Using Mechanical System Dynamics to Model Time-Discounting in Behavioural Economics

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Abstract

Time-discounting in behavioural economics is modelled using mechanical system dynamics through the economic engineering framework. The economic engineering framework is being developed at the Delft Center for Systems and Control, and uses mechanical system dynamics to model economic processes and systems. Time-discounting is the calculation of the present value of the received utility from future consumption. Presently behavioural economists have not been able to reach a consensus on how to model time-discounting behaviour. Two theories dominate economic literature: exponential discounting theory and hyperbolic discounting theory. These theories are treated separately by economists and have separate fields of application. Exponential discounting theory and hyperbolic discounting theory are shown to be related through the dynamics of the damped harmonic oscillator.

Exponential and hyperbolic discounting theory are linked to the dynamics of the critically damped and overdamped mechanical system respectively. The dynamics of the underdamped mechanical system are linked to the time-discounting behaviour of a trader. Moreover, the parameters of the damped harmonic oscillator are interpreted economically, resulting in the following analogues: the natural frequency is analogous to the risk-free discount rate, the damping ratio is analogous to time-preference, and the real part of the eigenvalues are analogous to the exponential discount rate. Modelling time-discounting using mechanical system dynamics therefore results in a time-discounting model based on economic first principles.

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Chapter 1

Introduction

Philosophers, economists, psychologists, and policymakers have discussed the topics preceding time-discounting throughout history [6]. Socrates speaks of man 'ruling over himself' in Plato's *Gorgias*: "that a man should be temperate and master of himself, and ruler of his own pleasures and passions" [7, p. 284]. Adam Smith linked this quality to the success of national economies in *The Wealth of Nations* in 1776 [8]. Psychologist George Ainslie invented the term "Picoeconomics" in 1992 to describe economics within the individual and its relation to consumption behaviour [9].

1-1 Time-Discounting and Economic Engineering

The field of behavioural economics is committed to studying how consumers make real-world choices. Behavioural economists develop predictions about people's choices, and study the discrepancies with the purely "rational" consumer [3, p. 566]. Behavioural economists study time-discounting to understand how people make decisions across time, called intertemporal choices [10].

Time-discounting and time-preference are fundamental concepts in behavioural economics [10, 11, 12, 6]. Academic interest these topics has continuously increased over the past decades (see Figure 1-1). Time-discounting is the present valuation of utility to be received from future consumption [10]. Time-preference is a measure for the preference for sooner rewards over delayed rewards [10, 13]. The concepts time-discounting and time-preference are further demonstrated in Section 2-1.

Behavioural economists have thus far been unable to reach consensus on a model for time-discounting [6, 14]. Currently over 20 different functions have been proposed and regressed on empirical data [15]. However, unaccountable anomalies are found for each of these discount functions [10, 16]. Section 2-2 and Section 2-3 demonstrate two discount functions that dominate economic literature.

In 1949 A.W. Phillips built a hydraulic machine, called the Moniac, to model a national economy. The machine, shown in Figure 1-2, simulates a national economy by use of pipes,

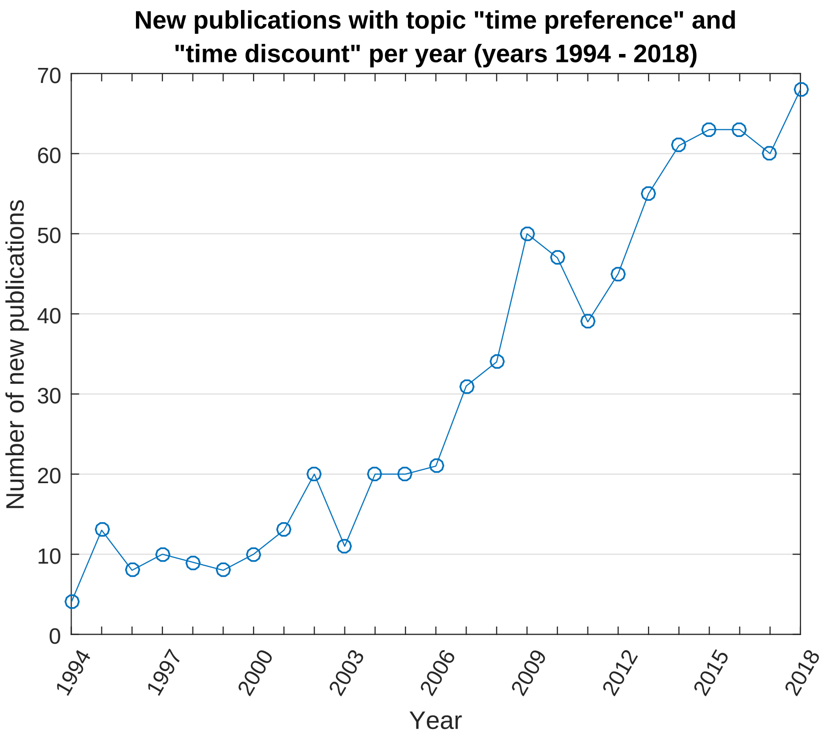

Figure 1-1: Graph illustrating the rising interest in time-preference and time-discounting; data retrieved from ISI Web of Knowldedge.

pumps, valves and water reservoirs. The design objective was to build an analogue computer that would solve differential equations from an economic model. The main purpose of the machine was to provide a visual understanding of the differential equations that modelled the economic processes [1, 17, 18]. The machine, in the words of Morgan and Boumans, "... remains one of the few "objects" which the history of economic science can boast, ..." [19].

The economic engineering group at the Delft Center for Systems and Control (DCSC) studies economics from the viewpoint of a system and control engineer. Economic processes and systems are analysed comparable to the Moniac: analogous mechanical system models provide insight into the dynamical behaviours of the economic processes and systems [20]. The economic engineering analogy is presented in Chapter 3.

The economic engineering analogy does not yet provide a modelling framework for time-discounting [20]. The analogy must be expanded to be able to interpret and provide a first principles model for time-discounting. The research frontier can be summarised as follows:

- Current time-discounting theories solely rely on regression models and are unable to account for empirical data.
- Existing time-discounting theories are presented as conflicting, but conclusive evidence has not been demonstrated.
- The economic engineering analogy does not provide a straightforward way to model time-discounting.

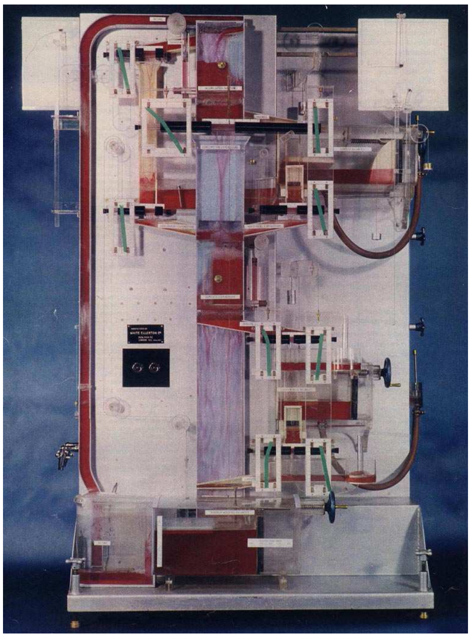

Figure 1-2: The Moniac analogue hydraulic computer. Image from Fortune Magazine's 1952 March issue [1, p. 100].

1-2 Research Effort and Contributions

The research effort is aimed at unifying economic time-discounting theories by modeling time-discounting using mechanical system dynamics. Hyperbolic discounting theory is linked to the dynamics of the overdamped second-order system in Chapter 4. In Chapter 5, exponential discounting theory is linked to the dynamics of the critically damped second-order system.

In Chapter 4 I model hyperbolic discounting theory using mechanical system dynamics. This is done as follows: In Section 4-1 a 'mechanical' discount function is constructed. This allows for a comparison to be made between economic theory and the dynamics of the analogous mechanical system. The mechanical discount function based on the dynamics of the over-damped second-order system is subsequently compared to hyperbolic and quasi-hyperbolic discounting theory in Section 4-2.

The parameters of the damped harmonic oscillator are interpreted for time-discounting in Section 4-4. The economic engineering analogy provides a first principles framework to interpret economics. I extend the analogy for time-discounting by a qualitative comparison between the parameters of mechanics and concepts from economic theory.

In Chapter 5 the dynamics of the critically damped harmonic oscillator are linked to exponential discounting theory in economics. Furthermore, an interpretation of the underdamped system is provided. In Section 5-3 I visually show how the eigenvalues of the system reveal the type of time-discounting that is present.

The contributions can be summarised as follows:

- Exponential and hyperbolic discounting theory are connected through the dynamics of the damped harmonic oscillator.

- A first principles model for time-discounting is constructed.
- The economic engineering analogy is extended to include time-discounting.

1-3 Model Predictive Control in Behavioural Economics

Consumer theory is based on the optimisation of utility under a certain endowment [3]. I develop ideas for future work in Chapter 6. A consumer is to be modelled as a model predictive controller and plant. A model for time-discounting is critical as it allows the valuation of future time-steps.

Chapter 2

Time-Discounting in Economics

This chapter introduces the concept of time-discounting in economics. Time-discounting and its mathematical application in economics are demonstrated in Section 2-1. Although many discounting methods have been proposed [15], this chapter restricts its exposition to two dominant discounting methods: exponential discounting and hyperbolic discounting. These methods form the basis of time-discounting theory [10]. Fundamental differences between these two theories are discussed. The chapter is concluded with a summary of the standing issues that behavioural economists are facing concerning time-discounting.

2-1 Time-Discounting and Time-Preference

Time-preference is a measure for the preference for sooner rewards over delayed rewards [10, 13]. A high time-preference is associated with demographics such as younger age, and lower income and education. It is also associated with impulsive behaviours such as relationship infidelity, smoking, a higher body mass index, and high credit card debt [21, 22, 23].

Time-discounting is the present valuation of a future cost or benefit [10]. Time-discounting differs from time-preference in that it is quantified by economists. This quantification is done through a discount function that relates the discount to time. Financial industries apply an exponential discount function, which is presented in Section 2-2 [15]. Regressing empirical data on various functions shows that consumers are best described by a hyperbolic discount function, which is presented in Section 2-3 [24, 25].

I define the terms ‘intertemporal choice’, ‘time-preference’, and ‘time-discounting’ to be used in this thesis:

Definition 2.1. *Intertemporal choices are choices of consumption over time.*

Definition 2.2. *Time-preference is a measure for the preference for immediate utility over delayed utility.*

Definition 2.3. *Time-Discounting is the process of determining the present valuation of a future benefit.*

Consumption is the destruction of an owned asset, such that the consumer receives utility from consumption. I define utility as a "numeric measure of a person's happiness" [3, p. 54]. I adopt the definition of time-preference as used by Frederick, Loewenstein and O'Donoghue in [10, p. 3], stated in Definition 2.2. High time-preference shows myopia of an individual, whereas low time-preference indicates an individual has a higher level of indifference for the future versus the present.

The Mathematics of Time-Discounting

A consumer faces a stream of consumption $c(t)$ over a specified time-period, $0 \leq t \leq T$. The expected utility U that is extracted from the consumption over the given time-period is then given by:

$$U = \int_0^T f(t)\, u(c(t), t)\; dt \tag{2-1}$$

Here, $u(c(t), t)$ is the instantaneous utility function that assigns a utility value $u(t)$ to the consumption stream $c(t)$ at every time t. Time-discounting is carried out by multiplying the instantaneous utility function with a discount function $f(t)$. The discount function assigns a weighting value, a discount factor, that depends on the time-distance between a future time t and the present time instant $t = 0$ [10, 26].

It is assumed that when an individual is faced with a choice between differing consumption plans $c_1(t)$, $c_2(t)$, ... , $c_N(t)$ at time $t = 0$, he will choose the consumption plan with the highest value of expected utility, $\max(U_1, U_2, \ldots, U_N)$. This is in essence, the dynamic utility maximization problem that defines the problem of intertemporal choice [26].

The focus of this thesis is the time-discounting process. Time-discounting is the determination of the present value of a future benefit (Definition 2.3). The mathematical representation of time-discounting is the argument of the integral in Equation 2-1 [5, 10]. The present valuation PV from consumption at time τ within the time-period $0 \leq t \leq T$ is given by:

$$PV(c(\tau)) = f(\tau)\, u(c(t), \tau) \tag{2-2}$$

Time-discounting is not restricted to consumption planning alone. For example, to calculate the Net Present Value (NPV) of a bond, future payments are time-discounted and added up to determine its market price [3]. Another example is the Discounted Cash Flow (DCF) method to valuate a business, by adding together time-discounted expected future profits over a set time-horizon [27]. I simplify Equation 2-2 so that both consumption discounting and financial discounting are included in a general equation:

$$X(t) = f(t)\, X(0) \tag{2-3}$$

The present valuation of a future economic event $X(t)$, be it consumption, a payment, profit or otherwise, is equal to the time-discounted valuation of the event taking place at the present $X(0)$. Here, $f(t)$ once again represents the discount function.

The question that remains is: What does the discount function look like? This question has extensively been researched by Strotz [26] and Koopmans [28]. Among other postulates, deductions, and definitions, the bounds of the function are defined: $f(t) \in [0, 1]$ where

$t \in [0, \infty)$. Over 20 different discount function have been proposed; [15] presents an overview and analysis. In the following sections, I present the exponential discount function and the hyperbolic discount function [14].

2-2 Exponential Discounting

In 1937 Paul Samuelson proposed the first discount function; this has come to be known as the exponential discount function [12]. In his paper *A Note on the Measurement of Utility* the discount function was presented as an assumption to understand utility measurement. The exponential discount function has however been widely adopted, and remains the standard time-discounting method in financial industry [10, 15].

The exponential discount function is characterised by a constant discount rate. It is formulated in Equation 2-4, and a graph of the function is shown in Figure 2-1.

$$f(t) = e^{-rt} \tag{2-4}$$

The instantaneous, continuous-time, discount rate is denoted by r, and is measured in per time-units 1/[time]. The instantaneous discount rate is calculated from discrete-time rates by $r = \ln(1 + d)$, where d denotes the discrete-time discount rate in per time-unit. An example application is carried out in Example 2.1.

