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The Avalanche Distribution of the Abelian Sandpile Model on the Bethe Lattice for any Branching Degree

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"The Avalanche Distribution of the Abelian Sandpile Model on the Bethe Lattice for any Branching Degree"

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Chapter 1

Introduction

The abelian sandpile model is a model first introduced by the physicists Bak, Tang and Wiesenfeld in 1986. Originally, the sandpile model was developed to study the concept of Self Organised Criticality (SOC). If we have a system that has SOC, this means that over time, it tends to evolve to its critical state. An example of a critical state is the phase transition from water to ice and vice versa, which happens at exactly $0°C$ under normal pressure circumstances. In nature, a system of water with no external influences does not lower its temperature to zero degrees, for this, the temperature needs to be fine-tuned. The same holds for ice, it does not turn into water without adding energy to the system. Therefore, a system of only water does not have SOC. When a system does admit to SOC, it means that the system tends to this transition phase. The abelian sandpile model is a model with very simple dynamics which have SOC.

Bak, Tang and Wiesenfeld's model concerns a graph where each node has its own value (of number of sand grains) which can be passed to its neighbouring nodes when it has more sand grains then it has neighbours, called toppling. This is triggered by adding sand grains to nodes randomly. This simple rule can lead to complex behaviour, especially when the graph gets large. When a node passes its grains to its neighbours, it can happen that a neighbour exceeds its maximum value causing a domino effect as a consequence of the initial toppling, we call this an avalanche. On finite sized graphs, there is dissipation due to the boundary of the graph. That means that grains fall off the graph at the boundary. Probability mass functions on various kinds of behaviour can be analysed, for example, the avalance sizes. In SOC, such probabilities tend to follow a power law, which is seen in nature often. This model has been studied by D. Dhar and S. Majumdar [4]. It can easily be seen that the sandpile model admits to SOC. Suppose we have a graph with some cluster of nodes somewhere in the graph such that average height of the clusters node is higher then outside this cluster, then toppling causes the nodes outside the cluster to gain more nodes as the chosen cluster has a high toppling probability. From the perspective outside the cluster, it starts with a little amount of grains. Since the cluster is more likely to topple, its grains are passed to the nodes neighbouring this cluster, therefore, the outside of the cluster increases in height.

For the sandpile model in general, the operation of adding sand grains in combination with its configurations it admits to an Abelian group. This means that the order or adding grains commutes. Work has been done to find the distribution for avalanches of length n on various types of graphs. In the SOC state, this asymptotic behaviour of avalanches usually follow a power law $n^{-\tau}$. Bak, Tang and Wiesenfeld have done numerical simulations on the $\mathbb{Z}^d$ lattice, they found a power law with coefficient $\tau \approx 1.22$. On $\mathbb{Z}^d$ where $d \geq 3$, upper and lower bounds for τ have been calculated by Jarai, Redig and Saada (2007). On Bethe lattices there are already some results, especially for Bethe lattices generated by binary trees. Dhar and Majumdar have found a power law with exponent $\tau = 1.5$. The model is also analysed on Erdös-Renyi networks. This is a graph where each node has a fixed probability of being connected to each other. Bonabeau has studied the sandpile model on this type of graph and found a power law with exponent of $\tau = 1.5$ [5]. Other than this, there are not too many results since analytical results are hard to obtain.

There are multiple questions that are still unanswered about the sandpile model on $\mathbb{Z}^d$. A few examples are: What happens when a configuration in a finite subset of $\mathbb{Z}^d$ is fixed and sand grains are always dropped on a single node? It is known by simulations that this results to fractal geometries, but analytical results have not been found. Another distribution of the sandpile model that requires more research is the probability distribution of height of points on finite subsets of $\mathbb{Z}^d$. It is even not known that the sandpile model can be defined on the entire set $\mathbb{Z}^d$.

The power law distribution is frequently seen in real-world situations. For example in neuroscience, a research team from Bethesda, Maryland, has been researching the brains as a sandpile [8]. A brain contains about 86 billion neurons or nerve cells, these neurons are connected to one another by axons. Mathematically, one can see the brain as a graph where the nerve cells are vertices and the axons are edges. Nerve cells work using electrical potential, when the brain is processing, nerve cells are electrically excited and when they reach high values of potential, the potential of the neurons is released, firing to neurons that are connected by an axon. From an evolutionary point of view, it can be said that this graph of a healthy brain is optimized in such a way making it efficient for its tasks. When we observe for example a person that suffers from Alzheimer's disease, the amount of nerve cells of this person decrease throughout the brain making the brain less efficient changing the parameters of the avalanche distribution. The authors from [8] have studied the avalanche distribution using electrodes on a rat's brain. They conclude that the avalanches of a healthy rats brain does follow the power law from the sandpile model. Another application concerns the movement of the Earth's crust but it can also be applied to the stock market and observing traffic jams. When a traffic jam is too long, the cars go to other highways that lead to the same destination in order to get there faster, distributing the cars over the roads. In this application, the different highways can be seen as vertices [1].

In this thesis, the approach by Dhar and Majumdar will be elaborated in more detail. In their paper, Dhar and Majumdar have shown the calculations for Bethe lattices of degree $q = 2$. This calculation is done again in more detail and some of the techniques will be used treating the model on Bethe lattices of higher branching degrees. The goal is to give a clear proof of avalanche distribution function for any value of q. It will be shown that in the special case $q = 2$ some factors drop that do occur in the general case, this does not happen. Almost all analysis will be done on finite Bethe lattices, since the concept of SOC, with large avalanches only appear in very large systems such that power laws can exist. We are interested in the infinite volume limit of the model. In chapter 2, it will be shown that the model is a Markov chain and that it has a unique stationary distribution. The fact that there is a unique stationary distribution allows us to find the avalanche size distribution. Because we are interested in the infinite volume limit, it must be shown that this stationary distribution also exists in the limit.

It will be determined what types of configurations are allowed by setting a set of rules that configurations need to follow in order to be in the SOC state. The allowed configurations are distinguished in the later defined weak and strong, such that a relation can be found between the two, the asymptotic behaviour of the ratio between the total number of weakly and strongly allowed subconfigurations is important for making the calculations possible, therefore, an other approach is used. Doing so, allows to find the number of possible clusters and the number of clusters containing some chosen subconfiguration. Next, the generating function of the number of possible avalanches with a fixed length as seen from the origin or the lattice, is calculated using a recursive expression. In the case $q = 2$, this is solved using the quadratic equation, but for arbitrary valued of q, this is no longer possible. Combining these results allows us to find a probability mass function for avalanche sizes. To do so, the number of clusters of a certain size is divided by the total number of possible recurrent configurations. Multiplying this with the number of clusters that are precisely of length n yields the unique stationary distribution. This is followed by some analysis of its asymptotic behaviour resulting in a power law.

Chapter 2

The abelian sandpile model

2.1 Height configurations

Consider a finite set of nodes V, where the vertices $u \in V$ are connected by the edges $e \in E$ generating the graph $\mathscr{T} = (V, E)$. Each node in V gets an integer valued height. A height configuration η is a state where each $u \in V$ has its own value and is denoted by $\eta \in \mathbb{N}^{\mathscr{T}}$ where $\mathbb{N}^{\mathscr{T}}$ is the set of all possible height configurations with integer valued height values. Let q_u be the number of neighbours of vertex u, two nodes are neighbours if there exists an edge $e \in E$ that connects these two nodes. For $\eta \in \mathbb{N}^{\mathscr{T}}$ and $u \in \mathscr{T}$, $\eta_u \in \mathbb{N}$ denotes the height at node $u \in \mathscr{T}$. A height configuration $\eta \in \mathbb{N}^{\mathscr{T}}$ is called *stable* if for all $u \in V$ we have that $\eta_u \leq q_u$. The set of stable configurations of $\mathscr{T}$ is called $\Omega_{\mathscr{T}} \subseteq \mathbb{N}^{\mathscr{T}}$. If there exists a node $v \in V$ such that $\eta_v > q_v$, then the graph is called *unstable*.

We define *toppling* as the following. Assume we have a stable height configuration. If increasing the height by 1 at $u \in V$ causes the graph to become unstable, then the node u topples. This is defined by the *toppling operator* T_u as

$$(T_u(\eta))_v = \eta_v - \Delta_{uv} \tag{2.1}$$

where Δ_{uv} is the *toppling matrix*. This matrix has the values $\Delta_{uu} = q_u$ and $\Delta_{uv} = -1$ if u and v are neighbouring nodes. In words, one could say that u gives each of its neighbours a sand grain. We distinguish two types of topplings, when $\eta_u > q_u$, then the toppling T_u is *legal* and it is not legal when $\eta_u \leq q_u$. The *stabilisation* of a configuration η is defined by the unique stable configuration $\mathscr{T}(\eta) \in \Omega_{\mathscr{T}}$ as a consequence of a sequence of legal topplings $T_{u_n} \circ \cdots \circ T_{u_1}$. In this sequence of n legal topplings, every u_i, $i \in \{1, \ldots, n\}$ can be a different nodes.

2.2 Sandpile model as a Markov chain

Until now, the model is entirely deterministic. Consider again the finite graph $\mathscr{T} = (V, E)$ with V and the set E consisting of edges connecting nodes in V. One can increase the value of an arbitrary node u by 1. For this, the *addition operator* is defined as a mapping $a_u : \Omega_{\mathscr{T}} \to \Omega_{\mathscr{T}}$ by

$$a_u \eta = \mathscr{T}(\eta + \delta_u) \tag{2.2}$$

where $\delta_u \in \{0, 1\}^{\mathscr{T}}$ such that $\delta_u(u) = 1$ and $\delta_u(v) = 0$ when $u \neq v$ and $u, v \in V$. This is followed by a stabilisation. If $\eta_u + 1 \leq q_u$ then $\mathscr{T}(\eta + \delta_u) = \eta + \delta_u$ because no toppling is induced. If the addition causes $\eta_u > q_u$, then $\mathscr{T}(\eta + \delta_u)$ stabilises to an allowed configuration. It is known from other literature [6] that the addition operator is abelian, that is $a_u a_v \eta = a_v a_u \eta$ for all $\eta \in \Omega_{\mathscr{T}}$. It can also be seen that the configuration after a sand grain is dropped only depends on the configuration on which a grain is dropped, this yields the Markov property. A consequence is that the abelian sandpile model is a discrete-time Markov chain. This holds because the set V is finite and all nodes can have a finite number of values, causing the set $\Omega_{\mathscr{T}}$ to be finite. The Markov chain is then $\{\eta(n), n \in \mathbb{N}\}$ which is defined by

$$\eta(n) = \prod_{i=1}^{n} a_{X_i} \eta(0) \tag{2.3}$$

where X_i are i.i.d uniform random variables choosing nodes of $\mathscr{T}$ and $\eta(0)$ the starting configuration. $\eta(n)$ is nothing different than the addition operator applied n times. An *avalanche* is defined as a consequence of the addition operator. If increasing the value η_u by 1 causes a series of topplings, then we define the set $A(u,\eta) \in \mathscr{T}$ as the set of nodes that have toppled by applying the addition operator to u on height configuration η.

2.3 Recurrent configurations and stationary measure

A *recurrent* configuration is a configuration such that the probability $\mathcal{P}(\eta(n) = \eta|\eta(0) = \eta) > 0$ for all $n \in \mathbb{N}$. The set of all these recurrent configurations form a subset of the set of all stable configurations. We call a configuration $\eta \in \Omega_{\mathscr{T}}$ a *forbidden subconfiguration* (FSC) if there exists a subset $S \subseteq \mathscr{T}$ such that for all $u \in S$, η_u is less or equal than the number of neighbours of u. For clarification, pick some subconfiguraton in this configuration, the value of q_u depends on the neighbours that are also in the subconfiguration and not the ones that are outside. If a configuration does not contain a FSC, then we call the configuration allowed. Define $\mathscr{R}_{\mathscr{T}} = \{\eta \in \Omega_{\mathscr{T}} : \eta \text{ is recurrent}\}$. In [6] it is stated that this set coincides with the set of allowed configurations. By the ergodic theorem of Markov chains we know that there exists a unique probability density. The Markov chain has a stationary measure which is precisely the uniform probability measure. In the paper it is given that the stationary probability measure is given by

$$\mu_{\mathscr{T}} = \frac{1}{|\mathscr{R}_{\mathscr{T}}|} \sum_{\eta \in \mathscr{R}_{\mathscr{T}}} \delta_\eta. \tag{2.4}$$

Here, δ_n denotes the Dirac measure on configuration η. That is, $\delta_\eta(\eta') = 1$ if $\eta = \eta'$ and $\delta_\eta(\eta') = 0$ if $\eta \neq \eta'$. This allows us to find the probability mass function of the avalanche size by finding the total amount of avalanches of size n and dividing that by the total amount of allowed configurations.

2.4 The abelian sandpile model on infinitele trees

In the previous sections the sandpile model and some of its properties have been elaborated for finite graphs. Later when treating the asymptotic behaviour of the avalanche distribution on graphs of infinite size, it must first be shown that the model exists on such graphs. Another important property that needs to be shown is that there exists a unique stationary distribution. To do so, paper [3] is used. Define S as the infinite rootless binary tree, a binary tree means a tree with a branching degree $q = 2$. Let V be a finite subset of S. We denote $\Omega_V = \{\eta : V \to \{1,2,3\}^V\}$ as the set of all stable configurations on V and Ω as the set of all stable configurations on S denoted by $\Omega = \{\eta : S \to \{1,2,3\}^S\}$. Endowing Ω with the product topology makes it into a compact metric space.

