

Analysis and Control of Uncertain Systems by Using Robust Semi-Definite Programming

S.G. Dietz

Analysis and Control of Uncertain Systems by Using Robust Semi-Definite Programming

Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus prof.dr.ir. J.T. Fokkema,
voorzitter van het College van Promoties,
in het openbaar te verdedigen op woensdag 15 oktober 2008 om 12:30 uur
door Sjoerd Gerard DIETZ,
ingenieur luchtvaart- en ruimtevaarttechniek,
geboren te Eindhoven.

Dit proefschrift is goedgekeurd door de promotor:
Prof. dr. C.W. Scherer

Samenstelling promotiecommissie:

Rector Magnificus	voorzitter
Prof.dr. C.W. Scherer	Technische Universiteit Delft, promotor
Prof.dr. J.C. Willems	Katholieke Universiteit Leuven
Prof.dr. A. van der Schaft	Universiteit Groningen
Prof.ir. O.H. Bosgra	Technische Universiteit Delft
Prof.dr.ir M. Steinbuch	Technische Universiteit Eindhoven
Dr. S. Weiland	Technische Universiteit Eindhoven
Dr.ir. M.M.J. van de Wal	Philips Applied Technologies
Prof.dr.ir. D. J. Rixen	Technische Universiteit Delft (reservelid)



This thesis has been completed in partial fulfillment of the requirements of the Dutch Institute for Systems and Control (DISC). The work was supported by the Technology Foundation STW, applied science division of NWO and the technology programme of the Ministry of Economic Affairs.

Published and distributed by: S.G. Dietz

Copyright © 2008 by S.G. Dietz

All rights reserved. No part of the material protected by this copyright notice may be reproduced or utilized in any form or by any means, electronic or mechanical, including photocopying, recording or by any information storage and retrieval system, without written permission of the author.

Printed in The Netherlands

Preface

Having accomplished the goal of writing this PhD thesis feels like reaching the summit of a high mountain never climbed before. It has been a tough climb and an outstanding experience I will not forget. In my view, the analogy between mountaineering and working one's way up to a PhD degree is evident. A climber is marked by his passion, discipline, concentration and perseverance, all characteristics that will be recognized by research fellows. Getting up the summit also requires patience, which again applies to a PhD student, as one typically has to wait for the right moment when the conditions are fine. Further, both mountaineers and researchers are usually working together, or even in teams, in order to face new challenges and achieve their goals.

I owe most thanks to Carsten Scherer, my promoter and daily supervisor. For the past five years I have been delighted to work with such a great mathematician, who is driven by tough and challenging problems that are still open. With his excellent educational skills, he could guide me through the scientific field by discussing the essential details. But above all I thank him for his kindness, and his encouragements during hard times. It strengthened my confidence, and helped me to continue till the end.

I also would like to thank Hakan K orođlu, who worked on the same project, for the many fruitful discussions and his excellent reviewing work on the papers that were written. All colleagues at the Delft Center of Systems and Control should be mentioned as well, for all the nice personal conversations and interesting scientific discussions.

Concerning this document, special thanks goes to Will Cuypers, for giving me valuable feedback on my writing skills in the vital parts of the thesis, as well as to Marianne Nijhuis for designing the cover. I also acknowledge the Pallottiner-nunnery in Limburg a.d. Lahn, where I was able to perfectly isolate myself in order to zoom out and concentrate on the contents of the thesis report.

I finally thank my parents and my sister, for their love and understanding and their support in everyday life.

Sjoerd G. Dietz,
Delft, October 2008.

Contents

Preface	v
1 Introduction	xi
1.1 Control engineering: an overview	xi
1.1.1 The controller design problem	xii
1.1.2 From classical to modern controller design	xiv
1.2 Analysis and controller design in the LMI framework	xv
1.3 Aims of the thesis.	xix
1.4 Outline and contributions	xxi
2 Robust semi-definite programming and LMI relaxation schemes	1
2.1 Motivation: μ -upper bound computation	3
2.1.1 Improved upper bounds	5
2.1.2 Stability analysis of uncertain systems	6
2.2 Direct relaxation approach based on matrix sum-of-squares	8
2.2.1 When is a polynomial positive ?	8
2.2.2 An existing relaxation approach using matrix sum-of-squares	12
2.2.3 An alternative relaxation approach	14
2.2.4 From rational to polynomial dependence	19
2.3 Multiplier-based relaxations	20
2.3.1 Relaxation schemes based on convexity arguments	23
2.3.2 Multiplier relaxations using Pólya's Theorem	24
2.3.3 Sum-of-squares relaxations	25
2.4 Estimating the relaxation gap	25
2.4.1 Computing lower bounds by gridding the parameter set	26
2.4.2 Verifying exactness for multiplier relaxations	26
2.4.3 Numerical example: μ -upper bound computation	31
2.5 Solving polynomial systems by linear algebra	37
2.5.1 A new algorithm	41
2.5.2 Numerical example	49
2.6 Summary	53

3	Analysis with integral quadratic constraints	55
3.1	Robust quadratic performance	59
3.2	Robust \mathcal{H}_2 -performance analysis	61
3.3	IQC-analysis for time-varying parameters	64
3.4	Numerical example	67
3.5	Summary	70
4	Analysis of discrete-time LPV systems	71
4.1	Lyapunov stability analysis	73
4.1.1	A survey on parameter dependent Lyapunov functions	75
4.1.2	Non-conservative Lyapunov-based analysis	75
4.1.3	Stability of LPV systems using a lifting approach	78
4.1.4	Asymptotic exactness of the lifted approach to stability analysis of LPV systems	81
4.1.5	Proving instability via periodic parameter sequences	86
4.1.6	A numerical procedure for computing stability margins	87
4.1.7	Evaluation of the proposed analysis method	91
4.2	Performance analysis	92
4.2.1	Quadratic performance	92
4.2.2	\mathcal{H}_2 -performance	94
4.2.3	Lifting of discrete-time LPV systems	97
4.2.4	Asymptotically exact performance analysis	101
4.3	Numerical example: l_2 -gain analysis	105
4.3.1	Stability analysis	105
4.3.2	l_2 -gain analysis	106
4.4	Summary	110
5	A convex robust synthesis solution for specific generalized plant structures	111
5.1	Motivating example: uncertainty in the disturbance model	112
5.2	Problem formulation	115
5.3	Robust synthesis via LMIs	116
5.3.1	From analysis to controller synthesis	116
5.3.2	A new characterization of nominal stability	117
5.3.3	Derivation of the convex synthesis conditions	120
5.3.4	Robust \mathcal{H}_2 -synthesis	127
5.3.5	Elimination of the controller parameters	128
5.3.6	Other interconnections of uncertain systems	130
5.4	Case study: parameter dependent disturbance filter	136
5.5	Summary	140

6	Scheduled controller synthesis	141
6.1	LPV synthesis by robust semi-definite programming	142
6.1.1	Elimination of parameters	147
6.2	Numerical example	149
6.3	Summary	155
7	Conclusions	157
7.1	Towards a systematic and practical design procedure	161
A	Analysis of LTI systems through LMIs	163
A.1	Stability	163
A.2	Performance measures	164
A.2.1	\mathcal{H}_∞ -norm	164
A.2.2	\mathcal{H}_2 -norm	165
A.2.3	Quadratic performance	166
A.3	LMI characterization of stability and performance	167
B	Linear fractional transformations	171
C	Equivalence of the sum-of-squares approximations	177
D	Asymptotic exactness in \mathcal{L}_2-gain analysis	181
	Bibliography	183
	Summary	197
	Samenvatting	199
	Curriculum Vitae	201

Chapter 1

Introduction

Control engineering arises in many practical systems. The growing complexity of dynamical systems has led to the development of mathematical tools that support control design. In this thesis, new results in the field of robust and scheduled control are developed in terms of semi-definite programming problems.

1.1 Control engineering: an overview

The goal of control engineering is to improve the performance of a dynamical system by making use of sensors, actuators and controllers. For instance, the current trend of reducing the mass of commercial aircrafts in order to save fuel costs and diminish environmental pollution has led to higher demands on the control system technology. In fact, the policy of lowering the aircraft's mass is mainly supported by the expectation that a reduction in the stiffness of the aircraft's structure can be effectively compensated for by the use of feedback control. A similar trend is seen in the production of electronic components, where piezo elements are installed in order to actively dampen pick-and-place units.

The diagram in Figure 1.1 represents a control system, in which the system to be controlled might for example be an aircraft, a CD-player, a hard disk or a power plant. An essential feature of a control system is the feedback mechanism, in which the measured signals are used to modify the system via the controller and the actuators. Some systems are controlled without taking on-line measurements of the system and are usually referred to as an open-loop control system. The interconnected system as depicted in Figure 1.1 will be referred to as the *closed-loop control system*. Also indicated are the disturbances that affect the system, the command signals that are typically defined by a human operator, and the signals used for system monitoring and fault diagnosis.

A feedback strategy changes the behavior of the system. It can improve performance and stabilize systems that are open-loop unstable. This thesis is concerned with the analysis of control systems as well as the design of a feedback controller, also

referred to as a control law. The controller consists of an algorithm that describes how the actuator signals are generated from the sensor and command signals. The design of a controller, also known as controller synthesis, plays an fundamental role in the design of a control system as a whole. A large variety of design methods is now being used in practice. We will be concerned with methods that are based on a mathematical model of the system.

The success of a control system design is not only determined by the control law. An important factor that acts on the achievable performance is the selection and positioning of actuators and sensors, known as the *control configuration*. Typical actuators are valves, DC motors or heating devices, whereas typical measurements involve position, temperature or pressure.

Besides fundamental limitations of the system to be controlled, the mathematical model that is used influences the achievable closed-loop performance. Building such a model includes a description of the signals that act on the system, i.e. noise- and disturbance signals, as well as the command signals that are generated by the operator or some trajectory planning device.

Mathematical models vary in complexity and fidelity. A simple model is preferred from the designer's perspective, as it simplifies the design, simulation or analysis of the control system. On the other hand, by leaving out too many details, it might no longer be possible to achieve satisfactory results. In this thesis, all mathematical models are built from ordinary differential equations. These models typically result from laws of physics (e.g. Newton's equations) but can also be obtained by identification procedures. The tools in this thesis allow to effectively account for the possible mismatch between the real-life system and its mathematical description.

In the next section the main scope of this thesis is presented, which is the design of a (feedback) controller. Note that, ideally, a control engineer should also be involved in the design of the system itself, the control configuration and the modeling phase. Although this fact is well-recognized when it concerns the development of advanced mechatronic systems, it often happens that the control task is formulated after the system has been manufactured.

1.1.1 The controller design problem

The purpose of designing a control system is to achieve a certain desired behavior. This behavior is characterized in terms of specifications on the performance and robustness of the closed-loop system. Performance specifications describe how the closed-loop system should perform and are defined on certain input-output channels. Examples of performance specifications are:

- Good rejection of disturbances
Disturbances cause the system response to deviate from the desired trajectory or set point. Examples of external disturbances are ground vibrations as

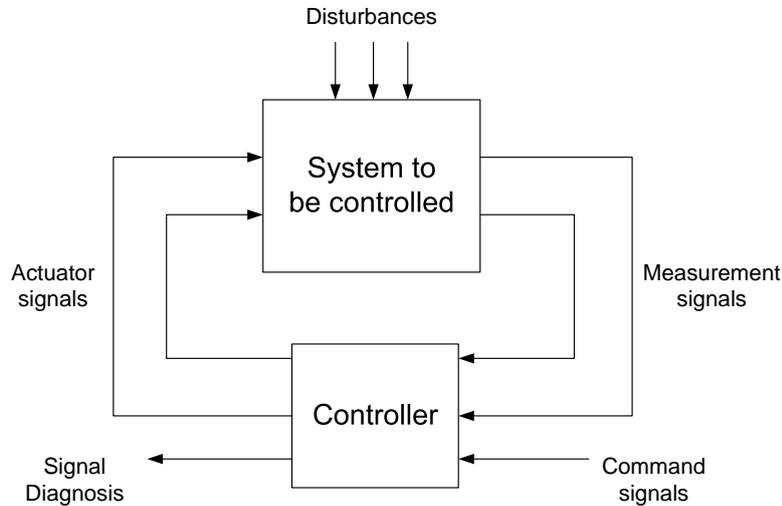


Figure 1.1: Abstract representation of a general closed-loop control system.

seen in servo-mechanical systems or wind gusts that act on aircrafts. Measurement noise is also treated as an exogenous disturbance, as it degrades the performance in an indirect way.

- Command following
This concerns the response of the system due to a new set-point or pre-defined reference trajectory. For instance, the design of a flight control system always involves several requirements regarding the response due to a certain pilot input.
- Avoidance of actuator saturation
In order to make sure that a controller performs as expected, the commands to the actuator should respect physical limitations. For instance, a servo-mechanical actuator has a maximal possible deflection and a finite bandwidth.

The dynamics of a system is not constant during operation. Typical quantities like friction- or temperature coefficients are affected by aging and change over the life time of the system. Moreover, the efficiency of the actuators or the noise level that act on the sensor measurements are not constant. Whenever system characteristics vary over time in a way that is practically impossible to put into a mathematical model, we are motivated to add so-called *robustness specifications*. These account for imperfections in the mathematical model. Robustness analysis has gained much interest over the last three decades, resulting in powerful and practically useful mathematical analysis tools.

Finally, one might need to add certain requirements on the controller itself. In industrial applications, controllers are often required to be of a specific structure,

e.g. PID, which means that modern optimization-based control algorithms cannot be applied immediately. If the number of actuators and sensors is large, a particular structure is likely to be preferred. In a decentralized control configuration, each individual actuator signal depends on one sensor signal only, which simplifies the controller implementation. In this thesis, no such demands are put on the controller and the goal will always be to construct a single control element that interacts with the system.

For a given mathematical model of the system, the control design problem consists of computing a controller that yields satisfactory performance. Like all design problems, the controller synthesis problem involves trade-offs. The most obvious trade-off arises between performance demands on the one hand and robustness requirements on the other hand. In fact, the more a system should be robust against variations in the plant, the less performance can be achieved. Other trade-offs are not so obvious.

Any controller that fulfills the specified design goals is said to be suitable. If it can be shown that no such controller exists, one either has to relax the design specifications or redesign the system, for instance by adding or relocating sensors and actuators.

1.1.2 From classical to modern controller design

Early controller design approaches were derived for Linear Time Invariant (LTI) systems, mainly single-input-single-output (SISO) systems, and were based on frequency-domain techniques (Bode, Nyquist). With these methods, satisfactory performance could be achieved by suppressing certain frequency bands that are known to represent the disturbance characteristics, at the cost of becoming more sensitive at frequencies where no excitation is to be expected. The robustness of the closed-loop system against small variations in the plant was expressed in terms of gain and phase margins. These classical design tools are of limited use, when designing controllers for general multi-input multi-output (MIMO) control systems. In fact, the robustness of general MIMO control systems cannot be captured in terms of the gain- and phase margins of all individual SISO control channels.

In the 1960s, methods based on state-space system descriptions were introduced and many control problems were rephrased in the context of mathematical optimization. With the development of the linear quadratic regulator and Kalman filter theory, LQG control (which later became known as \mathcal{H}_2 -synthesis) provided an effective tool for making a general MIMO system insensitive to unknown additive noise sources, of which the spectral content is known. Disturbances can often be very well described by random processes, which explains why LQG optimal control has been often applied in practice. Nevertheless, as shown in [65], LQG control has no guaranteed robustness margins. In order to effectively incorporate plant model

uncertainty, additional tools were needed.

In the same period, the so-called \mathcal{H}_∞ -optimal control theory was introduced. In combination with the structured singular value theory [66, 10], the \mathcal{H}_∞ -norm performance measure could adequately capture robustness margins of MIMO systems, in particular for specific classes of time-invariant uncertainties on the plant model. A Matlab toolbox for μ -analysis and robust \mathcal{H}_∞ -synthesis has been developed [10], which has been successfully applied on various practical applications.

In optimal \mathcal{H}_∞ -controller synthesis, one minimizes the \mathcal{H}_∞ -norm of the weighted closed loop transfer matrix. Initial solution approaches to this problem started in the frequency domain, see for example [74]. The development of state-space solutions in [67] showed that \mathcal{H}_∞ -synthesis could be solved in terms of a Riccati matrix equation, similar as was demonstrated earlier for the LQG control problem. Moreover, a solution to the Riccati matrix equation could be found by modern convex optimization based control, and semi-definite programming in particular, see [190, 28, 68, 44] to list a few.

In the next section, we will introduce semi-definite programming as the computational framework that forms the basis of the developed tools in this thesis. In Section 1.3, several limitations of the existing controller synthesis algorithms are listed, from which we will extract our main goals. Finally, a brief summary of the contributions and the thesis outline are given in Section 1.4.

1.2 Analysis and controller design in the LMI framework

Matrix Riccati equations play a fundamental role in \mathcal{H}_2 - and \mathcal{H}_∞ - optimal control problems and variations thereof, see [191, 140]. The relation between Riccati equations and linear matrix inequalities (LMIs) has been known for long, see [178]. It got renewed interest with the observation that solving LMIs is a convex problem for which fast interior point algorithms are accessible in commercial toolboxes [78] or as freeware on the internet [170]. Linear optimization subject to LMI constraints is also known as semi-definite programming.

During the last couple of decades, intensive research efforts on the applications of LMI's in control has lead to a large amount of literature, as cited e.g. in [29, 69]. Many classical problems in control were reformulated in terms of LMIs and the use of semi-definite programming for solving new control tasks was explored. Other fields of engineering were shown to benefit as well from the convex optimization tools, as is seen in recent books [30, 17], and the references therein.

Although Riccati equations play an important a role, it is dissipation theory for linear and non-linear systems that lies at the basis of modern controller design in the LMI framework. The notion of dissipative systems, first presented in [179], has proven to be extremely powerful in characterizing all sorts of input-output properties in a unified setting. For instance, if both the input w and output z are measured

in terms of energy, a typical performance indicator is defined as the smallest γ for which $\|z\|_2 \leq \gamma\|w\|_2$ holds true for any input w that has finite energy. This quantity is known as the induced \mathcal{L}_2 -gain. Consider the LTI system

$$\dot{x} = Ax + Bw, \quad z = Cx + Dw.$$

One can show that this system is stable and the induced \mathcal{L}_2 -gain is bounded by γ if the following LMIs are feasible:

$$X \succ 0, \quad \begin{pmatrix} A^T X + XA & XB & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} \prec 0. \quad (1.1)$$

We emphasize that X enters in a linear fashion, which explains the terminology *linear* matrix inequality. The constraint $X \succ 0$ indicates that all eigenvalues of X are positive, $X \prec 0$ would correspond to having all eigenvalues of X negative. Without going into details, it can be shown that the minimum γ for which a solution X satisfies these two LMI constraints is equal to the \mathcal{H}_∞ -norm of the transfer matrix $G(s) = D + C(sI - A)^{-1}B$.

When using the LMI characterization (1.1) for controller design purposes, the system matrices will depend on the to-be-designed controller variables which renders the second matrix inequality in (1.1) bilinear in the controller variables and the Lyapunov matrix X . As shown in [159, 122], a full solution of the (nominal) controller synthesis problem is obtained by using a general non-linear variable transformation. Multiple performance conditions of different nature (mixed $\mathcal{H}_\infty, \mathcal{H}_2$) can be combined in order to design multi-objective controllers and other quadratic performance measures can be included as well.

Control of uncertain systems

A successful control system design achieves a desired performance level while being also tolerant against variations in the system. By exploiting structural knowledge of the uncertainties, the performance of complex dynamical systems can be increased. Such knowledge is often present. In a mechanical system for instance, the damping and spring coefficients are known up to a certain level, which motivates to consider the model class that is formed by all possible damping and spring coefficient values

In general, the mismatch of a mathematical model with the real system can be captured by using an *uncertain model* rather than a single mathematical model. In other words, we cover a complex dynamical system by a family of relatively simple models. The resulting *robust* controller synthesis problem consists of finding a suitable controller that satisfies performance for all admissible uncertainties.

As will be extensively discussed in Chapter 7 of this thesis, robustness analysis plays an essential role in the controller design process. In order to guarantee that the system achieves the desired performance, one has to verify whether all models that are contained in the uncertain model, are stable and satisfy performance. This is known as robustness analysis. If the performance specifications cannot be met, one either tries to improve the quality of the model, or one has to relax the design goals.

In this thesis, uncertain models appear in two different forms that are well-known in the robust control community. The first is based on the so-called generalized plant framework and can handle all sorts of time-varying parametric and non-parametric, linear or non-linear phenomena, at least conceptually. The framework was introduced with the by now well-established \mathcal{H}_∞ -controller synthesis technique. The uncertain system is modeled as the feedback interconnection of a fixed nominal plant, which we will assume to be LTI, and an uncertain element Δ specified to be contained in some set $\Delta \in \mathbf{\Delta}$.

A second class of uncertain systems consists of the so-called Linear Parameter Varying (LPV) models, which are marked by the fact that only parametric uncertainties are involved. If (reliable) online measurements of the parameters are available, one typically aims at designing a controller that schedules with the parameter, rather than a robust one. Let us discuss in more detail these two different uncertain model classes that are addressed in this thesis.

Generalized plant framework

A large variety of robust control design problems can be addressed in the so-called generalized plant framework, see [10] for a reference. Here, each system component is modeled as a fixed (nominal) linear plant model and a collection of uncertain elements that are due to e.g. actuator saturation, backlash, dead-zones, time delays or quantization.

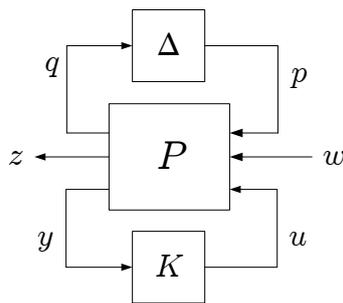


Figure 1.2: Generalized plant

In a general interconnection of subsystems, uncertainty will affect each individual component. A procedure known as “pulling out the uncertainties” enables the construction of a single fixed plant model P , the generalized plant and a (generally

structured) set of operators $\mathbf{\Delta}$ that represent all the uncertainties. The result of this procedure is illustrated in Figure 1.2 and naturally leads to a structured operator $\Delta \in \mathbf{\Delta}$, see also Appendix B. Hence, the uncertain system can be written as

$$\begin{pmatrix} q \\ z \\ y \end{pmatrix} = P \begin{pmatrix} p \\ w \\ u \end{pmatrix}, \quad p = \Delta(q), \quad \Delta \in \mathbf{\Delta}, \quad u = Ky.$$

The system model P and the feedback controller K are typically chosen to be LTI systems. The philosophy of adding non-linear or time-varying uncertain elements to a nominal LTI plant has proven to be very useful in describing uncertain systems. In fact, it goes back to the absolute stability problem in [144], in which an LTI system is interconnected with a single unknown static non-linear element that is characterized by certain sector bounds.

As will be extensively discussed in Chapter 3, the generalized plant framework lies at the basis of analyzing general interconnections of uncertain systems. The machinery for rendering the robustness analysis problem numerically tractable is provided by so-called integral quadratic constraints (IQC), [123, 98]. For some recent contributions on IQCs, see for instance [75, 101, 102, 100, 1].