Example 2.1. *An individual's discount rate for an expected transaction is estimated to be at* 10 % *per year (due to interest, opportunity cost, risk, etc.). Using Equation 2-4, this individual's exponential discount function for the expected transaction is thus (displayed in Figure 2-1):*

$$f(t) = e^{-\ln(1+0.1)\,t}$$

Payment of a thousand euros is expected in six months (half a year). The exponentially discounted present value (PV_{exp}) of the transaction is then calculated using Equation 2-3.

$$PV_{exp} = X_{0.5} = f(0.5)\,X_0 = e^{-ln(1.1)\cdot 0.5} \cdot 1000 = 953,46 \text{ €}$$

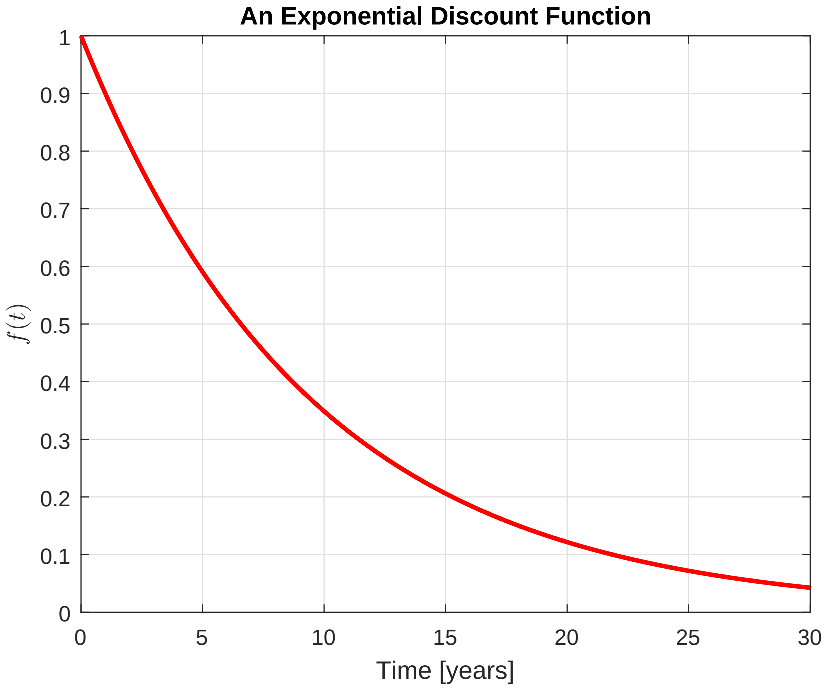

Figure 2-1: An exponential discount function discounting at a rate of $10\ \%$ per year

The exponential discount function has gained a lot of popularity because of its simplicity [10]. It is equivalent to the exponential radio-active decay function, and is indifferent to the near future versus the time-instants farther away. In other words, it time-discounts linearly in time. This makes the function particularly suitable for financial industries as many institutions strive to be indifferent to time [15].

2-3 Hyperbolic Discounting

Empirical research shows that humans and animals are not indifferent to time [24, 25]. This means that the exponential discount function is not a suitable function to model consumer behaviour. A discount function that deals with this critique is the hyperbolic discount function [10].

The hyperbolic discount function is characterized by a steep initial discount and a flat discount in the far away future. The hyperbolic discount function is given in Equation 2-5, and a graph of the function is shown in Figure 2-2 [3, p. 575]

$$f(t) = \frac{1}{1+kt} \tag{2-5}$$

Here, the hyperbolic discount rate is denoted by k, and is measured in per time-units $1/[\text{time}]$. As opposed to the exponential discount rate (where a certain percentage is discounted over time), the hyperbolic discount rate does not provide an intuitive meaning. An example application of the hyperbolic discount function follows in Example 2.2.

Example 2.2. *Similar to Equation 2.1 a future payment is expected. The hyperbolic discount rate for this individual is estimated from historical data on similar transactions. The rate is set at* 0.25 *per year. Using Equation 2-5, the hyperbolic discount function for this individual and transaction is thus (displayed in Figure 2-2):*

$$f(t) = \frac{1}{1 + 0.25\,t}$$

Payment of a thousand euros is expected in six months (half a year). The hyperbolically discounted present value (PV_{hyp}) of the transaction is then calculated using Equation 2-3.

$$PV_{hyp} = f(0.5)\,X_0 = \frac{1}{1 + 0.125} \cdot 1000 = 888,89\ €$$

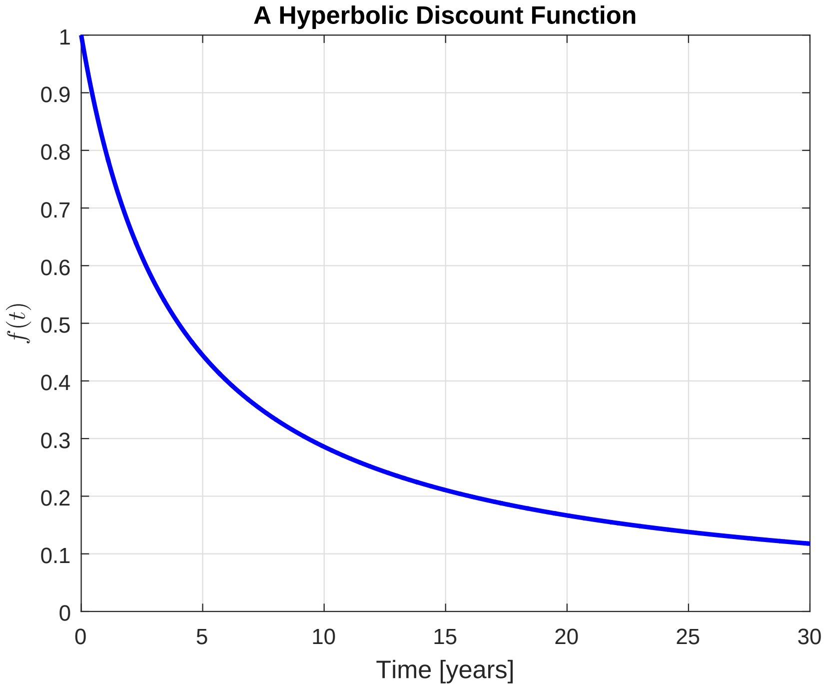

Figure 2-2: A hyperbolic discount function with a discount-rate of 0.25 per year

Hyperbolic discounting is described by economists as time-discounting with a diminishing discount rate. In Figure 2-2 we can observe that the initial discount rate is steep, and flattens out as time continues into the future.

2-4 Comparing Exponential and Hyperbolic Discount Functions

Two discount functions - an exponential and a hyperbolic functions - are plotted in Figure 2-3. The exponential discount function has a yearly (discrete-time) discount rate of 10 % per year; the hyperbolic discount function has a continuous-time hyperbolic discount rate of 0.25 per year. The discount rates have been chosen arbitrarily to make a qualitative comparison between the two functions.

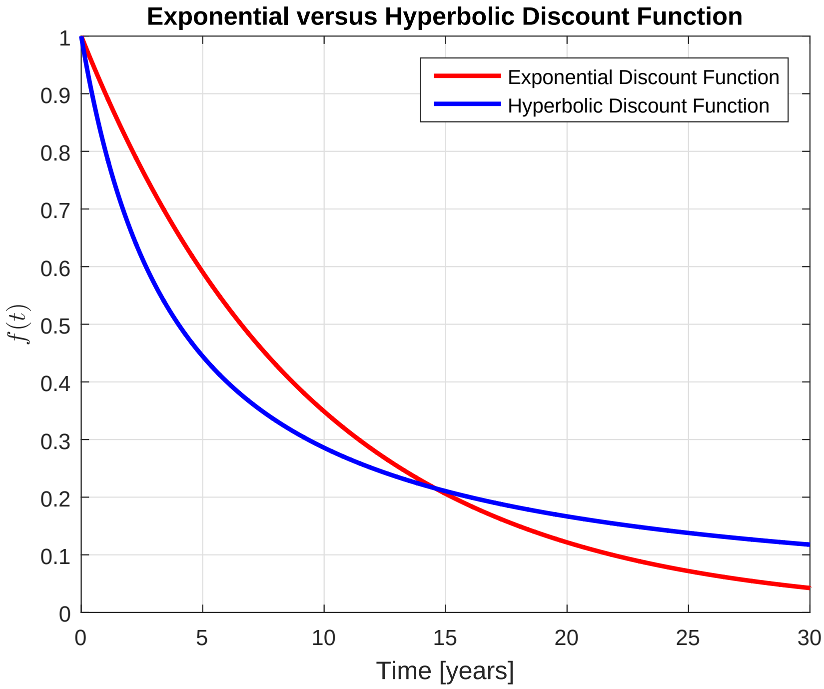

Figure 2-3: An illustrative example of the differences in characteristics between exponential and hyperbolic discount functions

It can be observed that the hyperbolic discount function discounts at a declining rate. Initially, it discounts much faster than the exponential discount function. This also becomes clear from the quantitative examples in Examples 2.1 and 2.2. The hyperbolically discounted present value is much lower than the exponentially discounted present value. It can be observed from Figure 2-3 that hyperbolic discounting assigns higher value to long-term events (longer than 15 years) than exponential discounting.

Integrating the exponential and hyperbolic discount functions over limitless time-horizon, leads to an interesting insight. Whereas the integral converges for the exponential discount function, it goes of to infinity for the hyperbolic discount function (see Equation 2-6).

$$\int_0^\infty e^{-r\,t}\,dt = \frac{1}{r} \qquad\qquad \int_0^\infty \frac{1}{1+kt}\,dt = \infty \tag{2-6}$$

As the hyperbolic discount function discounts at very low rates at time-instants far away, and its integral does not converge, economists argue that the hyperbolic discount function should be used when valuating intergenerational policies like pensions and climate-policy [29].

Another discount function has been developed to bridge between the exponential and hyperbolic discount function. The method is known as '$\beta - \delta$' discounting, or 'quasi-hyperbolic' discounting, and is discussed in Appendix A. This function was first proposed in 1968 by Phelps and Pollak [30], and has gained popularity through the work of David Laibson in 1997. The discount function is a modification of the discrete-time exponential discount function with a constant discount rate. It mimics the hyperbolic discount function, but remains linear in time [5].

2-4-1 Dynamic Consistency

There exists a qualitative difference between the two time-discounting methods in economic terms — economists call this dynamic (in)consistency [10, 26]. The concept is demonstrated at the hand of Example 2.3.

Example 2.3. *This example illustrates the concept of dynamic consistency (copied from [25]).*

1. *Choose between:*
   (a) *One apple today.*
   (b) *Two apples tomorrow.*
2. *Choose between:*
   (a) *One apple in one year.*
   (b) *Two apples in one year plus one day.*

A consumer makes a decision for both problems. If given the possibility to alter his decision in exactly one year, the consumer will stick with his initial plan if he time-discounts exponentially. The time-difference between options *(a)* and *(b)* is equal. Under a constant exponential discount rate, this will result in exactly the same decision. Therefore, economists call this consumer dynamically consistent [25].

Dynamic-inconsistency occurs when different options are chosen for both problems. When applying a hyperbolic discount method, the consumer might valuate an apple at the present to be worth more than two apples the next day. The declining discount rate will however always result in the choice for *(b)* in problem *2*. Redeciding in one year might result in a change of choice, which makes the consumer dynamically inconsistent by the standard of economists [26].

As dynamically inconsistency is observed in actual behaviour, economists assume that consumers apply a type of hyperbolic discount method [25].

2-5 Standing Issues in the Study of Time-Discounting

The exponential discounting method is an inadequate model to describe behaviour [10, p. 14]. Empirical data shows that exponential discount rates are not constant, but are in fact declining. This is captured in a hyperbolic discount function by economists [24].

The hyperbolic discount function deals with critique on the exponential discount function [10, 26]. Experiments on pigeons results in data that shows a strong correlation with a hyperbolic discount function [24]. The exponential and hyperbolic discount functions have been compared in further trials; results speak in favour of the hyperbolic discount function for consumer behaviour [31].

Exponential discounting is unable to explain dynamically inconsistent behaviour, because of its constant discount rate. Hyperbolic discounting is however able to explain and predict dynamic-inconsistency.

Measuring time-discounting is a complicated task. Experimental data is heavily influenced by the type of elicitation method used, and besides that real-world data is hard to collect [29, 32]. The estimations of the discount function therefore vary widely among studies [10]. As a consequence, predictions based on these estimations result in poor performance compared to real-world behaviour.

Economists have thus far been unable to reach consensus on a general model for time-discounting. Over 20 different discount functions have currently been proposed [15], but many economists find empirical data to be anomalous with their predictions based on these functions [10, 16]. Therefore, new methods should be developed to unify the existing theories on time-discounting.

Chapter 3

The Economic Engineering Analogy

The philosophy of approach of this thesis in presented in 3-1. The mathematical similarity between the dynamics of a first-order system and exponential discounting is demonstrated. This results in various questions that are answered by the economic engineering analogy. An exposition of the economic engineering analogy follows. The chapter concludes with the research frontier of the economic engineering framework.

3-1 Philosophy of Approach: Exponential Discounting as First-Order Dynamics

Exponential time-discounting function shows similarity with the radio-active decay function. The philosophy of approach is to understand time-discounting as for example, radio-active decay. Where the radio-active decay is analogous to the depreciating value over time.

Exponential time-discounting is captured by the following equation (see Section 2-2):

$$X(t) = e^{-rt}\,X(0) \tag{3-1}$$

Where r is the exponential discount rate; $X(t)$ is the present valuation of the economic event at time t; $X(0)$ is the valuation of the economic event if it were to take place instantaneously.

Equation 3-1 resembles the solution of a first-order differential equation. In this section I investigate if, and how, this mathematical resemblance can lead to a useful mechanical analogue. I start the investigation with a first-order mechanical system, consisting of a spring and a damper.

Figure 3-1 depicts a first-order mechanical system. It consists of a linear spring and a linear damper where the spring force F_{spring} and the friction force F_{friction} (exerted by the damper) are given by Equation 3-2. The terms and their corresponding units are summarised in Table 3-1.

$$F_{\text{spring}} = k\,x \qquad\qquad F_{\text{friction}} = b\,\dot{x} \tag{3-2}$$

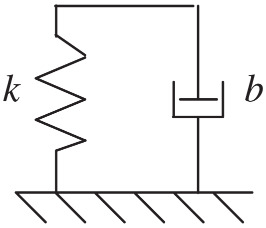

Figure 3-1: A first-order mechanical system, consisting of a linear spring and a linear damper (modified from [2, p. 223])

x	generalised position	$[m]$	k	spring constant	$\left[\frac{\text{kg}}{s^2}\right]$
$\dot{x}$	generalised velocity	$\left[\frac{m}{s}\right]$	b	damping coefficient	$\left[\frac{\text{kg}}{s}\right]$
F	force	$\left[\frac{\text{kg}\,m}{s^2}\right]$			

Table 3-1: Relevant terms system Figure 3-1

The equation of motion of the system in Figure 3-1 — a first-order spring-damper system without external forcing — is given by Equation 3-3 [2].

$$b\,\dot{x} + k\,x = 0 \tag{3-3}$$

From the equations of motion, the solution for $x(t)$ can be derived. The solution is given in Equation 3-4 [2, p. 224].

$$x(t) = x(0)\,e^{-\frac{k}{b}\,t} \tag{3-4}$$

By comparing Equation 3-1 and Equation 3-4 we can recognise that the function $x(t)$ is analogous to the exponential discounting process. $x(t)$ is the exponentially discounted valuation of the event that is t time-units in the future. $x(0)$ is the valuation of the event occurring at the planning instant. We can further conclude that the eigenvalue of the mechanical system is analogous to the exponential discount rate. This means that the exponential discount rate r is analogous to k/b [2].

The mechanical system can be interpreted as a discounting system as follows: The system is initialised with an instantaneous valuation $x(0)$, and the time is set to $t = 0$ when time-discounting is applied. The present valuation PV of the event occurring in the future is then equal to the solution $x(t)$ that follows from the equations of motion. This can be seen as a form of mental accounting, where a consumer takes the influence of time into consideration for his valuation [33].