Fix $\eta \in \Omega$, then we call the restriction of η to V, η_V. Recall the set $\mathscr{R}_V$ of recurrent configurations in V, it is easily seen that $\mathscr{R}_V \subset \Omega_V$. Define the set

$$\mathscr{R} = \{\eta \in \Omega : V \text{ is finite}, \eta_V \in \mathscr{R}_V\} \tag{2.5}$$

We call a function f *local* if there exists a finite $V \subset S$ such that for a restriction of η and η' to V with $\eta_V = \eta'_V$ implies $f(\eta) = f(\eta')$. This means that the values outside of V, in V^c, do not affect the value of $f(\eta)$. For $\eta, \zeta \in \Omega$ let $\eta_V \zeta_{V^c}$ denote the configuration whose restriction to V is given by η_V. So for any $\xi, \zeta \in \Omega$ we have $f(\eta_V \xi_{V^c}) = f(\eta_V \zeta_{V^c})$. The smallest set $V \subset S$ such that a function f is local is called the *minimal* and is denoted with D_f. A local function can be seen as a function on Ω_V for all $V \supset D_f$ and every function on Ω_V. Local functions are continuous on their domain [7].

Definition 1. *Let $\mathscr{S} = \{V \subset S,\ V \text{ finite}\}$ and $f : \mathscr{S} \to (K, d)$ with d the standard metric. Then*

$$\lim_{V \uparrow S} f(V) = \kappa \tag{2.6}$$

if for all $\varepsilon > 0$, there exists a $V_0 \in S$ such that for all $V \supset V_0$, $d(f(V), \kappa) < \varepsilon$.

This definition is satisfied since $D_f \subset V$ and $\kappa = f(S)$. Let $\varepsilon > 0$, since f is local we know that $f(V) = f(S) = \kappa$ and thus $d(f(V), \kappa) = 0 < \varepsilon$. Define N_V as the collection with all probability measures ν_V on Ω_V. We say that N_V is a Cauchy net is for any local function f, for all $\varepsilon > 0$ there exists a V_0 such that $D_f \subset V_0$ and for any $V, V' \supset V_0$ we have

$$\left| \int f(\eta) d\nu_V(\eta) - \int f(\eta) d\nu_{V'}(\eta) \right| \leq \varepsilon \tag{2.7}$$

Since $D_f \subset V_0 \subset V, V'$ and f is local, the integrals are equal, as a result, the collection of measures is a Cauchy net. A Cauchy net converges to a probability measure ν in the following sense. Let Ψ be the mapping of a local function f to the integral

$$\Psi : \mathscr{L} \to \mathbb{R}, \qquad f \mapsto \Psi(f) = \lim_{V \uparrow S} \int f d\nu_v, \tag{2.8}$$

where $\mathscr{L}$ is the set of all local functions, defines a linear functional. Since f is continuous on Ω and Ω is a compact topology, it follows that f is bounded. Now as a consequence of the Stone-Weierstrass theorem, $\mathscr{L}$ is dense in the set $C(\Omega)$ of continuous functions on Ω. Ψ satisfies the conditions $\Psi(f) \geq 0$ for any $0 \leq f \in \mathscr{L}$ since ν_V is a probability measure. And we have $\Psi(1) = 1$ as $\int d\nu_V = 1$. The Riesz representation theorem tells us that there exists a unique probability measure such that $\Psi(f) = \int f d\nu$ on Ω. We say that ν_V converges to the infinite volume limit ν, $\nu_V \to \nu$. A result of this is that the sandpile model can be extended to infinite sized graphs.

From here, everything states is proven in [6]. For $\eta \in \Omega$, $x \in S$, define the $C_3(x, \eta)$ as the nearest neighbour connected cluster of sites containing x having height 3. To show that that the infinite volume limit exists we state two theorems, the proofs are found in [6]. Proving the infinite volume dynamics requires the following theorems:

Theorem 1. *For the set $\mathscr{R}$ we have*

1. *$\mathscr{R}$ is compact,*
2. *The interior of $\mathscr{R}$ is empty,*
3. *For all $\eta \in \mathscr{R}$ there exists a non-constant sequence $(\eta_n)_{n \geq 1}$, $\eta_n \in \mathscr{R}$, such that $\eta_n \to \eta$ when $n \to \infty$.*

Theorem 2. *The finite volume stationary measures μ_V, defined in (2.4) form a Cauchy net. Their infinite volume limit satisfies the following:*

1. *$\mu(\mathscr{R}) = 1$,*
2. *μ is invariant under tree automorphisms and mixing,*
3. *$\mu(\eta : |C_3(0, \eta)| < \infty) = 1$,*
4. *$\int |C_3(0, \eta)| \mu(d\eta) = \infty$.*

This shows that μ is a probability measure. We extend the addition operator to the whole set Ω by $a_{x,V} : \Omega \to \Omega$ where $\eta \mapsto (a_{x,V} \eta_V)_V \eta_{V^c}$. This extension holds on a subset Ω' such that $\mu(\Omega') = 1$ and the addition operator still commutes and leaves the measure from the previous theorem invariant. Furthermore, it is shown in [3] that this extension is a Markov process with a unique stationary distribution. The proof of the existence of the stationary distribution can be generalised to infinite trees with an arbitrary branching degree $q \geq 2$ by changing all twos into q. This causes for example $\Omega_V = \{\eta : V \to \{1, 2, 3\}^V\}$, the rest follows.

It is now known that the sandpile model exists in infinite sized trees of any branching degree and that it is a Markov process with a invariant stationary distribution given by equation (2.4). The fact that the expressions of the distributions are the same on finite graphs as on infinite graphs is very convenient when analysing the asymptotic behaviour. Using this, we can proceed to the main focus of this thesis; the avalanche distribution.

Chapter 3

Avalanche distribution on the Bethe lattice with degree 2

First a special type of graph will be introduced on which we will analyse the sandpile model.

3.1 Allowed and forbidden states

The sandpile model will be analysed on a special type of graphs. To define this graph, we must first define homogeneous trees, more specifically, *binary trees*. A binary tree starts with a single node, this is a binary tree of zero generations, which generates the tree by one rule. To make a binary tree of generation $n+1$, every node of the nth generation of a binary tree of n generations becomes a parent node of 2 new nodes which are only neighbours of their parent node. A *Bethe lattice* of degree 2 is a node, the origin, which is connected by an edge to three binary trees. An example of a binary tree is seen in figure 3.2 and a Bethe lattice of degree 2 is seen in 3.1. First, we will look at the possibilities and the impossibilities for height configurations on binary trees in order to derive a recursive formula for the ratio between the number of weakly allowed and the strongly allowed configurations. What weakly and strongly allowed means will be defined later. Using this ratio is possible to find the asymptotic behaviour of the ratio. This number is used to find the total number of allowed configuration on the Bethe lattice observed from a single node. Using a transfer matrix, a matrix that describes the behaviour of two consecutive nodes, allows to look at larger clusters. After obtaining that result, the total number of allowed configurations is computed. Dividing the number of configurations containing a cluster by the total number of clusters yields the probability of finding a configuration containing a specific cluster.

When adding a sand grain to a node causing an avalanche, then we call the set of each node that has toppled as a consequence of that sand grain in that avalanche a cluster. In height configurations, one

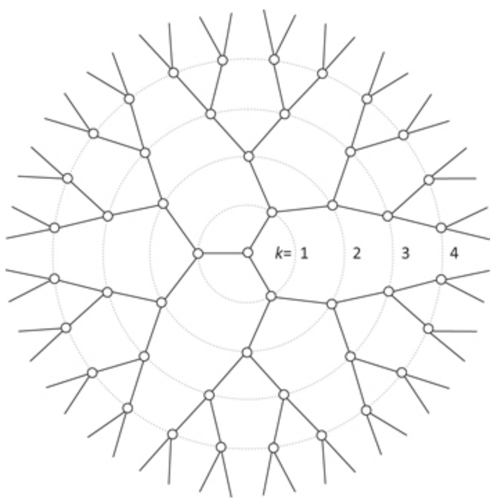

Figure 3.1: A Bethe lattice, k is the shell number (Source: iopscience.iop.org)

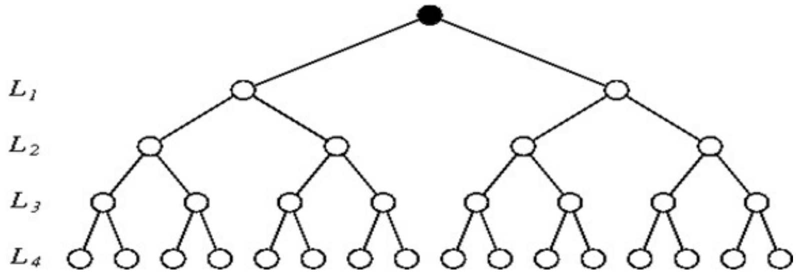

Figure 3.2: An example of a binary tree of 4 generations. The L_i, $i \in \{1, 2, 3, 4\}$ denotes the ith generation. The black node is the root of the binary tree. (Source: www.interviewcake.com)

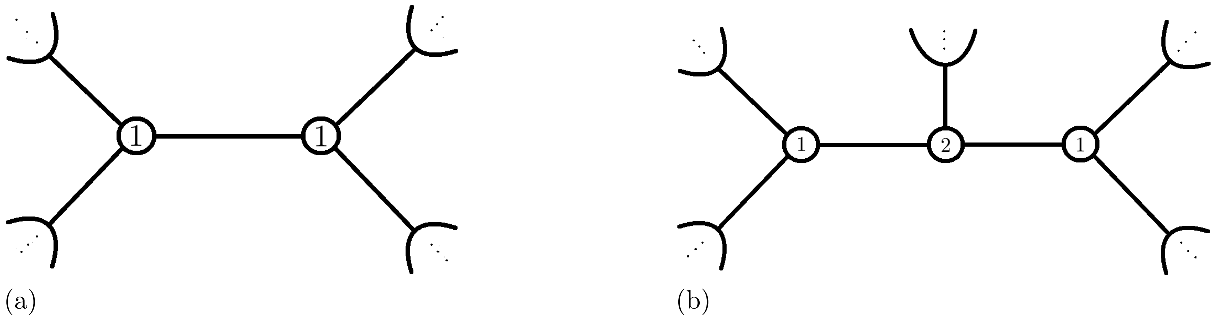

Figure 3.3: Two states which are forbidden in SOC state.

can observe smaller parts of the graph, we call these subconfigurations. These subconfigurations can be divided into two classes.

Definition 2. *Let T be a subtree with root a and let T be connected to the rest of the graph via node b. Consider an allowed subconfiguration C on T. Define C' as the subconfiguration where T is connected to node b ($T' = T \cup b$) through a. If C' is allowed when $\eta_b = 1$, then we call C' a strongly allowed subconfiguration. If setting $\eta_b = 1$ makes C' forbidden, then it is called a weakly allowed subconfiguration.*

In a SOC state there are some situations which are not allowed. For example, observe the situation in figure 3.3 (a). To create the first state with two neighbouring nodes with a height value of one, both nodes have to topple at some time. If a neighbour node topples, the value of the neighbouring node is atleast 2 which is always larger than 1. The state of a node with value 2 and two neighbours with value 1 is forbidden . This is because both 1-valued nodes have toppled. There are two options, the first option is that the node with value two has not toppled, then its value must be at least 3, the second option is that it has toppled, but if that is the case, at least one of the neighbours cannot have value 1.

Consider a binary tree T with root a. We define $N_w(T, i)$ as the number of weakly allowed subconfigurations of T with $\eta_a = i$ and $i \in \{1, 2, 3\}$. Similarly, $N_s(T, i)$ is the number of strongly allowed subconfigurations with height i. Let

$$\begin{cases} N_w(T) = \sum_{i=1}^{3} N_w(T, i), \\ N_s(T) = \sum_{i=1}^{3} N_s(T, i). \end{cases} \tag{3.1}$$

When we observe the lattice where T_1 and T_2 with roots a_1 and a_2 respectively are connected by their root nodes through node a, the following relations can be found.

$$N_w(T, 1) = N_s(T_1)N_s(T_2), \tag{3.2}$$
$$N_w(T, 2) = N_w(T_2)N_s(T_1) + N_w(T_1)N_s(T_2), \tag{3.3}$$
$$N_w(T, 3) = N_w(T_1)N_w(T_2), \tag{3.4}$$
$$N_s(T, 1) = 0, \tag{3.5}$$
$$N_s(T, 2) = N_s(T_1)N_s(T_2), \tag{3.6}$$
$$N_s(T, 3) = N_s(T_1)N_s(T_2) + N_w(T_1)N_s(T_2) + N_w(T_2)N_s(T_1). \tag{3.7}$$

Recall that $N_d(T,i)$ denotes the number of possible configurations on the tree T with root value i, where $d = s, w$. Suppose all trees that contribute to $N_d(T',i)$ and $N_d(T'',i)$ can be connected to $T = T' \cup T''$. Pick one tree from $N_d(T',i)$, for this tree, any tree that contributes to $N_d(T'',i)$ gives a possibility for $N_d(T,i)$. Since this holds for all trees from $N_d(T',i)$ the result is a product of the two numbers. Now each of the relationships will be clarified.

- (3.2). In this case, T has a root with value one, attaching an arbitrary weakly allowed subconfiguration results in two neighbouring nodes with value one. Therefore, T_1 and T_2 must both be strong.
- (3.3). As mentioned on page 8, attaching two subtrees with a root of value one, gives the configuration where we have the forbidden state seen in figure (3.3,b). If two strongly allowed subconfigurations are connected to the tree, the tree becomes strongly allowed. Since we observe a weakly allowed state, only one strongly and one weakly allowed subconfiguration can be used.
- (3.4). The situation where a node has value 3 and its three neighbouring nodes all have value 1 is a forbidden state by the same reason as the subconfiguration in (3.3,b). Having one weakly and one strongly allowed subconfiguration results in a strongly allowed configuration for T.
- (3.5). It is clear by definition that if the root of tree T has value one, it can never be strongly allowed.
- (3.6). Having one or two weakly allowed subconfigurations for T_1 and T_2 result in forbidden states.
- (3.7). Same argument for (3.4) except that we are now observing a strongly allowed configuration.