Remark 1.1 *Similar to the observation that “pulling out the delta’s” leads to a block-structured Δ , the to-be-designed controller block K can represent a mixture of local feed-forward and feed-back compensators that arise in the underlying interconnected system. At present, it is however unknown how to turn the design of structured controllers into an LMI optimization problem. For some recent developments on this problem, the reader is referred to [93, 153, 109, 46] and references therein.*

Linear parameter varying systems and scheduled control

Building a model by a feedback interconnection of a fixed LTI plant with an uncertain operator Δ is not the most obvious approach if the uncertainties are parametric. In fact, an LTI model very often involves a set of physical quantities like temperature, altitude or pressure. The fact that these parameters may vary in time naturally leads to a so-called *linear parameter varying* system, which is described as

$$\begin{aligned} \dot{x} &= A(\delta(t))x + B_1(\delta(t))w + B_2(\delta(t))u \\ z &= C_1(\delta(t))x + D_{11}(\delta(t))w + D_{12}(\delta(t))u \\ y &= C_2(\delta(t))x + D_{21}(\delta(t))w + D_{22}(\delta(t))u \end{aligned} \tag{1.2}$$

in which $x(t)$ denotes the state, $w(t)$ the disturbance, $z(t)$ the controlled output and $\delta(t)$ the (time-varying) parameters, all of which can be vector-valued signals. The family of admissible parameter trajectories is defined in terms of the relation $\delta(t) \in \mathbf{\delta}$ for all t , for some compact set $\mathbf{\delta}$. A more precise characterization is obtained by adding certain bounds on $\dot{\delta}(t)$. We will see in Chapter 4 and 6 that it is

convenient to assume that the system matrices in (1.2) are given as a linear fractional representation. Then, the LPV system can also be described as a generalized plant with a parametric uncertainty $\Delta = \Delta(\delta)$, see also Appendix B.

Recall the LMI condition (1.1) which characterizes a bound on the \mathcal{L}_2 -gain of an LTI system. This characterization of performance has a natural extension to the class of LPV systems. In fact, based on dissipation theory [178], one can show that the LPV system is stable and a bound γ on the \mathcal{L}_2 -gain of the channel $w \rightarrow z$ is provided by the existence of an X that satisfies

$$X \succ 0, \quad \begin{pmatrix} A(\delta)^T X + X A(\delta) & X B_1(\delta) & C_1(\delta)^T \\ B_1(\delta)^T X & -\gamma I & D_{11}(\delta)^T \\ C_1(\delta) & D_{11}(\delta) & -\gamma I \end{pmatrix} \prec 0 \quad (1.3)$$

for all admissible parameter values $\delta \in \delta$. The second matrix inequality has a semi-infinite nature, since it must hold at an infinite number of parameter values. Optimization that involves such parameter-dependent LMI constraints, or robust LMIs, is referred to as robust semi-definite programming, and is computationally hard in general.

1.3 Aims of the thesis.

In view of the key role of LMIs in solving control problems, there is a need for reliable and efficient algorithms for solving (robust) LMI problems. The numerous examples in [30, 17] and references therein show the success of applying existing algorithms to \mathcal{H}_∞ - or \mathcal{H}_2 -optimal control problems and variations thereof.

The analysis of uncertain systems and of LPV systems in particular involves robust LMI constraints, an argument that applies to the robust and scheduled controller design problem as well. Our first objective is motivated by the fact that there are still many challenging control problems for which a characterization in terms of robust LMI constraints is expected to exist. Among many other interesting design problems, for instance the controller synthesis problem of fixed order or structured controllers, this thesis is concerned with the design of robust and scheduled output-feedback controllers and the analysis of LPV systems in particular.

Objective 1: Improve the usability of the LMI framework for solving robust control problems.

Robust LMIs are expected to play an important role in handling future control design problems, as can be concluded from the books [29, 69]. Here, we aim at developing new and improved algorithms in two fields.

- *Lyapunov-based analysis and controller synthesis of LPV systems:*
Based on quadratic-in-the-state Lyapunov functions, one can easily formulate

sufficient conditions for analyzing the stability and performance of an LPV system (1.2). However, despite the fact that conservatism is (somewhat) reduced by using a parameter dependent Lyapunov matrix in (1.3), it is unknown how to systematically improve the analysis conditions in general. Specific classes of non-quadratic Lyapunov functions have been shown to be non-restrictive (the so-called “converse theorems”), though an efficient numerical implementation of these results is lacking.

- *Robust controller synthesis in the generalized plant framework:*

The robust output-feedback controller synthesis problem amounts to designing an optimal LTI controller for an uncertain system. At present, an LMI solution to this problem is lacking. Heuristic (iterative) procedures typically do not provide globally optimal solutions. Based on the controller parameter transformation technique in [159, 122], the nominal output-feedback controller synthesis problem has been formulated as an LMI optimization problem. It is expected that the robust controller synthesis problem can be rendered convex for particular problem classes, by exploiting the problem structure.

Although robust LMI constraints are convex in the decision variables they are generally non-tractable and can only be approximately solved by so-called relaxation schemes. Such schemes are usually conservative, in the sense that the computed solutions are feasible, but not necessarily optimal for the original robust LMI problem. In order to fully benefit from robust controller design methods based on robust LMI optimization, it is essential that relaxation schemes are constructed in a systematic fashion. This leads us to our second objective.

Objective 2: Develop a unified framework for constructing LMI relaxations.

With a strong focus on control challenges that have been formulated as a robust LMI optimization problem, our main goal is to provide an easy and flexible environment for evaluating and comparing different relaxation methods. Important research questions that should be addressed are the following:

- How to estimate the level of conservatism (relaxation gap)?
- How to modify relaxation schemes in order to reduce conservatism?
- How to exploit the problem structure?

Associated with these technical and, to some extent, theoretical issues, a numerical implementation of the relaxation methods should be developed. In order to fully explore the flexibility of the framework as well as to save time, it is also needed to build user-friendly tools that automatically construct relaxation schemes for a specified problem.

1.4 Outline and contributions

Chapter 2 describes the construction of suitable relaxation schemes for approximating a robust LMI constraint. At the heart of any relaxation scheme lies the issue of verifying the positivity of polynomials, a topic that will take up a large part of this chapter. It is shown that robust LMI constraints that are rational in the uncertain parameters, can always be transformed into an equivalent polynomial one. The contributions in this chapter can be summarized as follows:

- A new implementation of relaxations based on matrix-sum-of-squares.
- A condition for verifying whether a computed S-procedure based relaxation is exact. In contrast to the original paper [157], a generalization of the exactness test regarding the case of multiple robust SDP constraints is given, which amounts to solving a polynomial system.
- A new algorithm for solving systems of polynomials, which is elegant and conceptually simple, since it only makes use of linear algebraic operations. It forms an extension of the so-called Stetter's method [168].
- A Matlab Toolbox that automates the construction of relaxation schemes, see [55].

With the computational tools of Chapter 2 available, the remaining chapters focus on the analysis of uncertain systems and the design of robust and scheduled controllers. In each individual chapter, it is shown how to translate the problem into optimization subject to (robust) LMI constraints. Numerical examples are included and illustrate the use of relaxation schemes in different contexts.

Chapter 3 starts with the problem setup in Figure 1.2. We revisit the analysis approach based on integral quadratic constraints. In this method, a parameterized class of multipliers is properly chosen such that it captures a given set of uncertain operators $\Delta \in \mathbf{\Delta}$. Although the IQC methodology applies to general non-linear and time-varying uncertainties, emphasis is put on parametric uncertainties, for which suitable multiplier classes can be described in terms of robust LMI constraints. A numerical example is included, in which upper bounds on the worst-case \mathcal{L}_2 -gain are computed for a given LPV system.

In Chapter 4 we consider the class of LPV systems in discrete-time. Based on a quadratic-in-the-state Lyapunov function, analysis conditions for stability can be developed in the form of robust LMI constraints. In general, these conditions do not lead to exact computations (even if the relaxation gap is zero) and it is unknown how to systematically estimate the level of conservatism.

As an alternative to analysis tests based on non-quadratic Lyapunov functions, we present a framework for stability and performance analysis of general discrete-time LPV systems that employs a well-known lifting technique. As one of our

main contributions, it is shown that the level of conservatism can be reduced to zero by increasing the lifting horizon. The family of conditions for stability and performance analysis of LPV systems is therefore called asymptotically exact. The potential benefit of our approach lies in the fact that the numerical complexity of the constructed schemes does not depend on the state-dimension, contrary to what is typically seen in an approach based on higher-order-in-the state Lyapunov functions. We will emphasize the key role played by N -periodic parameter trajectories, leading to a systematic construction of destabilizing or worst-case parameter trajectories. A list of the contributions of Chapter 4 reads as follows:

- An alternative approach for the analysis of LPV systems in discrete-time with general parameter variation bounds, based on a well-known lifting technique.
- A proof of the fact that the constructed family of robust SDP conditions is asymptotically exact.
- For the stability analysis problem, the asymptotically exact family of analysis conditions can be viewed as a generalization of the joint spectral radius for switched systems to LPV systems with general parameter variation bounds.
- Analysis conditions that characterize the induced l_2 -gain and \mathcal{H}_2 -performance of LPV systems are derived in terms of the N -lifted system, resulting in robust LMI constraints.
- A comparison is made between the Lyapunov-based and IQC-based analysis method in the context of l_2 -gain analysis for an LPV system.

In Chapter 5 we turn to the robust controller synthesis problem in the generalized plant framework. Motivated by a particular design problem in which the disturbance signals are characterized by an uncertain input filter, we confine ourselves to a particularly structured generalized plant. The robust synthesis result is found in Section 5.3 and exploits the plant structure, resulting in convex synthesis conditions for the robust output-feedback design problem. The contributions in this chapter can be summarized as:

- A new proof of a recently developed state-space characterization of nominal stability in the context of IQC-analysis with dynamic multipliers.
- A complete solution to the robust output-feedback controller synthesis problem for generalized plants of a certain structure.

The success of the proposed algorithm is shown by means of a numerical example, in which the robust disturbance-rejection problem is solved by considering an uncertain disturbance filter at the plant input.

Finally, in Chapter 6 we consider the design of scheduled controllers for LPV systems. The presented LPV synthesis approach is taken from the literature, see [184], and sum-of-squares relaxation tools are employed instead of the usual convex

hull arguments. This will lead to an improvement in the closed loop performance and also gives us the ability to include more realistic parameter regions described by polynomial inequalities. Our final contribution in the thesis can be formulated as:

- A numerical example in which sum-of-squares relaxations are employed to solve the LPV controller synthesis problem.

Chapter 7 concludes this thesis and gives suggestions for future research. Background material on the theory of LTI systems and LFT calculus has been included in Appendix A and B.

Chapter 2

Robust semi-definite programming and LMI relaxation schemes

This chapter develops a framework for solving robust LMI optimization problems. As will be shown in the next chapters, questions in robust stability and performance analysis, as well as robust or scheduled controller synthesis naturally lead to such robust LMI problems.

A robust LMI is a matrix inequality that depends on so-called decision variables $y = (y_1, \dots, y_{n_d}) \in \mathbb{R}^{n_d}$, as well as on parameters x_1, \dots, x_s that are assumed to lie within a typically compact set $\mathcal{X} \subseteq \mathbb{R}^s$. Formally, a robust LMI reads as

$$P_0(x) + \sum_{i=1}^{n_d} P_i(x)y_i \prec 0 \quad \text{for all } x \in \mathcal{X}, \quad (2.1)$$

in which $P_i(x)$ are Hermitian-valued mappings, assumed rationally dependent and well-defined on $x \in \mathcal{X}$. Computing a feasible point $y \in \mathbb{R}^{n_d}$ of (2.1), or minimization of a linear cost functional over the feasible set described by (2.1) are referred to as robust semi-definite programming (SDP) problems. Note that by taking the P_0, \dots, P_{n_d} as constant matrices, this formulation is seen to include genuine LMI constraints on y as well. For a brief introduction on LMIs, the reader is referred to Appendix A.

In this thesis, the following alternative representation of robust LMI constraint (2.1) will be used:

$$F(x)'J(y)F(x) \prec 0 \quad \text{for all } x \in \mathcal{X}, \quad (2.2)$$

in which $J(y)$ is affine in y . The equivalence with (2.1) follows if defining the matrix

functions $F(x), J(y)$ as

$$F(x) = \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n_d}(x) \\ I \end{pmatrix} \quad \text{and} \quad J(y) = \frac{1}{2} \begin{pmatrix} 0 & 0 & \dots & I \\ 0 & & & y_1 I \\ & & & \vdots \\ : & \ddots & & y_{n_d} I \\ I & y_1 I & \dots & y_{n_d} I & 0 \end{pmatrix}.$$

Although multiple LMI constraints can always be combined into a single one, it is often convenient to explicitly take multiple constraints into account. Hence, adding also a linear cost functional to the problem, the general robust SDP optimization problem can be formulated as follows:

$$\gamma_{\text{opt}} = \inf \left\{ \langle c, y \rangle : y \in \mathbb{R}^{n_d}, \quad F_i(x)' J_i(y) F_i(x) \prec 0 \quad \text{for all } x \in \mathcal{X}, \right. \\ \left. i = 1, \dots, n_c \right\} \quad (2.3)$$

in which $c \in \mathbb{R}^{n_d}$ is a fixed vector that defines the cost, $y \in \mathbb{R}^{n_d}$ are the decision variables, $J_i(y)$ are affine in y and map into the space of symmetric matrices and $F_i(x)$ are allowed to be rational in x . Throughout this thesis, the existence of a feasible point satisfying (2.3) is assumed, from which it follows that $\gamma_{\text{opt}} < \infty$. Note that the parameter domain \mathcal{X} need not be the same for each of the robust LMI constraints. In order not to make our notation more cumbersome, we have assumed $\mathcal{X}_1 = \mathcal{X}_2 = \dots = \mathcal{X}_{n_c} = \mathcal{X}$.

Even though a robust LMI constraint is convex in finitely many decision variables $y \in \mathbb{R}^{n_d}$, optimization over robust LMIs is numerically intractable, apart from some specific cases. The difficulty of solving the robust SDP (2.3) largely depends on the specified parameter region \mathcal{X} as well as the functional dependence of the matrix valued maps F_1, \dots, F_{n_c} .

In the remaining part of this chapter, we will show how to compute approximate solutions $y \in \mathbb{R}^{n_d}$ of the robust SDP (2.3). We are particularly interested in those solutions for which the (non-tractable) robust SDP constraint is guaranteed to hold. An LMI approximation of problem (2.3) for which the resulting $y \in \mathbb{R}^{n_d}$ always satisfies all semi-infinite constraints in (2.3) is said to be a *relaxation scheme*. As we will see later, the formulation (2.2) rather than (2.1) turns out to be suitable for the construction of relaxation schemes.

In general, relaxation schemes only provide upper bound values $\gamma_{\text{rel}} \geq \gamma_{\text{opt}}$. In order to interpret our numerical results, we are thus faced with the issue of estimating the relaxation gap $\gamma_{\text{rel}} - \gamma_{\text{opt}}$. For specific relaxations one can give a priori bounds on the relaxation gap see [124, 131], though this cannot be done in general. A straightforward and ad hoc method to gain insight in the quality of relaxation schemes is to compute a lower bound value $\gamma_{\text{lb}} \leq \gamma_{\text{opt}}$, by sampling the robust SDP constraint on a finite grid, and solving the resulting standard LMI problem. In

[128, 90, 158] a more systematic approach based on a gridding technique is proposed.

The essential theory from which relaxation schemes can be constructed is contained in Section 2.2. The key question is how to verify positivity of a given polynomial on some specified set. We will derive a family of relaxations for which the relaxation gap $\gamma_{\text{rel}} - \gamma_{\text{opt}}$ can be rendered arbitrarily small. For a specific class of relaxation schemes in Section 2.3, it is even possible to detect whether a computed relaxation is exact. We will elaborate on the subject of verifying exactness in Section 2.4. The corresponding test for verifying exactness amounts to finding a solution of a system of polynomials. Motivated by this problem, a new algorithm for solving systems of polynomial equations has been developed and is contained in Section 2.5. The proposed relaxation schemes of Section 2.3 hence provide a systematic procedure for extracting worst-case parameters from the computed relaxation scheme. Before we start the development of relaxation methods, let us first provide the reader with a motivating example.

2.1 Motivation: μ -upper bound computation

This section illustrates why robust SDPs are a relevant problem class, which naturally follows from the stability analysis problem for uncertain systems. We first discuss an essential robust linear algebra problem, and show its relation to uncertain systems in Section 2.1.2.

Suppose that matrix $A \in \mathbb{C}^{n \times n}$ and a set \mathcal{B} of structured complex matrices of size $n \times n$ are given. Then, computing the largest real number r for which

$$\det(I - AB) \neq 0 \quad \text{for all } B \in r\mathcal{B} \tag{2.4}$$

is non-singular is a problem that has been intensively studied, see for example [10, 132]. In fact, it resulted in the notion of 'structured singular value' μ , formally defined as

$$\mu_{\mathcal{B}}(A) = \frac{1}{\sup\{r \mid \det(I - AB) \neq 0 \quad \text{for all } B \in r\mathcal{B}\}}.$$

In general, finding the exact value of $\mu_{\mathcal{B}}(A)$ is a non-tractable problem and only upper bound values can be computed. A (typically rough) upper bound is provided by the largest singular value of A , an argument commonly referred to as small-gain. Since \mathcal{B} is a set of structured matrices, less conservative results can be obtained by exploiting this structure, which explains why μ is called structured singular value. Suppose D is an invertible matrix with the following commuting property:

$$DB - BD = 0 \quad \text{for all } B \in \mathcal{B}. \tag{2.5}$$

It then follows that

$$D^{-1}(I - AB)D = I - D^{-1}ADB$$

from which the value $\|D^{-1}AD\| \leq \|A\|$ is also an upper bound value of $\mu_{\mathcal{B}}(A)$. The least upper bound on $\mu_{\mathcal{B}}(A)$ as it is found by using the D -scales amounts to a standard LMI optimization problem. In fact, if we introduce $R = D^T D$, the norm bound $\|D^{-1}AD\| < r$, i.e.

$$D^{-1T} A^T D^T D A D^{-1} - \frac{1}{r^2} I \prec 0,$$

is equivalent to

$$R \succeq 0 \quad \text{and} \quad r^2 A^T R A - R \prec 0. \quad (2.6)$$

For a fixed r , the search for $R \succeq 0$ that satisfies (2.6) is a standard LMI feasibility problem. The best possible approximation of $\mu_{\mathcal{B}}(A)$ corresponds to $\frac{1}{r_{max}}$, when r_{max} denotes the maximal r for which (2.6) is feasible. From the solution R , the scalings D can be obtained from any Choleski factorization $R = D^T D$.

Remark 2.1 *The maximal r for which (2.14) holds is obtained by bisection on r .*

Example 2.1 *For some real-valued parameter $p \in [0, 1]$, let matrix A be given as*

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 2p & 0 & p & 0 & 0 & 0 \\ 0 & -2p & 0 & -p & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-p & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-p \end{pmatrix} \quad (2.7)$$

and let B be a structured matrix in the set

$$\mathcal{B} = \left\{ \begin{pmatrix} x_1 I & 0 & 0 \\ 0 & x_2 I & 0 \\ 0 & 0 & x_3 I \end{pmatrix} \mid x_1, x_2, x_3 \in \mathbb{R}, |x_i| \leq 1, i = 1, 2, 3 \right\}, \quad (2.8)$$

in which I denotes the identity matrix of size two. The values $\|A\|$ as a function of p are plotted in Figure 2.1 together with the least upper bounds corresponding to the D -scalings test.

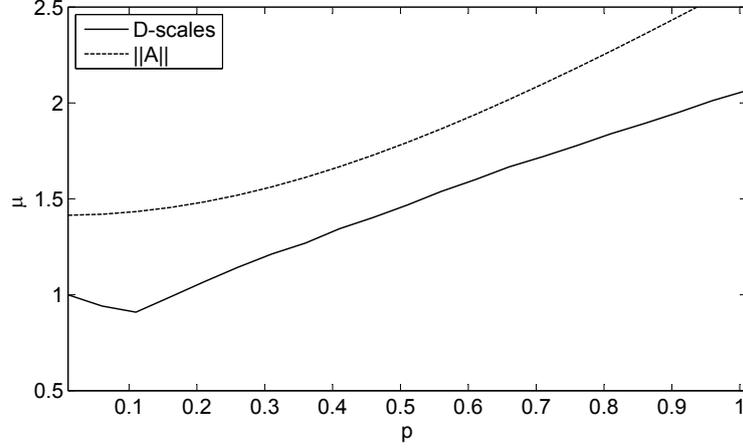


Figure 2.1: Upper bound values on $\mu_{\mathcal{B}}(A)$ for different parameters p in (2.7).

2.1.1 Improved upper bounds

Using D -scales for computing $\mu_{\mathcal{B}}(A)$ is a well known approach within the control community, see for example [131, 124]. However, the obtained upper bound values can be far from accurate which is why less conservative relaxations are desired. In order to do so, we first transform the condition (2.4) in two matrix inequality conditions, one of which becomes semi-infinite.

Lemma 2.1 *Let A and B be given matrices in $\mathbb{R}^{n \times n}$. Then, $I - AB$ is non-singular if*

$$\begin{pmatrix} I \\ A \end{pmatrix}' \Pi \begin{pmatrix} I \\ A \end{pmatrix} \prec 0. \quad (2.9)$$

and

$$\begin{pmatrix} B \\ I \end{pmatrix}' \Pi \begin{pmatrix} B \\ I \end{pmatrix} \succeq 0. \quad (2.10)$$

for some matrix $\Pi = \Pi' \in \mathbb{R}^{2n \times 2n}$.

Proof. Suppose that $I - AB$ is singular. Then, there exists $z \neq 0$ that satisfies $z = ABz$. Define the (nonzero) vector $y = Bz$. Finally, left-and right multiplication of (2.9) with y^T, y and left-and right multiplication of (2.10) with z^T, z , leads to the following facts

$$\begin{pmatrix} y \\ z \end{pmatrix}' \Pi \begin{pmatrix} y \\ z \end{pmatrix} \prec 0, \quad \begin{pmatrix} y \\ z \end{pmatrix}' \Pi \begin{pmatrix} y \\ z \end{pmatrix} \succeq 0,$$

which finishes the proof. ■

If this lemma is used for computing upper bounds on $\mu_{\mathcal{B}}(A)$, conditions (2.10) must hold for all $B \in \mathcal{B}$, which turns it into a semi-infinite LMI constraint. Let the elements $B \in \mathcal{B}$ be parameterized as by $x \in \mathcal{X}$, i.e. $B = \Delta(x)$, with parameter vector $x = (x_1, \dots, x_s) \in \mathcal{X} \subset \mathbb{R}^s$ and $\Delta(\cdot)$ linear, e.g. as was done in (2.8). Then, the existence of $\Pi = \Pi'$ that satisfies (2.9) and

$$\begin{pmatrix} \Delta(x) \\ I \end{pmatrix}' \Pi \begin{pmatrix} \Delta(x) \\ I \end{pmatrix} \succeq 0, \quad \forall x \in \mathcal{X} \quad (2.11)$$

implies that $I - AB$ is non-singular for all $B \in \mathcal{B}$. The latter matrix inequality is a robust LMI constraint, and is inherently non-tractable as it should hold for all $x \in \mathcal{X}$. Observe that it perfectly matches with the general form (2.2), if we parameterize $\Pi = J(y)$ by $y \in \mathbb{R}^{n_d}$ and choose the outer factor $F(x)$ accordingly.

It now becomes relatively easy to construct relaxations of various complexity. In fact, by imposing the following different structures of the multiplier Π :

$$\Pi_1 = \begin{pmatrix} -I & 0 \\ 0 & r^2 I \end{pmatrix} \quad \text{or} \quad \Pi_2 = \begin{pmatrix} -R & 0 \\ 0 & r^2 R \end{pmatrix}, \quad R \succeq 0, \quad (2.12)$$

the semi-infinite LMI constraint (2.11) is automatically fulfilled. As a consequence, by substituting either $\Pi = \Pi_1$ or $\Pi = \Pi_2$ into (2.9), we obtain a sufficient condition for $I - A\Delta$ to be non-singular for all $x \in \mathcal{X}$. Note that feasibility of (2.9) corresponds to the norm bound $\|A\| < r$ in case of choosing $\Pi = \Pi_1$, whereas it corresponds to feasibility of the D -scalings test (2.6) if setting $\Pi = \Pi_2$. As will be shown in Chapter 3, the upper bounds on μ can be improved by using full-block multipliers Π or even other relaxation methods.

