partial v}{\partial\tau} = \frac{1}{D}\left(\frac{\partial p}{\partial y} - \Lambda_{21}\frac{\partial\bar{u}}{\partial y} - \Lambda_{22}\frac{\partial\bar{p}}{\partial y}\right) + \Lambda_{41}\frac{\partial\bar{v}}{\partial\tau} \qquad (3.2.5b)$$

It is to be remembered here that in (3.2.5b) the derivatives to τ are taken while travelling with the shock. Hence when expressed in the co-ordinate system chosen behind the shock the derivatives to τ are to be replaced by

$$\partial/\partial\tau + m\,\partial/\partial x.$$

The relations (3.2.5) in their present form are, however, not suitable to describe the boundary condition at the shock. In §3.3 we shall use these relations together with the linearized equations of motion (2.2.16)-(2.2.18) to derive a single condition in p to be applied at the shock plane.

Shock-aerofoil intersection

For the region behind the shock, on the aerofoil (y = 0) at the shock location ($x = m\tau$), the tangency condition (3.2.3) gives

$$v = m_1 f'\{(m + m_1)\tau\}, \qquad (3.2.6)$$

or

$$v_1 = (W + U)\, f'\{(m + m_1)\tau\}.$$

Again, for the region ahead of the shock, we can write for the normal component of velocity at $y = 0$, $x = m\tau$

$$v_o = W\, f'\{(m + m_1)\tau\}.$$

Using these values of v_1 and v_o in the shock relation (cf. Eqs. 2.3.7)

$$v_1 = v_o = -\, U\psi_y,$$

we obtain at $y = 0$, $x = m\tau$ $\qquad f' = -\,\psi_y.$

Thus we see that the tangency condition in front of and behind the shock is satisfied together with the requirement that the shock is perpendicular to the surface of the aerofoil.

Now we can use the shock condition (3.2.5b) together with the relation (3.2.6), and obtain a condition on $\partial p/\partial y$ at $x = m\tau$, $y = 0$

$$\frac{\partial p}{\partial y} = D\, m_1(m + m_1)\, f''\{(m + m_1)\tau\} + \left\{\Lambda_{21}\frac{\partial \bar{u}}{\partial y} + \Lambda_{22}\frac{\partial \bar{p}}{\partial y} - D\,\Lambda_{41}\frac{\partial \bar{v}}{\partial \tau}\right\}_{\text{at } y = 0}. \qquad (3.2.7)$$

From Eqs. (3.2.4) and (3.2.7) we may notice that when on the aerofoil surface the shock is approached and when along the shock the aerofoil is approached, the two limits of $\partial p/\partial y$ are different in general. This non-uniformity has probably no physical significance, but is a reflection of the simplifications in the assumed model, viz. the shock has zero thickness, the boundary layer is omitted, the conditions on the aerofoil are satisfied at $y = 0$ while those at the disturbed shock are applied at its undisturbed location.

At infinity

For the disturbance field behind the shock we can prescribe that all the perturbations vanish at infinity, i.e. for $x \leqslant m\tau$, $y > 0$

as $x \to -\infty$, $y \to \infty$ $\qquad p(x, y, \tau)$ and its derivatives $\to 0$. $\qquad$ (3.2.8)

With the initial and boundary conditions discussed above and the governing Eqs. (2.1.16)-(2.1.18) and (2.1.24) (with z-derivatives omitted), the formulation of the problem is complete.

3.3. The Lorentz transformation

The boundary conditions may be clearly visualized if the problem is presented in the $xy\tau$-space. As shown in Fig. 6, the boundary condition on the wing is prescribed in the plane $y = 0$ and those at the shock are stipulated on the plane $x = m\tau$ which is perpendicular to the first plane. It will be convenient if the shock boundary conditions to be satisfied at $x = m\tau$ are transferred to a co-ordinate plane, say $\bar{x} = 0$. This together with the fact that the equation to be solved is a wave equation suggests the application of a Lorentz transformation of the type

$$\bar{x} = (x - m\tau)/(1 - m^2)^{\frac{1}{2}}, \qquad \bar{\tau} = (\tau - mx)/(1 - m^2)^{\frac{1}{2}}. \tag{3.3.1}$$

This transformation leaves the wave equation invariant.

The plane $\bar{x} = 0$ now corresponds to the shock plane $x = m\tau$, and the wave Eq. (2.1.24) gives

$$\frac{\partial^2 p}{\partial \bar{x}^2} + \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 p}{\partial \bar{\tau}^2} = 0. \tag{3.3.2}$$

The initial conditions (3.2.1) can be expressed as

$$p(\bar{x}, y, \bar{\tau}) = \partial p(\bar{x}, y, \bar{\tau})/\partial \bar{\tau} = 0, \qquad \text{for } \bar{\tau} \leqslant 0. \tag{3.3.3}$$

For the boundary condition (3.2.4) on the wing, we can write for $\bar{\tau} > 0$, $y = 0$, $-\infty < \bar{x} < 0$

$$\partial p/\partial y = A_o\, f''\{\bar{a}(\bar{\tau} + \lambda_o \bar{x})\}, \tag{3.3.4}$$

with $f\{\bar{a}(\bar{\tau} + \lambda_o x)\} = 0$, for $(\bar{\tau} + \lambda_o \bar{x}) \leqslant 0$,

$A_o = -m_1^2$, $\bar{a} = (m + m_1)/(1 - m^2)^{\frac{1}{2}}$ and $\lambda_o = (1 + mm_1)/(m + m_1)$.

Condition at the shock: Using the Lorentz transformation (3.3.1), we can write

$$\partial/\partial\tau + m\, \partial/\partial x = (1 - m^2)^{\frac{1}{2}}\, \partial/\partial\bar{\tau}.$$

Hence the shock relation (3.1.5b) gives, at $\bar{x} = 0$

$$\frac{\partial v}{\partial \bar{\tau}} = \frac{1}{D(1 - m^2)^{\frac{1}{2}}} \left(\frac{\partial p}{\partial y} - \Lambda_{21} \frac{\partial \bar{u}}{\partial y} - \Lambda_{22} \frac{\partial \bar{p}}{\partial y}\right) + \Lambda_{41} \frac{\partial \bar{v}}{\partial \bar{\tau}}. \tag{3.3.5}$$

The equations of motion, (2.1.16) together with (2.1.18) and the first of Eqs. (2.1.17), yield upon transformation

$$\frac{\partial p}{\partial \bar{\tau}} - m \frac{\partial p}{\partial \bar{x}} + \frac{\partial u}{\partial \bar{x}} - m \frac{\partial u}{\partial \bar{\tau}} + (1 - m^2)^{\frac{1}{2}} \frac{\partial v}{\partial y} = 0,$$

and

$$\frac{\partial u}{\partial \bar{\tau}} - m \frac{\partial u}{\partial \bar{x}} = - \left(\frac{\partial p}{\partial \bar{x}} - m \frac{\partial p}{\partial \bar{\tau}}\right).$$

We first eliminate $\partial u/\partial \bar{x}$ from these two equations, then differentiating the resulting equation with respect to $\bar{\tau}$ we are led to

$$\frac{\partial^2 p}{\partial \bar{x}\, \partial \bar{\tau}} + \frac{\partial^2 u}{\partial \bar{\tau}^2} + \frac{m}{(1 - m^2)^{\frac{1}{2}}} \frac{\partial^2 v}{\partial y\, \partial \bar{\tau}} = 0. \tag{3.3.6}$$

Making use of the shock relations (3.2.5a) and (3.3.5), we can substitute for u and $\partial v/\partial \bar{\tau}$ in (3.3.6). Thus we obtain a single condition in terms of p at the shock $\bar{x} = 0$, $0 < y < \infty$

$$\frac{\partial^2 p}{\partial \bar{x}\, \partial \bar{\tau}} + \frac{1}{A} \frac{\partial^2 p}{\partial \bar{\tau}^2} + \frac{m}{1 - m^2} \frac{1}{D} \frac{\partial^2 p}{\partial y^2}$$

$$= \frac{1}{A} \frac{\partial^2}{\partial \bar{\tau}^2} (B\bar{u} + C\bar{p}) + \frac{m}{1 - m^2} \frac{1}{D} \frac{\partial^2}{\partial y^2} (\Lambda_{21}\bar{u} + \Lambda_{22}\bar{p})$$

$$- \frac{m}{(1 - m^2)^{\frac{1}{2}}} \Lambda_{41} \frac{\partial^2 \bar{v}}{\partial y\, \partial \bar{\tau}}. \tag{3.3.7}$$

Noting that $\frac{1 - m^2}{m} D = - 2m, \quad \frac{1 - m^2}{m} \frac{D}{A} = - (1 + 1/M^2),$

the condition (3.3.7) can be simplified to yield, at $\bar{x} = 0$, $0 < y < \infty$

$$- \frac{\partial^2 p}{\partial y^2} + 2m \frac{\partial^2 p}{\partial \bar{x}\, \partial \bar{\tau}} + (1 + 1/M^2) \frac{\partial^2 p}{\partial \bar{\tau}^2}$$

$$= (1 + 1/M^2) \frac{\partial^2}{\partial \bar{\tau}^2} (B\bar{u} + C\bar{p}) - \frac{\partial^2}{\partial y^2} (\Lambda_{21}\bar{u} + \Lambda_{22}\bar{p})$$

$$- \frac{2m^2}{(1 - m^2)^{\frac{1}{2}}} \Lambda_{41} \frac{\partial^2 \bar{v}}{\partial y\, \partial \bar{\tau}}. \tag{3.3.8}$$

The upstream perturbation parameters $\bar{u}$, $\bar{v}$, $\bar{p}$ at the shock location $x = m\tau$ are given by the expressions (2.4.17). In terms of new variables they can be expressed as follows, at $\bar{x} = 0$

$$\left.\begin{aligned}\bar{u} &= (k_1/\beta)\ f'\{\bar{a}(\bar{\tau} - \bar{b}y)\},\\ \bar{v} &= -\ \beta\bar{u},\quad \bar{p} = -\ (k_2/k_1)\bar{u},\end{aligned}\right\} \tag{3.3.9}$$

where $\bar{b} = \beta/\bar{a}$, and $\bar{a}$ is defined in (3.3.4). Substituting these expressions in (3.3.8) we obtain for the shock condition, at $\bar{x} = 0$, $0 < y < \infty$

$$-\frac{\partial^2 p}{\partial y^2} + 2m\frac{\partial^2 p}{\partial\bar{x}\,\partial\bar{\tau}} + (1 + 1/M^2)\frac{\partial^2 p}{\partial\bar{\tau}^2} = \frac{K}{\bar{b}}\frac{\partial}{\partial\bar{\tau}}\ f''\{\bar{a}(\bar{\tau} - \bar{b}y)\}, \tag{3.3.10}$$

where K is a constant given by

$$K = -\ (1 + 1/M^2)(Bk_1 - Ck_2) + (\Lambda_{21}k_1 - \Lambda_{22}k_2)\bar{b} + \frac{2m^2}{1 - m^2}\Lambda_{41}k_1\bar{a}\bar{b}^2.$$

Here we may notice that the right hand side of (3.3.10) is due to the disturbances ahead of the shock, and by definition of f it vanishes for

$$(\bar{\tau} - \bar{b}y) \leqslant 0, \quad \text{or} \quad y \geqslant (m + m_1)\tau/\beta.$$

The upstream disturbances along the shock are thus over a finite section IF (Fig. 5b), as should be expected.

Equation (3.3.10) is a second-order differential condition in p to be specified at the shock plane. This is to be supplemented by the condition at the shock-aerofoil intersection given by (3.2.7). The condition (3.2.7) in the new variables together with (3.3.9) leads to

$$\text{at } \bar{x} = 0,\ y = 0 \qquad \partial p/\partial y = B_o f''(\bar{a}\bar{\tau}) \tag{3.3.11}$$

where B_o is another constant given by

$$B_o = -\ \{2m^2/(1 - m^2)^{\frac{1}{2}}\}(m_1 - \Lambda_{41}k_1)\bar{a} + (\Lambda_{21}k_1 - \Lambda_{22}k_2).$$

Finally the condition at infinity (3.2.8) can be expressed, for $\bar{x} < 0$, $y > 0$

$$\text{as } \bar{x} \to -\infty,\ y \to \infty \qquad p \text{ and its derivatives} \to 0. \tag{3.3.12}$$

Summarizing, the wave Eq. (3.3.2) with the initial conditions (3.3.3) and the boundary conditions (3.3.4), (3.3.10), (3.3.11) and (3.3.12) represent the complete formulation in terms of Lorentz variables. In the sequel we seek the solution to this formulation by using integral transforms.

3.4. Analytic solution

In view of the time dependence of the pressure p, we first introduce a Laplace transform with respect to $\bar{\tau}$ defined by (cf. Sneddon 1951)

$$p^{*}(\bar{x}, y, s) = L\{p(\bar{x}, y, \bar{\tau})\} = \int_0^{\infty} p(\bar{x}, y, \bar{\tau}) \exp\{-s\bar{\tau}\}d\bar{\tau}. \qquad (3.4.1)$$

Applying this transform to the wave Eq. (3.3.2) and using the initial conditions (3.3.3) we are led to

$$\frac{\partial^2 p^{*}}{\partial \bar{x}^2} + \frac{\partial^2 p^{*}}{\partial y^2} - s^2 p^{*} = 0. \qquad (3.4.2)$$

The boundary condition (3.3.4) gives, for $y = 0$, $-\infty < \bar{x} < 0$

$$\partial p^{*}/\partial y = A_0 \exp\{s\lambda_0 \bar{x}\} \; G(s), \qquad (3.4.3)$$

where $$G(s) = L\{f''(\bar{a}\bar{\tau})\}.$$

The condition at the shock (3.3.10) together with the initial conditions (3.3.3) will give at $\bar{x} = 0$, $0 < y < \infty$

$$-\frac{\partial^2 p^{*}}{\partial y^2} + 2ms \frac{\partial p^{*}}{\partial \bar{x}} + (1 + 1/M^2)s^2 p^{*} = \frac{K}{\bar{b}} s \exp\{-\bar{b}sy\}. \qquad (3.4.4)$$

The supplementary condition (3.3.11) yields

$$\text{at } \bar{x} = 0,\ y = 0 \qquad \partial p^{*}/\partial y = B_0 \, G(s) \qquad (3.4.5)$$

The condition (3.3.12) now becomes

$$\text{as } \bar{x} \to -\infty,\ y \to \infty \qquad p^{*} \text{ and its derivatives} \to 0. \qquad (3.4.6)$$

Next, since the boundary conditions at $y = 0$ are given for $\partial p^{*}/\partial y$, we can introduce a Fourier cosine transform with respect to y defined by (cf. Sneddon 1951)

$$p^{**}(\bar{x}, \alpha, s) = F_c\{p^{*}(\bar{x}, y, s)\} = \int_0^{\infty} p^{*}(\bar{x}, y, s) \cos(\alpha y) \, dy. \qquad (3.4.7)$$

Applying Fourier cosine transform to the Eq. (3.4.2), and employing the conditions (3.4.3) and (3.4.6), we obtain

$$\frac{\partial^2 p^{**}}{\partial \bar{x}^2} - \lambda^2 p^{**} = A_o \exp\{s\lambda_o \bar{x}\}\, G(s), \tag{3.4.8}$$

valid for $-\infty < \bar{x} < 0$, where $\lambda^2 = \alpha^2 + s^2$.
Further applying the transform (3.4.7) to the condition (3.4.4), and using the conditions (3.4.5) and (3.4.6), we deduce, at $\bar{x} = 0$

$$(\lambda^2 + s^2/M^2)p^{**} + 2ms\,\frac{\partial p^{**}}{\partial \bar{x}} = - B_o\, G(s) + K\,\frac{s^2}{\alpha^2 + \bar{b}^2 s^2}\, G(s). \tag{3.4.9}$$

Finally the condition (3.4.6) gives

$$\text{as } \bar{x} \to -\infty \qquad p^{**} \to 0. \tag{3.4.10}$$

Thus we have reduced the complicated problem to an ordinary non-homogeneous differential equation (3.4.8), to be solved subject to the conditions (3.4.9) and (3.4.10). The complete solution of (3.4.8) can be written as

$$p^{**} = E_1 \exp\{\lambda\bar{x}\} + E_2 \exp\{-\lambda\bar{x}\} - \frac{A_o}{\lambda^2 - \lambda_o^2 s^2} \exp\{s\lambda_o\bar{x}\}\, G(s). \tag{3.4.11}$$

In view of the condition (3.4.10), the co-efficient of $\exp\{-\lambda\bar{x}\}$ must vanish. The co-efficient E_1 is then determined by using the condition (3.4..9) at $\bar{x} = 0$. Hence we obtain

$$p^{**} = \left(A_o\,\frac{\lambda}{\lambda^2 - \lambda_o^2 s^2} - 2mA_o\,\frac{\lambda}{H(\lambda)}\,\frac{s}{\lambda + \lambda_o s} - B_o\,\frac{\lambda}{H(\lambda)} + K\,\frac{\lambda}{H(\lambda)}\,\frac{s^2}{\alpha^2 + \bar{b}^2 s^2}\right)\frac{\exp\{\lambda\bar{x}\}}{\lambda}\, G(s) - \frac{A_o}{\lambda^2 - \lambda_o^2 s^2}\exp\{s\lambda_o\bar{x}\}\, G(s), \tag{3.4.12}$$

where $H(\lambda) = \lambda^2 + 2ms\lambda + s^2/M^2$, which can be represented as

$$H(\lambda) = (\lambda + \lambda_2 s)(\lambda + \lambda_3 s),$$

λ_2 and λ_3 being the roots (real, distinct and positive) of the quadratic equation

$$\lambda_i^2 - 2m\lambda_i + 1/M^2 = 0. \qquad (3.4.13)$$

Then (3.4.12) can be simplified and expressed in the form

$$p^{**} = \tfrac{1}{2}A_o\left(\frac{1}{\lambda - \lambda_o s}\frac{\exp\{\lambda\bar{x}\}}{\lambda} - \frac{2}{\lambda^2 - \lambda_o^2 s^2}\exp\{s\lambda_o\bar{x}\}\right) G(s)$$

$$+ \frac{1}{2}\sum_{i=1}^{5} A_i \frac{1}{\lambda + \lambda_i s}\frac{\exp\{\lambda\bar{x}\}}{\lambda} G(s), \qquad (3.4.14)$$

where $A_1 = A_o H(\lambda_o)/H(-\lambda_o)$, $H(\lambda_o) = \lambda_o^2 + 2m\lambda_o + 1/M^2$,

$$A_2 = \frac{-2\lambda_2}{\lambda_2 - \lambda_3}\left(B_o + \frac{2mA_o}{\lambda_o - \lambda_2} - \frac{K}{\lambda_2^2 - \lambda_4^2}\right),$$

$$A_3 = \frac{2\lambda_3}{\lambda_2 - \lambda_3}\left(B_o + \frac{2mA_o}{\lambda_o - \lambda_3} - \frac{K}{\lambda_3^2 - \lambda_4^2}\right),$$

$$A_4 = K/\{(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)\}, \quad A_5 = K/\{(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)\},$$

$$\lambda_1 = \lambda_o, \quad \lambda_5 = -\lambda_4 \quad \text{and} \quad \lambda_4^2 = (1 - \bar{b}^2) > 0.$$

The constant λ_o has been defined under Eq. (3.3.4),

$$\lambda_o = \lambda_1 = (1 + mm_1)/(m + m_1)$$

Making use of (2.3.13) and (2.5.1), λ_o(or λ_1) may be determined for given M and M′. It may be noticed that for real shock (M > 1) and initially supersonic obstacles (M′> 1), λ_o is always less than unity. Further λ_2 and λ_3 are determined by Eq. (3.4.13). They are given by

$$\lambda_2, \lambda_3 = m \pm (m^2 - 1/M^2)^{\frac{1}{2}}.$$

Using (2.3.13) we see that λ_2 and λ_3 are functions of M alone, and they are always less than unity for real shock (M > 1). Finally for $\bar{b} > 1$, λ_4 is always real and less than unity. It is given by

$$\lambda_4 = \{1 - \beta^2(1 - m^2)/(m + m_1)^2\}^{\frac{1}{2}}$$

a function of M and M′. The variations of $\lambda_o = \lambda_1$, λ_2, λ_3 and λ_4 with M and M′ are illustrated in Fig. 7.

The real λ_4, however, implies that

$$(m + m_1)/\beta > (1 - m^2)^{\frac{1}{2}}, \quad \text{or} \quad IF > BF,$$

the intersection of the plane shock with the original Mach wave of the body lies outside the sonic circle (cf. Fig. 5).

Now, to obtain the pressure p we seek the inversion of (3.4.14)

$$p(\bar{x}, y, \bar{\tau}) = L^{-1}[F_c^{-1}\{p^{**}(\bar{x}, \alpha, s)\}].$$

The inversion of (3.4.14) is considered in Appendix (1.1). Thus we obtain for the pressure field

$$p(\bar{x}, y, \bar{\tau}) = \frac{1}{\pi}\Bigg(-A_o\int_0^{\bar{\tau}} d\mu \int_{-\infty}^{0} \frac{f''\{\bar{a}(\mu + \lambda_o\xi)\}}{\{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}}\, d\xi$$

$$+ \sum_{i=1}^{4} A_i \int_0^{\bar{\tau}} d\mu \int_{-\infty}^{0} \frac{f''\{\bar{a}(\mu + \lambda_i\xi)\}}{\{(\bar{\tau} - \mu)^2 - (\bar{x} + \xi)^2 - y^2\}^{\frac{1}{2}}}\, d\xi$$

$$- A_5 \int_0^{\bar{\tau}} d\mu \int_{-\infty}^{0} \frac{f''\{\bar{a}(\mu + \lambda_4\xi)\}}{\{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}}\, d\xi\Bigg)$$

$$+ \frac{A_5}{\bar{a}\bar{b}}\, f'\{\bar{a}(\bar{\tau} + \lambda_4\bar{x} - \bar{b}y)\}, \qquad (3.4.15)$$

The integrals for i = 1,2, 3 & 4 can also be written as

$$\sum_{i=1}^{4} A_i \int_0^{\bar{\tau}} d\mu \int_0^{\infty} \frac{f''\{\bar{a}(\mu - \lambda_i\xi)\}}{\{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}}\, d\xi.$$

Also the last two terms in (3.4.15) can be combined (cf. Appendix 1.2) to yield

$$A_5 \int_0^{\bar{\tau}} d\mu \int_0^{\infty} \frac{f''\{\bar{a}(\mu + \lambda_4\xi)\}}{\{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}}\, d\xi.$$

Thus we can express the result (3.4.15) in a more compact form as follows

$$p(\bar{x}, y, \bar{\tau}) = \frac{1}{\pi}\Big(-A_o\int_0^{\bar{\tau}} d\mu \int_{\xi_2}^{0} \frac{1}{r_c} f''\{\bar{a}(\mu + \lambda_o\xi)\}\, d\xi$$

$$+ \sum_{i=1}^{5} A_i \int_0^{\bar{\tau}} d\mu \int_0^{\xi_1} \frac{1}{r_c} f''\{\bar{a}(\mu - \lambda_i\xi)\}\Big)\, d\xi \qquad (3.4.16)$$

where $\xi_1 = \bar{x} + \{(\bar{\tau} - \mu)^2 - y^2\}^{\frac{1}{2}}$, $\xi_2 = \bar{x} - \{(\bar{\tau} - \mu)^2 - y^2\}^{\frac{1}{2}}$, $r_c = \{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}$ and $\lambda_5 = -\lambda_4$.

Shock intersection inside sonic circle

To arrive at the above results from (3.4.14) we assumed λ_4 to be real, i.e. $\bar{b} < 1$. When $\bar{b} = 1$, we notice that IF = BF, the shock intersection I coincides with the point B of the sonic circle. The region IBDI of Fig. 5 now disappears, and the curved portion of the incident shock is confined to BF. In this case then $\lambda_4 = 0$, and the results for the interaction field are still given by (3.4.15) or (3.4.16)

In case $\bar{b} > 1$, IF < BF, the shock intersection I then lies below point B of the sonic circle, Fig. 8. The shock is curved below B and there are contact discontinuities along approximate positions BO and IO. However, analytically this situation does not pose a different problem, since we formulate the condition at the undisturbed shock plane for $0 < y < \infty$ (cf. Eq. 3.3.10), and the fact that the perturbations of region (2) are confined from I to F (now less than BF) along the shock is automatically taken care of by the right hand side of the condition (3.3.10). The above analysis is then valid as such except that now λ_4 and λ_5 become imaginary. Under these conditions we consider the last two terms (for i = 4 & 5) in (3.4.14) as follows

$$I^{**}_{4,5} = \frac{1}{2}\left(\frac{A_4}{\lambda + \lambda_4 s} + \frac{A_5}{\lambda - \lambda_4 s}\right) \frac{\exp\{\lambda\bar{x}\}}{\lambda}\, G(s)$$

$$= (Q_1\lambda + Q_2 s)\, \frac{1}{\alpha^2 + \bar{b}^2 s^2}\, \frac{\exp\{\lambda\bar{x}\}}{\lambda}\, G(s), \qquad (3.4.17)$$

where $Q_1 = \dfrac{K(\lambda_2\lambda_3 + \lambda_4^2)}{(\lambda_2^2 - \lambda_4^2)(\lambda_3^2 - \lambda_4^2)}$, $Q_2 = \dfrac{- K(\lambda_2 + \lambda_3)\lambda_4^2}{(\lambda_2^2 - \lambda_4^2)(\lambda_3^2 - \lambda_4^2)}$,

both are real. The inversion of (3.4.17) is achieved in Appendix (1.3). Thus we obtain

$$I_{4,5}(\bar{x}, y, \bar{\tau}) = L^{-1}[F_c^{-1}\{I_{4,5}^{**}(\bar{x}, \alpha, s)\}]$$

$$= \frac{1}{\pi\bar{b}} \int_0^{\bar{\tau}} d\mu \int_0^{\infty} f''\{\bar{a}(\mu - \bar{b}\zeta)\} \Big(Q_1\{N_1(\mu, \zeta) + N_1(\mu, -\zeta)\}$$

$$+ Q_2\{N_2(\mu, \zeta) + N_2(\mu, -\zeta)\}\Big) \, d\zeta, \qquad (3.4.18)$$

where the funtions N_1 and N_2 are given by

$$N_1(\mu, \zeta) = \frac{\bar{x}}{\bar{x}^2 + (y - \zeta)^2} \frac{\bar{\tau} - \mu}{\{(\bar{\tau} - \mu)^2 - \bar{x}^2 - (y - \zeta)^2\}^{\frac{1}{2}}},$$

$$N_2(\mu, \zeta) = \{(\bar{\tau} - \mu)^2 - \bar{x}^2 - (y - \zeta)^2\}^{-\frac{1}{2}}.$$

Hence in case $\bar{b} > 1$, to obtain the pressure field the last two terms (for i = 4 & 5) in (3.4.16) should be replaced by $I_{4,5}$ (Eq. 3.4.18).

It may be remarked here that for the case $\bar{b} > 1$, we can also evaluate the interaction pressure field behind the shock from (3.4.16) provided we consider the real parts of the expressions involving λ_4 and λ_5. To write the real parts of these expressions, in their present form, does not seem feasable. We can, however, do so after the integrals in (3.4.16) have been evaluated for a specific example. This is done in §3.6 and the results checked with those evaluated by the direct integration of (3.4.18).

The solution of the flow problem discussed above, (3.4.16) and (3.4.18), satisfies the governing equation and the prescribed initial and boundary conditions. By using the transformation (3.3.1) $p(\bar{x}, y, \bar{\tau})$ can be expressed in terms of original variables (x, y, τ). It is, however, easier to carry out the integrations in (3.4.16) and (3.4.18) for a given f (the profile of an aerofoil) first and then express the results in original variables.

Once the pressure p is known, all the other flow variables, viz.

velocity components, entropy, density, can be determined in the non-uniform flow region behind the shock, including the shock shape. This can be accomplished by using the linearized equations of motion (2.2.17)-(2.2.19), and the shock relations (2.3.10) together with the upstream parameters given by (2.4.17). In the following we shall deduce the entropy and density field, and the form of the shock front.

Entropy and density field

From §2.2 it is known that the entropy of the fluid flow remains fixed with the fluid particles on crossing the shock. Hence to obtain the entropy we can use the basic relation (2.1.9)

$$s(x,\ y) = p(x,\ y,\ \tau) - \rho(x,\ y,\ \tau) \tag{3.4.19}$$

at the instant the shock passes ($\tau = x/m$). The shock condition (2.3.12) gives, at $\tau = x/m$

$$\rho = E\,p + F\,\bar{u} + G\,\bar{p} \tag{3.4.20}$$

Then from (3.4.19) and (3.4.20), we obtain for the entropy

$$s(x,\ y) = (1 - E)\ p(x,\ y,\ \tau{=}x/m) - (F\,\bar{u} + G\,\bar{p}).$$

Substituting for $\bar{u}$ and $\bar{p}$ from (2.4.17) we are led to

$$s(x,\ y) = (1 - E)\ p(x,\ y,\ \tau{=}x/m) + \frac{1}{\beta}(k_1F - k_2G)\ f'\{(x - \bar{c}y)(m + m_1)/m\}, \tag{3.4.21}$$

where $\bar{c} = \beta m/(m + m_1)$.

With the entropy determined, the density field is immediately given by (3.4.19). Thus

$$\rho(x,\ y,\ \tau) = p(x,\ y,\ \tau) + (E - 1)\ p(x,\ y,\ \tau{=}x/m) - \frac{1}{\beta}(k_1F - k_2G)\ f'\{(x - \bar{c}y)(m + m_1)/m\}. \tag{3.4.22}$$

Shock front

The form of the incident shock on diffraction is given by

$$x = m\tau + \psi(y,\ \tau) + O(\varepsilon^2), \tag{3.4.23}$$

in the co-ordinate system chosen behind the shock. The shock displacement ψ can be determined by integrating the shock relation (2.3.10b).

The shock relation (2.3.10b) together with (2.4.17) gives

$$\psi_\tau = \frac{1}{\Pi_{21}} \left[p(x=m\tau,\ y,\ \tau) - \frac{1}{\beta}\,(k_1\Lambda_{21} - k_2\Lambda_{22})\ f'\{(m + m_1)\tau - \beta y\}\right].$$

On integration and taking the initial condition $\psi = 0$ for $\tau = 0$

$$\psi(y,\ \tau) = \frac{1}{\Pi_{21}}\left(\int_0^\tau p(x=m\mu,\ y,\ \mu)d\mu - \frac{k_1\Lambda_{21} - k_2\Lambda_{22}}{\beta(m + m_1)}\ f\{(m + m_1)\tau - \beta y\}\right). \tag{3.4.24}$$

This completes the solution for the non-uniform flow field under consideration.

3.5. Properties of the solution and discussion

The pressure integrals (3.4.16) can be interpreted as Possio type integrals (cf. Possio 1937). The numerator of the integrand represents the 'source strength' and the denominator represents the 'pseudo distance' between the source point $(\xi,\ 0,\ \mu)$ and the field point $(\bar{x},\ y,\ \bar{\tau})$. The domain of integration is that region of the ξ-μ plane in which the denominator is real. This is the hyperbolic area

$$(\bar{\tau} - \mu) \geqslant \{(\bar{x} - \xi)^2 + y^2\}^{\frac{1}{2}},$$

obtained by intersecting the forecone whose vertex is at $(\bar{x},\ y,\ \bar{\tau})$ with the ξ-μ plane (cf. Fig. 9). It is readily seen that only points inside the so defined ξ-μ domain may contribute to a disturbance at the point $(\bar{x},\ y,\ \bar{\tau})$. On the other hand the contributing points are limited to the area in which $f''\{\bar{a}(\mu \pm \lambda_i\xi)\}$ is different from zero. Thus a sector of the hyperbolic area is cut off.

We notice from (3.4.16) that the sources

$$g_1 = A_o f''\{\bar{a}(\mu + \lambda_o\xi)\}\ , \qquad g_2 = -\sum_{i=1}^{5} A_i f''\{\bar{a}(\mu - \lambda_i\xi)\},$$

are spread in the plane of the aerofoil, $y = 0$. The source strength g_1 is in fact given by the boundary condition on the wing (cf. Eq. 3.3.4) with $\bar{x}$ replaced by ξ and $\bar{\tau}$ by μ. We can think of the conditions at the shock being replaced by an equivalent condtion on the aerofoil, at $y = 0$

$$\partial p/\partial y = - \sum_{i=1}^{5} A_i \; f''\{\bar{a}(\bar{\tau} - \lambda_i \bar{x})\}. \tag{3.5.1}$$

Then the source strength g_2 can be given by this condition with $\bar{x}$ replaced by ξ and $\bar{\tau}$ by μ.

It is wothwhile to point out here that Ting & Ludloff (1952) have assumed the solution for the problem of diffraction of plane shock by stationary aerofoils in the form of Possio integral, viz.

$$p(\bar{x}, y, \bar{\tau}) = -\frac{1}{\pi} \iint_D \frac{p_y(\xi, 0, \mu)}{\{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}} d\xi \; d\mu.$$

They argue that such a solution will be applicable provided $\partial p/\partial y$ is prescribed all along the entire plane $y = 0$. However $\partial p/\partial y$ (at $y = 0$) is known only for the half plane $\bar{x} < 0$. Hence they proceed to formulate a reflection principle, using a tedious procedure, by which they replace the conditions at the shock by a condition on the extended plane of the aerofoil beyond the shock plane, i.e. $\partial p/\partial y$ at $y = 0$, for $\bar{x} > 0$. Such a reflection principle on the other hand is implicit in our solution and can be obtained as a corollary (cf. Eq. 3.5.1).

Pressure field

We shall consider the pressure field given by (3.4.15) or (3.4.16) in the form

$$\begin{aligned} p(\bar{x}, y, \bar{\tau}) = \frac{1}{\pi} \Big(&-A_o \int_0^{\bar{\tau}} d\mu \int_{\xi_2}^{0} \frac{1}{r_c} \; f''\{\bar{a}(\mu + \lambda_o \xi)\} \; d\xi \\ &+ \sum_{i=1}^{4} A_i \int_0^{\bar{\tau}} d\mu \int_0^{\xi_1} \frac{1}{r_c} \; f''\{\bar{a}(\mu - \lambda_i \xi)\} \; d\xi \\ &- A_5 \int_0^{\bar{\tau}} d\mu \int_{\xi_2}^{0} \frac{1}{r_c} \; f''\{\bar{a}(\mu + \lambda_4 \xi)\} \; d\xi \Big) \\ &+ A_5 (1/\beta) \; f'\{\bar{a}(\bar{\tau} + \lambda_4 \bar{x} - \bar{b}y)\} \end{aligned} \tag{3.5.2}$$

with ξ_1, ξ_2 and r_c defined in (3.4.16). From the first six terms it is seen that six domains exist in which f is different from zero. Note that since $f(x_1)$, with $f(x_1) = 0$ for $x_1 \leqslant 0$, describes the aerofoil surface, $x_1 = 0$ fixes the leading edge of the aerofoil. While $\xi = -\mu/\lambda_o$ describes the motion of the real leading edge, $\xi = \mu/\lambda_i$ (for i = 1, 2, 3 & 4) and $\xi = \mu/\lambda_5 = -\mu/\lambda_4$ indicate the motion of five "fictitious leading edges". These lines in the ξ-μ plane separate six domains in which the sources $A_i f''(\xi, \mu)$ assume the following values

$$-\mu/\lambda_o < \xi < 0, \quad A_o\, f''\{\bar{a}(\mu + \lambda_o\xi)\},$$

$$0 < \xi < \mu/\lambda_i, \quad A_i\, f''\{\bar{a}(\mu - \lambda_i\xi)\}, \quad \text{for } i = 1, 2, 3 \,\&\, 4,$$

$$-\mu/\lambda_4 < \xi < 0, \quad A_5\, f''\{\bar{a}(\mu + \lambda_4\xi)\}.$$

The conical surface $x^2 + y^2 = \tau^2$, $x \leqslant m\tau$, remains a conical surface in the Lorentz variables, namely $\bar{x}^2 + y^2 = \bar{\tau}^2$, $\bar{x} \leqslant 0$, Fig. 10. Any point $(\bar{x}, y, \bar{\tau})$ lying on this conical surface is influenced by the sources in the ξ-μ plane which lie inside the sector bounded by the straight lines $\xi = \pm\mu/\lambda_i$ and by the hyperbola

$$\mu^2 - \xi^2 - 2\bar{\tau}\mu + 2\bar{x}\xi = 0, \tag{3.5.3}$$

which passes through the origin, Fig. 11. We can see that the integration area does not include the sources $A_i\, f''\{\bar{a}(\mu - \lambda_i\xi)\}$, for i = 1, 2, 3 & 4, since for $\mu > 0$ the hyperbola lies in the second quadrant, while the straight lines $\xi = \mu/\lambda_i$ (i = 1, 2, 3 & 4) lie in the first quadrant (cf. Fig. 11b); hence no sector is cut off by these bounding lines with the hyperbola. Thus for the points on the conical surface, the integrals for i = 1, 2, 3 & 4 in (3.5.2) do not contribute. Also for the points outside the Mach cone for which $\bar{x}^2 + y^2 > \bar{\tau}^2$, the corresponding hyperbola is shifted to the left and the integrals for i = 1, 2, 3 & 4 in (3.5.2) do not contribute.

Again for the points on the conical surface, the hyperbola (3.5.3) will or will not intersect the bounding lines $\xi = -\mu/\lambda_o$ and $\xi = -\mu/\lambda_4$, depending on the location of the point $(\bar{x}, y, \bar{\tau})$. The intersection of the hyperbola with the line $\xi = -\mu/\lambda_o$ will occur if

$|\bar{x}/(\bar{x}^2 + y^2)^{\frac{1}{2}}| > \lambda_o$. Thus the line $y/\bar{x} = -(1 - \lambda_o^2)^{\frac{1}{2}}/\lambda_o$ separates the points $(\bar{x}, y, \bar{\tau})$ on the conical surface which furnish the intersection from those that do not, and it hits the Mach circle at the point of contact of Mach line AC, as indicated in Fig. 10. For any point lying between B and C on the conical surface (Fig. 10) the hyperbola will be located as shown in Fig. 11b, i.e. the first integral in (3.5.2) will be zero here. However for points lying between E and C the hyperbola will be located as shown in Fig. 11c, i.e. the source intensity is different from zero in parts of the integration area. For points on or outside the conical surface to the left of line FC (Fig. 10), the first integral in (3.5.2) then integrates to give

$$I_o = -\frac{A_o}{\bar{a}(1 - \lambda_o^2)^{\frac{1}{2}}} f'[\bar{a}\{\bar{\tau} + \lambda_o\bar{x} - (1 - \lambda_o^2)^{\frac{1}{2}} y\}], \tag{3.5.4}$$

which vanishes for $\bar{\tau} + \lambda_o\bar{x} - (1 - \lambda_o^2)^{\frac{1}{2}} y \leqslant 0$, i.e. for the points on or to the left of AC, which is a Mach wave from the leading edge of the aerofoil and tangent to the Mach circle. The expression (3.5.4) thus represents the solution in the region ACE due to the supersonic flow, $(W + U) > a_1$, over the aerofoil and is the same as obtained by the usual linearized theory of steady supersonic flow.