By analysing the exponential discount function through the mechanical system, the discount rate has been decoupled into a spring- and a damping element. To successfully develop a theory, answers to the following questions are however required:

- What are the economical analogues of the generalised position x, and the generalised velocity $\dot{x}$?

- What are the economical analogues of the constituents of the mechanical system (the spring coefficient k and the damping coefficient b)?

3-2 The Economic Engineering Framework

The economic engineering framework extends the mapping between the electrical and mechanical domain to the economic domain. This allows the application of modelling and control techniques from engineering science [20]. I discuss the following mechanical phenomena of the mechanical-economic analogy: inertia, force, and energy. These phenomena are relevant to the understanding of time-discounting.

3-2-1 Inertia Follows from Demand

The economic engineering framework is built around the economic analogies of generalised position, generalised velocity, and momentum. The position variable q is defined as the asset stock, in this thesis handled using the generic unit [#]. In practice, the asset stock is measured either in number of units, meters of textile, or kilograms of resource. The generalised velocity is analogous to a flow of assets; positive flow increases the asset stock (out of production or purchases), negative flow decreases the asset stock (because of consumption or sales). Economic engineering defines price as analogous to momentum. Price is measured in euros per unit [€/#]. Table 3-2 presents an overview of the central assumptions of economic engineering [20].

	Mechanics		Economics	
q	position	m	asset	$\#$
$\dot{q}$	velocity	$\frac{m}{s}$	asset flow	$\frac{\#}{\text{yr}}$
p	momentum	$\frac{\text{kg}\, m}{s}$	price	$\frac{€}{\#}$

Table 3-2: Central assumptions economic engineering

In classical mechanics, the linear momentum p is a function of velocity; $p = p(\dot{q})$. The momentum is defined as the product of particle mass and its velocity, see Equation 3-5 [20, 34].

$$p = m\dot{q} \tag{3-5}$$

In economics, price and flow are connected through demand. Graphically, this is presented through the demand curve, see Figure 3-2. The demand curve links the quantity demanded to the reservation price. The reservation price is a consumers' maximum willingness to pay, per unit [3]. Economic engineering treats the quantity demanded as a flow of goods (quantity demanded over time $\dot{q}$) [20]. As a consequence, the reservation price is analogous to momentum $p(\dot{q})$.

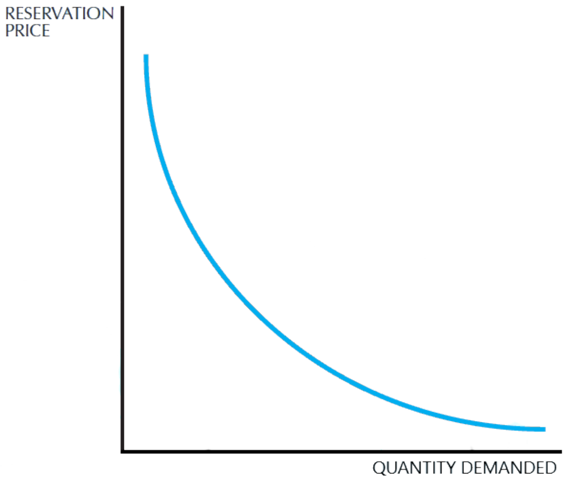

Figure 3-2: A typical demand curve found in economics textbooks (modified from [3, p. 6])

A central assumption in Newtonian mechanics is that mass is constant [35]. Taking into account Equation 3-5, this implies that mass of a particle is equal to the change in momentum divided by the velocity change:

$$m = \frac{\partial p}{\partial \dot{q}} = \frac{dp}{d\dot{q}} \tag{3-6}$$

The demand curve in Figure 3-2 violates the equality in Equation 3-6. To ensure a constant mass in the economic analogy, I infer Assumption 3.1 throughout this thesis. This assumption is central in economic engineering [20].

Assumption 3.1. *The demand curve is linear; the slope of the demand curve is thus constant.*

Figure 3-3 depicts a linear demand curve. The downward sloping line implies a negative value for "economic mass"; this is a misinterpretation however. When an exchange of trade occurs, the supply price and the demand price have to be in opposite directions to fulfil Newton's third law of motion [36]. Furthermore, the velocities must also run in opposite directions to correspond to either sales or purchases. These considerations are ignored in economics, as these quantities are not treated as vectors in economics. Economic engineering does include these considerations [20]. Table 3-3 formalises the definition of mass in the mechanical-economic framework.

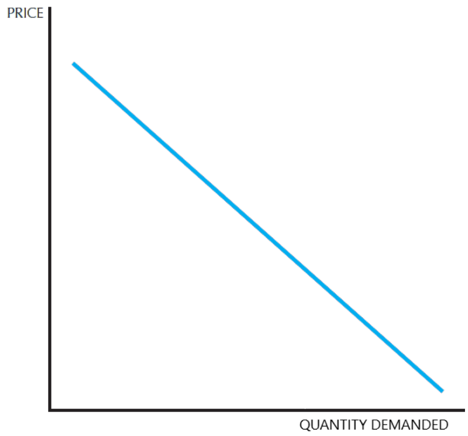

Figure 3-3: A linear demand curve (modified from [3, p. 271])

	Mechanics		Economics	
m	mass	kg	slope of the demand curve	$\frac{€\,\mathrm{yr}}{\#^2}$

Table 3-3: Mass is analogues to the slope of the demand curve

3-2-2 Force as a Cost

Force is known as a cost in the economic engineering. Costs are measured in monetary units per asset per time-unit. Acceleration is analogous to the time-derivative of the asset flow [20].

The total force acting on a particle is defined as the rate of change of its momentum. This is known as Newton's second law of motion. As mass is constant in both classical mechanics and in the mechanical-economic analogy, force is equal to the product of mass m and acceleration, see Equation 3-7 [34, 36].

$$F = \dot{p} = \frac{d}{dt}(m\dot{q}) = m\ddot{q} \tag{3-7}$$

The cost and acceleration analogy is summarised in Table 3-4.

	Mechanics		Economics	
$\ddot{q}$	acceleration	$\frac{m}{s^2}$	flow change	$\frac{\#}{\mathrm{yr}^2}$
F	force	$\frac{\mathrm{kg}\,m}{s^2}$	cost	$\frac{€}{\#\,\mathrm{yr}}$

Table 3-4: Force in economic engineering

Springs and Dampers

The economic engineering analogues of springs and dampers are rent and depreciation respectively. The analogue of the spring constant k is the rental rate. The damping coefficient b is analogous to the depreciation coefficient [20].

The first-order system in Section 3-1 consists of a linear spring and a linear damper. A spring is a store of distance, a damper dissipates energy from movement [20].

$$F_{\text{spring}} = k\,q$$
$$F_{\text{friction}} = b\,\dot{q}$$

The spring force is interpreted as a storage cost or the benefit of owning an asset (depending on the sign). A damper is interpreted as dissipating value from a closed economic system. For example, a sales tax dissipates value from the system only if a flow of goods $\dot{q}$ is present [20]. Table 3-5 summarises.

The rental rate k and depreciation coefficient b are not measurable quantities in economics. This limits the application of the Newtonian method to model to real-world economic systems using the economic engineering analogy [20].

	Mechanics		**Economics**	
k	stiffness	$\frac{\text{kg}}{s^2}$	rental rate	$\frac{€}{\#^2\,\text{yr}}$
b	damping coefficient	$\frac{\text{kg}}{s}$	depreciation coefficient	$\frac{€}{\#^2}$

Table 3-5: Mechanical elements in economic engineering

3-2-3 Energy as a Cash Flow

A damped second-order mechanical system exhibits three types of energy: kinetic energy, potential energy, and heat. In economics these represent various types of cash flows. They are discussed in this section.

The energy of a mechanical system is defined as the time-derivative of its force multiplied with its velocity, see Equation 3-8 [2]. Cash flow is the economic analogue of energy, and is measured in monetary units per year, see Table 3-6 [20].

$$E \equiv \int F\,\dot{q}\,dt \tag{3-8}$$

	Mechanics		**Economics**	
U	energy	$\frac{\text{kg}\,m^2}{s^2}$	cash flow	$\frac{€}{\text{yr}}$

Table 3-6: The economic engineering energy analogy

Kinetic Energy

The definition of kinetic energy follows from expanding Equation 3-8, see Equation 3-9 [2, p. 21]. It becomes apparent that the kinetic energy is defined by the variation of the momentum, or the price in economics [20].

$$E_{\text{kin}} = \int F\,\dot{q}\,dt = \int \frac{dp}{dt} \cdot \dot{q}\,dt = \int \dot{q}\,dp \tag{3-9}$$

The economic analogue of kinetic energy is consumer surplus. Consumer surplus is the total utility received minus the nominal price value of the good [3, ch. 14]). Figure 3-4 depicts a demand curve. The curve represents a consumers price valuation of the good per flow quantity. At price p, the surplus from consumption is then represented by the shaded area in Figure 3-4.

Figure 3-4: Kinetic energy is analogous to the surplus from consumption (modified from [3, p. 256])

Standard economic practice is to measure the return of consumption in utility units, named utils [3, ch. 4]. However, money metric utility measures utility in monetary units. Where one util is equal to one monetary unit. Therefore, consumer surplus is a virtual cash flow (the extra utility gained from consumption, measured in monetary units) [37, ch. 10].

Potential Energy

Potential energy is stored in distance q. Expanding Equation 3-8 leads to the representation of the potential in Equation 3-10. Example 3.1 provides intuition on the cash flow that is analogous to potential energy.

$$E_{\text{pot}} = \int F\,\dot{q}\,dt = \int F_{\text{spring}}\,dq \tag{3-10}$$

Example 3.1 (Parking Delft City Centre)**.** *A parking permit in the city centre of Delft costs* 165 €/*yr for the first car. A permit for a second car and above will cost the resident* 474 €/*yr per car [38].*

Summarising this information in the economic engineering framework results in the following equation for the cost F_{spring}*:*

$$F_{spring}(q) = \begin{cases} 165\ \frac{€}{\#\,yr} & for\ 0 < q \leq 1 \\ 474\ \frac{€}{\#\,yr} & for\ q > 1 \end{cases}$$

The cash flow to be paid for having two parking permits in the city center of Delft then follows from Equation 3-10 (where the integration takes place over the two parking spots q).

$$E_{pot} = \int_0^2 F_{spring}\, dq = 639\ \frac{€}{yr}$$

Heat

Friction forces result in dissipation in a mechanical system [36]. The energy exits the system in the form of heat, and is directly dependent on the velocity, see Equation 3-11 [20].

$$Q = \int F_{\text{damper}}\, \dot{q}\, dt \tag{3-11}$$

Depreciation is the economic engineering analogue of dissipation. Heat is then analogous to the depreciated value [20].

3-3 Research Frontier of Economic Engineering

Economic engineering provides a basis to model economic phenomena and processes using engineering modelling methods. Previous theses within the economic engineering group have modelled the labour market, leasing companies, and monetary policy [39, 40, 41]. Others have researched economic applications of Lagrangian and Hamiltonian mechanics [42, 43].

The economic engineering framework does not provide a straightforward method to model time-discounting, even though it answers the questions that have come forward in Section 3-1. The following points need to be addressed:

- The economic engineering framework must find a way to reproduce existing discount functions.
- The economic engineering analogies k and b as the rental rate and depreciation coefficient do not represent economic quantities. Example 3.1 uses the analogue of a spring force as it represents an economic quantity, namely the price of a parking place per year. Defining time-discounting will thus require different parameters that allow a practical economic interpretation.

Chapter 4

Time-Discounting as Mechanical System Dynamics

This chapter applies the economic engineering framework the hyperbolic time-discounting theory. A mechanical discount function is proposed in Section 4-1 to make the economic engineering framework compatible with economic theory. This allows a comparison between the economic engineering approach and economic theory. The overdamped second-order system exhibits hyperbolic behaviour that can be seen in hyperbolic discounting; this is investigated in Section 4-2.

The economic engineering framework is limited as its analogues for the spring constant and damping coefficient are not measurable quantities in economics. In Section 4-3 the dynamics of the damped harmonic oscillator are proposed as the economic engineering representative for time-discounting. This allows the various parameters of the damped harmonic oscillator to be interpreted economically, which is done in Section 4-4. Section 4-5 concludes.

4-1 A Mechanical Discount Function Based on System Dynamics

Economists regress empirical data on discount functions [44]. The economic engineering analogy reproduces economic data through dynamical system models [20]. I synchronise the two approaches by constructing a 'mechanical' discount function based on system dynamics.

The time-discounting process is governed by the following equation (see Chapter 2):

$$X(t) = f(t)\,X(0) \tag{4-1}$$

X(t) is the present valuation of an economic event in the future. X(0) is the valuation of the same event occurring at $t = 0$. The discount function $f(t)$ relates the two. Rearranging this equation results in:

$$f(t) = \frac{X(t)}{X(0)} \tag{4-2}$$

Momentum in physics is analogous to the reservation price economics (see Chapter 3). Under the assumption of money metric utility the present valuation $X(t)$ is analogous to momentum $p(t)$. Therefore, the mechanical discount function for the economic engineering framework is:

$$f_{\text{mech}}(t) \equiv \frac{p(t)}{p(0)} \tag{4-3}$$

The mechanical discount function is defined as the dynamics of the reservation price (all valuations of the economic event taking place in the future) divided by the reservation price at $t = 0$. Or simply put: the normalised reservation price.

4-2 Hyperbolic Discounting as Overdamped Second-Order Dynamics

The mechanical second-order system is depicted in Figure 4-1. The system consists of a mass, spring, and a damper. The economic engineering analogies of these elements are the price elasticity, rental rate, and depreciation constant respectively (see Chapter 3.

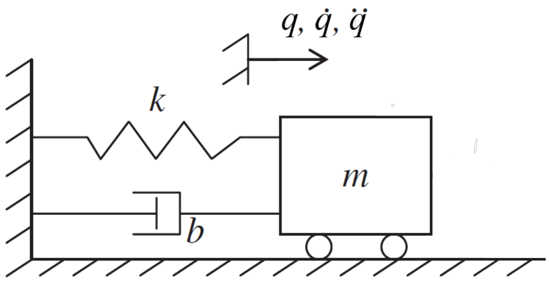

Figure 4-1: The second-order mechanical system (modified from [2, p. 246])

The system dynamics are captured by the equation of motion. The equation of motion of the second-order system is presented in Equation 4-4 [2].

$$m\,\ddot{q} + b\,\dot{q} + k\,q = 0 \tag{4-4}$$

The progression of the reservation price $p(t)$ is derived from the equation of motion. The derivation is carried out in Appendix B. The progression of the reservation price is given in Equation 4-5.

$$p(t) = e^{-\beta t}\left(p_0 \cosh\left(\omega_{\text{od}}\, t\right) + \left(m\, q_0\, \omega_{\text{od}} - \beta\, \frac{m\, \beta\, q_0 + p_0}{\omega_{\text{od}}}\right) \sinh\left(\omega_{\text{od}}\, t\right)\right) \tag{4-5}$$

For ease of notation β is defined in Equation 4-6. Furthermore, the overdamped frequency ω_{od} is defined in Equation 4-7. The initial conditions are given by $q(0) = q_0$ and $p(0) = p_0$ respectively.

$$\beta \equiv \frac{b}{2m} \tag{4-6}$$

$$\omega_{\text{od}} \equiv \sqrt{\beta^2 - k/m} \tag{4-7}$$

From Equation 4-5, it can immediately be observed that hyperbolic effects are present in the system. Both the hyperbolic sine and hyperbolic cosine are found in the equation. This strengthens the hypothesis that the overdamped second-order system can be used as an analogue for hyperbolic time-discounting.