Now that the equalities have been elaborates properly, the sums can be observed. This is done to find a ratio between $N_w(T)$ and $N_s(T)$.

$$\begin{aligned} N_w(T) &= \big(N_s(T_1) + N_w(T_1)\big)\big(N_s(T_2) + N_w(T_2)\big), \\ N_s(T) &= 2N_s(T_1)N_s(T_2) + N_w(T_1)N_s(T_2) + N_s(T_1)N_w(T_2). \end{aligned} \tag{3.8}$$

Let $X(T) = \frac{N_w(T)}{N_s(T)}$, applying this to equation 3.8 yields

$$X(T) = \frac{N_w(T)}{N_s(T)} = \frac{\big(1 + X(T_1)\big)\big(1 + X(T_2)\big)}{2 + X(T_1) + X(T_2)}. \tag{3.9}$$

We call a tree where each node has two children a binary tree. If this tree consists of n generation, we denote this tree as B_n for $n \in \mathbb{N}$. Note that if two trees B_n are connected through the roots by a new point a, the new tree will be B_{n+1}. With this, a recursion formula can be found for $X(B_n)$ using equation 3.9. Take $T_1 = T_2 = B_{n-1}$, then

$$X(B_n) = \frac{(1 + X(B_{n-1})(1 + X(B_{n-1})}{2(1 + X(B_n))} = \frac{1 + X(B_{n-1})}{2}. \tag{3.10}$$

To solve this recursion, we have to find $X(B_0)$ to generate the next terms of the sequence $X(B_n)$. It is easy to see that $X(B_0) = \frac{1}{2}$ because $N_w(B_0)) = 1$ because there is only one possibility to make a tree of a single point with height 1 and two possibilities of having height higher than 1, assuming we have a binary tree. Writing out the first few terms results in

$$X(B_1) = \frac{3}{4}, \quad X(B_2) = \frac{7}{8}, \quad X(B_3) = \frac{15}{16}, \quad X(B_4) = \frac{31}{32}. \tag{3.11}$$

It can be shown by induction that

$$X(B_n) = 1 - 2^{-n-1}. \tag{3.12}$$

Conclude $X(B_n) \to 1$ as $n \to \infty$. The asymptotic behaviour is what we call "being deep in the lattice", it means that the smallest distance to one of the surface nodes tends to infinity. So in such a large lattice we have

$$N_w(T_1) + N_w(T_2) + N_w(T_3) = N_s(T_1) + N_s(T_2) + N_s(T_3). \tag{3.13}$$

To find a relation between the values between each of the components of equation (3.13), we use equations (3.2)-(3.7).

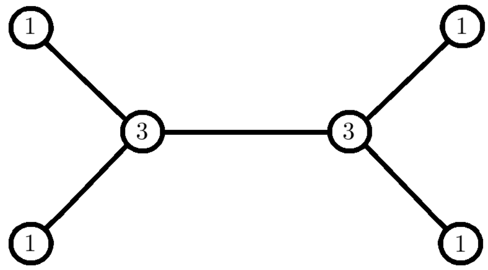

Figure 3.4: This state is forbidden because the states required to obtain this state are forbidden.

$N(i,j)$	$i=1$	$i=2$	$i=3$
$j=1$	0	1	3
$j=2$	1	5	10
$j=3$	3	10	15

Table 3.1: The number of allowed possibilities for neighbouring nodes with height i and j.

3.2 Height distribution on pairs of nodes

We are interested in the probability mass function of two points having specific values. Consider two neighbouring nodes, a_1 and a_2 and let $N(i,j)$ be the number of the possible configurations having when the nodes a_1 and a_2 have values i and j respectively. These two points have four subtrees as seen in figure 3.4, except the nodes with height one are replaced with subconfigurations. All combinations of i and j will be observed to see how many of which type of subconfiguration makes the whole configuration allowed. The first case $i=j=1$. As mentioned on page 8, a state with two neighbouring points having height 1 is forbidden in the SOC state. Therefore it will not occur and thus $N(1,1)=0$. In the case $i=1$, $j=2$ all connected subtrees must be strongly allowed subconfigurations. This is true because if one subconfiguration is weakly allowed, then there are either have two neighbouring nodes with value one, or there is a node with value 2 with two nodes with value 1 as neighbours as seen in 3.3. We conclude that all connected subtrees must be strongly allowed subconfigurations. The other situations are treated by the same arguments. The only special case is $i=j=3$. Here, four weakly allowed subconfigurations are not allowed, this is because all root nodes of the subtrees have toppled since they are allowed to have value 1 in order to get the subconfiguration in figure 3.4. In this case however, there must have been a 1-1 configuration which is forbidden. The result of the number of possibilities is seen in table 3.1. Because there is a total of 48 possibilities, the probability of each of these states is the number of possibilities divided by 48. Equation (3.8) can be used to derive a relation between two consecutive nodes. Expanding the equation and choosing the subtrees as in figure 3.6.

$$\begin{aligned} N_w(T_{k+1}) &= N_s(U_{k+2})\big[(1+x_{k+2})N_w(T_k)+(1+x_{k+2})N_s(T_k)\big] \\ N_s(T_{k+1}) &= N_s(U_{k+2})\big[N_w(T_k)+(2+x_{k+2})N_s(T_k)\big] \end{aligned} \tag{3.14}$$

where $x_{k+2}=\dfrac{N_w(U_{k+2})}{N_s(U_{k+2})}$, the ratio of weakly and strongly allowed subconfigurations. If the lattice size approaches infinity we get $x_{k+2}=1$ for all k as seen in equation (3.12). Writing the previous equality as

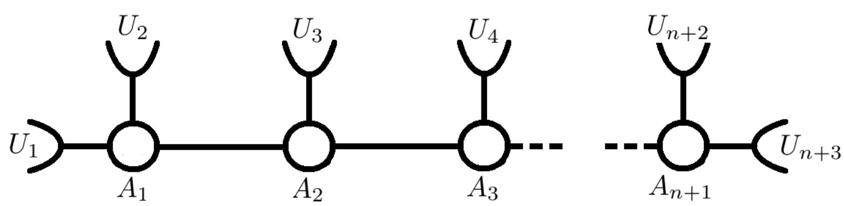

Figure 3.5

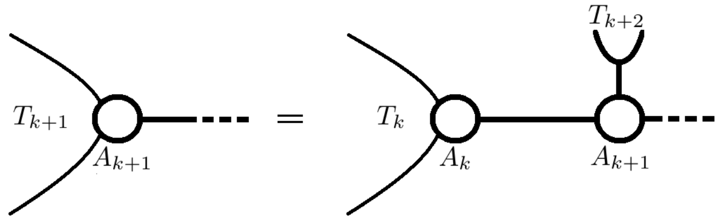

Figure 3.6: The subtree T_k with the node A_k can be seen as simplify T_{k+1}, this recursion idea is why we use the transfer matrix approach.

a matrix equation yields

$$\begin{pmatrix} N_w(T_{k+1}) \\ N_s(T_{k+1}) \end{pmatrix} = N_s(U_{k+2}) \begin{pmatrix} 1+x_{k+2} & 1+x_{k+2} \\ 1 & 2+x_{k+2} \end{pmatrix} \begin{pmatrix} N_w(T_k) \\ N_s(T_k) \end{pmatrix}. \tag{3.15}$$

When applying the transfer equation (3.15), the subtree T_{k+1} becomes T_k at its root A_k, See figure 3.6. Repeating this for the whole chain allows us to see what happens at two nodes which are at arbitrary distance to each other. By applying the equation (3.15) $n-1$ times and assuming that the chain is located deep in the lattice such that $x_k = 1$ it follows

$$\begin{pmatrix} N_w(T_n) \\ N_s(T_n) \end{pmatrix} = \prod_{k=1}^{n-1} \left(N_s(U_{k+2}) \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \right) \begin{pmatrix} N_w(T_1) \\ N_s(T_1) \end{pmatrix} = \left[\prod_{k=1}^{n-1} N_s(U_{k+2}) \right] \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}^{n-1} \begin{pmatrix} N_w(T_1) \\ N_s(T_1) \end{pmatrix}. \tag{3.16}$$

This is motivated by figure 3.5. Using the eigenvector decomposition of the matrix $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, its powers can be expressed as

$$\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}^n = \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4^n & 0 \\ 0 & 1^n \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4^n+2 & 2\cdot 4^n - 2 \\ 4^n - 1 & 2 \cdot 4^n + 1 \end{pmatrix}. \tag{3.17}$$

Using equation (3.16), $N_s(T_n)$ can be evaluated to

$$\begin{aligned} N_s(T_n) &= \left[\prod_{k=1}^{n-1} N_s(U_{k+2}) \right] \cdot \frac{1}{3} \big((4^{n-1}-1) N_w(T_1) + (2\cdot 4^{n-1}+1) N_s(T_1) \big) \\ &\overset{\text{Eqn. (3.13)}}{=} \left[\prod_{k=1}^{n-1} N_s(U_{k+2}) \right] 4^{n-1} N_s(T_1). \end{aligned} \tag{3.18}$$

This will be used to evaluate the total number of possible clusters.

Suppose there is a node with the subtrees connected, T_n, U_{n+1}, U_{n+2}. Next, the total number of allowed configurations is calculated. Consider a single node with height $i=1$, then there is only one possibility for these subtrees, they all must be strong to be in an allowed state. For $i=2$ there are the options, all strongly allowed subtrees and the situation where there are 2 strongly allowed subtrees and one weakly allowed, which gives four possibilities. And last, $i=3$, here we allow all possibilities with 0, 1 and 2 weakly allowed subconfigurations, resulting in 7 possibilities.

$$\begin{aligned} N_{total}(1) &= N_s(T_n) N_s(U_{n+2}) N_s(U_{n+1}), \\ N_{total}(2) &= 4 N_s(T_n) N_s(U_{n+2}) N_s(U_{n+1}), \\ N_{total}(3) &= 7 N_s(T_n) N_s(U_{n+2}) N_s(U_{n+1}). \end{aligned}$$

Then the total number of configurations is equal to

$$N_{total} = \sum_{i=1}^{3} N_{total}(i) = 12 N_s(T_n) N_s(U_{n+2}) N_s(U_{n+1}). \tag{3.19}$$

Earlier an expression for $N_s(T_n)$ was found, using that result from equation (3.18) gives

$$N_{total} = 12 \left[\prod_{k=3}^{n+1} N_s(U_k) \right] 4^{n-1} N_s(T_1) N_s(U_{n+2}) N_s(U_{n+1}) = 3 \cdot 4^n \prod_{k=1}^{n+3} N_s(U_k), \tag{3.20}$$

T_1 is located at the end of the chain, since the last node has one extra connected subconfiguration, T_1 exists out of two subconfigurations, we call these U_1 and U_2. Now that we know how many possible clusters exist, we are interested in how many clusters exist of a certain size. This will be elaborated in the next section.

3.3 Number of allowed clusters on the Bethe lattice

Consider a cluster C containing n nodes, that is the set of nodes which are connected by edges which are involved in an avalanche. If this cluster contains n nodes, then there are $n+2$ subconfigurations connected to it. This is because when we have one node, then there are $3 = 1+2$ subtrees and every new node that is connected to the cluster we get a total of three new subtrees to that node. The cluster take one connection, leaving two unused, one of which is a new one and the other subtree already existed. So adding a new node gives one extra subtree, resulting that each cluster consisting of n nodes has $n+2$ subtrees connected. For all these subtrees there are two restrictions that need to be taken into account

1. Not all U_i, $i \in \{1, \dots, n+2\}$ can be weakly allowed,
2. No subtree can have a root node with height 3.

To verify the first rule, let C be an allowed cluster consisting of one node, a. In order to be a part of this cluster, it must be true that $\eta_a = 3$ before the toppling, otherwise it would not topple when gaining a grain. There can be three subconfigurations attached to a. Assume they can all be weakly allowed. In this case there exists a subset in C which has exactly three neighbours. This subset is the set containing only the node a. Then by the definition on page 4, C is forbidden because $3 \not> 3$. As a non-trivial case, consider a new allowed cluster C' containing n nodes and assume that all connected subconfigurations are weakly allowed. Since C' is allowed, there exists by definition a node a such that $\eta_a > q_a$. In order to be part of the cluster is must hold that $\eta_a = 3$. But by the assumption that C' is allowed it must hold that $\eta_a > q_a = 3$. Because in a Bethe lattice of degree 2, a node has at most three neighbours so this contradicts the assumption that C' is allowed. We conclude that not all subconfigurations can be weakly allowed. Using these rules we can determine the number of configurations $N(C)$ containing a certain cluster C. For this, we want the sum of all possibilities of products where each subconfiguration U_i is chosen as weak of strong precisely once. First let

$$\bar{N}(C) = \prod_{i=1}^{n+2} \left[N_s(U_i, 2) + N_w(U_i, 1) + N_w(U, i, 2) \right]. \tag{3.21}$$

The term $\bar{N}(C)$ also counts the configurations where each subconfiguration that is connected to C is weakly allowed. To compensate, the options with only weakly allowed subconfigurations should be subtracted. This results in

$$N(C) = \bar{N}(C) - \prod_{i=1}^{n+2} \left[N_w(U_i, 1) + N_w(U_i, 2) \right]. \tag{3.22}$$

Using the relations previously found in equation (3.13) makes it possible to write $N(C)$ in terms of only $N_s(U_i)$.

$$N(C) = \prod_{i=1}^{n+2} \left[1 \cdot N_s(U_i) \right] - \prod_{i=1}^{n+2} \left[\frac{3}{4} \cdot N_s(U_i) \right] = \left[1 - \left(\frac{3}{4} \right)^{n+2} \right] \prod_{i=1}^{n+2} N_s(U_i) \tag{3.23}$$

The probability $U_c(n)$ is obtained by dividing the clusters containing C by the total amount of clusters, hence

$$U_c(n) = \frac{N(C)}{N_{total}} = \frac{1 - \left(\frac{3}{4}\right)^{n+2}}{3 \cdot 4^n} \frac{\prod_{i=1}^{n+2} N_s(U_i)}{\prod_{i=1}^{n+2} N_s(U_i)} = \frac{1}{3} \left[1 - \left(\frac{3}{4} \right)^{n+2} \right] 4^{-n}. \tag{3.24}$$

It can be concluded that the larger a cluster is, the less likely it is for such a cluster to occur in the SOC state. Note that the value $U_c(n)$ is independent of the shape of the chosen cluster C and is only dependent on the number of nodes involved. Let a_n be the number of distinct clusters of size n. For each cluster that contributes to a_n, this shape-independency holds. Therefore, multiplying $U_c(n)$ with a_n yields the probability of obtaining a cluster of size n.