Similarly for the sixth integral in (3.5.2) we may conclude that for the points on or outside the conical surface, the integral vanishes everywhere except in the region DGE where it gives

$$I_{51} = -A_5\,(1/\beta)\, f'\{\bar{a}(\bar{\tau} + \lambda_4\bar{x} - \bar{b}y)\}. \tag{3.5.5}$$

It vanishes for $\bar{\tau} + \lambda_4\bar{x} - \bar{b}y \leqslant 0$, i.e. for the points on or to the left of GD which is tangent to the conical surface at D. However the last term in (3.5.2), I_{52}, is finite in the region IFG (Fig. 10), and vanishes for the points on or to the left of IG which is tangent to the conical surface at D. Thus we see that the last two terms in (3.5.2) while taken together will vanish for the points lying between E and D on the conical surface, while in the region IDB they yield

$$I_5 = A_5\,(1/\beta)\, f'\{\bar{a}(\bar{\tau} + \lambda_4\bar{x} - \bar{b}y)\}$$

which vanishes on or to the left of ID, another Mach wave.

From the above discussion it will be clear that the hyperbola corresponding to any point in domain I (Fig. 10) with large co-ordinates $\bar{x}$ or y will be shifted considerably in the direction of negative ξ or negative μ. In other words, $p(\infty) = 0$, as stipulated in §3.2. Further it can be seen that if the point under consideration moves continuously from outside to inside across the sonic circle, the respective hyperbola moves continuously into the direction of positive ξ, the domain which is covered by the sources $A_i\ f''\{\bar{a}(\mu - \lambda_i\xi)\}$, for i = 1, 2, 3 & 4. Since the source intensity is finite, it follows that p will vary continuously across the Mach circle.

Density field

We consider the expression (3.4.22) for the density variation, where $p(x, y, \tau=x/m)$ can be obtained form $p(x, y, \tau)$, Eq. (3.5.2), by setting $\tau = x/m$. Thus from the previous discussion we may notice that the first six terms in $p(x, y, \tau=x/m)$ will vanish for

$$\{x^2 + y^2 \geqslant \tau^2\}_{\text{at } \tau=x/m}, \quad \text{or } x(1 - m^2)^{\frac{1}{2}} \leqslant my,$$

i.e. for the points on or to the left of line BO (Fig. 5b), while the last term in $p(x, y, \tau=x/m)$ yields

$$A_5\ (1/\beta)\ f'\{(x - c\bar{y})(m + m_1)/m\}\ , \quad \text{with } \bar{c} = \beta\ m/(m + m_1),$$

which vanishes for $x \leqslant \bar{c}y$, i.e. for the points on or to the left of line IO. Also we may notice that the last term in (3.4.22) vanishes for $x \leqslant \bar{c}y$.

Taking these into account it follows from (3.4.22) that the lines of constant density (isopycnics) will coincide with those of constant pressure (isobars) for $x \leqslant \bar{c}y$, but will deviate from the latter for $x > \bar{c}y$, i.e. inside the area IOF, Fig. 5b. In other words the flow inside IOF is non-isentropic and rotational, while outside it is isentropic, irrotational. Since across the reflected shock BCDE the pressure varies continuously, as noted above, the particles do not acquire the vorticity on crossing the reflected shock, but only by crossing the curved portion of the incident shock IF. These particles stay inside the area swept over by IF, i.e. inside IOF.

The lines IO and BO are thus the lines of contact discontinuity across which the density or its derivatives, depending on the form of f, are discontinuous though the pressure remains continuous.

It is clear from the preceding discussion of the pressure and density fields that the flow pattern for the non-uniform region behind the shock, which was discussed in §1.3, essentially emerges from the solution.

Flow pattern classification

For the sake of completeness we may classify the interaction flow picture at an instant $t > 0$, on the geometrical consideration according to whether (cf. Fig. 12)

$$\phi_1 < \phi_o, \quad \phi_1 > \phi_o > \phi_2, \quad \text{or} \quad \phi_o < \phi_2, \tag{3.5.6}$$

where $\phi_o = \arctan \{1/M'^2 - 1)^{\frac{1}{2}}\}$, $\phi_1 = \arctan \{1/(m_1^2 - 1)^{\frac{1}{2}}\}$,

and $\phi_2 = \arctan \{(1 - m^2)^{\frac{1}{2}}/(m + m_1)\}$.

When $\phi_1 < \phi_o$, the Mach waves AC and ID intersect each other, Fig. 12a; when $\phi_1 > \phi_o > \phi_2$, the Mach waves AC and ID do not intersect, Fig. 12b; when $\phi_o < \phi_2$, the shock-shock intersection lies inside the sonic circle and there is Mach wave AC only, Fig. 12c. These have been termed in the literature as intersecting tangents, non-intersecting tangents, and single tangent cases. From (3.5.6) we may deduce that these three cases will occur according to whether

$$M' < F_1(M), \quad F_1(M) > M' > F_2(M), \quad \text{or} \quad M' < F_2(M), \tag{3.5.7}$$

where

$$F_1(M) = 2(M^2 - 1)/\left[\{2\gamma M^2 - (\gamma - 1)\}^{\frac{1}{2}}\{(\gamma - 1)M^2 + 2\}^{\frac{1}{2}} - (\gamma - 1)M\right],$$

$$F_2(M) = \left[M + \{\ell^2 + \ell(M^2 - 1)\}^{\frac{1}{2}}\right]/(\ell^2 - 1), \text{ with } \ell = \frac{M^2 - 1}{M^2}\,\frac{(\gamma - 1)M^2 + 2}{\gamma + 1}.$$

Figure 13 illustrates the range of M' (the obstacle Mach number) and M (the shock Mach number) for which the three different cases arise. From this figure we may infer that $\phi_o > \phi_2$ (for points M, M' to the left of curve II) for all M and values of M' for which the usual linearized theory yields satisfactory results. It is only for higher values of M' that $\phi_o < \phi_2$ (for points M, M' to the right of curve II). Since the linearized theory only provides for infinitely weak shocks, i.e. small $M'(> 1)$, for all practical calculations we may consider that $\phi_o > \phi_2$, i.e. the shock intersection lies outside the sonic circle (Figs. 12a & b).

3.6. Application

As an example of the theory we shall apply the results of §3.4 to a thin wedge of semi-vertex angle ε. The surface of such a wedge can be expressed as

$$f(x_1) = \varepsilon x_1.$$

Then $$f'(x_1) = \varepsilon H(x_1), \quad \text{and} \quad f''(x_1) = \varepsilon\delta(x_1), \tag{3.6.1}$$

where H and δ represent the unit step and delta functions respectively. Using (3.6.1) in the expression (3.4.16), we obtain for the pressure field

$$p(\bar{x}, y, \bar{\tau}) = \frac{\varepsilon}{\pi\bar{a}}\Big(-A_o\int_{\omega_o}^{0}\{(\bar{\tau} + \lambda_o\xi)^2 - (\bar{x} - \xi)^2 - y^2\}^{-\frac{1}{2}}\,d\xi$$

$$+ \sum_{i=1}^{5} A_i\int_{0}^{\omega_i}\{(\bar{\tau} - \lambda_i\xi)^2 - (\bar{x} - \xi)^2 - y^2\}^{-\frac{1}{2}}\,d\xi\Big), \tag{3.6.2}$$

where $\omega_o = [\bar{x} + \lambda_o\bar{\tau} - \{(\bar{\tau} + \lambda_o\bar{x})^2 - (1 - \lambda_o^2)y^2\}^{\frac{1}{2}}]/(1 - \lambda_o^2)$

$\omega_i = [\bar{x} - \lambda_i\bar{\tau} + \{(\bar{\tau} - \lambda_i\bar{x})^2 - (1 - \lambda_i^2)y^2\}^{\frac{1}{2}}]/(1 - \lambda_i^2)$.

Now (3.6.2) can be easily evaluated and it yields

$$p(\bar{x}, y, \bar{\tau}) = \frac{\varepsilon}{\pi\bar{a}}\Big\{\frac{-A_o}{(1 - \lambda_o^2)^{\frac{1}{2}}}\arccos\Big(\frac{\lambda_o\bar{\tau} + \bar{x}}{\{(\bar{\tau} + \lambda_o\bar{x})^2 - (1 - \lambda_o^2)y^2\}^{\frac{1}{2}}}\Big)$$

$$+ \sum_{i=1}^{5}\frac{A_i}{(1 - \lambda_i^2)^{\frac{1}{2}}}\arccos\Big(\frac{\lambda_i\bar{\tau} - \bar{x}}{\{(\bar{\tau} - \lambda_i\bar{x})^2 - (1 - \lambda_i^2)y^2\}^{\frac{1}{2}}}\Big)\Big\}, \tag{3.6.3}$$

for λ_is' less than unity. Expressing (3.6.3) in the original variables by using the transformation (3.3.1) and simplifying we obtain

$$p(x, y, \tau) = \frac{\varepsilon}{\pi\bar{a}}\left\{\bar{A}_o \arccos\left(\frac{\gamma_o\tau + x}{\{(\tau + \gamma_o x)^2 - (1 - \gamma_o^2)y^2\}^{\frac{1}{2}}}\right)\right.$$

$$\left. + \sum_{i=1}^{5} \bar{A}_i \arccos\left(\frac{\gamma_i\tau - x}{\{(\tau - \gamma_i x)^2 - (1 - \gamma_i^2)y^2\}^{\frac{1}{2}}}\right)\right\}, \qquad (3.6.4)$$

where $\bar{A}_o = - A_o/(1 - \lambda_o^2)^{\frac{1}{2}}$, $\gamma_o = (\lambda_o - m)/(1 - \lambda_o m)$,

and $\bar{A}_i = A_i/(1 - \lambda_i^2)^{\frac{1}{2}}$, $\gamma_i = (\lambda_i + m)/(1 + \lambda_i m)$, for $i = 1, 2, \ldots 5$.

The above result (3.6.4) can also be expressed in terms of conical (physical) variables defined by $X = x/\tau$, $Y = y/\tau$. Thus we obtain

$$p(X, Y) = \frac{\varepsilon}{\pi\bar{a}}\left\{\bar{A}_o \arccos\left(\frac{\gamma_o + X}{\{(1 + \gamma_o X)^2 - (1 - \gamma_o^2)Y^2\}^{\frac{1}{2}}}\right)\right.$$

$$\left. + \sum_{i=1}^{5} \bar{A}_i \arccos\left(\frac{\gamma_i - X}{\{(1 - \gamma_i X)^2 - (1 - \gamma_i^2)Y^2\}^{\frac{1}{2}}}\right)\right\}. \qquad (3.6.5)$$

This easily yields the results for the pressure on the wedge face ($Y = 0$) and along the shock ($X = m$), which can be shown by laborious calculations (cf. Appendix 2) to be the same as the expressions obtained by Smyrl (1963).

For the above calculations we considered $\bar{b} < 1$. In case $\bar{b} > 1$, the last two terms in (3.6.2) must be replaced by the expression (using (3.6.1) in (3.4.18))

$$I_{4,5}(\bar{x}, y, \bar{\tau}) = \frac{\varepsilon}{\pi\bar{a}\bar{b}}\int_0^\infty \Big(Q_1\{n_1(y, \zeta) + n_1(-y, \zeta)\}$$

$$+ Q_2\{n_2(y, \zeta) + n_2(-y, \zeta)\}\Big)\, d\zeta, \qquad (3.6.6)$$

where $n_1(y, \zeta) = \dfrac{\bar{x}}{\bar{x}^2 + (y - \zeta)^2}\,\dfrac{\bar{\tau} - \bar{b}\zeta}{\{(\bar{\tau} - \bar{b}\zeta)^2 - \bar{x}^2 - (y - \zeta)^2\}^{\frac{1}{2}}}$

and $n_2(y, \zeta) = \{(\bar{\tau} - \bar{b}\zeta)^2 - \bar{x}^2 - (y - \zeta)^2\}^{-\frac{1}{2}}$.

On integrating (3.6.6) and by further simplification we are led to

$$I_{4,5}(\bar{x}, y, \bar{\tau}) = \frac{\varepsilon}{\pi \bar{a}\bar{b}}\left\{- Q_1 \arctan\left(\frac{2\bar{x}\bar{N}}{\bar{x}^2 + \lambda_4'^2\bar{\tau}^2 - \bar{N}^2}\right) + \frac{Q_2}{2\lambda_4'} \ln\left(\frac{\bar{x}^2 + (\bar{N} + \lambda_4'\bar{\tau})^2}{\bar{x}^2 + (\bar{N} - \lambda_4'\bar{\tau})^2}\right)\right\}, \tag{3.6.7}$$

with $\bar{N} = \bar{b}(\bar{\tau}^2 - x^2 - y^2)^{\frac{1}{2}}$ and $\lambda_4' = (\bar{b}^2 - 1)^{\frac{1}{2}}$.

We may notice that the first term in (3.6.7) vanishes for $\bar{N} = 0$, while the second term vanishes either for $\bar{\tau} = 0$ (cf. the first initial condition 3.3.3) or for $\bar{N} = 0$. Thus the expression (3.6.7) vanishes for $(\bar{\tau}^2 - \bar{x}^2 - y^2)^{\frac{1}{2}} \leqslant 0$, i.e. for points on or outside the sonic wavelet BCE, Fig. 8. It may further be pointed out that the expression (3.6.7) is also the real part of the last two terms in (3.6.3), as was noted in §3.4. On expressing (3.6.7) in the original variables (x, y, τ) we obtain

$$I_{4,5}(x, y, \tau) = \frac{\varepsilon}{\pi \bar{a}\bar{b}}\left\{- Q_1 \arctan\left(\frac{2(x - m\tau)N}{(x - m\tau)^2 + \lambda_4'^2(\tau - mx)^2 - N^2}\right) + \frac{Q_2}{2\lambda_4'} \ln\left(\frac{(x - m\tau)^2 + \{N + \lambda_4'(\tau - mx)\}^2}{(x - m\tau)^2 + \{N - \lambda_4'(\tau - mx)\}^2}\right)\right\}, \tag{3.6.8}$$

with $N = \bar{b}(1 - m^2)^{\frac{1}{2}}(\tau^2 - x^2 - y^2)^{\frac{1}{2}}$.

Density field

Using (3.6.1) in (3.4.22), we can write for the density variation for the wedge

$$\rho(x, y, \tau) = p(x, y, \tau) + (E - 1)\, p(x, y, \tau{=}x/m) - (\varepsilon/\beta)(k_1F - k_2G)\, H(x - \bar{c}y), \tag{3.6.9}$$

where $p(x, y, \tau)$ is given by (3.6.4), together with (3.6.8), and $p(x, y, \tau{=}x/m)$ is obtained from $p(x, y, \tau)$ by substituting $\tau = x/m$. Thus

$$p(x,\ y,\ \tau=x/m) = \frac{\varepsilon}{\pi\bar{a}} \sum_{i=0}^{3} \bar{A}_i \arccos\left(\frac{\lambda_i(1 - m^2)^{\frac{1}{2}}x}{\{(1 - m^2)x^2 - m^2(1 - \lambda_i^2)y^2\}^{\frac{1}{2}}}\right)$$

$$+ \frac{\varepsilon}{\pi\beta}(A_4 - A_5) \arccos\left(\frac{\lambda_4(1 - m^2)^{\frac{1}{2}}x}{\{(1 - m^2)x^2 - m^2\bar{b}^2y^2\}^{\frac{1}{2}}}\right)$$

$$+ \frac{\varepsilon}{\beta} A_5\, H(x - \bar{c}y), \tag{3.6.10}$$

for $\bar{b} < 1$. When $\bar{b} > 1$, the last two terms in (3.6.10) must be replaced by $I_{4,5}(x,\ y,\ \tau=x/m)$ to be obtained from (3.8.6)

$$I_{4,5}(x,\ y,\ \tau=x/m) = \frac{\varepsilon}{\pi\beta} \frac{Q_2}{2\lambda_4'} \ln\left(\frac{[\bar{b}\{x^2 - y^2m^2/(1 - m^2)\}^{\frac{1}{2}} + \lambda_4'x]^2}{[\bar{b}\{x^2 - y^2m^2/(1 - m^2)\}^{\frac{1}{2}} - \lambda_4'x]^2}\right). \tag{3.6.11}$$

We may notice that the first five terms in $p(x,\ y,\ \tau=x/m)$, Eq. (3.6.10), vanish for $x \leqslant ym/(1 - m^2)^{\frac{1}{2}}$, i.e. to the left of BO, Fig. 5b, and are continuous there. The last term in (3.6.10) vanishes for $x \leqslant \bar{c}y$, i.e. to the left of IO and is discontinuous there.

Since $I_{4,5}(x,\ y,\ \tau)$ vanishes for $\tau^2 - x^2 - y^2 \leqslant 0$, $I_{4,5}(x,\ y,\ \tau=x/m)$, Eq. (3.6.11), must vanish for $x \leqslant ym/(1 - m^2)^{\frac{1}{2}}$, i.e. to the left of BO, Fig. 8, and is continuous there.

Finally the last term in (3.6.9) vanishes for $x \leqslant \bar{c}y$, i.e. to the left of IO and is discontinuous there.

The lines IO and BO are thus the contact discontinuities obtained in the flow field. The nature of these lines is further discussed in §7.1.

Shock displacement

Making use of (3.6.4) in (3.4.24), we obtain for the shock displacement

$$\psi(y,\ \tau) = \tau \frac{1}{\Pi_{21}} \Bigg[p(x = m\tau,\ y,\ \tau)$$

$$+ \frac{\varepsilon}{\pi\bar{a}(1 - m^2)^{\frac{1}{2}}} \Bigg\{A_0\, y \arctan\left(\frac{1 + \gamma_0 m}{\gamma_0 + m} \frac{\{(1 - m^2)\tau^2 - y^2\}^{\frac{1}{2}}}{y}\right)$$

$$- \sum_{i=1}^{5} A_i\, y \arctan\left(\frac{1 - \gamma_i m}{\gamma_i - m} \frac{\{(1 - m^2)\tau^2 - y^2\}^{\frac{1}{2}}}{y}\right)\Bigg\}$$

$$+ \frac{\varepsilon}{\beta}(k_1\Lambda_{21} - k_2\Lambda_{22})\Bigg]. \tag{3.6.12}$$

Chapter 4

AXISYMMETRIC SLENDER BODIES

4.1. Introductory remarks

In the preceding development of shock-on-shock interaction theory we have dealt with two-dimensional planar obstacles, i.e. the problems in which the boundary conditions on the obstacle have been expressible to the required degree of accuracy, in an x-z plane. For sufficiently slender bodies of revolution it is possible to derive an analogous method by specifying the boundary conditions in the vicinity of the body axis. The present chapter is confined to the development of this phase of axial symmetry.

The flow pattern due to the interaction of a plane shock with supersonically moving slender pointed bodies of revolution has been discussed in Chapter 1 (§1.3). In Chapter 2 we have discussed the governing equations of motion (Eqs. 2.2.21-2.2.23, and 2.2.25), the conditions at the disturbed shock (Eqs. 2.3.11a & 2.3.11d) together with the upstream perturbation parameters (Eqs. 2.4.18), which will be used to describe the non-uniform flow field under consideration. In what follows the formulation of the problem will be completed (§§4.2 & 4.3), and the solution will be sought (§4.4) by similar techniques as employed in the previous chapter. The results of the theory will be finally applied (§4.5) to calculate the interaction field for a slender conical projectile.

4.2. Initial and boundary conditions

The initial conditions to be assigned for the disturbed region behind the shock are

$$p(x, r, \tau) = \partial p(x, r, \tau)/\partial\tau = 0, \quad \text{for } \tau \leq 0. \qquad (4.2.1)$$

On the disturbed shock front

All along the shock plane $x = m\tau$, $0 < r < \infty$, we can use the conditions (2.3.11a) and (2.3.11d) which are

$$u = \frac{1}{A}\,(p - B\bar{u} - c\bar{p}), \qquad (4.2.2a)$$

$$\frac{\partial q}{\partial \tau} = \frac{1}{D}\,\left(\frac{\partial p}{\partial r} - \Lambda_{21}\,\frac{\partial \bar{u}}{\partial r} - \Lambda_{22}\,\frac{\partial \bar{p}}{\partial r}\right) + \Lambda_{41}\,\frac{\partial \bar{q}}{\partial \tau}, \qquad (4.2.2b)$$

where the derivatives to τ are taken while travelling with the shock.

On the body surface

Owing to the singularities on the body axis ($r = 0$), it is necessary to formulate the linearized boundary condition at the body surface carefully. Following §2.4.3, for the flow over smooth slender bodies of revolution we can write for the region behind the shock near the body surface, as $r \to 0$

$$rq_1 \to (W + U)\ f\{x + (W + U)t\}\ f'\{x + (W + U)t\},$$

in the co-ordinate system (x, r, t) assumed. In dimensionless form it gives for $x < m\tau$ and $r \to 0$

$$rq \to m_1\ f(x + m_1\tau)\ f'(x + m_1\tau). \qquad (4.2.3)$$

Since the flow is undisturbed to the left of the body nose, we can have for $x < -m_1\tau$ and $r \to 0$

$$rq \to 0.$$

Further since $f(x + m_1\tau) = 0$, for $x < -m_1\tau$, for pointed bodies (cf. §2.4), we can use the condition (4.2.3) all along $-\infty < x \leqslant m\tau$.

Using the second momentum Eq. (2.2.22) it follows that, as $r \to 0$

$$r\ \partial p/\partial r \to -\ m_1^2\ F(x + m_1\tau), \qquad (4.2.4)$$

for $-\infty < x < m\tau$. Here the funtion F is the same as defined in (2.4.11) and is given by

$$F(\xi) = \begin{cases} 0, & \xi \leqslant 0 \\ f'^2(\xi) + f(\xi)\ f''(\xi), & \xi > 0. \end{cases}$$

We may note that the condition (4.2.4) is not valid at $x = m\tau$, since it is deduced by using the momentum equation which leads to infinite values at the shock. To obtain the condition at the shock-body intersection we must invoke the shock relations.

Ahead of and behind the shock, the normal component of velocities on the body surface at the shock location ($x = m\tau$) are given by

$$q_o \approx W f'\{(m + m_1)\tau\},$$

and

$$q_1 \approx (W + U) f'\{(m + m_1)\tau\}.$$

Using these in the shock relation (2.3.10f) which states

$$q_1 - q_o = - U\psi_r,$$

we are led to

$$\psi_r = - f',$$

at the shock-body intersection. Hence the shock intersects the body surface orthogonally.

From (4.2.3) we can obtain, on the body surface ($r \to 0$) along the shock ($x = m\tau$)

$$rq \to m_1 \ f\{(m + m_1)\tau\} \ f'\{(m + m_1)\tau\}. \qquad (4.2.5)$$

Using this relation in the shock condition (4.2.2b) it follows that, at $x = m\tau$, as $r \to 0$

$$r \ \partial p/\partial r \to D \ m_1(m + m_1) \ F\{(m + m_1)\tau\} + \lim_{r\to 0} (\Lambda_{21} \ r \ \partial\bar{u}/\partial r + \Lambda_{22} \ r \ \partial\bar{p}/\partial r - D \ \Lambda_{41} \ r \ \partial\bar{q}/\partial\tau). \qquad (4.2.6)$$

We may note that at the intersection of the shock and the body surface there is a discontinuity in the derivatives of p characterized by Eqs. (4.2.4) and (4.2.6). The reasons for this non-uniformity are similar to the one's discussed in §3.2 for the two-dimensional case.

At infinity

For the region behind the shock ($x < m\tau$) we can prescribe that all the perturbations vanish at infinity, hence

$$p(x, r, \tau) = 0, \qquad \text{for } x \to -\infty, \ r \to \infty, \qquad (4.2.7)$$

together with its derivatives.

4.3. Formulation in Lorentz variables

We introduce new independent variables ($\bar{x}$, r, $\bar{\tau}$) related to the variables (x, r, τ) by the Lorentz transformation defined by (3.3.1). The shock plane $x = m\tau$ now becomes the plane $\bar{x} = 0$. The governing wave Eq. (2.2.25) remains invariant, viz.

$$\frac{\partial^2 p}{\partial \bar{x}^2} + \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} - \frac{\partial^2 p}{\partial \bar{\tau}^2} = 0. \qquad (4.3.1)$$

The initial conditions (4.2.1), and the boundary condtions (4.2.7) and (4.2.4) can be written in new variables as

$$\text{for } \bar{\tau} \leq 0,\ \bar{x} < 0, \qquad p(\bar{x}, r, \bar{\tau}) = \partial p(\bar{x}, r, \bar{\tau})/\partial\bar{\tau} = 0, \qquad (4.3.2)$$

$$\bar{\tau} > 0,\ \bar{x} < 0, \qquad \text{as } \bar{x} \to -\infty,\ r \to \infty$$

$$p(\bar{x}, r, \bar{\tau}) \text{ and its derivatives} \to 0, \qquad (4.3.3)$$

$$\bar{\tau} > 0, \quad -\infty < \bar{x} < 0, \quad \text{as } r \to 0$$

$$r\,\partial p/\partial r \to A_o\ F\{\bar{a}(\bar{\tau} + \lambda_o \bar{x})\}, \qquad (4.3.4)$$

where A_o, $\bar{a}$ and λ_o are the same constants as defined under (3.3.4).

Using the Lorentz transformation (3.3.1), we obtain from the basic Eqs. (2.2.21) together with (2.2.23), and the first of Eqs. (2.2.22)

$$\left(\frac{\partial}{\partial\bar{\tau}} - m\frac{\partial}{\partial\bar{x}}\right)p + \left(\frac{\partial}{\partial\bar{x}} - m\frac{\partial}{\partial\bar{\tau}}\right)u + (1 - m^2)^{\frac{1}{2}}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)q = 0,$$

and
$$m\frac{\partial u}{\partial\bar{x}} = \frac{\partial u}{\partial\bar{\tau}} + \left(\frac{\partial}{\partial\bar{x}} - m\frac{\partial}{\partial\bar{\tau}}\right)p.$$

Eliminating $\partial u/\partial\bar{x}$ and differentiating with respect to $\bar{\tau}$ it follows that

$$\frac{\partial^2 p}{\partial\bar{x}\partial\bar{\tau}} + \frac{\partial^2 u}{\partial\bar{\tau}^2} + \frac{m}{(1 - m^2)^{\frac{1}{2}}}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{\partial q}{\partial\bar{\tau}} = 0. \qquad (4.3.5)$$

Using the shock conditions (4.2.2) we can substitute for u and $\partial q/\partial\bar{\tau}$ and thus obtain a differential expression for p which must be satisfied at $\bar{x} = 0$, $0 < r < \infty$

$$\frac{\partial^2 p}{\partial\bar{x}\partial\bar{\tau}} + \frac{1}{A}\frac{\partial^2 p}{\partial\bar{\tau}^2} + \frac{m}{1 - m^2}\frac{1}{D}\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r}\right)$$

$$= \frac{1}{A}\frac{\partial^2}{\partial\bar{\tau}^2}(B\bar{u} + C\bar{p}) + \frac{m}{1 - m^2}\frac{1}{D}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)(\Lambda_{21}\bar{u} + \Lambda_{22}\bar{p})$$

$$- \frac{m}{(1 - m^2)^{\frac{1}{2}}}\Lambda_{41}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{\partial\bar{q}}{\partial\bar{\tau}}. \qquad (4.3.6)$$

The upstream perturbation parameters $\bar{u}$, $\bar{q}$, $\bar{p}$ at the shock are given by (2.4.18), which in terms of Lorentz variables may be expressed as follows

$$\left.\begin{aligned} \bar{u} &= -k_1 \int_0^{\bar{\tau}-\bar{b}r} \frac{F(\bar{a}\mu)}{\{(\bar{\tau} - \mu)^2 - \bar{b}^2 r^2\}^{\frac{1}{2}}}\, d\mu, \\ \bar{q} &= k_1 \frac{\bar{a}}{r} \int_0^{\bar{\tau}-\bar{b}r} \frac{(\bar{\tau} - \mu)\, F(\bar{a}\mu)}{\{(\bar{\tau} - \mu)^2 - \bar{b}^2 r^2\}^{\frac{1}{2}}}\, d\mu, \\ \bar{p} &= k_2(-\bar{u}/k_1), \end{aligned}\right\} \qquad (4.3.7)$$

where $k_1 = M'/M$, $k_2 = M'^2$, and $\bar{b} = \beta/\bar{a}$.

Substituting for $\bar{u}$, $\bar{q}$, $\bar{p}$ from (4.3.7) in (4.3.6) and simplifying, we are led to, at $\bar{x} = 0$, $0 < r < \infty$

$$-\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r}\right) + 2m\frac{\partial^2 p}{\partial\bar{x}\partial\bar{\tau}} + (1 + 1/M^2)\frac{\partial^2 p}{\partial\bar{\tau}^2}$$

$$= K\frac{\partial^2}{\partial\bar{\tau}^2}\int_0^{\bar{\tau}-\bar{b}r}\frac{F(\bar{a}\mu)}{\{(\bar{\tau} - \mu)^2 - \bar{b}^2 r^2\}^{\frac{1}{2}}}\, d\mu, \qquad (4.3.8)$$

where K is as defined under (3.3.10). The condition (4.3.8) must be supplemented by (4.2.6), which together with (4.3.7) yields at $\bar{x} = 0$, as $r \to 0$

$$r\,\partial p/\partial r \to B_o\, F(\bar{a}\bar{\tau}), \qquad (4.3.9)$$

where B_o is as defined under (3.3.11).

With the wave Eq. (4.3.1) and the initial- and boundary-conditions (4.3.2)-(4.3.4), (4.3.8) and (4.3.9), the formulation of the problem is complete in the $(\bar{x}, r, \bar{\tau})$ variables.

4.4. Solution

To obtain the solution to the above formulation we first introduce the Laplace transfrom with respect to $\bar{\tau}$ as in §3.3. Next in view of the axial symmetry of the problem we shall apply the Hankel transform with respect to r. These transforms are defined as follows (cf. Sneddon 1951)

$$p^*(\bar{x}, r, s) = L\{p(\bar{x}, r, \bar{\tau})\} = \int_0^\infty p(\bar{x}, r, \bar{\tau}) \exp\{-s\bar{\tau}\} \, d\bar{\tau}, \tag{4.4.1}$$

$$p^{**}(\bar{x}, \alpha, s) = H\{p^*(\bar{x}, r, s)\} = \int_0^\infty r \, p^*(\bar{x}, r, s) \, J_o(\alpha r) \, dr, \tag{4.4.2}$$

where J_o is a Bessel function of the first kind of order zero.

Application of the Laplace transform to the wave Eq. (4.3.1) together with the initial condtions (4.3.2) gives

$$\frac{\partial^2 p^*}{\partial \bar{x}^2} + \frac{\partial^2 p^*}{\partial r^2} + \frac{1}{r}\frac{\partial p^*}{\partial r} - s^2 p^* = 0. \tag{4.4.3}$$

The boundary condtions (4.3.3), (4.3.4), (4.3.8) together with the initial conditions (4.3.2), and (4.3.9) reduce to

$$\text{as } \bar{x} \to -\infty, \ r \to \infty \qquad p^* \text{ and its derivatives} \to 0, \tag{4.4.4}$$

$$\text{for } -\infty < \bar{x} < 0, \text{ as } r \to 0 \qquad r \, \partial p^*/\partial r \to A_o \exp\{s\lambda_o \bar{x}\} \, G(s), \tag{4.4.5}$$

at $\bar{x} = 0$, $0 < r < \infty$

$$-\left(\frac{\partial^2 p^*}{\partial r^2} + \frac{1}{r}\frac{\partial p^*}{\partial r}\right) + 2ms \frac{\partial p^*}{\partial \bar{x}} + (1 + 1/M^2) \, s^2 p^* = K s^2 \, G(s) \, K_o(\bar{b} r s), \tag{4.4.6}$$

$$\bar{x} = 0, \text{ as } r \to 0 \qquad r \, \partial p^*/\partial r \to B_o G(s), \tag{4.4.7}$$

where $G(s) = L\{F(\bar{a}\bar{\tau})\}$ and K_o is the modified Bessel function of the second kind of order zero.

Now we apply the Hankel transform (4.4.2) to the Eq. (4.4.3), which together with the conditions (4.4.4) and (4.4.5), yields

$$\frac{\partial^2 p^{**}}{\partial \bar{x}^2} - \lambda^2 p^{**} = A_o \{\exp s\lambda_o \bar{x}\} \, G(s), \tag{4.4.8}$$

where $\lambda^2 = \alpha^2 + s^2$. Also the condition (4.4.6), together with the conditions (4.4.4) and (4.4.7), gives at $\bar{x} = 0$

$$(\lambda^2 + s^2/M^2)p^{**} + 2ms \frac{\partial p^{**}}{\partial \bar{x}} = (K \frac{s^2}{\alpha^2 + \bar{b}^2 s^2} - B_o)\ G(s). \qquad (4.4.9)$$

Finally the condition (4.4.4) at infinity gives

$$\text{as } \bar{x} \to -\infty \qquad p^{**}_{-} \to 0. \qquad (4.4.10)$$

We note here that the formulation (4.4.8)-(4.4.10) is the same as obtained in §3.4, Eqs. (3.4.8)-(3.4.10). The complete solution of the present formulation is then given by (3.4.14), viz.

$$p^{**}(\bar{x}, \alpha, s) = \tfrac{1}{2}A_o\left(\frac{1}{\lambda - \lambda_o s}\frac{\exp\{\lambda\bar{x}\}}{\lambda} - \frac{2}{\lambda^2 - \lambda_o^2 s^2}\exp\{s\lambda_o\bar{x}\}\right) G(s)$$
$$+ \frac{1}{2}\sum_{i=1}^{5} A_i \frac{1}{\lambda + \lambda_i s}\frac{\exp\{\lambda\bar{x}\}}{\lambda}\ G(s), \qquad (4.4.11)$$

where A_is' and λ_is' are the same as defined under (3.4.14). The above expression when expressed in integral form (cf. Appendix 1.1) can be written as

$$p^{**}(\bar{x}, \alpha, s) = \frac{1}{2}\Big(-A_o\int_{-\infty}^{0} \exp\{s\lambda_o\xi\}\ G(s)\ \frac{1}{\lambda}\exp\{-\lambda(|\bar{x} - \xi|)\}\ d\xi$$
$$+ \sum_{i=1}^{4} A_i\int_{-\infty}^{0} \exp\{s\lambda_i\xi\}\ G(s)\ \frac{1}{\lambda}\exp\{\lambda(\bar{x} + \xi)\}\ d\xi$$
$$- A_5\int_{-\infty}^{0} \exp\{s\lambda_4\xi\}\ G(s)\ \frac{1}{\lambda}\exp\{-\lambda(|\bar{x} - \xi|)\}\ d\xi\Big)$$
$$+ A_5\{1/(\alpha^2 + \bar{b}^2 s^2)\}\exp\{s\lambda_4\bar{x}\}\ G(s). \qquad (4.4.12)$$

The inversion of (4.4.12) in this case (cf. Appendix 3.1) leads us to

$$p(\bar{x}, r, \bar{\tau}) = L^{-1}\left[H^{-1}\{p^{**}(\bar{x}, \alpha, s)\}\right]$$
$$= \frac{1}{2}\Big(-A_o\int_{-\infty}^{0} \frac{1}{\bar{R}}\ F\{\bar{a}(\bar{\tau} - \bar{R} + \lambda_o\xi)\}\ d\xi$$
$$+ \sum_{i=1}^{4} A_i\int_{-\infty}^{0} \frac{1}{\bar{R}_1}\ F\{\bar{a}(\bar{\tau} - \bar{R}_1 + \lambda_i\xi)\}\ d\xi\ -$$

$$- A_5\int_{-\infty}^{0} \frac{1}{\bar{R}} F\{\bar{a}(\bar{\tau} - \bar{R} + \lambda_4\xi)\}\, d\xi\Big)$$

$$+ A_5 \int_{0}^{\bar{\tau}+\lambda_4\bar{x}-\bar{b}r} \frac{F(\bar{a}\mu)}{\{(\bar{\tau} + \lambda_4\bar{x} - \mu)^2 - \bar{b}^2r^2\}^{\frac{1}{2}}}\, d\mu, \tag{4.4.13}$$

where $\bar{R} = \{(\bar{x} - \xi)^2 + r^2\}^{\frac{1}{2}}$, and $\bar{R}_1 = \{(\bar{x} + \xi)^2 + r^2\}^{\frac{1}{2}}$.
Alternatively, we can express (4.4.13) as follows

$$p(\bar{x}, r, \bar{\tau}) = \frac{1}{2}\Big(- A_o\int_{-\infty}^{0} \frac{1}{\bar{R}} F\{\bar{a}(\bar{\tau} - \bar{R} + \lambda_o\xi)\}\, d\xi$$

$$+ \sum_{i=1}^{5} A_i\int_{0}^{\infty} \frac{1}{\bar{R}} F\{\bar{a}(\bar{\tau} - \bar{R} - \lambda_i\xi)\}\, d\xi\Big), \tag{4.4.14}$$

with $\lambda_5 = - \lambda_4$. Using the definition of F (cf. Eq. 4.2.4) the last expression can also be written as

$$p(\bar{x}, r, \bar{\tau}) = \frac{1}{2}\Big(- A_o\int_{\omega_o}^{0} \frac{1}{\bar{R}} F\{\bar{a}(\bar{\tau} - \bar{R} + \lambda_o\xi)\}\, d\xi$$

$$+ \sum_{i=1}^{5} A_i\int_{0}^{\omega_i} \frac{1}{\bar{R}} F\{\bar{a}(\bar{\tau} - \bar{R} - \lambda_i\xi)\}\, d\xi\Big), \tag{4.4.15}$$

where $\omega_o = \bar{x} + \lambda_o\bar{\tau} - \{(\bar{\tau} + \lambda_o\bar{x})^2 - (1 - \lambda_o^2)r^2\}^{\frac{1}{2}}/(1 - \lambda_o^2)$,

and $\omega_i = \bar{x} - \lambda_i\bar{\tau} + \{(\bar{\tau} - \lambda_i\bar{x})^2 - (1 - \lambda_i^2)r^2\}^{\frac{1}{2}}/(1 - \lambda_i^2)$.

Finally by means of the transformation (3.3.1), $p(\bar{x}, r, \bar{\tau})$ can be expressed in terms of original variables (x, r, τ).

In deducing the above results we assumed $\bar{b} < 1$. As discussed in §3.4, this implies that the shock intersection I lies outside the sonic circle (cf. Fig. 2). In case the shock intersection lies inside the sonic circle, $\bar{b} > 1$; λ_4 and λ_5 in the above analysis then become imaginary. Under these conditions we can consider the last two terms in (4.4.11) as follows (cf. Eq. 3.4.17)

$$I^{**}_{4,5} = (Q_1\lambda + Q_2 s)\,\frac{1}{\alpha^2 + \bar{b}^2 s^2}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda}\,G(s), \tag{4.4.16}$$

where Q_1 and Q_2 are the same as defined in (3.4.17). The inversion of (4.4.16) can be expressed in the form (cf. Appendix 3.2)

$$I_{4,5} = \frac{1}{\bar{b}}\,(Q_1\,\frac{\partial}{\partial\bar{x}} + Q_2\,\frac{\partial}{\partial\bar{\tau}})\int_0^{\bar{\tau}} F\{\bar{a}(\bar{\tau} - \mu)\}\,E(\mu)\,d\mu \tag{4.4.17}$$

with $E(\mu) = \int_0^{\mu} d\mu_1 \int_0^{\infty} \sin(\alpha\mu/\bar{b})\,J_0\big[\alpha\{(\mu - \mu_1)^2 - \bar{x}^2\}^{\frac{1}{2}}\big]\,J_0(\alpha r)\,d\alpha.$

To evaluate (4.4.17) one must in general resort to numerical methods which will not be pursued here further in view of the remarks made at the end of §3.5.