2.1.2 Stability analysis of uncertain systems

A classical problem in the field of robust control is the stability analysis of the interconnection in Figure 2.2, see for example [10, 66]. Let $M \in RH_{\infty}^{n \times n}$ be a given stable transfer matrix of size $n \times n$ and let Δ be the uncertain LTI element $\Delta \in \mathbf{\Delta}$ in which the set $\mathbf{\Delta}$ is defined as

$$\mathbf{\Delta} := \{\Delta(s) \in RH_{\infty}^{n \times n} \mid \Delta(i\omega) \in \mathbf{\Delta}_c \text{ for all } \omega \in \mathbb{R} \cup \{\infty\}\}.$$

The set of (structured) complex matrices $\mathbf{\Delta}_c$ is assumed to satisfy $\|\Delta_c\| < 1$ for all $\Delta_c \in \mathbf{\Delta}_c$, i.e. its elements are unit norm bounded.

The goal in the stability analysis problem is to compute the largest $r \in \mathbb{R}$ for which the loop is stable for all $\Delta \in r\mathbf{\Delta}$. If M and Δ are both stable, it can be shown that stability of the interconnected system in Figure 2.2 amounts to $I - M(s)\Delta(s)$ having a proper and stable inverse. If we further assume that $\mathbf{\Delta}_c$ is star-shaped, i.e. $\tau\Delta_c \in \mathbf{\Delta}_c$ for all $\tau \in [0, 1]$ and all $\Delta_c \in \mathbf{\Delta}_c$, it can be shown that $I - M\Delta$ has a

proper and stable inverse for all $\Delta \in \mathbf{\Delta}$ if

$$I - M(i\omega)\Delta(i\omega) \quad \text{is non-singular} \quad \forall \omega \in \mathbb{R} \cup \{\infty\}, \forall \Delta \in r\mathbf{\Delta}. \quad (2.13)$$

In order to numerically implement this condition, one typically introduces a frequency grid and computes for each fixed frequency $\bar{\omega}$, denoting $A = M(i\bar{\omega})$, the largest r for which

$$I - A\Delta_c \quad \text{is non-singular for all } \Delta_c \in r\mathbf{\Delta}_c. \quad (2.14)$$

Note that (2.14) is identical to (2.4) if replacing the inclusion $\Delta_c \in \mathbf{\Delta}_c$ by $B \in \mathcal{B}$. If we now combine all previous arguments, the loop of M with Δ is proven to be robustly stable once $\|\Delta\|_\infty < \frac{1}{\bar{r}}$ if \bar{r} is defined as

$$\bar{r} = \min_{\omega \in \mathbb{R} \cup \{\infty\}} \frac{1}{\mu_{\mathbf{\Delta}_c}(M(i\omega))}.$$

It is stressed that the frequency can formally be treated as uncertainty as well, avoiding the need to grid over frequency, see [158].

Remark 2.2 *Lemma 2.1 touches on a fundamental stability theorem with integral quadratic constraints. We will see in Chapter 3 that a generalization of (2.9)-(2.10) enables one to prove robust stability for the interconnection of general time-varying or nonlinear operators Δ .*

Remark 2.3 *The resulting robustness margins in this section hold for linear time-invariant uncertainties, as opposed to more general uncertainties that will be treated in Chapter 5.*

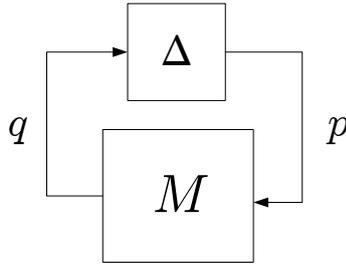


Figure 2.2: The $M - \Delta$ loop for robust stability analysis

2.2 Direct relaxation approach based on matrix sum-of-squares

As was mentioned in the previous section, polynomials lie at the basis of constructing relaxations schemes for the robust SDP (2.3). In this section, we therefore initially let the matrices $F_i(x)$ be polynomial matrix functions. In Section 2.2.4 we will show that any robust SDP constraint with rationally dependent $F_i(x)$ can be easily transformed into an equivalent polynomial.

The discussion starts with the question of verifying positivity of polynomials in the global sense. Then, we will present some classical results that can detect whether a polynomial is positive on some specified region. From these results we will be able to construct relaxation schemes for general polynomially dependent matrix valued constraints. As we will point out several times, a whole family of relaxation schemes can be constructed, by which one can (systematically) modify or re-construct a relaxation scheme in order to arrive at less conservative results. Moreover, the proposed family of relaxation schemes is asymptotically exact. That is, the relaxation gap can be brought to zero by adding auxiliary variables.

2.2.1 When is a polynomial positive ?

With the purpose of constructing relaxation schemes for robust SDP (2.3), we first concentrate on the very basic question of verifying positivity of multivariate polynomials. This issue has recently received new interest in the field of systems and control, see for example [88] and references therein.

Let p be a polynomial in the variables x_1, \dots, x_s . Then p is said to be (globally) positive if $p(x) > 0$ for all $x \in \mathbb{R}^s$ and (globally) non-negative if the inequality is non-strict. A sufficient condition for a polynomial to be non-negative is the existence of some N and polynomials s_1, \dots, s_N for which

$$p(x) = \sum_{j=1}^N s_j(x)^2.$$

Any p with such a decomposition is called a *sum-of-squares* polynomial. The German mathematician Hilbert had proven that not every non-negative polynomials admits such a decomposition. For example, the polynomial

$$p(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$$

is globally nonnegative but cannot be represented as a sum-of-squares of polynomials, see [149]. He expected though, that any non-negative rational function is a sum-of-squares of rational functions, i.e. any rational function $p(x)$ can be written as

$$p(x) = \left(\frac{q_1(x)}{s_1(x)}\right)^2 + \dots + \left(\frac{q_N(x)}{s_N(x)}\right)^2.$$

for some polynomials $q_j(x), s_j(x), j = 1, \dots, N$. In fact, he formulated this at the International Congress of Mathematicians in Paris in 1900, and it became known as Hilbert's 17th problem. A proof that this fact holds true was provided by Artin in [6]. Let us now focus on the question of verifying positivity on a pre-specified region, which brings us one step closer to the actual construction of relaxation schemes.

Positivity on polytopic regions

In the context of solving the robust SDP (2.3), one is typically interested in restricted positivity. Suppose that our aim is to verify whether a given polynomial p in s variables is positive on \mathcal{X} for some given set $\mathcal{X} \subset \mathbb{R}^s$. The difficulty of performing such a test depends on p as well as on \mathcal{X} . In case \mathcal{X} is described as the convex hull of finitely many points $x^1 \in \mathbb{R}^s, \dots, x^q \in \mathbb{R}^s$, that is

$$\mathcal{X} = \text{co}\{x^1, \dots, x^q\}, \quad (2.15)$$

an immediate solution is available if p is a concave function. By requiring $p(x^i) > 0$ for $i = 1, \dots, q$, it follows that $p(x) > 0$ on $\text{co}\{x^1, \dots, x^q\}$.

The question whether a general polynomial p is positive on a polytopic region \mathcal{X} of the form (2.15) can be addressed by using Pólya's theorem, which applies to homogeneous polynomials f that are positive on the unit simplex. Recall that a polynomial f in the variables $\lambda_1, \dots, \lambda_s$ is homogeneous of degree k if $f(\alpha\lambda) = \alpha^k f(\lambda)$ for all $\lambda \in \mathbb{R}^s$. The unit simplex is defined as

$$\Delta_S := \{\lambda \in \mathbb{R}^s \mid \sum_{i=1}^s \lambda_i = 1, \lambda_i \geq 0, 1 \leq i \leq s\}. \quad (2.16)$$

Theorem 2.1 (Pólya) *Let f be a homogeneous polynomial in the variables $\lambda_1, \dots, \lambda_s$. Suppose that*

$$f(\lambda) > 0 \quad \text{for all } \lambda \in \Delta_S. \quad (2.17)$$

Then, for some non-negative integer d , the (homogeneous) polynomial

$$\left(\sum \lambda_i\right)^d f(\lambda)$$

has positive coefficients.

Proof. A proof can be found in [149] and references therein. The presence of the term $(\sum \lambda_i)^d$ is not difficult to understand, if one realizes that on the simplex we have that $f(\lambda) \equiv (\sum \lambda_i)^d f(\lambda)$. A full proof of the fact that an integer d exists such that the homogeneous polynomial $(\sum \lambda_i)^d f(\lambda)$ has positive coefficients, can also be found in [145], where bounds on the required degree d are provided as well. We stress that if f is only non-negative on Δ_S , Pólya's Theorem no longer holds true. ■

Hence, Pólya's Theorem is concerned with homogeneous polynomials on the unit simplex. Nevertheless, it can be used for an arbitrary polynomial on any bounded polytopic region \mathcal{X} . In order to see why this is true, we first note that any given polynomial can be rendered homogeneous, without changing its values on the simplex. For example, the polynomial $p(x) = x_1^2 x_2 + x_2$ and the homogeneous one $\tilde{p}(x) = x_1^2 x_2 + (x_1 + x_2)^2 x_2$ are identical on the simplex.

Second, for a given polytopic region $\mathcal{X} = \text{co}\{x^1, \dots, x^q\}$, any $x \in \mathcal{X}$ can be expressed as a convex combination of the generators, which motivates us to define

$$f(\lambda) := \tilde{p}\left(\sum_{j=1}^s \lambda_j x^j\right)$$

in the new variable $\lambda \in \mathbb{R}^s$. By construction, λ is an element of Δ_S and we have thus shown that

$$p(x) > 0 \quad \text{for all } x \in \mathcal{X} \tag{2.18}$$

can be transformed into a condition of the form (2.17) for an arbitrary polynomial p and set polytopic region \mathcal{X} .

Hence, a sufficient condition for p to be positive on \mathcal{X} is obtained by first constructing the homogeneous polynomial $f(\lambda)$ and then verify whether all coefficients of $(\sum \lambda_i)^d f(\lambda)$ are positive for some fixed integer d . In Section 2.3.2, we will work out the details for the matrix-valued condition (2.11) in the μ -analysis problem.

Remark 2.4 *Theorem 2.1 provides an explicit proof to Hilbert's 17th problem for the family of homogeneous polynomials that are strictly positive on \mathcal{X} , see [83].*

Positivity on semi-algebraic sets

A parameter domain \mathcal{X} is often described implicitly by a number of polynomial inequalities. For given polynomials g_1, \dots, g_n in the variables x_1, \dots, x_s , consider sets of the form

$$\mathcal{X} = \{x \in \mathbb{R}^s \mid g_1(x) \leq 0, g_2(x) \leq 0, \dots, g_n(x) \leq 0\}. \tag{2.19}$$

Such sets are called *semi-algebraic*. As will be shown below, positivity of matrix valued polynomials on semi-algebraic sets can be verified by making use of sum-of-squares arguments. The following property turns out to be important. A given semi-algebraic set \mathcal{X} , described through $g_1(x), \dots, g_n(x)$, is said to satisfy the constraint qualification if there exists a positive $r \in \mathbb{R}$ and sum-of-squares polynomials $s_1(x), \dots, s_n(x)$ such that

$$r - \|x\| + s_1(x)g_1(x) + \dots + s_n(x)g_n(x) \text{ is a sum-of-squares.} \tag{2.20}$$

Assumption 2.1 (Constraint Qualification) *The set \mathcal{X} of the form (2.19) satisfies the constraint qualification, i.e. property (2.20) holds.*

Theorem 2.2 *Let \mathcal{X} in (2.19) be given and let Assumption 2.1 hold. If a polynomial p in the variables x_1, \dots, x_s is positive on \mathcal{X} , there exist sum-of-squares-polynomials $s_0(x), \dots, s_n(x)$ for which*

$$p(x) + s_1(x)g_1(x) + \dots + s_n(x)g_n(x) = s_0(x). \quad (2.21)$$

Proof. It follows immediately that (2.21) implies $p(x) \geq 0$. For a complete proof, the reader is referred to [146]. ■

In order to construct relaxation schemes for (2.3) based on this fact, let us generalize the arguments on scalar polynomials to polynomial matrices. A polynomial matrix $P(x)$ of size $p \times p$ with the indeterminate variables $x = (x_1, \dots, x_s)$ is called *matrix sum-of-squares* if there exists some (typically tall) polynomial matrix $S(x)$ such that

$$P(x) = S(x)^T S(x).$$

Once $P(x)$ admits such a matrix sum-of-squares decomposition, it is globally positive semi-definite since all eigenvalues of $P(x)$ are non-negative. Let us denote by $\Pi_d^{p \times q}$ the space of $p \times q$ matrices with polynomial entries having a total degree of at most d .

Decomposing a symmetric polynomial matrix $P(x)$ into a sum-of-squares is done by first representing $P(x)$ as

$$P(x) = W(x)^T \tilde{P} W(x), \quad (2.22)$$

with a symmetric matrix \tilde{P} and some monomial matrix $W(x)$. Denoting the total degree of $P(x)$ by $2d$, the elements in $W(x)$ can always be chosen to have total degree at most d . Hence, if the columns of $W(x)$ span the space of polynomial matrices of size $p \times p$ and total degree d in the sense that

$$\{LW(x) \mid L \in \mathbb{R}^{p \times n_W}\} =: \Pi_d^{p \times p}, \quad (2.23)$$

we are guaranteed to find the factorization (2.22). Moreover, it suffices to choose $n_W = \binom{s+d}{d}$, though for sparse polynomials an a priori reduction of the required monomials in $W(x)$ is possible by applying Newton-polytope techniques, see [171].

With the factorization (2.22) we can introduce the subspace \mathcal{K} of all symmetric matrices K for which

$$W(x)^T K W(x) \text{ is the zero polynomial matrix,} \quad (2.24)$$

and observe that a basis of \mathcal{K} can easily be computed. It then follows that $P(x)$ is matrix sum-of-squares if and only if there exists some $K \in \mathcal{K}$ for which $\tilde{P} + K \succeq 0$, which is a standard LMI feasibility problem.

The following result is an extension of Theorem 2.2 that allows to verify restricted positivity of the matrix-valued $P(x)$.

Theorem 2.3 *Let \mathcal{X} be defined as in (2.19) and let the constraint qualification (2.20) hold. If $P(x)$ is positive definite on \mathcal{X} there exists sum-of-squares-matrices $S_0(x), \dots, S_n(x) \in \Pi_d^{p \times p}$ for which*

$$P(x) + S_1(x)g_1(x) + \dots + S_n(x)g_n(x) = S_0(x) \quad (2.25)$$

holds.

Proof. Again, the decomposition (2.25) implies $P(x) \succeq 0$. For the full proof, the reader is referred to [160]. ■

Note that Theorem 2.3 concerns $P(x) \succ 0$ and not $P(x) \succeq 0$. This is the reason why, in the sequel, we rather consider sum-of-squares decompositions of the shifted matrix polynomial $P(x) - \epsilon I$, for some small $\epsilon > 0$, just to make sure that Theorem 2.3 applies.

Since multiple constraints can always be combined into a single one, we continue our discussion by considering the robust SDP of the form

$$\begin{aligned} & \text{inimize } c^T y \\ & \text{subject to } P(x, y) \succ 0 \quad \text{for all } x \text{ with } G(x) \preceq 0, \end{aligned} \quad (2.26)$$

of which the optimal value is denoted by γ_{opt} . Again, the decision variable $y \in \mathbb{R}^{n_d}$ can be further specified to lie within any region described by LMIs, since such constraints can always be assumed to be included in the condition $P(x, y) \succ 0$. Both $P(x, y)$ and $G(x)$ are polynomial in x , Hermitian-valued of dimension $p \times p$ and $q \times q$ respectively, while $P(x, y)$ is affine in $y \in \mathbb{R}^{n_d}$. In view of the constraint qualification needed in Theorem 2.2, the following assumption does not come as a surprise. A proof for the fact that this is the correct matrix-valued extension of (2.20) can be found in [160].

Assumption 2.2 (Constraint Qualification) *There exists $r > 0$ and sum-of-squares polynomials $S_1(x), S_0(x)$ such that*

$$r - \|x\|^2 + \text{trace}(S_1(x)G(x)) = S_0(x). \quad (2.27)$$

2.2.2 An existing relaxation approach using matrix sum-of-squares

This section provides a family of numerically tractable relaxation schemes for the robust SDP (2.26), that has been recently proposed in [160]. First, the robust SDP (2.26) is reformulated into an unconstrained problem in which a particular polynomial matrix must be a sum-of-squares. Since it involves auxiliary functional

variables in an infinite dimensional space, the problem remains non-tractable at first. By a suitable parametrization of the sum-of-squares functions, a standard LMI optimization problem is obtained.

Let us introduce the bilinear mapping

$$(\cdot, \cdot)_p : \mathbb{R}^{p \times q} \times \mathbb{R}^{p \times q} \rightarrow \mathbb{R}^{p \times p}, \quad (A, B)_p = \text{Tr}_p((I_p \otimes A)B)$$

with

$$\text{Tr}_p(C) := \begin{pmatrix} \text{Tr}(C_{11}) & \cdots & \text{Tr}(C_{1p}) \\ \vdots & \ddots & \vdots \\ \text{Tr}(C_{p1}) & \cdots & \text{Tr}(C_{pp}) \end{pmatrix} \quad (2.28)$$

for $C \in \mathbb{R}^{pq \times pq}$, $C_{jk} \in \mathbb{R}^{q \times q}$ for $j, k = 1, \dots, p$. Consider the following optimization problem:

$$\begin{aligned} & \text{infimize } c^T y \\ & \text{such that } P(x, y) - \epsilon I + (G(x), S(x))_p \\ & \text{and } S(x) \text{ is sum-of-squares in } x \text{ and } \epsilon > 0. \end{aligned} \quad (2.29)$$

Since any feasible y of (2.29) is automatically feasible for (2.26), the optimal value of (2.29) is always an upper bound on γ_{opt} , the optimal value of the original robust SDP (2.26). Moreover, without any restriction on $S(x)$, both problems are identical, as shown in [160]. An alternative proof will be given in Section 2.2.3.

Let us now fix the parametrization of $S(x)$ as follows:

$$S(x) = \begin{pmatrix} C_1 & \cdots & C_N \end{pmatrix} \left(I \otimes \begin{pmatrix} \mu_1(x) \\ \vdots \\ \mu_N(x) \end{pmatrix} \right), \quad (2.30)$$

in which the symbol \otimes denotes the Kronecker product, $C_i \in \mathcal{S}^{2q}$ are real symmetric matrices. Then, condition (2.29) amounts to an LMI optimization problem, a so-called *sum-of-squares relaxation*, in the decision variables y and $C_i, i = 1, \dots, N$. For some sufficiently rich monomial basis matrix $W(x)$, one then imposes the linear equations that are implied by the identity

$$P(x, y) - \epsilon I + (G(x), S(x))_p = W(x)' S_0 W(x), \quad (2.31)$$

in combination with the LMI constraints $S_0 \succeq 0, C_i \succeq 0$ for $i = 1, \dots, N$. Hence, a sufficient condition for verifying whether the expression $P(x, y) - \epsilon I + (G(x), S(x))_p$ is matrix sum-of-squares is obtained in terms of a set of linear equation constraints and LMI constraints in the decision variables y, S_0 and C_1, \dots, C_N .

The next section presents an alternative implementation of a sum-of-squares relaxation. It avoids equation constraints, which is advantageous when using available LMI solvers that were not developed to efficiently handle combined equal-

ity/inequality constraint. Moreover, a more explicit formulation could open the way for taking particular information on the problem structure into account. A third reason for developing an alternative implementation of sum-of-squares relaxations comes from the fact that existing LMI solvers handle strict LMIs only, and that positive definite solutions S_0, C_1, \dots, C_N satisfying (2.31) do not exist. With the linear equation constraints pulled out of the optimization problem, one might gain a better understanding about how to enforce strict feasibility of the LMIs that are involved in building relaxation schemes.

2.2.3 An alternative relaxation approach

Recently, an alternative approach to approximate the robust SDP (2.26) based on matrix sum-of-squares has been proposed in our paper [61]. With polynomial matrices $T_1(x), \dots, T_M(x)$ of dimension $q \times p$, consider the following optimization problem

$$\begin{aligned} & \text{inimize } c^T y \\ & \text{such that } P(x, y) - \epsilon I + \sum_{j=1}^M T_j(x)' G(x) T_j(x) \\ & \hspace{15em} \text{is sum-of-squares in } x \text{ and } \epsilon > 0. \end{aligned} \quad (2.32)$$

Again, the optimal value of (2.32) is always an upper bound for γ_{opt} . In the sequel the argument x is occasionally left out in order to avoid cumbersome notation.

In this section, we show how to transform condition (2.32) into a standard LMI problem, by introducing a suitable parametrization of $T_1(x), \dots, T_M(x)$.

Translation into LMIs

The approximation of the original robust SDP by the sum-of-squares problem (2.32) provides upper bounds on the genuine optimal value γ_{opt} , though without fixing the number and degree of basis matrices T_j in (2.32), the problem remains non-tractable. By a suitable parametrization of T_1, \dots, T_M , a relaxation scheme will be derived as an LMI problem, at the cost of introducing conservatism.

First, express both $P(x, y)$ and $G(x)$ in the form

$$P(x, y) = U(x)^T \tilde{P}(y) U(x) \quad \text{and} \quad G(x) = V(x)^T \tilde{G} V(x) \quad (2.33)$$

in which $\tilde{P}(y)$ is affine in y and $U(x)$ has total degree d , if the total degree of $P(x, y)$, for fixed $y \in \mathbb{R}^{n_d}$ is $2d$. Then choose basis matrices $T_1(x), \dots, T_M(x)$ of total degree l and parameterize them with a monomial basis $B_1(x), \dots, B_N(x)$ of $\Pi_l^{q \times p}$ as

$$T_j(x) = \sum_{\nu=1}^N \alpha_\nu^j B_\nu(x), \quad j = 1, \dots, M. \quad (2.34)$$

Substituting the description (2.33) into the constraint in (2.32) we get:

$$U(x)^T \tilde{P}(y) U(x) - \epsilon I + \sum_{j=1}^M T_j(x)^T V(x)^T \tilde{G} V(x) T_j(x) \text{ is sum-of-squares in } x. \quad (2.35)$$

With X defined as

$$X = \sum_{j=1}^M \begin{pmatrix} \alpha_1^j \\ \vdots \\ \alpha_N^j \end{pmatrix} \begin{pmatrix} \alpha_1^j \\ \vdots \\ \alpha_N^j \end{pmatrix}^T \succeq 0, \quad (2.36)$$

it follows that

$$\begin{aligned} \sum_{j=1}^M T_j(x)^T V(x)^T \tilde{G} V(x) T_j(x) &= \\ &= \sum_{j=1}^M \left(\sum_{\nu=1}^N \alpha_{\nu j} B_\nu(x)^T \right) V(x)^T \tilde{G} V(x) \left(\sum_{\mu=1}^N \alpha_{\mu j} B_\mu(x) \right) \\ &= \sum_{\nu=1}^N [B_\nu(x)^T V(x)^T \sum_{j=1}^M (\alpha_{\nu j}^2) \tilde{G} V(x) B_\nu(x)] \\ &= \begin{pmatrix} V(x)B_1(x) \\ \vdots \\ V(x)B_N(x) \end{pmatrix}^T [X \otimes \tilde{G}] \begin{pmatrix} V(x)B_1(x) \\ \vdots \\ V(x)B_N(x) \end{pmatrix}. \end{aligned}$$

Similar as was argued in (2.23), one can find a tall monomial matrix $W(x)$, of large enough total degree, for which there exist matrices $L_0 = L_0^T$, L_U and L_V that satisfy

$$I = W^T(x) L_0 W(x), \quad U(x) = L_U W(x), \quad \text{and} \quad \begin{pmatrix} V(x)B_1(x) \\ \vdots \\ V(x)B_N(x) \end{pmatrix} = L_V W(x). \quad (2.37)$$

Hence, the matrix expression appearing in (2.35) equals

$$W(x)^T \left(L_U^T \tilde{P}(y) L_U - \epsilon L_0 + L_V^T [X \otimes \tilde{G}] L_V \right) W(x). \quad (2.38)$$

Finally define the subspace \mathcal{K} of all K for which $W(x)^T K W(x)$ is the zero polynomial matrix. This allows us to formulate the *sum-of-squares relaxation* of the robust SDP

(2.26) with optimal value γ_{rel} :

$$\begin{aligned} & \text{infimize } c^T y \\ & \text{subject to } \epsilon > 0, \quad X \succeq 0, \quad K \in \mathcal{K}, \\ & \quad L_U^T \tilde{P}(y) L_U - \epsilon L_0 + L_V^T [X \otimes \tilde{G}] L_V + K \succeq 0. \end{aligned} \quad (2.39)$$

Feasibility of (2.39) implies that condition (2.35) holds true for some suitably chosen T_1, \dots, T_M . From a Cholesky factorization of the solution matrix X in (2.36), the coefficients α_ν^k can be extracted, which defines the $T_j(x)$ via (2.34).