In the next chapter, this concept will be generalised for Bethe lattices of any degree. Once $U_c(n)$ is found for all branching degrees, an expression for a_n will be derived.

Chapter 4

Avalanche distribution on the Bethe lattice for any branching degree

When observing Bethe lattices of any degree $q \geq 2$ requires more work then the case done in the previous chapter. In order to find a recursion relationship the following rules need to be followed.

1. Two neighbouring nodes cannot both have height 1.
2. The state where one node of height two has two neighbours with height one is forbidden.
3. If a node has height $c \in \{1, 2, \ldots, q+1\}$, then it can have at most $c-1$ neighbours with height one in order to be allowed.

Rules one and two have been elaborated before. The last rule can be seen as a generalisation of the first two. Suppose we fix a node of value 1 with $q+1$ neighbours which all have a height of non-one, otherwise the state would not be allowed by rule 1. In order for all these nodes to reach a value of one, they must topple at least once. The toppling of these nodes may not cause the fixed node to topple, otherwise they get a value of two, when we are only interested in heights of 1. Suppose that more than $c-1$ nodes topple, then the fixed node has a value higher than c or has toppled. When $c-1$ nodes topple, this is not the case. This verifies rule three.

The first few relationships between the value of the root and the root of the subconfiguration will be derived using these rules, then a general expression will be deduced. Let T be some binary tree with root a. It is clear that $N_s(T,1) = 0$ because this configuration is strongly allowed, it can be connected with its root to a new node with height one. Because this would lead to a forbidden state the solution must be zero. When the root node has two grains, to prevent us from breaking the second rule there may not be any weakly allowed configurations connected to this node. Since it is a strongly allowed state it would have two ones connected, violating rule 2. Therefore $N_s(T,2) = \prod_{i=1}^{q} N_s(T_i)$. For the sake of notation, let $x_j = \frac{N_w(T_j)}{N_s(T_j)}$. Using the same treatment for the case $i = 2$, one can argue that $N_s(T,3) = \left[1 + \sum_{j=1}^{q} x_j\right]\left[\prod_{i=1}^{q} N_s(T_i)\right]$. Looking at these solutions carefully, it can be seen that there is a relation between the value of the root node and the height. Every possibility that does not violate rule three gives a possibility that contributes to the value $N_s(T,i)$. Summing over all possibilities with less or equal to $i-2$ weakly allowed subconfigurations connected. Denote $\mathcal{I}_i^q$ as the collection of all disjoint index sets with length i containing only elements from $\{1, 2, \ldots, q\}$. Define the collection

$$\mathcal{I}_i^q = \{A \in \mathscr{P}(\{1, 2, \ldots, q\}) \mid \#A = i\}, \tag{4.1}$$

where $q \in \mathbb{N}_{\geq 2}$ and $\mathscr{P}(\{1, 2, \ldots, q\})$ denotes the power set of $\{1, 2, \ldots, q\}$. In words, it is a collection of index sets containing elements from $\{1, 2, \ldots, q\}$ with cardinality i. By convention let $\sum_{k \in \mathcal{I}_0^q} \prod_{\ell \in k} x_\ell = 1$. Using this, the number of strongly allowed subconfigurations with root height of i is expressed as

$$N_s(T,i) = \left[\sum_{\eta=0}^{i-2} \sum_{k \in \mathcal{I}_\eta^q} \prod_{\ell \in k} x_\ell\right] \prod_{m=1}^{q} N_s(T_m). \tag{4.2}$$

An example of what this looks like will be given in equations (4.6) and (4.7). For $N_w(T,i)$ the treatment is slightly easier. Here, there can be only one fixed number depending on i of weakly allowed subconfigurations attached to the observed node. We want a root node with value i and q subtrees connected to that node. The root a of T is also connected to another node b. From the q subtrees, we want the highest number of weakly allowed subconfigurations connected to the node such that the configuration is still allowed if $\eta_a \neq 1$ and if η_b becomes one, it becomes forbidden. This happens when we have exactly $i-1$ weak subconfigurations, if there are less, the configurations becomes strong and one more weak configuration makes it forbidden. By following this argument, it is easy to realise that $N_w(T,1) = \prod_{i=1}^{q} N_s(T_i)$ because this has exactly $1-1=0$ weakly allowed subtrees. In general:

$$N_w(T,i) = \left[\sum_{k\in\mathcal{I}_{i-1}^q} \prod_{\ell\in k} x_\ell\right] \prod_{m=1}^{q} N_s(T_m) \tag{4.3}$$

since for a weakly allowed state with height i there must be precisely $i-1$ weakly allowed subconfigurations connected to the root. Because, to get the desired result, each T_i will be substituted with B_{n-1} as in the $q=2$ case, we are only interested in how many disjoint index sets there are for all values i since we will be able to compute the value of x_ℓ. For the set $\mathcal{I}_n^{q+1}$, there are $q+1$ values a node can take and i values are used, there are $q+1-i$ values that are not in the set. This coincides with the combinatorial interpretation of the binomial coefficient so the collection $\mathcal{I}_i^{q+1}$ contains $\binom{q+1}{i}$ sets. The total number of weakly and strongly allowed subconfigurations is given by

$$N_s(T) = \sum_{i=2}^{q+1} N_s(T,i) = \left[\sum_{i=2}^{q+1}\sum_{\eta=0}^{i-2}\sum_{k\in\mathcal{I}_\eta^q} \prod_{\ell\in k} x_\ell\right] \prod_{j=1}^{q} N_s(T_j) \tag{4.4}$$

$$N_w(T) = \sum_{i=1}^{q+1} N_w(T,i) = \left[\sum_{i=1}^{q+1}\sum_{k\in\mathcal{I}_{i-1}^q} \prod_{\ell\in k} x_\ell\right] \prod_{j=1}^{q} N_s(T_j) \tag{4.5}$$

Since these equalities look rather complicated it does no harm to simplify them. First a small example will be given to make the formulas above more readable. Set $q=3$ and take the term $i=4$. The collection $\mathcal{I}_1^3$ consists of all sets of length 1 containing only elements of the set $\{1,2,\ldots,q+1\}$. It follows that $\mathcal{I}_1^3 = \{\{1\},\{2\},\{3\}\}$ so the cardinality of this collection $\#\mathcal{I}_1^3 = 3 = \binom{q}{1}$. $\mathcal{I}_2^3 = \{\{1,2\},\{1,3\},\{2,3\}\}$ containing $\binom{q}{2} = 3$ elements. Applying this to equation (4.4) yields

$$\begin{aligned} N_s(T,4) &= \left[\sum_{\eta=0}^{i-2}\sum_{k\in\mathcal{I}_\eta^{q+1}} \prod_{\ell\in k} x_\ell\right] \prod_{i=1}^{3} N_s(T_i) = \prod_{\ell\in\mathcal{I}_0^3} x_\ell + \left[\sum_{k\in\mathcal{I}_1^3} \prod_{\ell\in k} x_\ell\right] \prod_{i=1}^{3} N_s(T_i) + \left[\sum_{k\in\mathcal{I}_2^3} \prod_{\ell\in k} x_\ell\right] \prod_{i=1}^{3} N_s(T_i) \\ &= N_w(T_1)N_s(T_2)N_s(T_3) + N_s(T_1)N_w(T_2)N_s(T_3) + N_s(T_1)N_s(T_2)N_w(T_3) \\ &+ N_w(T_1)N_w(T_2)N_s(T_3) + N_w(T_1)N_s(T_2)N_w(T_3) + N_s(T_1)N_w(T_2)N_w(T_3) \\ &+ N_s(T_1)N_s(T_2)N_s(T_3). \end{aligned} \tag{4.6}$$

Continuing in this manner for higher values of q and i will always give binomial cardinalities for the index collection. The parts between the brackets in equation (4.5) is simply the sum of all different products with a fixed amount of terms representing weakly allowed configurations. When we consider the case $q=3$ and $i=2$ for $N_w(T,i)$ we get

$$\begin{aligned} N_w(T,2) &= \sum_{k\in\mathcal{I}_2^q} \prod_{\ell\in k} x_\ell \prod_{i=1}^{3} N_s(T_i) \\ &= N_s(T_1)N_w(T_2)N_w(T_3) + N_w(T_1)N_s(T_2)N_w(T_3) + N_w(T_1)N_w(T_2)N_s(T_3). \end{aligned} \tag{4.7}$$

Recall the definition of $X(T)$ from page 9. Dividing by $\prod_{i=1}^{q} N_s(T_i)$ leaves us with

$$\left[\prod_{i=1}^{q} N_s(T_i)\right]^{-1} N_w(T) = \left[\prod_{i=1}^{q} N_s(T_i)\right]^{-1} \sum_{i=1}^{q+1} \left[\sum_{k\in\mathcal{I}_{i-1}^q} \prod_{\ell\in k} x_\ell\right] \prod_{i=1}^{q} N_s(T_i) \tag{4.8}$$

$$= \sum_{i=1}^{q+1} \sum_{k\in\mathcal{I}_{i-1}^q} \prod_{\ell\in k} X(T_\ell). \tag{4.9}$$

Intuitively, this is just the sum of all possible products of $X(T_i)$ where each distinct term at most once in the sum of each possible product. To simplify this even further, the following lemma will be helpful.

Lemma 1. *The product* $\prod_{n=1}^{q}(1+a_n)$ *is the sum of every possible product of* a_i*, with each product occuring precisely once.*

Proof. This is proven using induction. The product of two terms is calculated first, $(1+a_1)(1+a_2) = 1+a_1+a_2+a_1a_2$. Set the induction hypothesis that $\prod_{n=1}^{q}(1+a_n)$ is a sum as described in the statement. Multiplying this by $1+a_{q+1}$ yields

$$\left[\prod_{n=1}^{q}(1+a_n)\right](1+a_{q+1}) = 1\cdot\prod_{n=1}^{q}(1+a_n) + a_{q+1}\prod_{n=1}^{q}(1+a_n). \tag{4.10}$$

For each term in the sum there are two options: it gets the new factor a_{q+1} or it does not. We assumed that we started with a sum containing each product precisely once, multiplying by $(1+a_{q+1})$ results in a sum with that same property except it has one more possible term in its product. Looking at equation (4.10), this is exactly what happens. □

Using this lemma we can write equation (4.9) as a simple product

$$N_w(T) = \sum_{i=1}^{q+1} N_w(T,i) = \sum_{i=1}^{q+1}\sum_{k\in\mathcal{I}_{i-1}^{q}}\prod_{\ell\in k} X(T_\ell) = \prod_{i=1}^{q}\big(1+X(T_i)\big). \tag{4.11}$$

since the left hand side is the sum of every possible product of distinct terms. We proceed to simplify $N_s(T)$.

$$\begin{aligned}\left[\prod_{i=1}^{q} N_s(T_i)\right]^{-1} N_s(T) &= \left[\prod_{i=1}^{q} N_s(T_i)\right]^{-1}\sum_{i=1}^{q+1}\left[\sum_{\eta=0}^{i-2}\sum_{k_\eta\in\mathcal{I}_\eta^q}\prod_{\ell\in k_\eta} x_\ell\right]\prod_{i=1}^{q} N_s(T_i)\\ &= \sum_{i=2}^{q+1}\sum_{\eta=0}^{i-2}\sum_{k_\eta\in\mathcal{I}_\eta^q}\prod_{\ell\in k_\eta} x_\ell = \sum_{i=0}^{q-1}\sum_{\eta=0}^{i}\sum_{k\in\mathcal{I}_\eta^q}\prod_{\ell\in k} x_\ell\\ &\overset{(\dagger)}{=} \sum_{i=1}^{q} i\binom{q}{i} X(B_{n-1})^{q-i}.\end{aligned} \tag{4.12}$$

In the step (†) the order of summation has been changed and T_ℓ is substituted with B_{n-1} for all $i \in \{1,\ldots,q\}$. Here, B_n is defined as the binary tree of n generations. The substitution $T_i = B_{n-1}$ is done because we are observing a tree from the origin with $q+1$ identical subtrees attached to it. This expression is reduced by using Newton's binomial theorem. Let $f(x) = (1+x)^q$, then by the binomial theorem it follows that $f(x) = \sum_{i=0}^{q}\binom{q}{i}x^i$. Then $xf'(x) = \sum_{i=0}^{q} i\binom{q}{i}x^i$, from this we conclude that $xf'(x^{-1}) = \sum_{i=1}^{q} i\binom{q}{i}x^{-i} = \frac{q(1+x^{-1})^q}{x(1+x^{-1})}$. It follows that $\sum_{i=1}^{q} i\binom{q}{i}x^{q-i} = q(1+x)^{q-1}$. Letting $x = X(B_{n-1})$ yields the expression for $N_s(T)$. Now that some useful expressions have been found for both the number of weak and strongly allowed states, a recursive relation will be derived which can be used to solve $X(B_n)$.