It can be recalled from Section 2-3 that the hyperbolic discount function is given by Equation 4-8, where k is the hyperbolic discount rate. The function is plotted for various values of the discount rate in Figure 4-2.

$$f_{\text{hyp}}\left(t\right) = \frac{1}{1 + k\,t} \tag{4-8}$$

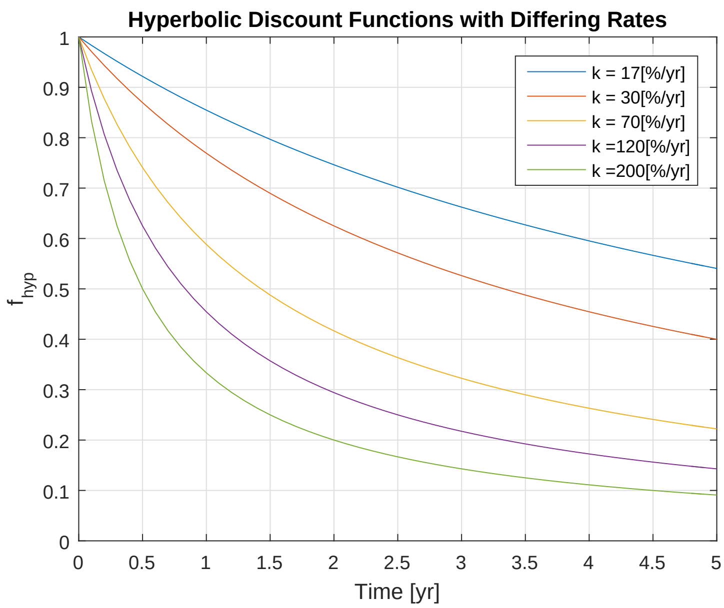

Figure 4-2: Hyperbolic discount functions with differing hyperbolic discount rates

The proposed mechanical discount function is plotted against the hyperbolic discount function in Figure 4-3. The system parameters have been chosen such that the system is overdamped ($\beta > k/m$ [2]). The system parameters are:

- $m = 1\ \frac{€\,\text{yr}}{\#^2}$
- $b = 10\ \frac{€}{\#^2}$
- $k = 5\ \frac{€}{\#^2\,\text{yr}}$

The initial reservation price is set at $p_0 = -3\ \frac{€}{\#}$. The initial asset stock is set at $q_0 = 3\ \#$. The hyperbolic discount rate for the economic discount function is set at 200 %/yr.

We observe in Figure 4-3 that the mechanical discount function initially discounts quickly, and then transitions into a slower discount around $t = 0.4$ yr. This can be explained by the

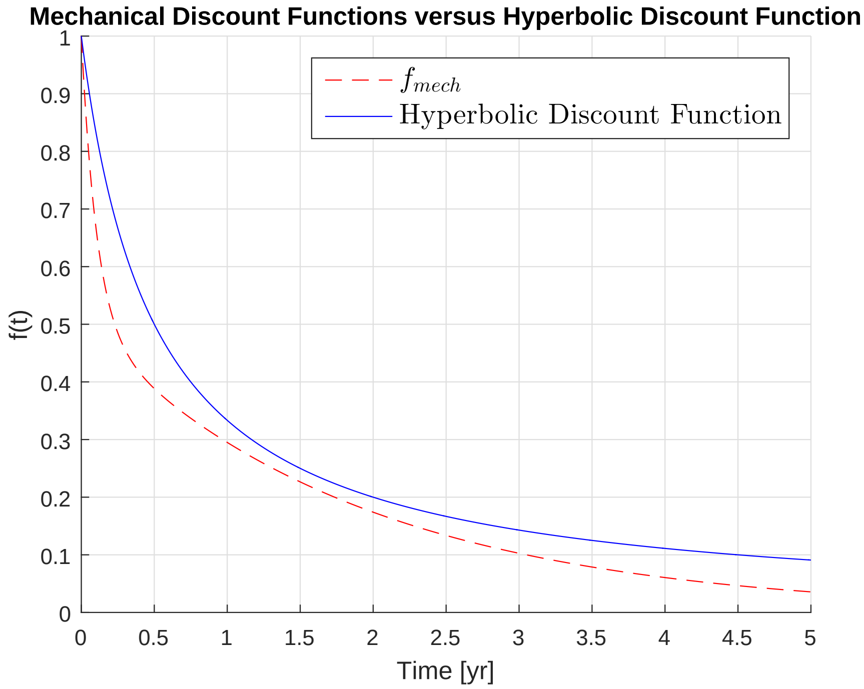

Figure 4-3: The proposed mechanical discount function versus the hyperbolic discount function

fact that the overdamped second-order system has a fast and a slow eigenvalue; initially the fast eigenvalue is dominant, later the slow eigenvalue takes over.

The economic hyperbolic discount function initially discounts faster, and transitions into a slower discount later on. The transition is smoother than the mechanical discount function. This can be explained by the fact that the discounting behaviour is determined by a single discount rate, whereas the mechanical discount function relies on two different eigenvalues (one fast short term, and one slow longer term eigenvalue).

In Figure 4-4 the mechanical discount function is plotted against the $\beta - \delta$-discount function (see Appendix A) which was briefly discussed in Chapter 2. The discrete-time discount function is given in Equation 4-9. The $\beta - \delta$-function in Figure 4-4 has the following parameters: $\beta = 0.5$ (dimensionless); $\delta = 0.8$ (dimensionless); the time-step is $\Delta t = 1/2$ yr.

$$\begin{aligned} f(\tau) &= \beta\,\delta^{\tau} \qquad \text{where:} \quad \tau = \{0, 1, 2, \dots\} \\ t &= \tau \Delta t \end{aligned} \tag{4-9}$$

The discrete-time $\beta - \delta$-function discounts with a factor β in the first time-step, and additionally discounts δ for all time-steps. The β-parameter thus acts as the fast initial discount, whereas the δ-parameter is a slow discount rate that is linear in time. This further strengthens the idea that the hyperbolic time-discounting process consists of two eigenvalues (one fast, one slow).

The mechanical discount function is able to mimic both the hyperbolic discount function and the $\beta - \delta$-discount function. This is based on the hand-tuned system. Minimum variance parameter estimation will result in further convergence. The overdamped second-order system

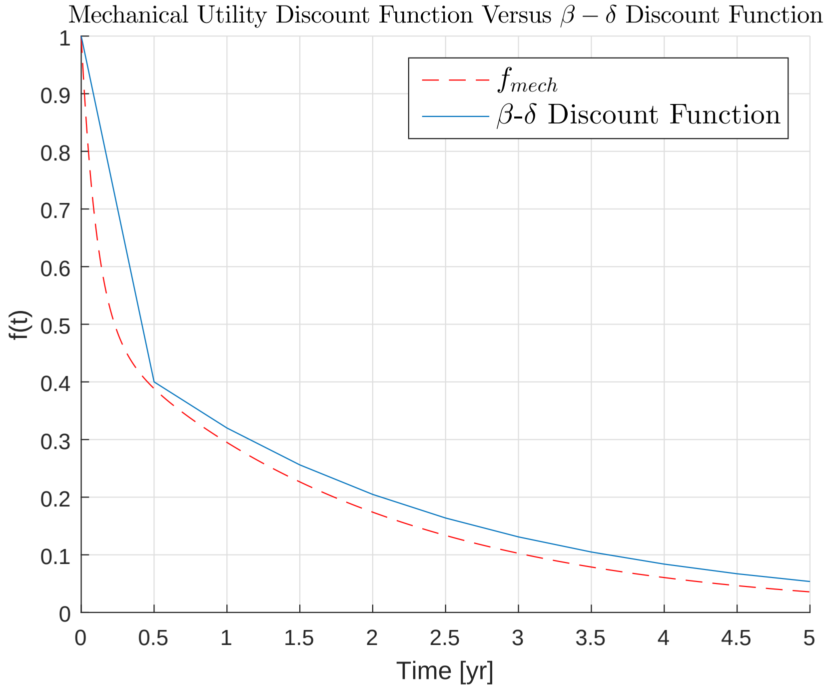

Figure 4-4: The mechanical utility discount function resembles the $\beta - \delta$ discount function

is therefore an effective analogue for hyperbolic time-discounting. By applying the economic engineering framework, time-discounting dynamics can now be provided by an analogous mechanical second-order system.

4-3 The Damped Harmonic Oscillator as a Time-Discounting System

This section addresses the following issue: The economic engineering variables b and k (the depreciation coefficient and the rental rate) are not readily available in economics. In this section, I therefore switch to the damped harmonic oscillator representation of the second-order system. Doing so allows extension of the economic engineering framework by finding the correct economic engineering analogues of the parameters of the damped harmonic oscillator. This extension is carried out in Section 4-4.

The harmonic oscillator is, similar to the second-order system, a system that moves to restore to its equilibrium position when a displacement is applied. It is of interest as the system is described through, among other parameters, its natural frequency. Economic dynamic behaviour is described in rates, which should be interpreted as frequencies [20]. The equation of motion of the damped harmonic oscillator is given in Equation 4-10 [2].

$$\ddot{q} + 2\zeta\omega_n \dot{q} + \omega_n^2 q = 0 \tag{4-10}$$

The system is described through the damping ratio ζ, and the undamped natural frequency ω_n, which is the undamped oscillation rate. The natural frequency is defined in Equation

4-11 [2].

$$\omega_n \equiv \sqrt{\frac{k}{m}} \tag{4-11}$$

The damping ratio is defined in Equation 4-12 [2]. By substituting the definitions of the damping ratio and the natural frequency into Equation 4-10, the original equation of motion of the second-order system in Section 4-2 is retrieved.

$$\zeta \equiv \frac{b}{2\,m\,\omega_n} = \frac{b}{2\,\sqrt{m\,k}} \tag{4-12}$$

The characteristic equation of the equation of motion of the damped harmonic oscillator is given in Equation 4-13 [2].

$$s^2 + 2\zeta\omega_n\,s + \omega_n^2 = 0 \tag{4-13}$$

The eigenvalues of the system follow by solving the quadratic Equation 4-13:

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2\,\omega_n^2 - \omega_n^2} \tag{4-14}$$

4-4 Extending the Mechanical-Economic Framework

In this section the economic engineering framework is extended. The damped harmonic oscillator is considered as a time-discounting system. This requires the damping ratio ζ and eigenfrequency ω_n to be linked to economic variables.

4-4-1 The Eigenvalues Provide the Exponential Discount Rate

The eigenvalues of the second-order time-discounting system system are economically interpreted in this subsection. In Section 3-1 it was shown that the eigenvalue of a first-order mechanical system was analogous to the instantaneous exponential discount rate from exponential discounting theory. This subsection investigates if a similar analogy counts for the second-order mechanical system.

The eigenvalues of the damped harmonic oscillator are given in Equation 4-14. They can be further expanded to be defined for the under-, critically-, and overdamped system respectively, see Equation 4-15.

$$s_{1,2} = \begin{cases} -\zeta\omega_n \pm i\,\omega_n\sqrt{1-\zeta^2} & \text{if } \zeta < 1 \\ -\zeta\omega_n \pm \varepsilon & \text{if } \zeta = 1 \\ -\zeta\omega_n \pm \epsilon\,\omega_n\sqrt{\zeta^2 - 1} & \text{if } \zeta > 1 \end{cases} \tag{4-15}$$

Here, i is the complex number with the property $i^2 = -1$; ε is the dual number with the property $\varepsilon^2 = 0$; and ϵ is the double number with the property $\epsilon^2 = 1$. They correspond to circular-, parabolic-, and hyperbolic geometry respectively [45].

The solution for the reservation price progression over time, takes the form of Equation 4-16; where P_1 and P_2 follow from the initial conditions of the system [2].

$$p(t) = P_1\, e^{s_1\, t} + P_2\, e^{s_2\, t} \tag{4-16}$$

The eigenvalues in Equation 4-15 have a special structure where the eigenvalues for all three domains (under-, critically-, and overdamped) are complex-, double-, or dual conjugate with equal real part. This is summarised in Equation 4-17; where y denotes either the complex-, double-, or dual number depending on the domain of the system.

$$s_{1,2} = a \pm b\, y \qquad\qquad a, b \in \mathrm{R} \tag{4-17}$$

Recognizing the structure of the eigenvalues allows the expansion of the form of the reservation price in Equation 4-16. The real part of the eigenvalues are separated from the circular-, parabolic-, and hyperbolic effects of the system, see Equation 4-18.

$$p(t) = e^{a\, t}\left(P_1\, e^{b\, y\, t} + P_2\, e^{-b\, y\, t}\right) \tag{4-18}$$

$$= e^{-\zeta\omega_n\, t} \times (\text{Second Order Effects}) \tag{4-19}$$

By doing so, the first-order effects have effectively been separated from the second-order effects of the system, see Equation 4-19. The first-order effect is one of a constant discount rate, whereas the second-order effect can result in oscillatory or hyperbolic behaviour.

From Section 3-1, we know that the eigenvalue of the first-order system is analogous to the exponential discount rate. From the structure in Equation 4-19 it becomes clear that for the second-order system the exponential discount rate is analogous to the real part of the eigenvalues. Table 4-1 summarises.

	Mechanics		**Economics**	
$\zeta\omega_n$	decay rate/envelope	yr^{-1}	exponential discount rate	yr^{-1}

Table 4-1: Exponential discount rate analogy

4-4-2 The Damping Ratio as the Time-Preference

The damping ratio ζ is economically interpreted in this subsection. The damping ratio is a dimensionless parameter that varies for underdamped- ($\zeta < 1$), overdamped ($\zeta > 1$), or critically damped ($\zeta = 1$) second-order systems. The influence of the damping ratio can be investigated by varying the value of the damping ratio while other parameters are kept constant.

Figure 4-5 shows the hyperbolic discount function for increasing hyperbolic discount rates. Figure 4-6 shows the mechanical discount function for increasing values of the damping ratio ζ. Both Figures are presented on page 30. Where $m = 5\ \frac{€\,\mathrm{yr}}{\#^2}$, $\omega_n = 0.08\ \mathrm{yr}^{-1}$, and with initial conditions $q(0) = 3$ and $p(0) = -1$. The mechanical discount function is defined as in Section 4-1: $f_{\mathrm{mech}} = \frac{p(t)}{p(0)}$.

The dynamics of the reservation price $p(t)$ for the overdamped harmonic oscillator are derived in Appendix C-2, and is given by:

$$p(t) = e^{-\zeta\omega_n t}\left(p_0 \cosh\left(\omega_{\text{od}}\, t\right) + \left(m\, q_0\, \omega_{\text{od}} - \zeta\omega_n\, \frac{m\,\zeta\omega_n\, q_0 + p_0}{\omega_{\text{od}}}\right) \sinh\left(\omega_{\text{od}}\, t\right)\right) \tag{4-20}$$

The definition of time-preference can be recalled from Chapter 2 (Definition 2.2): Time-preference is a measure for the preference for immediate utility over delayed utility. This means that an increased time-preference translates to a steeper discount function [10]. In Figure 4-5 this means that a higher time-preference is linked to a higher discount rate.