Remark 2.5 *The construction of the described LMI relaxation can be performed for arbitrary polynomial matrices $B_1(x), \dots, B_N(x)$ and any monomial matrix $W(x)$, provided that the representation (2.37) holds. It is however unknown how to systematically pick $B_1(x), \dots, B_N(x)$ and $W(x)$ in order to arrive at good-quality LMI relaxations of small size.*

We now prove that the relaxation gap can be rendered arbitrary small by increasing M as well as the total degree of T_1, \dots, T_M . In other words, an asymptotically exact family of approximation schemes can be deduced from (2.29) or (2.32).

Asymptotic exactness of the relaxation family

This section contains a proof of the fact that the relaxation gap $\gamma_{\text{rel}} - \gamma_{\text{opt}}$ goes to zero when increasing both M and the degree of T_j in (2.32) in a systematic fashion. In other words, without restricting M or the degree of $T_j(x)$, the optimal value of (2.32) and the original robust SDP (2.26) are equal. As a consequence, we can approximate (2.26) by choosing M large enough and including all possible basis matrices $B_\nu(x)$ in the construction of the LMI scheme the previous section.

Theorem 2.4 *Let $\gamma_{\text{opt}}, \gamma_{\text{rel}}$ be the optimal values of the robust SDP (2.26) and the matrix sum-of-squares reformulation (2.32) respectively. Then $\gamma_{\text{rel}} \geq \gamma_{\text{opt}}$. If the constraint qualification (2.27) is satisfied, there exists for any $\epsilon > 0$ some M and polynomial matrices $T_1(x), \dots, T_M(x)$ for which $\gamma_{\text{rel}} \leq \gamma_{\text{opt}} + \epsilon$.*

Proof. The first statement is elementary to prove. Indeed suppose that \hat{y} is feasible for (2.32). This implies that

$$P(x, \hat{y}) - \epsilon I \succeq - \sum_{j=1}^M T_j(x)^T G(x) T_j(x)$$

for all x , since sum-of-squares matrices are globally non-negative semi-definite. If we now choose an arbitrary x for which $G(x) \preceq 0$, we infer

$$- \sum_{j=1}^M T_j(x)^T G(x) T_j(x) \succeq 0$$

and hence $P(x, \hat{y}) \succeq \epsilon I$. Since $\epsilon > 0$, this reveals that \hat{y} is feasible for (2.26). We have shown that the set of feasible points y in (2.32) is contained in that of (2.26), which indeed implies that $\gamma_{\text{rel}} \geq \gamma_{\text{opt}}$.

The proof for the second statement is somewhat more involved and strongly resembles the proof in [160]. Given $\epsilon > 0$, choose some \hat{y} which is feasible for (2.26) and which satisfies $c^T \hat{y} < \gamma_{\text{opt}} + \epsilon$. This implies $P(x, \hat{y}) \succ 0$ for all x with $G(x) \preceq 0$. As shown in [160], there exist unit vectors v_1, \dots, v_{N_0} such that

$$v_i^T G(x) v_i \leq 0, \quad i = 1, \dots, N_0 \quad \Rightarrow \quad P(x, \hat{y}) \succ 0.$$

Hence, by Theorem 2.3 we infer that there exist sum-of-squares matrices $S_1(x), \dots, S_{N_0}(x)$ and $\epsilon > 0$ for which

$$P(x, \hat{y}) - \epsilon I + \sum_{i=1}^{N_0} S_i(x) v_i^T G(x) v_i \quad \text{is a matrix sum-of-squares.}$$

As sum-of-squares matrices $S_i(x)$ can be written as $S_i(x) = \sum_{k=1}^{r_i} t_k^i(x) (t_k^i(x))^T$ with polynomial column vectors $t_k^i(x)$, we obtain

$$\begin{aligned} P(x, \hat{y}) - \epsilon I + \sum_{i=1}^{N_0} S_i(x) v_i^T G(x) v_i &= \\ &= P(x, \hat{y}) - \epsilon I + \sum_{i=1}^{N_0} \sum_{k=1}^{r_i} t_k^i(x) v_i^T G(x) v_i (t_k^i(x))^T. \end{aligned}$$

With the $M = r_1 + \dots + r_{N_0}$ rank-one polynomial matrices $v_i (t_k^i(x))^T$, $i = 1, \dots, N_0$, $k = 1, \dots, r_i$ (whose degrees are determined from those of $S_1(x), \dots, S_{N_0}(x)$), we have proven that \hat{y} is feasible for (2.32). Therefore the optimal value of (2.32) is not larger than $c^T \hat{y}$ which is in turn smaller than $\gamma_{\text{opt}} + \epsilon$. Hence, assuming that the constraint qualification (2.27) holds, we have shown that the sum-of-squares problem (2.32) approximates the polynomial robust SDP (2.26) by any desired accuracy and can thus be seen as a reformulation in terms of matrix sum-of-squares. ■

Alternative methods for handling matrix valued polynomials

Instead of the derived LMI problem (2.39), other implementations have been considered as well. Scalar polynomials and sum-of-squares relaxations have been considered in [136, 117, 37]. As shown in for example [92], matrix valued polynomials can always be suitably transformed into scalar ones, that is: $P(x, y) \succeq 0$ for all x satisfying $G(x) \preceq 0$ is implied by $p(x, y, z) = z^T P(x, y) z \geq 0$ for all (x, z) that satisfy $g(x, z) = z^T G(x) z \leq 0$. By Theorem 2.2, a sufficient condition for (2.26) is the existence of a sum-of-squares polynomial $s(x, z)$ for which $p(x, y, z) + s(x, z)g(x, z)$ is a sum-of-squares.

Another alternative implementation can be found in [160], where a family of relaxation schemes was constructed from (2.29) rather than from (2.32). As shown in Appendix C, this implementation is equivalent, in the sense that without restrictions on the total degree of the polynomial matrices $S(x)$ in (2.29) or polynomial matrices $T_1(x), \dots, T_M$ in (2.32), the optimal value of both problems are the same. Note that, as mentioned earlier, the conditions from [160] involve linear equation constraints.

Adding redundant constraints

The sum-of-squares reformulations (2.29) and (2.32) both lead to an asymptotically exact family of relaxation schemes for the robust SDP (2.26). However, there is an additional freedom that hasn't been mentioned or exploited yet, which may possibly improve the numerical behavior and reduce conservatism in practical problem instances.

In order to derive relaxation schemes, that are different from the ones we have discussed so far, let the set \mathcal{X} be given as

$$\mathcal{X} = \{x \mid g_1(x) \leq 0, g_2(x) \leq 0, \dots, g_n(x) \leq 0\}. \quad (2.40)$$

Then, any inequality $\tilde{g}(x) = -g_j(x)g_i(x) \leq 0$ can be added to the description of \mathcal{X} without changing it. We stress though, due to our constraint qualification, that such redundant constraints are not needed for proving that the family of relaxations is asymptotically exact. The following example illustrates how it leads us to an alternative family of asymptotically exact relaxations.

Example 2.2 *Let $\mathcal{X} = \{x \mid g_1(x) \leq 0, g_2(x) \leq 0\}$. Then, by Theorem 2.2 it holds that p is positive on \mathcal{X} implies that there exist some sum-of-squares polynomials s_0, s_1, s_2 for which*

$$p = s_0 - g_1s_1 - g_2s_2. \quad (2.41)$$

By introducing redundant constraints it also holds that p is positive on \mathcal{X} implies that there exist sum-of-squares polynomials $s_j, j = 0, 1, 2, \dots$ for which

$$p = s_0 + g_1s_1 + g_2s_2 - g_1g_2s_3 - g_1^2s_4 - g_2^2s_5 + g_1^3s_6 + g_1^2g_2s_7 + \dots \quad (2.42)$$

For a given level of available computational power, one can therefore either decide to increase the total degree of s_0, s_1, s_2 in (2.41), or one can add redundant constraints while keeping the total degree of s_j in (2.41) small.

In a similar fashion, one construct a modified family of relaxation schemes from (2.29). Assuming that the redundant constraints are combined with $G(x)$ by placing both constraints on the diagonal, adding redundant constraints would increase the dimension of $T_1(x), \dots, T_M(x)$.

We conclude this section with the following interesting fact, which concerns the

case in which the constraint qualification (2.27) is not satisfied, e.g. if considering non-compact parameter regions. Then, a family of asymptotically exact relaxations schemes can still be constructed, provided that the additional (redundant) constraints are included. In some sense, the constraint qualification is enforced by adding sufficiently many redundant constraints. The result holds for the sum-of-squares problem with a scalar polynomial on a semi-algebraic set of the form (2.40).

Consider the set \mathcal{X} in (2.40), and the set of polynomials $\mathcal{F} = \{-g_1, \dots, -g_n\}$. Define the set \mathcal{M} , called the multiplicative monoid generated by \mathcal{F} , as

$$\mathcal{M} := \{f_1 f_2 \cdots f_m \mid f_i \in \mathcal{F} \text{ for } i = 1, \dots, m, m \geq 1\}.$$

Theorem 2.5 (Schmüdgen) *Let \mathcal{X} be compact, let r be some positive integer, and defined in (2.40). Let the cone generated by \mathcal{F} be defined as*

$$\mathcal{P}(\mathcal{F}) := \{s_0 + \sum_{i=1}^r s_i b_i \mid s_i \text{ is sum-of-squares, } b_i \in \mathcal{M}\}.$$

Then, p is positive on \mathcal{X} implies $p \in \mathcal{P}(\mathcal{F})$.

Proof. The result is based on the Positivstellensatz, see [164, 136]. ■

Thus, the set of all redundant constraints consists of a cone. A general version of Theorem 2.5 that also includes equality constraints can be found in the nicely written paper [136].

2.2.4 From rational to polynomial dependence

In case that F_1, \dots, F_{n_c} are rational functions in x , we can derive an equivalent version of the SDP constraints in which x enters polynomially as follows. Consider a single constraint of the form

$$F(x)' J(y) F(x) \prec 0 \quad \forall x \in \mathcal{X}, \quad (2.43)$$

and let $F(x)$ have n_{col} columns. Then, let $d_j(x)$ be the polynomial of lowest degree for which the j^{th} column of $d_j(x)F(x)$ is polynomial. By construction, the matrix function $\tilde{F}(x) = F(x)T(x)$ has polynomial entries only if defining

$$T(x) = \begin{pmatrix} d_1(x) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & d_{n_{\text{col}}}(x) \end{pmatrix}.$$

Applying to (2.43) the congruence transformation with $T(x)$ renders the expression (2.43) polynomial in x , provided that $d_j(x) \neq 0$ for all $x \in \mathcal{X}$, $j = 1, \dots, n_{\text{col}}$. Hence, assuming without loss of generality that there exists some $\hat{x} \in \mathcal{X}$ for which $d_j(\hat{x}) > 0$

as the *full-block S-procedure* argument, see [155, 97]. We stress the fact that there exist many variations of the S-procedure, some of which are elementary to prove. In particular, it is often seen in relation to positivity of quadratic functions on quadratically constrained sets for which the argument is simple, though this version is not immediately linked to the Lemma that is presented next.

Lemma 2.2 *Let J be a Hermitian matrix and \mathcal{X} be compact. Then*

$$\begin{pmatrix} \Delta(x)(I - A\Delta(x))^{-1}B \\ I \end{pmatrix}' J \begin{pmatrix} \Delta(x)(I - A\Delta(x))^{-1}B \\ I \end{pmatrix} \prec 0 \quad (2.45)$$

holds for all $x \in \mathcal{X}$ if and only if there exists a Hermitian multiplier Π that satisfies

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}' \Pi \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + J \prec 0 \quad (2.46)$$

and which is related to \mathcal{X} by

$$\begin{pmatrix} \Delta(x) \\ I \end{pmatrix}' \Pi \begin{pmatrix} \Delta(x) \\ I \end{pmatrix} \succeq 0 \quad \forall x \in \mathcal{X}. \quad (2.47)$$

Proof. Define

$$F(x) = \begin{pmatrix} \Delta(x)(I - A\Delta(x))^{-1}B \\ I \end{pmatrix}$$

and observe that

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix} F(x) = \begin{pmatrix} \Delta(x) \\ I \end{pmatrix} (I - A\Delta(x))^{-1}B.$$

If we now left-and right multiply condition (2.46) with $F(x)'$, $F(x)$ respectively, one arrives at condition (2.45), since (2.47) holds for any $x \in \mathcal{X}$. The converse of the proof can be found in [155, 97]. ■

Remark 2.6 *The 'if'-part in Lemma 2.2 holds without assuming compactness of the set \mathcal{X} .*

In order to construct relaxation schemes for problem (2.3) based on the S-procedure, let us be given the matrix functions $J_i(y)$ and

$$F_i(x) = \begin{pmatrix} \Delta_i(x)(I - A_i\Delta_i(x))^{-1}B_i \\ I \end{pmatrix} \quad (2.48)$$

for some matrices A_i, B_i and $\Delta_i(x) \in \mathbb{R}^{p_i \times q_i}$, which is linear in x , for $i = 1, \dots, n_c$. If $F_i(x)$ are given as the LFR $\tilde{F}_i(x) = D_i + C_i\Delta_i(x)(I - A\Delta_i(x))^{-1}B_i$ for some

matrices A_i, B_i, C_i, D_i , we can easily arrive at the representation (2.45) in Lemma 2.2 by substituting

$$J_i(y) \rightarrow \begin{pmatrix} C_i & D_i \end{pmatrix}' J_i(y) \begin{pmatrix} C_i & D_i \end{pmatrix} \quad i = 1, \dots, n_c.$$

In order to render the semi-infinite LMI constraint (2.47) numerically tractable, specific sets $\mathbf{\Pi}$ of block structured multipliers Π are chosen such that the semi-infinite constraint (2.47) holds for each $\Pi \in \mathbf{\Pi}$. We have already seen in the example of Section 2.1 that $\mathbf{\Pi}$ could be chosen as the set of structured matrices in (2.12).

In order to be able to describe more general multiplier sets $\mathbf{\Pi}$ than those obtained by only imposing a certain block structure, we make use of the following characterization

$$\mathbf{\Pi}_i := \{ \Pi_i \in \mathcal{V} \mid G_i(\Pi_i) \preceq 0 \}, \quad (2.49)$$

in which $\mathcal{V} \subset \mathcal{S}^{q_i+p_i}$ is a subspace of (e.g. block structured) symmetric matrices of dimension $q_i + p_i$, and the affine matrix functions $G_i(\cdot)$ for $i = 1, \dots, n_c$ are suitably chosen such that

$$\begin{pmatrix} \Delta_i(x) \\ I \end{pmatrix}' \Pi_i \begin{pmatrix} \Delta_i(x) \\ I \end{pmatrix} \succeq 0 \quad \forall x \in \mathcal{X}, \forall \Pi_i \in \mathbf{\Pi}_i. \quad (2.50)$$

This allows us to formulate the *S-procedure-* or *multiplier relaxation* for problem (2.3):

$$\gamma_{\text{rel}} = \inf \left\{ \langle c, y \rangle : y \in \mathbb{R}^{n_d}, \quad \Pi_i \in \mathbf{\Pi}_i, \right. \\ \left. \begin{pmatrix} I & 0 \\ A_i & B_i \end{pmatrix}' \Pi_i \begin{pmatrix} I & 0 \\ A_i & B_i \end{pmatrix} + J_i(y) \prec 0, \quad i = 1, \dots, n_c \right\}. \quad (2.51)$$

The optimal value of (2.51) satisfies $\gamma_{\text{rel}} \geq \gamma_{\text{opt}}$ since for any tuple $(y, \Pi_1, \dots, \Pi_{n_c})$ that is feasible, the vector y is guaranteed to be a feasible solution of all the n_c robust SDP constraints in (2.3), just by applying Lemma 2.2.

Finally, the applicability of Lemma 2.2 depends on whether the matrix functions F_1, \dots, F_{n_c} are given as linear fractional representations. This includes the assumption that, for $i = 1, \dots, n_c$, the LFR is well-posed, i.e.

$$\det(I - A_i \Delta_i(x)) \neq 0 \quad \text{for all } x \in \mathcal{X}. \quad (2.52)$$

This condition can be viewed as the analogue to the scalar constraints $d_j(x) > 0$ obtained in (2.44). Contrary to the direct method of Section 2.2, the existence of a feasible tuple $(y, \Pi_1, \dots, \Pi_{n_c})$ of (2.51) for which the left-upper blocks of $J_1(y), \dots, J_{n_c}(y)$ are all positive semi-definite, ensures well-posedness of the LFR.

Example 2.3 (Well-posedness) From Lemma 2.1 it follows that well-posedness (2.52) holds if there exists $\Pi_i \in \mathbf{\Pi}_i$ that satisfies (2.50) as well as

$$\begin{pmatrix} I \\ A_i \end{pmatrix}' \Pi_i \begin{pmatrix} I \\ A_i \end{pmatrix} \prec 0. \quad (2.53)$$

Note that feasibility of the LMI constraint in (2.51) implies

$$\begin{pmatrix} I \\ A_i \end{pmatrix}' \Pi_i \begin{pmatrix} I \\ A_i \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix}' J_i(y) \begin{pmatrix} I \\ 0 \end{pmatrix} \prec 0.$$

If the left-upper blocks of $J_i(y)$ is positive semi-definite, condition (2.53) obviously holds. The argument can be repeated for $i = 1, \dots, n_c$ in order to infer well-posedness of the LFR of $F_1(x), \dots, F_{n_c}(x)$.

The observation that was made in this example is used in many analysis and controller synthesis problems, e.g. in Section 4.2.3 when computing \mathcal{L}_2 -gain upper bounds of LPV systems. In general, however, the right lower blocks of $J_1(y), \dots, J_{n_c}(y)$ are not all positive semi-definite and condition (2.52) needs to be imposed as a separate constraint.

In this section, we have seen that the S-procedure disentangles the rational dependence in robust LMI constraint (2.3) and forms a basis for so-called multiplier relaxations. Due to the quadratic dependence in (2.50), the S-procedure is particularly useful in combination with polytopic parameter domains \mathcal{X} , as is discussed next.

2.3.1 Relaxation schemes based on convexity arguments

Referring to the robust SDP constraint in (2.3) with the LFR of $F_i(x)$ given as (2.48), let \mathcal{X} be a convex polytope defined by a finite number of points, i.e. $\mathcal{X} = \text{co}\{x^1, \dots, x^q\}$. Introduce the notation

$$E_i(x, \Pi) := \begin{pmatrix} \Delta_i(x) \\ I \end{pmatrix}' \Pi \begin{pmatrix} \Delta_i(x) \\ I \end{pmatrix}, \quad (2.54)$$

and

$$C(\Pi) := \begin{pmatrix} I \\ 0 \end{pmatrix}' \Pi \begin{pmatrix} I \\ 0 \end{pmatrix}. \quad (2.55)$$

Note that requiring $C(\Pi) \preceq 0$ enforces $E_i(x, \Pi)$ to be a concave function in x . Thus, verifying whether $E_i(x, \Pi)$ is positive on \mathcal{X} is then reduced to verifying whether it is positive on the generators of \mathcal{X} only. The so-called *convex hull relaxation* for robust SDP (2.3) by solving (2.51) with the sets $\mathbf{\Pi}_i$ defined as

$$\mathbf{\Pi}_i := \{\Pi_i \in \mathcal{V} \mid C(\Pi_i) \preceq 0, E_i(x^\nu, \Pi_i) \succeq 0 \text{ for } \nu = 1, \dots, q\}. \quad (2.56)$$

Remark 2.7 *If compared with matrix sum-of-squares relaxations, there are no clear ways to build an asymptotically exact family of relaxation schemes based on convex hull arguments. As shown in [5], one can reduce conservatism by exploiting multi-convexity arguments. Another possible approach is to sequentially apply the S-procedure, increasing the number of multiplier variables.*

2.3.2 Multiplier relaxations using Pólya's Theorem

An alternative and often less conservative relaxation for polytopes $\mathcal{X} = \text{co}\{x^1, \dots, x^q\}$ is based on Pólya's theorem. Recall the definition of $E_i(x, \Pi)$ in (2.54). Since any $x \in \mathcal{X}$ can be expressed as a convex combination of the generators x^1, \dots, x^q , the i^{th} robust SDP constraint in (2.50) is equivalent to

$$E_i\left(\sum_{\nu=1}^q \lambda_\nu x^\nu, \Pi_i\right) \succeq 0 \quad \text{for all } \lambda \in \Delta_S, \quad \forall \Pi_i \in \mathbf{\Pi}_i, \quad (2.57)$$

in which Δ_S denotes the unit simplex as defined in (2.16). Note that for any λ on the unit simplex this matrix polynomial in λ is homogeneous of degree 2. This is easily seen from the identity

$$E_i\left(\sum_{\nu=1}^q \lambda_\nu x^\nu, \Pi_i\right) = \left(\begin{array}{c} \Delta_i(\sum_{\nu=1}^q \lambda_\nu x^\nu) \\ \sum \lambda_\nu I \end{array} \right)' \Pi_i \left(\begin{array}{c} \Delta_i(\sum_{\nu=1}^q \lambda_\nu x^\nu) \\ \sum \lambda_\nu I \end{array} \right),$$

and the fact that $\Delta_i(\cdot)$ is a linear mapping. For fixed degree d , one can thus extract the Hermitian-valued mappings $\Lambda_{(\beta_1, i, \dots, \beta_{q, i})}^i(\Pi_i)$ for which

$$(\lambda_1 + \lambda_2 + \dots + \lambda_q)^d E_i\left(\sum_{\nu=1}^q \lambda_\nu x^\nu, \Pi_i\right) = \sum_{\beta_1, i + \dots + \beta_{q, i} = 2} \Lambda_{(\beta_1, i, \dots, \beta_{q, i})}^i(\Pi_i) \lambda_1^{\beta_1, i} \dots \lambda_q^{\beta_{q, i}} \quad (2.58)$$

holds for $i = 1, \dots, n_c$. When the sets $\mathbf{\Pi}_i$ are defined as

$$\mathbf{\Pi}_i = \{\Pi_i \in \mathcal{S}^{p_i + q_i} \mid \Lambda_{(\beta_1, i, \dots, \beta_{q, i})}^i(\Pi_i) \succ 0 \quad \text{for } i = 1, \dots, n_c\} \quad (2.59)$$

with $\Lambda_{(\beta_1, i, \dots, \beta_{q, i})}^i(\Pi_i)$ defined in (2.58), the relaxation (2.51) is called *Pólya's (multiplier) relaxation of degree d* . Denoting the optimal value γ_{rel}^d , one can show that

$$\lim_{d \rightarrow \infty} \gamma_{\text{rel}}^d = \gamma_{\text{opt}}$$

The convergence follows from a matrix-extension of Pólya's theorem [51]: If (2.57) holds, there always exists some (possibly large) degree d , for which all of the coefficients $\Lambda_{(\beta_1, i, \dots, \beta_{q, i})}^i(\Pi_i)$ are positive definite for $i = 1, \dots, n_c$. As a consequence, the relaxation gap $\gamma_{\text{rel}} - \gamma_{\text{opt}}$ can be reduced to zero by increasing d . A complete proof for the fact that the family of Pólya relaxations is asymptotically exact can be found in [157, 145]. The latter reference also provides bounds on the required value of d .