$$X(B_n) = \frac{N_w(B_n)}{N_s(B_n)} = \frac{\big(1+X(B_{n-1})\big)^q}{q\big(1+X(B_{n-1})\big)^{q-1}} = \frac{1+X(B_{n-1})}{q}. \tag{4.13}$$

Using these results, the ratio of weakly and strongly allowed subconfigurations in a Bethe lattice containing n generations can finally be calculated. Applying the argument used in the previous chapter it is clear that $N_w(B_0) = 1$ and $N_s(B_0) = q$. Writing out the first three values results in

$$x_1 = \frac{1+\frac{1}{q}}{q} = q^{-1}+q^{-2}, \qquad x_2 = \frac{1+\frac{1+\frac{1}{q}}{q}}{q} = q^{-1}+q^{-2}+q^{-3}, \qquad x_3 = \frac{1+\frac{1+\frac{1+\frac{1}{q}}{q}}{q}}{q} = \sum_{k=1}^{4} q^{-k}. \tag{4.14}$$

It is easily shown by induction that $x_n = \sum_{k=1}^{n+1} q^{-k}$. Suppose it is true, then

$$x_{n+1} = \frac{1+\sum_{k=1}^{n+1} q^{-k}}{q} = q^{-1} + \sum_{k=2}^{n+2} q^{-k} = \sum_{k=1}^{n+2} q^{-k},$$

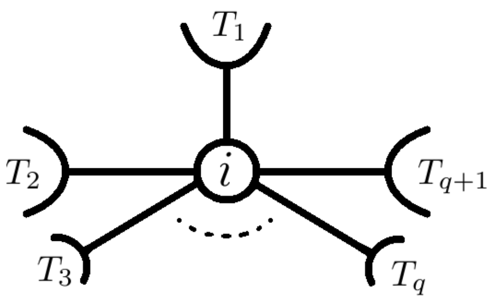

Figure 4.1: The node with height z_i has $q+1$ neighbours, q children and a parent node.

which verifies the claim. Since $q > 1$ this is a finite geometric series, it follows

$$x_n = \sum_{k=1}^{n+1} \left(q^{-1}\right)^k = \frac{1-q^{-(n+2)}}{1-q^{-1}} - \frac{1-q^{-1}}{1-q^{-1}} = \frac{1-q^{-n-1}}{q-1}. \tag{4.15}$$

Because we are interested in the asymptotic behaviour of this system we take the limit of the sequence x_n.

$$\lim_{n\to\infty} \frac{1-q^{-n-1}}{q-1} = \frac{1}{q-1}. \tag{4.16}$$

We conclude that the ratio of weakly and strongly allowed subconfigurations is approaches $\frac{1}{q-1}$ as the lattice gets larger. This asymptotic behaviour will be used to derive the behaviour on different nodes.

4.1 The total numbers of allowed clusters

In order to find the probability mass function of the avalanches of size n the total number of configurations observed from a single node is required. This value is calculated by observing the possibilities per root node height. Denote the number of allowed configurations with a origin value of i with $N(i)$. If the root has a value of i, every configuration that contributes to $N(i-1)$ is valid for $N(i)$. The sum of all these configurations is the total amount of possibilities $N_{total} = \sum_{i=1}^{q+1} N(i)$. The value $N(i)$ is the sum of all weakly and strongly allowed configurations for that particular value i. Since i is an origin value, both the parent and children nodes need to be considered, hence, there are $q+1$ subconfigurations that can be both strong or weak instead of q, see figure 4.1. Let

$$\begin{aligned} N(i) = N_w(T,i) + N_s(T,i) &= \left[\sum_{k\in I_{i-1}^{q+1}} \prod_{\ell\in k} x_\ell\right] \prod_{m=1}^{q+1} N_s(T_m) + \left[\sum_{n=0}^{i-2} \sum_{k\in I_n^{q+1}} \prod_{\ell\in k} x_\ell\right] \prod_{m=1}^{q+1} N_s(T_m) \\ &= \left[\sum_{n=0}^{i-1} \sum_{k\in I_n^{q+1}} \prod_{\ell\in k} x_\ell\right] \prod_{m=1}^{q+1} N_s(T_m). \end{aligned}$$

Using that $x_\ell = \frac{1}{q-1}$ deep in the lattice and $\#\mathcal{I}_n^{q+1} = \binom{q+1}{n}$, the value of $\sum_{n=0}^{i-1} \sum_{k\in\mathcal{I}_n^{q+1}} \prod_{\ell\in k} x_\ell$ is calculated.

$$\sum_{n=0}^{i-1} \sum_{k\in I_n^{q+1}} \prod_{\ell\in k} x_\ell = \sum_{n=0}^{i-1} \sum_{k\in I_n^{q+1}} \prod_{\ell\in k} \frac{1}{q-1} = \sum_{n=0}^{i-1} \sum_{k\in I_n^{q+1}} \left(\frac{1}{q-1}\right)^n = \sum_{n=0}^{i-1} \binom{q+1}{n} \left(\frac{1}{q-1}\right)^n.$$

summing over all $i \in \{1,\dots,q+1\}$ yields

$$\begin{aligned} \sum_{i=1}^{q+1} \sum_{n=0}^{i-1} \binom{q+1}{n} \left(\frac{1}{q-1}\right)^n &= \sum_{i=0}^{q} \sum_{n=0}^{i} \binom{q+1}{n} \left(\frac{1}{q-1}\right)^n \\ &= \sum_{i=0}^{q} (q+1-i) \binom{q+1}{i} \left(\frac{1}{q-1}\right)^i = \left(\frac{q}{q-1}\right)^q (q+1), \end{aligned}$$

which results in

$$N_{total} = \left(\frac{q}{q-1}\right)^q (q+1) \prod_{i=1}^{q+1} N_s(T_i), \tag{4.17}$$

using equation (4.12). Now clusters of nodes can be analysed instead of a single node. This is done using transfer matrices.

4.2 Recursive behaviour in a chain of nodes

Recall equation (4.5), this equation is exactly the sum of each possible product of length q containing the terms $N_d(T_i)$, $d = s, w$, and each i occurs precisely once per product. Using lemma 1, $N_w(T)$ can be rewritten as

$$N_w(T) = \prod_{i=1}^{q} \big[N_w(T_i) + N_s(T_i)\big]. \tag{4.18}$$

From this point, let $T = T_{k+1}$, $T_1 = T_k$ and $T_\nu = U_{k+2,\nu}$ for $\nu \in \{2, \cdots, q\}$. Then

$$\begin{aligned} N_w(T_{k+1}) &= \left[\prod_{i=2}^{q} \big[N_w(T_i) + N_s(T,i)\big]\right] \big[N_w(T_k) + N_s(T_k)\big] \\ &= \left[\prod_{i=2}^{q} N_s(U_{k+2,i})\right] \left[\prod_{i=2}^{q} [x_i + 1]\right] \big[N_w(T_k) + N_s(T_k)\big] \\ &= \left[\prod_{i=2}^{q} N_s(U_{k+2,i})\right] \left(\frac{q}{q-1}\right)^{q-1} \big[N_w(T_k) + N_s(T_k)\big], \end{aligned} \tag{4.19}$$

allowing us to write $N_w(T_{k+1})$ in terms of $N_w(T_k)$, and $N_s(T_k)$. The same can be done for $N_w(T_{k+1})$ in the following. Let $\varphi = \prod_{i=2}^{q} N_s(T_i)$. Using equation (4.4) we get

$$\begin{aligned} N_s(T) &= \left[\sum_{i=2}^{q+1}\sum_{n=0}^{i-2}\sum_{k\in\mathcal{I}_n^q}\prod_{\ell\in k} x_\ell\right]\prod_{i=1}^{q} N_s(T_i) = \left[\sum_{i=0}^{q-1}\sum_{n=0}^{i}\sum_{k\in\mathcal{I}_n^q}\prod_{\ell\in k} x_\ell\right] N_s(T_1)\varphi \\ &= \left[\sum_{i=0}^{q-1}(q-i)\sum_{k\in\mathcal{I}_i^q}\prod_{l\in k} x_\ell\right] N_s(T_1)\varphi. \end{aligned}$$

The goal is to express $N_s(T_{k+1})$ in terms of $N_s(T_k)$ and $N_w(T_k)$ so a relation between two nodes can be found at an arbitrary distance. To do so we have to split cases. Define the sets $\mathcal{I}_n^{q+} = \{A \in \mathcal{I}_n^q : 1 \in A\}$ and $\mathcal{I}_n^{q-} = \{A \in \mathcal{I}_n^q : 1 \notin A\}$ such that $\mathcal{I}_n^{q+} \cup I_n^{q-} = \mathcal{I}_n^q$. Note that in the case $\mathcal{I}_n^{q-}$ there are $q-1$ values to choose for n positions, therefore $\#\mathcal{I}_n^{q-} = \binom{q-1}{n}$ and by Pascal's identity it can be concluded that $\mathcal{I}_n^{q+} = \#\mathcal{I}_n^q - \#\mathcal{I}_n^{q-} = \binom{q}{n} - \binom{q-1}{n} = \binom{q-1}{n-1}$. Note that the collection $\mathcal{I}_0^{q+}$ does not exist, as it cannot contain sets with no elements since they contain $\{1\}$ by definition. Now $N_s(T)$ can be split the following way

$$N_s(T) = \left[\sum_{i=0}^{q-1}(q-i)\sum_{\psi\in\mathcal{I}_i^q}\prod_{l\in\psi} x_\ell\right] N_s(T_k)\varphi' = \left[\sum_{i=0}^{q-1}(q-i)\sum_{\psi\in\mathcal{I}_n^{q+}\cup\mathcal{I}_n^{q-}}\prod_{l\in\psi} x_\ell\right] N_s(T_k)\varphi' \tag{4.20}$$

$$= \left[q\sum_{i=0}^{q-1}\sum_{\psi\in\mathcal{I}_n^{q+}}\prod_{l\in\psi\setminus\{1\}} x_\ell\right] N_w(T_k)\varphi' + \left[q\sum_{i=0}^{q-1}\sum_{\psi\in\mathcal{I}_n^{q-}}\prod_{l\in\psi} x_\ell\right] N_s(T_k)\varphi' -$$

$$\left[\sum_{i=0}^{q-1} i\sum_{\psi\in\mathcal{I}_n^{q+}}\prod_{l\in\psi\setminus\{1\}} x_\ell\right] N_w(T_k)\varphi' - \left[\sum_{i=0}^{q-1} i\sum_{\psi\in\mathcal{I}_n^{q-}}\prod_{l\in\psi} x_\ell\right] N_s(T_k)\varphi'$$

$$= \left[q\sum_{i=0}^{q-1}\binom{q-1}{i-1}\left(\frac{1}{q-1}\right)^{i-1}\right] N_w(T_k)\varphi' + \left[q\sum_{i=0}^{q-1}\binom{q-1}{i}\left(\frac{1}{q-1}\right)^{i}\right] N_s(T_k)\varphi' - \tag{1}$$

$$\left[\sum_{i=0}^{q-1} i\binom{q-1}{i-1}\left(\frac{1}{q-1}\right)^{i-1}\right] N_w(T_k)\varphi' - \left[\sum_{i=0}^{q-1} i\binom{q-1}{i}\left(\frac{1}{q-1}\right)^{i}\right] N_s(T_k)\varphi' \tag{2}$$

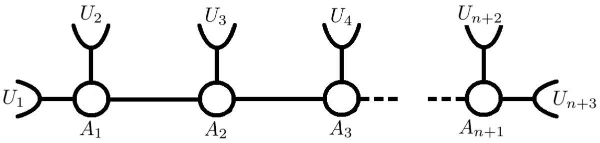

Figure 4.2: A chain of nodes, U_i, $i \in \{2, \ldots, n+2\}$ all consist of $q-1$ subconfigurations. For $i = 1$ and $i = n+3$ there is only one subconfiguration.

When we let $x_\ell = \frac{1}{q-1}$ and use the order or the collections $I_n^{q\pm}$, the sums in line (4.20, 2) can be calculated in the same way as done in equation (4.12). To solve (4.20, 1) the binomial theorem is used. The solution i

$$N_s(T) = \left(\frac{q}{q-1}\right)^{q-3} N_w(T_k)\varphi' + \left(\frac{q}{q-1}\right)^{q-1} \frac{q^2-q+1}{q} N_s(T_k)\varphi', \tag{4.21}$$

where $\varphi' = \prod_{i=2}^{q'} N_s(U_i)$, q' depends on the number of unused connections to the corresponding node, and $U_{k,i}$ the ith subconfiguration of the kth node. In the following case a chain of nodes are observed with starting node A_1 ending with the node A_n. Each node A_i, $i \in \{2, \cdots, q\}$ has $q-1$ nodes that are not "part of the chain" as seen in figure 4.2. The ith subconfiguration at U_j is denoted by $U_{j,i}$. Because of this, the nodes at the end of the chain has one more neighbour than a node that is not at the end contributing to φ. This is seen in equation (4.23). Using (4.19) and (4.21) we get

$$\begin{pmatrix} N_w(T_{k+1}) \\ N_s(T_{k+1}) \end{pmatrix} = \varphi \begin{pmatrix} \left(\frac{q}{q-1}\right)^{q-1} & \left(\frac{q}{q-1}\right)^{q-1} \\ \left(\frac{q}{q-1}\right)^{q-2} & \left(\frac{q}{q-1}\right)^{q-1} \frac{q^2-q+1}{q} \end{pmatrix} \begin{pmatrix} N_w(T_k) \\ N_s(T_k) \end{pmatrix}. \tag{4.22}$$

With this equality, a relationship between T_n and T_1 is found.

