The Use of Precautionary Loss Functions in Risk Analysis

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Key Words — **Risk analysis, Loss function, Bayes estimation, Decision theory.**

Summary & Conclusions — **Risk analysis is discussed within a Bayes framework. Traditionally, Bayes parameter estimation is based on a quadratic loss-function. This paper introduces an alternative asymmetric precautionary loss-function, derives its main features, and presents a general class of precautionary loss-functions with the quadratic loss-function as a special case. These loss functions approach infinity near the origin to prevent underestimates and thus give conservative estimates, especially when, for example, low failure rates are being estimated. The conservative estimates make these loss functions useful when the consequences are major and under-estimation is serious. They are intuitively appealing and easy to calculate.**

1. INTRODUCTION

Risk analysis is used in several industrial disciplines, *eg*, chemical-process industry and offshore industry. In risk analysis, both the *potentiality* of an undesired event and its consequences are investigated. This *potentiality* is usually measured by either a probability or a failure rate. The Bayes approach is widely applied to estimate this probability (failure rate). Some examples on the Bayes approach are in [1].

When dealing with disastrous consequences, it can be worse to underestimate the *potentiality* of an event than to overestimate it. This is important when risk-level is the basis of a risk-reducing initiative, either by reducing the *potentiality* or the consequences. An erroneously low estimated risk-level can lead to the absence of necessary initiative to reduce the risk level. It is unreasonable to use a loss function that allows one to estimate a failure probability of zero. A positive loss function at the origin allows estimating zero, and in a risk analysis, estimating zero failure probability simply means that no risk is anticipated. Hence, a precautionary loss function is defined, a specific class of loss functions that are precautionary is developed. Some examples are given. The uses of Bayes models are described in [2, 3, 5, 6].

Notation

Ω	sample space
x	$(x_1,\ldots x_n)$: $n\times 1$ vector of s-independent observations of the r.v.; $x\in\Omega$
Θ	space of all possible θ describing the *potentiality* of the event
A	space of estimates $a\in A$ of the unknown $\theta\in\Theta$
$l(\theta,a)$	loss function: cost of estimating a when θ is the value
$\pi(\theta)$	prior pdf$\{\theta\}$
$f(x_i\|\theta)$	pdf$\{X_i\|\theta\}$
$f(x\|\theta)$	$\prod_i f(x_i\|\theta)$: joint pdf$\{X\}$
$\pi(\theta\|x)$	posterior pdf$\{\theta\|x\}$
k	precautionary index.

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

2. LOSS FUNCTION

Optimal policy selection has traditionally been discussed in relation to symmetric (and often quadratic) loss functions. By using non-symmetric loss functions one is able to deal with cases where it is more damaging to miss the target on one side than the other. According to [3], should $l(\theta,a)$ be continuous, quasi-convex[1] and attain the lowest value at $a=\theta$. This implies that the loss function increases as a moves away from θ.

A loss function is (for any $\epsilon > 0$):

- *downside damaging* if $l(a-\epsilon,a) \geq l(a+\epsilon,a)$,
- *upside damaging* if $l(a-\epsilon,a) \leq l(a+\epsilon,a)$,
- *symmetric* if the loss function is both downside and upside damaging.

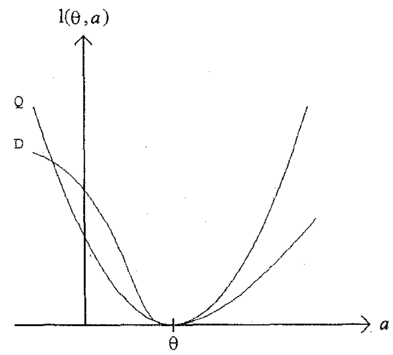

Figure 1. Functional Form of an Asymmetric Downside-Damaging Loss Function (*D*) *vs* a Quadratic Loss Function (*Q*)

Figure 1 is an example of a downside-damaging loss function compared with a quadratic loss function. The downside-

[1] A real function $h(x)$ is quasi-convex if, for any given real number r, the set of all x such that $h(x) \leq r$ is convex. Any convex function is quasi-convex, but the converse is not necessarily true.

damaging loss function gives higher costs for an underestimate compared to the quadratic loss function.

2.1 Precautionary Loss Functions

The $l(\theta,a)$ is a precautionary loss function iff:

1. $l(\theta,a)$ is downside damaging, and
2. for each fixed θ, $l(\theta,a) \to \infty$ when $a \to 0$.

Figure 2 shows some typical precautionary loss functions for $k > 0$.