Density and shock displacement

Integrating (2.2.23) and using the shock condition (2.3.12) we can obtain the density variation in the disturbance field behind the shock. This is given by

$$\begin{aligned}\rho(x, r, \tau) &= p(x, r, \tau) + (E - 1)\,p(x, r, \tau{=}x/m) \\ &\quad - (Fk_1 - Gk_2)\int_0^{\bar{x}-\bar{c}r} \frac{F(\beta\mu/\bar{c})}{\{(x - \mu)^2 - \bar{c}^2 r^2\}^{\frac{1}{2}}}\,d\mu,\end{aligned} \tag{4.4.18}$$

with $\bar{c} = \beta m/(m + m_1)$.

The shock displacement $\psi(r, \tau)$ can be obtained by integrating the shock relation (2.3.10b). Thus we have

$$\begin{aligned}\psi(r, \tau) &= \frac{1}{\Pi_{21}}\Big(\int_0^{\tau'} p(x{=}m\mu, r, \mu)\,d\mu + (\Lambda_{21}k_1 - \Lambda_{22}k_2) \\ &\quad \times \int_0^{\tau} d\mu \int_0^{(m+m_1)\mu-\beta r} \frac{F(\xi)}{\big[\{(m + m_1)\mu - \xi\}^2 - \beta^2 r^2\big]^{\frac{1}{2}}}\,d\xi\Big).\end{aligned} \tag{4.4.19}$$

Properties of the solution

The pressure integrals (4.4.13) or (4.4.14) can be interpreted as 'retarded potential' integrals. The numerator of the integrands represent the 'temporary sources' distributed along the ξ-axis, and $\bar{R}$ is the real distance between the source point (ξ, 0) and the field point ($\bar{x}$, r) at a time $\bar{\tau}$ caused by the value of the source function at ξ at a time $\bar{\tau} - (R \pm \lambda_i \xi)$. That is, a spherical wave moves outwards from its origin at a unit velocity inducing disturbances in the flow field only at those points that are momentarily on the spherical surface itself.

In a similar way as for the two-dimensional case (§3.5), it may be concluded that for the points on or outside the spherical wavelet $\bar{x}^2 + r^2 \geqslant \bar{\tau}^2$, (BCDE, Fig. 2), the first integral in (4.4.13) is finite in the region ACE, while it vanishes elsewhere, and is given by

$$I_o = -A_o \int_0^{\bar{\tau}+\lambda_o\bar{x}-(1-\lambda_o^2)^{\frac{1}{2}}r} \frac{F(\bar{a}\mu)}{\{(\bar{\tau} + \lambda_o\bar{x} - \mu)^2 - (1 - \lambda_o^2)r^2\}^{\frac{1}{2}}} d\mu. \qquad (4.4.20)$$

This vanishes for $\bar{\tau} + \lambda_o\bar{x} - (1 - \lambda_o^2)^{\frac{1}{2}}r \leqslant 0$, i.e. for the points on or to the left of AC, the Mach conoid with vertex at A (the nose of the axisymmetric body) and tangent to the wavelet BCDE at C, Fig. 2. Again the fifth integral in (4.4.13) behaves exactly as the first integral with λ_o replaced by λ_4. Further the last term in (4.4.13) vanishes for $\bar{\tau} + \lambda_4\bar{x} - \bar{b}r \leqslant 0$, i.e. for the points on or to the left of a surface of revolution with I (the shock intersection) as the vertex and tangent to the spherical wavelet BCDE (Fig. 2) at D and extending to the body surface. Then the last two terms in (4.4.13) when taken together will vanish outside the spherical wavelet everywhere except in the region IDB, where they reduce to yield

$$I_5 = A_5 \int_0^{\bar{\tau}+\lambda_4\bar{x}-\bar{b}r} \frac{F(\bar{a}\mu)}{\{(\bar{\tau} + \lambda_4\bar{x} - \mu)^2 - \bar{b}^2r^2\}^{\frac{1}{2}}} d\mu. \qquad (4.4.21)$$

Thus ID is another Mach conoid from the shock intersection I, Fig. 2. The remaining four integrals (for i = 1, 2, 3 & 4) in (4.4.13) vanish for the points on or outside the sonic wavelet BCDE.

Now we consider the expression (4.4.18) for the density variation, where p(x, r, τ=x/m) is given by (4.4.13) at the instant the fluid particles cross the shock (i.e. $\tau = x/m$). Hence it follows from the above discussion that the first five terms in $p(x, r, \tau = x/m)$ will vanish for

$$\{\bar{x}^2 + r^2 \geqslant \bar{\tau}^2\}_{\text{at } \tau = x/m}, \quad \text{or} \quad x(1 - m^2)^{\frac{1}{2}} \leqslant mr,$$

i.e. to the left of the surface BO, Fig. 2. The last term in $p(x, y, \tau = x/m)$ yields

$$A_5 \int_0^{x - \bar{c}r} \frac{F(\beta\mu/\bar{c})}{\{(x - \mu)^2 - \bar{c}^2 r^2\}^{\frac{1}{2}}}\, d\mu$$

which vanishes for $x \leqslant \bar{c}r$, i.e. for the points on or to the left of the surface IO, Fig. 2. Also the last expression in (4.4.18) vanishes for $x \leqslant \bar{c}r$. Considering these it follows from (4.4.18) that the lines of constant density coincide with the lines of constant pressure for $x \leqslant \bar{c}r$ but deviate from the latter for $x > \bar{c}r$, i.e. the flow inside the region IOF (Fig. 2) will be rotational while outside irrotational.

It follows again that the flow pattern, which was discussed in §1.3, can be seen to emerge from the solution of the problem.

4.5. Application

We specialize the results of §4.4 for interaction with a slender conical projectile of nose angle 2ε. From (4.4.15) we then obtain for the pressure

$$p(\bar{x}, r, \bar{\tau}) = \tfrac{1}{2}\varepsilon^2\left(- A_0 \int_{\omega_0}^{0} \frac{d\xi}{\{(\bar{x} - \xi)^2 + r^2\}^{\frac{1}{2}}} + \sum_{i=1}^{5} A_i \int_0^{\omega_i} \frac{d\xi}{\{(\bar{x} - \xi)^2 + r^2\}^{\frac{1}{2}}}\right)$$

$$= \tfrac{1}{2}\varepsilon^2\left(Q \operatorname{arcsinh}\{\bar{x}/r\} - \sum_{i=0}^{5} A_i \operatorname{arcsinh}\{(\bar{x} - \omega_i)/r\}\right), \quad (4.5.1)$$

where $Q = \sum_{i=0}^{5} A_i = 2(A_0 - B_0)$, and ω_is' are defined in (4.4.15).

Transforming to the original variables (x, r, τ)

$$p(x, r, \tau) = \tfrac{1}{2}\varepsilon^2\left\{Q \operatorname{arcsinh}\left(\frac{x - m\tau}{(1 - m^2)^{\frac{1}{2}} r}\right) - \sum_{i=0}^{5} A_i \operatorname{arcsinh}\left(\frac{x - m\tau - \Omega_i}{(1 - m^2)^{\frac{1}{2}} r}\right)\right\}, \quad (4.5.2)$$

where $\Omega_o = \{(1 + \gamma_o m)/(1 - \gamma_o^2)\}[(x + \gamma_o\tau) - \{(\tau + \gamma_o x)^2 - (1 - \gamma_o^2)r^2\}^{\frac{1}{2}}]$,

$\Omega_i = \{(1 - \gamma_i m)/(1 - \gamma_i^2)\}[(x - \gamma_i\tau) + (\tau - \gamma_i x)^2 - (1 - \gamma_i^2)r^2\}^{\frac{1}{2}}]$,

with $\gamma_o = (\lambda_o - m)/(1 - \lambda_o m)$, $\gamma_i = (\lambda_i + m)/(1 + \lambda_i m)$, for $i = 1, 2 \ldots 5$.

When expressed in terms of conical variables defined by

$$\sigma = x/\tau, \quad \eta = r/\tau,$$

the result (4.5.2) becomes

$$p(\sigma, \eta) = \tfrac{1}{2}\,\varepsilon^2 \left\{Q \operatorname{arcsinh}\left(\frac{\sigma - m}{(1 - m^2)^{\frac{1}{2}}\eta}\right) - \sum_{i=0}^{5} A_i \operatorname{arcsinh}\left(\frac{\sigma - m - \Omega_i(\sigma, \eta)}{(1 - m^2)^{\frac{1}{2}}\eta}\right)\right\}, \tag{4.5.3}$$

where $\Omega_i(\sigma, \eta) = (1/\tau)\,\Omega_i(x, r, \tau)$.

At the surface of the conical projectile, where $\eta = \varepsilon(m_1 + \sigma)$, $\Omega_i(\sigma, \eta)$ approximates to

$$\Omega_o \simeq -(1 - \sigma)(1 + \gamma_o m)/(1 + \gamma_o),$$

and $$\Omega_i \simeq (1 + \sigma)(1 - \gamma_i m)/(1 + \gamma_i), \qquad \text{for } i = 1, 2 \ldots 5.$$

The expression (4.5.3) then simplifies and yields at the cone surface

$$p = \tfrac{1}{2}\,\varepsilon^2 \left\{-Q \operatorname{arcsinh}\left(\frac{1}{\varepsilon(1 - m^2)^{\frac{1}{2}}}\,\frac{m - \sigma}{m_1 + \sigma}\right) - A_o \operatorname{arcsinh}\left(\frac{1}{\varepsilon(1 - m^2)^{\frac{1}{2}}}\,\frac{1 - m}{1 + \gamma_o}\,\frac{1}{m_1}\right)\right.$$
$$\left. + \sum_{i=1}^{5} A_i \operatorname{arcsinh}\left(\frac{1}{\varepsilon(1 - m^2)^{\frac{1}{2}}}\,\frac{1 + m}{1 + \gamma_i}\,\frac{1 - \gamma_i\sigma}{m_1 + \sigma}\right)\right\}. \tag{4.5.4}$$

Now for the density variation, we obtain from (4.4.18) for the cone under consideration

$$\rho = p(x, r, \tau) + (E - 1)\,p(x, r, \tau{=}x/m) - (Fk_1 - Gk_2)\varepsilon^2 \operatorname{arccosh}\{x/\bar{c}r\}. \tag{4.5.5}$$

Here using (4.5.1) we can obtain for

$$p(x, r, \tau{=}x/m) = \tfrac{1}{2}\,\varepsilon^2\{A_o \operatorname{arcsinh} \nu_o - \sum_{i=1}^{3} A_i \operatorname{arcsinh} \nu_i$$
$$- (A_4 - A_5) \operatorname{arcsinh} \nu_4\} + \varepsilon^2 A_5 \operatorname{arccosh}\{x/(\bar{c}r)\},$$

with $$\nu_i = \frac{\lambda_i(1 - m^2)^{\frac{1}{2}}x - \{(1 - m^2)x^2 - m^2(1 - \lambda_i^2)r^2\}^{\frac{1}{2}}}{m(1 - \lambda_i^2)r^2}, \quad \text{for } i = 0, 1, \ldots 4.$$

Now we may notice that the first five terms in $p(x, y, \tau=x/m)$ vanish for $x \leqslant rm/(1 - m^2)$, i.e. to the left of surface BO (Fig. 2), and are continuous there. Further the last term in $p(x, y, \tau=x/m)$ and the last expression in (4.5.5) vanish for $x \leqslant \bar{c}r$, i.e. to the left of IO and are continuous there. The nature of these surfaces is discussed in Chapter 7.

Using (4.5.2) in (4.4.19), we can obtain the shock displacement $\psi(r, \tau)$ for interaction with the cone. Noting that $\psi(r, \tau)$ should remain continuous at B (the intersection of the reflected shock with the main shock, Fig. 2), on integrating (4.4.15) we are led to

$$\begin{aligned}\psi(r, \tau) = \frac{1}{\Pi_{21}}\Bigg\{&\tau\, p(x=m\tau, r, \tau) \\ &+ \tfrac{1}{2}\,\varepsilon^2\Bigg(\frac{A_o}{1 + m\gamma_o}\Big[(m + \gamma_o)\tau - \{(1 + m\gamma_o)^2\tau^2 - (1 - \gamma_o^2)r^2\}^{\frac{1}{2}}\Big] \\ &- \sum_{i=1}^{5} \frac{A_o}{1 - m\gamma_i}\Big[(m - \gamma_i)\tau + \{(1 - m\gamma_i)^2\tau^2 - (1 - \gamma_i^2)r^2\}^{\frac{1}{2}}\Big]\Bigg) \\ &+ \varepsilon^2(k_1\Lambda_{21} - k_2\Lambda_{22})\Big[\tau\,\operatorname{arccosh}\{(m + m_1)\tau/\beta r\} \\ &\qquad - \{\tau^2 - \beta^2 r^2/(m + m_1)^2\}^{\frac{1}{2}}\Big]\Bigg\}. \qquad (4.5.6)\end{aligned}$$

Chapter 5

THREE-DIMENSIONAL PLANAR WINGS

5.1. Introductory remarks

This chapter deals with the diffraction of a shock by supersonically moving three-dimensional thin wings. The general three-dimensional problem can be divided into two parts, the symmetrical and anti-symmetrical problems. The problem is regarded symmetrical if the wing under consideration is a symmetrical wing at zero incidence, and antisymmetrical if the wing is antisymmetrical, i.e. a wing with zero thickness but at an incidence to the oncoming flow. The different physical cases present analytical problems of varying degrees of difficulty. The simplest case is that of the interaction of a plane shock with a symmetrical wing, or an antisymmetrical wing with supersonic leading edges for which the principle of independence is valid.

The main features of the flow pattern developed behind the shock as a result of the interaction are essentially the same as discussed in Chapter 1 for an axisymmetric body, except that now a three-dimensional flow picture is to be visualized. Figures 14 and 15 show the flow configurations for time $t \leqslant 0$ and $t > 0$ respectively. The cartesian co-ordinates (x, y, z) are fixed in the undisturbed flow behind the shock, Fig. 15. The assumptions to be made have been discussed in §1.3.

The solution to the three-dimensional problem is a generalization of that already achieved for two-dimensional and axisymmetric cases. The governing equations of motion to be considered now are Eqs. (2.2.16)-(2.2.19), and (2.2.24), while the conditions at the disturbed shock are now (2.3.11a)-(2.3.11c) together with the upstream perturbations at the shock given by (2.4.19). The additional conditions will be formulated to complete the solution of the problem.

It is shown that the solution to the three-dimensional wave Eq. (2.2.24) subject to the appropriate boundary conditions leads us to the formulas, which may be regarded as requiring either the evaluation of integrals, or the solution of an integral equation, depending upon whether the boundary value problem is symmetrical or antisymmetrical. In general the integral equation obtained for the antisymmetrical problem is very complicated (cf. §5.5), the solution of which will not be pursued here. However, it is indicated that a certain class of antisymmetrical problems can be handled without recourse to the solution of the integral equation (cf. §5.6).

The material presented in the sequel is divided into two parts. In the first part (§§5.2-5.4) the symmetrical problem is treated, while the second part (§§5.5&5.6) is concerned with the antisymmetrical problem. In the last section (§5.7), some applications of the theory are illustrated.

Part I Symmetrical problem

5.2. Initial and boundary conditions

At the instant the shock just coincides with the wing vertex, we can write the initial conditions for the disturbed region behind the shock, $x < m\tau$

$$p(x, y, z, \tau) = \partial p/\partial \tau = 0, \quad \text{for } \tau \leqslant 0. \tag{5.2.1}$$

At infinity

Since all the perturbations must vanish at infinity, we can write for $x < m\tau$

$$\text{as } x \to -\infty,\ y \to \pm\infty,\ z \to \pm\infty \quad p \text{ and its derivatives} \to 0. \tag{5.2.2}$$

On the wing

For the symmetrical problem the flow pattern on the two sides of the wing will be symmetrical, and it is sufficient to consider the flow problem, say, for $y > 0$. The vertical component of the perturbation velocity is then odd in the y co-ordinate. Hence for the region behind the shock we can prescribe on the wing, y = 0

$$v_1 = \begin{cases} (W + U)\ \eta'\{x + (W + U)t,\ z\}, & \text{on the wing projection } \Sigma \\ 0, & \text{elsewhere} \end{cases}$$

where the prime denotes the differentiation with respect to the first argument. This becomes in dimensionless form

$$v = \begin{cases} m_1\ \eta'\{(x + m_1\tau),\ z\}, & \text{on } \Sigma \\ 0, & \text{elsewhere,} \end{cases} \tag{5.2.3}$$

Using the second momentum Eq. (2.2.17) it follows that, at y = 0

$$\partial p/\partial y = -\ m_1^2\ F_1\{(x + m_1\tau),\ z\}, \tag{5.2.4}$$

where F_1 is the same as defined under (2.4.15a)

$$F_1(x_1,\ z) = \begin{cases} \eta''(x_1,\ z), & \text{on } \Sigma \\ 0, & \text{elsewhere.} \end{cases}$$

The condition (5.2.4) again fails at the shock location, as pointed out earlier for the two-dimensional case (cf. §3.2). Hence the shock relations must be invoked to obtain ∂p/∂y at the shock-wing intersection.

On the wing at the shock location ($y = 0$, $x = m\tau$), Eq. (5.2.3) gives us

$$v = \begin{cases} m_1\ \eta'\{(m + m_1)\tau\ ,\ z\}, & \text{on } \Sigma \\ 0, & \text{elsewhere.} \end{cases}$$

This relation can be used in the shock condition (2.3.11b). Thus we obtain, at $x = m\tau$, $y = 0$

$$\partial p/\partial y = D\ m_1(m + m_1)\ F_1\{(m + m_1)\tau,\ z\} + \left\{\Lambda_{21}\ \partial\bar{u}/\partial y + \Lambda_{22}\ \partial\bar{p}/\partial y - D\ \Lambda_{41}\ \partial\bar{v}/\partial\tau\right\}_{\text{at } y=0}. \tag{5.2.5}$$

The Eqs. (5.2.4) and (5.2.5) lead to a discontinuity in ∂p/∂y at the shock-wing intersection behind the shock. The nature of such a non-uniformity is the same as discussed for the two-dimensional case.

On the shock

The shock conditions (2.3.11a)-(2.3.11c) shall be used in conjunction with the equations of motion (2.2.16)-(2.2.18) to derive a single differential condition to be prescribed at the shock front.

5.3. Formulation in Lorentz variables

As in §3.3, we introduce new variables $(\bar{x}, y, z, \bar{\tau})$ related to the variables (x, y, z, τ) by the Lorentz transformation defined by (3.3.1). The shock plane $x = m\tau$ will then correspond to the plane $\bar{x} = 0$, and the wave Eq. (2.2.24) remains unchanged, viz.

$$\frac{\partial^2 p}{\partial \bar{x}^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - \frac{\partial^2 p}{\partial \bar{\tau}^2} = 0. \qquad (5.3.1)$$

The initial conditions (5.2.1) and the conditions (5.2.2) at infinity can now be expressed as

for $\bar{\tau} \leqslant 0$, $\bar{x} < 0$, $y > 0$, $-\infty < z < \infty$

$$p(\bar{x}, y, z, \bar{\tau}) = \partial p/\partial \bar{\tau} = 0, \qquad (5.3.2)$$

$\bar{\tau} > 0$, $\bar{x} < 0$, $y > 0$ as $\bar{x} \to -\infty$, $y \to \infty$, $z \to \pm\infty$

$$p(\bar{x}, y, z, \bar{\tau}) \text{ and its derivatives} \to 0, \qquad (5.3.3)$$

At the shock front

Using the Lorentz transformation (3.3.1), the basic Eqs. (2.2.16) and (2.2.18), and the first of Eqs. (2.2.17) can be combined to yield the relation

$$\frac{\partial^2 p}{\partial \bar{x}\, \partial \bar{\tau}} + \frac{\partial^2 u}{\partial \bar{\tau}^2} + \frac{m}{(1 - m^2)^{\frac{1}{2}}} \left(\frac{\partial^2 v}{\partial y\, \partial \bar{\tau}} + \frac{\partial^2 w}{\partial z\, \partial \bar{\tau}}\right) = 0. \qquad (5.3.4)$$

Employing the shock relations (2.3.11a)-(2.3.11c) we can substitute for u, $\partial v/\partial \bar{\tau}$ and $\partial w/\partial \bar{\tau}$ in (5.3.4). Thus we are led to a differential condition in p, at $\bar{x} = 0$, $0 < y < \infty$, $-\infty < z < \infty$

$$\frac{\partial^2 p}{\partial \bar{x}\, \partial \bar{\tau}} + \frac{1}{A} \frac{\partial^2 p}{\partial \bar{\tau}^2} + \frac{m}{1 - m^2} \frac{1}{D} \left(\frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right)$$
$$= \frac{1}{A} \frac{\partial^2}{\partial \bar{\tau}^2} (B\bar{u} + C\bar{p}) + \frac{m}{1 - m^2} \frac{1}{D} \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\Lambda_{21}\, \bar{u} + \Lambda_{22}\, \bar{p})$$
$$- \frac{m}{(1 - m^2)^{\frac{1}{2}}} \Lambda_{41} \frac{\partial}{\partial \bar{\tau}} \left(\frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right). \qquad (5.3.5)$$

Note that for the upstream perturbations ahead of the shock (cf. §2.4)

$$\bar{p} = - (k_2/k_1)\, \bar{u},$$

and
$$\partial \bar{v}/\partial y + \partial \bar{w}/\partial z = \beta^2\, \partial \bar{u}/\partial x_1$$
$$= \bar{a}\, \bar{b}^2\, \partial \bar{u}/\partial \bar{\tau} \text{ (at the shock } \bar{x} = 0)$$

where $k_1 = M'/M$, $k_2 = M'^2$, and x_1 refers to the streamwise co-ordinate attached to the wing vertex. Then (5.3.5) can be simplified and we obtain

$$D[p] = (1 + 1/M^2)(B\, k_1 - C\, k_2) \frac{1}{k_1} \frac{\partial^2 \bar{u}}{\partial \bar{\tau}^2}$$
$$- (\Lambda_{21}\, k_1 - \Lambda_{22}\, k_2) \frac{1}{k_1} \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \bar{u}$$
$$+ (1 - m^2)^{\frac{1}{2}}\, D\, \Lambda_{41}\, \bar{a}\, \bar{b}^2 \frac{\partial^2 \bar{u}}{\partial \bar{\tau}^2}, \qquad (5.3.6)$$

where
$$D[p] \equiv - \left(\frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right) + 2m \frac{\partial^2 p}{\partial \bar{x}\, \partial \bar{\tau}} + (1 + 1/M^2) \frac{\partial^2 p}{\partial \bar{\tau}^2}.$$

The upstream parameter $\bar{u}$ at the shock is given by the relation (2.4.19a) for the symmetrical wings. Thus in the new variables

$$\bar{u} = - \frac{k_1}{\pi} \int_{-\infty}^{\infty} d\zeta \int_0^{\mu_1} \frac{F_1(\bar{a}\mu,\, \zeta)}{[(\bar{\tau} - \mu)^2 - \bar{b}^2\{(z - \zeta)^2 + y^2\}]^{\frac{1}{2}}}\, d\mu, \qquad (5.3.7)$$

where $\mu_1 = \bar{\tau} - \bar{b}\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}$ and F_1 is defined under (2.4.15a). Substituting for $\bar{u}$ from (5.3.7) in (5.3.6), we obtain for the condition at the shock $\bar{x} = 0$, $0 < y < \infty$, $-\infty < z < \infty$

$$D[p] = \frac{K}{\pi} \frac{\partial^2}{\partial \bar{\tau}^2} \int_{-\infty}^{\infty} d\zeta \int_0^{\mu_1} \frac{F_1(\bar{a}\mu,\, \zeta)}{[(\bar{\tau} - \mu)^2 - \bar{b}^2\{(z - \zeta)^2 + y^2\}]^{\frac{1}{2}}}\, d\mu, \qquad (5.3.8)$$

where the constant K is the same as defined in (3.3.10).

On the wing

The condition (5.2.4) on the wing gives in the transformed variables, for $\bar{\tau} > 0$, $-\infty < z < \infty$, $\bar{x} > 0$, $y = 0$

$$\partial p/\partial y = A_o F_1\{\bar{a}(\bar{\tau} + \lambda_o \bar{x}), z\}, \tag{5.3.9}$$

where A_o and λ_o are the constants defined under (3.3.4). Also the condition (5.2.5) at the shock-wing intersection, together with (5.3.7), yields at $\bar{x} = 0$, $y = 0$, $-\infty < z < \infty$

$$\partial p/\partial y = B_o F_1(\bar{a}\bar{\tau}, z), \tag{5.3.10}$$

where the constant B_o is the same as defined in (3.3.11). The condition (5.3.10) is used to supplement the condition (5.3.8) at the shock.

The wave Eq. (5.3.1), the initial conditions (5.3.2), the boundary condition (5.3.9) on the wing surface, the boundary condition (5.3.8) at the shock together with (5.3.10), and the conditions (5.3.3) at infinity completes the formulation of the symmetrical problem. In the following the solution is sought once again by the application of the integral transforms.

5.4. Solution

We introduce the following transforms (cf. Sneddon 1951)

$$p^*(\bar{x}, y, z, s) = L\{p(\bar{x}, y, z, \bar{\tau})\} = \int_0^\infty p(\bar{x}, y, z, \bar{\tau}) \exp\{-s\bar{\tau}\}\, d\bar{\tau}, \tag{5.4.1}$$

$$p^{**}(\bar{x}, \alpha, z, s) = F_c\{p^*(\bar{x}, y, z, s)\} = \int_0^\infty p^*(\bar{x}, y, z, s) \cos(\alpha y) dy, \tag{5.4.2}$$

$$p^{***}(\bar{x}, \alpha, \nu, s) = F\{p^{**}(\bar{x}, \alpha, z, s)\} = \int_{-\infty}^\infty p^{**}(\bar{x}, \alpha, z, s) \exp\{-i\nu z\}\, dz, \tag{5.4.3}$$

the Laplace transform, the Fourier cosine transform and the exponential Fourier transform with respect to $\bar{\tau}$, y and z respectively.

Application of the Laplace transform to the wave Eq. (5.3.1), together with the initial conditions (5.3.2), yields

$$\frac{\partial^2 p^*}{\partial \bar{x}^2} + \frac{\partial^2 p^*}{\partial y^2} + \frac{\partial^2 p^*}{\partial z^2} - s^2 p^* = 0. \tag{5.4.4}$$

The conditions (5.3.9), (5.3.8) together with (5.3.2), (5.3.10) and (5.3.3) reduce to, for $-\infty < z < \infty$

$\bar{x} < 0,\ y = 0$ $$\partial p^*/\partial y = A_o \exp\{s\lambda_o \bar{x}\}\ G_1(s,\ z), \tag{5.4.5}$$

$\bar{x} = 0,\ y > 0$ $$-\left(\frac{\partial^2 p^*}{\partial y^2} + \frac{\partial^2 p^*}{\partial z^2}\right) + 2ms\,\frac{\partial p^*}{\partial \bar{x}} + (1 + 1/M^2)\ s^2 p^* = \frac{K}{\pi} s^2 \int_{-\infty}^{\infty} G_1(s,\ \zeta)\ K_o\left[\bar{b}s\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}\right], \tag{5.4.6}$$

$\bar{x} = 0,\ y = 0$ $$\partial p/\partial y = B_o\ G_1(s,\ z), \tag{5.4.7}$$

$\bar{x} < 0,\ y > 0$ as $\bar{x} \to -\infty,\ y \to \infty,\ z \to \pm\infty$

$$p^* \text{ and its derivatives} \to 0, \tag{5.4.8}$$

where $G_1(s,\ z) = L\{F_1(\bar{a}\tau,\ z)\}$, and K_o is the modified Bessel function of the second kind.

Next using the Fourier cosine transform (5.4.2) for y, the Eq. (5.4.4) together with the conditions (5.4.5) and (5.4.8) yields, for $\bar{x} < 0,\ -\infty < z < \infty$

$$\frac{\partial^2 p^{**}}{\partial \bar{x}^2} + \frac{\partial^2 p^{**}}{\partial z^2} - (\alpha^2 + s^2)p^{**} = A_o \exp\{s\lambda_o \bar{x}\}\ G_1(s,\ z). \tag{5.4.9}$$

The conditions (5.4.6), (5.4.7) and (5.4.8) will lead to, at $\bar{x} = 0$, $-\infty < z < \infty$

$$\alpha^2 p^{**} - \frac{\partial^2 p^{**}}{\partial z^2} + 2ms\,\frac{\partial p^{**}}{\partial \bar{x}} + (1 + 1/M^2)\ s^2 p^{**} = \tfrac{1}{2} Ks^2 \int_{-\infty}^{\infty} G_1(s,\ \zeta)\ \frac{\exp\{-\ (z - \zeta)(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}}}{(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}}}\ d\zeta - B_o G_1(s,\ \zeta), \tag{5.4.10}$$

and as $\bar{x} \to -\infty,\ z \to \infty$ $$p^{**} \text{ and its derivatives} \to 0. \tag{5.4.11}$$

Finally we apply the exponential Fourier transform (5.4.3) for z. The Eqs. (5.4.9)-(5.4.11) become

for $\bar{x} < 0$ $$\frac{\partial^2 p^{***}}{\partial \bar{x}^2} - \lambda^2 p^{***} = A_o \exp\{s\lambda_o \bar{x}\}\ H_1(s,\ \nu), \tag{5.4.12}$$

with the conditions, at $\bar{x} = 0$

$$(\lambda^2 + s^2/M^2)p^{***} + 2ms \frac{\partial p^{***}}{\partial \bar{x}} = (K \frac{s^2}{\lambda^2 - \lambda_4^2 s^2} - B_o) H_1(s, \nu), \quad (5.4.13)$$

and as $\bar{x} \to -\infty$ $\qquad p^{***} \to 0, \qquad (5.4.14)$

where $\lambda^2 = \alpha^2 + \nu^2 + s^2$, $H_1(s, \nu) = F\{G_1(s, \nu)\}$ and $\lambda_4^2 = (1 - \bar{b}^2) > 0$.

The complete solution of such a formulation (5.4.12)-(5.4.14) has already been achieved in §3.4, and can be given by (3.4.14) with G(s) replaced by $H_1(s, \nu)$, viz.

$$p^{***}(\bar{x}, \alpha, \nu, s) = \tfrac{1}{2} A_o\left(\frac{1}{\lambda - \lambda_o s} \frac{\exp\{\lambda\bar{x}\}}{\lambda} - \frac{2}{\lambda^2 - \lambda_o^2 s^2} \exp\{s\lambda_o\bar{x}\}\right)H_1(s, \nu)$$

$$+ \frac{1}{2} \sum_{i=1}^{5} A_i \frac{1}{\lambda + \lambda_i s} \frac{\exp\{\lambda\bar{x}\}}{\lambda} H_1(s, \nu), \qquad (5.4.15)$$

where the constants A_is' and λ_is' are functions of M and M′, and are the same as deduced in §3.4. The above expression can be further written as (cf. Eq. 4.4.12)

$$p^{***}(\bar{x}, \alpha, \nu, s) = \frac{1}{2}\Big(- A_o \int_{-\infty}^{0} \exp\{s\lambda_o\xi\} H_1(s, \nu) \frac{1}{\lambda} \exp\{-\lambda(|\bar{x} - \xi|)\} d\xi$$

$$+ \sum_{i=1}^{4} A_i \int_{-\infty}^{0} \exp\{s\lambda_i\xi\}H_1(s, \nu) \frac{1}{\lambda} \exp\{\lambda(\bar{x} + \xi)\} d\xi$$

$$- A_5\int_{-\infty}^{0} \exp\{s\lambda_4\xi\} H_1(s, \nu) \frac{1}{\lambda} \exp\{-\lambda(|\bar{x} - \xi|)\} d\xi\Big)$$

$$+ A_5\{1/(\alpha^2 + \nu^2 + \bar{b}^2 s^2)\} \exp\{s\lambda_4\bar{x}\} H_1(s, \nu) \quad (5.4.16)$$

This is a suitable form for seeking the inversions. The inversions are obtained in Appendix (4.1) with the aid of standard tables (cf. Erdelyi et al. 1954). Thus we are led to

$$p(\bar{x}, y, z, \bar{\tau}) = L^{-1} F_c^{-1}[F^{-1}\{p^{***}(\bar{x}, \alpha, \nu, s)\}]$$

$$= \frac{1}{2\pi}\Big(- A_0\int_{-\infty}^{0} d\xi \int_{-\infty}^{\infty} \frac{1}{\bar{R}} F_1\{\bar{a}(\bar{\tau} - \bar{R} + \lambda_0 \xi), \zeta\}\, d\zeta$$

$$+ \sum_{i=1}^{4} A_i \int_{-\infty}^{0} d\xi \int_{-\infty}^{\infty} \frac{1}{\bar{R}_1} F_1\{\bar{a}(\bar{\tau} - \bar{R}_1 + \lambda_i \xi), \zeta\}\, d\zeta$$

$$- A_5 \int_{-\infty}^{0} d\xi \int_{-\infty}^{\infty} \frac{1}{\bar{R}} F_1\{\bar{a}(\bar{\tau} - \bar{R} - \lambda_4 \xi), \zeta\}\, d\zeta\Big)$$

$$+ A_5 \frac{1}{\pi}\int_{-\infty}^{\infty} d\xi \int_{0}^{\mu_4} \frac{F_1(\bar{a}\mu, \zeta)}{[(\bar{\tau} + \lambda_4 \bar{x} - \mu)^2 - \bar{b}^2\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}]}\, d\mu, \tag{5.4.17}$$

where $\bar{R} = \{(\bar{x} - \xi)^2 + (z - \zeta)^2 + y^2\}^{\frac{1}{2}}$, $\bar{R}_1 = \bar{R}(-\xi)$

and $\mu_4 = \bar{\tau} + \lambda_4 \bar{x} - \bar{b}\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}$.

Alternatively, (5.4.17) can also be expressed as

$$p(\bar{x}, y, z, \bar{\tau}) = \frac{1}{2\pi}\Big(- A_0 \int_{-\infty}^{0} d\xi \int_{-\infty}^{\infty} \frac{1}{\bar{R}} F_1\{\bar{a}(\bar{\tau} - \bar{R} + \lambda_0 \xi), \zeta\}\, d\zeta$$

$$+ \sum_{i=1}^{5} A_i \int_{0}^{\infty} d\xi \int_{-\infty}^{\infty} \frac{1}{\bar{R}} F_1\{\bar{a}(\bar{\tau} - \bar{R} - \lambda_i \xi), \zeta\}\, d\zeta\Big), \tag{5.4.18}$$

with $\lambda_5 = -\lambda_4$. Finally the pressure p can be obtained in original variables (x, y, z, τ) by using the transformation (3.3.1).

The integrals occuring in (5.4.17) or (5.4.18) can be reorganized once again as 'retarded potential' integrals. The numerator of the integrands represent the 'temporary sources' spread in the plane of the wing $y = 0$, and $\bar{R}$ is the real distance between the source point $(\xi, 0, \zeta)$ and the field point $(\bar{x}, y, z)$.

Similar to the case of axisymmetric bodies, it can be once again concluded that for the points on or outside the spherical wavelet

$\bar{x}^2 + y^2 + z^2 \geqslant \bar{\tau}^2$, the first integral in (5.4.17) is finite in the region ACE (Fig. 15), while it vanishes everywhere else, and is given by

$$-\frac{A_o}{\pi}\int_{-\infty}^{\infty} d\zeta \int_0^{\mu_o} \frac{F_1(\bar{a}\mu, \zeta)}{[(\bar{\tau} + \lambda_o\bar{x} - \mu)^2 - (1 - \lambda_o^2)\{(z - \zeta)^2 + y^2\}]^{\frac{1}{2}}} d\mu,$$

where $\mu_o = \bar{\tau} + \lambda_o\bar{x} - (1 - \lambda_o^2)^{\frac{1}{2}}\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}$. This represents the solution in the field outside the spherical wavelet due to the supersonic flow, $(W + U) > a_1$, over the wing and is the same as obtained by the usual linearized theory. The surface of the Mach conoid emanating from the vertex of the wing is

$$\bar{\tau} + \lambda_o\bar{x} - (1 - \lambda_o^2)^{\frac{1}{2}}(y^2 + z^2)^{\frac{1}{2}} = 0,$$

which is tangent to the wavelet $\bar{x}^2 + y^2 + z^2 = \bar{\tau}^2$. Again the last two terms in (5.4.17), while taken together, vanish for the points on or outside the spherical wavelet except in the region IDB (Fig. 15) where they yield

$$\frac{A_5}{\pi}\int_{-\infty}^{\infty} d\zeta \int_0^{\mu_4} \frac{F_1(\bar{a}\mu, \zeta)}{|(\bar{\tau} + \lambda_4\bar{x} - \mu)^2 - \bar{b}^2\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}} d\mu,$$

where $\mu_4 = \bar{\tau} + \lambda_4\bar{x} - \bar{b}\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}$.

The remaining integrals (for i = 1, 2, 3 & 4) in (5.4.17) vanish for the points on or outside the spherical wavelet. Thus we see that the solution (5.4.17) leads to the various fields representation.

To arrive at the above results (5.4.17), we assumed λ_4 to be real, i.e. $\bar{b} < 1$. This again implies that the intersection of the plane shock with the Mach conoid of the wing lies outside the spherical wavelet (cf. Fig. 15). In case $\bar{b} > 1$, this intersection will lie inside the sonic circle, λ_4 and λ_5 in the above will then become imaginary. In that case we can consider the last two terms in (5.4.15) as follows (cf. 3.4.17)

$$I^{***}_{4,5} = (Q_1\lambda + Q_2 s)\frac{1}{\alpha^2 + \nu^2 + \bar{b}^2 s^2}\frac{\exp\{\lambda\bar{x}\}}{\lambda} H_1(s, \nu), \qquad (5.4.19)$$

where $\lambda = (\alpha^2 + \nu^2 + s^2)^{\frac{1}{2}}$, and Q_1 and Q_2 are the same as defined under (3.4.17). The inversion of (5.4.19) is considered in Appendix (4.2). Thus we obtain

$$I_{4,5} = \frac{1}{2\pi^2}\left(Q_1 \frac{\partial}{\partial \bar{x}} + Q_2 \frac{\partial}{\partial \bar{\tau}}\right)\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} dy_1 \int_{0}^{\bar{\tau}} \frac{F_1\{\bar{a}(\mu - r_1)\}}{r_1\{(\bar{\tau} - \mu)^2 - \bar{b}^2 r_2^2\}^{\frac{1}{2}}}\, d\mu, \tag{5.4.20}$$

with $r_1 = \{(\bar{x} - \xi)^2 + (\zeta - \zeta_1)^2 + y_1^2\}^{\frac{1}{2}}$ and $r_2 = \{(y - y_1)^2 + (z - \zeta_1)^2\}^{\frac{1}{2}}$.