2.3.3 Sum-of-squares relaxations

Let us finally show how to use sum-of-squares relaxations for characterizing the sets of multipliers Π_i for relaxation (2.51). Recall that sets \mathcal{X} are assumed to be described as

$$\mathcal{X} = \{x \in \mathbb{R}^s \mid V(x)^T \tilde{G} V(x) \preceq 0\}.$$

Note that the semi-infinite constraint (2.50) is already specified in the standard form $P_i(x, y) = U_i(x)^T \tilde{P}_i(y) U_i(x)$ that was used in Section 2.2.3, with $\tilde{P}(y)$ being the multiplier Π_i . Following the derivation of (2.39), one first fixes a parametrization of the matrices $T_1(x), \dots, T_M(x)$ in (C.1) for each of the constraints $i = 1, \dots, n_c$. Then, for a polynomial matrix $W(x)$ chosen in order to satisfy 2.38, the following set of admissible multipliers can be constructed:

$$\begin{aligned} \Pi_i = \{ \Pi_i \in \mathcal{S}^{p_i+q_i} \mid & X \succeq 0, K \in \mathcal{K}_i, \\ & L_{U,i}^T \Pi_i L_{U,i} - \epsilon L_{0,i} + L_{V,i}^T [X \otimes \tilde{G}] L_{V,i} + K \succeq 0 \}, \end{aligned}$$

in which $L_{U,i}, L_{0,i}, L_{V,i}$ and the sets of matrices \mathcal{K}_i for $i = 1, \dots, n_c$ follow immediately from the choice of $T_1(x), \dots, T_M(x)$ in (C.1) and $W(x)$ in (2.38).

Note that the multiplier $\Pi_i \in \mathcal{S}^{p_i+q_i}$ was chosen to be a full symmetric matrix in both the Pólya and sum-of-squares relaxations, as opposed to the structured matrices used in (2.56). These methods do not need any structure for obtaining tractable LMI relaxation schemes.

2.4 Estimating the relaxation gap

In the previous sections, two approaches for constructing relaxations for the robust SDP (2.3) have been developed. We already indicated the fact that the relaxations based on matrix sum-of-squares and Pólya's Theorem can be constructed for any desired accuracy by systematically increasing Pólya's degree d in (2.58), or alternatively, the number and order of the monomial basis functions that parameterize $T_j(x)$ in (C.1).

Despite the knowledge that we are, in principle, able to solve the robust LMI problem by computing a sequence of relaxations of growing complexity, it is yet unknown how to predict or evaluate the accuracy of a particular relaxation scheme. Due to the expected difficulty of deriving a priori bounds on the approximation quality, see for a discussion [17] and references therein, our focus is on estimating the relaxation gap $\gamma_{\text{rel}} - \gamma_{\text{opt}}$ after a LMI relaxation scheme has been solved.

In the next section, we will compute lower bounds on the optimal value by gridding the parameter set. A more elegant approach that tries to verify the relaxation's exactness, i.e. $\gamma_{\text{rel}} = \gamma_{\text{opt}}$, is presented in Section 2.4.2. It makes use of certain dual variables in the optimization algorithm, and extracts worst-case parameter values.

2.4.1 Computing lower bounds by gridding the parameter set

A classical way to handle robust LMIs is to sample the family of LMI constraints on a finite grid of parameter values, at the risk of possibly missing crucial ones. Gridding techniques always provide lower bound values $\gamma_{\text{lb}} \leq \gamma_{\text{opt}}$, as part of the constraints are dropped which possibly increases the set of feasible $y \in \mathbb{R}^{n_d}$. The main challenge in a gridding approach is to generate suitable grids of modest size that lead to good lower bound values. In addition, the question arises for which set of parameter values the optimal value of the genuine robust SDP problem coincides with its sampled version. Any set that has this property will be called a *representative set of parameter values* for the set of robust SDP constraints. It turns out that a representative set always exists and that the number of required grid points is bounded by the dimension of the decision space \mathbb{R}^{n_d} , see [158, 150].

Approximation schemes for computing lower bounds of (2.3) are constructed as follows. First, select for the i^{th} constraint q_i the gridpoints $\{x^{i,1}, \dots, x^{i,q_i}\}$ and define the set of indices

$$\Omega = \{(i, \nu) \mid i = 1, \dots, n_c, \nu = 1, \dots, q_i\}. \quad (2.60)$$

Then, the (finite) LMI problem

$$\gamma_{\text{lb}} = \inf \left\{ \langle c, y \rangle : y \in \mathbb{R}^{n_d}, F_i(x^{i,\nu})' J_i(y) F_i(x^{i,\nu}) \prec 0, \forall (i, \nu) \in \Omega \right\} \quad (2.61)$$

obtained by substituting these parameter values into the semi-infinite constraints yields a lower bound value $\gamma_{\text{lb}} \leq \gamma_{\text{opt}}$. By increasing the number of gridpoints q_i , the resulting lower bound γ_{lb} does not get worse. A representative set of parameter values in this context would be any set of $x^{i,\nu}$'s for which $\gamma_{\text{lb}} = \gamma_{\text{opt}}$. We assume that the grid is suitably chosen in order to make sure that the optimal value is finite, i.e. $-\infty < \gamma_{\text{lb}} \leq \gamma_{\text{opt}}$.

Remark 2.8 *Robust LMI constraints can also be handled in a probabilistic sense, in which the construction of the parameter grid is based on randomization. Bounds on the probability of constraint violation can be derived as shown in the recent book [173]. In this thesis randomized algorithms are not discussed and (2.3) is solved in a deterministic (worst-case) sense. Lower bound computations are needed in combination with upper bound computations in order to gain accuracy information on the value γ_{opt} , as well as for the construction of worst-case parameter values.*

2.4.2 Verifying exactness for multiplier relaxations

So far, the developed tools have enabled us to compute approximate solutions to the robust SDP (2.3). Unfortunately, a priori error bounds can be given for specific sets \mathcal{X} and affine $F_i(x)$ only, see [80, 18, 70]. For general problem instances it is not known how to estimate the relaxation gap. This section shows an algorithm for verifying whether a computed multiplier relaxation scheme is exact, i.e. whether

$\gamma_{\text{rel}} = \gamma_{\text{opt}}$. The ideas have first been proposed in [157, 158] and involve Lagrange dual optimal multipliers that are automatically obtained when computing γ_{rel} using primal-dual interior point solvers and relaxations of the form (2.51).

The key step in deriving the exactness test is to connect the Lagrange dual problems of both the lower- and upper bound approximations from Sections 2.4.1 and 2.3 respectively. Referring to the original robust SDP (2.3), let us assume without loss of generality the decomposition

$$J_i(y) = J_i^0 + \hat{J}_i(y)$$

where $\hat{J}_i(y)$ is further expressed as

$$\hat{J}_i(y) = J_i^1 y_1 + \dots + J_i^n y_n, \quad i = 1, \dots, n_c,$$

and J_i^0, \dots, J_i^n are symmetric matrices.

Since the LMI constraints in the sampled version of the problem are strictly feasible, as also assumed for the robust problem (2.3), we can dualize it without gap. With Lagrange multipliers $Z^{i,\nu} \succeq 0$ corresponding to the constraint for the gridpoint $x^{i,\nu} \in \mathcal{X}$ with $(i, \nu) \in \Omega$, define the following map:

$$L_{i,\nu}(Z^{i,\nu}) = F_i(x^{i,\nu}) Z^{i,\nu} F_i(x^{i,\nu})'. \quad (2.62)$$

The Lagrangian of (2.61) thus becomes

$$\begin{aligned} \langle c, y \rangle + \sum_{\nu=1}^{q_i} \sum_{i=1}^{n_c} \langle Z^{i,\nu}, F_i(x^{i,\nu})' J_i(y) F_i(x^{i,\nu}) \rangle &= \\ &= \langle c, y \rangle + \sum_{\nu=1}^{q_i} \sum_{i=1}^{n_c} \langle Z^{i,\nu}, F_i(x^{i,\nu})' \hat{J}_i(y) F_i(x^{i,\nu}) \rangle + \sum_{\nu=1}^{q_i} \sum_{i=1}^{n_c} \langle L_{i,\nu}(Z^{i,\nu}), J_i^0 \rangle. \\ &= \langle c + \sum_{\nu=1}^{q_i} \sum_{i=1}^{n_c} \hat{J}_i(L_{i,\nu}(Z^{i,\nu})), y \rangle + \sum_{\nu=1}^{q_i} \sum_{i=1}^{n_c} \langle L_{i,\nu}(Z^{i,\nu}), J_i^0 \rangle \end{aligned} \quad (2.63)$$

By strong duality and since γ_{lb} was assumed to be finite, the dual problem of (2.61) becomes

$$\gamma_{\text{lb}} := \begin{cases} \text{maximize} & \sum_{\nu=1}^{q_i} \sum_{i=1}^{n_c} \langle L_{i,\nu}(Z^{i,\nu}), J_i^0 \rangle \\ \text{subject to} & c_j + \sum_{\nu=1}^{q_i} \sum_{i=1}^{n_c} \langle J_i^j, L_{i,\nu}(Z^{i,\nu}) \rangle = 0, \quad \text{for } j = 1, \dots, n \\ & Z^{i,\nu} \succeq 0, \quad \text{for } (i, \nu) \in \Omega \end{cases} \quad (2.64)$$

In order to be able to derive the Lagrange dual problem for the multiplier relaxation, we make the parametrization of the multipliers Π_i explicit by introducing auxiliary

variables $\xi_i \in \mathbb{R}^{N_i}$, i.e.

$$\mathbf{\Pi}_i = \{\Pi_i(\xi_i) \mid G_i(\xi_i) \preceq 0\}, \quad i = 1, \dots, n_c.$$

Then, with the structured matrices U_i defined as

$$U_i = \begin{pmatrix} I & 0 \\ A_i & B_i \end{pmatrix}, \quad i = 1, \dots, n_c,$$

multiplier relaxation (2.51) is reformulated as:

$$\gamma_{\text{rel}} = \inf \left\{ \langle c, y \rangle : \quad y \in \mathbb{R}^{n_d}, \quad G_i(\xi_i) \preceq 0, \right. \\ \left. U_i' \Pi_i(\xi_i) U_i + J_i(y) \prec 0, \quad i = 1, \dots, n_c \right\}. \quad (2.65)$$

Similar as has been done for the approximation (2.61), Lagrange multiplier variables $\Phi_1, \dots, \Phi_{n_c}, \Psi_1, \dots, \Psi_{n_c}$ are introduced in order to form the Lagrangian of the multiplier relaxation (2.65):

$$\begin{aligned} \langle c, y \rangle + \sum_{i=1}^{n_c} \langle \Phi_i, U_i' \Pi_i(\xi_i) U_i + J_i(y) \rangle + \langle \Psi_i, G_i(\xi_i) \rangle \\ = \langle c + \sum_{i=1}^{n_c} \hat{J}_i^*(\Phi_i), y \rangle + \sum_{i=1}^{n_c} \langle J_i^0, \Phi_i \rangle + \sum_{i=1}^{n_c} \langle \xi_i, \Pi_i^*(U_i \Phi_i U_i') + G_i^*(\Psi_i) \rangle. \end{aligned} \quad (2.66)$$

By the assumption that there exists a feasible tuple $(\xi_1, \dots, \xi_{n_c}, y)$ for relaxation (2.65) for which $G_i(\xi) \prec 0$, by strong duality the dual of (2.65) becomes

$$\gamma_{\text{rel}} := \begin{cases} \text{maximize} & \sum_{i=1}^{n_c} \langle J_i^0, \Phi_i \rangle \\ \text{subject to} & \Phi_i, \Psi_i \succeq 0, \\ & c_j + \sum_{i=1}^{n_c} \langle J_i^j, \Phi_i \rangle = 0, \quad j = 1, \dots, n, \\ & G_i^*(\Psi_i) + \Pi_i^*(U_i \Phi_i U_i') = 0, \quad i = 1, \dots, n_c. \end{cases} \quad (2.67)$$

The following exactness result for multiplier relaxations of the form (2.51) is obtained by combining (2.64) with (2.67).

Theorem 2.6 *Let $\Phi_i, \Psi_i, i = 1, \dots, n_c$ be some dual optimal multipliers of (2.67). Suppose there exist $\bar{x}^{i,1}, \dots, \bar{x}^{i,q_i} \in \mathcal{X}$ for $i = 1, \dots, n_c$ such that the matrices $\Phi_1, \dots, \Phi_{n_c}$ can be written as*

$$\Phi_i = \sum_{\nu=1}^{q_i} F(\bar{x}^{i,\nu}) \bar{Z}^{i,\nu} F(\bar{x}^{i,\nu})' \quad (2.68)$$

for some $\bar{Z}^{i,\nu} \succeq 0$. Then the relaxation (2.51) is exact, i.e. $\gamma_{\text{rel}} = \gamma_{\text{opt}}$.

Proof. Given dual optimal (Φ_i, Ψ_i) , $i = 1, \dots, n_c$, note that by using the maps $L_{i,\nu}(\cdot)$ as defined in (2.62), the matrices in the summation (2.68) are actually equal to $L_{i,\nu}(\bar{Z}^{i,\nu})$. Hence, for $i = 1, \dots, n_c$, by using $\bar{\delta}^{i,\nu}, \dots, \bar{\delta}^{i,q_i}$ to sample the semi-infinite constraints, the matrices $\bar{Z}^{i,1} \succeq 0, \dots, \bar{Z}^{i,q_i} \succeq 0$ are feasible solutions to the lower bound computation (2.64), and the value of this problem equals $\sum_{i=1}^{n_c} \langle J_i^0, \sum_{j=1}^{q_i} L_{i,\nu}(\bar{Z}^{i,\nu}) \rangle = \sum_{i=1}^{n_c} \langle J_i^0, \Phi_i \rangle = \gamma_{\text{rel}}$. This implies $\gamma_{\text{lb}} \geq \gamma_{\text{rel}}$ and hence the equality $\gamma_{\text{rel}} = \gamma_{\text{lb}}$ holds. Moreover, any set $\{\bar{\delta}^{i,\nu} : (i,\nu) \in \Omega\}$ satisfying (2.68) for some $\bar{Z}^{i,\nu} \succeq 0$ is a representative set of parameter values, i.e. for the corresponding sampled problem (2.61) we have $\gamma_{\text{lb}} = \gamma_{\text{rel}} = \gamma_{\text{opt}}$. ■

Theorem 2.6 is applied as follows. Once dual optimal multipliers $\Phi_1, \dots, \Phi_{n_c}$ have been computed in solving the relaxation (2.51) and its dual (2.67), one must find variables $\bar{Z}^{i,\nu}$, and $\bar{x}^{i,\nu} \in \delta$ that satisfy (2.68). Parametrization of the matrices $\bar{Z}^{i,\nu}$ should be done in accordance with the rank of Φ_i . With rank revealing decomposition

$$\bar{Z}^{i,\nu} = \sum_{j=1}^{r_{i,\nu}} z_{j,i,\nu} z_{j,i,\nu}^T,$$

this means that the vectors $z_{j,i,\nu}$ are the unknowns and $q_i, r_{i,\nu}$ are a priori fixed parameters such that the relation

$$\text{rank}(\Phi_i) = \sum_{\nu=1}^{q_i} r_{i,\nu} \quad i = 1, \dots, n_c$$

holds. For any such parametrization for $\bar{Z}^{i,\nu}$, condition (2.68) is a matrix rational in the unknowns $(\delta, \bar{Z}^{i,\nu})$. By vectorizing (2.68) and multiplying each polynomial with the common denominator polynomial of $(I - A_i \Delta_i(x))^{-1} B_i$ a polynomial system is obtained, though even for moderate sized SDP constraints, this straightforward application of condition (2.68) results in a high number of polynomials. Therefore, it is a suggestion for future research to develop more efficient techniques that can directly handle (2.68) in the case of general robust SDP problems.

There are already two situations in which the complexity of the exactness test can be significantly reduced. For instance, we can extend the observation made in [157] for the case of having a single robust SDP constraint, and infer that (2.68) is solvable if there exists solutions $x^1, \dots, x^{n_c} \in \mathcal{X}$ for which

$$L_i(x^i) = \begin{pmatrix} I & -\Delta_i(x^i) \end{pmatrix} \begin{pmatrix} I & 0 \\ A_i & B_i \end{pmatrix} \Phi_i = 0, \quad i = 1, \dots, n_c$$

holds. Moreover, for a specific class of multiplier relaxation schemes, it can be shown that the relaxation is automatically exact in case that $\text{rank}(\Phi_i)=1$ for $i = 1, \dots, n_c$, see again [157]. This sufficient condition for exactness of the relaxation can be implemented by verifying whether $\|L_i(x^i)\| \approx 0$ for $i = 1, \dots, n_c$, which amounts to

verifying whether there exists $x^1, \dots, x^{n_c} \in \mathcal{X}$ such that

$$\begin{pmatrix} \alpha I & L_i(x^i)' \\ L_i(x^i) & \alpha I \end{pmatrix} \succeq 0 \quad \text{for } i = 1, \dots, n_c, \quad (2.69)$$

for some $\alpha \approx 0$. The latter condition amounts to a standard LMI problem if \mathcal{X} is given as an LMI region.

Another useful insight that provides an efficient implementation of (2.68) concerns the case in which the B_i -matrices in (2.48) have only one column. Then, the (scalar) variables $\bar{Z}^{i,\nu}$ can be eliminated as follows: With $(\tilde{C}_i \ \tilde{D}_i)$ chosen such that their rows form a basis of the left-kernel of Φ_i , solvability of (2.68) implies

$$\begin{aligned} & (\tilde{C}_i \ \tilde{D}_i) F_i(x^i) \bar{Z}^{i,\nu} F_i(x^i)' (\tilde{C}_i \ \tilde{D}_i)' = 0 \\ \iff & (\tilde{C}_i \ \tilde{D}_i) F_i(x^i) \sum_{j=1}^{r_{i\nu}} z_{j,i,\nu} z'_{j,i,\nu} F_i(x^i)' (\tilde{C}_i \ \tilde{D}_i)' = 0 \\ \iff & \left\| (\tilde{C}_i \ \tilde{D}_i) F_i(x^i) \begin{pmatrix} z_{1,i,\nu} & z_{2,i,\nu} & \cdots & z_{r_{i\nu},i,\nu} \end{pmatrix} \right\| = 0 \\ \iff & (\tilde{C}_i \ \tilde{D}_i) F_i(x^i) \begin{pmatrix} z_{1,i,\nu} & z_{2,i,\nu} & \cdots & z_{r_{i\nu},i,\nu} \end{pmatrix} = 0 \end{aligned}$$

in case the B_i -matrices in (2.48) have only one column, the $z_{j,i,\nu}$ for $j = 1, \dots, r_{i\nu}$ become scalar variables and thus the latter condition implies

$$(\tilde{C}_i \ \tilde{D}_i) F_i(x^i) = 0. \quad (2.70)$$

By multiplying the expression with the denominator of $F_i(x)$ a polynomial system is obtained that depends only on x . Problems in which B_i have only one column represent robust linear programming problems, a problem class for which numerous references can be found, e.g. [138, 16, 176, 79].

Remark 2.9 *Note that (2.70) is only necessary for (2.68), and there might exist solutions $x^i = \bar{x}^i$ for which no solution pair $(\bar{x}^i, \bar{Z}^{i,\nu})$ of (2.68) exists. Thus, in order to be sure about exactness of the relaxation, one has to either solve (2.68) for $Z^{i,\nu}$ or compute a lower bound $\gamma_{lb} \leq \gamma_{opt}$ using a grid defined by the computed solutions $x^i = \bar{x}^i$ from (2.70).*

Remark 2.10 *Note that the success of applying Theorem 2.6 depends upon the fact that primal-dual LMI solvers return Lagrange multipliers Φ_i of the desired structure, if they exist.*

Remark 2.11 *Conditions for exactness of relaxations can alternatively be posed in terms of so-called moment matrices, see [90] for scalar polynomials, and [89] for matrix polynomials.*

2.4.3 Numerical example: μ -upper bound computation

Let us revisit the example from Section 2.1, which addressed the computation of upper bounds for μ . We will implement the sufficient conditions (2.9)-(2.10) for verifying whether $I - AB$ is invertible for all $B \in \mathcal{B}$, by constructing several relaxation schemes for the semi-infinite constraint (2.10). Moreover, we will explore the use of structured multipliers Π . In all relaxations, similar as was done for the D -scales in Section 2.1, the best upper bound on μ is obtained by bisection on r in order to find the maximal r for which $I - AB$ is invertible for all $B \in r\mathcal{B}$. Note that the dependence of matrix $A(p)$ on parameter $p \in [0, 1]$ is omitted for notational convenience. As we have already mentioned, the elements $B \in \mathcal{B}$ are parameterized by $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, making use of the linear mapping $\Delta : \mathbb{R}^3 \rightarrow \mathbb{R}^{6 \times 6}$, which reads as

$$\Delta(x) = \begin{pmatrix} x_1 I & 0 & 0 \\ 0 & x_2 I & 0 \\ 0 & 0 & x_3 I \end{pmatrix},$$

in which I denotes the identity matrix of size 2. With the set \mathcal{X} defined as

$$\mathcal{X} = \{x \in \mathbb{R}^3 \mid |x_i| < 1 \text{ for } i = 1, \dots, 3\},$$

any $B \in \mathcal{B}$ corresponds to an $x \in \mathcal{X}$ and vice versa, through the relation $B = \Delta(x)$.

Convex hull relaxation

The first relaxation scheme is based on the convexity arguments in Section 2.3.1. With a full block multiplier Π , the LMI conditions for relaxation 'Full block' become

$$\begin{pmatrix} I \\ A \end{pmatrix}' \Pi \begin{pmatrix} I \\ A \end{pmatrix} \prec 0, \quad (2.71)$$

and

$$\begin{pmatrix} I \\ 0 \end{pmatrix}' \Pi \begin{pmatrix} I \\ 0 \end{pmatrix} \prec 0, \quad \begin{pmatrix} \Delta(x) \\ I \end{pmatrix}' \Pi \begin{pmatrix} \Delta(x) \\ I \end{pmatrix} \succeq 0, \quad (2.72)$$

the latter of which is evaluated at the 8 extreme points x^1, \dots, x^8 of \mathcal{X} . As mentioned earlier, the value of μ is obtained by considering the scaled region $r\mathcal{X}$ and performing a bi-section argument on $r \in \mathbb{R}$. With a total number of $1+12 \cdot 13/2=79$ LMI variables, significant improvements are seen if compared to the D -scalings test, see Figure 2.3.