Increasing the damping ratio results in a steeper discount function, this can be observed in figure 4-5. This is effect is similar to increasing the hyperbolic discount rate, which can be concluded from Figure 4-6. It can thus be concluded that the damping ratio is in fact analogous to the time-preference of the economic system. This is summarised in Table 4-2.

	Mechanics		**Economics**	
ζ	damping ratio	$-$	time-preference factor	$-$

Table 4-2: Time-preference analogy

4-4-3 The Natural Frequency as the Risk-Free Discount Rate

The natural frequency is economically interpreted in this subsection. The critically damped system is considered for this purpose, which means that the damping ratio is set to $\zeta = 1$. This allows for a qualitative comparison between the properties of the system and economic properties.

The critically damped system will return to the equilibrium position in the shortest amount of time [46]. This means that the economic system will discount only for exogenous variables, and not for behavioural effects. The exogenous effect that needs to be discounted for is that of interest [11]. The discounted reservation price will take the following form (by substituting $\zeta = 1$ into Equation 4-19):

$$p(t) = e^{-\omega_n\, t} \times (\text{Second Order Effects}) \tag{4-21}$$

By eliminating the behavioural effects (damping ratio) from the equation it becomes clear that the critically damped system only discounts the natural frequency. The natural frequency thus compensates for the rate of interest. The discount rate that compensates for the interest rate is known as the risk-free rate, or risk-free discount rate [3]. Table 4-3 summarises.

	Mechanics		**Economics**	
ω_n	natural frequency	yr^{-1}	risk-free discount rate	yr^{-1}

Table 4-3: Natural frequency analogy

4-5 Conclusion: Economic Engineering Models Hyperbolic Discounting

The economic engineering framework has been made compatible with existing economic time-discounting theory by defining a mechanical discount function in Section 4-1. This has allowed a comparison between the overdamped second-order system as a time-discounting system and both the hyperbolic and $\beta - \delta$ discount functions in Section 4-2. From this comparison it was concluded that the overdamped second-order system is a valid representative for hyperbolic time-discounting dynamics.

To increase understanding of time-discounting within the economic engineering framework the damped harmonic oscillator was adopted in Section 4-3. This allowed for a practical economic interpretation of the parameters of the mechanical time-discounting system. This was carried out in 4-4 and is summarised in Table 4-4.

The focus of this chapter has been the application of the economic engineering framework to time-discounting theory. The straightforward method was to use the analogy between hyperbolic discounting theory and the overdamped second-order system. With known economic interpretations of the parameters of the mechanical time-discounting system, the underdamped and critically damped systems can be investigated. This will be carried out in Chapter 5.

	Mechanics		Economics	
ζ	damping ratio	–	time-preference factor	–
ω_n	natural frequency	yr^{-1}	risk-free discount rate	yr^{-1}
$\zeta\omega_n$	decay rate/envelope	yr^{-1}	exponential discount rate	yr^{-1}

Table 4-4: Extensions of economic engineering

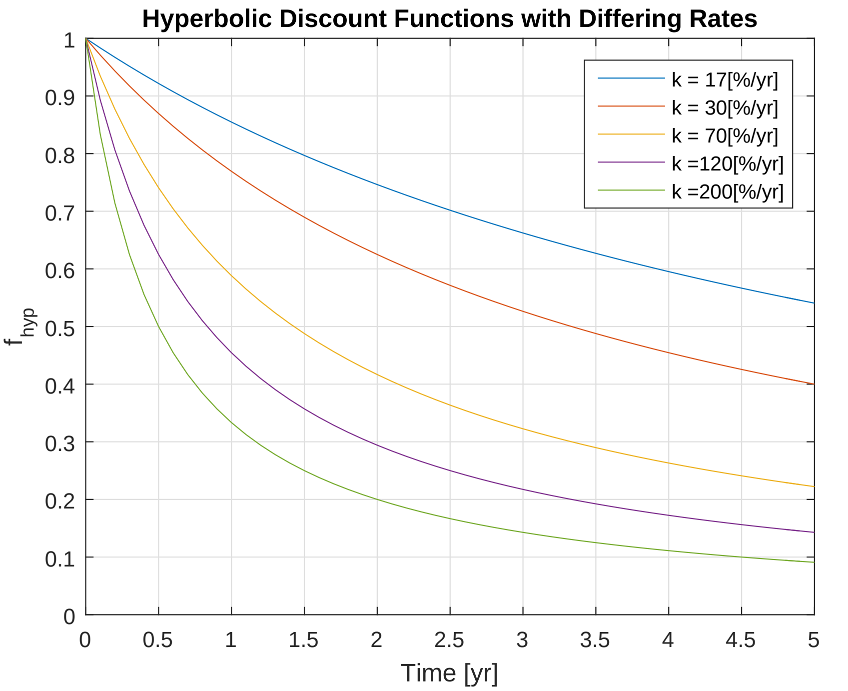

Figure 4-5: Hyperbolic discount functions with differing hyperbolic discount rates

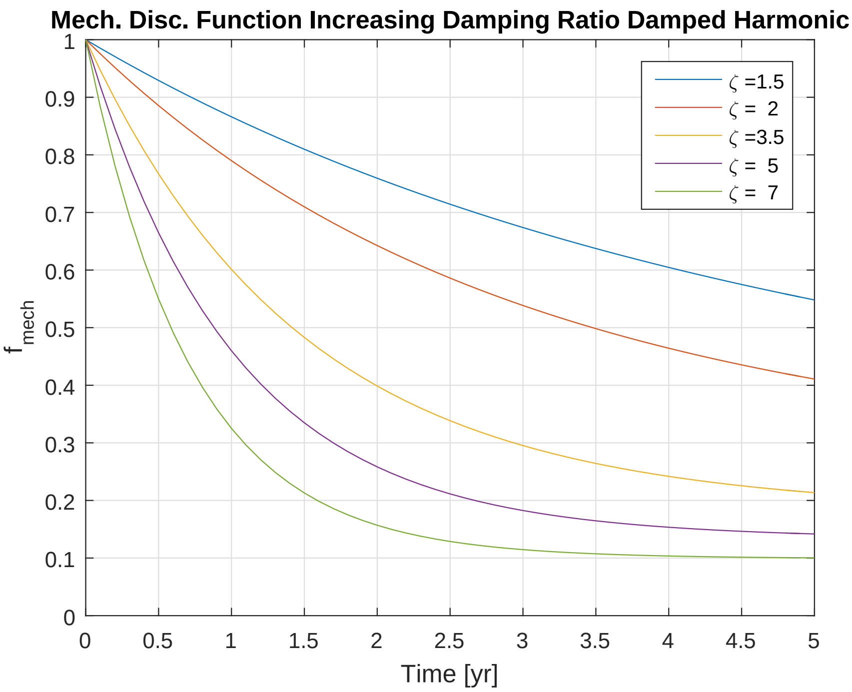

Figure 4-6: Various values for damping ratio for the damped harmonic oscillator

Chapter 5

The Damped Harmonic Oscillator as an Economic Representative

The goal of this thesis is to unify economic time-discounting theory by applying the economic engineering framework. The hypothesis is that the second-order damped harmonic oscillator is able to provide the dynamics of the discounting paths described in economic theory. The two dominant economic time-discounting theories are exponential and hyperbolic discounting [10]. The dynamics of hyperbolic discounting theory were reproduced in the previous chapter. This chapter is dedicated to the dynamics of exponential discounting, represented by the critically damped system, and the economic interpretation of the underdamped system.

Section 5-1 contains the investigation of the hypothesis that the critically damped system provides the exponentially discounted dynamics. Section 5-2 is dedicated to the economic interpretation of the remaining underdamped harmonic oscillator dynamics. Section 5-3 further investigates the role of the eigenvalues of the economic engineering time-discounting system. Section 5-4 concludes.

5-1 Exponential Discounting as Critically Damped Dynamics

A critically damped system returns to its equilibrium position along the shortest possible trajectory [46]. From Section 4-4-3 we know that this translates to time-discounting taking place without behavioural influences. Therefore, it can be stated that $\zeta = 1$ is time-preference neutral behaviour.

Financial industries apply time-preference neutral discounting [15]. Therefore, an exponential discount function is applied. The choice for this discount function relies on two qualitative properties: its simplicity and the fact that it is dynamically consistent (see Section 2-4-1) [10, 12]. This section investigates the hypothesis that the critically damped time-discounting system represents the form of time-discounting that the exponential discount function exerts.

That exponential time-discounting function is formulated as follows (see Chapter 2):

$$X(t) = e^{-rt}\,X(0) \tag{5-1}$$

The present valuation of future consumption $X(t)$ is equal to the discounted instantaneous consumption $X(0)$. The exponential discount function is given by $f(t) = e^{-rt}$, where r is the instantaneous exponential discount rate. Figure 5-1 shows the exponential discount function for differing yearly discount rates.

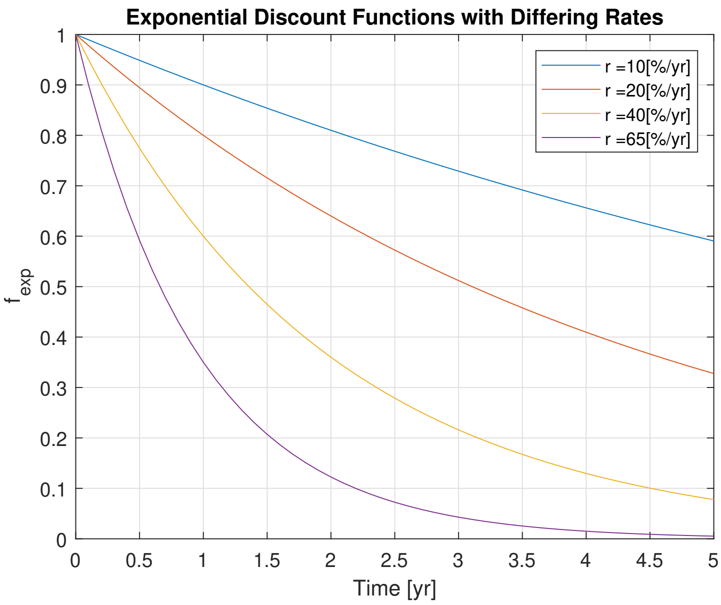

Figure 5-1: Exponential discount functions with differing exponential discount rates

The critically damped harmonic oscillator results in the following definition for the reservation price (derivation in Appendix C):

$$p(t) = e^{-\omega_n t}\Big(p_0 - \omega_n(m\omega_n q_0 + p_0)\,t\Big) \tag{5-2}$$

It can be observed that the system exponentially discounts two terms: the valuation at $t = 0$ and a time-dependent term. Comparing Equations 5-1 and 5-3 it should be noted that both are discounting the valuation at $t = 0$. The risk-free discount rate ω_n and the exponential discount rate r are not equivalent in economic definition however.

$$p(t) = \underbrace{e^{-\omega_n t}}_{\text{exp. disc.}}\Big(p_0 \underbrace{-\omega_n(m\omega_n q_0 + p_0)\,t}_{\text{2nd order effects}}\Big) \tag{5-3}$$

Figure 5-2 shows the exponential discount function plotted against the mechanical discount function from Chapter 4 ($f_{\text{mech}} = \frac{p(t)}{p(0)}$). A discount is applied to compensate for 10 % annually. The parameters and initial conditions are given in the table below.

It can be observed that the resulting discount functions are almost equivalent for the given parameters. Even though exponential discounting is a first-order process and the damped

$\omega_n = -\ln 0.9\ \frac{1}{\text{yr}}$	$q_0 = 3\ \#$
$m = 3\ \frac{€\,\text{yr}}{\#^2}$	$p_0 = 1\ \frac{€}{\#}$
$r = -\ln 0.9\ \frac{1}{\text{yr}}$	

Table 5-1: Parameters and initial conditions dynamics plotted in Figure 5-2

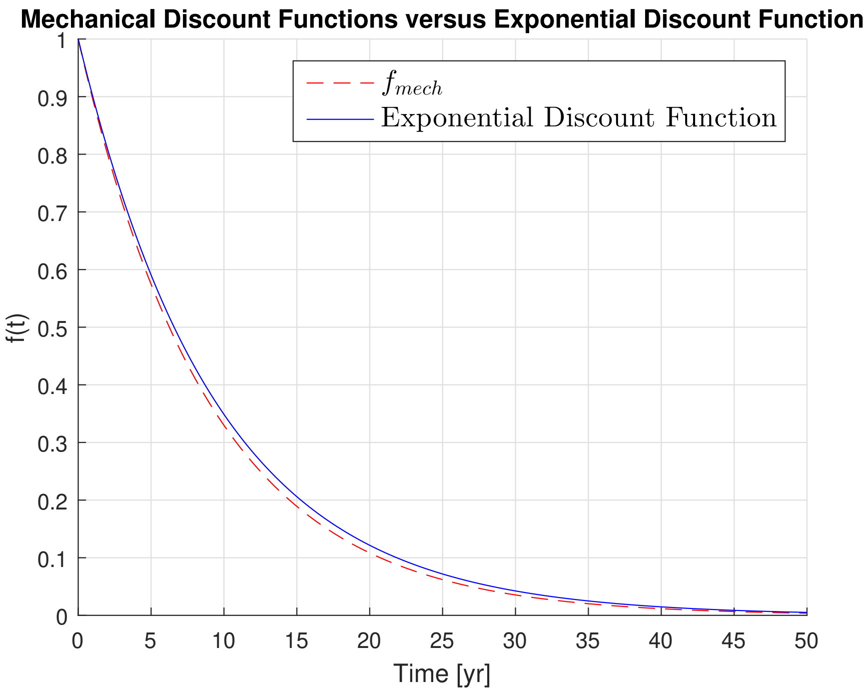

Figure 5-2: The mechanical discount function versus the exponential discount function

harmonic oscillator is a second order system. From this it can be concluded that the critically damped harmonic oscillator at least under some conditions is able to represent exponential discounting.

The damped harmonic oscillator is able to represent hyperbolic discounting when it is overdamped, and exponential discounting when it is critically damped.

5-2 Trading Behaviour as Underdamped Dynamics

In this section, the underdamped dynamics of the damped harmonic oscillator are investigated and interpreted in economic terms. Whereas the critically- and overdamped systems where linked to the exponential and hyperbolic discounting theories, I have found no economic literature on sinusoidal discounting. The hypothesis is that there is an economic interpretation to the underdamped time-discounting system despite missing economic literature.

The damped harmonic oscillator is underdamped for damping ratios/time-preference factors $0 \leq \zeta < 1$ [2]. The reservation price dynamics are derived in Appendix C and shown in

Equation 5-4.

$$p(t) = e^{-\zeta\omega_n t}\left(p_0 \cos(\omega_\mathrm{d}\, t) - \left(m\, q_0\, \omega_\mathrm{d} + \zeta\omega_n \frac{m\,\zeta\omega_n\, q_0 + p_0}{\omega_\mathrm{d}}\right) \sin(\omega_\mathrm{d}\, t)\right) \tag{5-4}$$

here, ω_d is called the damped frequency, and is defined as [2]:

$$\omega_\mathrm{d} \equiv \omega_n \sqrt{1 - \zeta^2} \tag{5-5}$$

Figure 5-3 shows the mechanical discount function for the underdamped system. The parameters are: $m = 3\,\frac{€\,\mathrm{day}}{\#^2}$, $\omega_n = 100\,\frac{\%}{\mathrm{day}}$, $\zeta = 0.02$. The initial conditions are: $q_0 = 0\,\#$ and $p_0 = 20\,\frac{€}{\#}$.