Once the pressure p is known, all the other flow variables and the shock shape can be determined by using the basic Eq. (2.2.17)-(2.2.19) and the shock relations (2.3.10). In particular for the density variation behind the shock, we can obtain

$$\rho(x, y, z, \tau) = p(x, y, z, \tau) + (E - 1)\, p(x, y, z, \tau = x/m)$$
$$- (Fk_1 - Gk_2)\,\frac{1}{\pi}\int_{-\infty}^{\infty} d\zeta \int_{0}^{\mu_2} \frac{F_1(\beta\mu/\bar{c},\, \zeta)}{[(x - \mu)^2 - \bar{c}^2\{(z - \zeta)^2 + y^2\}]^{\frac{1}{2}}}\, d\mu, \tag{5.4.21}$$

with $\mu_2 = x - \bar{c}\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}$ and $\bar{c} = \beta m/(m + m_1)$.

Part II Antisymmetrical problem

5.5. General case

For the general antisymmetrical problem, we must obtain the solution of the wave Eq. (5.3.1) subject to the appropriate conditions. The initial conditions and the conditions at infinity can be prescribed by (5.3.2) and (5.3.3). The condition at the shock can also be given by (5.3.6). The upstream parameter $\bar{u}$ in (5.3.6), should now be given by (2.4.19b), which in Lorentz variables give

$$\bar{u} = -\frac{k_1}{\pi}\frac{\partial}{\partial y}\int_{-\infty}^{\infty} d\zeta \int_{0}^{\mu_1} \frac{F_2(\bar{a}\mu,\, \zeta)}{[(\bar{\tau} - \mu)^2 - \bar{b}^2\{(z - \zeta)^2 + y^2\}]^{\frac{1}{2}}}\, d\mu, \tag{5.5.1}$$

where $\mu_1 = \bar{\tau} - \bar{b}\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}$, and F_2 is defined under (2.4.15b). Using (5.5.1) in (5.3.6), we obtain at the shock $\bar{x} = 0$, $0 < y < \infty$, $-\infty < z < \infty$

$$D[p] = \frac{K}{\pi}\frac{\partial^2}{\partial \bar{\tau}^2}\frac{\partial}{\partial y}\int_{-\infty}^{\infty} d\zeta \int_{0}^{\mu_1} \frac{F_2(\bar{a}\mu,\, \zeta)}{[(\bar{\tau} - \mu)^2 - \bar{b}^2\{(z - \zeta)^2 + y^2\}]^{\frac{1}{2}}}\, d\mu, \tag{5.5.2}$$

where D [p] is defined in (5.3.6). This differential condition will be prescribed at the shock front.

For the antisymmetrical case the flow on the two sides of the wing is no longer symmetrical. Then for a given antisymmetrical wing with subsonic leading edges in the region behind the shock, the vertical component of perturbation velocity v (and hence $\partial p/\partial y$) in the plane of the wing ($y = 0$) will no longer be zero outside the wing projection (in the area between the wing and the Mach cone). However the perturbation pressure p, at $y = 0$, will be zero everywhere outside the wing projection. The pressure p can then be regarded odd in the y co-ordinate, and again it is sufficient to consider for $y > 0$. Hence for the region behind the shock we can specify a condition on the wing in p itself, say, at $y = 0$, $-\infty < \bar{x} \leqslant 0$, $-\infty < z < \infty$

$$p = f_1(\bar{x}, z, \bar{\tau}) \tag{5.5.3}$$

where $$f_1 = \begin{cases} p(\bar{x}, 0, z, \bar{\tau}), & \text{on the wing projection } \Sigma \\ 0, & \text{elsewhere.} \end{cases}$$

This function is however unknown and can be determined only after the solution of the problem is completed. We verify that f_1 has to be determined from a complicated integral equation.

Summarizing, the wave Eq. (5.3.1), the initial conditions (5.3.2), the conditions (5.3.3) at infinity, the boundary condition (5.5.2) at the shock front, and the boundary condition (5.5.3) on the wing surface completes the formulation. The solution of this formulation will be again obtained by means of the integral transforms.

We introduce the integral transforms as in §5.4, except that now we replace the Fourier cosine transform by the Fourier sine transform for y as defined below

$$\begin{aligned} p^{*}(\bar{x}, y, z, s) &= L\{p(\bar{x}, y, z, \bar{\tau})\}, \\ p^{**}(\bar{x}, \alpha, z, s) &= F_s\{p^{*}(\bar{x}, y, z, s)\} = \int_0^{\infty} p^{*}(\bar{x}, y, z, s) \sin(\alpha y)\, dy, \\ p^{***}(\bar{x}, \alpha, \nu, s) &= F\{p^{**}(\bar{x}, \alpha, z, s)\}. \end{aligned} \tag{5.5.4}$$

Application of these transforms reduce the above formulation to obtaining the solution of the ordinary differential equation

$$\frac{\partial^2 p^{***}}{\partial \bar{x}^2} - \lambda^2 p^{***} = - \alpha\, h(\bar{x}, \nu, s), \tag{5.5.5}$$

subject to the conditions, at $\bar{x} = 0$

$$(\lambda^2 + s^2/M^2)p^{***} + 2ms \frac{\partial p^{***}}{\partial \bar{x}} = \alpha\, h(\nu\ , s) - K \frac{s^2}{\lambda^2 - \lambda_4^2 s^2}\, \alpha\, H_2(\nu\ , s), \tag{5.5.6}$$

and $$\text{as } \bar{x} \to -\infty \qquad p^{***} \to 0, \tag{5.5.7}$$

where $\lambda^2 = \alpha^2 + \nu^2 + s^2$, $h(\bar{x}, \nu, s) = F[L\{f_1(\bar{x}, \bar{\tau}, z)\}]$,

$h(\nu\ , s) = \{h(\bar{x}, \nu, s)\}_{\text{at } \bar{x} = 0}$, $H_2(\nu\ , s) = F[L\{F_2(\bar{x}, \bar{\tau}, z)\}]$

and $\lambda_4^2 = (1 - \bar{b}^2) > 0$.

The complete solution of Eq. (5.5.5) can be written as

$$p^{***} = A_1 \exp\{\lambda\bar{x}\} + A_2 \exp\{-\lambda\bar{x}\} - \frac{\alpha}{2}\frac{\exp\{\lambda\bar{x}\}}{\lambda}\int_{\infty}^{\bar{x}} h(\xi, \nu, s) \exp\{-\lambda\xi\}\, d\xi$$
$$+ \frac{\alpha}{2}\frac{\exp\{-\lambda\bar{x}\}}{\lambda}\int_{-\infty}^{\bar{x}} h(\xi, \nu, s) \exp\{\lambda\xi\}\, d\xi. \tag{5.5.8}$$

Here A_2 vanishes in view of the condition (5.5.7), while A_1 is determined by using the condition (5.5.6)

$$A_1 = - \frac{\alpha}{2}\int_0^{\infty} h(\xi, \nu, s) \frac{\exp\{-\lambda\xi\}}{\lambda}\, d\xi - \frac{H(-\lambda)}{H(\lambda)}\frac{\alpha}{2}\int_{-\infty}^{0} h(\xi, \nu, s) \frac{\exp\{\lambda\xi\}}{\lambda}\, d\xi$$
$$+ \frac{1}{H(\lambda)}\, \alpha\, h(\nu, s) - \frac{K}{H(\lambda)} \frac{s^2}{\lambda^2 - \lambda_4^2 s^2}\, \alpha\, H_2(\nu, s), \tag{5.5.9}$$

where $H(\lambda) = \lambda^2 + 2ms + s^2/M^2$. Substituting for A_1 and A_2 in (5.5.8) we are led to

$$p^{***} = \frac{\alpha}{2}\int_{-\infty}^{0} h(\xi, \nu, s) \frac{1}{\lambda} \exp\{-\lambda(|\bar{x} - \xi|)\}\, d\xi$$
$$- \frac{H(-\lambda)}{H(\lambda)}\frac{\alpha}{2}\int_0^{\infty} h(-\xi, \nu, s) \frac{1}{\lambda} \exp\{-\lambda(\xi - \bar{x})\}\, d\xi$$
$$+ \frac{1}{H(\lambda)} \exp\{\lambda\bar{x}\}\, \alpha\, h(\nu, s) - \frac{K}{H(\lambda)} \frac{s^2}{\lambda^2 - \lambda_4^2 s^2} \exp\{\lambda\bar{x}\}\, \alpha\, H_2(\nu, s). \tag{5.5.10}$$

This can further expressed as

$$p^{***} = \frac{1}{2}\Big(\alpha\int_{-\infty}^{0} h(\xi,\ \nu,\ s)\ \frac{1}{\lambda}\ \exp\{-\lambda(|\bar{x} - \xi|)\}\ d\xi$$

$$- \alpha\int_{0}^{\infty} h(-\xi,\ \nu,\ s)\ \frac{1}{\lambda}\ \exp\{-\lambda(\xi - \bar{x})\}\ d\xi$$

$$- 2ms\sum_{i=2}^{3} B_i\ \frac{1}{\lambda + \lambda_i s}\ \alpha\int_{0}^{\infty} h(-\xi,\ \nu,\ s)\ \frac{1}{\lambda}\ \exp\{-\lambda(\xi - \bar{x})\}\ d\xi$$

$$- \sum_{i=2}^{3} B_i\ \frac{1}{\lambda + \lambda_i s}\ \frac{\exp\{\lambda\bar{x}\}}{\lambda}\ \alpha\ h(\nu\ ,\ s)$$

$$- \sum_{i=2}^{5} C_i\ \frac{1}{\lambda + \lambda_i s}\ \frac{\exp\{\lambda\bar{x}\}}{\lambda}\ \alpha\ H_2(\nu\ ,\ s)\Big), \qquad (5.5.11)$$

where

$$B_2 = \frac{-2\lambda_2}{\lambda_2 - \lambda_3}, \qquad C_2 = \frac{-2\lambda_2}{\lambda_2 - \lambda_3}\ \frac{K}{\lambda_2^2 - \lambda_4^2}, \qquad C_5 = \frac{K}{(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)},$$

$$B_3 = \frac{2\lambda_3}{\lambda_2 - \lambda_3}, \qquad C_3 = \frac{2\lambda_3}{\lambda_2 - \lambda_3}\ \frac{K}{\lambda_2^2 - \lambda_4^2}, \qquad C_6 = \frac{K}{(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)},$$

λ_2 and λ_3 are the roots (real, distinct and positive) of the quadratic equation $\lambda_i^2 - 2m\lambda_i + s^2/M^2 = 0$, and $\lambda_5 = -\lambda_4$.

The inversion of (5.5.11) is obtained in Appendix (4.3), which gives

$$p = L^{-1}\Big(F_s^{-1}[F^{-1}\{p^{***}(\bar{x},\ \alpha,\ \nu,\ s)\}]\Big)$$

$$= \frac{1}{2\pi}\Big\{-\ \frac{\partial}{\partial y}\int_{-\infty}^{0} d\xi\int_{-\infty}^{\infty} \frac{1}{\bar{R}}\ f_1\{\xi, \zeta,\ (\bar{\tau} - \bar{R})\}\ d\zeta$$

$$+ \frac{\partial}{\partial y}\int_{0}^{\infty} d\xi\int_{-\infty}^{\infty}\ \frac{1}{\bar{R}}\ f_1\{-\xi,\ \zeta,\ (\bar{\tau} - \bar{R})\}\ d\zeta$$

$$+ \frac{\partial}{\partial y}\ \frac{\partial}{\partial\bar{\tau}}\int_{0}^{\infty} d\xi_1\int_{0}^{\infty} d\xi\int_{-\infty}^{\infty}\ \frac{1}{\bar{R}_2}\Big(\sum_{i=2}^{3}\ 2mB_i\ f_1\{-\xi,\ \zeta,\ (\bar{\tau} - \bar{R}_2 - \lambda_i\xi)\}\Big)\ d\zeta\ +$$

$$+ \frac{\partial}{\partial y}\int_0^\infty d\xi \int_{-\infty}^\infty \frac{1}{\bar{R}}\Big(\sum_{i=2}^{3} B_i \; f_1\{\zeta, (\bar{\tau} - \bar{R} - \lambda_i\xi)\}\Big)\, d\zeta$$

$$+ \frac{\partial}{\partial y}\int_0^\infty d\xi \int_{-\infty}^\infty \frac{1}{\bar{R}}\Big(\sum_{i=2}^{5} C_i \; F_2\{\zeta, \bar{a}(\bar{\tau} - \bar{R} - \lambda_i\xi)\}\Big)\, d\zeta\Big\}, \qquad (5.5.12)$$

where $\bar{R} = \{(\bar{x} - \xi)^2 + (z - \zeta)^2 + y^2\}^{\frac{1}{2}}$

and $\bar{R}_2 = \{(\bar{x} - \xi - \xi_1)^2 + (z - \zeta)^2 + y^2\}^{\frac{1}{2}}$.

By virtue of Eq. (5.5.3) valid at $y = 0$, we may notice that (5.5.12) is an integral equation, which must be solved to determine f_1. We must remark that the pressure in the region ACE (Fig. 15) can be given by the usual linearized potential theory. The integral Eq. (5.5.12) is, however, very complicated, and it will not be pursued further.

5.6. Special case

In the above presentation we have seen that the solution of the general antisymmetrical problem leads us to the solution of a complicated integral equation. The matter, however, can be simplified for the anti-symmetrical wing with definitely supersonic leading edges with respect to the flow, $(W + U) > a_1$, behind the shock. In that case, in the non-uniform region behind the shock, the flow pattern on the two sides of the wing will be independent of each other. One can then treat such a problem as effectively symmetrical.

In particular, if the lifting wing under consideration has supersonic leading edges with respect to the flow behind as well as ahead of the shock (for time $t > 0$), the problem is effectively the same as has been treated in part I.

Suppose now that one or both the leading edges of an antisymmetrical wing are subsonic before it meets the shock. After the wing has penetrated the shock, both the leading edges become supersonic in the region behind the shock. Then once again for the region behind the shock we can prescribe the conditions (5.2.4) and (5.2.5) on the wing surface, which give in the transformed variables, for $\bar{\tau} > 0$, $-\infty < z < \infty$, $y = 0$

$\bar{x} > 0 \qquad \partial p/\partial y = A_o\, F_1\{\bar{a}(\bar{\tau} + \lambda_o \bar{x}),\, z\},$ (5.6.1)

$\bar{x} = 0 \qquad \partial p/\partial y = B_{o1}\, F_1(\bar{a}\bar{\tau},\, z)$

$$+ \left\{\Lambda_{21}\frac{\partial \bar{u}}{\partial y} + \Lambda_{22}\frac{\partial \bar{p}}{\partial y} + D\,\Lambda_{41}\,(1 - m^2)^{\frac{1}{2}}\frac{\partial \bar{v}}{\partial \bar{\tau}}\right\}_{\text{at } y=0}, \qquad (5.6.2)$$

where A_o and λ_o are the same as defined under (3.3.4), and $B_{o1} = D\, m_1(m + m_1)$.

We shall assume that the perturbations ahead of the shock will be governed by the velocity potential for antisymmetrical wing, and $\bar{u}$ is given by (2.4.19b) or (5.5.1). The boundary condition at the shock front will then be given by (5.5.2), at $\bar{x} = 0$, $0 < y < \infty$, $-\infty < z < \infty$

$$D[p] = \frac{K}{\pi}\frac{\partial^2}{\partial \bar{\tau}^2}\frac{\partial}{\partial y}\int_{-\infty}^{\infty} d\zeta \int_0^{\mu_1} \frac{F_2(\bar{a}\mu,\, \zeta)}{\left[(\bar{\tau} - \mu)^2 - \bar{b}^2\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}\right]}\, d\mu. \qquad (5.6.3)$$

This condition will be supplemented by the condition (5.6.2), which together with (5.5.1) yields, at $\bar{x} = 0$, $y = 0$, $-\infty < z < \infty$

$$\partial p/\partial y = B_{o1}\, F_1(\bar{a}\bar{\tau},\, z) + B_{o2}\, F_3(\bar{a}\bar{\tau},\, z), \qquad (5.6.4)$$

where $\quad B_{o2} = (\Lambda_{21}\, k_1 - \Lambda_{22}\, k_2) - (1 - m^2)^{\frac{1}{2}}\, D\,\Lambda_{41}\, k_1 \bar{a},$

and $\quad F_3(\bar{a}\bar{\tau},\, z) = -\dfrac{\bar{b}^2}{\pi}\displaystyle\int_{-\infty}^{\infty} d\zeta \int_0^{\bar{\tau}-\bar{b}(z-\zeta)} \frac{F_2(\bar{a}\mu,\, \zeta)}{\{(\bar{\tau} - \mu)^2 - \bar{b}^2(z - \zeta)^2\}^{3/2}}\, d\mu.$

Now for this case we consider the wave Eq. (5.3.1), to be solved subject to the initial conditions (5.3.2), the conditions (5.3.3) at infinity, the boundary condition (5.6.1) on the wing, and the boundary condition (5.6.3) on the shock together with the condition (5.6.4). To obtain the solution we proceed exactly as in §5.4. Application of integral transforms (5.4.1)-(5.4.3) reduce the problem to finding the solution of the Eq. (5.4.12), viz. for $\bar{x} < 0$

$$\frac{\partial^2 p^{***}}{\partial \bar{x}^2} - \lambda^2\, p^{***} = A_o\, \exp\{s\lambda_o \bar{x}\}\, H_1(s,\, \nu), \qquad (5.6.5)$$

with the conditions, at $\bar{x} = 0$

$$(\lambda^2 + s^2/M^2)p^{***} + 2ms\,\frac{\partial p^{***}}{\partial \bar{x}}$$

$$= -B_{o1}\,H_1(s,\ \nu) - B_{o2}\,H_3(s,\ \nu) + K\{s^2/(\lambda^2 - \lambda_4^2 s^2)\}\,H_3(s,\ \nu), \tag{5.6.6}$$

and $$\text{as } \bar{x} \to -\infty \qquad p^{***} \to 0, \tag{5.6.7}$$

where $$H_3(s,\ \nu) = F\left[L\{F_3(\bar{a}\bar{\tau},\ z)\}\right].$$

The complete solution of the Eqs. (5.6.5)-(5.6.7) can once again be written as

$$p^{***}(\bar{x},\ \alpha,\ \nu,\ s) = \frac{1}{2}\Bigg\{A_o\left(\frac{1}{\lambda - \lambda_o s}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda} - \frac{2}{\lambda^2 - \lambda_o^2 s^2}\exp\{s\lambda_o\bar{x}\}\right) H_1(s,\ \nu)$$

$$+ \sum_{i=1}^{3} D_i\,\frac{1}{\lambda + \lambda_i s}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda}\,H_1(s,\ \nu)$$

$$+ \sum_{i=2}^{5} E_i\,\frac{1}{\lambda + \lambda_i s}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda}\,H_3(s,\ \nu)\Bigg\}, \tag{5.6.8}$$

Where $D_1 = A_1 = H(\lambda_o)/H(-\lambda_o)$,

$$D_2 = \frac{-2\lambda_2}{\lambda_2 - \lambda_3}\left(B_{o1} + \frac{2mA_o}{\lambda_o - \lambda_2}\right), \qquad E_2 = \frac{-2\lambda_2}{\lambda_2 - \lambda_3}\left(B_{o2} - \frac{K}{\lambda_2^2 - \lambda_4^2}\right),$$

$$D_3 = \frac{2\lambda_3}{\lambda_2 - \lambda_3}\left(B_{o1} + \frac{2mA_o}{\lambda_o - \lambda_3}\right), \qquad E_3 = \frac{2\lambda_3}{\lambda_2 - \lambda_3}\left(B_{o2} - \frac{K}{\lambda_3^2 - \lambda_4^2}\right),$$

$$E_4 = K/\{(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)\}, \qquad E_5 = K/\{(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)\}.$$

Upon inversion, (5.6.8) leads us to

$$p(\bar{x},\ y,\ z,\ \bar{\tau}) = L^{-1}\,F_c^{-1}\left[F^{-1}\{p^{***}(\bar{x},\ \alpha,\ \nu,\ s)\}\right]$$

$$= \frac{1}{2\pi}\Bigg\{-A_o\int_{-\infty}^{0} d\xi\int_{-\infty}^{\infty}\frac{1}{\bar{R}}\,F_1\{\bar{a}(\bar{\tau} - \bar{R} + \lambda_o\xi),\ \zeta\}\,d\zeta$$

$$+ \int_{0}^{\infty} d\xi\int_{-\infty}^{\infty}\frac{1}{\bar{R}}\Bigg(\sum_{i=1}^{3} D_i\,F_1\{\bar{a}(\bar{\tau} - \bar{R} - \lambda_i\xi),\ \zeta\}$$

$$+ \sum_{i=2}^{5} E_i\,F_3\{\bar{a}(\bar{\tau} - \bar{R} - \lambda_i\xi),\ \zeta\}\Bigg)d\zeta\Bigg\}. \tag{5.6.9}$$

where $$\bar{R} = \{(\bar{x} - \xi)^2 + (z - \zeta)^2 + y^2\}^{\frac{1}{2}}.$$

5.7. Applications

5.7.1. As an example of §5.4 we consider the interaction with a symmetrical pointed delta wing at zero incidence. Let the wing has the semi-apex angle $\theta = \arctan(1/n)$ and both the upper and lower surfaces have a constant slope ε, say, along the streamwise direction, so that they may be represented by the planes

$$y = \pm\ (x_1 - nz)\varepsilon, \quad \text{for } z > 0$$
$$= \pm\ (x_1 + nz)\varepsilon, \quad \text{for } z < 0,$$

with respect to the co-ordinates fixed at the apex of the wing. Hence in the assumed co-ordinates behind the shock, we can write for the upper surface of the wing

$$y = (x + m_1\tau \mp nz)\varepsilon. \tag{5.7.1}$$

This in transformed variables becomes

$$y = \{\bar{a}(\bar{\tau} + \lambda_o\bar{x} \mp \bar{n}z)\}, \tag{5.7.2}$$

where $\bar{n} = n/\bar{a}$.

Specializing the results (5.4.18) for the symmetrical delta wing, we obtain after simplification

$$p = \frac{\varepsilon}{2\pi\bar{a}}\Bigg\{-A_o\int_{\omega_o}^{0}\Big([(\bar{\tau} + \lambda_o\xi - \bar{n}z)^2 - (1 - \bar{n}^2)\{(\bar{x} - \xi)^2 + y^2\}]^{-\frac{1}{2}}$$
$$+ [(\bar{\tau} + \lambda_o\xi + \bar{n}z)^2 - (1 - \bar{n}^2)\{(\bar{x} - \xi)^2 + y^2\}]^{-\frac{1}{2}}\Big)d\xi$$
$$+ \sum_{i=1}^{5} A_i\int_{0}^{\omega_i}\Big([(\bar{\tau} - \lambda_i\xi - \bar{n}z)^2 - (1 - \bar{n}^2)\{(\bar{x} - \xi)^2 + y^2\}]^{-\frac{1}{2}}$$
$$+ [(\bar{\tau} - \lambda_i\xi + \bar{n}z)^2 - (1 - \bar{n}^2)\{(\bar{x} - \xi)^2 + y^2\}]^{-\frac{1}{2}}\Big)d\xi\Bigg\}, \tag{5.7.3}$$

where $\omega_o = [\bar{x} + \lambda_o\bar{\tau} - \{(\bar{\tau} + \lambda_o\bar{x})^2 - (1 - \lambda_o^2)(y^2 + z^2)\}^{\frac{1}{2}}]/(1 - \lambda_o^2)$,

and $\omega_i = [\bar{x} - \lambda_i\bar{\tau} + \{(\bar{\tau} - \lambda_i\bar{x})^2 - (1 - \lambda_i^2)(y^2 + z^2)\}]/(1 - \lambda_i^2)$.

Let us consider the integral

$$\int_0^{\omega_i} [(\bar{\tau} - \lambda_i\xi - \bar{n}z)^2 - (1 - \bar{n}^2)\{(\bar{x} - \xi)^2 + y^2\}]^{-\frac{1}{2}} d\xi,$$

which integrates to yield

$$\frac{1}{\{\bar{n}^2 - (1 - \lambda_i^2)\}^{\frac{1}{2}}} \operatorname{arccosh}\left(\frac{|(1 - \bar{n}^2)\bar{x} - \lambda_i(\bar{\tau} - \bar{n}z)|}{(1 - \bar{n}^2)^{\frac{1}{2}}[(\bar{\tau} - \bar{n}z - \lambda_i\bar{x})^2 + \{\bar{n}^2 - (1 - \lambda_i^2)\}y^2]^{\frac{1}{2}}}\right),$$

$$\text{for } (1 - \lambda_i^2)^{\frac{1}{2}} < \bar{n},$$

(5.7.4)

$$\frac{1}{\{(1 - \lambda_i^2) - \bar{n}^2\}^{\frac{1}{2}}} \arccos\left(\frac{|(1 - \bar{n}^2)\bar{x} - \lambda_i(\bar{\tau} - \bar{n}z)|}{(1 - \bar{n}^2)^{\frac{1}{2}}[(\bar{\tau} - \bar{n}z - \lambda_i\bar{x})^2 - \{(1 - \lambda_i^2) - \bar{n}^2\}y^2]^{\frac{1}{2}}}\right),$$

$$\text{for } (1 - \lambda_i^2)^{\frac{1}{2}} > \bar{n}.$$

Here we may note that

for $i = 0$ or 1 $\qquad (1 - \lambda_i^2)^{\frac{1}{2}} = (m_1^2 - 1)^{\frac{1}{2}}/\bar{a}$,

and for $i = 4$ or 5 $\qquad (1 - \lambda_i^2)^{\frac{1}{2}} = \beta/\bar{a}$.

Then the following cases of physical interest may arise depending upon whether $(1 - \lambda_o^2)^{\frac{1}{2}}$ and $(1 - \lambda_4^2)^{\frac{1}{2}}$ are greater or less than $\bar{n}$.

(a) When both $(1 - \lambda_4^2)^{\frac{1}{2}}$ and $(1 - \lambda_o^2)^{\frac{1}{2}}$ are greater than $\bar{n}$ -

$(1 - \lambda_4^2)^{\frac{1}{2}} > \bar{n}$ implies $\beta > n$, or $\theta > \phi_o$,

and $(1 - \lambda_o^2)^{\frac{1}{2}} > \bar{n}$ implies $(m_1^2 - 1)^{\frac{1}{2}} > n$, or $\theta > \phi_1$,

with $\phi_o = \arctan(1/\beta)$ and $\phi_1 = \arctan\{1/(m_1^2 - 1)^{\frac{1}{2}}\}$.

In other words for given M, M′ and θ the wing has supersonic leading edges ahead of as well as behind the shock for time $t > 0$, Fig. 16a.

(b) When both $(1 - \lambda_4^2)^{\frac{1}{2}}$ and $(1 - \lambda_o^2)^{\frac{1}{2}}$ are less than $\bar{n}$ -

$$\theta < \phi_o \quad \text{and} \quad \theta < \phi_1,$$

i.e. the wing has subsonic leading edges ahead of as well as behind the shock, Fig. 16b.

(c) When $(1 - \lambda_4^2)^{\frac{1}{2}} < \bar{n}$, and $(1 - \lambda_o^2)^{\frac{1}{2}} > \bar{n}$ - The wing has subsonic leading edges ahead of the shock, and supersonic leading edges behind the shock, Fig. 16c.

(d) When $(1 - \lambda_4^2)^{\frac{1}{2}} > \bar{n}$, and $(1 - \lambda_o^2)^{\frac{1}{2}} < \bar{n}$ - The wing has supersonic leading edges ahead of the shock, and subsonic leading edges behind the shock, Fig. 16d.

Now by using (5.7.4), the pressure integral (5.7.3) can be evaluated for the various physical cases described above.

5.7.2. We consider now a flat plate delta wing at an incidence ε to the oncoming flow. Let it has semi-apex angle θ and has supersonic leading edges before as well as after penetrating the shock, i.e.

$$\theta > \arctan(1/\beta) \qquad \text{and} \qquad \theta > \arctan\{1/(m_1^2 - 1)^{\frac{1}{2}}\}.$$

The slope of the upper surface of the delta wing can be expressed as

$$\eta' = \varepsilon \, H\{x + m_1\tau \mp nz\} \qquad (5.7.5)$$

where H is a unit step function. The results of §5.4 can once again be used for this case. The pressure integral (5.4.18) then leads us to (5.7.3) for the flat delta wing. Thus on integrating (5.7.3), changing to the original variables (x, y, z, τ) and expressing the results in terms of physical conical variables defined by

$$X = x/\tau, \quad Y = y/\tau \quad \text{and} \quad Z = z/\tau$$

we obtain

$$p(X, Y, Z) = \frac{\varepsilon}{2\pi\bar{a}} \sum_{i=0}^{5} \bar{A}_i\Big(\{N_{1i}(X, Y, Z) + N_{1i}(X, Y, -Z)\} - \{N_{2i}(X, Y, Z) + N_{2i}(X, Y, -Z)\}\Big), \qquad (5.7.6)$$

where $\bar{A}_o = - A_o/\Gamma_o$, $\Gamma_o = (1 - \lambda_o^2 - \bar{n}^2)^{\frac{1}{2}}$,

$N_{10}(X, Y, Z) = \arccos\left[\{\lambda_o\Omega_o + \Gamma_o^2(X - m)\}/\eta_o\right]$,

$N_{20}(X, Y, Z) = \arccos\left[\{\bar{n}^2\lambda_o\Omega_o - \Gamma_o^2\lambda_o\bar{n}(1 - m^2)^{\frac{1}{2}} Z + \Gamma_o^2\zeta_o\}/\{(1 - \lambda_o^2)\eta_o\right]$,

with $\Omega_o = (1 - m\lambda_o) - (m - \lambda_o)X - \bar{n}(1 - m^2)^{\frac{1}{2}} Z$,

$\eta_o = (1 - \bar{n}^2)^{\frac{1}{2}}\{\Omega_o^2 - \Gamma_o^2(1 - m^2)Y^2\}^{\frac{1}{2}}$,

$\zeta_o = [\{(1 - m\lambda_o) - (m - \lambda_o)X\}^2 - (1 - \lambda_o^2)(1 - m^2)(Y^2 + Z^2)]^{\frac{1}{2}}$;

for $i = 1, 2 \ldots 5$, $\bar{A}_i = A_i/\Gamma_i$, $\Gamma_i = (1 - \lambda_i^2 - \bar{n}^2)^{\frac{1}{2}}$,

$N_{1i}(X, Y, Z) = \arccos[\{\lambda_i\Omega_i - \Gamma_i^2(X - m)\}/\eta_i]$,

$N_{2i}(X, Y, Z) = \arccos[\{\bar{n}^2\lambda_i\Omega_i - \Gamma_i^2\lambda_i\bar{n}(1 - m^2)^{\frac{1}{2}} Z + \Gamma_i^2\zeta_i\}/\{(1 - \lambda_i^2)\eta_i\}]$,

with $\Omega_i = (1 + m\lambda_i) - (m + \lambda_i)X - \bar{n}(1 - m^2)^{\frac{1}{2}} Z$,

$\eta_i = (1 - \bar{n}^2)^{\frac{1}{2}}\{\Omega_i^2 - \Gamma_i^2(1 - m^2)Y^2\}^{\frac{1}{2}}$,

$\zeta_i = [\{(1 + m\lambda_i) - (m + \lambda_i)X\}^2 - (1 - \lambda_i^2)(1 - m^2)(Y^2 + Z^2)]^{\frac{1}{2}}$;

and $\bar{a}(1 - \lambda_i^2)^{\frac{1}{2}} > n$, for $i = 0, 1 \ldots 5$.

The expression (5.4.21) for the density variation gives for the flat delta wing

$$\rho = p(X, Y, Z) + (E - 1)\, p(m, Y', Z') + (Fk_1 - Gk_2)\frac{\varepsilon}{\pi(\beta^2 - n^2)^{\frac{1}{2}}}(\upsilon + \bar{\upsilon}) \qquad (5.7.7)$$

with $\upsilon = \arccos[(nX - \bar{c}\beta Z)/\{(\beta X - \bar{c}nZ)^2 - \bar{c}^2(\beta^2 - n^2)Y^2\}^{\frac{1}{2}}]$ and $\bar{\upsilon} = \upsilon(-Z)$.

Here $p(m, Y', Z')$ is obtained from $p(X, Y, Z)$ (5.7.6) by replacing X by m, Y by Y' and Z by Z'; Y' and Z' are now given by $Y' = mY/X$ and $Z' = mZ/X$. Then we can obtain for

$$p(m, Y', Z') = \frac{\varepsilon}{2\pi\bar{a}}\Big(\sum_{i=0}^{3} \bar{A}_i\{(n_{1i} + \bar{n}_{1i}) - (n_{2i} + \bar{n}_{2i})\} + (\bar{A}_4 - \bar{A}_5)\{(n_{14} + \bar{n}_{14}) - (n_{24} + \bar{n}_{24})\}\Big) + \frac{\varepsilon}{\pi(\beta^2 - n^2)^{\frac{1}{2}}}(\upsilon + \bar{\upsilon})$$

with $n_{1i} = \arccos(\lambda_i\Omega_i'/\eta_i')$,

$n_{2i} = \arccos[\bar{n}^2\lambda_i\Omega_i' - \Gamma_i^2\lambda_i\bar{n}(1 - m^2)^{\frac{1}{2}} Z + \Gamma_i^2\zeta_i'\}/\{(1 - \lambda_i^2)\eta_i'^2\}]$,

$$\Omega_i' = (1 - m^2) - \bar{n}(1 - m^2)^{\frac{1}{2}} Z',$$

$$\eta_i' = (1 - \bar{n}^2)^{\frac{1}{2}}\{\Omega_i'^2 - \Gamma_i^2(1 - m^2)Y'^2\}^{\frac{1}{2}},$$

and $$\zeta_i' = (1 - m^2)^{\frac{1}{2}}\{(1 - m^2) - (1 - \lambda_i^2)(Y'^2 + Z'^2)\}^{\frac{1}{2}}, \quad \text{for } i = 0, 1 \ldots 4,$$

$$\bar{n}_{1i} = n_{1i}(-Z') \quad \text{and} \quad \bar{n}_{2i} = n_{2i}(-Z').$$

We can show that the first five terms (for $i = 0, 1, \ldots 4$) in $p(m, Y', Z')$ will vanish for

$$X \leqslant (Y^2 + Z^2)^{\frac{1}{2}} m/(1 - m^2)^{\frac{1}{2}},$$

on or to the left of the conical surface with centre at 0 and its generators passing through B_1 and B_2 (also B, B′), Fig. 15. Again the last term in $p(m, Y', Z')$ and the last expression in (5.7.7) vanish for

$$X \leqslant \bar{c}(Y^2 + Z^2)^{\frac{1}{2}},$$

on or to the left of another conical surface with centre at 0 and its generators passing through I_1 and I_2 (also I, I′), Fig. 15. These conical surfaces are the contact surfaces to be obtained in the non-uniform flow field behind the shock. It may be noted that the similar features were also observed for the interaction with a conical projectile (cf. §4.5).

Chapter 6

THIN YAWED AEROFOILS

6.1. Introduction

We consider the interaction of a plane shock of arbitrary strength which encounters a yawed (with respect to the shock plane), thin two-dimensional aerofoil moving with supersonic speed. This case differs from those of head-on interaction of a plane shock with supersonically moving pointed bodies (Chapters 3, 4 & 5), since in the present case we cannot indicate a moment when the interaction begins. The incident shock is being disturbed at all times by the aerofoil and its associated field. However by considering the flow in a suitable reference frame the time may entirely be eliminated. The problem is then posed in such a way that the method of solution developed in Chapter 3 still applies. It is the intention to develop this chapter somewhat independently to start with.

Reference frame and assumptions

Let the plane shock be moving with a velocity V into a medium at rest imparting a velocity U to the fluid behind it; the density, pressure and the sonic velocity ahead of and behind the shock are R_o, P_o, a_o and R_1, P_1, a_1. The aerofoil is moving in a direction opposite to the shock with a supersonic velocity normal to the leading edge, $W > a_o$. The leading edge makes an angle χ with the shock front, Fig. 17. The point O, the intersection of the shock front with the leading edge of the aerofoil, travels with a velocity V_o whose direction makes an angle χ' with the shock front (cf. Fig. 17), where

$$V_o = (V^2 + W^2 + 2\ VW \cos\chi)^{\frac{1}{2}} \operatorname{cosec}\chi, \tag{6.1.1}$$

and $$\sin\chi' = V/V_o.$$

Assume an equal and opposite velocity $\underline{V}_o$ be superimposed on the entire flow field. The point 0 is then brought to rest and the flow is reduced to a steady state. The undisturbed rgion ahead of the shock, at rest before, has now acquired the velocity $\underline{V}_o$. Also the uniform flow behind the shock has now velocity $\underline{V}_1 = \underline{V}_o + \underline{U}$, whose magnitude is V_1 and makes and angle μ with the shock front (cf. Fig. 18a), where

$$\left.\begin{aligned} V_1 &= [(V - U)^2 + \{(W + V\cos\chi)/\sin\chi\}^2]^{\frac{1}{2}}, \\ \text{and} \qquad \tan\mu &= (V - U)\sin\chi/(W + V\cos\chi). \end{aligned}\right\} \tag{6.1.2}$$

Expressed in dimensionless form, (6.1.2) and (6.1.1) give after simplification

$$\left.\begin{aligned} \bar{M} &= V_1/a_1 = m/\sin\mu, \\ \tan\mu &= m\sin\chi/\{(a_o/a_1)M' + M\cos\chi\}, \\ \text{and} \qquad \tan\chi' &= M\sin\chi/(M' + M\cos\chi), \end{aligned}\right\} \tag{6.1.3}$$

where $M = V/a_o$ the shock Mach number, $M' = W/a_o$ the Mach number of the aerofoil, $m = (V - U)/a_1$ defined in (2.3.13), and a_o/a_1 given by (2.3.3).

We fix the cartesian co-ordinates (x, y, z) with origin at 0 such that the z-axis is in the direction of V_1, Fig. 18a. The aerofoil is assumed to lie in the x-z plane. The subsequent treatment relies on the fact that the flow is supersonic behind the shock, i.e.

$$V_1 > a_1. \tag{6.1.4}$$

This condition is fulfilled provided $\chi_1 > \chi > (\pi - \chi_2)$, where

$$\chi_1, \chi_2 = \arctan\left(\frac{M_1(1 - m^2)^{\frac{1}{2}} \pm M'(a_o/a_1)\{M_1^2 + (1 - m^2) - (M'a_o/a_1)^2\}^{\frac{1}{2}}}{(1 - m^2) - (M'a_o/a_1)^2}\right);$$

with $M_1 = U/a_1$, a_o/a_1 and m being functions of M (cf. Chapter 2). Furthermore χ should always be less than $\pi/2$. In Fig. 19, χ against M is plotted with M' as a parameter. This means that for any fixed aerofoil speed M', the point (M, χ) must lie to the left of the appropriate curve for the condition (6.1.4) to be fulfilled.