With computational complexity being often the limiting factor, one might prefer to reduce the number of variables in Π . Although imposing structure on Π will generally introduce conservatism, it might not influence the results in particular problem instances. The example here has been constructed in such a way that the

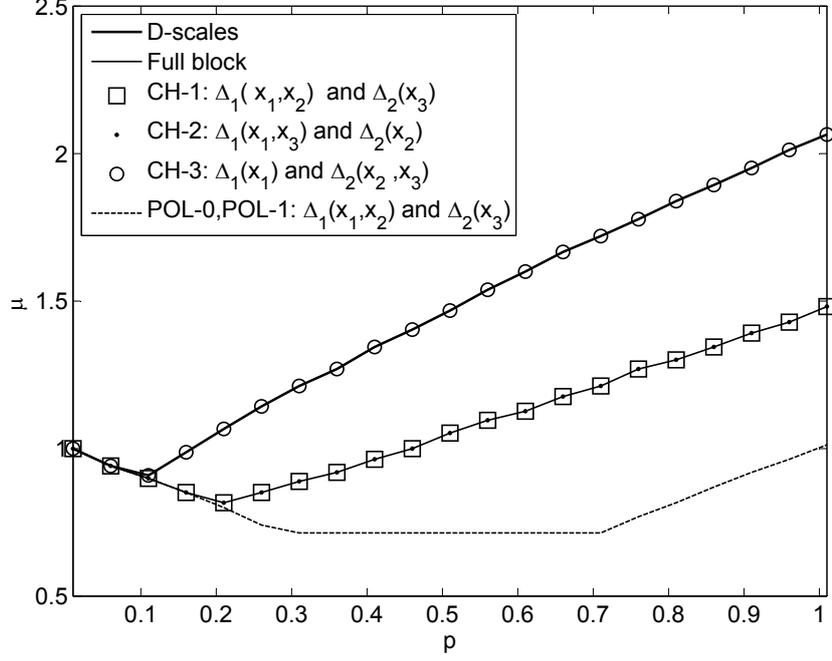


Figure 2.3: Least upper bounds of $\mu_{\Delta_c}(A)$ for different parameters p in (2.7) using convex hull and Pólya relaxation methods.

number of multiplier variables can be a priori reduced. In fact, if we define

$$\Delta_1(x) = \begin{pmatrix} x_1 I & 0 \\ 0 & x_2 I \end{pmatrix}, \quad \Delta_2(x) = x_3 I, \quad (2.73)$$

and

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0.5 & 0 & 0.5 & 0 \\ 2p & 0 & p & 0 \\ 0 & -2p & 0 & -p \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1-p & 0 \\ 0 & 1-p \end{pmatrix}, \quad (2.74)$$

it is easily seen that $I - A\Delta(x)$ is invertible if and only if both $I - A_1\Delta_1(x)$ and $I - A_2\Delta_2(x)$ are. Therefore, $I - A\Delta(x)$ is non-singular for all $x \in \mathcal{X}$ if there exist Π_1, Π_2 such that for $i = 1, 2$

$$\begin{pmatrix} I \\ A_i \end{pmatrix}' \Pi_i \begin{pmatrix} I \\ A_i \end{pmatrix} \prec 0, \quad \begin{pmatrix} \Delta_k(x) \\ I \end{pmatrix}' \Pi_i \begin{pmatrix} \Delta_k(x) \\ I \end{pmatrix} \succeq 0 \quad (2.75)$$

holds for all $x \in \mathcal{X}$. It is not difficult to see that a multiplier relaxation for the two robust LMI constraints (2.75) corresponds to a relaxation for (2.72) when using the

structured multiplier

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} & 0 & \Pi_{14} & \Pi_{15} & 0 \\ \Pi'_{12} & \Pi_{22} & 0 & \Pi_{24} & \Pi_{25} & 0 \\ 0 & 0 & \Pi_{33} & 0 & 0 & \Pi_{36} \\ \Pi'_{14} & \Pi'_{24} & 0 & \Pi_{44} & \Pi_{45} & 0 \\ \Pi'_{15} & \Pi'_{25} & 0 & \Pi'_{45} & \Pi_{55} & 0 \\ 0 & 0 & \Pi'_{36} & 0 & 0 & \Pi_{66} \end{pmatrix}. \quad (2.76)$$

The corresponding convex hull relaxation is denoted by CH-1 and involves a total of $8*9/2+5*4/2=46$ multiplier variables. As expected, the upper bound of relaxation CH-1 is not worse than “Full block”, as shown in Figure 2.3.

In a similar fashion, other multiplier structures are obtained by separating the constraint for x_2 from x_1, x_3 or the constraint for x_3 from x_1, x_2 . The convex hull relaxations corresponding to these structures are denoted by CH-2 and CH-3 respectively. While the multiplier structure in CH-3 yields worse results than the other convex hull relaxations, CH-2 does lead to the same upper bounds as CH-1, i.e. the upper bound corresponding to the full block multiplier.

Remark 2.12 *The case in which Π_{12}, Π_{15} and Π_{45} are all zero boils down to the so-called D-G scalings, see [124]. In our numerical example, the D-scalings upper bounds were not improved by D-G scalings.*

Approximation	# LMI vars	# LMI constraints	γ_{rel} for $p = 0.6$
D-scales	13	*	1.57
Full block	79	$1 + (8+1) = 10$	1.13
CH-1	47	$1 + (4+1) + (2+1) = 9$	1.13
CH-2	47	$1 + (4+1) + (2+1) = 9$	1.13
CH-3	47	$1 + (4+1) + (2+1) = 9$	1.57
POL-0	47	$1+13=14$	0.71
POL-1	47	$1+24=25$	0.71
SOS-1	1099	$1+1+3 = 5$	0.71
MIX	229	7	0.71

Table 2.1: Comparison of upper bounds γ_{rel} on $\mu_{\Delta_c}(A)$ with the parameter value $p = 0.6$ in (2.7). The D-scales upper bound has been computed using

Remark 2.13 *For large sizes of Δ , it is more relevant to reduce the number of decision variables by using structured multipliers. It is an interesting topic of future research to see whether one can start with a diagonal multiplier (D-scales) and extract from the computed relaxation the information needed for improving the relaxation by adding multiplier variables in the most efficient way.*

Pólya relaxations

Let us now apply Pólya's theorem with a structured multiplier similarly as in (2.76). That is, a relaxation is constructed for robust SDP constraints (2.75), in which A_1, A_2 are given in (2.74). The LMI constraints that characterize the set of admissible multipliers Π_i , referring to (2.58), are formed by imposing all $\Lambda_{(\beta_{1,i}, \dots, \beta_{q,i})}^i(\Pi_i) \succeq 0$ for $i = 1, 2$. In case that Pólya's degree for both semi-infinite constraints in (2.75) is chosen as $d = 0$ for $i = 1, 2$, the relaxation is denoted as POL-0, whereas POL-1 corresponds to the case in which $d = 1$ for $i = 1, 2$. As shown in Figure 2.3, the Pólya relaxations outperforms the convex hull relaxations, at the cost of an increased number of constraints. As shown in the figure, POL-1 is not better than POL-0, neither was the Pólya relaxation with a higher degree d .

The upper bounds obtained with Pólya are in fact exact. This follows from lower bound computations that were computed with `mussv.m` in Matlab. Exactness could not be proven with the exactness tests in Section 2.4.2.

Sum-of-squares relaxations

We finally construct relaxation schemes based on matrix sum-of-squares. This is done for multiplier-based relaxations only. Similar as we did for the convex hull relaxations, we start with a full-block multiplier. For fixed parameter bounds $|x_i| \leq r \in \mathbb{R}$, the uncertainty set \mathcal{X} is now described in implicit form as

$$G_i(x) = V(x_i)^T \tilde{G} V_i(x_i) = \begin{pmatrix} 1 \\ x_i \end{pmatrix}^T \begin{pmatrix} -r^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x_i \end{pmatrix} \leq 0$$

for $i = 1, \dots, 3$. Using previous results, it follows that

$$P(x, \Pi) = \begin{pmatrix} \Delta(x) \\ I \end{pmatrix}^T \Pi \begin{pmatrix} \Delta(x) \\ I \end{pmatrix} \succeq 0 \quad \text{on } \mathcal{X} \quad (2.77)$$

is implied by

$$P(x, \Pi) - \epsilon I + \sum_{i=1}^3 \sum_{j=1}^M T_{i,j}(x)^T G_i(x) T_{i,j}(x) \quad \text{is sum-of-squares in } x \text{ for some } \epsilon > 0. \quad (2.78)$$

Since $P(x, \Pi)$ is of size 6×6 and $G_i(x)$ are scalar valued, the basis matrices $T_{i,j}$ have dimension 1×6 . Taking all 0th and 1st order monomials into account, each constraint involves a total of $M = 6 \times 4 = 24$ basis functions. Following the procedure from

Section 2.2.3, a sufficient LMI condition for (2.78) is given by

$$L_U^T \Pi L_U - \epsilon L_0 + \sum_{i=1}^3 L_{V,i}^T [X_i \otimes \tilde{G}] L_{V,i} + K \succeq 0. \quad (2.79)$$

for some fixed matrices $L_0, L_U, L_{V,i}$ and with matrix variable $K \in K$. The relaxation denoted by SOS-1 thus consists of the LMI constraints $X_i \succeq 0$, for $i = 1, \dots, 3$, together with (2.71) and (2.79), which is the relaxation for semi-infinite constraint (2.72). The decision variables consist of X_i for $i = 1, \dots, 3$, having dimension 24×24 , the multiplier Π and the matrix $K \in K$. The resulting upper bounds on μ are shown in Figure 2.4.

In order to illustrate the flexibility of the framework, let us finally use the structured multiplier in (2.76) and combine two different relaxation methods for the two semi-infinite constraints in (2.75). The relaxation denoted as MIX is constructed by using a convex hull argument for the semi-infinite constraint that depends on $\Delta_2(x_3)$, while using a sum-of-squares relaxation scheme for the semi-infinite constraint that depends on $\Delta_1(x_1, x_2)$. For the latter relaxation method, the matrix functions $T_j(x)$ in (2.32) are chosen such that all 0th and 1st monomial basis matrices are included. As shown in Figure 2.4, the resulting optimal values are equal to the ones obtained by Pólya relaxations, see also Table 2.1. Exactness could not be proven using the test in Section 2.4.2, though developments for more powerful exactness tests are ongoing.

Interim summary

The relaxation tools as presented in this chapter have been applied to the μ -analysis problem introduced in Section 2.1. It was shown that using full block multipliers reduces conservatism, as compared to standard computations based on D - (or D/G -) scalings. With Pólya and matrix sum-of-squares techniques, we were able to prove robust stability for all uncertain parameters in the unit box. All relaxation schemes in this section were constructed with the recently developed Matlab toolbox [55], that is associated with this thesis work. The flexibility of the framework has been illustrated by combining two different relaxation methods (sum-of-squares and convex hull) into a single scheme.

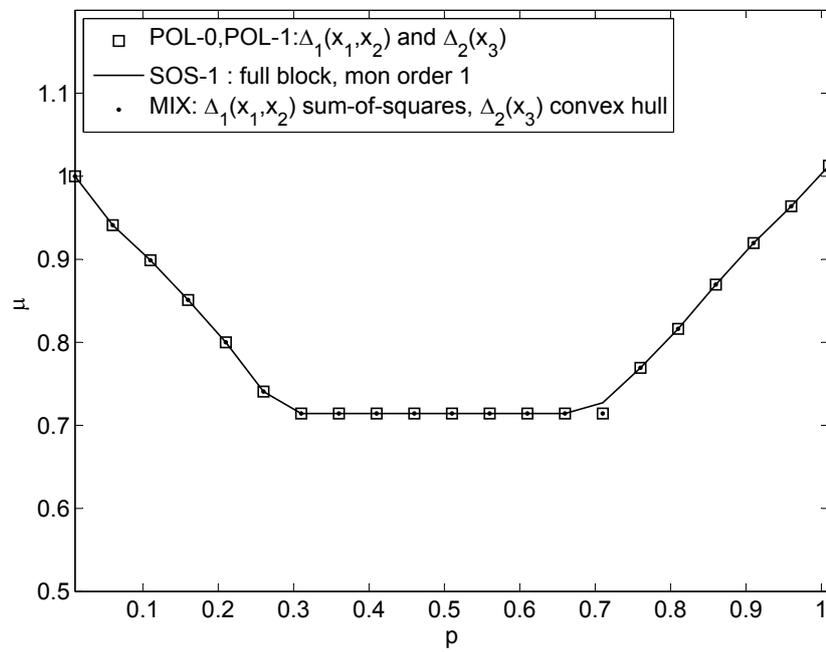


Figure 2.4: Computed least upper bounds of $\mu_{\Delta_c}(A)$ using sum-of-squares relaxation, for different values of p .

2.5 Solving polynomial systems by linear algebra

Motivated by the exactness test in the previous section let us finally consider the problem of finding all common zeros for given polynomials p_1, \dots, p_l in the indeterminate variables $x = (x_1, \dots, x_s)$. Hence, it is our goal to compute common solutions to the polynomial system

$$\begin{aligned} p_1(x_1, \dots, x_s) &= 0 \\ &\vdots \\ p_l(x_1, \dots, x_s) &= 0. \end{aligned} \tag{2.80}$$

The set of zeros $z \in \mathbb{C}^s$ for which $p_i(z) = 0$ for $i = 1, \dots, l$, is denoted by

$$Z(P) = \{z^1, \dots, z^m\}$$

in which $P = \{p_1, \dots, p_l\}$. Note that the zero set is assumed finite. We will start this section by introducing the necessary algebraic notions in order to explain the solution approach taken in [168, 40]. An extension of this method will be presented, which can also be found in our paper [60].

Instead of looking only at the set of polynomials P , one should consider a much larger set, called the ideal of polynomials generated by P , formally defined as

$$\mathcal{I} = \langle P \rangle = \left\{ \sum_{j=1}^l q_j(x) p_j(x) : q_j \in \mathcal{P}^s \right\} \tag{2.81}$$

where \mathcal{P}^s denotes the algebra of all complex polynomials in the variables $x = (x_1, \dots, x_s)$. It is obvious that $\langle P \rangle$ vanishes on $Z(P)$. Moreover, the algebraic variety defined by \mathcal{I} , denoted by $V(\mathcal{I})$, is defined as the set of joint zeros of all elements in \mathcal{I} , i.e.

$$V(\mathcal{I}) = \{z \mid p(z) = 0, \quad \forall p \in \mathcal{I}\}.$$

It is easy to show that $Z(P) = V(\langle P \rangle)$.

A fundamental step in the analysis of polynomial systems is to introduce the factor space $\mathcal{P}^s \setminus \mathcal{I}$ with elements denoted by $[p(x)]$. In the sequel we often leave out the argument x , thus writing $[p] \in \mathcal{P}^s \setminus \mathcal{I}$. These elements are equivalence classes modulo \mathcal{I} , meaning that for any $p, q \in \mathcal{P}^s$

$$[p] = [q] \quad \iff p - q \in \mathcal{I}.$$

It can be proven that $\mathcal{P}^s \setminus \mathcal{I}$ is a vector space over \mathbb{C} of dimension m (see [168], Theorem 2.4). Moreover, defining the multiplication

$$[p][q] = [pq] \quad p, q \in \mathcal{P}^s$$

equips $\mathcal{P}^s \setminus \mathcal{I}$ with the structure of a commutative ring. In order to treat vectors of

polynomials we introduce the abbreviation $\mathcal{I}^n = \{\text{col}(r_1, \dots, r_n) : r_1, \dots, r_n \in \mathcal{I}\}$.

Proposition 2.1 *Let $b = \text{col}(b_1 \cdots b_m)$ be a vector of polynomials for which $\mathcal{B} = \{[b_1], \dots, [b_m]\}$ forms a basis of $\mathcal{P}^s \setminus \mathcal{I}$. Then there exist matrices M_1, \dots, M_s , satisfying*

$$[x_i b(x)] = M_i [b(x)], \quad i = 1, 2, \dots, s, \quad (2.82)$$

often called the multiplication matrices corresponding to the basis vector b .

Proof. For any polynomial $q \in \mathcal{P}^s$, $[qb_k]$ is again an element in \mathcal{P}^s which implies there exist scalars α_{kj} for which

$$[qb_k] = \sum_{j=1}^m \alpha_{kj} [b_j] = (\alpha_{k1} \cdots \alpha_{km}) \begin{pmatrix} [b_1] \\ \vdots \\ [b_m] \end{pmatrix}, \quad k = 1, \dots, m.$$

Matrix M_i is found by choosing $q = x_i$ and using the resulting α_{kj} as the $(k, j)^{\text{th}}$ element of matrix M_i . The proof is also given in [42] Proposition 4.7. ■

It is surprisingly simple to construct $Z(P)$ once the so-called multiplication maps for each of the monomials x_1, \dots, x_s , as represented by M_1, \dots, M_s , are known. As pointed out in [168], the construction of a basis \mathcal{B} , also called *normal set*, is what causes trouble, in particular for polynomial systems in higher dimensions and of higher degree. Most of the available algorithms are based on first determining a Gröbner basis of the ideal of polynomials. There exists a vast amount of literature on how to efficiently compute Gröbner bases. As we will see, our procedure is applicable even if a priori knowledge on \mathcal{B} (or a Gröbner basis) is absent.

Before presenting the classical result of Stetter, let us recall the following fact.

Lemma 2.3 *Let M be a lower block triangular matrix, i.e.*

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

such that the eigenvalues of A and C are disjoint, and M commutes with matrix N . Then, N is block triangular with the same structure.

Proof. See [41, Proposition 4]. ■

Lemma 2.4 *Let $b = \text{col}(b_1 \cdots b_m)$ be a vector of polynomials for which the set $\mathcal{B} = \{[b_1], \dots, [b_m]\}$ forms a basis of $\mathcal{P}^s \setminus \mathcal{I}$, where \mathcal{I} is defined in (2.81). With the zero set $Z(P) = V([\mathcal{I}]) = \{z^1, \dots, z^m\}$ we have*

$$b(z^j) \neq 0 \quad \text{for } j = 1, \dots, m. \quad (2.83)$$

Proof. see [42, Theorem 2.10]. ■

We now discuss an algorithm to compute $V(\mathcal{I})$. By transforming matrices M_1, \dots, M_s into upper block triangular form, the zeros can be extracted from the eigenvalues of the individual blocks. In contrast to the approach in [168], joint eigenvectors of the multiplication matrices M_1, \dots, M_s are only required in the proof and not in the algorithm. Given a matrix A , we say that the similarity T transforms A into *block root-subspace form* if

$$T^{-1}AT = \text{diag}(A_1, \dots, A_{n_r}) \quad \text{where} \quad \sigma(A_j) = \{\lambda_j\} \quad \text{for } j = 1, \dots, k$$

and

$$\lambda_u \neq \lambda_v, \quad \text{for all } u, v = 1, \dots, k, \quad u \neq v.$$

Algorithm 2.1 *Construct s similarity transformations (applied to all matrices M_i simultaneously) in the following iterative fashion.*

Step 1. *Suppose that $\lambda_1^1, \dots, \lambda_1^{k_1}$ is the list of all pairwise different eigenvalues of M_1 . We can then transform M_1 into block root-subspace form*

$$M_1 = \text{diag}(M_1^1, \dots, M_1^{k_1})$$

with $\sigma(M_1^j) = \{\lambda_1^j\}$ for $j = 1, \dots, k_1$. Since M_2, \dots, M_s commute with M_1 , the transformation applied to all M_i results in

$$M_i = \text{diag}(M_i^1, \dots, M_i^{k_1}), \quad i = 1, \dots, s.$$

where we used Lemma 2.3.

Step 2. *For any $j \in \{1, \dots, k_1\}$ consider the block M_2^j . As in Step 1, we can also transform this matrix into block root-subspace form, i.e.*

$$M_2^j = \text{diag}(M_2^{j,1}, \dots, M_2^{j,l_j}),$$

where each block on the diagonal has only one eigenvalue, i.e. $\sigma(M_2^{j,l_j}) = \{\lambda_2^{j,l_j}\}$, and for different blocks these eigenvalues are different. By Lemma 2.3, all other blocks necessarily admit the same block-diagonal structure

$$M_i^j = \text{diag}(M_i^{j,1}, \dots, M_i^{j,l_j}) \quad \text{for } i = 1, \dots, s.$$

This refinement into l_j sub-blocks is performed for all $j = 1, \dots, k_1$. The total number of blocks in M_i is, up to this point, given by $k_2 := l_1 + \dots + l_{k_1}$:

$$M_i = \text{diag}(M_i^1, \dots, M_i^{k_2}), \quad i = 1, \dots, s$$

and we record the singletons

$$\sigma(M_i^j) = \{\lambda_i^j\} \text{ for } j = 1, \dots, k_2, \quad i = 1, 2.$$

After s steps. Putting all similarity transformation together we end up with

$$M_i = \text{diag}(M_i^1, \dots, M_i^{k_s})$$

and

$$\sigma(M_i^j) = \{\lambda_i^j\}, \quad j = 1, \dots, k_s, \quad i = 1, \dots, s.$$

where k_s is the total number of blocks resulting after s steps.

Once the singletons λ_i^j have been computed, $V(\mathcal{I})$ is given by the next theorem.

Theorem 2.7 *With the notation as in Algorithm 2.1,*

$$\{(\lambda_1^j, \dots, \lambda_s^j) : j = 1, \dots, k_s\} = V(\mathcal{I}).$$

Proof. We assume that b has been transformed such that (2.82) is valid for the block-diagonal matrices M_i resulting from Algorithm 2.1.

Proof of \subset . Fix $j \in \{1, \dots, k_s\}$. Since M_i^j , $i = 1, \dots, s$, are a commuting family of matrices, there exists a common left eigenvector $v_j \neq 0$. Since $M_i = \text{diag}(M_i^1, \dots, M_i^{k_s})$, we can extend v_j with zero components to obtain some $v \neq 0$ satisfying

$$v^* M_i = v^* \lambda_i^j \quad \text{for } i = 1, \dots, s. \quad (2.84)$$

For any multi-index $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}_0^s$, let us define

$$(M_1, \dots, M_s)^\alpha = M_1^{\alpha_1} M_2^{\alpha_2} \dots M_s^{\alpha_s}.$$

This operation extends in a natural fashion to an arbitrary polynomial $q(x) = \sum_\alpha c_\alpha x^\alpha$ in \mathcal{P}^s as

$$q(M_1, \dots, M_s) = \sum_\alpha c_\alpha (M_1, \dots, M_s)^\alpha.$$

Since M_1, \dots, M_s are pairwise commuting, it is straightforward to check that (2.82) implies

$$[q][b] = [qb] = q(M_1, \dots, M_s)[b].$$

Therefore, by using (2.84), we infer

$$v^*[q][b] = v^*q(M_1, \dots, M_s)[b] = v^*q(\lambda_1^j, \dots, \lambda_s^j)[b] = [v^*b]q(\lambda_1^j, \dots, \lambda_s^j). \quad (2.85)$$

Now note that $[v^*b] \neq 0$ since the components of $[b]$ are linearly independent and

$v \neq 0$. If we choose $q \in \mathcal{I}$, we have $[q] = 0$, and we can hence conclude

$$q(\lambda_1^j, \dots, \lambda_s^j) = 0.$$

Since $q \in \mathcal{I}$ was arbitrary, we have proved $(\lambda_1^j, \dots, \lambda_s^j) \in V(\mathcal{I})$. ■

Proof of \supset . Take a zero $z = (z_1, \dots, z_s) \in V(\mathcal{I})$. Since $[x_i b - M_i b] = 0$ we infer

$$x_i b(x) - M_i b(x) \in \mathcal{I}^m \quad \text{for } i = 1, \dots, s.$$

Evaluation at z implies

$$z_i b(z) - M_i b(z) = 0 \quad \text{for } i = 1, \dots, s.$$

Now partition $b(z) = \text{col}(b^1(z), \dots, b^{k_s}(z))^T$ according to the block-diagonal structure of M_i . Since $b(z) \neq 0$ by Lemma 2.4, there exists some j with $b^j(z) \neq 0$, and we infer

$$z_i b^j(z) = M_i^j b^j(z) \quad \text{for } i = 1, \dots, s.$$

It follows that $z_i = \lambda_i^j$ for all $i = 1, \dots, s$. ■

Remark 2.14 *The order in which the algorithm addresses M_1, \dots, M_s can be chosen differently, when permuting the components $z_1^j = \lambda_1^j, \dots, z_s^j = \lambda_s^j$ of the constructed zeros $z^1, \dots, z^{k_s} \in V(\mathcal{I})$ accordingly.*

Although Algorithm 2.1 transforms all M_i into block diagonal form (as required in the proof of Theorem 2.7), it actually suffices to construct unitary transformations rendering the M_i block triangular. The reason for this comes from the fact that the coordinate changes turning a block triangular M_i into block diagonal form leaves the eigenvalues of each sub-block M_i^j invariant. Hence, Algorithm 2.1 does not only avoid the computation of eigenvectors, but it is not even required to compute all joint invariant subspaces of M_1, \dots, M_s .