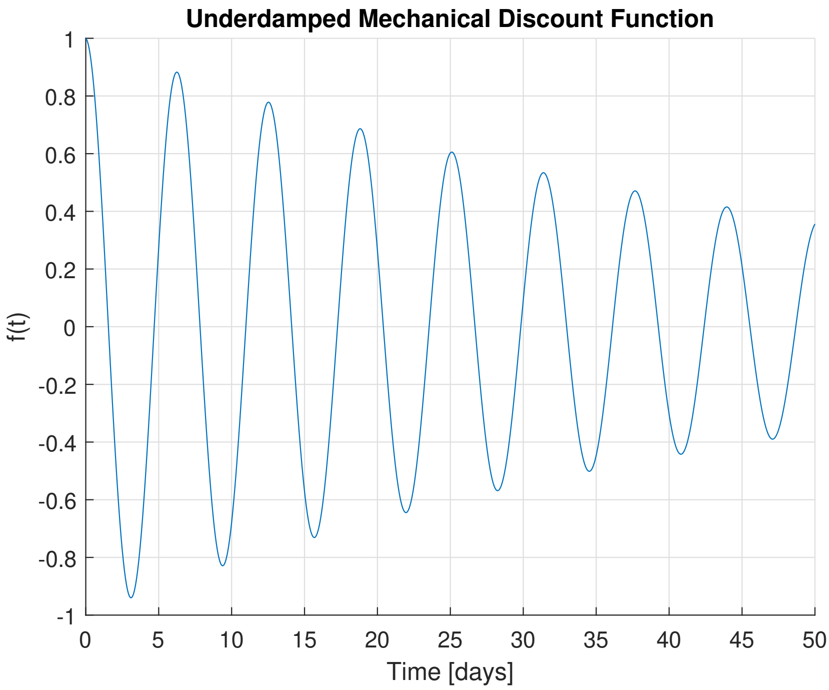

Figure 5-3: Discounting behaviour of a trader

It can be observed that the underdamped discount function shows oscillations with decreasing amplitude. Furthermore, the discount function oscillates around 0, and thus also takes negative values. This conflicts with the assumptions economists have made about the discount function, namely that $f(t) \in [0, 1]$ [26, 28]. I relax this assumption however, so that the mechanical discount function may take negative values as well.

The underdamped dynamics can be interpreted as the behaviour of a trader. A trader wants to sell (negative discount function values) goods when they are overpriced and purchase (positive discounting values) goods when they are cheap [47]. The underdamped mechanical discount function is particularly fit to model trading in a cyclical or volatile market. In a such a market price dynamics oscillate around a underlying "true value" price [48, 49].

Figure 5-4 highlights the discounting envelope of the discount function. The discounting envelope is equal to:

$$\text{Discouting Envelope} = e^{-\zeta \omega_n t} \tag{5-6}$$

The discounting envelope is an exponential, which translates economically to the trader discounting the amplitude of the oscillations linearly in time.

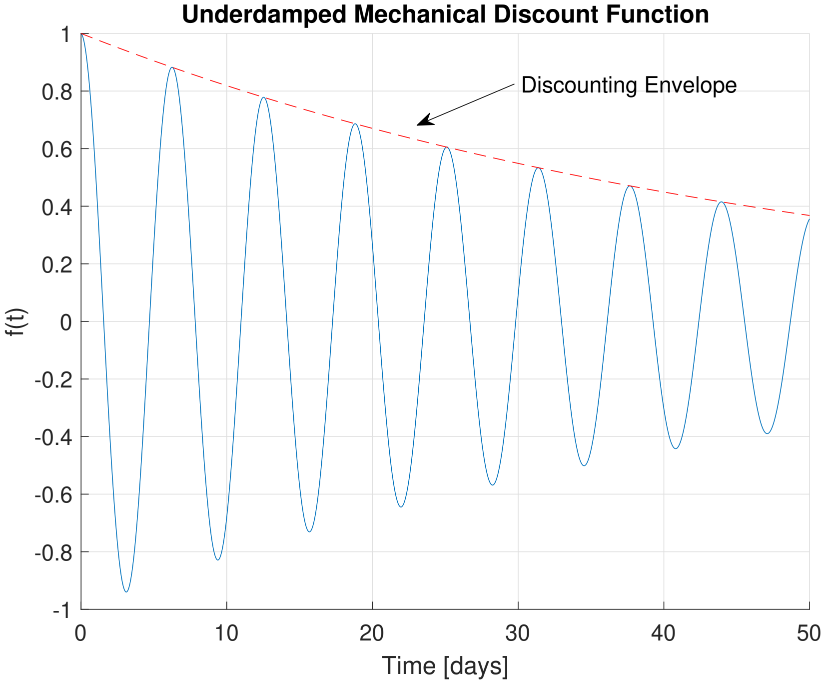

Figure 5-4: The underdamped discounting envelope is an exponential

The damped frequency in Equation 5-5 is the rate of the oscillations of the system. The damped frequency is dependent on the time-preference factor ζ and the natural discount rate ω_n (see Equation 5-5). A time-preference closer to 0 implies a higher damped frequency; a time-preference closer to 1 implies a lower damped frequency.

A trader wants to match his discounting frequency with the frequency of the price movements of securities. Econometric science has developed several techniques to detrend price movements resulting in its underlying sinusoidal movements [48]. These sinusoids are then synchronised with the discounting behaviour to maximise capital gains [49].

Figure 5-5 shows the time period of the damped oscillation. The trader matches this time period with the market cycle, which allows him to make a profit. The damped frequency is thus analogous to the frequency of the market cycle (formalised in Table 5-2).

	Mechanics		**Economics**	
ω_d	damped frequency	yr^{-1}	volatility rate/market cycle frequency	yr^{-1}

Table 5-2: Damped frequency analogy

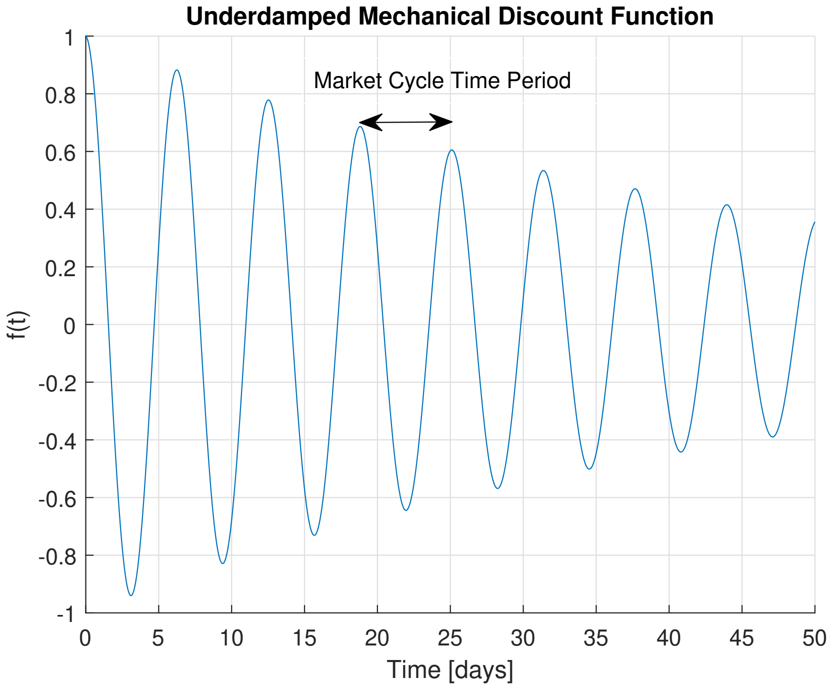

Figure 5-5: The duration of a market cycle is analogous to the time period of the damped oscillation

5-3 The Eigenvalues Reveal the Discounting Behaviour

The economic interpretation of the dynamics of the underdamped harmonic oscillator completes the understanding of the dynamics of the damped harmonic oscillator as an economic representative. The economic engineering time-discounting system can be summarised as follows:

- Underdamped dynamics: Trading in cyclical or volatile markets (see Section 5-2).
- Critically damped dynamics: Exponential discounting (see Section 5-1).
- Overdamped dynamics: Hyperbolic discounting (see Section 4-2).

The type of economic behaviour is a consequence of the time-preference factor ζ. By plotting the eigenvalues in a three dimensional system (real-, imaginary-, and hyperbolic axis), the various behaviours can clearly be distinguished. The eigenvalues for the damped harmonic oscillator are given by:

$$s_{1,2} = \begin{cases} -\zeta\omega_n \pm i\,\omega_n\sqrt{1-\zeta^2} & \text{if } \zeta < 1 \\ -\zeta\omega_n \pm \varepsilon & \text{if } \zeta = 1 \\ -\zeta\omega_n \pm \epsilon\,\omega_n\sqrt{\zeta^2-1} & \text{if } \zeta > 1 \end{cases} \tag{5-7}$$

Figure 5-6 plots the normalised eigenvalues $(s_{1,2}/\omega_n)$ of the damped harmonic oscillator for the values $\zeta \in [0, 4]$. It can be observed that:

- The overdamped system, representing hyperbolic discounting, has eigenvalues that lie on an actual hyperbola. The asymptotes are in the real-hyperbolic plane: $\pm\zeta\omega_n\,\epsilon$.
- The eigenvalues meet at $\zeta = 1$, and make a $90\,^\circ$ angle out of the complex plane when increasing ζ,

Exponential discounting takes place when $\zeta = 1$, and the eigenvalues meet at the point -1 on the real axis. It can be recalled from the previous section that the discounting envelope for the underdamped system is also a single exponential path. It can be inferred that with eigenvalues in the complex plane, the system is exponentially discounted.

In economic terms, it can be said that on the hyperbolic axis is where consumer behaviour takes place. A large distance between the eigenvalues implies a higher time-preference. Financial institutions operate within the complex plane; a constant discount is used to calculate the present value of future gains, and a trader matches his or her trading frequency with the rate of market cycles.

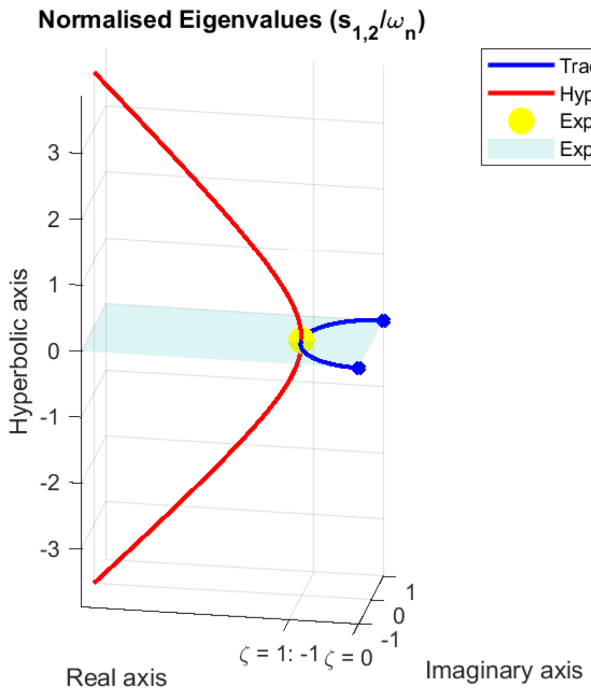

Figure 5-6: Normalised eigenvalues of the mechanical discounting system

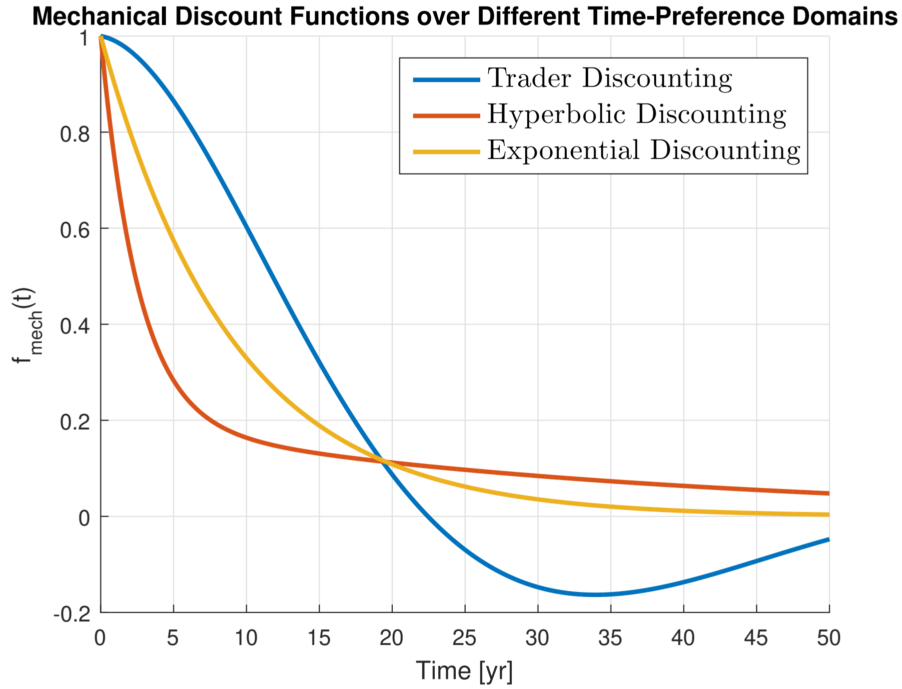

Figure 5-7: Discounting behaviours of the damped harmonic oscillator

5-4 Conclusion: The Damped Harmonic Oscillator Represents Economic Behaviour

The exponential discounting theory has been shown to be represented by the critically discounted harmonic oscillator under certain conditions. Economic engineering therefore provides a link between the hyperbolic (overdamped) and exponential (critically damped) discounting theories.

Furthermore, an economic interpretation has been given of the dynamics of the underdamped harmonic oscillator. The discounting envelope is discounted exponentially, whereas the discounting itself is a sine that oscillates between the envelope boundaries.

The damped harmonic oscillator allows inspection of the eigenvalues, which reveal the discounting behaviour. Looking at a visual representation, the different discounting domains (trader, exponential discounting, hyperbolic discounting) become apparent. Figure 5-7 shows how the same system represents all three behaviours. Economic engineering has thus provided a time-discounting system that unifies existing time-discounting theories.

Chapter 6

Discussion and Future Research

6-1 Discussion

Time-discounting is captured by mechanical system dynamics through extension of the economic engineering analogy. Previous models for time-discounting rely on regression of data on predefined functions, whereas the extended economic engineering framework provides a model based on economic first principles [10, 20].