Behind the plane shock if the fluid velocity normal to the leading edge of the aerofoil is supersonic, i.e.

$$(W + U \cos\chi) > a_1 \qquad (6.1.5)$$

the flow pattern above and below the aerofoil will be independent of each other. This condition is met if

$$\chi < \arccos\{(1 - M' a_o/a_1)/M_1\} .$$

Figure 20 illustrates the range of χ with M and M' as a parameter. One may note that while the condition (6.1.4) is sufficient, the condition (6.1.5) is necessary at least when the lifting yawed aerofoils are under consideration. However Figs. 19 and 20 show that for a wide range of χ, M and M' both the conditions are fulfilled. Under these conditions then it is sufficient to consider the solution, say, for $y > 0$.

The region of non-uniform flow behind the advancing shock is bounded below by the aerofoil and ahead by the shock. A Mach conoid emanates with the vertex at 0, its axis along the z-axis, and makes a semi-vertex angle $\bar{\alpha} = \arcsin(1/\bar{M})$, Fig. 18. Due to the supersonic velocity normal to the leading edge, $(W + U \cos\chi) > a_1$, a Mach plane is attached to the leading edge of the aerofoil, which is tangent to the Mach conoid. A similar weak shock is attached to the portion of the leading edge which lies ahead of the advancing shock, due to the velocity $W > a_o$; this separates the perturbed region (2) from region (0), which is at rest ahead of the plane shock. The weak shock between regions (0) and (2) meets the main shock along a line l through the origin 0 (Fig. 18). The tangent plane from l to the Mach conoid is another weak shock front. The plane containing l and the z-axis is the approximate position of a contact discontinuity. Another contact discontinuity lies approximately along the plane passing through the intersection of the Mach conoid and the main shock, and the z-axis. The portion of the incident shock between the line l and the surface of the aerofoil is disturbed.

If we consider a section parallel to the x-y plane on the z-axis, the flow picture obtained is the same as for the case of unyawed aerofoils, (cf. Chapter 3). For the two-dimensional unyawed aerofoils the flow picture was seen to be growing with respect to time t, while in the present case it can be regarded as growing with the space variable z. The assumptions introduced in §1.3 hold for this case too, except that now we shall use the equations of motion of steady rotational flow behind the shock. The physical parameters which define the problem are once again V, W, P_o, R_o (or M and M′), the shape of the aerofoil, and the angle of yaw χ.

In the following we shall present the equations of motion (§6.2.1) and deduce the relations at the disturbed shock (§6.2.2). In deducing the shock relations we shall follow the procedure of Chester (1954) who considers the diffraction of a plane shock by a stationary yawed aerofoil. The additional boundary conditions are discussed in §6.3. In §6.4 the complete formulation is given in Lorentz variables followed by its solution. In §6.5 the results are applied to calculate the interaction field with a yawed symmetric wedge.

6.2. Fundamental relations

6.2.1 Equations of motion

In the region of non-uniform flow we use the equations of adiabatic steady, three-dimensional flow, i.e. Eqs. (2.2.1), (2.2.2), (2.2.5) and (2.2.7) omitting the time derivatives. This field can be linearized on the assumption that the flow variables differ by small quantities from their values in region (1), Fig. 18, of uniform flow. We assume the following expansions

$$\left.\begin{aligned} R &= R_1 + \varepsilon\,\rho_1(x, y, z) + O(\varepsilon^2), \\ P &= P_1 + \varepsilon\,p_1(x, y, z) + O(\varepsilon^2), \\ S &= S_1 + \varepsilon\,s_1(x, y, z) + O(\varepsilon^2), \\ V &= V_1 + \varepsilon\,v_1(x, y, z) + O(\varepsilon^2), \end{aligned}\right\} \qquad (6.2.1)$$

with $\underline{v}_1 = i\,u_1 + j\,v_1 + k\,w_1$. The parameter ε may be interpreted as the deviation of the aerofoil surface from its mean plane. The perturbation parameters can be expressed in dimensionless form as

$$\rho = \rho_1/R_1, \qquad P = p_1/\gamma p_1, \qquad \underline{v} = \underline{v}_1/a_1, \qquad s = \gamma s_1/c_v.$$

The linearized equations of motion for the disturbed field in frame of reference (x, y, z) can then be written as

continuity, $\bar{M}\,\partial\rho/\partial z = -(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z),$ (6.2.2a)

momentum, $\left.\begin{array}{l}\bar{M}\,\partial u/\partial z = -\,\partial p/\partial x, \quad \bar{M}\,\partial v/\partial z = -\,\partial p/\partial y, \\ \bar{M}\,\partial w/\partial z = -\,\partial p/\partial z,\end{array}\right\}$ (6.2.2b)

together with $\partial\rho/\partial z = \partial p/\partial z,$ (6.2.2c)

and $s = (p - \rho),$ (6.2.2d)

where $\bar{M} = V_1/a_1$.

From the last two of Eqs. (6.2.2) it is easily seen that $\partial s/\partial z = 0$, i.e. the entropy remains attached to the fluid particles.

It is convenient to introduce a scale transformation defined by

$$z = \bar{\beta}\, z'. \tag{6.2.3}$$

where $\bar{\beta} = (\bar{M}^2 - 1)^{\frac{1}{2}} > 0$. The Eqs. (6.2.2a)-(6.2.2c) together with (6.2.3) can be combined to yield a wave equation in p

$$\partial^2 p/\partial x^2 + \partial^2 p/\partial y^2 - \partial^2 p/\partial z'^2 = 0 \tag{6.2.4}$$

This equation will be solved subject to a suitable set of boundary conditions, which will be formulated in the sequel.

6.2.2 <u>Conditions at the disturbed shock</u>

The undisturbed part of the shock front lies in the plane

$$x = \tan\mu\ z = \bar{m}z' \tag{6.2.5}$$

where $\bar{m} = \bar{\beta}\tan\mu$. Let the displacement of the shock from its undisturbed location be denoted by $\psi(y, z)$; it is assumed to be uniformly small. Consequently we take the equation of the disturbed shock to be

$$x = \tan\mu\ z + \psi(x, y) + O(\varepsilon^2),$$

or

$$x - \tan\mu\ z - \psi(x, y) = 0, \tag{6.2.6}$$

to the first order. The direction cosines of the normal to the shock front are then proportional to

$$1, -\psi_y, -\tan\mu - \psi_z.$$

If $\underline{n}$ is a unit vector normal to the shock we have

$$\underline{n} = \{\cos\mu(1 - \sin\mu\cos\mu\,\psi_z), -\cos\mu\,\psi_y, -\sin\mu\cos^3\mu\,\psi_z\}, \tag{6.2.7}$$

to the first order.

Now, the shock transition relations will depend on the normal component of the velocity in front of the shock. In fact the relations for the stationary shock may be written as

$$\left.\begin{aligned} \underline{V}_1 - \underline{V}_2 &= \frac{2}{\gamma + 1}\underline{V}_n\,(a_2^2/V_n^2 - 1), \\ P &= \frac{2}{\gamma + 1}R_2(V_n^2 - \frac{\gamma - 1}{\gamma + 1}a_2^2), \\ R &= \frac{\gamma + 1}{\gamma - 1}R_2/(1 + \frac{2}{\gamma - 1}a_2^2/V_n^2, \end{aligned}\right\} \tag{6.2.8}$$

where R, P, $\underline{V}_1$ being the density, the pressure and the fluid velocity behind the shock, R_2, a_2, $\underline{V}_2$, the density, the sonic velocity and the fluid velocity ahead of the shock, $\underline{V}_n$ the component of $\underline{V}_2$ normal to the shock, and γ the adiabatic index of the medium.

Since the advancing shock does not disturb the flow ahead of it, we can assume that the flow in region (2), Fig. 18, is isentropic. Let the density and pressure in this region be expressed as

$$\left.\begin{aligned} R_2 &= R_o + \rho_o + O(\varepsilon^2), \\ P_2 &= P_o + p_o + O(\varepsilon^2), \end{aligned}\right\} \tag{6.2.9}$$

where R_o and P_o being the density and pressure in the still region (0), and ρ_o and p_o the perturbation values of density and pressure in region (2). Then

$$a_2^2 = \gamma P_2/R_2 = a_o[1 + \{(\gamma - 1)/2\gamma\}p_o/P_o]$$

If (u_o, v_o, w_o) be the velocity perturbations in region (2) in the assumed co-ordinate system, the velocity ahead of the shock $\underline{V}_2$ is then

$$\underline{V}_2 = \{-|V_o|\sin(\chi' - \mu) + u_o, \quad v_o, \quad |V_o|\cos(\chi' - \mu) + w_o\} \qquad (6.2.10)$$

Since $\underline{V}_n = (\underline{V}_2 \cdot \underline{n})\underline{n}$, it follows with the help of (6.2.7) that

$$\underline{V}_n = \{-V(1 + \cot\gamma' \cos^2\mu \; \psi_z) + u_o\cos\mu - w_o\sin\mu\}\underline{n} \qquad (6.2.11)$$

Making use of (6.2.9), (6.2.10) and (6.2.11) togehter with (6.2.7) in (6.2.8), we can deduce the following relations which apply at the disturbed shock front, viz.

$$\rho = \frac{-4}{(\gamma - 1)M^2 + 2}\Big[(\bar{u}\cos\mu - \bar{w}\sin\mu) + \frac{1}{2}\{(\gamma - 1) - \frac{(\gamma - 1)M^2 + 2}{2}\}\bar{p} - \cot\gamma'\cos^2\mu \; \psi_z\Big],$$

$$p = \frac{4}{\gamma + 1}(R_o/R_1)(a_o/a_1)^2 M^2\{-(\bar{u}\cos\mu - \bar{w}\sin\mu) + (1/2M^2)(M^2 - \frac{\gamma - 1}{2})\bar{p} + \cot\gamma'\cos^2\mu \; \psi_z\},$$

$$u = \frac{2}{\gamma + 1}(a_o/a_1)\cos\mu\Big[-\frac{1 + M^2}{M}(\bar{u}\cos\mu - \bar{w}\sin\mu) + \frac{\gamma + 1}{2}M\sec\mu\;\bar{u} - \frac{\gamma - 1}{M}\bar{p} + \{\frac{1}{M}\cos(\chi' - \mu) + M\cos(\chi' + \mu)\}\mathrm{cosec}\chi'\cos\mu \; \psi_z\Big],$$

$$v = (a_o/a_1)M\bar{v} + \frac{2}{\gamma + 1}(a_o/a_1)\frac{1 - M^2}{M}\cos\mu \; \psi_y,$$

$$w = \frac{2}{\gamma + 1}(a_o/a_1)\Big[\frac{1 + M^2}{M}(\bar{u}\cos\mu - \bar{w}\sin\mu) + \frac{\gamma + 1}{2}M\bar{w} + \frac{\gamma - 1}{M}\sin\mu\;\bar{p} + \{\frac{1}{M}\sin(\chi' - \mu) - M\sin(\chi' + \mu)\}\,\mathrm{cosec}\chi'\cos^2\mu \; \psi_z\Big], \qquad (6.2.12)$$

where $\rho = \rho_o/R_o$, $p = p_o/\gamma P_o$, $\bar{u} = u/V$, $\bar{v} = v/V$ and $\bar{w} = w/V$.

Thus the expressions (6.2.12) relate the upstream perturbation parameters ρ, p, (u, v, w) with the downstream parameters $\bar{p}$, $(\bar{u}, \bar{v}, \bar{w})$ at the shock via the derivatives of the shock displacement ψ_y and ψ_z. The shock relations (6.2.12) will be used at the undisturbed location of the shock ($x = \tan\mu \; z = \bar{m}\; z'$) in consistence with the linearization.

Let the upper surface of the aerofoil be given by

$$y = f(x_1), \quad \text{with } f(x_1) = 0, \quad \text{for } x_1 \leqslant 0,$$

where x_1 being perpendicular to the leading edge of the aerofoil.

Then for the supersonic flow, $W > a_o$, past the aerofoil, the perturbations in velocity and pressure can be expressed as

$$\left.\begin{aligned} u_o &= -(W/\beta)\cos(\chi-\mu)\, f'(x_1-\beta y), \\ w_o &= -(W/\beta)\sin(\chi-\mu)\, f'(x_1-\beta y), \\ v_o = W f'(x_1-\beta y),\quad p_o &= R_o(W^2/\beta)\, f'(x_1-\beta y), \end{aligned}\right\} \tag{6.2.13}$$

where $\beta = (M'^2 - 1)^{\frac{1}{2}}$, and the prime represents the differentiation with respect to the argument.

In the co-ordinate system (x, y, z) assumed, x_1 then becomes

$$x\cos(\chi-\mu) + z\sin(\chi-\mu),$$

which at the shock location $x = \bar{m}\, z'$ gives

$$\bar{\beta}\, z' \sin\chi \sec\mu.$$

Then the perturbations given by (6.2.13) can be prescribed as the known upstream parameters in region (2). When expressed in dimensionless form, (6.2.13) gives at the shock $x = \bar{m}\, z'$

$$\left.\begin{aligned} \bar{u} &= -\frac{1}{\beta}\frac{M'}{M}\cos(\chi-\mu)\, f'(\xi), & \bar{v} &= \frac{M'}{M} f'(\xi), \\ \bar{w} &= -\frac{1}{\beta}\frac{M'}{M}\sin(\chi-\mu)\, f'(\xi), & \bar{p} &= \frac{M'^2}{\beta} f'(\xi), \end{aligned}\right\} \tag{6.2.14}$$

where $\xi = \bar{a}\{(1-\bar{m}^2)^{\frac{1}{2}} z' - \bar{b}\, y\}$,

with $\bar{a} = \bar{\beta}\sin\chi\sec\mu/(1-\bar{m}^2)^{\frac{1}{2}}$ and $\bar{b} = \beta/\bar{a}$.

Making use of (6.2.14) and (6.2.3), we obtain from the shock relations (6.2.12), at $x = \bar{m}\, z'$

$$\rho = \Lambda_1 f'(\xi) + \Pi_1 \psi_{z'} \tag{6.2.15a}$$

$$p = \Lambda_2 f'(\xi) + \Pi_2 \psi_{z'} \tag{6.2.15b}$$

$$u = \Lambda_3 f'(\xi) + \Pi_3 \psi_{z'} \tag{6.2.15c}$$

$$v = \Lambda_4 f'(\xi) + \Pi_4 \psi_y \tag{6.2.15d}$$

$$w = \Lambda_5 f'(\xi) + \Pi_5 \psi_{z'} \tag{6.2.15e}$$

where

$$\Lambda_1 = \frac{4}{(\gamma - 1)M^2 + 2}\,\frac{1}{\beta}\left[\frac{M'}{M}\cos\chi + \left\{\frac{\gamma - 1}{2} - \frac{(\gamma - 1)M^2 + 2}{4}\right\}M'^2\right],$$

$$\Lambda_2 = \frac{2}{2\gamma M^2 - (\gamma - 1)}\,\frac{1}{\beta}(2MM'\cos\chi + M^2M'^2 - \frac{\gamma - 1}{2}M'^2),$$

$$\Lambda_3 = \frac{2}{\gamma + 1}(a_o/a_1)\,\frac{\cos\mu}{\beta}\{(1 + 1/M^2)M'\cos\chi - \frac{\gamma + 1}{2}M'\sec\mu\cos(\chi - \mu) - (\gamma - 1)M'^2/M\},$$

$$\Lambda_4 = M'a_o/a_1,$$

$$\Lambda_5 = \frac{-2}{\gamma + 1}(a_o/a_1)\,\frac{\sin\mu}{\beta}\{(1 + 1/M^2)M'\cos\chi - \frac{\gamma + 1}{2}M'\mathrm{cosec}\,\mu\sin(\chi - \mu) - (\gamma - 1)M'^2/M\},$$

$$\Pi_1 = \frac{4}{(\gamma - 1)M^2 + 2}(a_o/a_1)\,\frac{m\cos^3\mu}{M(m^2 - \sin^2\mu)^{\frac{1}{2}}},$$

$$\Pi_2 = \frac{4}{\gamma + 1}\,\frac{m^2\cos^3\mu}{(m^2 - \sin^2\mu)^{\frac{1}{2}}},$$

$$\Pi_3 = \frac{2}{\gamma + 1}\,\frac{1}{\cos\chi'}\{\cos(\chi' + \mu) + (1/M^2)\cos(\chi' - \mu)\}\,\frac{m\cos^3\mu}{(m^2 - \sin^2\mu)^{\frac{1}{2}}},$$

$$\Pi_4 = -M_1\cos\mu,$$

$$\Pi_5 = \frac{-2}{\gamma + 1}\,\frac{1}{\cos\chi'}\{\sin(\chi' + \mu) - (1/M^2)\sin(\chi' - \mu)\}\,\frac{m\cos^3\mu}{(m^2 - \sin^2\mu)^{\frac{1}{2}}},$$

and a_o/a_1 and M_1 are given in (2.3.3).

From the above relations we notice that $\psi_{z'}$ can be eliminated from (6.2.5b) and (6.2.5c), while ψ can be eliminated from (6.2.5b) and (6.2.5d) by cross-differentiation. Thus we deduce at $x = mz'$

$$u = \{p - B\,f'(\bar{\xi})\}/A, \tag{6.2.16a}$$

and

$$\frac{\partial v}{\partial z'} = \frac{1}{D}\left\{\frac{\partial p}{\partial y} - \Lambda_2\,\frac{\partial f'(\bar{\xi})}{\partial y}\right\} + \Lambda_4\,\frac{\partial f'(\bar{\xi})}{\partial z'}, \tag{6.2.16b}$$

where $A = \Pi_2/\Pi_3$, $B = \Lambda_2 - A\Lambda_3$ and $D = \Pi_2/\Pi_4$.

These conditions will be used later to describe a single condition at the shock $x = \bar{m}z'$.

6.3. Boundary conditions

For the disturbed region behind the shock, $x < \bar{m}z'$, we can prescribe two starting conditions as

$$p(x, y, z') = \partial p/\partial z' = 0, \quad \text{for } z' \leqslant 0. \tag{6.3.1}$$

At infinity, since all the perturbations must vanish, we can have for $x < \bar{m}z'$, $z' > 0$

$$\text{as } \bar{x} \to -\infty, \ y \to \infty \quad p \text{ and its derivatives} \to 0. \tag{6.3.2}$$

For the region $x < \bar{m}z'$, $z' > 0$, the tangency condition gives on the aerofoil $(y = 0)$

$$v_1 = (W + U\cos\chi)\, f'\{x\cos(\chi - \mu) + z\sin(\chi - \mu)\} \tag{6.3.3}$$

Here we may notice that by the definition of f, v_1 vanishes to the left of the leading edge of the aerofoil. Expressing (6.3.3) in dimensionless form, using the second momentum Eq. (6.2.2b) and noticing that

$$V_1 \sin(\chi - \mu) = (W + U\cos\chi),$$

it follows that at $y = 0$

$$\partial p/\partial y = -\bar{m}_1^2\, f''\{x\cos(\chi - \mu) + z'\bar{\beta}\sin(\chi - \mu)\}, \tag{6.3.4}$$

where $\quad \bar{m}_1 = (W + U\cos\chi)/a_1 = M'a_o/a_1 + M_1\cos\chi.$

The condition (6.3.4), however, fails at the shock location, since it has been obtained by using the second momentum equation. Hence to obtain a condition on $\partial p/\partial y$ at the aerofoil-shock intersection, we invoke the shock condition (6.2.6b) which gives at $x = \bar{m}z'$, $y = 0$

$$\frac{\partial p}{\partial y} = D\,\frac{\partial v}{\partial z'} - \{\Lambda_2\,\bar{a}\bar{b} + D\,\Lambda_4\,\bar{a}(1 - \bar{m}^2)^{\frac{1}{2}}\}f''\{\bar{a}(1 - \bar{m}^2)^{\frac{1}{2}}\, z'\}.$$

Here v can be obtained from (6.3.3) which gives at $x = \bar{m}z'$, $y = 0$

$$v = \bar{m}_1\, f'\{\bar{a}(1 - \bar{m}^2)^{\frac{1}{2}}\, z'\}$$

Thus we obtain at the shock-aerofoil intersection, $x = \bar{m}z'$, $y = 0$

$$\partial p/\partial y = B_o\, f''\{\bar{a}(1 - \bar{m}^2)^{\frac{1}{2}}\, z'\}, \tag{6.3.5}$$

where $$B_o = D(1 - \bar{m}^2)^{\frac{1}{2}} (\bar{m}_1 - \Lambda_4)\bar{a} - \Lambda_2\bar{a}\bar{b}.$$

6.4. Formulation in Lorentz variables and solution

We introduce Lorentz transformation defined by (cf. §3.3)

$$\bar{x} = (x - \bar{m}z')/(1 - \bar{m}^2)^{\frac{1}{2}}, \quad \bar{z} = (z' - \bar{m}x)/(1 - \bar{m}^2)^{\frac{1}{2}}. \tag{6.4.1}$$

The plane $\bar{x} = 0$ then corresponds to the shock plane, and the wave Eq. (6.2.4) remains invariant, viz.

$$\partial^2 p/\partial\bar{x}^2 + \partial^2 p/\partial y^2 - \partial^2 p/\partial\bar{z}^2 = 0. \tag{6.4.2}$$

The boundary conditions (6.3.1), (6.3.2), (6.3.4) and (6.3.5) now become, for $\bar{x} < 0$, $y > 0$

$$p(\bar{x}, y, \bar{z}) = \partial p/\partial\bar{z} = 0, \quad \text{for } \bar{z} \leqslant 0, \tag{6.4.3}$$

for $\bar{z} > 0$, as $\bar{x} \to -\infty$, $y \to \infty$ $\quad$ p and its derivatives $\to 0$, $\qquad$ (6.4.4)

$$\bar{x} < 0,\ y = 0 \qquad \partial p/\partial y = A_o f''\{\bar{a}(\bar{z} + \lambda_o\bar{x})\}, \tag{6.4.5}$$

$$\bar{x} = 0,\ y = 0 \qquad \partial p/\partial y = B_o f''(\bar{a}\bar{z}), \tag{6.4.6}$$

where $A_o = -\bar{m}_1^2$ and $\lambda_o = (\cos\chi + m\,\bar{m}_1)/(\bar{\beta}\sin\chi)$.

Using the transformations (6.2.3) and (6.4.1), the basic Eqs. (6.2.2a)-(6.2.2c) can be combined to yield

$$(\bar{\beta}/\bar{M})\frac{\partial^2 p}{\partial\bar{x}\,\partial\bar{z}} + \frac{\partial^2 u}{\partial\bar{z}^2} + \frac{\bar{m}}{(1 - \bar{m}^2)^{\frac{1}{2}}}\frac{\partial^2 v}{\partial y\,\partial\bar{z}} = 0$$

We can substitute for u and $\partial v/\partial\bar{z}$ from the shock conditions (6.2.6), and thus obtain at, $\bar{x} = 0$, $y > 0$, $\bar{z} > 0$

$$-\frac{\partial^2 p}{\partial y^2} + 2a\frac{\partial^2 p}{\partial\bar{x}\,\partial\bar{z}} + (1 + b)\frac{\partial^2 p}{\partial\bar{z}^2} = \frac{K}{\bar{b}}\frac{\partial}{\partial\bar{z}} f''\{\bar{a}(\bar{z} - \bar{b}\,y)\}, \tag{6.4.7}$$

where $$a = -\frac{1 - \bar{m}^2}{2m}(\bar{\beta}/\bar{M})\,D, \quad 1 + b = -\frac{1 - \bar{m}^2}{\bar{m}}\frac{D}{A},$$

and $$K = \{(1 + b)B - \Lambda_2\bar{b}^2 - (1 - \bar{m}^2)^{\frac{1}{2}} D\Lambda_4\bar{b}\}\bar{a}\bar{b}.$$

The condition (6.4.8) is a differential condition in p, which will be used at the shock front.

The governing Eq. (6.4.2), together with the boundary conditions (6.4.3)-(6.4.7) completes the formulation of the problem. This can be compared with the formualation of the unyawed case (§3.3, Eqs. 3.3.2-3.3.4, 3.3.10-3.3.12). It is obvious that the two formulations are exactly the same except that $\bar{\tau}$ (in §3.3) now reads $\bar{z}$, and the different constants here are functions of M, M′ and χ instead of M and M′ alone. However all the constants involved in the present case are expressed in a form which clearly indicate the limit as $\chi \to 0$. Hence following §3.4, we can write the solution of the above formulation (Eqs. 6.4.2-6.4.7) immediately as follows (cf. 3.4.16)

$$p(\bar{x}, y, \bar{z}) = \frac{1}{\pi}\Big(-A_o\int_0^{\bar{z}} d\mu \int_{\xi_1}^{0} \frac{f''\{\bar{a}(\mu + \lambda_o\xi)\}}{\{(\bar{z} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}}\, d\xi$$

$$+ \sum_{i=1}^{5} A_i \int_0^{\bar{z}} d\mu \int_0^{\xi_2} \frac{f''\{\bar{a}(\mu - \lambda_i\xi)\}}{\{(\bar{z} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}}\, d\xi\Big) \qquad (6.4.8)$$

with $\xi_1 = \bar{x} - \{(\bar{z} - \zeta)^2 - y^2\}^{\frac{1}{2}}$ and $\xi_2 = \bar{x} + \{(\bar{z} - \zeta)^2 - y^2\}^{\frac{1}{2}}$.

Here $\lambda_1 = \lambda_o$, λ_2 and λ_3 are the roots (real, distinct and positive) of the quadratic equation

$$\lambda_i^2 - 2a\lambda_i + b = 0.$$

$\lambda_4 = (1 - \bar{b}^2)^{\frac{1}{2}} > 0$, and $\lambda_5 = -\lambda_4$. The coefficients A_is' are now given by

$$A_1 = A_o H(\lambda_o)/H(-\lambda_o), \quad H(\lambda_o) = \lambda_o^2 + 2a\lambda_o + b,$$

$$A_2 = \frac{-2\lambda_2}{\lambda_2 - \lambda_3}\Big(B_o + \frac{2aA_o}{\lambda_o - \lambda_2} - \frac{K}{\lambda_2^2 - \lambda_4^2}\Big),$$

$$A_3 = \frac{2\lambda_3}{\lambda_2 - \lambda_3}\Big(B_o + \frac{2aA_o}{\lambda_o - \lambda_3} - \frac{K}{\lambda_3^2 - \lambda_4^2}\Big),$$

$$A_4 = K/\{(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)\}, \quad A_5 = K/\{(\lambda_2 + \lambda_4)(\lambda_3 + \lambda_4)\}.$$

In deducing the above result (6.4.8), once again we assumed $\bar{b} < 1$. This condition is satisfied provided

$$M' < \frac{M + \ell^2(M^2 - 1)}{\ell - 1}\cos\chi, \quad \text{with } \ell = (1 - m^2)(a_o/a_1)^2, \qquad (6.4.9)$$

a function of M and χ. Figure 20 illustrates the variation of M' with M for various χ i.e. for any fixed angle of yaw χ, the point (M', M) must lie to the left of the appropriate curve for the condition (6.4.9) to be fulfilled. This condition, however, implies that the shock intersection lies outside the Mach conoid emanating from 0, Fig. 18b.

In case $\bar{b} > 1$, the shock intersection will be located inside the Mach conoid; λ_4 and λ_5 in (6.4.8) then become imaginary. The last two terms in (6.4.8) should then be replaced by (cf. 3.4.18)

$$I_{4,5} = \frac{1}{\pi\bar{b}}\int_0^{\bar{z}} d\mu \int_0^{\infty} f''\{\bar{a}(\mu - \bar{b}\zeta)\}\Big(Q_1\{N_1(\mu, \zeta) + N_1(\mu, -\zeta)\} + Q_2\{N_2(\mu, \zeta) + N_2(\mu, -\zeta)\}\Big)\, d\zeta, \qquad (6.4.10)$$

where Q_1 and Q_2 are constants given by

$$Q_1 = K(\lambda_2\lambda_3 + \lambda_4^2)/q, \quad Q_2 = K(\lambda_2 + \lambda_3)\lambda_4^2/q, \quad \text{with} \quad q = (\lambda_2^2 - \lambda_4^2)(\lambda_3^2 - \lambda_4^2),$$

and the functions N_1 and N_2 are

$$N_1(\mu, \zeta) = \frac{\bar{x}}{\bar{x}^2 + (y - \zeta)^2}\,\frac{\bar{z} - \mu}{\{(\bar{z} - \mu)^2 - \bar{x}^2 - (y - \zeta)^2\}^{\frac{1}{2}}},$$

$$N_2(\mu, \zeta) = \{(\bar{z} - \mu)^2 - \bar{x}^2 - (y - \zeta)^2\}^{-\frac{1}{2}}.$$

6.5. Application

We apply the result (6.4.8) to a symmetric wedge of semi-vertex angle ε, which is yawed by an angle χ with respect to the shock plane. Then on integration (6.4.8) yields

$$p(\bar{x}, y, \bar{z}) = \frac{\varepsilon}{\pi\bar{a}}\left\{\bar{A}_o \arccos\left(\frac{\lambda_o\bar{z} + \bar{x}}{\{(\bar{z} + \lambda_o\bar{x})^2 - (1 - \lambda_o^2)y^2\}^{\frac{1}{2}}}\right) + \sum_{i=1}^{5} \bar{A}_i \arccos\left(\frac{\lambda_i\bar{z} - \bar{x}}{\{(\bar{z} - \lambda_i\bar{x})^2 - (1 - \lambda_i^2)y^2\}^{\frac{1}{2}}}\right)\right\}, \qquad (6.5.1)$$

where $\bar{A}_o = -A_o/(1 - \lambda_o^2)^{\frac{1}{2}}$, $\bar{A}_i = A_i/(1 - \lambda_i^2)^{\frac{1}{2}}$, and λ_is' < 1 for $i = 0, 1, 2...5$. In case λ_2 or $\lambda_3 > 1$, a minor modification in (6.5.1) is necessary, since in that case the corresponding integrals in (6.4.8) will integrate to yield

$$\frac{A_i}{(\lambda_i^2 - 1)^{\frac{1}{2}}} \operatorname{arccosh}\left(\frac{\lambda_i\bar{z} - \bar{x}}{\{(\bar{z} - \lambda_i\bar{x})^2 + (\lambda_i^2 - 1)y^2\}^{\frac{1}{2}}}\right), \quad \lambda_i > 1. \tag{6.5.2}$$

Using the transformations (6.4.1) and (6.2.3), p can also be expressed in terms of original variables (x, y, z). Then (6.5.1) yields on simplification

$$p(x, y, z) = \frac{\varepsilon}{\pi\bar{a}}\left\{\bar{A}_o \arccos\left(\frac{\gamma_o(z/\bar{\beta}) + x}{\{(z/\bar{\beta} - \gamma_o x)^2 - (1 - \gamma_o^2)y^2\}^{\frac{1}{2}}}\right)\right.$$

$$\left. + \sum_{i=1}^{5} \bar{A}_i \arccos\left(\frac{\gamma_i(z/\bar{\beta}) - x}{\{(z/\bar{\beta} - \gamma_i x)^2 - (1 - \gamma_i^2)y^2\}^{\frac{1}{2}}}\right)\right\}, \tag{6.5.3}$$

where $\gamma_o = (\lambda_o - \bar{m})/(1 - \bar{m}\lambda_o)$ and $\gamma_i = (\lambda_i + \bar{m}\lambda_i)/(1 + \bar{m}\lambda_i)$.

Expressing the above result (6.5.3) in terms of conical variables now defined by $X = \bar{\beta}x/z$, $Y = \bar{\beta}y/z$, we obtain

$$p(X, Y) = \frac{\varepsilon}{\pi\bar{a}}\left\{\bar{A}_o \arccos\left(\frac{\gamma_o + X}{\{(1 + \gamma_o X)^2 - (1 - \gamma_o^2)Y^2\}^{\frac{1}{2}}}\right)\right.$$

$$\left. + \sum_{i=1}^{5} \bar{A}_i \arccos\left(\frac{\gamma_i - X}{\{(1 - \gamma_i X)^2 - (1 - \gamma_i^2)Y^2\}^{\frac{1}{2}}}\right)\right\}. \tag{6.5.4}$$

In case $\bar{b} > 1$, the last two terms in (6.5.4) must be replaced by (integrating Eq. 6.4.10)

$$I_{4,5} = \frac{\varepsilon}{\pi\bar{a}\bar{b}}\left\{Q_1 \arctan\left(\frac{2(\bar{m} - X)N}{(\bar{m} - X)^2 + \lambda_4'^2(1 - \bar{m}X)^2 - N^2}\right)\right.$$

$$\left. + \frac{Q_2}{2\lambda_4'} \ln\left(\frac{(\bar{m} - X)^2 + \{N + \lambda_4'(1 - \bar{m}X)\}^2}{(\bar{m} - X)^2 + \{N - \lambda_4'(1 - \bar{m}X)\}^2}\right)\right\}. \tag{6.5.5}$$

with $N = \bar{b}(1 - \bar{m}^2)^{\frac{1}{2}} (1 - X^2 - Y^2)^{\frac{1}{2}}$ and $\lambda_4' = (\bar{b}^2 - 1)^{\frac{1}{2}}$.

It is in order to point out that (6.5.5) is also the real part of the last two terms in (6.5.4).

The expressions for the density variation $\rho(x, y, z)$ and the shock displacement $\psi(y, z)$ could easily be written by analogy with the case of unyawed wedge (§3.6).

Chapter 7

NUMERICAL RESULTS AND DISCUSSION

In the first section of this chapter we shall present the numerical results for the examples considered in Chapters 3-6. The results for the case of yawed wedge are compared with those obtained by Smyrl (1963). For the case of slender conical projectile a comparison is drawn with the results of Blankenship (1965), which were obtained by numerical methods.

In §7.2 we shall conclude the work by pointing out where further detailed exploration is necessary for which the theory is either inadequate or does not give any information. The treatment of the non-linearities of the problem for which our theory gives erroneous information are, however, outside the scope of the present work.

7.1. Numerical results

Two-dimensional wedge

Pressure distributions on the wedge face ($Y = 0$) have been calculated for a number of cases using the results of §§3.6 and 6.5. They are illustrated to demonstrate the effect of the wedge speed M', Fig. 22, the shock speed M, Fig. 23, and the yaw χ, Fig. 24. The results are only computed for the non-uniform Mach-reflection region EF (Fig. 5b) along the wedge face. The value of the pressure along AE (Fig. 5b) is constant given by the value at E. We adjust the scale so as to make the distance EF the same in all cases. The numerical results illustrated above were first given by Smyrl (1963).

In Fig. 25 we have presented the complete pressure and density field using the results of §3.6, in the form of isobars and isopycnics in the Mach-reflection region BCDEF (Fig. 5b) for $M = 2.0$ and $M' = 2.0$;

we have also plotted the shape of the shock front for semi-vertex angle $\varepsilon = 0.05$. The pressure in regions ACE and IDB (Fig. 5) is constant. There are discontinuities along the Mach lines AC and ID. The several isobars end at the location of the discontinuities C and D on the sonic circle. Due to the constant pressure in the region IDB the shock is also displaced by a constant displacement from I to B accompanied by a discontinuity at I. Again there is a discontinuity in density along IO which separates the region of rotational flow IOF from that of irrotational flow (to the left of line IO). It is interesting to note that on traversing the line BO each isopycnic suddenly splits off and there is a steep density gradient adjoining BO, while there is no appreciable pressure gradient there.

In Fig. 26 we compare the results with those of Smyrl (1963) for the pressure distribution on the wedge face for $M' = 2.0$, $M = 2.0$ and $\chi = 0$, 0.5, 1.25 and 1.5 radians. Obviously, our results do not agree with those obtained by Smyrl except when $\chi = 0$, for which the two analyses yield the identical results as was noted in §3.6. The discrepancy in the results when $\chi > 0$, may be attributed to the fact that Smyrl calculated wrongly the direction of the unit vector $\underline{k}$ (Smyrl 1963, p. 235, bottom line), which should be in our notations,

$$\{\tan\phi_o \cos(\chi - \mu),\ -1,\ \tan\phi_o \sin(\chi - \mu)\},\ \text{with}\ \tan\phi_o = 1/\beta,$$

and this affects his subsequent results for the case of yaw.

Slender cone

Pressure distributions on the surface of the cone have been calculated for the Mach reflection region EF for the various conditions by using the results of §4.5. We consider a cone of semi-nose angle $\varepsilon = 0.025$ with $M = 2.0$ and 4.0 for various values of M' and with $M' = 1.5$ and 2.5 for various values of M, thus demonstrating the effect of the cone speed and shock strength respectively. The results are shown in Figs. 27 & 28. The scale is again adjusted to make the distance EF the same in all cases.

In Fig. 29 we have presented the isobars and isopycnics in the entire Mach-reflection region BCDEF (Fig. 2) and also the shock shape for $\varepsilon = 0.05$, $M' = 2.0$, $M = 2.0$. This figure has to be rotated 360° about the axis of symmetry for the physical picture. It may be noted that the isobars in the regions ACE and IDB (Fig. 2) are conical surfaces in this case. There are no longer discontinuities along AC and ID and thereby at C and D, as compared with the case of wedge. There is only adverse density gradient along BO and IO.

In Fig. 30 a comparison is drawn with the results of Blankenship (1965) for $\varepsilon = 0.025$, $M' = 2.5$ and $M = 11.25$ & 6.25. Two distinct features emerge from it. First, the starting values of the pressure at E are not the same in the two cases, Blankenship's values being higher. This may be due to a computational mistake in the latter's results. Since Blankenship starts from the well-known conical solution in the region AEC Fig. 2, while the author obtains the same conical solution from his analysis, the values of the pressure at E should have been the same in the two cases. Secondly the behaviour of the curves given by the two procedures is different. The discrepancy in the behaviour can be attributed to the fact that Blankenship considers the value of $r(\partial p/\partial r)$ on the cone surface to be constant from E to F (Fig. 2), which in fact is the case but he does not consider any jump in $r(\partial p/\partial r)$ at F. While in our formulation when the shock is approached along the body and when the body is approached along the shock, the two limits of $r(\partial p/\partial r)$ are different at F, thereby providing a jump in $r\, \partial p/\partial r$, though the assumption that the flow is tangential to the body even at the foot of the shock still holds. Incidentally, if we consider the value of $r(\partial p/\partial r)$ constant along the cone surface from E to F without any jump (which amounts to considering B_o as A_o in our analysis), the behaviour of the results thus obtained is closer to that given by Blankenship. These results are also plotted alongside the other results in Fig. 30.

The complete pressure field in the Mach-reflection region is plotted in Fig. 31 for $\varepsilon = 0.025$, $M' = 2.5$ and $M = 11.25$, which allows fuller comparison with Blankenship (1965, Fig. 6). The isobars in the two cases agree fairly well except in the vicinity of the cone surface as one would expect in the light of the preceding discussion.

Flat delta wing

As a numerical example of §5.7, we consider a flat delta wing having semi-apex angle $\theta = 45^{\circ}$ ($n = 1$). The pressure distributions are calculated for the disturbed region inside the spherical wavelet behind the shock for $M' = 2.0$ and $M = 10.0$. The results are plotted in the form of isobars on the plane of the wing $Y = 0$, on the root chord plane $Z = 0$ and on the plane of the shock $X = m$, Fig. 32; thus a three-dimensional physical picture can be easily visualized. We have also calculated the pressure variation on the wing along the root chord ($Y = 0$, $Z = 0$) for various cases to demonstrate the effect of the wing speed, the shock speed and the apex angle. The results are shown in Figs. 33, 34 & 35.

Figures 22-35 are characteristics of all the results obtained and demonstrate the application of the theory.