2.5.1 A new algorithm

In the previous section we have seen how to extract all zeros $z \in V(\mathcal{I})$ from the multiplication matrices M_1, \dots, M_s in (2.82). Motivated by the alternative proof that was given for Theorem 2.7, this section contains an extension of Algorithm 2.1 and determines $V(\mathcal{I})$ without knowing a basis of $\mathcal{P}^s \setminus \mathcal{I}$ a priori. The result has also been written down in [60, 58].

Let us now no longer assume to know some basis \mathcal{B} for $\mathcal{P}^s \setminus \mathcal{I}$. We will rather assume that $\mathcal{B} = \{[b_1] \cdots [b_n]\}$ is a spanning set, which implies that for any $[p] \in$

$\mathcal{P}^s \setminus \mathcal{I}$, written as $[p] = \sum \alpha_j [b_j]$, there exists a matrix N_q such that

$$[qp] = (\alpha_1 \cdots \alpha_n) N_q \begin{pmatrix} [b_1] \\ \vdots \\ [b_n] \end{pmatrix}.$$

For a given set of polynomials $P = \{p_1, \dots, p_l\}$, we define vector $p = (p_1 \cdots p_l)^T$ and fix some monomial vector $b = (b_1 \cdots b_n)^T$ with $b_1 = 1$ as well as monomial matrices $V_i, i = 1, \dots, s$, each consisting of n rows. First, it is required to verify whether the system of linear equations

$$x_i b(x) - N_i b(x) = V_i(x) C_i p(x), \quad i = 1, \dots, s \quad (2.86)$$

in the matrix variables N_i and C_i is solvable. In case (2.86) is not solvable, monomial terms should be added to b as well as to V_1, \dots, V_s . It is not difficult to see that the mere solvability of this equation does indeed imply that the components of $[b]$ span $\mathcal{P}^s \setminus \mathcal{I}$, which is the content of the next proposition. The following lemma helps us in proving that b is a spanning set once (2.86) is solvable.

Lemma 2.5 *Let $b = (b_1 \ b_2 \ \cdots \ b_n)^T$ be a vector of monomials with $b_1 = 1$ and suppose (2.86) is solvable. Then, for any given monomial $\mu \notin \{b_1, \dots, b_n\}$*

$$[\mu] \in \{[b_1], \dots, [b_n]\}.$$

Proof. For any given monomial μ , let us first prove that there exists a matrix N_μ such that

$$\mu b - N_\mu b \in \mathcal{I}^n. \quad (2.87)$$

Indeed, solvability of (2.86) implies

$$x_i b(x) = N_i b(x) + \hat{r}(x) \quad i = 1, \dots, s$$

where $\hat{r}(x) \in \mathcal{I}^n$. Thus for arbitrary $i, j \in \{1, 2, \dots, s\}$ we have

$$\begin{aligned} x_j x_i b(x) &= x_j (N_i b(x) + \hat{r}(x)) \\ &= N_j (N_i b(x) + \hat{r}(x)) + \tilde{r}(x) \\ &= N_j N_i b(x) + r(x) \end{aligned}$$

where the components of $r(x) = N_j \hat{r}(x) + \tilde{r}(x) \in \mathcal{I}^n$ (using the properties of ideal \mathcal{I}). Hence, for any monomial $\mu(x) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_s^{\alpha_s}$ with multi-degree $\alpha = (\alpha_1 \cdots \alpha_s)$, the matrix $N_\mu = N_1^{\alpha_1} N_2^{\alpha_2} \cdots N_s^{\alpha_s}$ satisfies (2.87).

Let us turn again to the original statement in the Lemma. Looking at the first row v_1^* of N_μ in (2.87) it follows that there exists $r \in \mathcal{I}$ such that

$$\mu(x) \cdot 1 = v_1^* b(x) + r(x)$$

and taking the remainders modulo \mathcal{I} implies

$$[\mu(x)] = v_1^*[b]$$

Thus, we have proven that $[\mu(x)]$ is a linear combination of $\{[b_1(x)], \dots, [b_n(x)]\}$. ■

Remark 2.15 Note that multiplication with monomial $\mu(x) = x_1^{\alpha_1} x_2^{\alpha_2} \dots, x_s^{\alpha_s}$ can result in different multiplication matrices N_μ if these matrices do not commute, i.e. the matrices N_μ are not uniquely defined by the multi-degree $\alpha \in \mathbb{N}_0^s$ of $\mu(x)$. For example, $N_\mu = N_2 N_1 N_2$ or $N_\mu = N_1 N_2 N_2$ are both valid matrices for the multiplication map that belongs to monomial $\mu(x) = x_1 x_2^2$.

Using this lemma, we now show that the components of $[b]$ span $\mathcal{P}^s \setminus \mathcal{I}$ once (2.86) is solvable. This result plays an essential role in proving that $V(\mathcal{I})$ can be extracted from the matrices N_1, \dots, N_s .

Proposition 2.2 Let $b = \text{col}(b_1 \ b_2 \ \dots \ b_n)$ be a vector of monomials with $b_1 = 1$ and suppose that (2.86) has a solution. Then $\text{span}\{[b_1], \dots, [b_n]\} = \mathcal{P}^s \setminus \mathcal{I}$.

Proof. Recall the fundamental fact that $\mathcal{P}^s \setminus \mathcal{I}$ is m -dimensional as a vector space over \mathbb{C} and admits a monomial basis. If

$$\mathcal{V}_0 = \text{Span}\{[b_1], \dots, [b_n]\} \subsetneq \mathcal{P}^s \setminus \mathcal{I} \quad (2.88)$$

then there certainly exist monomials $\mu_1, \mu_2, \dots, \mu_d$ such that

$$\text{Span}\{[b_1], \dots, [b_n], [\mu_1], \dots, [\mu_d]\} = \mathcal{P}^s \setminus \mathcal{I}. \quad (2.89)$$

Now define

$$\mathcal{V}_j := \text{Span}\{[b_1], \dots, [b_n], [\mu_1], \dots, [\mu_j]\}$$

so that we get the chain of vector spaces

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \dots \subset \mathcal{V}_d.$$

Note that (2.88)-(2.89) can only be true if for some $k \leq d$ we have

$$\mathcal{V}_0 = \mathcal{V}_1 = \dots = \mathcal{V}_{k-1} = \mathcal{V}_k \subsetneq \mathcal{V}_{k+1}$$

or equivalently

$$[\mu_{k+1}(x)] \notin \text{Span}\{[b_1], \dots, [b_n], [\mu_1], \dots, [\mu_k]\}.$$

This is in contradiction with Lemma 2.5 as $[\mu] \in \text{Span}\{[b_1], \dots, [b_n]\}$ for any given monomial $\mu(x)$ in case (2.86) is solvable. ■

Proposition 2.2 implies that it should be possible to construct a basis \mathcal{B} of $\mathcal{P}^s \setminus \mathcal{I}$ from the components of $[b]$. In fact, such a basis $\mathcal{B} = \{b_1, \dots, b_m\}$ corresponds to an m -dimensional subspace $\mathcal{V} \subset \mathbb{C}^n$ which is invariant under N_1, \dots, N_s . This theoretical insight is formulated first, before the main algorithm is presented.

Proposition 2.3 *Suppose that (2.86) has a solution for some chosen monomial basis in b and V_1, \dots, V_s . Then there exists a similarity transformation*

$$T = \begin{pmatrix} T_1 & T_2 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

for which

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} N_i \begin{pmatrix} T_1 & T_2 \end{pmatrix} = \begin{pmatrix} M_{11}^i & M_{12}^i \\ 0 & M_{22}^i \end{pmatrix}, \quad i = 1, \dots, s. \quad (2.90)$$

Then, with $[b_0] = R_1[b]$ the following relations hold:

$$[x_i b_0(x)] = M_{11}^i [b_0(x)], \quad i = 1, 2, \dots, s.$$

Thus, the components of $[b_0]$ form a basis \mathcal{B} of $\mathcal{P}^s \setminus \mathcal{I}$ and M_{11}^i are the multiplication matrices from Proposition 2.1.

Proof. With monomial vector b , let matrix K be such that

$$K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}, \quad K^{-1} = \begin{pmatrix} \bar{K}_1 \\ \bar{K}_2 \end{pmatrix} \text{ be such that } b = K \begin{pmatrix} b_0 \\ e \end{pmatrix}$$

where the components of $[b_0]$ form a basis of the m -dimensional space $\mathcal{P}^s \setminus \mathcal{I}$. Since $[b_0]$ is a basis of $\mathcal{P}^s \setminus \mathcal{I}$, there exists C such that $[e] = C[b_0]$, or equivalently, $\tilde{e} = e - Cb_0 \in \mathcal{I}^{n-m}$. From (2.86) we hence infer

$$\begin{pmatrix} I & 0 \\ -C & I \end{pmatrix} \left[x_i \begin{pmatrix} b_0 \\ e \end{pmatrix} - K^{-1} N_i K \begin{pmatrix} b_0 \\ e \end{pmatrix} \right] \in \mathcal{I}^n, \quad i = 1, 2, \dots, s$$

which reads

$$x_i \begin{pmatrix} b_0 \\ \tilde{e} \end{pmatrix} - \underbrace{\begin{pmatrix} I & 0 \\ -C & I \end{pmatrix} K^{-1} N_i K \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}}_{\begin{pmatrix} M_{11}^i & M_{12}^i \\ M_{21}^i & M_{22}^i \end{pmatrix}} \begin{pmatrix} b_0 \\ \tilde{e} \end{pmatrix} \in \mathcal{I}^n \quad i = 1, 2, \dots, s.$$

Therefore, $x_i b_0 - M_{11}^i b_0 - M_{12}^i \tilde{e} \in \mathcal{I}^m$ and, since $Y\tilde{e} \in \mathcal{I}^m$ for any matrix $Y \in \mathbb{R}^{m \times (n-m)}$, we get

$$x_i b_0 - M_{11}^i b_0 \in \mathcal{I}^m, \quad i = 1, 2, \dots, s.$$

Taking equivalence classes shows that M_{11}^i are multiplication matrices as in condition (2.82). Moreover, $x_i \tilde{e} - M_{21}^i b_0 - M_{22}^i \tilde{e} \in \mathcal{I}^{n-m}$ so that $M_{21}^i b_0 \in \mathcal{I}^{n-m}$ from which we get

$$[M_{21}^i b_0] = M_{21}^i [b_0] = 0, \quad i = 1, 2, \dots, s.$$

Since $[b_0]$ is a basis of $\mathcal{P}^s \setminus \mathcal{I}$, the components are linearly independent and as a consequence $M_{21}^i = 0$, $i = 1, \dots, s$. We thus conclude that N_1, \dots, N_s can be jointly transformed in upper block diagonal form and the restriction of N_i to subspace $\text{Im}(T_1)$ yields M_{11}^i for $i = 1, \dots, s$. Every $z \in V(\mathcal{I})$ can hence be found as an s-tuple of the eigenvalues of N_1, \dots, N_s . Defining

$$\begin{pmatrix} T_1 & T_2 \end{pmatrix} = \begin{pmatrix} K_1 + K_2 C & K_2 \end{pmatrix}, \quad \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} \bar{K}_1 \\ \bar{K}_2 - C \bar{K}_1 \end{pmatrix}$$

finishes the proof. ■

Let us summarize the results obtained so far. Once (2.86) is solvable, the components z_1, \dots, z_m of all zeros in $V(\mathcal{I})$ can be found as eigenvalues of the matrices N_1, \dots, N_s . The restriction of N_1, \dots, N_s to an (unknown) subspace \mathcal{V} are exactly the commuting matrices M_1, \dots, M_s from Proposition 2.1. As extensively discussed in [168], there exists no generically best algorithm for computing such a basis, which lead us to investigate alternative approaches. We will show that $V(\mathcal{I})$ can be obtained directly from the matrices N_1, \dots, N_s without explicitly computing the joint invariant subspace \mathcal{V} corresponding a basis \mathcal{B} of $\mathcal{P}^s \setminus \mathcal{I}$.

The following algorithm constructs candidate zeros, by iteratively identifying joint invariant subspaces. It constructs largest joint invariant subspaces of N_1, \dots, N_s , which therefore must also contain $\text{Im}(T_1)$ for T_1 defined as in (2.90). Contrary to existing methods, there is no need to compute all joint eigenvectors. The candidate zeros are constructed by sequentially applying similarity transformations to the N_1, \dots, N_s , and storing eigenvalues systematically.

Algorithm 2.2 *Suppose that N_1, \dots, N_s are solutions of (2.86). Then the following algorithm iteratively constructs s-lists of complex numbers $\lambda \in \mathbb{C} \cup \{\infty\}$ on the basis of a sequence of similarity transformations $T^{(j)}$, $j = 1, \dots, s$.*

Step $i = 1$

Choose nonsingular $T^{(1)} = (T_1, \dots, T_{k_1})$ and let $\text{col}(\hat{T}_1, \dots, \hat{T}_{k_1}) = (T^{(1)})^{-1}$ such that

$$\begin{pmatrix} \hat{T}_1 \\ \vdots \\ \hat{T}_{k_1} \end{pmatrix} N_1 (T_1 \ \cdots \ T_{k_1}) \tag{2.91}$$

is in block root-subspace form. Let λ_j denote the eigenvalue of the block $\hat{T}_j N_1 T_j$ for $j = 1, \dots, k_1$ and d_j its dimension and collect this information, with the all ones

row vector e_j of length j , as

$$\Lambda_1 = (\lambda_1 e_{d_1} \quad \dots \quad \lambda_{k_1} e_{d_{k_1}}).$$

In order λ_j to be the first component z_1 of zero $z \in V(\mathcal{I})$ there must exist a joint eigenvector that lies in the subspace $\text{Im}(T_j)$.

Step $i = 2$

For all $j = 1, \dots, k_1$, choose a basis matrix K_j of the largest subspace K_j that satisfies

$$N_2 K_j \subseteq K_j \quad \text{and} \quad K_j \subseteq \text{Im}(T_j).$$

Define

$$\hat{L}_j := \hat{T}_j K_j$$

and let L_j be a right inverse of \hat{L}_j . Extend L_j to the nonsingular matrix $(L_j \ M_j)$.

Since T_j generally consist of multiple columns, N_2 restricted to the subspace $\text{Im}(T_j)$ has multiple (possibly distinct) eigenvalues. Let us therefore denote, for $j = 1, \dots, k_1$, the r_j different eigenvalues of $P_j = \hat{L}_j (\hat{T}_j N_2 T_j) L_j$ by $\lambda_j^1, \dots, \lambda_j^{r_j}$ with

$$\text{algebraic multiplicity } d_j^1, \dots, d_j^{r_j}$$

and transform P_j as

$$\begin{pmatrix} \hat{Q}_j^1 \\ \vdots \\ \hat{Q}_j^{r_j} \end{pmatrix} P_j \begin{pmatrix} Q_j^1 & \dots & Q_j^{r_j} \end{pmatrix} \quad (2.92)$$

into block root-subspace form (with blocks of dimension $d_j^1, \dots, d_j^{r_j}$). Then define for $j = 1, \dots, k_1$

$$U_j^\alpha = (T_j L_j Q_j^\alpha), \quad W_j = T_j M_j, \quad \alpha = 1, \dots, r_j \quad (2.93)$$

Then, define the similarity transformation $T^{(2)}$ as

$$T^{(2)} = (U_1^1 \quad \dots \quad U_1^{r_1} \quad W_1 \quad \dots \quad U_{k_1}^1 \quad \dots \quad U_{k_1}^{r_{k_1}} \quad W_{k_1}) \quad (2.94)$$

with $r_1 + \dots + r_{k_1} + k_1$ blocks of column size (some of which can be empty)

$$d_1^1, \dots, d_1^{r_1}, \hat{d}^1, d_2^1, \dots, d_2^{r_2}, \hat{d}^2, \dots, d_{k_1}^1, \dots, d_{k_1}^{r_{k_1}}, \hat{d}^{k_1}$$

corresponding the dimensions of $U_j^1, \dots, U_j^{r_j}$ and W_j , $j = 1, \dots, k_1$. Similarly, we define

$$(T^{(2)})^{-1} = \text{col}(\hat{U}_1^1, \dots, \hat{U}_1^{r_1}, \hat{W}_1, \dots, \hat{U}_{k_1}^1, \dots, \hat{U}_{k_1}^{r_{k_1}}, \hat{W}_{k_1})$$

Hence, after the first 2 steps we know that

$$\text{Im}(U_1^1, \dots, U_1^{r_1}, \dots, U_{k_1}^1, \dots, U_{k_1}^{r_{k_1}})$$

is a joint invariant subspace of N_1, N_2 . As the proof below shows, we can securely drop the eigenvalues of $\tilde{W}_j N_i W_j$. Therefore, we introduce placeholder ∞ in accordance with the blocks W_j , $j = 1, \dots, k_1$, and augment Λ_1 as follows

$$\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 e_{d_1} & & \dots & & \lambda_{k_1} e_{d_{k_1}} \\ \left(\lambda_1^1 e_{d_1^1} \quad \dots \quad \lambda_{r_1}^1 e_{d_{r_1}^1} \quad \infty e_{\hat{d}_1} \right) & \dots & \left(\lambda_1^{k_1} e_{d_1^{k_1}} \quad \dots \quad \lambda_{r_{k_1}}^{k_1} e_{d_{r_{k_1}}^{k_1}} \quad \infty e_{\hat{d}_{k_1}} \right) \end{pmatrix}$$

recording the relevant and irrelevant eigenvalues with their corresponding multiplicities. The algorithm proceeds with the $k_2 = r_1 + \dots + r_{k_1}$ blocks

$$U_1^1, \dots, U_1^{r_1}, \dots, U_{k_1}^1, \dots, U_{k_1}^{r_{k_1}}, \quad (2.95)$$

for which we will use the symbols T_1, \dots, T_{k_2} .

Steps $i = 3, \dots, s$

These steps generate similarity transformations $T^{(3)}, \dots, T^{(s)}$. For each block $T_1, \dots, T_{k_{i-1}}$ of $T^{(i-1)}$, as constructed in step i , the largest N_{i+1} -invariant subspace that lies in $\text{Im}(T_j)$ is computed for $j = 1, \dots, k_{i-1}$. Transform P_j into the form (2.92) and construct matrices $U_j^\alpha, W_j, \alpha = 1, \dots, r_j$ as in (2.93), which also defines the new similarity transformation matrix $T^{(i)}$ with new blocks

$$T^{(i)} = (T_1 \quad \dots \quad T_{k_{i-1}}).$$

After s steps, the matrix

$$\Lambda = \begin{pmatrix} \Lambda_1 \\ \vdots \\ \Lambda_s \end{pmatrix} \quad (2.96)$$

is obtained which contains all information on the elements in $V(\mathcal{I})$.

Theorem 2.8 Let N_1, \dots, N_s satisfy (2.86), and run Algorithm 2.2 to obtain Λ in (2.96). Then each $z \in V(\mathcal{I})$ can be found as a column of Λ which does not contain ∞ .

Proof. Recall that by Proposition 2.3 and Theorem 2.7 there exists a joint eigenvector v of the N_i 's satisfying $N_i v = z_i v$, $i = 1, \dots, s$ for every $z = (z_1, \dots, z_s) \in V(\mathcal{I})$. Let us therefore choose some $z \in V(\mathcal{I})$ and let v be the corresponding joint eigenvector.

Consider Algorithm 2.2. In step one, suppose $\mathcal{V}_1 = \text{Im}(T_1)$ is the root subspace of N_1 corresponding to z_1 . We necessarily have $v \in \mathcal{V}_1$ and thus $\text{span}\{v\} \subset \mathcal{V}_1$. Since $\text{span}\{v\}$ is also N_2 -invariant and K_2 is defined as the largest N_2 -invariant subspace that lies in \mathcal{V}_1 we clearly have that

$$\text{span}\{v\} \subset K_2 \subseteq \mathcal{V}_1.$$

In fact, referring to (2.93), defining

$$\mathcal{V}_2 = \text{Im}(T_1 L_1 Q_1^j) \quad \text{for } j \in \{1, \dots, r_1\} \text{ such that } \sigma(N_2|_{\mathcal{V}_2}) = \{z_2\},$$

that is, the eigenvalue of N_2 restricted to subspace \mathcal{V}_2 is the second component of z . At each of the remaining step $i = 3, \dots, s$ we will be able to find a subspace of $\mathcal{V}_i \subset \text{im}(T^{(i)})$ for which $\sigma(N_i|_{\mathcal{V}_i}) = \{z_i\}$ which defines a sequence of subspaces satisfying

$$\mathcal{V}_1 \supseteq K_2 \supseteq \mathcal{V}_2 \supseteq K_3 \cdots \supseteq K_s \supseteq \mathcal{V}_s \quad \text{and} \quad v \in \mathcal{V}_i \quad \text{for all } i = 1, \dots, s.$$

This shows that the algorithm finds every $z \in V(\mathcal{I})$. Referring to the notation in Algorithm 2.2, the invariant subspaces $\mathcal{V}_i \subset \text{im}(T^{(i)})$ must be contained in the largest invariant subspace $\text{span}(U_1^1, \dots, U_1^{r_1}, \dots, U_{k_i-1}^1, \dots, U_{k_i-1}^{r_{k_i-1}})$, which is why the blocks W_j were disregarded. ■

Algorithm 2.2 reduces to the classical method of Theorem 2.7 when the elements of monomial vector b in (2.86) form a basis of $\mathcal{P}^s \setminus \mathcal{I}$. Indeed, the matrices N_1, \dots, N_s are then pairwise commuting. With $T^{(1)}$ turning N_1 into block root-subspace in step 1, we infer that $T^{(1)}$ actually turns all N_2, \dots, N_s into block root-subspace by Lemma 2.3. As a consequence, the N_i 's in (2.86) reduce to the M_i 's in (2.82). In addition, at each step i , the largest N_{i+1} invariant subspace equals $K_j = \text{Im}(T_j)$ for all $j = 1, \dots, k_i$, which means that the blocks W_j are void.

We emphasize that the developed procedure adds to the work done by Stetter and co-workers [168]. Rather than improving the numerical behavior of existing algorithms, the results of this section show that the zero set can be computed without knowing a basis of the quotient space.

Due to the iterative nature of the algorithm it is difficult to derive theoretical bounds on its computational complexity. Using a Gröbner basis approach usually becomes inefficient when the basis is large if compared to the number of isolated solutions of (2.80). For elementary problems, the determination of N_i in Algorithm 2.2 is computationally usually the most demanding step.

Remark 2.16 *Various other approaches exist for solving (2.80), many of which do not rely on algebraic operations of finding a Gröbner basis. For a good reference in this respect see [167], which focusses on homotopy methods, or [7], using the notion of the resultant of two polynomials. Recently, an LMI approach that uses homogeneous polynomials was developed in [37].*

2.5.2 Numerical example

In this section, we will construct a multiplier-based relaxation and verify its exactness by using Theorem 2.6. It will also be shown how to extract zeros of the resulting polynomial system along the lines of Algorithm 2.2.