The mechanical discount function takes the following form:

$$f_{\text{mech}}(t) = p(0)^{-1} \left(P_1 \; e^{s_1 \, t} + P_2 \; e^{s_2 \, t} \right) \tag{6-1}$$

where s_1 and s_2 are two conjugate complex valued eigenvalues of the economic system. This discount function describes time-discounting by the summation of two exponentials. Such a model has previously been proposed in behavioural economic literature, namely two exponential discounting [15, 50]. The two exponential discount function is given by:

$$f(t) = w \, e^{-r_1 \, t} + (1 - w) \, e^{-r_2 \, t} \tag{6-2}$$

where r_1 and r_2 are two real valued discount rates, and w is a weighting factor.

Two exponential discounting uses two arbitrary real-valued discount rates, whereas the economic engineering eigenvalues are conjugate, and are valued with a real component and a complex-, dual-, or double component. The eigenvalues are defined by the time-preference and the risk-free discount rate, whereas the two exponential discount function relies data-fitting two discount rates and a weighting factor [50].

A single discount rate defines the exponential and hyperbolic discount functions, whereas this thesis proposes the use of three parameters: mass, natural frequency, and damping ratio. Although the simplicity of the exponential and hyperbolic discount functions is favored by some scholars [14], previous research indicates that it is likely that more than one parameter will be required to accurately describe time-discounting [44].

Parameter estimation complicates practical application of the extended economic engineering framework. The time-preference factor (analogous to the damping ratio) is not an existing economic parameter with a cardinal scale, and will thus need to be investigated and calibrated by economists. Exact data about demand and the risk-free discount rate is not readily available in every situation and may also require estimation. Parameter estimation therefore also relies on data-fitting with the extended economic engineering framework.

Despite increasing academic interest, behavioural economists have made limited progress in describing the fundamentals of time-discounting and providing a generally accepted model thereof [10, 6]. The extension of economic engineering in this thesis bases time-discounting on economic first principles and captures exponential- and hyperbolic discounting behaviour with the same system.

6-2 Future Research: The Consumer as a Model Predictive Controller

This thesis is restricted to modelling time-discounting dynamics using the economic engineering framework. Economists study consumption planning through the utility maximisation problem. This is a maximisation problem where the consumer optimises for utility over consumption bundles that are affordable with the consumers' wealth level. The problem is given by [51, p. 50]:

$$\max_{x} \quad u(x) \tag{6-3}$$

$$\text{subject to} \quad p \cdot x \leq w \tag{6-4}$$

where u(x) is the utility function that returns the utility for the given consumption bundle x. The purchase price is denoted by p and the wealth level is denoted by w.

This optimisation is a static problem that only considers consumption in one time-period. Attempts have been made to model consumer behaviour by including future time-periods and considering income, savings and liquid assets [5]. In this model the consumer optimises his consumption path at every time-step. I therefore propose to extend the field of economic engineering by modelling the consumer as a Model Predictive Controller (MPC).

The time-discounting model developed in this thesis functions as the dynamic model of the plant. Outputs of the system are the level of utility, as well as the consumers liquid and illiquid assets. The consumer has a certain standard (views of its utility it deserves, and growth of his liquid and illiquid assets) which functions as the reference signal. The issue of self-control is then modelled as a tracking problem.

The consumer makes estimations of future utility levels and his (il)liquid assets based on the current time-step $\hat{y}(k+j|k)$ (where j is an arbitrary future time-step). These estimates follow from the consumers allocation of resources, who decides what and how much is to be consumed or saved. These are the inputs $u(k)$ to the system. A Model Predictive Controller considers the system evolution over a prediction period N_p [4]. Therefore the following in-, output and reference vectors are defined for the consumer:

$$\tilde{u}(k) = \begin{bmatrix} u^T(k) & \dots & u^T(k+N_p-1) \end{bmatrix}^T \tag{6-5}$$

$$\tilde{y}(k) = \begin{bmatrix} \hat{y}^T(k+1|k) & \dots & \hat{y}^T(k+N_p|k) \end{bmatrix}^T \tag{6-6}$$

$$\tilde{r}(k) = \begin{bmatrix} r^T(k) & \dots & r^T(k+N_p-1) \end{bmatrix} \tag{6-7}$$

The consumer optimises his consumption plan by minimising his cost function J for every time-step. In conventional MPC the cost function penalises both the tracking error and the control effort [4]. The cost function has the following form:

$$J(k) = \|\tilde{y}(k) - \tilde{r}(k)\|_Q^2 + \lambda\|\tilde{u}(k)\|_R^2 \tag{6-8}$$

where Q and R are positive definite matrices that penalise the tracking error and the control effort respectively. λ is a nonnegative integer which is used as a tuning parameter in MPC [4]. Figure 6-1 shows the control scheme of the MPC.

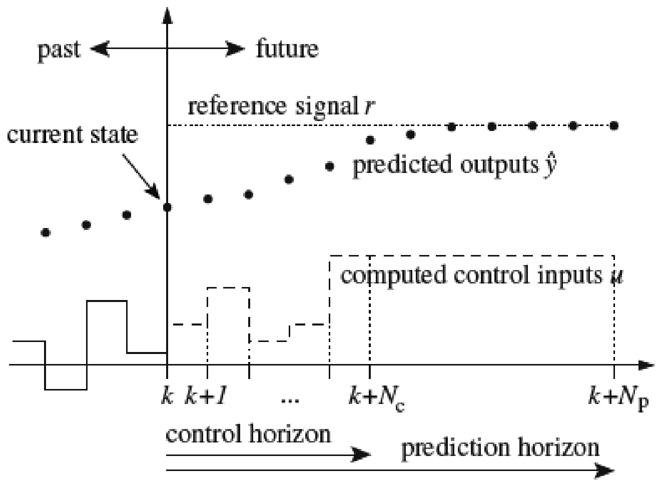

Figure 6-1: Representation of the MPC control scheme [4]

The MPC allows extra constraints to be imposed on the optimisation. This is advantageous since it allows known restrictions from economics to be implemented. Examples are (ignoring interest rates) [5]:

$$\text{Consumption } [€] \leq \text{Salary } [€] + \text{Liquid Assets } [€] \tag{6-9}$$

$$\text{Salary} + \text{Assets}\,(k-1) - \text{Consumption} = \text{Assets}\,(k) \tag{6-10}$$

The cost function allows for a real-life interpretation as it is a measure for how much a person struggles with his consumption path under his life conditions. If he feels like he is not receiving the amount of utility he deserves he will penalise the tracking error through the Q matrix (this can be caused by unrealistic views of his standards, the reference signal, or his own views of his model dynamics are inadequate). A (responsible) family with newborn children aims for stability, and will thus penalise the control effort through the R matrix.

The MPC consumer model will open up for disciplinary work with social sciences. Where are problems found with drug addictions? Are these individuals discounting irresponsibly? Are they no longer optimising their cost functions or do they use improper metrics (optimisation algorithm)? Is their prediction horizon too short, or do they have an unrealistic reference signal? The MPC based consumer model will lead to ample exciting research opportunities.

Chapter 7

Conclusions

This research captures time-discounting through the dynamics of a mechanical system. The mechanical system unifies existing time-discounting theories from economics. The overdamped harmonic oscillator provides hyperbolic and quasi-hyperbolic time-discounting dynamics. The critically damped harmonic oscillator provides exponential time-discounting dynamics.

Based on a qualitative comparison between mechanical systems theory and economic theory, the economic engineering framework has been extended. Economic analogues of the damping ratio, natural frequency, and damped frequency are included, such that the damped harmonic oscillator can be interpreted as a time-discounting system, with the system parameters based on economic first principles.

The economic engineering analogy is extended to include time-discounting: The natural frequency is analogous to the natural discount rate. The damping ratio is analogous to time-preference. The damped frequency is analogous to the market cycle frequency.

By analysing the time-domain dynamics of the under-, critically-, and overdamped harmonic oscillator, this thesis presents the damped harmonic oscillator as a time-discounting system. The underdamped system has oscillatory discounting dynamics, which is analogous to a trader, where the damped frequency is analogous to the frequency of the market price. The critically damped system discounts exponentially, which is analogous to financial institutions. The overdamped system discounts hyperbolically, and is analogous to a consumer.

By analysing the eigenvalues of the damped harmonic oscillator, it can be concluded that they reveal the discounting behaviour. When the eigenvalues lie within the complex plane, future events are discounted at a constant rate. The real part of the eigenvalues is analogous to the exponential discount rate; The complex part is responsible for the oscillatory behaviour. When the eigenvalues lie in the real-hyperbolic plane, the real part also discounts future events exponentially, but the second-order effects are responsible for hyperbolic movements.

Appendix A

Beta-Delta Discounting

Beta-delta, or quasi-hyperbolic, discounting was proposed by Phelps and Pollak in 1968 [30]. It has gained great popularity after David Laibson's paper in 1997. This paper has been cited numerous times (over 150 times), and the discount function has been used to predict and model human behaviour [5].

Beta-delta discounting uses a discrete-time discount function, and discounts utility according to the structure in Equation A-1. From this equation, the discrete-time discount function can be derived; this is presented in Equation A-2.

$$u_\tau(c_\tau) = \beta\,\delta^\tau\,u_0(c_\tau) \tag{A-1}$$

$$f(\tau) = \beta\,\delta^\tau \qquad \text{where:} \quad \tau = \{0, 1, 2, \dots\} \tag{A-2}$$

We can recognise that δ^τ is the discrete time equivalent of an exponent $e^{-(1-\delta)\,\tau}$. Beta-delta discounting can thus be summarised as an exponential discount function multiplied by a constant β. The β factor solves any issues with dynamic-inconsistency that exponential discounting encounters. David Laibson writes [5]:

> "When $0 < \beta < 1$, the discount structure in Equation A-2 mimics the qualitative property of the hyperbolic discount function, while maintaining most of the analytical tractibility of the exponential discount function."

The quasi-hyperbolic discount function is plotted against an exponential and a hyperbolic discount function in Figure A-1. The used discount rates are chosen arbitrarily, but can be found in [5].

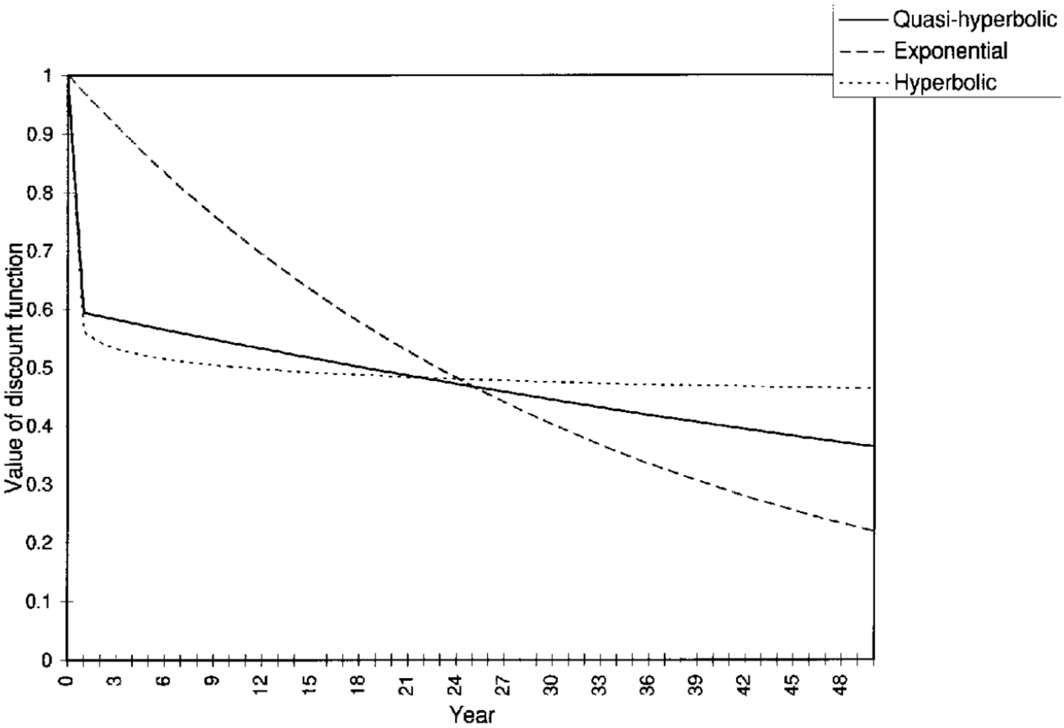

Figure A-1: Quasi-hyperbolic discount function versus exponential and hyperbolic discounting, image copied from [5]

Appendix B

Free-Response Solutions Overdamped Second-Order System

This appendix presents the derivation of the solutions of the free-response of the second-order system (Figure B-1). The system's equation of motion is given in Equation B-1. The progression of the asset stock $q(t)$ and the reservation price $p(t)$ are derived.

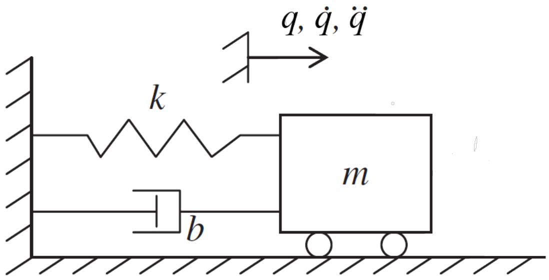

Figure B-1: A second-order mechanical system (modified from [2, p. 246])

$$m\,\ddot{q}(t) + b\,\dot{q}(t) + k\,q(t) = 0 \tag{B-1}$$

Eigenvalues

The characteristic equation of the system is given by [2]:

$$m\,s^2 + b\,s + k = 0$$

Subsequently, the eigenvalues are the solution of the quadratic equation:

$$\begin{aligned} s_{1,2} &= -\frac{b}{2m} \pm \frac{1}{2m}\sqrt{b^2 - 4\,m\,k} \\ &= -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}} \end{aligned}$$

For ease of notation, I define β as follows:

$$\beta \equiv \frac{b}{2m}$$

Such that the eigenvalues are given by Equation B-2.

$$s_{1,2} = -\beta \pm \sqrt{\beta^2 - \frac{k}{m}} \tag{B-2}$$

A second-order system is said to be overdamped when $\beta > \frac{k}{m}$ [2]. Subsequently, I define the overdamped frequency ω_{od} as:

$$\omega_{\text{od}} \equiv \sqrt{\beta^2 - \frac{k}{m}}$$

Such that the eigenvalues take the final hyperbolically conjugate form in Equation B-3.

$$s_{1,2} = -\beta \pm \omega_{\text{od}} \tag{B-3}$$

Homogeneous Solution

The solution for the progression of the asset stock $q(t)$ takes the form of Equation B-4, where Q_1 and Q_2 constants. Which can be rearranged into Equation B-5 due to the structure of the eigenvalues.

$$q(t) = Q_1\, e^{s_1\, t} + Q_2\, e^{s_2\, t} \tag{B-4}$$

$$= e^{-\beta\, t} \left(Q_1\, e^{\omega_{\text{od}}\, t} + Q_2\, e^{-\omega_{\text{od}}\, t}\right) \tag{B-5}$$

The asset flow $q(t)$ is consequently calculated by taking the time-derivative of $q(t)$. Applying the product rule, and rearranging results in Equation B-6.