$$\begin{pmatrix} N_w(T_n) \\ N_s(T_n) \end{pmatrix} = N_s(U_{k+3}) \prod_{k=1}^{n-1} \left[\prod_{i=2}^{q} N_s(U_{k+2,i})\right] \left(\frac{q}{q-1}\right)^{(n-1)(q-1)} \begin{pmatrix} 1 & 1 \\ \frac{q^{-2}}{(q-1)^{-2}} & q^{-1}(q^2-q+1) \end{pmatrix}^{n-1} \begin{pmatrix} N_w(T_1) \\ N_s(T_1) \end{pmatrix}. \tag{4.23}$$

The power of the matrix is easily calculated using the eigen decomposition. Its eigenvectors are q^{-1} and q with eigenvectors $\begin{pmatrix} \frac{-q}{q-1} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{q-1} \\ 1 \end{pmatrix}$ respectively. We find

$$\begin{pmatrix} 1 & 1 \\ \frac{q^{-1}}{(q-1)^{-2}} & q^{-1}(q^2-q+1) \end{pmatrix}^{n} = \begin{pmatrix} \frac{q^{1-n}+q^n}{q+1} & \frac{-q^{1-n}+q^{n+1}}{q^2-1} \\ \frac{(q-1)\left(q^n-q^{-n}\right)}{q+1} & \frac{q^{-n}+q^{n+1}}{q+1} \end{pmatrix}. \tag{4.24}$$

Note that $N_w(T_1) = \frac{1}{q-1} N_s(T_1)$. Substituting (4.24)) into (4.23 reduces $N_s(T_{k+1})$ to

$$\begin{aligned} N_s(T_n) &= N_s(U_{k+3}) \left[\prod_{k=1}^{n-1}\prod_{i=2}^{q} N_s(U_{k+2,i})\right] \left(\frac{q}{q-1}\right)^{(n-1)(q-1)} \left(\frac{(q-1)(q^{(n-1)}-q^{1-n})}{q+1} N_w(T_1) + \frac{q^{1-n}+q^n}{q+1} N_s(T_1)\right) \\ &= N_s(U_{k+3}) \left[\prod_{k=1}^{n-1}\prod_{i=2}^{q} N_s(U_{k+2,i})\right] \left(\frac{q}{q-1}\right)^{(n-1)(q-1)} q^{n-1} N_s(T_1) \\ &= N_s(U_{k+3}) \left[\prod_{k=1}^{n-1}\prod_{i=2}^{q} N_s(U_{k+2,i})\right] \frac{q^{(n-1)q}}{(q-1)^{(n-1)(q-1)}} N_s(T_1). \end{aligned} \tag{4.25}$$

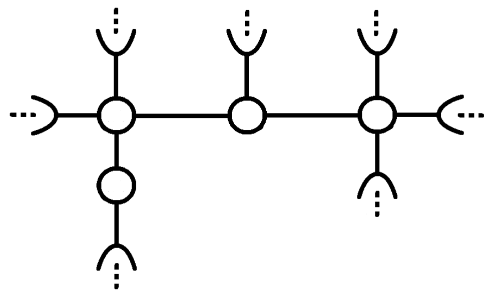

Figure 4.3: An example of a cluster containing 4 nodes, having a total of $4(q-1)+2$ subconfigurations. $U_2, .., U_{n+2}$ can be split into $q-1$ subconfigurations in the general case.

Observe the $(n+1)$-th node. In figure 4.2, T_1 represents the same tree as U_1. Substituting this in the total amount of possible configurations in equation (4.17) yields

$$N_{total} = \left(\frac{q}{q-1}\right)^q (q+1) \left[\prod_{i=2}^{q+1} N_s(U_{n+1,i})\right] N_s(T_n) \tag{4.26}$$

$$= \left(\frac{q}{q-1}\right)^q (q+1) \left[\prod_{i=2}^{q+1} N_s(U_{n+1,i})\right] \left[\prod_{k=1}^{n-1}\prod_{i=2}^{q} N_s(U_{k+2,i})\right] \frac{q^{(n-1)q}}{(q-1)^{(n-1)(q-1)}} N_s(U_1) \tag{4.27}$$

$$= \frac{q^{nq}(q^2-1)}{(q-1)^{n(q-1)+2}} \left[N_s(U_1)N_s(U_{n+3}) \prod_{i=1}^{n}\prod_{i=1}^{q-1} N_s(U_{k+2,i})\right] \tag{4.28}$$

$$= \frac{q^{nq}(q^2-1)}{(q-1)^{n(q-1)+2}} \left[\prod_{i=1}^{n(q-1)+2} N_s(U_i)\right], \tag{4.29}$$

where U_i number all the subconfigurations attached to the nodes of the cluster. For simplicity the numbering of the subconfigurations from equation (4.25) is changed.

4.3 Number of allowed configurations containing a given configuration

Consider a cluster C consisting of the nodes $a_1, .., a_n$ somewhere deep in the lattice such that $X(T) = \frac{1}{q-1}$. Each node has degree $q+1$, since they are connected. If $n=2$, then a pair of neighbouring nodes can have a total of $2q$ connections. In general, one can say that a cluster consisting of n nodes has $n(q-1)+2$ connection. This is true because one node has $q+1$ connections, and each node that is added to the tree takes one connection from the existing tree and from itself leaving $q-1$ unused. Consider such a cluster, visualised in figure 4.3.

Recall the rules from page 12:

1. Not all U_i, $i \in \{1, \ldots, n(q-1)+2\}$ can be weakly allowed,
2. No subtree can have a root node with height $q+1$.

The rules are also valid for the the case where $q \in \mathbb{N}_{\geq 2}$ is arbitrarily chosen. Rule 1 is a straight forward generalisation of the $q=2$ case since the argument still holds when changing 2 into q. Rule 2 holds because otherwise the choice of the cluster would be violated. We are interested in each product where the $n(q-1)+2$ subconfigurations are weakly or strongly allowed. These subconfigurations may only have root values which are at most q. If one subconfiguration has a root value or $q+1$ it would be part of the cluster. To satisfy rule 2, the situations where all subconfigurations are weakly allowed need to be subtracted. This results in

$$N(C) = \prod_{i=1}^{n(q-1)+2} \underbrace{\left[\sum_{j=1}^{q} N_w(U_i,j) + \sum_{j=2}^{q} N_s(U_i,j)\right]}_{\clubsuit} - \prod_{i=1}^{n(q-1)+2} \underbrace{\left[\sum_{j=1}^{q} N_w(U_i,j)\right]}_{\spadesuit}, \tag{4.30}$$

where U_i, $i \in \{1, \ldots, n(q-1)+2\}$ are the subconfigurations attached to the cluster. Remember that $N(C)$ is the number of allowed configurations containing C. It is useful to observe that in the ♣ part, by equations (4.2) and (4.3) it is clear that $\sum_{j=1}^{q} N_w(U_i, j) = N_s(U_i, q+1)$. resulting in

$$\sum_{j=1}^{q} N_w(U_i, j) + \sum_{j=2}^{q} N_s(U_i, j) = \sum_{j=2}^{q+1} N_s(U_i, j) = N_s(U_i) = \frac{q-1}{q-1} N_s(U_i). \tag{4.31}$$

For the ♠ part we have

$$\begin{aligned}\sum_{j=1}^{q} N_w(U_i, j) &= \frac{\sum_{j=1}^{q} N_w(U_i, j)}{N_s(U_i)} N_s(U_i) = \frac{\sum_{i=1}^{q} \binom{q}{i-1} \left(\frac{1}{q-1}\right)^i}{\sum_{i=0}^{q-1} (q-i) \binom{q}{i} \left(\frac{1}{q-1}\right)^i} N_s(U_i) \\ &= \left(\frac{1}{q-1}\right) \frac{q^q - 1}{q^q} N_s(U, i) \stackrel{*}{=} \left(\frac{1}{q-1}\right) (1 - q^{-q}) N_s(U_i).\end{aligned}$$

In the third expression above, the denominator is solved by substituting $j = i - 1$ and applying the binomial theorem. The denominator is treated analogue to equation (4.12). At step $*$, the equation is multiplied by $\frac{q^{-q}}{q^{-q}}$. This holds because

$$\sum_{j=1}^{q} N_w(U_i, j) = \sum_{i=0}^{q-1} \sum_{k \in \mathcal{I}_i^q} \prod_{\ell \in k} x_\ell = \left[\sum_{i=1}^{q} \binom{q}{i-1} \left(\frac{1}{q-1}\right)^{i-1}\right] = \left(\frac{1}{q-1}\right)^q (q^q - 1), \tag{4.32}$$

this result follows by substituting $j = i - 1$ and apply the binomial theorem. Furthermore

$$\begin{aligned}N_s(U_i) &= \sum_{j=1}^{q+1} N_s(U_i, j) = \sum_{j=1}^{q+1} \sum_{n=0}^{j-2} \sum_{k \in \mathcal{I}_n^q} \prod_{\ell \in k} x_\ell = \sum_{j=0}^{q-1} \sum_{n=0}^{i} \sum_{k \in \mathcal{I}_n^q} \prod_{\ell \in k} x_\ell \\ &= \sum_{j=0}^{q-1} (q-j) \binom{q}{j} \left(\frac{1}{q-1}\right)^j = \frac{q^q}{(q-1)^{q-1}}.\end{aligned}$$

This can be solved analogous to equation (4.12). Using these equalities, These results allow us to express $N(C)$ in terms of $\prod_{i=1}^{n(q-1)+2} N_s(U_i)$ by replacing the sums.

$$\begin{aligned}N(C) &= \prod_{i=1}^{n(q-1)+2} \frac{q-1}{q-1} N_s(U_i) - \prod_{i=1}^{n(q-1)+2} \frac{1-q^{-q}}{q-1} N_s(U_i) \\ &= \left(\frac{1}{q-1}\right)^{n(q-1)+2} \left((q-1)^{n(q-1)+2}) + (1-q^{-q})^{n(q-1)+2}\right) \prod_{i=1}^{n(q-1)+2} N_s(U_i).\end{aligned}$$

Dividing this by N_{total} results in probability $U_C(n)$ of a certain state C occurring.

$$U_C(n) = \frac{N(C)}{N_{total}} = \frac{(q-1)^{n(q-1)+2}) + (1-q^{-q})^{n(q-1)+2}}{(q^2-1)q^{nq}}. \tag{4.33}$$

Note that $U_c(n)$ no longer depends on the cluster C. All that is left to show is the number of distinct configurations on a lattice of n generations. Multiplying $U(n)$ with the number of animals of size n in the stationary state yields the probability of finding a cluster of precisely n nodes, see page 12. The next chapter is devoted to finding the quantity a_n.

Chapter 5

Analysis on binary trees

In this chapter the probability mass function for the length of avalanches on a binary tree will be derived. A binary tree is a tree generated by a root node, and every node has two children. An animal of size n is a subtree which consists of n nodes from the point of origin. To find this distribution we find how many disjoint n-sized animals exist. Consider a Bethe lattice, a graph with a root node and three binary trees connected at that node. Pick an arbitrary node on a binary tree, call this the origin. The subtree found by following a vertex is called an animal. Denote b_n, the amount of distinct n-sized animals with n nodes. Because there is only one way to make an animal of size one, we say $b_1 = 1$ and by convention we say that $b_0 = 1$. It is also clear $b_2 = 2$ and drawing every graph of length three yields $b_3 = 5$. We want $B(x)$ to be a generating function of these coefficients. To find an explicit expression for these coefficients a recurrent expression for $B(x)$ is used. Each b_n can be split into the amount of children, all neighbours excluding parent nodes, to the corresponding node. We get

$$\begin{aligned} B(x) &= \sum_{n=1}^{\infty} b_n x^n = x + \sum_{n=2}^{\infty} b_n x^n \overset{\ddagger}{=} x + \sum_{n=2}^{\infty}\sum_{k=0}^{n-1} b_k b_{n-k-1} x^{n-1} = x + x\sum_{n=1}^{\infty}\sum_{k=0}^{n} b_k b_{n-k} x^n \\ &= x\left(1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} b_k b_{n-k} x^n\right) = x\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} b_k b_{n-k} x^n\right) = x\left(\sum_{n=0}^{\infty} b_n x^n\right)^2 = x\left(1 + \sum_{n=1}^{\infty} b_n x^n\right)^2 \\ &= x\big(1 + B(x)\big)^2. \end{aligned} \tag{5.1}$$

To see why step ‡ holds we fix some integer n and note that the problem is being solved on a binary tree. This means that we have a root-node of our animal and two subtrees connected to this node. This is visualised in figure 5.1 Assume we have two subanimals which have k and $n-k-1$ nodes so together they have $n-1$ nodes (or n including the root-node). When we fix one of those subanimals with k nodes, then for that subtree, there are b_{n-k} possible subanimals on the other subtree. This holds for each animal of length k. This means that there are $b_k b_{n-k-1}$ possibilities to make a subtree of length $n-1$. When we do this for all k the amount of combinations for an animal of length n is sum all of these combinations $b_n = \sum_{k=0}^{n} b_k b_{n-k}$. The equation $B(x) = x\big(1+B(x)\big)^2$ can be solved using the quadratic formula. Doing so yields

$$B(x) = \frac{1 - 2x - \sqrt{1-4x}}{2x}. \tag{5.2}$$

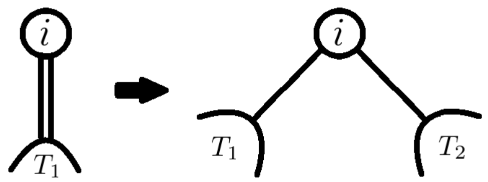

Figure 5.1: On the left side, there is a “bundle” of two subtrees, because we work on a binary tree they can be split into two, allowing to observe them individually.

When using the quadratic formula we have to deal with a $\pm$ sign. Because $B(0) = 0$ by equation (5.1), we must have a minus sign, this is because

$$\lim_{n\to 0} B(x) = \lim_{n\to\infty} \frac{1-2x-\sqrt{1-4x}}{2x} \overset{\text{L'Hôpital}}{=} \lim_{n\to\infty} \frac{-2+2(1-4x)^{-\frac{1}{2}}}{2} = 0.$$

This would not be the case if plus sign was used. Writing the solution for $B(x)$ as a power series by using the series of $\sqrt{1-4x}$ it is possible to find an expression for b_n. To do so, the Taylor series of the square root function is required. For this, the following lemma is helpful.