7.2. Conclusion and discussion

In conclusion it may be observed that we have succeeded in treating the problem of interaction of a plane shock of arbitrary strength with supersonically moving aerodynamic obstacles, based on the theory presented in this work. There is a definite unifying feature throughout the treatment. The disturbance pressure is used as a primary variable to formulate the problem in terms of initial and boundary values. A Lorentz transformation is employed to rotate the shock plane in the plane of the axes. The problem is amenable to the method of integral transforms, which is the important characteristic of our treatment. For every case considered, the application of the appropriate integral transforms reduce the mathematical problem to a fairly simple formulation (cf. Eqs. 3.4.8-3.4.10), i.e. a second order non-homogenous ordinary differential equation subject to two boundary conditions, which permits a simple solution. Finally one is left with obtaining the appropriate inversions. The inversions, it is shown, exist for the cases considered. The solution presented leads to the various field representations behind the shock.

The theory presented can be easily generalized for the diabatic flows, i.e. the inviscid non-heat conducting fluids with heat addition by means of sources (heat injection). It can also be modified to treat the diffraction of a shock by subsonically moving obstacles. It is to be hoped that the method of solution to the boundary value problem applied here makes possible solutions for more general physical conditions. For instance the solution of the gas motion could be found when the oncoming shock interacts with the unsteady disturbances generated by the vibration of the body caused by the shock impact, or the shock encounters an accelerating obstacle moving towards it, or the shock is accompanied by a non-uniform flow region behind it.

The solution presented can also be used to study the other interesting shock diffraction problems, viz. the diffraction of a shock by an interface of two medias, due to atmospheric turbulence, due to non-smooth walls in shock tube, etc.

In what follows we shall note some precarious items in the analysis, which each in themselves require further attention. The solution of the general antisymmetrical problem (cf. § 6.5) is not complete, since we have not been able to solve the integral equation obtained. This requires further study.

Further we consider those aspects of the flow for which the linearized theory gives erroneous information. The construction of the flow field outlined in the introduction points out the fact that the interaction was the result of the two fields - the attached shock generated by the body in region (0) and diffracted by the plane shock, and the shock generated by the body moving in region (1). It is clear that these two fields (ACE and IDB, Fig. 2) overlap depending upon the speed of the body and the strength of the shock. In the region of overlap, the two fields interact in a non-linear fashion. The construction based on the linearized theory (in which the pressure is given by the linear superposition of the two fields) does not take into account this interaction,and thus does not correctly represent the flow in the interacting region.

Again, the shape of the obstacle is assumed such as to produce small disturbances everywhere. It is well known that for the supersonic flow past the three-dimensional wings with subsonic leading edges, there are singularities along the leading edges, e.g. a symmetric wing at zero incidence has logrithmic singularities while flat delta wing at incidence has square root singularities. On interaction with a plane shock, these singularities, in general, will reappear behind the shock (cf. §5.7.1). Thus the pressure tends to become infinitely large in the neighbourhood of the subsonic leading edges and so also the shock displacement there. In real media this divergence is eliminated by the non-linear effects, and by the effects of viscosity and thermal conductivity. Consequently the linearized theory becomes inadequate near such regions.

In our discussion of interaction problems, we have assumed the obstacles to be semi-infinite in length in the flow direction. In case the obstacle is a finite one a rear shock or shocks may appear. It cannot be assumed that the flow is uniform behind the rear shock, even on linear theory. Thus the flow field ahead of the incident shock is affected by what is happening in the wake of the body; this in turn will influence the interaction flow field behind the shock thus altering the shock pattern.

No attempt is made to get the reflected shock (BCDE, Fig. 2) correct; it has been assumed to be a Mach wave. However, Tan (1951), Ting (1952) and Chester (1954) have extracted the strength of the reflected shock from the linearized solution for the problem of diffraction of plane shock by stationary two-dimensional aerofoils by applying the technique of Lighthill (1949b) for rendering the approximate solutions uniformly valid. The similar procedure could be carried out for determining the strength of the reflected shock from the linearized solution for shock-on-shock interaction problems.

Our theory does not give any information regarding the precise nature of the flow field at the triple point shock wave intersection (point B, Fig. 2), which seems another non-linear feature of our problem. One may refer to the work of Sternberg (1959) for some details of the flow structure in an analogous situation, where the incident shock is considered to be a weak shock and the Mach shock has two-dimensional structure (cf. also Sichel 1962, 1963).

The condition at the base of the shock on the body surface is complicated by the presence of the boundary layer and its changing behaviour due to change in the Reynolds number on penetrating the shock and due to interaction with the moving shock. The change in the boundary layer flow has been experimentally observed as a side effect of the shock interaction by Merrit & Aronson (1966). They have observed that the model boundary layer appears laminar as it approaches the shock but that a turbulent boundary layer is clearly visible on the portion of the model that has penetrated the shock. The presence of the boundary layer on the wall results in an altered boundary condition for the shock wave, and on the other hand the large pressure gradients associated with the incident shock strongly alter the boundary layer flow. In general the problem is a non-linear one, in which neither the simplification of boundary layer nor that of simple shock wave theory apply.

The construction of the flow field based on the present work is only valid for the idealized aerodynamic obstacles which present small disturbances to the main flow field. For real bodies with strong bow shocks that cannot be put into the small disturbance category, very little information is available. The computer experimentation offers the best possibility of solving the real body problem. In this connection the work of Ludloff & Friedmann (1955) may be mentioned who did two computations for the diffraction of a shock by a stationary wedge. This work may serve as the first step for solving the non-linear problem of shock-shock interaction.

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APPENDIX 1

1.1. Inversion of expression (3.4.14)

(a) We consider the first expression in (3.4.14)

$$I_o^{**} = \frac{1}{2}\left(\frac{1}{\lambda - \lambda_o s}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda} - \frac{2}{\lambda^2 - \lambda_o^2 s^2}\exp\{s\lambda_o\bar{x}\}\right)G(s) \qquad \text{(A 1.1)}$$

or,
$$I_o^{**} = \frac{1}{2}\left(\frac{1}{\lambda - \lambda_o s}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda} - \frac{1}{\lambda - \lambda_o s}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda}\exp\{-(\lambda - \lambda_o s)\bar{x}\}\right.$$
$$\left. - \frac{1}{\lambda + \lambda_o s}\,\frac{\exp\{-\lambda\bar{x}\}}{\lambda}\exp\{(\lambda + \lambda_o s)\bar{x}\}\right)G(s).$$

This can now be expressed in integral form as

$$I_o^{**} = \frac{1}{2}\left(\frac{\exp\{\lambda\bar{x}\}}{\lambda}\int_0^{\bar{x}}\exp\{-(\lambda - \lambda_o s)\xi\}\,d\xi\right.$$
$$\left. - \frac{\exp\{-\lambda\bar{x}\}}{\lambda}\int_{-\infty}^{\bar{x}}\exp\{(\lambda + \lambda_o s)\xi\}\,d\xi\right)G(s).$$

Here $\lambda = (\alpha^2 + s^2)^{\frac{1}{2}}$, and λ_o is always less than unity, so that $Re(\lambda \pm \lambda_o s)$ is always positive. I_o^{**} can be rewritten in the form

$$I_o^{**} = \frac{1}{2}\left(-\int_{\bar{x}}^{0}\exp\{s\lambda_o\xi\}\,\frac{1}{\lambda}\exp\{-\lambda(\xi - \bar{x})\}\,d\xi\right.$$
$$\left. - \int_{-\infty}^{\bar{x}}\exp\{s\lambda_o\xi\}\,\frac{1}{\lambda}\exp\{-\lambda(\bar{x} - \xi)\}\,d\xi\right)G(s).$$

These integrals can be taken together in the form

$$I_o^{**} = -\frac{1}{2}\int_{-\infty}^{0}\exp\{s\lambda_o\xi\}\,G(s)\,\frac{1}{\lambda}\exp\{-\lambda(|\bar{x} - \xi|)\}\,d\xi. \qquad \text{(A 1.2)}$$

Now we seek inversion of $I_o^{**}(\bar{x}, \alpha, s)$ first by applying the inverse cosine transform for α and then the inverse Laplace transform for s. The inversion can be achieved by using the standard tables of integral transforms (cf. Erdelyi et al. 1954).

$$I_o(\bar{x}, y, \bar{\tau}) = L^{-1}\left\{ F_c^{-1}\left(-\frac{1}{2}\int_{-\infty}^{0} \exp\{s\lambda_o\xi\}\, G(s)\, \frac{1}{\lambda} \exp\{-\lambda(|\bar{x} - \xi|)\}\, d\xi \right)\right\}$$

$$= -\frac{1}{2} L^{-1}\left\{\frac{2}{\pi}\int_0^{\infty} \cos(\alpha y) \left(\int_{-\infty}^{0} \exp\{s\lambda_o\xi\}\, G(s)\, \frac{1}{\lambda} \times \exp\{-\lambda(|\bar{x} - \xi|)\} d\xi\right) d\alpha \right\}.$$

Noting that $\lambda^2 = \alpha^2 + s^2$, and granting the possibility of change of order of integration, we can write

$$I_o = -\frac{1}{\pi} L^{-1}\left\{ \int_{-\infty}^{0} \exp\{s\lambda_o\xi\}\, G(s) \left(\int_0^{\infty} \cos(\alpha y)\, \frac{\exp\{-(\alpha^2 + s^2)^{\frac{1}{2}}(|\bar{x} - \xi|)\}}{(\alpha^2 + s^2)^{\frac{1}{2}}} d\alpha\right) d\xi\right\}$$

$$= -\frac{1}{\pi} L^{-1}\left\{ \int_{-\infty}^{0} \exp\{s\lambda_o\xi\}\, G(s)\, K_o\left(s\{(\bar{x} - \xi)^2 + y^2\}^{\frac{1}{2}}\right) d\xi\right\}. \qquad \text{(A 1.3)}$$

Here $L^{-1}\left\{K_o\left(s\{(\bar{x} - \xi)^2 + y^2\}^{\frac{1}{2}}\right)\right\}$

$$= \begin{cases} 0, & 0 < \bar{\tau} < \{(\bar{x} - \xi)^2 + y^2\}^{\frac{1}{2}} \\ \{\bar{\tau}^2 - (\bar{x} - \xi)^2 - y^2\}^{-\frac{1}{2}}; & \bar{\tau} > \{(\bar{x} - \xi)^2 + y^2\}^{\frac{1}{2}}, \end{cases}$$

and since $s\lambda_o\xi$ is always negative, we obtain from (A.1.3)

$$I_o = -\frac{1}{\pi}\int_{-\infty}^{0} d\xi \int_0^{\bar{\tau}} \frac{f''\{\bar{a}(\mu + \lambda_o\xi)\}}{\{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}}\, d\mu. \qquad \text{(A 1.4)}$$

(b) Next we consider the terms for i = 1, 2, 3 & 4 in (3.4.14)

$$I_i^{**} = \frac{1}{2}\,\frac{1}{\lambda + \lambda_i s}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda}\; G(s). \qquad \text{(A 1.5)}$$

This can be expressed in integral form as

$$I_i^{**} = \frac{1}{2}\int_{-\infty}^{0} \exp\{(\lambda + \lambda_i s)\xi\} \frac{\exp\{\lambda\bar{x}\}}{\lambda} G(s)\, d\xi,$$

where $Re(\lambda + \lambda_i s)$ is always positive. I_i^{**} can be rewritten as

$$I_i^{**} = \frac{1}{2}\int_{-\infty}^{0} \exp\{s\lambda_i\xi\}\, G(s)\, \frac{1}{\lambda} \exp\{\lambda(\bar{x} + \xi)\}\, d\xi. \qquad \text{(A 1.6)}$$

Here $s\lambda_i\xi$ and $(\bar{x} + \xi)$ are always negative. Hence to obtain the inversion of (A 1.6) we can proceed exactly as for (A 1.2). Then we obtain, for i = 1, 2, 3 & 4

$$I_i = L^{-1}\left[F_c^{-1}\left\{I_i^{**}\right\}\right]$$
$$= \frac{1}{\pi}\int_{-\infty}^{0} d\xi \int_{0}^{\bar{\tau}} \frac{f''\{\bar{a}(\mu + \lambda_i\xi)\}}{\{(\bar{\tau} - \mu)^2 - (\bar{x} + \xi)^2 - y^2\}^{1/2}}\, d\mu \qquad \text{(A 1.7)}$$

(c) Finally we consider the last term in (3.4.14)

$$I_5^{**} = \frac{1}{2}\,\frac{1}{\lambda + \lambda_5 s}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda}\, G(s). \qquad \text{(A 1.8)}$$

Since $\lambda_5 (= -\lambda_4)$ is always negative, we can not achieve the inversion of I_5^{**} in the same fashion as for (A 1.5). Hence we proceed as follows. We write

$$I_5^{**} = \frac{1}{2}\left(\frac{1}{\lambda - \lambda_4 s}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda} - \frac{2}{\lambda^2 - \lambda_4^2 s^2}\exp\{s\lambda_4\bar{x}\}\right) G(s)$$
$$+ \frac{1}{\lambda^2 - \lambda_4^2 s^2}\exp\{s\lambda_4\bar{x}\}\, G(s).$$

Following the same procedure as in (a) above we can express the first two terms in integral form. Then

$$I_5^{**} = -\frac{1}{2}\int_{-\infty}^{0} \exp\{s\lambda_4\xi\}\, G(s)\, \frac{1}{\lambda}\exp\{-\lambda(|\bar{x} - \xi|)\}d\xi + \frac{1}{\alpha^2 + \bar{b}^2 s^2}\exp\{s\lambda_4\bar{x}\}\, G(s)$$
$$\equiv I_{51}^{**} + I_{52}^{**} \qquad \text{(A 1.9)}$$

Since here $s\lambda_4\xi$ is always negative, the inversion of I_{51}^{**} can be written exactly as for (A 1.2). For the inversion of I_{52}^{**} we write

$$I_{52} = L^{-1}\left\{F_c^{-1}\left(\frac{1}{\alpha^2 + \bar{b}^2 s^2} \exp\{s\lambda_4\bar{x}\}\ G(s)\right)\right\}.$$

Since $F_c^{-1}\left(\frac{1}{\alpha^2 + \bar{b}^2 s^2}\right) = \frac{2}{\pi}\int_0^\infty \frac{1}{\alpha^2 + \bar{b}^2 s^2} \cos(\alpha y)\ d\alpha = \frac{\exp\{-\bar{b}sy\}}{\bar{b}s}$,

$$I_{52} = L^{-1}\left(\frac{1}{\bar{b}s} \exp\{-(\bar{b}y - \lambda_4\bar{x})s\}\ G(s)\right).$$

Here $(\bar{b}y - \lambda_4\bar{x})$ is always positive, then

$$I_{52} = \frac{1}{\bar{a}\bar{b}} f'\{\bar{a}(\bar{\tau} - \bar{b}y + \lambda_4\bar{x})\}\ , \quad \text{for } \bar{\tau} > (\bar{b}y - \lambda_4\bar{x}) \tag{A 1.10}$$

Thus the inversion of (A 1.9) gives

$$I_5 = -\frac{1}{\pi}\int_{-\infty}^{\bar{0}} d\xi \int_0^{\bar{\tau}} \frac{f''\{\bar{a}(\mu + \lambda_4\xi)\}}{\{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}}\ d\mu$$
$$+ \frac{1}{\bar{a}\bar{b}} f'\{\bar{a}(\bar{\tau} + \lambda_4\bar{x} - \bar{b}y)\}. \tag{A. 1.11}$$

1.2. Simplification of I_5

The last two terms in (3.4.15) (or I_5, Eq. A 1.11) can be expressed as

$$I_5 = \frac{1}{\pi}\int_0^{\bar{\tau}} d\mu \int_0^{\infty} \frac{f''\{\bar{a}(\mu + \lambda_4\xi)\}}{\{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}}\ d\xi$$
$$- \frac{1}{\pi}\int_0^{\bar{\tau}} d\mu \int_{-\infty}^{\infty} \frac{f''\{\bar{a}(\mu + \lambda_4\xi)\}}{\{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}}\ d\xi$$
$$+ \frac{1}{\bar{a}\bar{b}} f'\{\bar{a}(\bar{\tau} + \lambda_4\bar{x} - \bar{b}y)\}. \tag{A 1.12}$$

We consider the second integral in (A 1.12)

$$I_{5b} = -\frac{1}{\pi}\int_0^{\bar{\tau}} d\mu \int_{-\infty}^{\infty} \frac{f''\{\bar{a}(\mu + \lambda_4\xi)\}}{\{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}}\, d\xi. \qquad \text{(A 1.13)}$$

Here a new variable may be introduced to replace μ, say $\mu + \lambda_4\xi = q$. Then

$$I_{5b} = -\frac{1}{\pi}\int_0^{q_1} f''(\bar{a}q)\, dq \int_{\xi_2}^{\xi_1} \frac{d\xi}{[\{(\bar{\tau} - q)^2 - \bar{x}^2 - y^2\} + \{\bar{x} + \lambda_4(\bar{\tau} - q)\}\xi - \bar{b}^2\xi^2]^{\frac{1}{2}}}$$

where $\xi_1 = \left[\bar{x} + \lambda_4(\bar{\tau} - q) + \{(\bar{\tau} - q + \lambda_4\bar{x})^2 - \bar{b}^2y^2\}^{\frac{1}{2}}\right]/\bar{b}^2,$

$\xi_2 = \left[\bar{x} + \lambda_4(\bar{\tau} - q) - \{(\bar{\tau} - q + \lambda_4\bar{x})^2 - \bar{b}^2y^2\}^{\frac{1}{2}}\right]/\bar{b}^2,$

and $q_1 = \bar{\tau} + \lambda_4\bar{x} - \bar{b}y$. The above expression now integrates to yield

$$I_{5b} = -\frac{1}{\bar{a}\bar{b}} f'\{\bar{a}(\bar{\tau} + \lambda_4\bar{x} - \bar{b}y)\}. \qquad \text{(A 1.14)}$$

Hence from (A 1.12) we obtain

$$I_5 = \frac{1}{\pi}\int_0^{\bar{\tau}} d\mu \int_0^{\infty} \frac{f''\{\bar{a}(\mu + \lambda_4\xi)\}}{\{(\bar{\tau} - \mu)^2 - (\bar{x} - \xi)^2 - y^2\}^{\frac{1}{2}}}\, d\xi \qquad \text{(A 1.15)}$$

1.3. Inversion of $I^{**}_{4,5}$, Eq. (3.4.17)

$$I_{4,5}(\bar{x}, y, \bar{\tau}) = L^{-1}\left\{F_c^{-1}\left(Q_1 \frac{1}{\alpha^2 + \bar{b}^2s^2}\exp\{\lambda\bar{x}\}\, G(s)\right.\right.$$

$$\left.\left. + Q_2 \frac{s}{\alpha^2 + \bar{b}^2s^2}\frac{\exp\{\lambda\bar{x}\}}{\lambda} G(s)\right)\right\}, \qquad \text{(A 1.16)}$$

Since $F_c^{-1}\{1/(\alpha^2 + \bar{b}^2s^2)\} = (1/\bar{b}s)\exp\{-\bar{b}sy\}$

$$F_c^{-1}\left(\exp\{(\alpha^2 + s^2)^{\frac{1}{2}}\bar{x}\}\right) = \frac{2}{\pi} s\, \bar{x}(\bar{x}^2 + y^2)^{-\frac{1}{2}} K_1\{s(\bar{x}^2 + y^2)^{\frac{1}{2}}\},$$

by convolution (cf. Sneddon 1951), we obtain

$$F_c^{-1}\left(\{1/(\alpha^2 + \bar{b}^2s^2)\}\exp\{(\alpha^2 + s^2)^{\frac{1}{2}}\bar{x}\}\right)$$

$$= \frac{1}{\pi\bar{b}}\int_0^{\infty} \exp\{-\bar{b}s\zeta\}\{f_1(\zeta) + f_1(-\zeta)\}\, d\zeta, \qquad \text{(A 1.17)}$$

where $f_1(\zeta) = \bar{x}\{\bar{x}^2 + (y - \zeta)^2\}^{-\frac{1}{2}} K_1\left(s\{\bar{x}^2 + (y - \zeta)^2\}^{\frac{1}{2}}\right)$.

Similarly we can obtain

$$F_c^{-1}\left(\frac{s}{\alpha^2 + \bar{b}^2 s^2} \frac{\exp\{(\alpha^2 + s^2)^{\frac{1}{2}} \bar{x}\}}{(\alpha^2 + s^2)^{\frac{1}{2}}}\right)$$

$$= \frac{1}{\pi\bar{b}} \int_0^\infty \{f_2(\zeta) + f_2(-\zeta)\} \exp\{-\bar{b}s\zeta\} \, d\zeta, \qquad \text{(A 1.18)}$$

where $f_2(\zeta) = K_0\left(s\{\bar{x}^2 + (y - \zeta)^2\}^{\frac{1}{2}}\right)$.

Substituting (A 1.17) and (A 1.18) in (A 1.16) and further using the tables of inverse Laplace transforms we obtain

$$I_{4,5} = \frac{Q_1}{\pi\bar{b}} \int_0^{\bar{\tau}} d\mu \int_0^\infty \{N_1(\mu, \zeta) + N_1(\mu, -\zeta)\} f''\{\bar{a}(\mu - \bar{b}\zeta)\} \, d\zeta$$

$$+ \frac{Q_2}{\pi\bar{b}} \int_0^{\bar{\tau}} d\mu \int_0^\infty \{N_2(\mu, \zeta) + N_2(\mu, -\zeta)\} f''\{\bar{a}(\mu - \bar{b}\zeta)\} \, d\zeta, \qquad \text{(A 1.19)}$$

where $N_1(\mu, \zeta) = \dfrac{\bar{x}}{\bar{x}^2 + (y - \zeta)^2} \dfrac{\bar{\tau} - \mu}{\{(\bar{\tau} - \mu)^2 - \bar{x}^2 - (y - \zeta)^2\}^{\frac{1}{2}}}$,

and $N_2(\mu, \zeta) = \{(\bar{\tau} - \mu)^2 - \bar{x}^2 - (y - \zeta)^2\}^{-\frac{1}{2}}$.

APPENDIX 2

The results of Smyrl (1963, pp. 232 & 233) for the pressure on the wedge face and on the shock plane are respectively

$$p' = p_5' - \left[\sum_{i=1}^{4} C_i \arctan\left(\frac{\gamma_i - \sqrt{2}}{\gamma_i + \sqrt{2}}\right)^{\frac{1}{2}} t + C_5 \arctan\left(\frac{\gamma_3 + \sqrt{2}}{\gamma_3 - \sqrt{2}}\right)^{\frac{1}{2}} t \right.$$
$$\left. + C_6 \arctan\left(\frac{\gamma_4 + \sqrt{2}}{\gamma_4 - \sqrt{2}}\right)^{\frac{1}{2}} t\right], \qquad \text{(A 2.1)}$$

and

$$p' = p_5' - \frac{\pi}{2}(C_5 + C_6) - \frac{1}{2}\left[C_1 \arctan\left(\frac{\gamma_1^2 - 2}{\xi + 1}\right)^{\frac{1}{2}} + C_2 \arctan\left(\frac{\gamma_2^2 - 2}{\xi + 1}\right)^{\frac{1}{2}}\right.$$
$$\left. + (C_3 - C_5) \arctan\left(\frac{\gamma_3^2 - 2}{\xi + 1}\right)^{\frac{1}{2}} + (C_4 - C_6) \arctan\left(\frac{\gamma_4^2 - 2}{\xi + 1}\right)^{\frac{1}{2}}\right], \qquad \text{(A 2.2)}$$

where $t^2 = (1 + x)(1 - x_o)/\{(1 - x)(1 + x_o)\}$, $\xi = (y_o^2 + y^2)/(y_o^2 - y^2)$,

$$C_1 = \frac{4}{(\gamma_2 - \gamma_1)(\gamma_1^2 - 2)^{\frac{1}{2}}}\left(\frac{K_1}{\gamma_4^2 - \gamma_1^2} + \frac{K_2}{\gamma_3^2 - \gamma_1^2} + K_3\right),$$

$$C_2 = \frac{4}{(\gamma_1 - \gamma_2)(\gamma_2^2 - 2)^{\frac{1}{2}}}\left(\frac{K_1}{\gamma_4^2 - \gamma_1^2} + \frac{K_2}{\gamma_3^2 - \gamma_2^2} + K_2\right),$$

$$C_3 = \frac{(\gamma_1 + \gamma_3)(\gamma_2 + \gamma_3)}{(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_3)} C_5, \quad C_4 = \frac{(\gamma_1 + \gamma_4)(\gamma_2 + \gamma_4)}{(\gamma_1 - \gamma_4)(\gamma_2 - \gamma_4)} C_6,$$

$$C_5 = \frac{2}{\pi} p_5', \quad C_6 = -\frac{2}{\pi} p_3',$$

$$K_1 = -\frac{1}{\pi} p_3' (\gamma_1 + \gamma_4)(\gamma_2 + \gamma_4)(\gamma_4^2 - 2)^{\frac{1}{2}}\gamma_4,$$

$$K_2 = \frac{1}{\pi} p_5' (\gamma_1 + \gamma_3)(\gamma_2 + \gamma_3)(\gamma_3^2 - 2)^{\frac{1}{2}}\gamma_3,$$

$$K_3 = (v_4' - M_1 - M' c_o/c_1)\frac{y_o}{\pi B} \gamma_1\gamma_2 + \frac{p_3'}{\pi} (\gamma_1 + \gamma_2 + \gamma_4)(\gamma_4^2 - 2)^{\frac{1}{2}}$$
$$- \frac{p_5'}{\pi} (\gamma_1 + \gamma_2 + \gamma_3)(\gamma_3^2 - 2)^{\frac{1}{2}},$$

$$p_3' = (y_3 u_4' - x_o v_4')/(y_3 x_2 - x_o y_2),$$

$$p_5' = (M' c_o/c_1 + M_1)^2 / \{(M' c_o/c_1 + M_1)^2 - 1\},$$

in terms of the notations used by Smyrl.

Translating the above in terms of the constants of our analysis, we obtain

$$X_o \equiv m, \quad \sqrt{2}/\gamma_1 \equiv \lambda_2, \quad \sqrt{2}/\gamma_2 \equiv \lambda_3, \quad \sqrt{2}/\gamma_3 \equiv \lambda_o = \lambda_1, \quad \sqrt{2}/\gamma_4 \equiv \lambda_4 = -\lambda_5,$$

$$p_3' \equiv \varepsilon K/\{\bar{a}\bar{b}(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_4)\}, \quad p_5' \equiv -\varepsilon A_o/\{\bar{a}(1 - \lambda_o^2)^{\frac{1}{2}}\},$$

and $(v_4' - M_1 - M' c_o/c_1) y_o/B \equiv \bar{b} p_3' + B_o/\bar{a}$.

Hence

$$C_1 \equiv \frac{4\lambda_2}{(\lambda_2 - \lambda_3)(1 - \lambda_2^2)^{\frac{1}{2}}}\left(\frac{\varepsilon B_o}{\pi\bar{a}} + \frac{p_5'}{\pi}\frac{\lambda_2 + \lambda_3}{\lambda_2 - \lambda_4}(1 - \lambda_o^2)^{\frac{1}{2}} - \frac{p_3'}{\pi}\frac{\lambda_3 + \lambda_4}{\lambda_2 - \lambda_4}\bar{b}\right)$$

$$\equiv -2\varepsilon A_2/\{\pi\bar{a}(1 - \lambda_2^2)^{\frac{1}{2}}\}$$

$$C_2 \equiv -2\varepsilon A_3/\{\pi\bar{a}(1 - \lambda_3^2)^{\frac{1}{2}}\}, \quad C_3 \equiv -2\varepsilon A_1/\{\pi\bar{a}(1 - \lambda_1^2)^{\frac{1}{2}}\},$$

$$C_4 \equiv -2\varepsilon A_4/\{\pi\bar{a}(1 - \lambda_4^2)^{\frac{1}{2}}\}, \quad C_5 \equiv -2\varepsilon A_o/\{\pi\bar{a}(1 - \lambda_o^2)^{\frac{1}{2}}\},$$

$$C_6 \equiv -2\varepsilon A_5/\{\pi\bar{a}(1 - \lambda_5^2)^{\frac{1}{2}}\}.$$

Thus the expressions (A 2.1) and (A 2.2), when expressed in our notations yield

$$p = \frac{\varepsilon}{\pi\bar{a}}\left[\bar{A}_o \arccos\left(\frac{\gamma_o + X}{1 + \gamma_o X}\right) + \sum_{i=1}^{5} \bar{A}_i \arccos\left(\frac{\gamma_i - X}{1 - \gamma_i X}\right)\right] \qquad \text{(A 2.3)}$$

and

$$p = \frac{\varepsilon}{\pi\bar{a}}\left[\bar{A}_o \arccos\left(\frac{\gamma_o + m}{\{(1 + \gamma_o m)^2 - (1 - \gamma_o^2)Y^2\}^{\frac{1}{2}}}\right)\right.$$

$$\left. + \sum_{i=1}^{5} \bar{A}_i \arccos\left(\frac{\gamma_i - m}{\{(1 - \gamma_i m)^2 - (1 - \gamma_i^2)Y^2\}^{\frac{1}{2}}}\right)\right]. \qquad \text{(A 2.4)}$$

The expressions (A 2.3) and (A 2.4) are obviously the same as obtained from (3.6.5) by substituting Y = 0 and X = m respectively.

APPENDIX 3

3.1. Inversion of expression (4.4.12)

The inversion of (4.4.12) is obtained first by applying the inverse Hankel transform for α and then the inverse Laplace transform for s.

(a) We consider the first integral in (4.4.12)

$$I_o^{**} = \int_{-\infty}^{0} \exp\{s\lambda_o\xi\}\, G(s)\, \frac{1}{\lambda} \exp\{-\lambda(|\bar{x} - \xi|)\} d\xi, \qquad \text{(A 3.1)}$$

with $\lambda = (\alpha^2 + s^2)^{\frac{1}{2}}$. For its inversion we can then write

$$I_o(\bar{x}, r, \bar{\tau}) = L^{-1}\left[H^{-1}\{I_o^{**}(\bar{x}, \alpha, s)\}\right]$$

$$= L^{-1}\left[\int_{-\infty}^{0} \exp\{s\lambda_o\xi\}\, G(s)\left\{\int_{0}^{\infty} \alpha J_o(\alpha r)\, \frac{\exp\{-(\alpha^2 + s^2)^{\frac{1}{2}}(|\bar{x} - \xi|)\}}{(\alpha^2 + s^2)^{\frac{1}{2}}}\, d\alpha\right\} d\xi\right]$$

$$= L^{-1}\left\{\int_{-\infty}^{0} \exp\{s\lambda_o\xi\}\, G(s)\, \frac{\exp\{-s\bar{R}\}}{\bar{R}}\, d\xi\right\}$$

$$= L^{-1}\left\{\int_{-\infty}^{0} \frac{1}{\bar{R}} \exp\{-s(\bar{R} - \lambda_o\xi)\}\, G(s)\, d\xi\right\}, \qquad \text{(A 3.2)}$$

with $\bar{R} = \{(\bar{x} - \xi)^2 + r^2\}^{\frac{1}{2}}$. Here since $\lambda_o > 0$, $\xi < 0$, $(\bar{R} - \lambda_o\xi)$ is always positive. Thus we obtain

$$I_o = \int_{-\infty}^{0} \frac{1}{\bar{R}} F\{\bar{a}(\bar{\tau} - \bar{R} + \lambda_o\xi)\}\, d\xi. \qquad \text{(A 3.3)}$$

(b) Next we consider the integrals for i = 1, 2, 3 & 4 in (4.4.12)

$$I_i^{**} = \int_{-\infty}^{0} \exp\{s\lambda_i\xi\}\, G(s)\, \frac{1}{\lambda} \exp\{\lambda(\bar{x} + \xi)\}\, d\xi. \qquad \text{(A 3.4)}$$

Since in (A 3.4), $(\bar{x} + \xi)$ is always negative, we can proceed for its inversion as for (A 3.1) and obtain

$$I_i = L^{-1}\left\{\int_{-\infty}^{0} \frac{1}{\bar{R}_1} \exp\{-s(\bar{R}_1 - \lambda_i\xi)\}\, G(s)\, d\xi\right\}, \qquad \text{(A 3.5)}$$

with $\bar{R}_1 = \{(\bar{x} + \xi)^2 + r^2\}^{\frac{1}{2}}$. Again since $\lambda_i > 0$ (for $i = 1, 2, 3$ & 4), and $\xi < 0$, $(\bar{R}_1 - \lambda_i \xi)$ is always positive here, then we obtain from (A 3.5)

$$I_i = \int_{-\infty}^{0} \frac{1}{\bar{R}_1} F\{\bar{a}(\bar{\tau} - \bar{R}_1 + \lambda_i \xi)\} \, d\xi \qquad \text{(A 3.6)}$$

(c) Now we consider the sixth integral in (4.4.12)

$$I_{51}^{**} = \int_{-\infty}^{0} \exp\{s\lambda_4 \xi\} \, G(s) \, \frac{1}{\lambda} \exp\{-\lambda(|\bar{x} - \xi|)\} \, d\xi. \qquad \text{(A 3.7)}$$

This behaves exactly as (A 3.1), since $\lambda_4 > 0$ and $\xi < 0$. Hence its inversion is given by (A 3.3) with λ_o replaced by λ_4. Thus

$$I_{51} = \int_{-\infty}^{0} \frac{1}{\bar{R}} F\{\bar{a}(\bar{\tau} - \bar{R} + \lambda_4 \xi)\} \, d\xi. \qquad \text{(A 3.8)}$$

Finally we consider the last term in (4.4.12)

$$I_{52}^{**} = \frac{1}{\alpha^2 + \bar{b}^2 s^2} \exp\{s\lambda_4 \bar{x}\} \, G(s), \qquad \text{(A 3.9)}$$

For its inversion we can write

$$I_{52} = L^{-1}\left\{ \int_{0}^{\infty} \alpha \, J_o(\alpha r) \, \frac{1}{\alpha^2 + \bar{b}^2 s^2} \exp\{s\lambda_4 \bar{x}\} \, G(s) \, d\alpha \right\} \qquad \text{(A 3.10)}$$

$$= L^{-1}\Big(\exp\{s\lambda_4 \bar{x}\} \, G(s) \, K_o(\bar{b} s r) \Big)$$

$$= \int_{0}^{\mu_4} \frac{F(\bar{a}\mu)}{\{(\bar{\tau} + \lambda_4 \bar{x} - \mu)^2 - \bar{b}^2 r^2\}^{\frac{1}{2}}}, \quad \text{with } \mu_4 = \bar{\tau} + \lambda_4 \bar{x} - \bar{b} r. \qquad \text{(A 3.11)}$$

3.2. Inversion of $I_{4,5}^{**}$, Eq. (4.4.16)

The expression (4.4.16) can also be written as

$$I_{4,5}^{**} = \left(Q_1 \frac{\partial}{\partial x} + Q_2 s\right)\left(\frac{1}{\alpha^2 + \bar{b}^2 s^2} \, \frac{\exp\{\lambda \bar{x}\}}{\lambda} \, G(s)\right). \qquad \text{(A 3.12)}$$

We consider

$$I = L^{-1}\left\{H^{-1}\left(\frac{1}{\alpha^2 + \bar{b}^2 s^2}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda}\;G(s)\right)\right\},$$

$$= L^{-1}\left\{\int_0^\infty \alpha\;J_o(\alpha r)\,\frac{1}{\alpha^2 + \bar{b}^2 s^2}\,\frac{\exp\{(\alpha^2 + s^2)^{\frac{1}{2}}\bar{x}\}}{(\alpha^2 + s^2)^{\frac{1}{2}}}\;G(s)\;d\alpha\right\}. \qquad (A\ 3.13)$$

Since $L^{-1}\left(\frac{\alpha}{\alpha^2 + \bar{b}^2 s^2}\right) = \frac{1}{\bar{b}}\sin\left(\frac{\alpha}{\bar{b}}\,\bar{\tau}\right)$,

$$L^{-1}\left(\frac{\exp\{(\alpha^2 + s^2)^{\frac{1}{2}}\bar{x}\}}{(\alpha^2 + s^2)^{\frac{1}{2}}}\right) = \begin{cases} 0, & 0 < \bar{\tau} < \bar{x} \\ J_o\{\alpha(\bar{\tau}^2 - \bar{x}^2)^{\frac{1}{2}}\}, & \bar{\tau} < \bar{x}, \end{cases}$$

by convolution

$$L^{-1}\left(\frac{1}{\alpha^2 + \bar{b}^2 s^2}\,\frac{\exp\{(\alpha^2 + s^2)^{\frac{1}{2}}\bar{x}\}}{(\alpha^2 + s^2)^{\frac{1}{2}}}\,G(s)\right)$$

$$= \frac{1}{\bar{b}}\int_0^{\bar{\tau}} d\mu \int_0^{\mu} F\{\bar{a}(\bar{\tau} - \mu)\}\,\sin(\alpha\mu_1/\bar{b})\;J_o\left[\alpha\{(\mu - \mu_1)^2 - \bar{x}^2\}^{\frac{1}{2}}\right]d\mu_1.$$

Then, $$I = \frac{1}{\bar{b}}\int_0^{\bar{\tau}} F\{\bar{a}(\bar{\tau} - \mu)\}\;E(\mu)\;d\mu, \qquad (A\ 3.14)$$

with $E(\mu) = \int_0^{\mu} d\mu_1 \int_0^{\infty} \sin(\alpha\mu_1/\bar{b})\;J_o\left[\alpha\{(\mu - \mu_1)^2 - \bar{x}^2\}^{\frac{1}{2}}\right]\;J_o(\alpha r)\;d\alpha.$

Hence, $$I_{4,5} = L^{-1}\{H^{-1}(I^{**}_{4,5})\}$$

$$= \frac{1}{\bar{b}}\left(Q_1\,\frac{\partial}{\partial\bar{x}} + Q_2\,\frac{\partial}{\partial\bar{\tau}}\right)\int_0^{\bar{\tau}} F\{\bar{a}(\bar{\tau} - \mu)\}\;E(\mu)\;d\mu. \qquad (A\ 3.15)$$

APPENDIX 4

4.1. Inversion of expression (5.4.16)

The inversion is obtained first by applying the inverse exponential transform for ν, then the inverse Fourier cosine transform for α, and finally the inverse Laplace transform for s.