2.2 Class of Precautionary Loss Functions

A literature search for precautionary loss functions gave some variants of,

$$l(\theta,a) = (\ln(a/\theta))^2, \tag{1}$$

which is mentioned in [6]. The Bayes' estimate for (1) is usually difficult to calculate. The problem is to come up with a loss function which gives both simple calculations and is intuitively appealing, *eg*,

$$l(\theta,a) = (\theta-a)^2/a. \tag{2}$$

This satisfies the criterion of being precautionary. The Bayes' estimate for (2) is obtained from theorem 1.

Theorem 1. Let $l(\theta,a)$ be defined by (2). The Bayes' estimate is:

$$\hat{\theta}_p^2 = E_{\pi(\theta|x)}\{\theta^2\} = \int_{\theta\in\Theta} \theta^2\cdot\pi(\theta|x)\, d\theta \tag{3}$$

$$\hat{\theta}_p = f\cdot E_{\pi(\theta|x)}\{\theta\},$$

$$f^2 \equiv 1 + \mathrm{Var}_{\pi(\theta|x)}\{\theta\}/E^2_{\pi(\theta|x)}\{\theta\} > 1. \tag{4}$$

Theorem 1 is proved in [4]. There it is also shown that the s-expected squared estimation error is less than twice the posterior variance. The Bayes' estimate in (3) is a geometric mean; in (4) it is given as a function of the posterior s-expectation and the posterior variance. The f represents the state of being precautionary, and f becomes important if the s-expectation is close to zero or when the variance is large. It means that the Bayes' estimate is sensitive to the choice of the loss function when very uncertain conditions occur or when estimating low probabilities.

The loss function in (2) can be generalized to

$$l(\theta,a) = \frac{(\theta-a)^2}{a^k}\cdot w(\theta),\ 0 < k \le 2,\ w(\theta) > 0; \tag{5}$$

$w(\theta)$ is an arbitrary weight function. The $k \le 2$ ensures that the cost increases as the difference $a-\theta$ grows. k is a precautionary index since it regulates how downside damaging the loss function is. The loss function in (5) covers a spectrum of precautionary loss functions. Figure 2 plots (5) for various k.

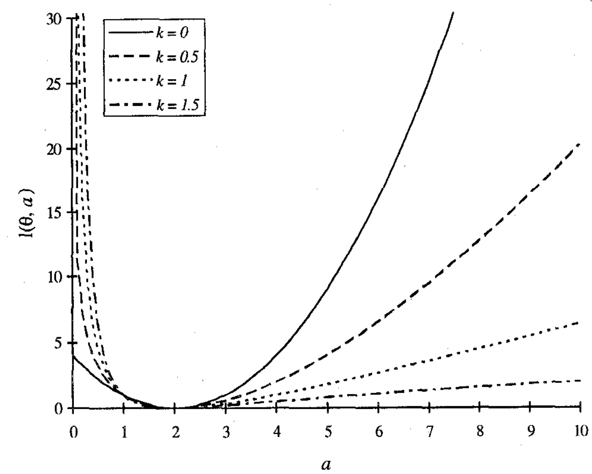

Figure 2. The Loss Function (5) *vs* k

In the limit, $k=0$, the loss function is the familiar quadratic loss function, and when $k=2$ it approaches $w(\theta)$ as $a\to\infty$. Thus, the loss function becomes more precautionary as k increases.

A way to determine k is to:

- select two values of a, $\theta-\epsilon$, and $\theta+c\cdot\epsilon$, where $c>1$, which have equal loss for the decision maker,
- solve $l(\theta,\ \theta-\epsilon) = l(\theta,\ \theta+c\cdot\epsilon)$ with respect to k.

Let $\epsilon=\theta/c$; the problem simplifies to:

$$l(\theta,\theta - \theta/c) = l(\theta, 2\theta);$$

solving this with respect to k gives:

$$k = 2\cdot\ln(c)/\ln[2c/(c-1)]. \tag{6}$$

Two special values of c are:

- $c=2$ gives $k=1$ as in (2),
- $c=3$ gives $k=2$.