Lemma 2. *The taylor series of the function* $f(x) = \sqrt{1+x}$ *is given by*

$$f(x) = 1 - \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(n!)^2(2n-1)4^n} x^n \tag{5.3}$$

and has a radius of convergence of 1.

Proof. Let $f(x) = \sqrt{1+x}$. Table 5.1 contains the derivatives evaluated at $x = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{\frac{1}{2}}$	1
1	$\frac{1}{2}(1+x)^{-\frac{1}{2}}$	$\frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-\frac{3}{2}}$	$-\frac{1}{4}$
3	$\frac{3}{8}(1+x)^{-\frac{5}{2}}$	$\frac{3}{8}$
4	$-\frac{15}{16}(1+x)^{-\frac{7}{2}}$	$-\frac{15}{16}$
5	$\frac{105}{32}(1+x)^{-\frac{9}{2}}$	$\frac{105}{32}$
6	$-\frac{945}{64}(1+x)^{-\frac{11}{2}}$	$-\frac{945}{64}$

Table 5.1: Table with multiple Taylor coefficients of $f(x) = \sqrt{1+x}$.

We now want to find a pattern to describe the coefficient for a general integers n. Observe that the denominator increases with 2^n, the series alternates and is positive when n is odd and negative when n is even. In the numerator, one can recognize the double-factorial.[1] It turns out that $(2n-3)!!$ matches the numerator. Using the recurrence that $(2n-1)\cdot(2n-3)!! = (2n-1)!!$ we can find the value of $(-3)!!$ and $(-1)!!$. It follows that $(-1)!! = \frac{(-1+2)!!}{-1+2} = 1!! = 1$, and $(-3)!! = \frac{(-3+2)!}{-3+2} = -(-1)!! = -1$. Now we want to manipulate the expression for the coefficients so it can be used in an easier way. First note that

$$2^n n! = \big(2\cdot 2\cdot 2\cdot \ldots\big)\big(n(n-1)(n-3)\ldots\big) = (2n)(2n-2)(2n-4)\cdots = (2n)!!.$$

By multiplying every element pairwise. This will be applied to find an expression for general Taylor coefficients of f.

$$(2n-3)!! = \frac{(2n-1)!!}{(2n-1)} = \frac{(2n)!}{(2n-1)(2n)!!} = \frac{(2n)!}{(2n-1)2^n n!}.$$

The taylor series is defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^{n+1}(2n-1)!!}{2^n n!} x^n = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(2n)!}{(n!)^2(2n-1)4^n} x^n \tag{5.4}$$

$$= 1 - \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(n!)^2(2n-1)4^n} x^n \tag{5.5}$$

It follows from the alternating series test that this series has a radius of convergence of at least one. This can be shown easily using the geometric series and noting that the series is alternating and the non-alternating part decreases to zero, this is because

$$(2n)! = (2n)!!(2n-1)!! < ((2n)!!)^2 = 4^n(n!)^2 \iff \frac{(2n)!}{(n!)^2(2n-1)4^n} < \frac{1}{2n-1} \overset{n\to\infty}{\longrightarrow} 0. \tag{5.6}$$

[1]The double factorial is a function such that $n!! = n\cdot(n-2)\ldots 4\cdot 2$ if n is even, and $n!! = n\cdot(n-2)\ldots 5\cdot 3$ if n is odd.

it follows that

$$\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{(n!)^2(2n-1)4^n}x^n < \sum_{n=1}^{\infty}x^n, \tag{5.7}$$

and the geometric series has a radius of convergence of 1. □

Now that we have a series expression for the function $f(y)=\sqrt{1+y}$ it can be applied to equation 5.2. To get the desired square root, we choose $y=-4x$. Then

$$\begin{aligned}
B(x) &= \frac{1}{2x}\left(1-2x-\sqrt{1-4x}\right)\\
&= \frac{1}{2x}\left(1-2x-\left(1-\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{(n!)^2(2n-1)4^n}(-4x)^n\right)\right),\\
&= -1+\frac{1}{2x}\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2(2n-1)}x^n
\end{aligned}$$

In the last step it is used that the term corresponding to $n=0$ is equal to -1. But $B(x)$ is defined as a sum that starts from $n=1$ so

$$-1+\sum_{n=1}^{\infty}\frac{1}{2}\frac{(2n)!}{(n!)^2(2n-1)}x^{n-1} = -1+\sum_{n=0}^{\infty}\frac{1}{2}\frac{(2n)!(2n+1)(2n+2)}{(n!)^2(n+1)^2(2n+1)}x^n = \sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2(n+1)}x^n.$$

The first term of the sum was equal to 1 so the -1 is cancelled. This results in the following:

$$b_n = \binom{2n}{n}\frac{1}{n+1} = C_n. \tag{5.8}$$

Here, C_n is nth Catalan number. These are the amount of n-sized animals on one subtree. But because each point has a degree of three we are interested in the generating function of the possible distinct n-sized animals from the origin. Note that not only the children are observed as we did for $B(x)$ but also the parent node. This function is denoted by $G(x)$ as a power series. Let $G(x)=\sum_{n=1}^{\infty}a_nx^n$, the coefficient a_n represents the amount of disjoint animals with the origin as the starting node. The equation will be derived the exact same way as $B(x)$ except that we have three neighbours instead of two, so the coefficients need to be split twice. After that, $B(x)$ is used to find an explicit expression for a_n.

$$G(x) = x+x\sum_{n=1}^{\infty}\sum_{k=0}^{n}b_kb_{n-k}x^n = x+x\sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{\ell=0}^{k}b_\ell b_{k-\ell}b_{n-k}x^n = x\big(1+B(x)\big)^3.$$

This might look hard to solve but the Catalan numbers are the solution to the sequence $(b_n)_{n\geq 1}$, $b_0=1$ and $b_{n+1}=\sum_{k=0}^{n}b_kb_{n-k}$. It follows that

$$\begin{aligned}
G(x) = x\big(1+B(x)\big)^3 &= x\left(\sum_{n=0}^{\infty}b_nx^n\right)^3 = x\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n}b_kb_{n-k}x^n\right)\left(\sum_{n=0}^{\infty}b_nx^n\right)\\
&= x\left(\sum_{n=0}^{\infty}b_{n+1}x^n\right)\left(\sum_{n=0}^{\infty}b_nx^n\right) = x\sum_{n=0}^{\infty}\sum_{k=0}^{n}b_kb_{n+1-k}x^n =\\
&= x\sum_{n=1}^{\infty}\sum_{k=1}^{n}b_kb_{n-k}x^{n-1} = x\sum_{n=1}^{\infty}\left[\left(\sum_{k=0}^{n}b_kb_{n-k}\right)-b_0b_n\right]x^{n-1}\\
&= \sum_{n=1}^{\infty}(b_{n+1}-b_n)x^n := \sum_{n=1}^{\infty}a_nx^n.
\end{aligned}$$

To write these terms more explicitly:

$$a_n = b_{n+1}-b_n = b_n\left(\frac{2(2n+1)-(n+2)}{(n+2)}\right) = \frac{(2n)!}{(n+1)(n!)^2}\frac{3n}{n+2} = \frac{3}{2n+1}\binom{2n+1}{n-1}.$$

There are $U_c(n) = \frac{1}{3}\left[1 - \left(\frac{3}{4}\right)^{n+2}\right] 4^{-n}$ possible trees when we have n nodes. Multiplying the number of n-sized animals with $U_c(n)$ from equation (3.24) gives the probability mass for the avalanche size as reasoned on page 12. The mass function of the avalanche size is given as

$$U_c(n)a_n := \mathcal{P}(n) = \left[1 - \left(\frac{3}{4}\right)^{n+2}\right] \frac{4^{-n}}{2n+1} \binom{2n+1}{n-1}, \tag{5.9}$$

where $\mathcal{P}(n)$ denotes the probability of an avalanche of length n occurring. It can be shown that the probabilities for avalanches of at least length 1 sums to $\sum_{n=1}^{\infty} \mathcal{P}(n) = \frac{7}{12}$, this can mean two things. Either there exist avalanches or the probability of having no avalanche after adding a grain is $\frac{5}{12}$. The latter is true, this can be concluded from the height distribution of single nodes, seen in Dhar's paper in equation (4.7) [4]. For large n a simple expression can be found that describes the asymptotic behaviour.

$$\begin{aligned}
U_c(n)a_n = \mathcal{P}(n) &= \left[1 - \left(\frac{3}{4}\right)^{n+2}\right] \frac{4^{-n}}{2n+1} \binom{2n+1}{n-1} \\
&= \left[1 - \left(\frac{3}{4}\right)^{n+2}\right] 4^{-n} \frac{(2n)!}{(n!)^2} \frac{n}{(n+1)(n+2)} \\
&\approx \left[1 - \left(\frac{3}{4}\right)^{n+2}\right] 4^{-n} \frac{\sqrt{4\pi n}(2n)^{2n} e^{-2n}}{2\pi n n^{2n} e^{-2n}} \frac{n}{(n+1)(n+2)} \\
&= \left[1 - \left(\frac{3}{4}\right)^{n+2}\right] \frac{1}{\sqrt{\pi n}} \frac{n}{(n+1)(n+2)}.
\end{aligned}$$

So

$$\frac{U_c(n)a_n}{n^{-\frac{3}{2}}} = \frac{\left[1 - \left(\frac{3}{4}\right)^{n+2}\right] \frac{1}{\sqrt{\pi n}} \frac{n}{(n+1)(n+2)}}{n^{-\frac{3}{2}}} \xrightarrow{n\to\infty} \frac{1}{\sqrt{\pi}}.$$

Thus the probability mass function of getting an avalanche of size n is proportional to $n^{-\frac{3}{2}}$ when n gets large. In the next chapter this concept will be generalized. Then instead of the fixed degree for each node 2, it can be any degree larger than one.

Chapter 6

Analysis on homogeneous trees of higher degree

Using the same approach for homogeneous trees where each node has q children we can use the same idea. To find a recursive expression for $B(x)$. For general q it is impossible to calculate the generating function directly from the implicit expression of $B(x)$, as done with the quadratic equation in the previous chapter. Note that b_n is now the number of disjoint animals on a subtree where each node has q children. A tree of degree q can be treated as a binary tree, except that its edges are "bound" together. The same trick used to acquire equation (5.1) can be applied to this case, except that it will be used multiple times. We will the split the "bundle" of subanimals into individual subanimals, see figure 6.1 for a visualisation.

Splitting the bundle is done by letting $b_n = \sum_{k=0}^{n} b_k b_{n-k}$, $(q-1)$ times. To recognize a pattern in the next calculation we first need to know about Cauchy products. The product of two infinite sums is

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k a_{n-k} x^n.$$

Repeating this $q-2$ times yields

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^q = \sum_{n=1}^{\infty}\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{q-1}=0}^{k_{q-2}} b_{k_{q-1}} \prod_{\ell=2}^{q} b_{k_\ell - k_{\ell-1}} x^n. \tag{6.1}$$

Now it is clear how to recognize powers of infinite series, the tree can be observed in a more general case.

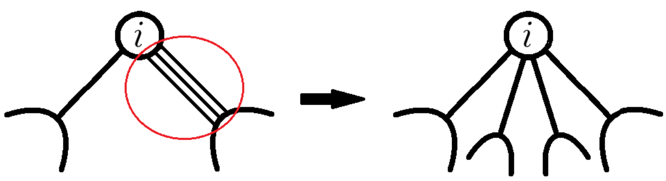

Figure 6.1: The same idea is used in the previous chapter, but this time it is done on a homogenous tree of degree $q = 4$. In this visualisation, we split the "bundle" two times.