(a) We consider the first integral in (5.4.16)

$$I_o^{***} = \int_{-\infty}^{0} \exp\{s\lambda_o\xi\}\ H_1(s,\nu)\ \frac{1}{\lambda} \exp\{-\lambda(|\bar{x} - \xi|)\}\ d\xi, \qquad \text{(A 4.1)}$$

where $\lambda = (\alpha^2 + s^2 + \nu^2)^{\frac{1}{2}}$. For its inversion we can then write

$$I_o(\bar{x},\ y,\ x,\ \bar{\tau}) = L^{-1}\Big(F_c^{-1}\big[F^{-1}\{I_o^{***}(\bar{x},\ \alpha,\ \nu,\ s)\}\big]\Big)$$

Since $\quad F^{-1}\{H_1(s,\nu)\} = G_1(s,z),$

and $\quad F^{-1}\Big(\frac{1}{\lambda} \exp\{-\lambda(|\bar{x} - \xi|)\}\Big)$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{i\nu z\}\ \frac{\exp\{-(\alpha^2 + \nu^2 + s^2)^{\frac{1}{2}}(|\bar{x} - \xi|)\}}{(\alpha^2 + \nu^2 + s^2)^{\frac{1}{2}}}\ d\nu$$

$$= \frac{1}{\pi} K_o\ \left[(\alpha^2 + s^2)^{\frac{1}{2}}\{(\bar{x} - \xi)^2 + z^2\}^{\frac{1}{2}}\right],$$

by convolution

$$F^{-1}\Big(H_1(s,\nu)\ \frac{1}{\lambda} \exp\{-\lambda(|\bar{x} - \xi|)\}\Big)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} G_1(s,\zeta)\ K_o\left[(\alpha^2 + s^2)^{\frac{1}{2}}\{(\bar{x} - \xi)^2 + (z - \zeta)^2\}^{\frac{1}{2}}\right].$$

Then

$$I_o = L^{-1}\left[F_c^{-1}\left\{\frac{1}{\pi} \int_{-\infty}^{0} \exp\{s\lambda_o\xi\}\Big(\int_{-\infty}^{\infty} G_1(s,\zeta)\ K_o\left[(\alpha^2 + s^2)^{\frac{1}{2}}\{(\bar{x} - \xi)^2 + (z - \zeta)^2\}^{\frac{1}{2}}\right]d\zeta\Big)\right.\right.$$

(A 4.2)

Further $F_c^{-1}\left(K_o\left[(\alpha^2 + s^2)^{\frac{1}{2}}\{(\bar{x} - \xi)^2 + (z - \zeta)^2\}^{\frac{1}{2}}\right]\right)$

$$= \frac{2}{\pi}\int_0^\infty \cos(\alpha y)\ K_o\left[(\alpha^2 + s^2)^{\frac{1}{2}}\{(\bar{x} - \xi)^2 + (z - \zeta)^2\}^{\frac{1}{2}}\right] d\alpha$$

$$= \frac{1}{\bar{R}} \exp\{-s\bar{R}\}, \quad \text{with} \quad \bar{R} = \{(\bar{x} - \xi)^2 + (z - \zeta)^2 + y^2\}^{\frac{1}{2}}.$$

Hence, $I_o = \frac{1}{\pi} L^{-1}\left\{\int_{-\infty}^{0} d\xi \int_{-\infty}^{\infty} \exp\{s\lambda_o\xi\}\ G_1(s,\zeta)\ \frac{1}{\bar{R}} \exp\{-s\bar{R}\}\ \ d\zeta\right\}$

$$= \frac{1}{\pi} L^{-1}\left\{\int_{-\infty}^{0} d\xi \int_{-\infty}^{\infty} \frac{1}{\bar{R}} \exp\{-s(\bar{R} - \lambda_o\xi)\}\ G_1(s,\zeta)\ d\zeta\right\}. \qquad \text{(A 4.3)}$$

Since $L^{-1}\{G_1(s,z)\} = F_1(\bar{a}\bar{\tau}, z)$, and $(R - \xi)$ is always positive, by shift rule we can obtain

$$I_o = \frac{1}{\pi} \int_{-\infty}^{0} d\xi \int_{-\infty}^{\infty} \frac{1}{\bar{R}}\ F_1\{\bar{a}(\bar{\tau} - \bar{R} + \lambda_o\xi),\zeta\}\ \ d\zeta, \qquad \text{(A 4.4)}$$

where $\qquad \bar{R} = \{(\bar{x} - \xi)^2 + (z - \zeta)^2 + y^2\}.$

(b) Next we consider the integrals for i = 1, 2, 3 & 4 in (5.4.16)

$$I_i^{***} = \int_{-\infty}^{0} \exp\{s\lambda_i\xi\}\ H_1(s,\nu)\ \frac{1}{\lambda} \exp\{\lambda(\bar{x} + \xi)\}\ d\xi. \qquad \text{(A 4.5)}$$

Since here $(\bar{x} + \xi)$ is always negative and $\lambda_i > 0$, we can proceed for the inversion of (A 4.5) as for (A 4.1) and thus obtain

$$I_i = \frac{1}{\pi} \int_{-\infty}^{0} d\xi \int_{-\infty}^{\infty} \frac{1}{\bar{R}_1} F_1\{\bar{a}(\bar{\tau} - \bar{R}_1 + \lambda_i\xi),\zeta\}\ d\zeta, \qquad \text{(A 4.6)}$$

with $\quad \bar{R}_1 = \{(\bar{x} + \xi)^2 + (z - \zeta)^2 + y^2\}^{\frac{1}{2}} = \bar{R}(-\xi).$

(c) Again, we consider the sixth integral in (5.4.16)

$$I_{51}^{***} = \int_{-\infty}^{0} \exp\{s\lambda_4\xi\}\ \ H_1(s,\nu)\ \frac{1}{\lambda} \exp\{-\lambda(|\bar{x} - \xi|)\}\ d\xi. \qquad \text{(A 4.7)}$$

Since $\lambda_4 > 0$, this behaves exactly like the first term (A 4.1). Hence its inversion can be given by (A 4.4) with λ_o replaced by λ_4.

Finally we consider the last term in (5.4.16)

$$I_{52}^{***} = (\alpha^2 + \nu^2 + \bar{b}^2 s^2)^{-1} \exp\{s\lambda_4\bar{x}\} \; H_1(s,\nu) \qquad \text{(A 4.8)}$$

Since $$F^{-1}\left(\frac{1}{\alpha^2 + \nu^2 + \bar{b}^2 s^2}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp\{i\nu z\}}{\alpha^2 + \nu^2 + \bar{b}^2 s^2}\, d\nu$$

$$= \frac{1}{2}(\alpha^2 + \bar{b}^2 s^2)^{-\frac{1}{2}} \exp\{-(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}} z\},$$

by convolution

$$F^{-1}\left(\frac{H_1(s,\nu)}{\alpha^2 + \nu^2 + \bar{b}^2 s^2}\right) = \frac{1}{2}\int_{-\infty}^{\infty} G_1(s,\zeta)\, \frac{\exp\{-(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}}(|z - \zeta|)\}}{(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}}}.$$

Hence,

$$I_{52} = L^{-1}\left[F_c^{-1}\left\{\frac{1}{2}\int_{-\infty}^{\infty} \exp\{s\lambda_4\bar{x}\}\, G_1(s,\zeta) \quad \frac{\exp\{-(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}}(|z - \zeta|)\}}{(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}}}\, d\zeta\right\}\right]$$

$$= \frac{1}{\pi} L^{-1}\left\{\int_{-\infty}^{\infty} \exp\{s\lambda_4\bar{x}\} \quad G_1(s,\zeta)\; K_o\left[\bar{b}s\{(z - \zeta)^2 + y^2\}\right]\right\}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} d\zeta \int_0^{\mu_4} \frac{F_1(a\mu,\zeta)}{[(\bar{\tau} + \lambda_4\bar{x} - \mu)^2 - \bar{b}^2\{(z - \zeta)^2 + y^2\}]^{\frac{1}{2}}}\, d\mu \qquad \text{(A 4.9)}$$

with $\quad \mu_4 = \bar{\tau} + \lambda_4\bar{x} - \bar{b}\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}$.

4.2. Inversion of $I_{4,5}^{***}$, Eq. (5.4.19)

The expression (5.4.19) can also be expressed as

$$I_{4,5}^{***} = (Q_1 \frac{\partial}{\partial x} + Q_2 s)\; \frac{1}{\alpha^2 + \nu^2 + \bar{b}^2 s^2}\; \frac{\exp\{\lambda\bar{x}\}}{\lambda}\; H_1(s,\nu), \qquad \text{(A 4.10)}$$

with $\quad \lambda = (\alpha^2 + \nu^2 + s^2)^{\frac{1}{2}}$. Then

$$I_{4,5}(\bar{x}, y, z, \bar{\tau}) = L^{-1}\left(F_c^{-1}\left[F^{-1}\{I_{4,5}^{***}(\bar{x}, \alpha, \nu, s)\}\right]\right)$$

We consider $\quad I = L^{-1}\left[F_c^{-1}\left\{F^{-1}\left(\frac{1}{\alpha^2 + \nu^2 + \bar{b}^2 s^2} \frac{\exp\{\lambda \bar{x}\}}{\lambda} H_1(s,\nu)\right)\right\}\right]$

Since $F^{-1}\{ H_1(s,\nu)\} = G_1(s,z)$,

$$F^{-1}\{1/(\alpha^2 + \nu^2 + \bar{b}^2 s^2)\} = \frac{1}{2} (\alpha^2 + \bar{b}^2 s^2)^{-\frac{1}{2}} \exp\{ -(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}} z\},$$

$$F^{-1}\left(\frac{\exp\{(\alpha^2 + \nu^2 + s^2)^{\frac{1}{2}}\bar{x}\}}{(\alpha^2 + \nu^2 + s^2)^{\frac{1}{2}}}\right) = \frac{1}{\pi} K_o\left[(\alpha^2 + s^2)^{\frac{1}{2}} \{(\bar{x} - \xi)^2 + z^2\}^{\frac{1}{2}}\right],$$

by convolution we obtain

$$I = L^{-1}\left[F_c^{-1}\left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} G_1(s,\zeta) K_o\left[(\alpha^2 + s^2)\{(\bar{x} - \xi)^2 + (\zeta_1 - \zeta)^2\}^{\frac{1}{2}}\right] \times \frac{\exp\{-(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}}(|z - \zeta_1|)\}}{(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}}} d\zeta_1\right\}\right] \quad (A\ 4.11)$$

Further, since

$$F_c^{-1}\left(K_o\{(\alpha^2 + s^2)^{\frac{1}{2}} a_1\}\right) = \frac{\exp\{-s(a_1^2 - y^2)^{\frac{1}{2}}\}}{(a_1^2 + y^2)^{\frac{1}{2}}}, \quad a_1 > 0$$

$$F_c^{-1}\left(\frac{\exp\{-(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}} b_1\}}{(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}}}\right) = \frac{2}{\pi} K_o\{\bar{b}s(y^2 + b_1^2)^{\frac{1}{2}}\}, \quad b_1 > 0$$

by convolution

$$F_c^{-1}\left(K_o\{(\alpha^2 + s^2)^{\frac{1}{2}} a_1\} \quad \frac{\exp\{-(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}} b_1\}}{(\alpha^2 + \bar{b}^2 s^2)^{\frac{1}{2}}}\right)$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{\exp\{-s(a_1^2 + y_1^2)^{\frac{1}{2}}\}}{(a_1^2 + y_1^2)^{\frac{1}{2}}} \left(K_o\left[\bar{b}s\{(y - y_1)^2 + b_1^2)\}^{\frac{1}{2}}\right] + K_o\left[\bar{b}s\{(y + y_1)^2 + b_1^2\}^{\frac{1}{2}}\right] \right) dy_1,$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\exp\{-s(a_1^2 + y_1^2)^{\frac{1}{2}}\}}{(a_1^2 + y_1^2)^{\frac{1}{2}}} K_o\left[\bar{b}s\{(y - y_1)^2 + b_1^2\}^{\frac{1}{2}}\right] dy_1.$$

Thus we obtain from (A 4.11)

$$I = \frac{1}{2\pi^2} L^{-1} \left\{\int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} G_1(s,\zeta) \frac{1}{r_1} \exp\{-sr_1\} K_o(\bar{b}sr_2) dy_1\right\},$$

with $r_1 = \{(\bar{x} - \xi)^2 + (\zeta - \zeta_1)^2 + y^2\}^{\frac{1}{2}}$, $r_2 = \{(y - y_1)^2 + (z - \zeta_1)^2\}^{\frac{1}{2}}$.

Then

$$I = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} dy_1 \int_0^{\bar{\tau}} \frac{F_1\{\bar{a}(\mu - r_1)\}}{r_1\{(\bar{\tau} - \mu)^2 - \bar{b}^2 r_2^2\}^{\frac{1}{2}}} d\mu . \qquad \text{(A 4.12)}$$

Hence we obtain the inversion of (A 4.10) as

$$I_{4,5} = \frac{1}{2\pi^2}\left(Q_1 \frac{\partial}{\partial \bar{x}} + Q_2 \frac{\partial}{\partial \bar{\tau}}\right) \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} d\zeta_1 \int_{-\infty}^{\infty} dy_1 \int_0^{\bar{\tau}} \frac{F_1\{\bar{a}(\mu - r_1)\}}{r_1\{(\bar{\tau} - \mu)^2 - \bar{b}^2 r_2^2\}^{\frac{1}{2}}} d\mu . \qquad \text{(A 4.13)}$$

4.3. Inversion of expression (5.5.11)

(a) Consider the first integral

$$I_o = L^{-1}\left[F_s^{-1}\left\{F^{-1}\left(\int_{-\infty}^{0} \alpha\, h(\xi, \nu, s) \frac{1}{\lambda} \exp\{-\lambda(|\bar{x} - \xi|)\} d\xi\right)\right\}\right], \qquad \text{(A 4.14)}$$

with $\lambda = (\alpha^2 + \nu^2 + s^2)^{\frac{1}{2}}$. Then

$$I_o = \frac{1}{\pi} L^{-1}\left[F_s^{-1}\left\{ \int_{-\infty}^{0} d\xi \int_{-\infty}^{\infty} \alpha\, g_1(\xi, \nu, s)\, K_o\left[(\alpha^2 + s^2)^{\frac{1}{2}} (\bar{x} - \xi)^2 + (z - \zeta)^2\}^{\frac{1}{2}}\right] d\zeta\right\}\right] \qquad \text{(A 4.15)}$$

where $g_1(\xi, \nu, s) = F^{-1}\{h(\xi, \nu, s)\}$.

Since, $F_s^{-1}\left(\alpha\, K_o\left[(\alpha^2 + s^2)^{\frac{1}{2}}\{(\bar{x} - \xi)^2 + (z - \zeta)^2\}^{\frac{1}{2}}\right]\right)$

$$= \frac{2}{\pi} \int_0^{\infty} \alpha \sin\alpha y\, K_o\left[(\alpha^2 + s^2)^{\frac{1}{2}}\{(\bar{x} - \xi)^2 + (z - \zeta)^2\}^{\frac{1}{2}}\right]$$

$$= -\frac{2}{\pi} \frac{\partial}{\partial y} \int_0^{\infty} \cos\alpha y\, K_o\left[(\alpha^2 + s^2)^{\frac{1}{2}}\{(\bar{x} - \xi)^2 + (z - \zeta)^2\}^{\frac{1}{2}}\right]$$

$$= -\frac{\partial}{\partial y}\left(\frac{1}{\bar{R}} \exp\{-s\bar{R}\}\right), \quad \text{with } \bar{R} = \{(\bar{x} - \xi)^2 + (z - \zeta)^2 + y^2\}^{\frac{1}{2}},$$

we obtain from (A 4.15)

$$I_o = -\frac{1}{\pi} L^{-1}\left(\frac{\partial}{\partial y} \int_{-\infty}^{0} d\xi \int_{-\infty}^{\infty} \frac{1}{\bar{R}} g_1(\xi, \zeta, s) \exp\{-s\bar{R}\} d\zeta\right)$$

$$= -\frac{1}{\pi}\frac{\partial}{\partial y} \int_{-\infty}^{0} d\xi \int_{-\infty}^{\infty} \frac{1}{\bar{R}} f_1\{\xi, \zeta, (\bar{\tau} - \bar{R})\} d\zeta. \qquad \text{(A 4.16)}$$

Similarly we can obtain the transform for the second integral in (5.5.11).

(b) Consider the integral

$$I^{***}_{2,i} = \frac{s\alpha}{\lambda + \lambda_i s} \int_{0}^{\infty} h(-\xi, \nu, s) \frac{1}{\lambda} \exp\{-\lambda(\xi - \bar{x})\} d\xi, \qquad \text{(A 4.17)}$$

for i = 2 & 3. This can be further expressed as

$$I^{***}_{2,i} = s\alpha \left(\int_{0}^{\infty} \exp\{-(\lambda + \lambda_i s)\xi_1\} d\xi_1\right) \int_{0}^{\infty} h(-\xi, \nu, s) \frac{1}{\lambda} \exp\{-\lambda(\xi - \bar{x})\} d\xi$$

$$= s\alpha \int_{0}^{\infty} d\xi_1 \int_{0}^{\infty} \exp\{-s\lambda_i\xi\} h(-\xi, \nu, s) \frac{1}{\lambda} \exp\{-\lambda(\xi - \bar{x})\} d\xi. \qquad \text{(A 4.18)}$$

Proceeding in the similar manner as in (a) above we can obtain for the inversion of (A 4.18)

$$I_{2,i} = L^{-1}\left[F_s^{-1}\{F^{-1}(I^{***}_{2,i})\}\right]$$

$$= -\frac{1}{\pi}\frac{\partial}{\partial y}\frac{\partial}{\partial \bar{\tau}} \int_{0}^{\infty} d\xi_1 \int_{0}^{\infty} d\xi \int_{-\infty}^{\infty} \frac{1}{\bar{R}_2} f_1\{-\xi, \zeta, (\bar{\tau} - \bar{R}_2 - \lambda_i\xi_1)\} d\zeta, \qquad \text{(A 4.19)}$$

where $\bar{R}_2 = \{(\bar{x} - \xi - \xi_1)^2 + (z - \zeta)^2 + y^2\}^{\frac{1}{2}}$.

(c) Now consider

$$I^{***}_{3,i} = \frac{1}{\lambda + \lambda_i s} \frac{\exp\{\lambda\bar{x}\}}{\lambda} \alpha h(\nu, s), \qquad \text{(A 4.20)}$$

for i = 2 & 3. This can be expressed in integral form as

$$I^{***}_{3,i} = \alpha \int_{0}^{\infty} \exp\{-\lambda_i\xi s\} h(\nu, s) \frac{1}{\lambda} \exp\{-\lambda(\xi - \bar{x})\} d\xi. \qquad \text{(A 4.21)}$$

The inversion of (A 4.21) yields

$$I_{3,i} = -\frac{1}{\pi}\frac{\partial}{\partial y}\int_0^\infty d\xi\int_{-\infty}^\infty \frac{1}{\bar{R}} f_1\{\zeta,(\bar{\tau} - \bar{R} - \lambda_i\xi)\}\, d\xi. \qquad \text{(A 4.22)}$$

Similarly we can obtain the inversion of the expressions

$$I^{***}_{4,i} = \frac{1}{\lambda + \lambda_i s}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda}\,\alpha\, H_2(s,\ \nu),$$

for $i = 2, 3$ & 4.

(d) Finally we consider the last term in (5.5.11) as

$$I^{***}_5 = \frac{1}{\lambda - \lambda_4 s}\,\frac{\exp\{\lambda\bar{x}\}}{\lambda}\,\alpha\, H_2(s,\ \nu), \qquad \text{(A 4.23)}$$

Since $\lambda_5 = -\lambda_4$. This can be expressed as follows (cf. Appendix 1.1 (c)

$$I^{***}_5 = -\int_{-\infty}^0 \exp\{s\lambda_4\xi\}\alpha H_2(s,\ \nu)\ \frac{1}{\lambda}\exp\{-\lambda(|\bar{x} - \xi|)\}\, d\xi$$

$$+ \{1/(\alpha^2 + \nu^2 + \bar{b}^2 s^2)\}\exp\{s\lambda_4\bar{x}\}\ \alpha\, H_2(s,\ \nu). \qquad \text{(A 4.24)}$$

The inversion of (A 4.24) now yields

$$I_5 = \frac{1}{\pi}\frac{\partial}{\partial y}\int_{-\infty}^0 d\xi\int_{-\infty}^\infty \frac{1}{\bar{R}}\ F_2\{\zeta,\ \bar{a}(\bar{\tau} - \bar{R} + \lambda_4\xi)\}\, d\zeta$$

$$-\frac{1}{\pi}\frac{\partial}{\partial y}\int_{-\infty}^\infty d\zeta\int_0^{\mu_4} \frac{F_2(\bar{a}\mu,\ \zeta)}{[(\bar{\tau} + \lambda_4\bar{x} - \mu)^2 - \bar{b}^2\{(z - \zeta)^2 + y^2\}]^{\frac{1}{2}}}, \qquad \text{(A 4.25)}$$

with $\mu_4 = \bar{\tau} + \lambda_4\bar{x} - \bar{b}\{(z - \zeta)^2 + y^2\}^{\frac{1}{2}}$.

On further simplification (A 4.25) can also be written as

$$I_5 = -\frac{1}{\pi}\frac{\partial}{\partial y}\int_0^\infty d\xi\int_{-\infty}^\infty \frac{1}{\bar{R}}\ F_2\{\zeta,\ \bar{a}(\bar{\tau} - \bar{R} + \lambda_4\xi)\}\ d\xi. \qquad \text{(A 4.26)}$$

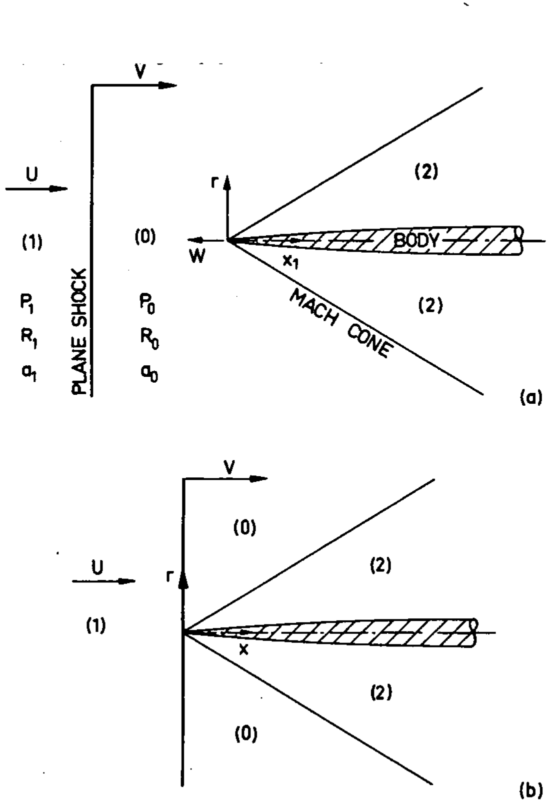

Fig.1. Flow pattern (a) for time t < 0, (b) for time t=0.

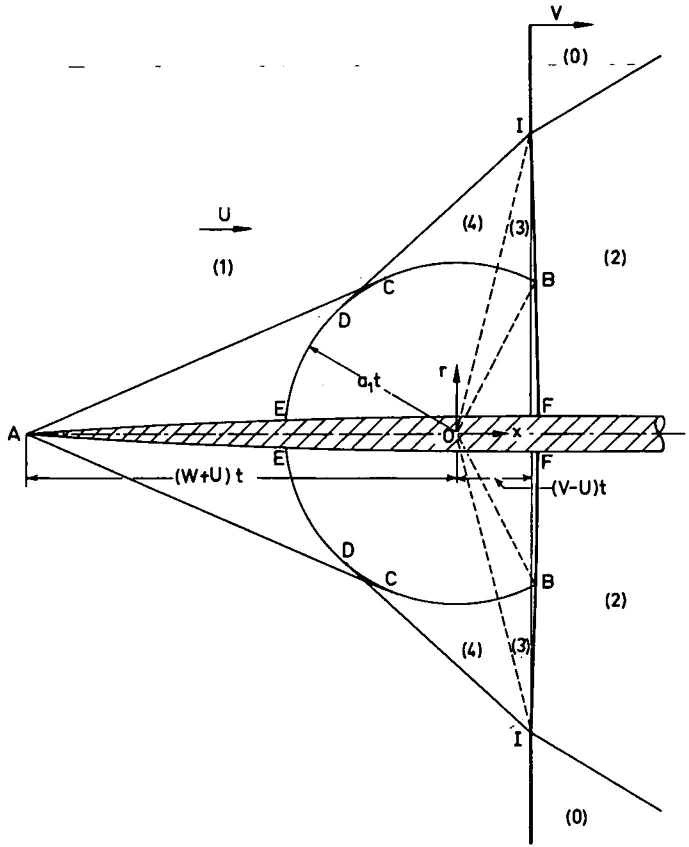

Fig.2. Interaction flow configuration for time t > 0.

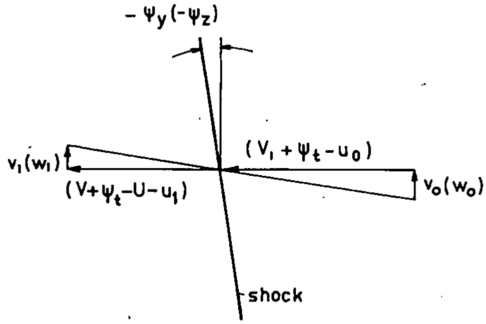

Fig.3. Local conditions at the perturbed shock.

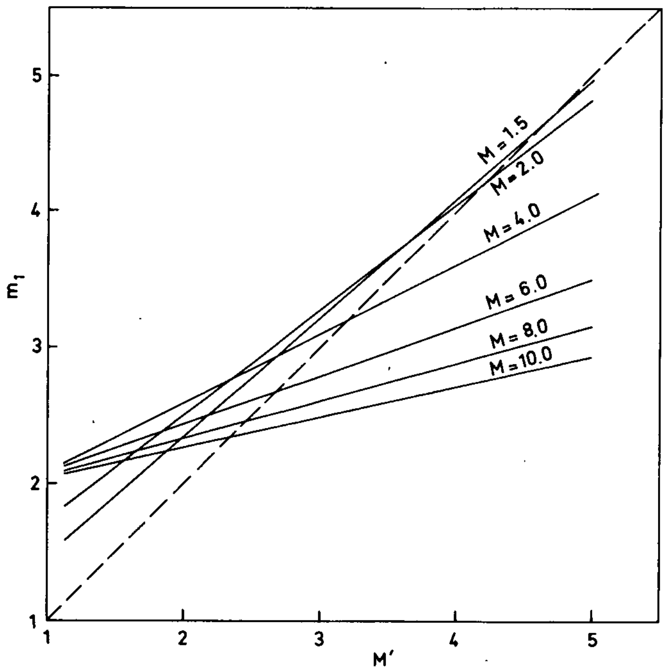

Fig.4. Obstacle Mach number change across the shock.

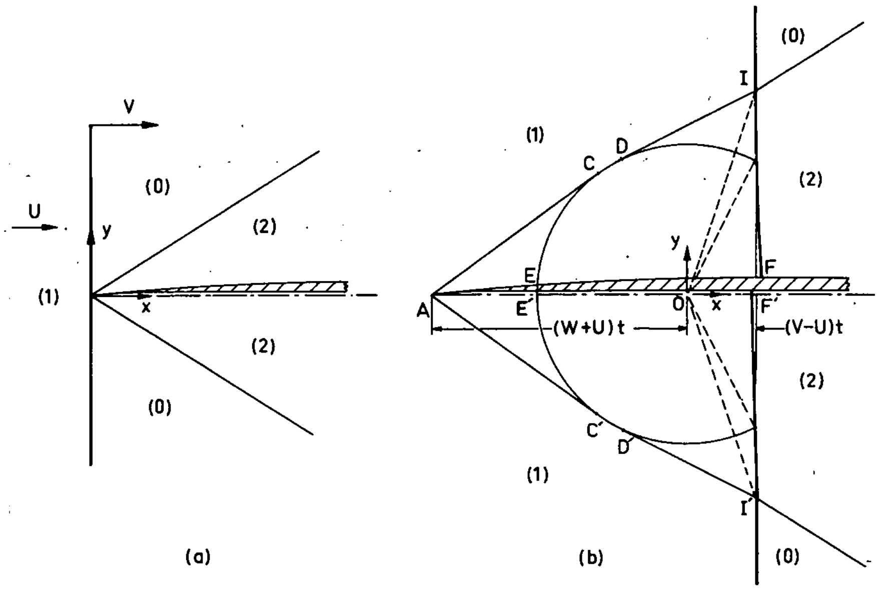

Fig. 5. Flow pattern for two-dimensional aerofoils; (a) for time $t \leq 0$, for time $t > 0$.

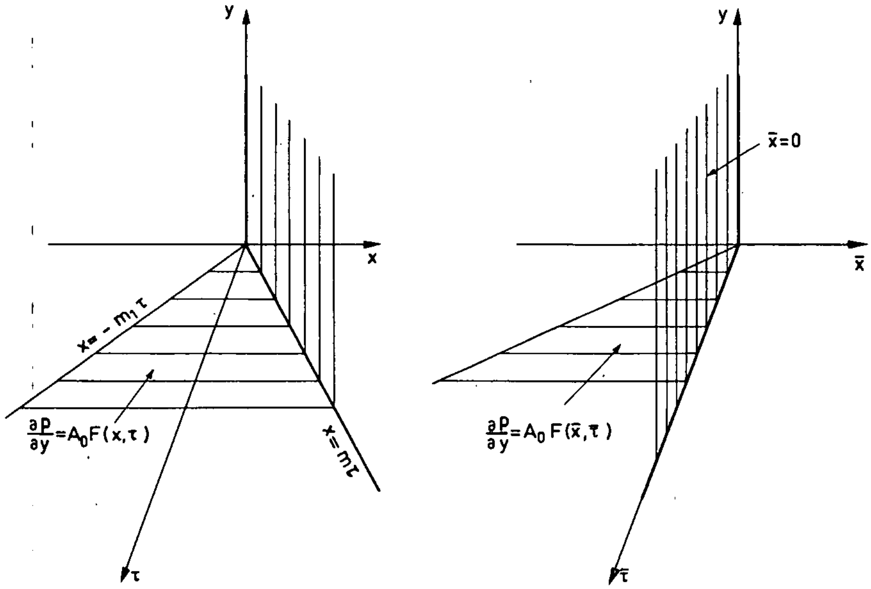

Fig.6. Boundary conditions; (a) $x\,y\,\tau$ - space, (b) $\bar{x}\,y\,\bar{\tau}$ - space.

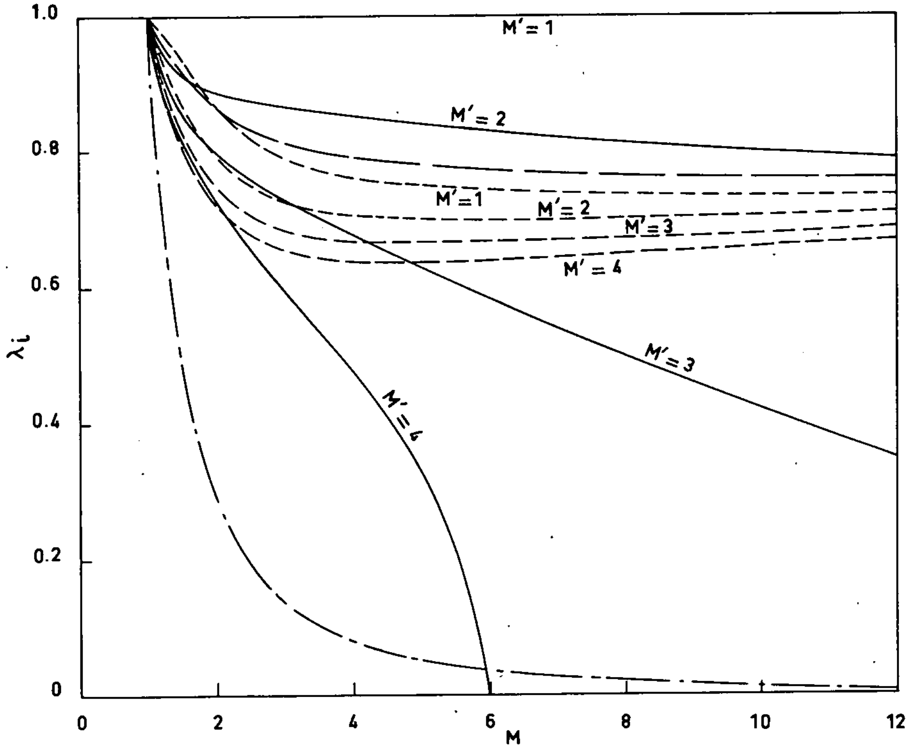

Fig.7. Variations of λ_is′ with M and M′; – – –, $\lambda_0 = \lambda_1$, —— —, λ_2, —·—, λ_3 and ———, λ_4.

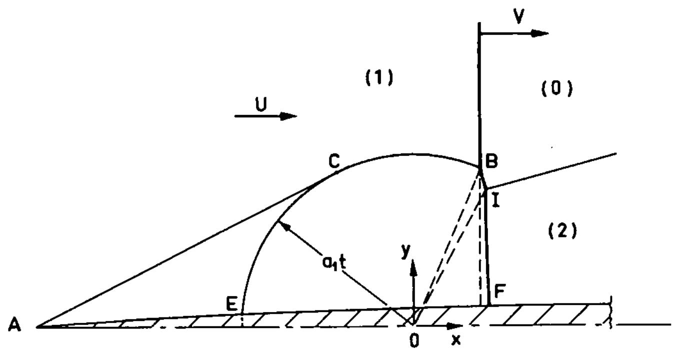

Fig.8. Flow pattern for time $t > 0$, when the shock intersection lies inside the sonic circle.

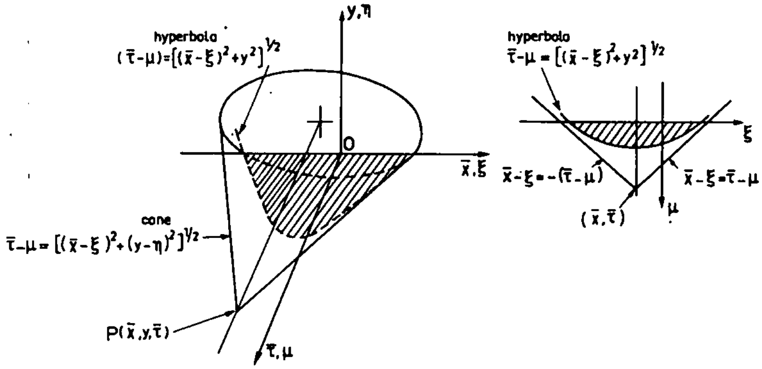

Fig.9. Cone of influence and domain of integration (shaded area).

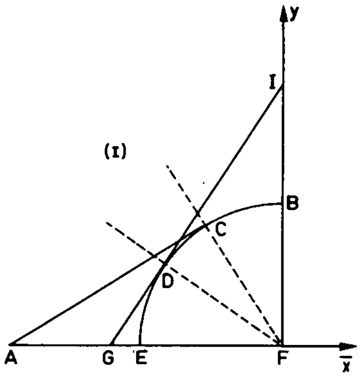

Fig. 10. Properties of the solution.

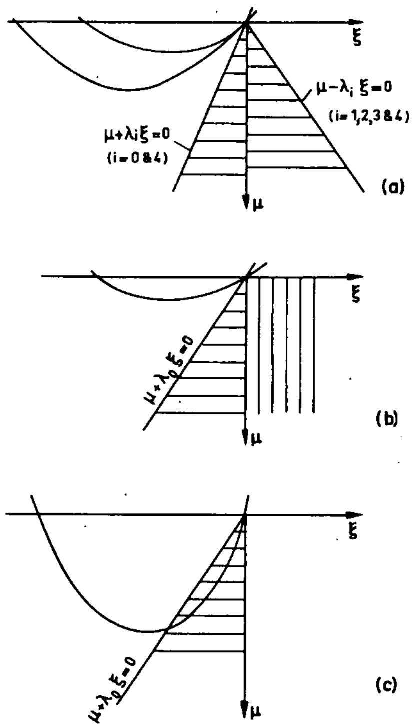

Fig.11. Properties of the solution.

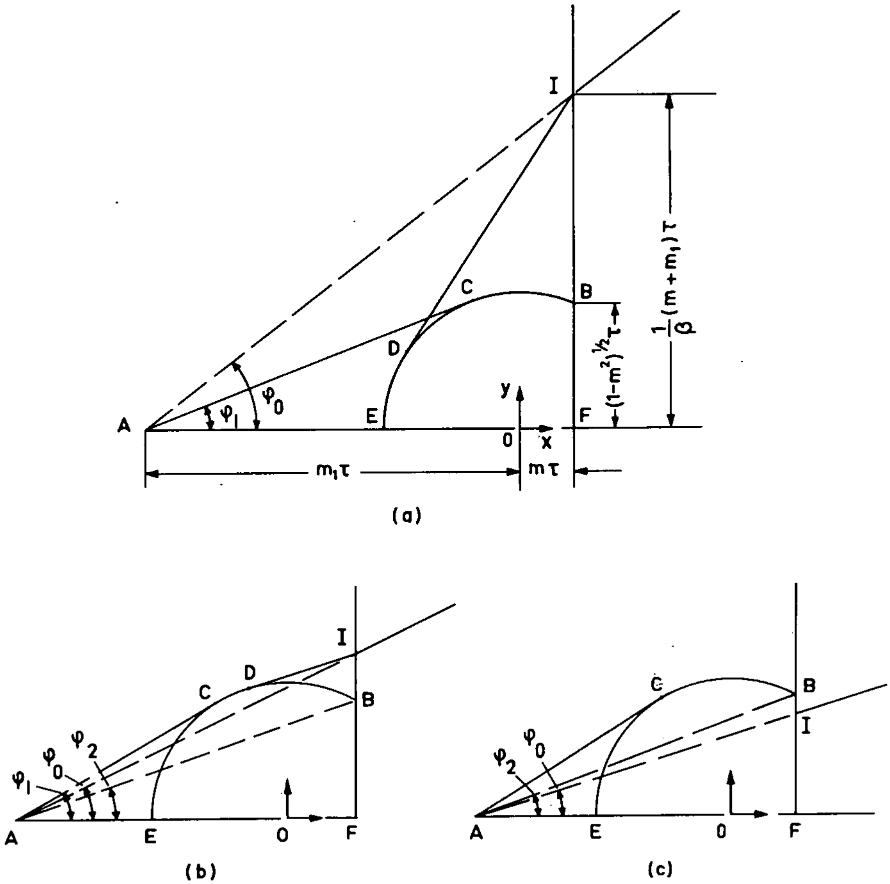

Fig.12. Flow pattern classification (two-dimensional aerofoils); (a) intersecting tangents,(b) non-intersecting tangents and (c) single tangent.

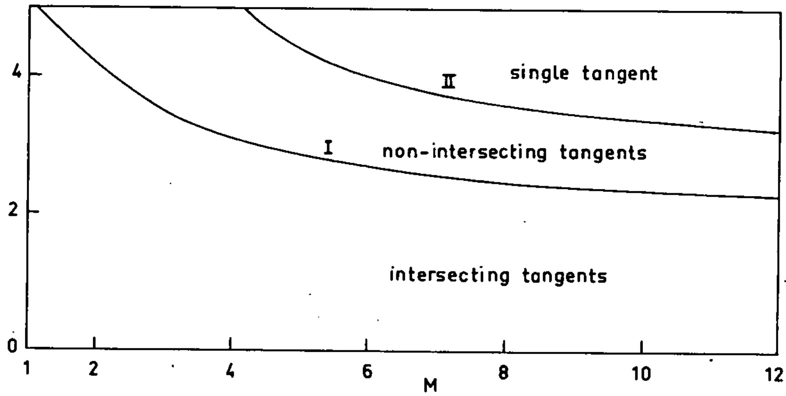

Fig. 13. Range of M′ and M for flow pattern classification.