Theorem 2. Let the loss function be defined by (5). Then the Bayes' estimate is:

$$\hat{\theta}_p = \frac{1}{\Psi_1(x)}\cdot[\bar{k} + \sqrt{\bar{k}^2 + \Psi_1(x)\cdot\Psi_2(x)}], \tag{7}$$

$$\Psi_1(x) \equiv (1+\bar{k})\cdot E_{\pi(\theta|x)}\{w(\theta)\}/E_{\pi(\theta|x)}\{\theta\cdot w(\theta)\},$$

$$\Psi_2(x) \equiv (1-\bar{k})\cdot E_{\pi(\theta|x)}\{\theta^2\cdot w(\theta)\}/E_{\pi(\theta|x)}\{\theta\cdot w(\theta)\},$$

$$\bar{k} \equiv 1 - k.$$ ◄

The proof is obtained by differentiating the posterior s-expected loss and equating it to zero; then solving this for a to find the critical points. Since $a>0$, the sign in front of the square root is positive. Since a is the action with minimum loss and therefore is the Bayes' estimate, it is denoted $\hat{\theta}_p$ and the theorem is proved.

Theorem 1 follows from theorem 2 for $k = 1$ and $w(\theta) = 1$.

3. EXAMPLES[2]

Two examples apply the precautionary loss function. Example 1 gives a theoretical illustration while example 2 is a numerical case. The estimates are compared with those obtained from the quadratic loss function. We consider the Poisson sample model and use only the precautionary loss function with $k=1$, $w(\theta)=1$, and a conjugate prior.

Notation

λ unknown failure rate, to be estimated.

3.1 Example 1: Poisson

The Poisson sampling model of s failures in t time is:

$$f(s|\lambda;t) = \text{poim}(s;\ \lambda\cdot t)$$

The conjugate prior in this case is the gamma pdf:

$$\pi(\lambda;\ \alpha,\beta) = \beta\cdot\text{gamd}(\beta\cdot\lambda;\ \alpha). \tag{8}$$

Bayes' theorem gives the posterior gamma pdf:

$$\pi(\lambda|s;t,\ \alpha,\beta) = (t+\beta)\cdot\text{gamd}((t+\beta)\cdot\lambda;\ s+\alpha), \tag{9}$$

given that s failures in time t are observed (here $\lambda\equiv\theta$). The posterior s-expectation & variance are:

$$\hat{\lambda}_q = (s+\alpha)/(t+\beta),$$

$$\text{Var} = (s+\alpha)/(t+\beta)^2.$$

The quadratic loss function gives the $\hat{\lambda}_q$ as the Bayes' estimate. This is compared with the Bayes' estimate obtained by using the *precautionary* loss function with $k=1$:

$$\hat{\lambda}_p = \frac{\sqrt{(s+\alpha)\cdot(s+\alpha+1)}}{t+\beta}$$

$$= \hat{\lambda}_q\cdot f, \tag{10}$$

$$f \equiv \sqrt{1 + \frac{1}{s+\alpha}};$$

$f \to 1$ as s increases — as shown in figure 3.

[2]The number of significant figures is not intended to imply any accuracy in the estimates, but to illustrate the arithmetic.

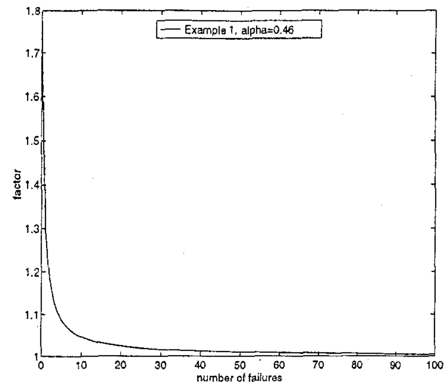

Figure 3. How $f\to 1$ as s Increases

3.2 Example 2

Some offshore reliability data are used to illustrate the effect of the precautionary loss function. When a prior data-set is available, empirical Bayes is used to model the failure frequency of a pump. This has resulted in a posterior gamma distribution with:

$$s+\alpha = 1.31,$$

$$t+\beta = 25300,$$

in (9). These values give the posterior mean and variance:

$$\hat{\lambda}_q = 51.7\cdot 10^{-6}/\text{hour},$$

$$\text{Var} = 20.5\cdot 10^{-10}/\text{hour}^2. \tag{11}$$

The Bayes estimate for the quadratic loss function is:

$$\hat{\lambda}_q = 51.7\cdot 10^{-6}/\text{hour}; \tag{12}$$

and the Bayes estimate from theorem 1 is, from (10):

$$\hat{\lambda}_p = f\cdot\hat{\lambda}_q = 1.33\cdot 51.7\cdot 10^{-6}/\text{hour}$$

$$= 68.7\cdot 10^{-6}/\text{hour} = 0.60/\text{year}. \tag{13}$$

This means that the Bayes estimate increases by 33% when the precautionary loss function is used instead of the quadratic loss function.

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