$$\begin{aligned}\dot{q}(t) &= -\beta e^{-\beta\, t} \left(Q_1\, e^{\omega_{\text{od}}\, t} + Q_2\, e^{-\omega_{\text{od}}\, t}\right) \quad \cdots \\ &\qquad \cdots \quad + e^{-\beta\, t} \left(Q_1\, \omega_{\text{od}}\, e^{\omega_{\text{od}}\, t} - Q_2\, \omega_{\text{od}}\, e^{-\omega_{\text{od}}\, t}\right) \\ &= e^{-\beta\, t} \left((\omega_{\text{od}} - \beta)\, Q_1\, e^{\omega_{\text{od}}\, t} - (\omega_{\text{od}} + \beta)\, Q_2\, e^{-\omega_{\text{od}}\, t}\right)\end{aligned} \tag{B-6}$$

The progression of the reservation price $p(t)$ is then calculated in Equation B-7.

$$\begin{aligned}p(t) &= m\, \dot{q}(t) \\ &= m\, e^{-\beta\, t} \left((\omega_{\text{od}} - \beta)\, Q_1\, e^{\omega_{\text{od}}\, t} - (\omega_{\text{od}} + \beta)\, Q_2\, e^{-\omega_{\text{od}}\, t}\right)\end{aligned} \tag{B-7}$$

Initial Value Problem

The constants Q_1 and Q_2 follow the initial value problem. When time-discounting (at $t = 0$) we have access to an initial stock and an instantaneous reservation price:

$$q(0) = q_0 \qquad\qquad p(0) = p_0$$

Substituting the initial values into Equation B-5 and Equation B-7 results in two independent conditions for Q_1 and Q_2. The first condition is given in Equation B-8.

$$\begin{aligned} q(0) &= e^{-\beta \cdot 0} \left(Q_1 \, e^{\omega_\mathrm{od} \cdot 0} + Q_2 \, e^{-\omega_\mathrm{od} \cdot 0} \right) \\ &= Q_1 + Q_2 = q_0 \end{aligned} \tag{B-8}$$

The second condition is given in Equation B-9.

$$\begin{aligned} p(0) &= m \, e^{-\beta \cdot 0} \left((\omega_\mathrm{od} - \beta) \, Q_1 \, e^{\omega_\mathrm{od} \cdot 0} - (\omega_\mathrm{od} + \beta) \, Q_2 \, e^{-\omega_\mathrm{od} \cdot 0} \right) \\ &= m \left((\omega_\mathrm{od} - \beta) \, Q_1 - (\omega_\mathrm{od} + \beta) \, Q_2 \right) = p_0 \\ \implies \qquad & (\omega_\mathrm{od} - \beta) \, Q_1 - (\omega_\mathrm{od} + \beta) \, Q_2 = \frac{p_0}{m} \end{aligned} \tag{B-9}$$

Rearranging the first condition, and subsequently substituting the parametrisation of Q_2 into the second condition results in the value of Q_1 in Equation B-10.

$$\begin{aligned} Q_2 &= q_0 - Q_1 \\ \implies \qquad (\omega_\mathrm{od} - \beta) \, Q_1 - (\omega_\mathrm{od} + \beta) \, (q_0 - Q_1) &= p_0/m \\ 2 \, \omega_\mathrm{od} \, Q_1 - (\omega_\mathrm{od} + \beta) \, q_0 &= p_0/m \\ Q_1 &= \frac{(\omega_\mathrm{od} + \beta) \, q_0}{2 \, \omega_\mathrm{od}} + \frac{p_0/m}{2 \, \omega_\mathrm{od}} \\ Q_1 &= \frac{q_0}{2} + \frac{\beta \, q_0 + p_0/m}{2 \, \omega_\mathrm{od}} \end{aligned} \tag{B-10}$$

Substituting Equation B-10 into the first condition, Equation B-8, results in the values for Q_1 and Q_2 (Equation B-11).

$$Q_1 = \frac{q_0}{2} + \frac{\beta \, q_0 + p_0/m}{2 \, \omega_\mathrm{od}} \qquad\qquad Q_2 = \frac{q_0}{2} - \frac{\beta \, q_0 + p_0/m}{2 \, \omega_\mathrm{od}} \tag{B-11}$$

Hyperbolic Functions

Substituting the newly acquired values for Q_1 and Q_2 into the equation for $q(t)$ (Equation B-5) results into the complete free-response solution. Rearranging leads to the structure in Equation B-12.

$$\begin{aligned} q(t) &= e^{-\beta \, t} \left(\left(\frac{q_0}{2} + \frac{\beta \, q_0 + p_0/m}{2 \, \omega_\mathrm{od}} \right) e^{\omega_\mathrm{od} \, t} + \left(\frac{q_0}{2} - \frac{\beta \, q_0 + p_0/m}{2 \, \omega_\mathrm{od}} \right) e^{-\omega_\mathrm{od} \, t} \right) \\ &= e^{-\beta \, t} \left(q_0 \left(\frac{e^{\omega_\mathrm{od} \, t} + e^{-\omega_\mathrm{od} \, t}}{2} \right) + \frac{\beta \, q_0 + p_0/m}{\omega_\mathrm{od}} \left(\frac{e^{\omega_\mathrm{od} \, t} - e^{-\omega_\mathrm{od} \, t}}{2} \right) \right) \end{aligned} \tag{B-12}$$

Take into account the following analytical definitions of the hyperbolic cosine and hyperbolic sine [52, p. 86]:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad\qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

Combining these definitions, and Equation B-12 leads to the final form of the asset stock progression $q(t)$ in Equation B-13.

$$q(t) = e^{-\beta t} \left(q_0 \, \cosh\left(\omega_{\text{od}}\, t\right) + \frac{\beta q_0 + p_0/m}{\omega_{\text{od}}} \, \sinh\left(\omega_{\text{od}}\, t\right) \right) \tag{B-13}$$

Taking the time-derivative results in $\dot{q}(t)$:

$$\begin{aligned}
\dot{q}(t) &= -\beta\, e^{-\beta\, t} \left(q_0 \, \cosh\left(\omega_{\text{od}}\, t\right) + \frac{\beta q_0 + p_0/m}{\omega_{\text{od}}} \, \sinh\left(\omega_{\text{od}}\, t\right) \right) \quad \cdots \\
&\qquad \cdots \quad + e^{-\beta\, t} \Big(\left(\beta\, q_0 + p_0/m\right) \, \cosh\left(\omega_{\text{od}}\, t\right) + q_0\, \omega_{\text{od}} \, \sinh\left(\omega_{\text{od}}\, t\right) \Big) \\
&= e^{-\beta\, t} \left(\frac{p_0}{m} \, \cosh\left(\omega_{\text{od}}\, t\right) \, + \, \left(q_0\, \omega_{\text{od}} - \beta \, \frac{\beta\, q_0 + p_0/m}{\omega_{\text{od}}} \right) \sinh\left(\omega_{\text{od}}\, t\right) \right)
\end{aligned} \tag{B-14}$$

Such that the progression of the reservation price takes its final form in Equation B-15.

$$\begin{aligned}
p(t) &= m\, \dot{q}(t) \\
&= e^{-\beta t} \left(p_0 \cosh\left(\omega_{\text{od}}\, t\right) + \left(m\, q_0\, \omega_{\text{od}} - \beta \, \frac{m\, \beta\, q_0 + p_0}{\omega_{\text{od}}} \right) \, \sinh\left(\omega_{\text{od}}\, t\right) \right)
\end{aligned} \tag{B-15}$$

Appendix C

Reservation Price Dynamics Damped Harmonic Oscillator

In this appendix, the reservation price dynamics are defined for the trader-, exponential, and hyperbolic discounting dynamics of the damped harmonic oscillator.

The eigenvalues for the damped harmonic oscillator are given by:

$$s_{1,2} = \begin{cases} -\zeta\omega_n \pm i\,\omega_n\sqrt{1-\zeta^2} & \text{if } \zeta < 1 \\ -\zeta\omega_n \pm \varepsilon & \text{if } \zeta = 1 \\ -\zeta\omega_n \pm \epsilon\,\omega_n\sqrt{\zeta^2-1} & \text{if } \zeta > 1 \end{cases} \tag{C-1}$$

where i is the complex number with the property $i^2 = -1$; ε is the dual number with the property $\varepsilon^2 = 0$; and ϵ is the double number with the property $\epsilon^2 = 1$. They correspond to circular-, parabolic-, and hyperbolic geometry respectively [45].

I define the corresponding damped frequencies by:

$$\omega_d = \begin{cases} i\,\omega_n\sqrt{1-\zeta^2} & \text{if } \zeta < 1 \\ \varepsilon & \text{if } \zeta = 1 \\ \epsilon\,\omega_n\sqrt{\zeta^2-1} & \text{if } \zeta > 1 \end{cases} \tag{C-2}$$

The dynamics of the position variable q takes the form (Equation B-4):

$$q(t) = Q_1\,e^{s_1\,t} + Q_2\,e^{s_2\,t} \tag{C-3}$$

where Q_1 and Q_2 are dependent on the initial conditions. The solution of the initial value problem in Equation B-11 can be generalised to:

$$Q_1 = \frac{q_0}{2} + \frac{\zeta\omega_n\,q_0 + p_0/m}{2\,\omega_d} \qquad Q_2 = \frac{q_0}{2} - \frac{\zeta\omega_n\,q_0 + p_0/m}{2\,\omega_d} \tag{C-4}$$

C-1 The Critically Damped System

The critically damped system has the property $\zeta = 1$. The position dynamics are given by:

$$q(t) = e^{-\omega_n t}\Big(Q_1\, e^{\varepsilon\, t} + Q_2\, e^{\varepsilon\, t}\Big) \tag{C-5}$$

Since $\varepsilon^2 = 0$, it follows that $\varepsilon^n = 0$ for any integer $n > 1$. Through the taylor series expansion we get [45]:

$$e^{\varepsilon\, t} = 1 + \varepsilon t \qquad\qquad e^{-\varepsilon\, t} = 1 - \varepsilon t \tag{C-6}$$

This results in the following position dynamics:

$$q(t) = e^{-\omega_n t}\left(\left(\frac{q_0}{2} + \frac{\omega_n\, q_0 + p_0/m}{2\,\varepsilon}\right)(1+\varepsilon t) + \left(\frac{q_0}{2} - \frac{\omega_n\, q_0 + p_0/m}{2\,\varepsilon}\right)(1-\varepsilon t)\right) \tag{C-7}$$

$$= e^{-\omega_n t}\left(q_0 + \left(\omega_n\, q_0 + \frac{p_0}{m}\right) t\right) \tag{C-8}$$

The velocity dynamics are given by the time-derivative of the position dynamics:

$$\dot{q}(t) = e^{-\omega_n t}\left(\frac{p_0}{m} - \omega_n \left(\omega_n q_0 + \frac{p_0}{m}\right) t\right) \tag{C-9}$$

The reservation price dynamics follow from $p(t) = m\dot{q}$:

$$p(t) = e^{-\omega_n t}\left(p_0 - \omega_n \left(m\omega_n q_0 + p_0\right) t\right) \tag{C-10}$$

C-2 The Overdamped System

The derivations of the overdamped second-order system can be found in Appendix B. Substituting the identity $\beta = \zeta\omega_n$ and redefining the overdamped frequency as $\omega_{\text{od}} = \omega_n\sqrt{\zeta^2 - 1}$ in Equations B-15, B-14, and B-15 results in the following dynamics for position, velocity, and reservation price:

$$q(t) = e^{-\zeta\omega_n t}\left(q_0\, \cosh\left(\omega_{\text{od}}\, t\right) + \frac{\zeta\omega_n q_0 + p_0/m}{\omega_{\text{od}}}\, \sinh\left(\omega_{\text{od}}\, t\right)\right) \tag{C-11}$$

$$\dot{q}(t) = e^{-\zeta\omega_n\, t}\left(\frac{p_0}{m}\, \cosh\left(\omega_{\text{od}}\, t\right) + \left(q_0\, \omega_{\text{od}} - \zeta\omega_n\, \frac{\zeta\omega_n\, q_0 + p_0/m}{\omega_{\text{od}}}\right) \sinh\left(\omega_{\text{od}}\, t\right)\right) \tag{C-12}$$

$$p(t) = e^{-\zeta\omega_n t}\left(p_0 \cosh\left(\omega_{\text{od}}\, t\right) + \left(m\, q_0\, \omega_{\text{od}} - \zeta\omega_n\, \frac{m\, \zeta\omega_n\, q_0 + p_0}{\omega_{\text{od}}}\right) \sinh\left(\omega_{\text{od}}\, t\right)\right) \tag{C-13}$$

C-3 The Underdamped System

The dynamics for the underdamped system can be calculated from the overdamped system. Using the identities $\cosh x = \cos ix$ and $\sinh x = -i \sin x$, Equation C-11 can be rewritten as:

$$q(t) = e^{-\zeta\omega_n t} \left(q_0 \, \cos\left(i\,\omega_{\text{od}}\,t\right) + \frac{\zeta\omega_n q_0 + p_0/m}{\omega_{\text{od}}} \left(-i\right) \, \sin\left(i\,\omega_{\text{od}}\,t\right) \right) \tag{C-14}$$

The overdamped ω_{od} and underdamped ω_{ud} frequencies are related as follows:

$$i\omega_{\text{od}} = i\,\omega_n\sqrt{\zeta^2 - 1} = \omega_n\sqrt{1 - \zeta^2} = \omega_{\text{ud}} \tag{C-15}$$

Substituting $\omega_{\text{od}} = -i\,\omega_{\text{ud}}$ in Equation C-14 results in the position dynamics for the underdamped system:

$$\begin{aligned} q(t) &= e^{-\zeta\omega_n t} \left(q_0 \, \cos\left(i\left(-i\right)\omega_{\text{ud}}\,t\right) + \frac{\zeta\omega_n q_0 + p_0/m}{\left(-i\right)\omega_{\text{ud}}} \left(-i\right) \, \sin\left(i\left(-i\right)\omega_{\text{ud}}\,t\right) \right) \\ &= e^{-\zeta\omega_n t} \left(q_0 \, \cos\left(\omega_{\text{ud}}\,t\right) + \frac{\zeta\omega_n q_0 + p_0/m}{\omega_{\text{ud}}} \, \sin\left(\omega_{\text{ud}}\,t\right) \right) \end{aligned} \tag{C-16}$$

This allows the dynamics of the underdamped velocities, and reservation price dynamics to be derived:

$$\dot{q}(t) = e^{-\zeta\omega_n\,t} \left(\frac{p_0}{m} \, \cos\left(\omega_{\text{ud}}\,t\right) \, - \, \left(q_0\,\omega_{\text{ud}} + \zeta\omega_n \, \frac{\zeta\omega_n\,q_0 + p_0/m}{\omega_{\text{ud}}} \right) \sin\left(\omega_{\text{ud}}\,t\right) \right) \tag{C-17}$$

$$p(t) = e^{-\zeta\omega_n t} \left(p_0 \cos\left(\omega_{\text{ud}}\,t\right) - \left(m\,q_0\,\omega_{\text{ud}} + \zeta\omega_n \, \frac{m\,\zeta\omega_n\,q_0 + p_0}{\omega_{\text{ud}}} \right) \, \sin\left(\omega_{\text{ud}}\,t\right) \right) \tag{C-18}$$

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