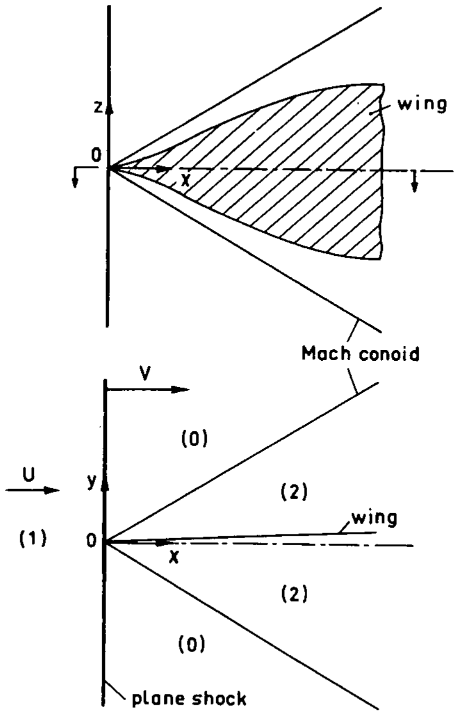

Fig. 14. Flow pattern for time $t \leq 0$.

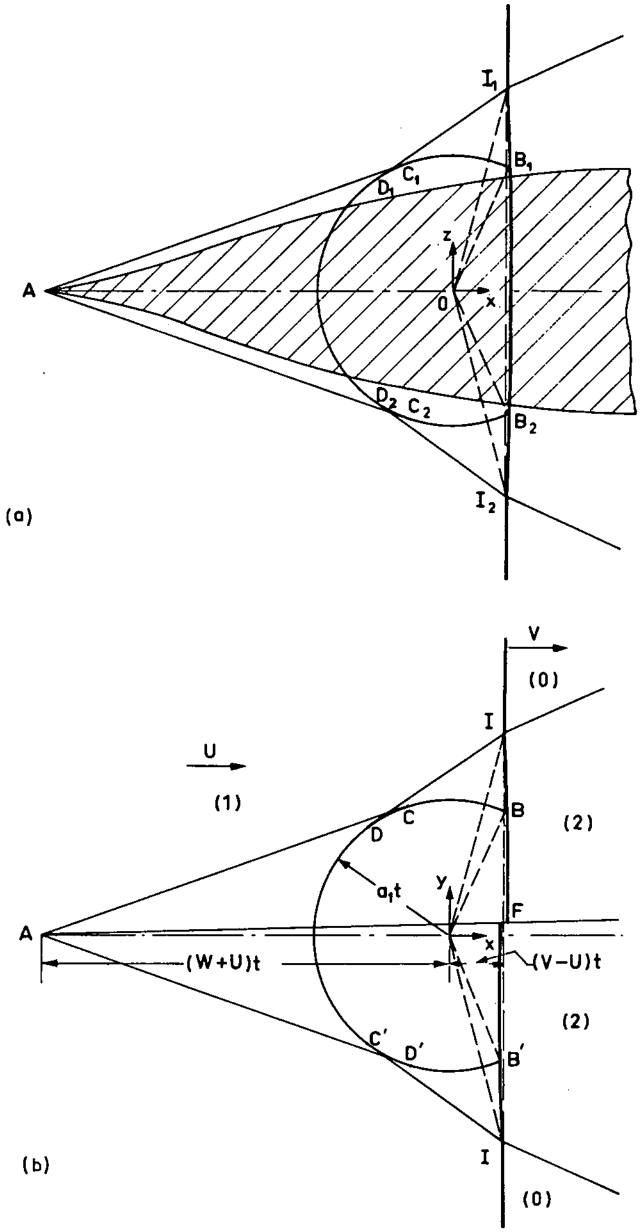

Fig.15. Interaction flow configuration for time $t > 0$,(a) on the wing plane $y = 0$,(b) on the root chord plane $z = 0$.

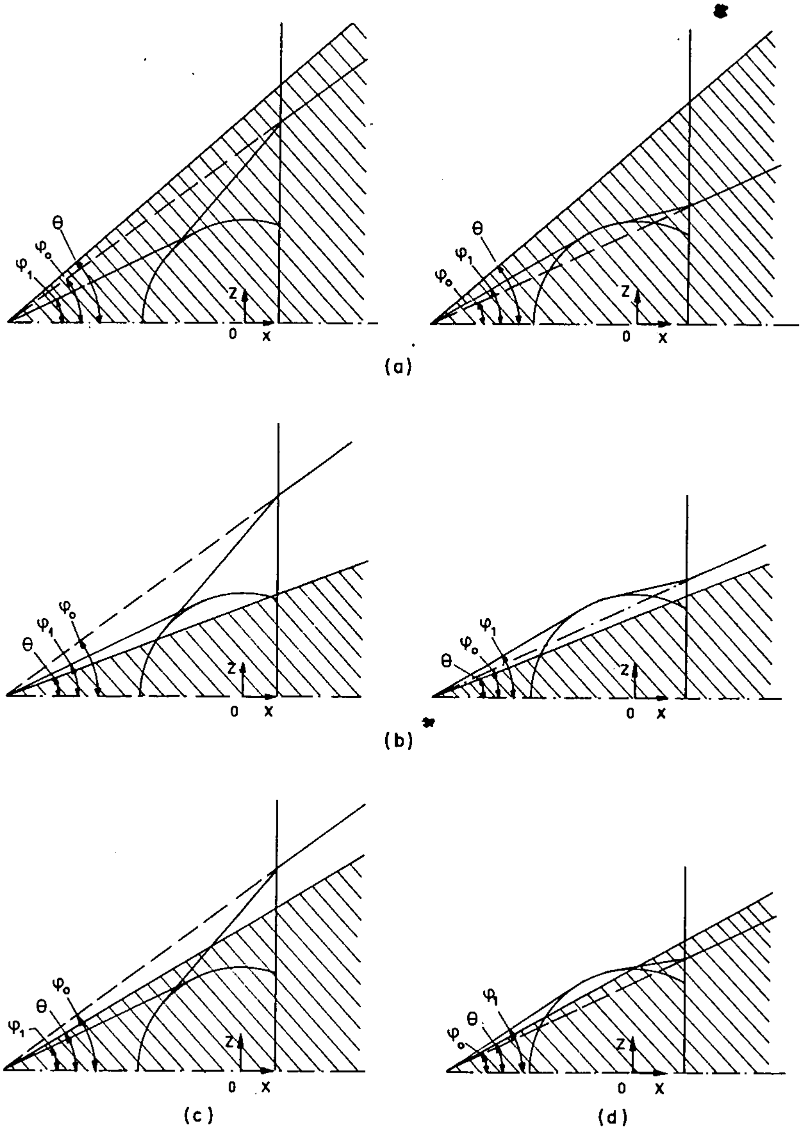

Fig.16. Flow configuration classification (three-dimensional wings)

(a) $\varphi_o < \theta > \varphi_1$, supersonic L.Es. ahead of as well as behind the shock,

(b) $\varphi_o > \theta < \varphi_1$, subsonic " " " " " " " " " ,

(c) $\varphi_o > \theta > \varphi_1$, subsonic L.Es. ahead of and supersonic L.Es. behind the shock,

(d) $\varphi_o < \theta < \varphi_1$, supersonic L.Es. " " " subsonic " " " " .

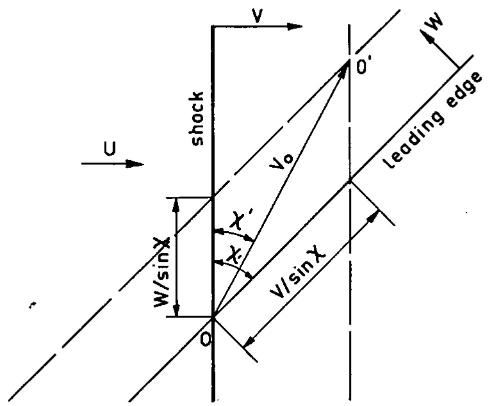

Fig. 17. Intersection of plane shock with yawed aerofoil.

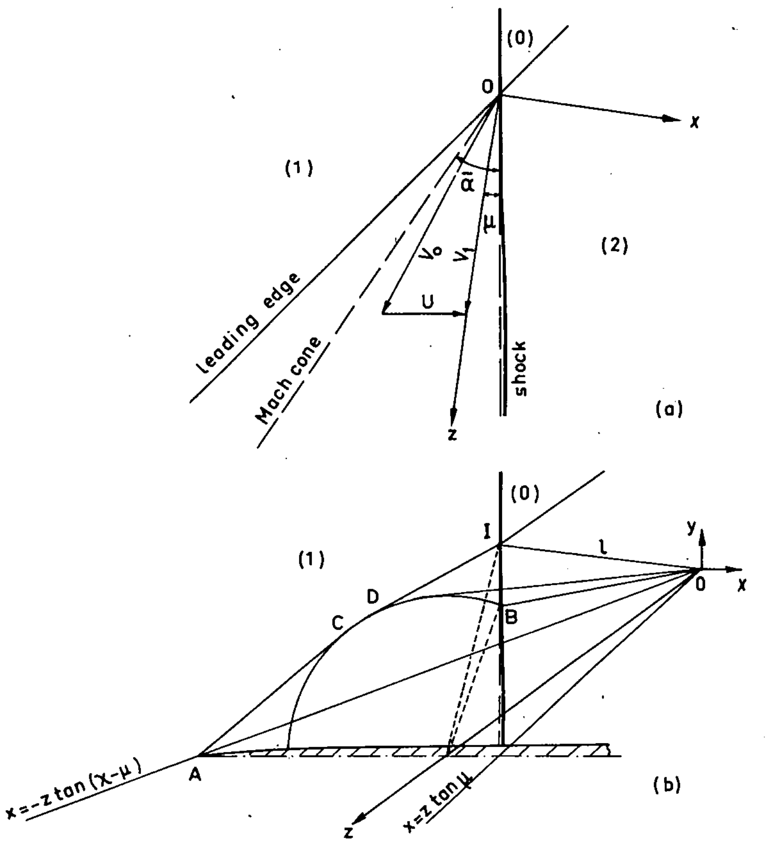

Fig.18. Flow configuration behind the shock, (a) in the x - z plane (b) in the x - y plane.

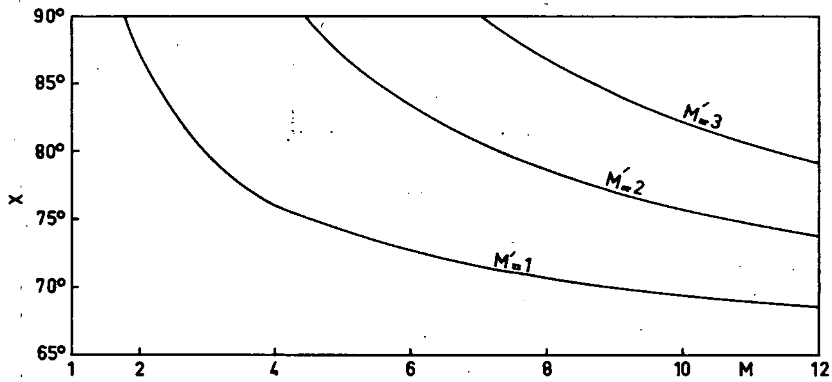

Fig.19. The range of χ for which the flow behind the shock is supersonic.

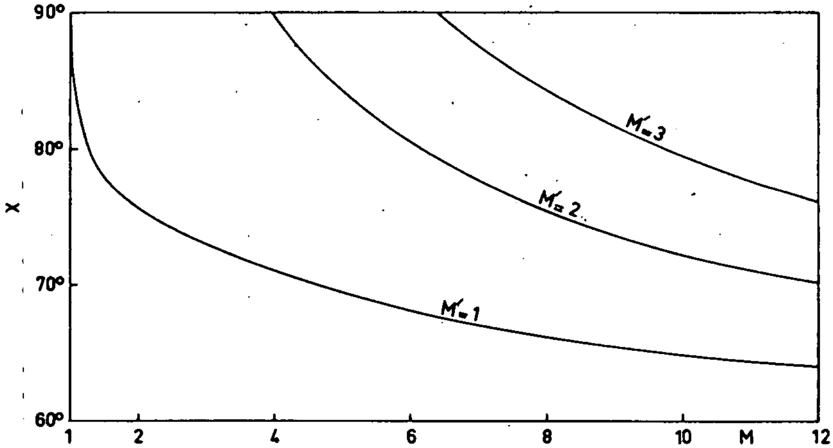

Fig.20. The range of χ for which the velocity component normal to the aerofoil L.E. behind the shock is supersonic.

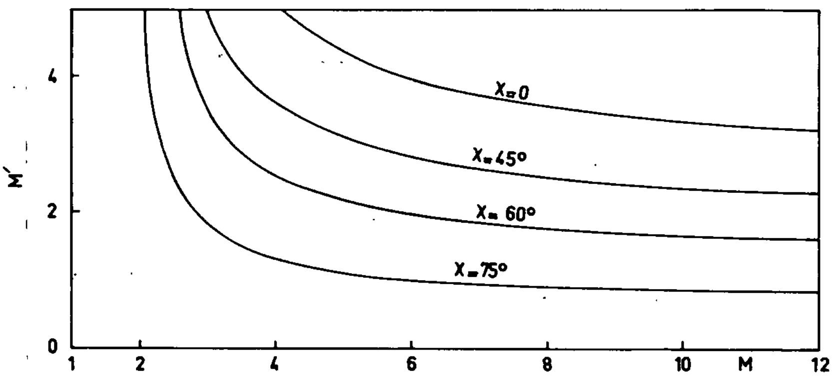

Fig.21. Range of M′ and M for the condition $\bar{b} \leq 1$.

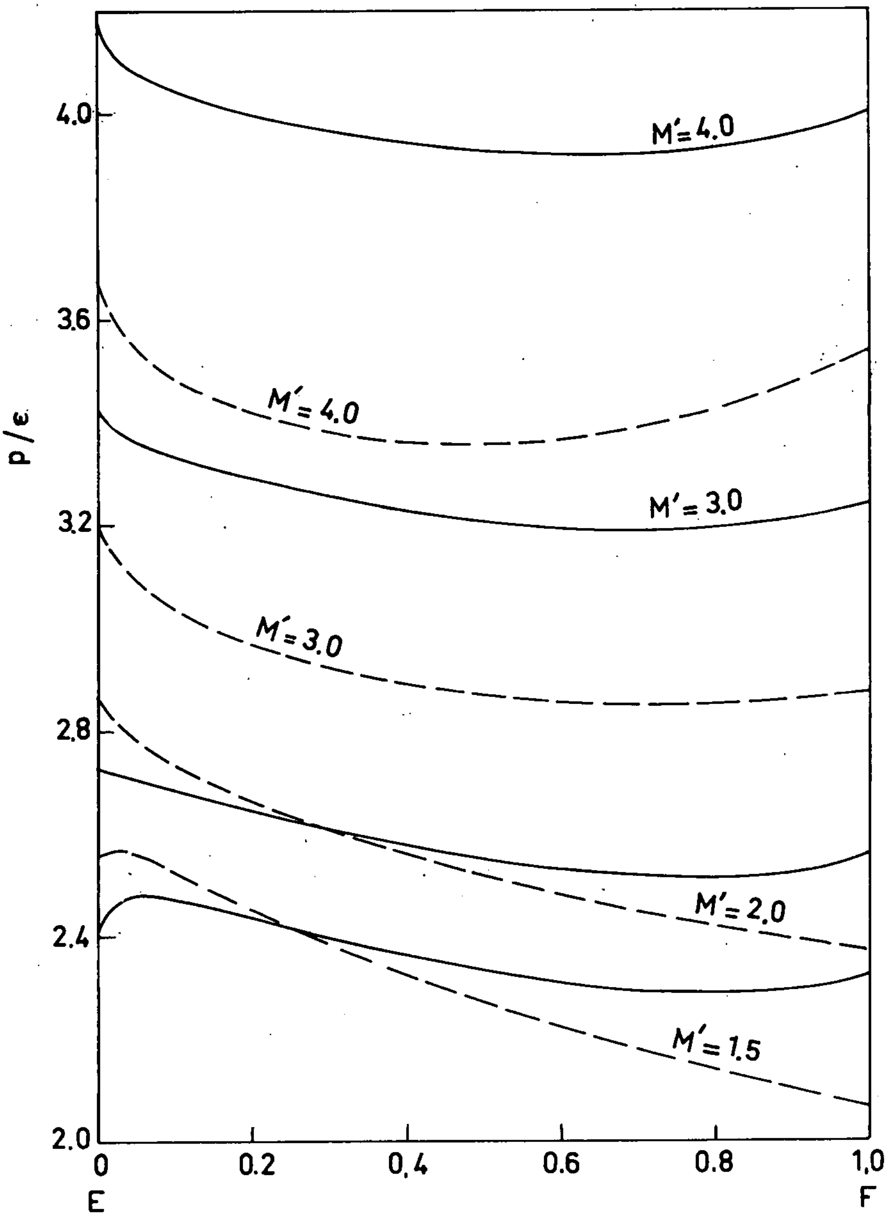

Fig. 22. Pressure distribution on the wedge for various values of M′. —— , M = 2.0 ; – – –, M = 4.0.

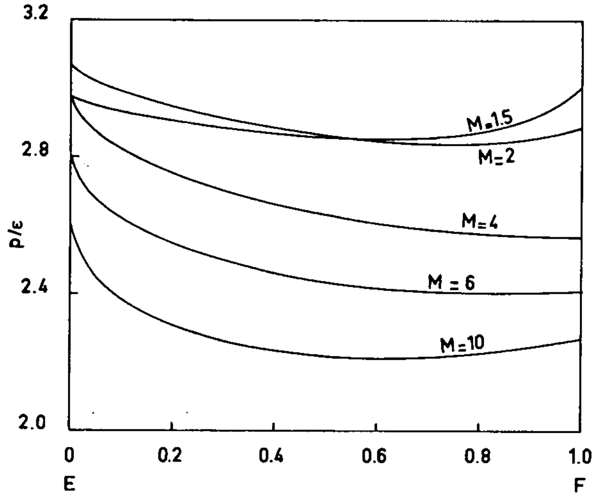

Fig.23. Pressure distribution on the wedge for $\chi = 0$, $M' = 2.5$ and for various values of M.

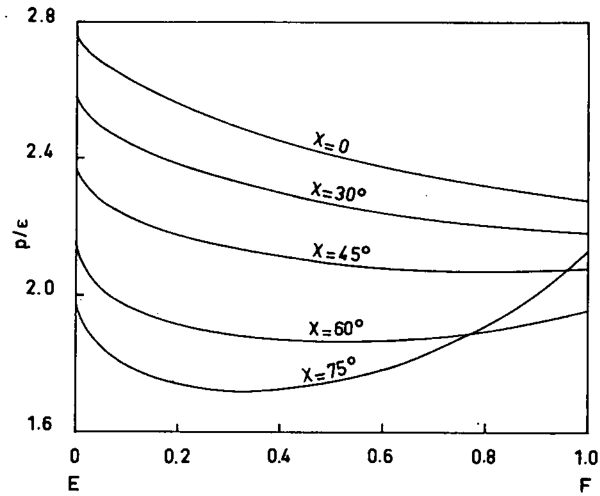

Fig.24. Pressure distribution on the wedge for $M' = 2.0$, $M = 4.0$ for various values of χ.

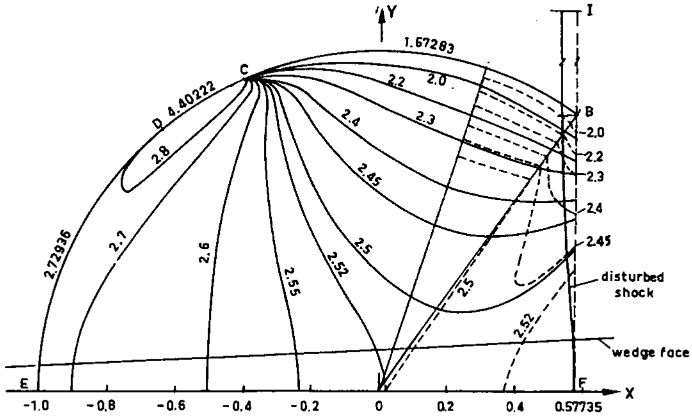

Fig. 25. Pressure and density variations in the Mach-reflexion region and the shock front for the wedge for $\varepsilon = 0.05$, $M' = 2.0$, $M = 2.0$, $\chi = 0$, ——, isobars (p/ε); ----, isopycnics (ρ/ε).

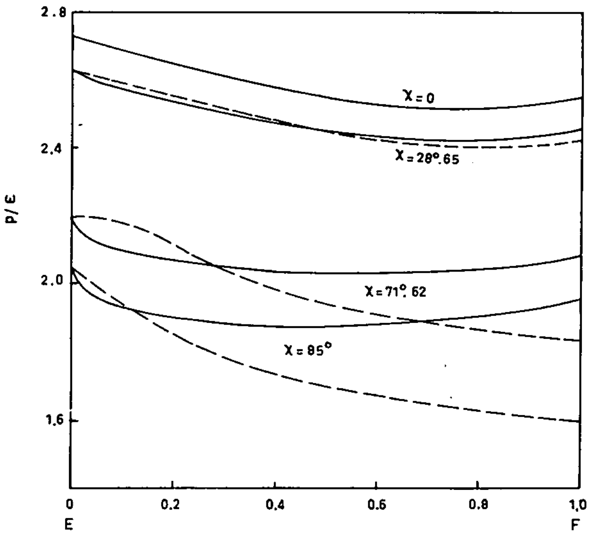

Fig. 26. Comparison of results – pressure distribution on the wedge for $M' = 2.0$, $M = 2.0$, $\chi = 0$, 0.5, 1.25, 1.5 radians; ——, present theory; ---, smyrl (1963).

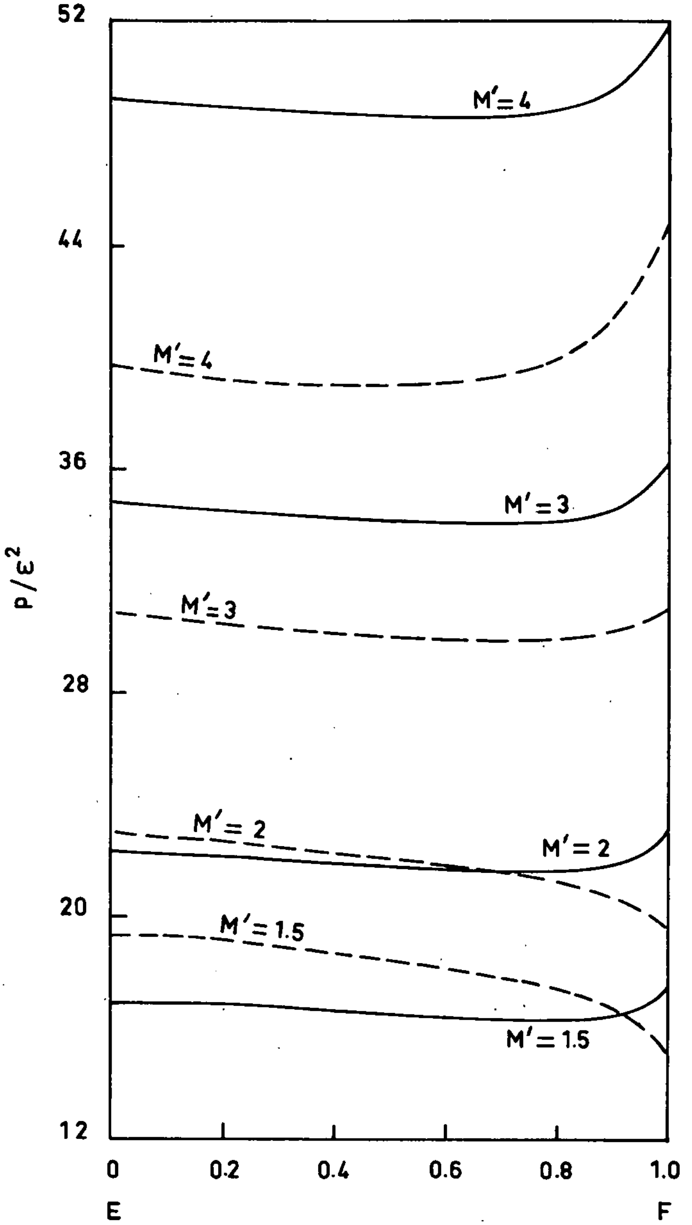

Fig.27. Pressure distribution on the cone surface for $\varepsilon = 0.025$ for various values of M′. ——, M = 2.0 ; – – –, M = 4.0 .

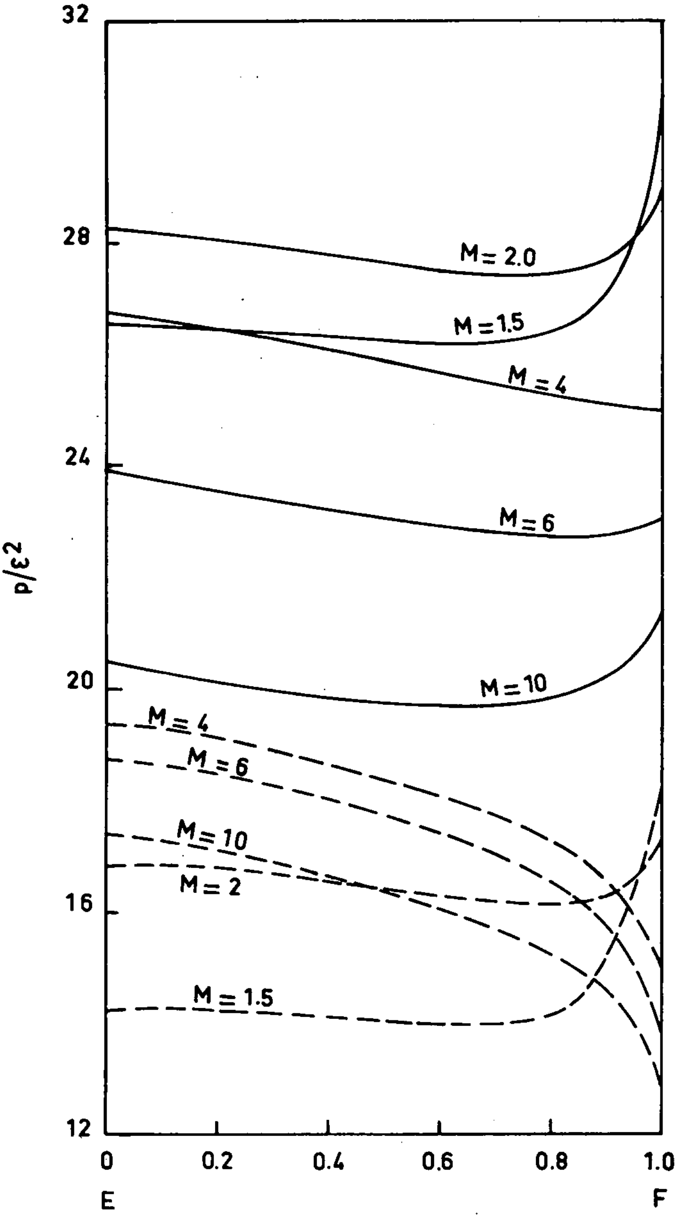

Fig.28. Pressure distribution on the cone surface for $\varepsilon = 0.025$ for various values of M. ——, $M' = 2.5$; - - -, $M' = 1.5$.

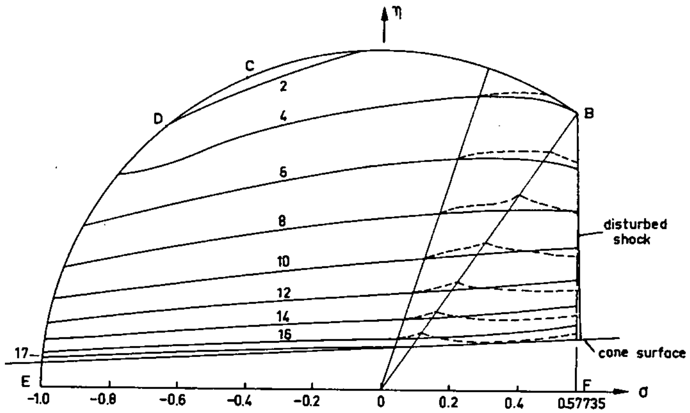

Fig. 29. Pressure and density variation in the Mach-reflexion region and the shock front for the cone for $\varepsilon = 0.05$, $M' = 2.0$, $M = 2.0$. —, isobars (p/ε^2); ---- isopycnics (ρ/ε^2).

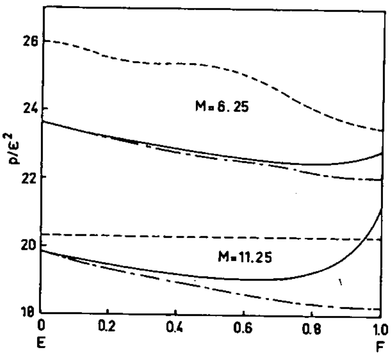

Fig.30. Comparison of results – pressure distribution on the cone surface for $\varepsilon = 0.025$, $M' = 2.5$, $M = 11.25$ and 6.25; —, present theory; —-—, present theory with B_0 taken as A_0; ---- Blankenship (1965).

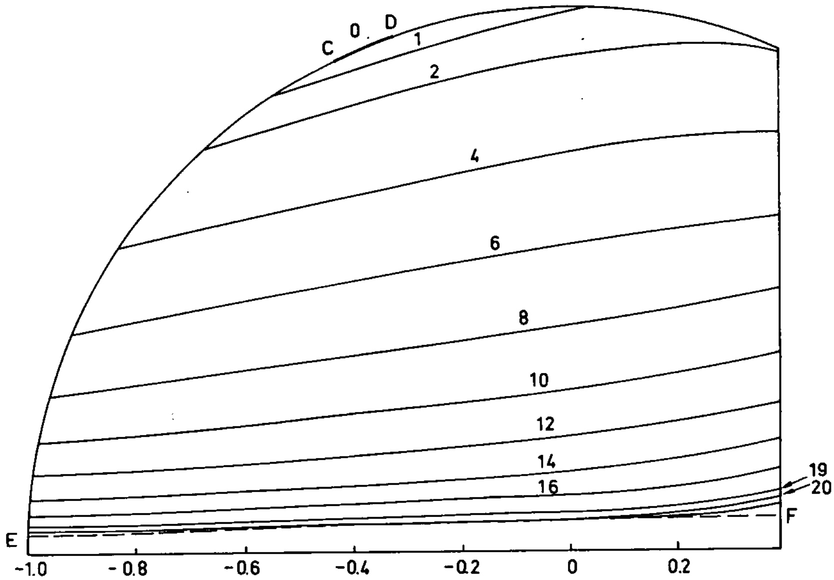

Fig. 31. Isobars (p/ε^2) for the cone for $\varepsilon = 0.025$, $M' = 2.5$, $M = 11.25$.

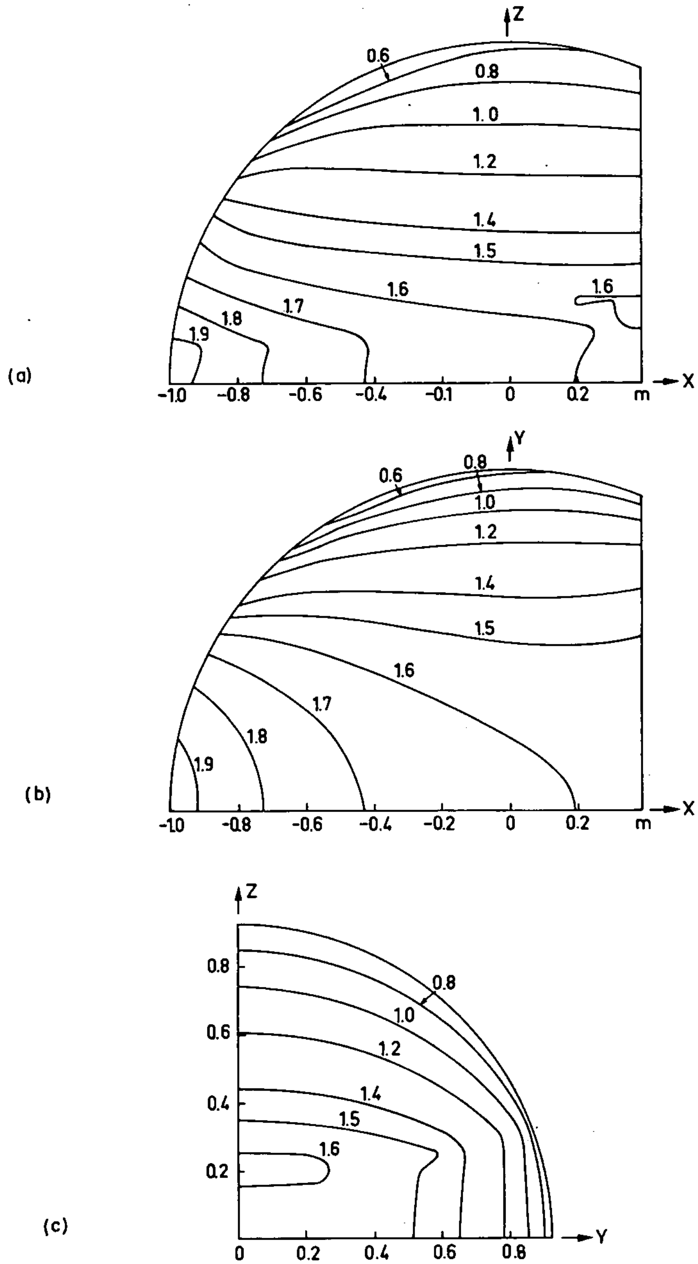

Fig. 32. Pressure variations in the Mach-reflexion region for the delta wing for $\theta = 45°$, $M' = 2.0$, $M = 10.0$ — isobars $(p/\mathcal{E})$, (a) on the wing plane, $Y = 0$, (b) on the root chord plane, $Z = 0$, (c) on the shock plane, $X = m$.

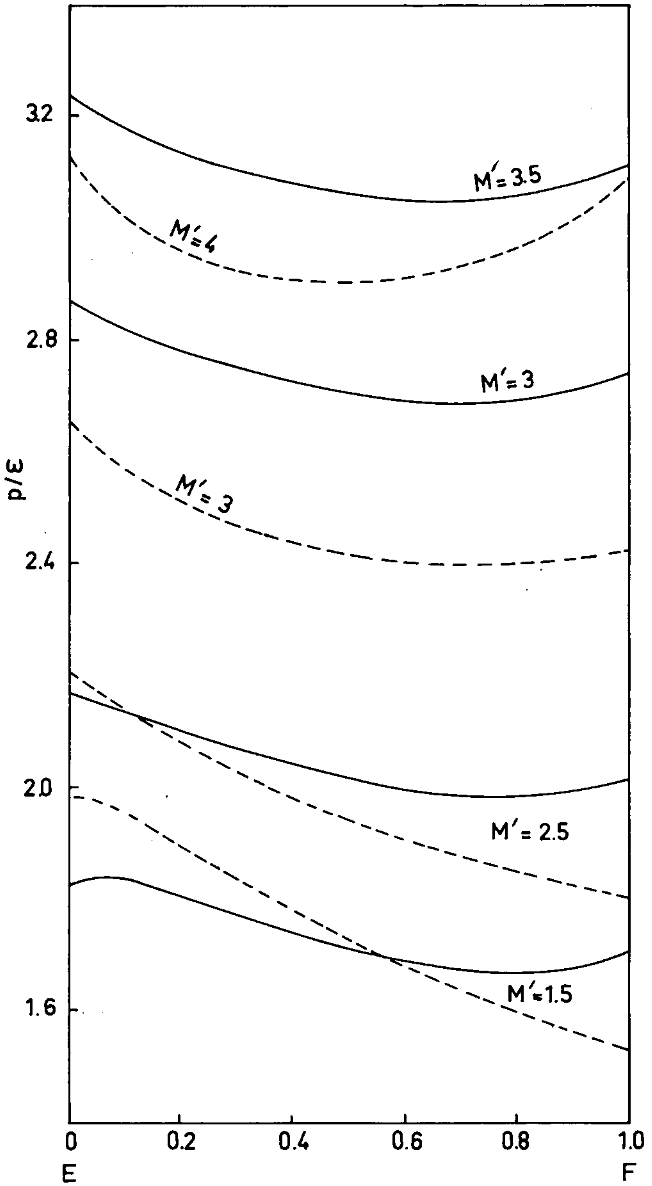

Fig.33. Pressure distribution on the wing root chord (Y=0, Z=0) for θ =45° for various values of M′
—, M = 2.0; ----, M=4.0.

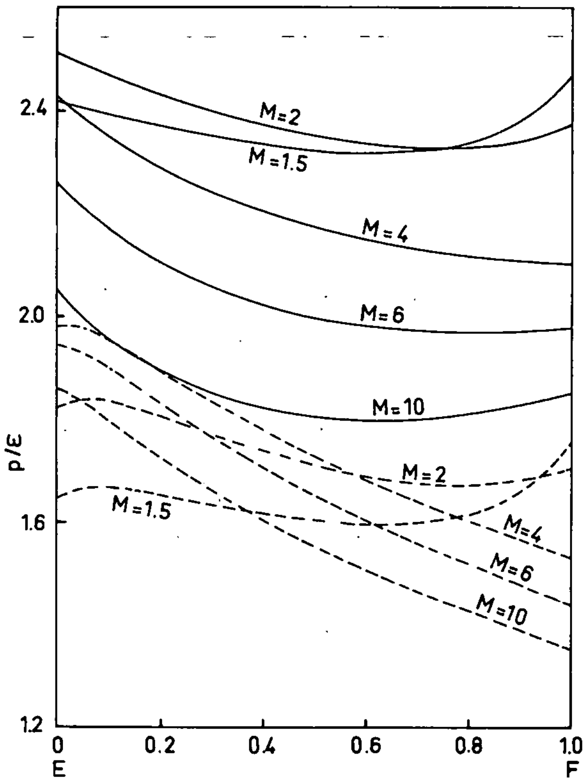

Fig.34. Pressure distribution on the wing root chord for θ=45° for various values of M. —, M′=2.5 ; ----, M′=1.5.

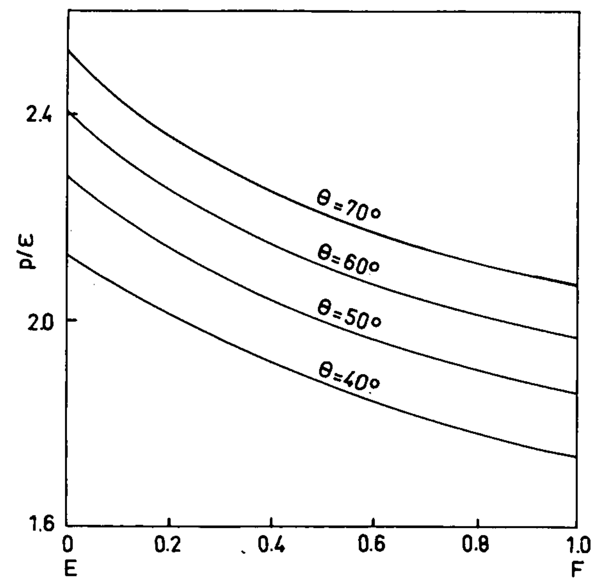

Fig.35. Pressure distribution on the wing root chord for M′=2.0, M=4.0 for various values of θ.