For simplicity some steps were skipped that have been seen in the case $q = 2$.

$$\begin{aligned}
B(x) &= \sum_{n=1}^{\infty} b_n x^n = x + \sum_{n=2}^{\infty} b_n x^n = x + \sum_{n=1}^{\infty}\sum_{k=0}^{n} b_k b_{n-k} x^{n+1} \qquad (6.2)\\
&= x + x\sum_{n=1}^{\infty}\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1} b_{k_2} b_{k_1-k_2} b_{n-k_1} x^n\\
&\vdots\\
&= x + x\sum_{n=1}^{\infty}\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{q-1}=0}^{k_{q-2}} b_{k_{q-1}} \prod_{\ell=2}^{q} b_{k_\ell - k_{\ell-1}} x^n\\
&= x\sum_{n=0}^{\infty}\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{q-1}=0}^{k_{q-2}} b_{k_{q-1}} \prod_{\ell=2}^{q} b_{k_\ell - k_{\ell-1}} x^n\\
&= x\big(1 + B(x)\big)^q,
\end{aligned}$$

where the last step used equation (6.1). The solution to this implicit equation is the generating function of the number of subtrees made using the children nodes. Solving this equality will be tricky. For this reason we make an educated guess. Recall the solution in equation (5.8) to the case $q = 2$. Observe the $\binom{2n}{n}$ term and note that every new node generates two new possible connections, while only one was made. This can be interpreted as $2n$ options and only n can be chosen. Using this reasoning, the solution for the general problem has a term $\binom{qn}{n}$ in it. It is possible to derive an implicit relation between the coefficients b_n

$$\begin{aligned}
B(x) = x\big(1+B(x)\big)^q = x(1+\sum_{n=1}^{\infty} b_n x^n)^q &= \sum_{n=0}^{\infty}\left(\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_q=0}^{k_{q-1}} b_{k_q} b_{n-k_1}\prod_{i=2}^{q} b_{k_{i-1}-k_1}\right) x^{n+1}\\
&= \sum_{n=1}^{\infty}\left(\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{k_1}\cdots\sum_{k_q=0}^{k_{q-1}} b_{k_q} b_{(n-1)-k_1}\prod_{i=2}^{q} b_{k_{i-1}-k_1}\right) x^{n}.
\end{aligned}$$

When two power series are equal for all x on some open set, the coefficients of both series must be equal, thus

$$b_n = \sum_{k_1=0}^{n-1}\sum_{k_2=0}^{k_1}\cdots\sum_{k_q=0}^{k_{q-1}} b_{k_q} b_{(n-1)-k_1}\prod_{i=2}^{q} b_{k_{i-1}-k_i}, \tag{6.3}$$

where $b_0 = b_1 = 1$. This sequence seems hard to solve, for this reason we guess and check whether the guess is correct. Remember the observation that there should be a factor $\binom{qn}{n}$ involved. Using a computer simulation of what the value should be, a candidate solution is found. Let

$$b_n = \frac{1}{(q-1)n+1}\binom{qn}{n} = \frac{1}{qn+1}\binom{qn+1}{n}. \tag{6.4}$$

This expression matched the outcome of the numerical simulation for all tested values. When taking powers of the hypothesised solution $\sum_{n=1}^{\infty} b_n x^n$ we need to deal with some cumbersome sums with binomial coefficients involved. One identity that allows us to solve such a sum is the Rothe-Hagen identity. This identity states the following:

$$\sum_{k=0}^{n} \frac{x}{x+kz}\binom{x+kz}{k}\frac{y}{y+(n-k)z}\binom{y+(n-k)z}{n-k} = \frac{x+y}{x+y+nz}\binom{x+y+nz}{n}, \tag{6.5}$$

for all $x, y, z \in \mathbb{C}$. Higher powers of $\sum_{n=0}^{\infty} b_n x^n$ are calculated using induction. Squaring $B(x)$ to show the first induction step yields

$$\left(1 + \sum_{n=1}^{\infty} b_n\right)^2 = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} b_k b_{n-k}\right] x^n, \tag{6.6}$$

where

$$\sum_{k=0}^{n} b_k b_{n-k} = \sum_{k=0}^{n} \frac{1}{qk+1}\binom{qk+1}{k} \frac{1}{q(n-k)+1}\binom{q(n-k)+1}{n-k} = \frac{2}{2+nq}\binom{2+nq}{n}, \tag{6.7}$$

using the Rothe-Hagen identity. For the next induction step assume that for m sums the following holds

$$\sum_{k_2=0}^{\ell} b_{\ell-k_2} \sum_{k_3=0}^{k_2} b_{k_3-k_2} \cdots \sum_{k_n=0}^{k_{m-1}} b_{k_m} b_{k_{m-1}-k_m} = \frac{m}{m+\ell q}\binom{m+\ell q}{\ell}. \tag{6.8}$$

Taking the power $m+1$ adds one more sum

$$\sum_{\ell=0}^{n} b_{k_0-\ell} \sum_{k_2=0}^{\ell} b_{k_2-k_1} \sum_{k_3=0}^{k_2} b_{k_3-k_2} \cdots \sum_{k_n=0}^{k_{m-1}} b_{k_m} b_{k_{m-1}-k_m} \overset{\text{Hyp}}{=} \sum_{\ell=0}^{n} b_{n-\ell} \frac{m}{m+\ell q}\binom{m+\ell q}{\ell}$$
$$= \sum_{\ell=0}^{n} \frac{m}{m+(n-\ell)q}\binom{m+(n-\ell)q}{n-\ell} \frac{m}{m+\ell q}\binom{m+\ell q}{\ell} = \frac{(m+1)}{(m+1)+nq}\binom{(m+1)+nq}{n}.$$

Conclude that arbitrary powers $k \in \mathbb{N}$, $\left(\sum_{n=0}^{\infty} b_n x^n\right)^k = \sum_{n=0}^{\infty} \frac{k}{k+nq}\binom{k+nq}{n} x^n$. The result will be used in the first step to verify the candidate solution of the coefficients.

$$x\big(1+B(x)\big)^q = \sum_{n=0}^{\infty} \frac{q}{q(n+1)}\binom{q(n+1)}{n} x^{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}\binom{qn}{n-1} x^n \tag{6.9}$$
$$= \sum_{n=1}^{\infty} \frac{1}{q(n-1)+1} \frac{(qn)!}{n!(q(n-1))!} x^n = \sum_{n=1}^{\infty} \frac{1}{q(n-1)+1}\binom{qn}{n} x^n = \sum_{n=1}^{\infty} b_n x^n = B(x).$$

This verifies the guess for b_n. Now we can continue determining the behaviour on the lattice observed from the origin. In this case we need to take all $q+1$ neighbours into account instead of only the q children. When observing the origin, which has $q+1$ neighbours, the children- and parent nodes, the problem is treated slightly different than $B(x)$. Let the function $G(x)$ be the generating function for the origin. It will be treated in the same way as $B(x)$ except that this origin has one more neighbour.

$$G(x) = x + x\sum_{n=1}^{\infty}\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1} b_{k_2} b_{k_1-k_2} b_{n-k_1} x^n \tag{6.10}$$
$$= x + x\sum_{n=1}^{\infty}\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_q=0}^{k_{q-1}} b_{k_q}\prod_{\ell=2}^{q+1} b_{k_\ell-k_{\ell-1}} x^n$$
$$= x\sum_{n=0}^{\infty}\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_q=0}^{k_{q-1}} b_{k_q}\prod_{\ell=2}^{q+1} b_{k_\ell-k_{\ell-1}} x^n$$
$$= x\big(1+B(x)\big)^{q+1}.$$

When observing the equations of $B(x)$ and $G(x)$, one can find a close relation between the two functions. we can express $G(x)$ in terms of $B(x)$ the following

$$G(x) = x\big(1+B(x)\big)^{q+1} = x\big(1+B(x)\big)^q\big(1+B(x)\big) \overset{\text{Eqn.(6.2)}}{=} B(x)\big(1+B(x)\big) \tag{6.11}$$

This relation can be applied to find the coefficients of $G(x)$. This results in

$$\sum_{n=1}^{\infty} a_n x^n : = G(x) = B(x)(1+B(x)) \left(\sum_{n=1}^{\infty} b_n x^n\right)^2 + \sum_{n=0}^{\infty} b_n x^n = \left(\sum_{n=1}^{\infty} b_n x^n - 1\right)^2 + \left(\sum_{n=0}^{\infty} b_n x^n - 1\right)$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} b_k b_{n-k} x^n - 2\sum_{n=1}^{\infty} b_n x^n + 1 + \left(\sum_{n=0}^{\infty} b_n x^n - 1\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} b_k b_{n-k} x^n - \sum_{n=0}^{\infty} b_n x^n$$
$$= \sum_{n=1}^{\infty} \underbrace{\sum_{k=0}^{n} b_k b_{n-k} - b_n}_{a_n} x^n. \tag{6.12}$$

Here $\sum_{k=0}^{n} b_k b_{n-k}$ occurs again, this time the result is calculated again using the identity of Rothe-Hagen. From equation (6.12) it is known that $a_n = \left(\sum_{k=0}^{n} b_k b_{n-k}\right) - b_n$, this can be evaluated further as following

$$\begin{aligned} a_n &= \left(\sum_{k=0}^{n} b_k b_{n-k}\right) - b_n = \left(\sum_{k=0}^{n} \frac{1}{qk+1}\binom{qk+1}{k}\frac{1}{q(n-k)+1}\binom{q(n-k)+1}{n-k}\right) - \frac{1}{qn+1}\binom{qn+1}{k} \\ &= \frac{2}{2+nq}\binom{2+nq}{n} - \frac{1}{qn+1}\binom{qn+1}{n} = \binom{qn+1}{n}\left(\frac{2}{(q-1)n+2} - \frac{1}{qn+1}\right) = \frac{q+1}{nq+1}\binom{nq+1}{n-1}. \end{aligned}$$

This sequence tells how many n-sized animals there are. At this point, all ingredients of the avalanche length probability distribution are known. When we divide this by the total amount of different subtrees, the probability mass function of avalanche sizes can be obtained, that is $U_c(n)a_n$. Now we want determine the asymptotic behaviour of that product. First approximate a_n for large n, for the factorials, Stirling's approximation is applied, the immediately drop the exponential terms that arise when applying the approximation because they cancel.

$$\begin{aligned} \frac{q+1}{nq+1}\binom{qn+1}{n-1} &= (q+1)\frac{(qn)!}{(n-1)!((q-1)n+2)!} \\ &\approx \frac{q+1}{nq+1}\frac{\sqrt{2\pi(nq+1)}(nq+1)^{n-1}(nq+1)^{n(q-1)+2}}{(n-1)^{n-1}\sqrt{2\pi(n-1)}((q-1)n+2)^{(q-1)n+2}\sqrt{2\pi((q-1)n+2)}} \\ &= \frac{q+1}{nq+1}\left(\frac{nq+1}{n-1}\right)^{n-1}\left(\frac{nq+1}{(q-1)n+2}\right)^{n(q-1)+2}\sqrt{\frac{nq+1}{n-1}}\frac{1}{\sqrt{2\pi(n(q-1)+2)}} \\ &\approx \frac{q+1}{nq+1}q^{nq+1}(q-1)^{-n(q-1)-2}\sqrt{q}\frac{1}{\sqrt{2\pi(n(q-1)+2)}}. \end{aligned}$$

Note that for large n and $q \geq 3$ it can said that

$$U_c(n) = \frac{(q-1)^{n(q-1)+2} - (1-q^{-q})^{n(q-1)+2}}{(q^2-1)q^{nq}} \approx \frac{(q-1)^{n(q-1)+2}}{(q^2-1)q^{nq}}.$$

Now, when we multiplying these two approximations it follows that

$$\begin{aligned} U_c(n)a_n &\approx \frac{(q-1)^{n(q-1)+2}}{(q^2-1)q^{nq}}\frac{q+1}{nq+1}q^{nq+1}(q-1)^{-n(q-1)-2}\sqrt{\frac{nq+1}{n-1}}\frac{1}{\sqrt{2\pi(n(q-1)+2)}} \\ &= \frac{q^{\frac{3}{2}}}{(q-1)}\frac{1}{nq+1}\frac{1}{\sqrt{2\pi(n(q-1)+2)}} \end{aligned}$$

It can be shown that

$$\frac{1}{n^{-\frac{3}{2}}}\frac{q^{\frac{3}{2}}}{(q-1)}\frac{1}{nq+1}\frac{1}{\sqrt{2\pi(n(q-1)+2)}} \overset{n\to\infty}{\longrightarrow} \frac{1}{2(q-1)}\sqrt{\frac{2q}{\pi(q-1)}}, \tag{6.13}$$

which is constant. We conclude that the avalanche size distribution $U_c(n)a_n \sim n^{-\frac{3}{2}}$. For small avalanches, the probability is strongly dependent on the branching degree q. However, when observing the tail of the probability mass functions, the distributions of distinct values of q tend to the same shape. In the tail, the distribution follows the power law $n^{-\frac{3}{2}}$.

Chapter 7

Conclusions and Discussion

In this thesis we analysed the avalanche distribution on the Bethe lattice of the Abelian sandpile model. This model can be used for modelling a variety of phenomena in the real world, for example the movement of the Earth's crust, the brains functionality or the stock market. The model concerns a graph in the SOC state such that only recurrent states are allowed. When for some $u \in V$, the height at u is larger than the number of its neighbours, then a toppling is induced at the node u. A sequence of topplings is an avalanche. Increasing height at uniformly random nodes makes the model stochastic. The goal of this thesis is to derive the probability mass function for the model on Bethe lattices for each branching degree q. By showing the described process is a Markov chain, the stationary distribution is found, which reads

$$\mu_{\mathscr{T}} = \frac{1}{|\mathscr{R}_{\mathscr{T}}|} \sum_{\eta \in \mathscr{R}_{\mathscr{T}}} \delta_\eta.$$

This equation holds for finite sized graphs. Since the goal of this thesis is to find the probability distribution of the avalanche size and to show that the asymptotic behaviour is independent of the branching degree the infinite dynamics are studied. For this it is shown that the model exists on infinite sized graphs. On such graphs the model is also Markov chain with a stationary distribution.

General expressions for the total number of configurations containing a chosen cluster consisting of n nodes $N(C)$ and for the total number of allowed configurations N_{total} are derived and calculated. Dividing $N(C)$ by N_{total} yields

$$\frac{N(C)}{N_{total}} = U_c(n) = \frac{(q-1)^{n(q-1)+2} - (1-q^{-q})^{n(q-1)+2}}{(q^2-1)q^{nq}},$$

the probability of obtaining a height configuration containing the cluster C. It is interesting to note that this quantity is only dependent on the size and not on the shape of the cluster. The quantity a_n denotes the number of possible clusters consisting of exactly n nodes. This number is given by

$$a_n = \frac{q+1}{qn+1}\binom{qn+1}{n-1}.$$

Both expressions strongly depend on the value of q. $U_c(n)$ is independent of the shape of the chosen cluster C and is only dependent on the number of nodes involved . For each cluster that contributes to a_n, this shape-independency holds. Therefore, multiplying $U_c(n)$ with a_n yields the probability of obtaining a cluster of size n. This yields the probability density function according to the stationary distribution.

$$\mathcal{P}(n) = U_c(n)a_n = \frac{(q-1)^{n(q-1)+2} - (1-q^{-q})^{n(q-1)+2}}{(q^2-1)q^{nq}} \frac{q+1}{qn+1}\binom{qn+1}{n-1} \sim n^{-\frac{3}{2}},$$

for large n. The asymptotic behaviour is found using Stirling's approximation and neglecting small terms. Note that for small values of n, the probability is strongly dependent on the branching degree q. This statement does not hold for the asymptotic behaviour where the avalanche probability is independent of the value of q.

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