Consider the following robust linear programming problem:

$$\begin{aligned} & \text{infimize} && \langle c, y \rangle \\ & \text{subject to} && y \in \mathbb{R}^2, \quad a_i(\delta)y - b_i < 0 \quad \text{for all } \delta \in \boldsymbol{\delta}, \quad i = 1, \dots, 4, \end{aligned} \quad (2.97)$$

in which $c = [-1, -1]$, $a_i(\delta)$ is the i^{th} row of $A(\delta) = A_0 + E(\delta)$, with

$$A_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E(\delta) = \begin{pmatrix} \frac{\delta_1}{5} & \frac{\delta_2}{5} \\ \frac{\delta_3}{5} & \frac{\delta_4}{5} \\ \frac{\delta_5}{5} & \frac{2\delta_6^2 - \delta_5}{10} \\ \frac{\delta_7}{5} & \frac{2\delta_8^2 - 5\delta_7}{10} \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

and the set $\boldsymbol{\delta}$ is a direct product of ellipsoids defined as

$$\boldsymbol{\delta} = \hat{\boldsymbol{\delta}} \times \hat{\boldsymbol{\delta}} \times \hat{\boldsymbol{\delta}} \times \hat{\boldsymbol{\delta}} \quad \text{with} \quad \hat{\boldsymbol{\delta}} = \{ \hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2) \in \mathbb{R}^2 \mid \|\hat{\delta}\| \leq 1 \}.$$

This example thus consists of 4 semi-infinite LP constraints and resembles the example in Section 5.1 of [33], where a new randomized approach was proposed for handling uncertain convex programs. Originally, ellipsoidal perturbations on the rows of the A were assumed which enables to recast the problem *exactly* as a tractable conic quadratic program, see [15]. In order not to have access to a priori tight approximation schemes, we have introduced polynomial dependence in $E(\delta)$.

Computing upper bounds

As sketched at the beginning of this chapter, each of the constraint will be rewritten in the form

$$F_i(\delta)' J_i(y) F_i(\delta) < 0 \quad \text{for all } \delta \in \boldsymbol{\delta} \quad i = 1, \dots, 4 \quad (2.98)$$

with $F_i(\delta) = \begin{pmatrix} \Delta_i(\delta)(I - A_i \Delta_i(\delta))^{-1} B_i \\ I \end{pmatrix}$. Let us denote the size of $\Delta_i(\cdot)$ by $d_i \times d_i$.

Then, applying the S -procedure argument from Section 2.3 turns the problem into the infimization of $\langle c, y \rangle$ subject to $y \in \mathbb{R}^2$, $\Pi_i \in \mathbb{R}^{2d_i \times 2d_i}$, and

$$\begin{pmatrix} I & 0 \\ A_i & B_i \end{pmatrix}' \Pi_i \begin{pmatrix} I & 0 \\ A_i & B_i \end{pmatrix} + J_i(y) < 0, \quad (2.99)$$

$$\begin{pmatrix} \Delta_i(\delta) \\ I \end{pmatrix}' \Pi_i \begin{pmatrix} \Delta_i(\delta) \\ I \end{pmatrix} \succeq 0, \quad \forall \delta \in \hat{\boldsymbol{\delta}}, \quad i = 1, 2, 3, 4. \quad (2.100)$$

Notice that semi-infinite constraints (2.100) characterizing the set of admissible scalings Π_i do not reflect the entire domain δ . In fact, the uncertainties enter the problem *constraint-wise*, see [15], which allows to independently relax the constraints $a_i(\delta) - b_i < 0$, $i = 1, \dots, 4$, and each of them only involves two parameters.

Since the regions $\hat{\delta}$ are semi-algebraic, the following relaxations are based on sum-of-squares techniques presented in [61]. We note that other implementations based on the same principle can be used as well, e.g. [160, 158, 110]. The first relaxation, denoted by REL-1, has the least computational complexity by using an parameter independent monomial basis $m(\hat{\delta}_1, \hat{\delta}_2) = 1$ for each of the constraints. The resulting upper bound value is $\gamma_{\text{rel}} = -1.570$. Inspection of the optimal dual multipliers of the LMIs in (2.99) reveals that only Φ_3, Φ_4 are nonzero, by which it follows that the constraints (2.99) are inactive for $i = 1, 2$. This motivates us to partially extend the monomial basis for sum-of-squares relaxation REL-2, as indicated in Table 2.2. The upper bound value has indeed improved to $\gamma_{\text{rel}} = -1.602$.

Approximation	monomial basis per constraint				CPU time (s)	γ_{rel}
	$i = 1$	$i = 2$	$i = 3$	$i = 4$		
REL-1	1	1	1	1	4.21	-1.570
REL-2	1	1	$1, \delta_5, \delta_6$	$1, \delta_7, \delta_8$	5.42	-1.602

Table 2.2: Upper bounds for γ_{opt} corresponding (2.97).

Let us verify exactness along the lines Section 2.4.2. The optimal dual multipliers Φ_3, Φ_4 both have rank 2 and read as

$$\Phi_3 = \begin{pmatrix} 0.0870 & 0.2730 & 0 & 0.2990 \\ 0.2730 & 0.8560 & 0 & 0.9350 \\ 0 & 0 & 0.9350 & 0 \\ 0.2990 & 0.9350 & 0 & 1.0220 \end{pmatrix}$$

and

$$\Phi_4 = \begin{pmatrix} 0.2910 & -0.2050 & 0 & -0.4110 \\ -0.2050 & 0.1440 & 0 & 0.2890 \\ 0 & 0 & 0.2890 & 0 \\ -0.4110 & 0.2890 & 0 & 0.5800 \end{pmatrix}.$$

As mentioned earlier, we can eliminate the $Z^{i,\nu}$ from (2.68) if the constraints are scalar-valued, and rather solve (2.70). From this polynomial system, the following solutions were found

$$\begin{aligned} (\bar{\delta}^{5,1}, \bar{\delta}^{6,1}) &= (0.2917, 0.9565), & (\bar{\delta}^{5,2}, \bar{\delta}^{6,2}) &= (0.2917, -0.9565), \\ (\bar{\delta}^{7,1}, \bar{\delta}^{8,1}) &= (-0.7083, -0.7059), & (\bar{\delta}^{7,2}, \bar{\delta}^{8,2}) &= (-0.7083, 0.7059). \end{aligned} \quad (2.101)$$

Note that we used indices 5,6,7,8 of the original problem. In order to verify exactness, we can either search for $Z^{i,\nu}$ that satisfy (2.68) or we can compute lower bound values γ_{lb} by sampling the original constraints making use of the parameter values (2.101).

Lower bound computations

Motivated by our the results obtained so far, we analyze lower bound values for three different grids. Let us first be somewhat ignorant and sample each of the constraints with 1257 parameter values, uniformly distributed over the unit disk, which leads to a total of 5028 constraints, denoted as GRID-1. Although the optimal value $\gamma_{\text{lb}} = -1.604$ is already close to the upper bound $\gamma_{\text{rel}} - 1.602$ and would suffice as a certificate for exactness of REL-2 in practice, we can obtain better results with much less computational effort. Using the fact that constraints 1 and 2 were inactive in REL-1 and REL-2, let us replace the corresponding grid by a singleton $(\delta_1, \delta_2) = (\delta_3, \delta_4) = (0, 0)$, which we refer to as GRID-2. As shown in Table 2.5.2 the same result is achieved with half of the computational complexity. Finally, GRID-3 further reduces the problem size by sampling constraint 3 and 4 on any parameter pair given in (2.101). We emphasize that due to the uncertainty entering *constraint-wise*, a single (generally non-unique) worst-case parameter pair always exist, see [15]. From the optimal values indicated in Table 2.5.2, the relaxation REL-2 is proven to be exact.

Approximation	# gridpoints per constraint				CPU time (s)	γ_{lb}
GRID-1	1257	1257	1257	1257	8.32	-1.604
GRID-2	1	1	1257	1257	4.27	-1.604
GRID-3	1	1	1	1	0.45	-1.602

Table 2.3: Lower bounds for γ_{opt} corresponding (2.97).

Solving the polynomial system using Algorithm 2.2

Let us finally illustrate how the solutions in (2.101) were obtained by following the procedure of Section 2.5.1. With the dual optimal multipliers Φ_3, Φ_4 , the polynomial system (2.70) for constraint 3, with notation $x = (x_1 \ x_2) = (\delta_5, \delta_6)$ becomes

$$\begin{aligned} p_1(x) &= 0.1411 + 0.8881x_1 - 0.4374x_2^2, \\ p_2(x) &= 0.6781 - 0.4086x_1 - 0.6109x_2^2 \end{aligned} \quad (2.102)$$

The first step is to solve the system of linear equations (2.86) with the choices

$$b_i(x) = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_2^2 \end{pmatrix}, \quad \text{and} \quad V_i(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_2^2 & 0 & 0 & 0 & x_2^2 \end{pmatrix},$$

for $i = 1, 2$, which leads to the solution matrices

$$N_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.2917 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.2917 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.3225 & 2.0305 & 0 & 0 \\ 0 & 0.9149 & 0 & 0 \end{pmatrix}.$$

Applying Algorithm 2.2, we first transform N_1 into block root-subspace form \tilde{N}_1 and apply the transformation to both N_2 leading to

$$\tilde{N}_1 = \begin{pmatrix} 0.2917 & 0 & 0 & 0 \\ 0 & 0.2917 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\tilde{N}_2 = \begin{pmatrix} -0.9565 & -0.0851 & 0 & 0 \\ 0 & 0.9565 & 0 & 0 \\ 0 & 0 & 0.6280 & -0.1318 \\ 0 & 0 & 0.5457 & -0.6280 \end{pmatrix}.$$

We are able to read off the zeros, and need not apply further transformations. There are two other candidate solutions of the form $(0, \cdot)$ but these do not satisfy (2.102). In a similar fashion, the parameter values corresponding the exactness test for constraint 4 have been computed.

2.6 Summary

A general framework has been presented for computing approximate solutions of robust SDP optimization problems. The robust SDP constraints are allowed to be rationally dependent on the parameters. Lower bound values are found by sampling the given semi-infinite constraint on a finite number of parameter values. For computing upper bound values, so-called relaxation schemes have been developed.

The basic theory from which all relaxation schemes are constructed concerns robust SDPs with a polynomial parameter dependence. For these type of problems, it has been shown how to derive a whole family of LMI relaxation schemes based on Pólya's theorem or matrix sum-of-squares decompositions. If compared to the LMI-relaxations as presented earlier in [93, 160], the derived LMI conditions are conceptually simpler and lead to semi-definite programs without any affine equation constraints.

In a second relaxation approach the robust SDP constraints were assumed to be specified in terms of linear fractional representations. By virtue of an S-procedure argument, any given robust SDP constraint can be alternatively described in terms of a semi-infinite constraint that is quadratic in the parameters, at the cost of introducing auxiliary multiplier variables. As compared to the direct sum-of-squares method, relaxations based on the S-procedure are especially suited for rationally dependent robust SDP constraints. In the numerical example of Section 2.4.3, these S-procedure based or so-called multiplier relaxations were constructed by employing convex hull arguments, Pólya's theorem and sum-of-squares decompositions. The flexibility of the proposed framework has been exploited in developing a Matlab toolbox, see [55], as well as in a numerical example.

Since the relaxation gap cannot be estimated a priori in general, a condition that verifies exactness is presented for the general case of having multiple robust SDP constraints. Motivated by the fact that this test amounts to finding a solution to a system of polynomial equations, a rather independent discussion on how to numerically compute such solutions has been given. Standard techniques in computational algebra, often referred to as Stetter's method, involve the computation of a Gröbner basis of the ideal generated by the polynomials and further require joint eigenvector computations in order to arrive at the zeros of the polynomial system. Our algorithm, presented in [60], does not require structural knowledge of the polynomial system, nor does it require computation of joint eigenvectors.

Chapter 3

Analysis with integral quadratic constraints

A powerful framework for the analysis of uncertain systems is based on so-called integral quadratic constraints (IQC). In this section, we briefly recapitulate this analysis approach which can handle all sorts of non-linear and time-varying uncertainties by making use of a so-called multiplier variable. Popov actually first used the notion of 'multiplier' in the context of feedback systems and considered the stability analysis problem for an LTI system with a single non-linearity in the loop, see [144]. It is by now well-known that IQC analysis generalizes the Popov criterion, circle criterion and many variations thereof, see [75]. It includes important stability principles such as those based on small gain or passivity arguments. In this chapter, we discuss the main theory on the analysis with IQCs, and show how to address parametric uncertainties. A numerical example will be given in Section 4.3.

Consider the interconnection as shown in Figure 3.1, in which

$$M = \begin{pmatrix} M_{qp} & M_{qw} \\ M_{zp} & M_{zw} \end{pmatrix}. \quad (3.1)$$

The uncertain element Δ maps signals from $\mathcal{L}_{2,e}^{n_q}$ to $\mathcal{L}_{2,e}^{n_p}$, in which $\mathcal{L}_{2,e}$ denotes the space of signals w for which all truncations $T_N(w)$ defined as

$$T_N(w) = \begin{cases} w(t) & \text{for } 0 \leq t \leq N \\ 0 & \text{for } t \geq N \end{cases}$$

have finite energy. Hence, if we let \mathcal{L}_2 denote the space of all functions $[0, \infty] \rightarrow \mathbb{R}^n$ of finite energy, the space $\mathcal{L}_{2,e}$ is defined as

$$\mathcal{L}_{2,e} := \{w : [0, \infty) \rightarrow \mathbb{R}^n \mid T_N(w) \in \mathcal{L}_2 \text{ for any } N \geq 0\}.$$

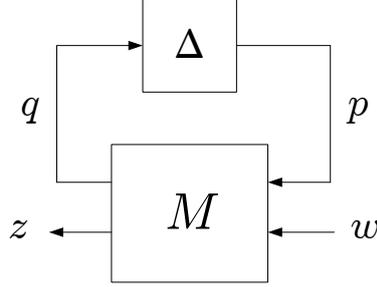


Figure 3.1: Interconnection for stability and performance analysis

The set of uncertain elements, denoted by Δ , represents all tolerable uncertainties and captures both the nature of the uncertainties (linear/nonlinear, time-invariant/time-varying, static/dynamic), their size (in terms of bounds on norm or gain) and their structure (block-diagonal, full-block). Whether the resulting robust stability analysis problem can be put into efficient algorithms clearly depends on the class of uncertainties that is considered. As will become clear in the next section, two properties of Δ are essential in order to establish a fundamental analysis result in IQC theory.

Assumption 3.1 (Causal, bounded) *The operators $\Delta \in \mathbf{\Delta} : \mathcal{L}_{2,e}^{n_q} \rightarrow \mathcal{L}_{2,e}^{n_p}$ are causal and have finite gain on the vector space \mathcal{L}_2 .*

The analysis of uncertain systems in the IQC framework is founded on the use of an auxiliary multiplier variable. Let a set $\mathbf{\Delta}$ of uncertain operators be given. Suppose that a matrix function $\Pi(\cdot)$ exists, which is Hermitian-valued and essentially bounded on the imaginary axis, and for which the following integral quadratic constraint (IQC) holds true for all elements $\Delta \in \mathbf{\Delta}$:

$$\int_{-\infty}^{\infty} \begin{pmatrix} \widehat{\Delta q}(i\omega) \\ \widehat{q}(i\omega) \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} \widehat{\Delta q}(i\omega) \\ \widehat{q}(i\omega) \end{pmatrix} \geq 0 \quad \forall q \in \mathcal{L}_2^{n_q}. \quad (3.2)$$

Here, \widehat{q} indicates the Fourier transform of a finite energy signal $q \in \mathcal{L}_2^{n_q}$ and $\widehat{\Delta q}(i\omega)$ indicates the Fourier transform of the signal $\Delta(q)$. Any matrix function Π for which (3.2) holds, will be called an *admissible* multiplier for the uncertainty set $\mathbf{\Delta}$. In fact, the LMI conditions that are derived in this section depend on the multiplier. In order to guarantee robust stability or performance of the interconnected system in Figure 3.1 with respect to the uncertain set $\mathbf{\Delta}$, the multiplier Π needs to satisfy (3.2), and any multiplier with this property is therefore referred to as being “admissible”.

Example 3.1 (Time-invariant real parameter) *Let Δ correspond to multiplication with a real scalar $\delta \in [-1, 1]$, i.e. $(\Delta q)(t) = \delta q(t)$. Then, an admissible*

multiplier is

$$\Pi(i\omega) = \begin{pmatrix} -X(i\omega) & Y(i\omega) \\ Y(i\omega)^* & X(i\omega) \end{pmatrix},$$

where $X(i\omega) = X(i\omega)^* \succeq 0$ and $Y(i\omega) + Y(i\omega)^* = 0$ for all $\omega \in \mathbb{R}$. In fact,

$$\begin{pmatrix} \delta\hat{q}(i\omega) \\ \hat{q}(i\omega) \end{pmatrix}^* \begin{pmatrix} -X(i\omega) & Y(i\omega) \\ Y(i\omega)^* & X(i\omega) \end{pmatrix} \begin{pmatrix} \delta\hat{q}(i\omega) \\ \hat{q}(i\omega) \end{pmatrix} = \hat{q}(i\omega)^*(X(i\omega) - \delta^2 X(j\omega) + \delta Y(i\omega) + \delta Y(i\omega)^*)\hat{q}(i\omega) \succeq 0 \quad (3.3)$$

holds for all $\delta \in [-1, 1]$. By integration, the IQC (3.2) is shown to be satisfied for any parameter value in the set.

The following theorem provides a sufficient condition for robust stability of the interconnected system in Figure 3.1. This interconnection of M with Δ is said to be well-posed if $I - M_{qp}\Delta$ has a causal and bounded inverse.

Theorem 3.1 Let $M(s) \in RH_\infty^{(n_q+n_z) \times (n_p+n_w)}$ and let Δ be a set of bounded causal operators $\Delta : \mathcal{L}_{2,e}^{n_q} \rightarrow \mathcal{L}_{2,e}^{n_p}$. Let $\Pi \in RH_\infty^{(n_q+n_p) \times (n_p+n_w)}$ be a proper matrix function without poles on the extended imaginary axis. Suppose that

- i) for every $\tau \in [0, 1]$ the interconnection of M_{qp} and $\tau\Delta$ is well-posed,
- ii) $\tau\Delta$ satisfies the IQC (3.2) defined by Π for every $\tau \in [0, 1]$,
- iii) M_{qp} satisfies

$$\begin{pmatrix} I \\ M_{qp}(i\omega) \end{pmatrix}^* \Pi(j\omega) \begin{pmatrix} I \\ M_{qp}(i\omega) \end{pmatrix} \prec 0, \quad \forall \omega \in \mathbb{R} \cup \{\infty\}. \quad (3.4)$$

Then, the feedback interconnection of M and Δ is stable.

Proof. A detailed discussion on IQC analysis results can be found in [123]. ■

The inequality (3.2) is often called the IQC defined by the multiplier Π . Obviously, there will be other elements $\Delta \notin \Delta$ that satisfy (3.2) for a particularly chosen multiplier. For this reason, the feasibility condition (3.4), as considered for a single multiplier Π , is usually not a precise analysis test for proving robust stability.

One way of obtaining more accurate results is by constructing a whole family of multipliers, denoted by $\mathbf{\Pi}$, each element of which satisfies (3.2) for the uncertainty set Δ under consideration. The stability condition in Theorem 3.1 then amounts to searching some $\Pi \in \mathbf{\Pi}$ for which the frequency domain inequality (3.4) is feasible. Roughly speaking, the larger the set $\mathbf{\Pi}$ is, the more accurate the analysis result will be. The design of suitable multiplier classes for practical problems has been considered in [147, 101, 87] and has led to the development of a Matlab toolbox [98]. Nevertheless, it is yet unknown how to estimate the level of conservatism.

Let us have a closer look at how to implement condition (3.4) in Theorem 3.1. In the remaining part of the chapter, the dynamic multiplier Π is assumed to be real-rational and bounded on the extended imaginary axis.

Assumption 3.2 *The dynamic multiplier $\Pi \in RH_\infty^{n_p+n_q}$ is chosen to be described as*

$$\Pi = \Psi^* Q \Psi, \quad \text{with } Q \in \mathcal{Q}, \quad (3.5)$$

in which Ψ is a fixed rational transfer matrix, the elements of which are proper and stable transfer functions. A suitably chosen set of matrices $Q \in \mathcal{Q}$ parameterizes a set of admissible multipliers Π .

The motivation for considering multipliers of the form (3.5) stems from the following fact. For any given Π there exists, by boundedness of Π , some α such that $\Pi + \alpha^2 I$ is positive definite on \mathbb{C}^0 . Hence, we can factorize $\Pi + \alpha^2 I = F^* F$ such that F and the inverse of F are both proper and stable. With the choices

$$\Psi = \begin{pmatrix} F \\ \alpha I \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

the structure in (3.5) follows.

Under the assumption that M is stable and the property i) and ii) in Theorem 3.1 hold, stability of the interconnected system defined by M and Δ is implied by the existence of $Q \in \mathcal{Q}$ such that $\Pi = \Psi^* Q \Psi$ satisfies (3.4). This frequency domain inequality (FDI) is a particular type of a robust SDP constraint in the single variable ω and can be recast as a genuine LMI by using the Kalman-Yakubovich-Popov Lemma, see Appendix A. In fact, let M_{qp} be realized as

$$M_{qp} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]. \quad (3.6)$$

Further, let $\Psi \in RH_\infty^{n_Q \times (n_q+n_p)}$ be partitioned according to the columns/rows of $\begin{pmatrix} I \\ M_{qp} \end{pmatrix}$, with minimal realization

$$\Psi = \left(\begin{array}{cc} \Psi_1 & \Psi_2 \end{array} \right) = \left[\begin{array}{c|cc} A_\Psi & B_{\Psi_1} & B_{\Psi_2} \\ \hline C_\Psi & D_{\Psi_1} & D_{\Psi_2} \end{array} \right] \quad (3.7)$$

and $A_\Psi \in \mathbb{R}^{n_\Psi \times n_\Psi}$. In order to numerically verify (3.4), we introduce the composed transfer matrix

$$(\Psi_1 \quad \Psi_2) \begin{pmatrix} I \\ M_{qp} \end{pmatrix} = (\Psi_2 M_{qp} + \Psi_1) = \left[\begin{array}{cc|c} A_\Psi & B_{\Psi_2}C & B_{\Psi_2}D + B_{\Psi_1} \\ 0 & A & B \\ \hline C_\Psi & D_{\Psi_2}C & D_{\Psi_2}D + D_{\Psi_1} \end{array} \right]. \quad (3.8)$$

By the KYP Lemma, it then follows that (3.4) is equivalent to the existence of $Q \in \mathcal{Q}$ and $\mathcal{X} = \mathcal{X}'$, partitioned as

$$\mathcal{X} = \begin{pmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{pmatrix},$$

that satisfy

$$(\dots)' \begin{pmatrix} 0 & 0 & \mathcal{X}_{11} & \mathcal{X}_{12} & 0 \\ 0 & 0 & \mathcal{X}_{21} & \mathcal{X}_{22} & 0 \\ \hline \mathcal{X}_{11} & \mathcal{X}_{12} & 0 & 0 & 0 \\ \mathcal{X}_{21} & \mathcal{X}_{22} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \hline A_\Psi & B_{\Psi_2}C & B_{\Psi_2}D + B_{\Psi_1} \\ 0 & A & B \\ \hline C_\Psi & D_{\Psi_2}C & D_{\Psi_2}D + D_{\Psi_1} \end{pmatrix} \prec 0. \quad (3.9)$$

In Figure 3.2, we have visualized the fact that IQC analysis with dynamic multipliers of the form $\Pi = \Psi^* Q \Psi$ involves the composed plant (3.8). Here, the signals \tilde{p}, \tilde{q} are defined as the filtered versions of p, q , that is

$$\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \Psi \begin{pmatrix} p \\ q \end{pmatrix}, \quad (3.10)$$

and the IQC (3.2) can alternatively be written as

$$\int_{-\infty}^{\infty} \left(\Psi(i\omega) \begin{pmatrix} \widehat{\Delta q}(i\omega) \\ \widehat{q}(i\omega) \end{pmatrix} \right)^* Q \left(\Psi(i\omega) \begin{pmatrix} \widehat{\Delta q}(i\omega) \\ \widehat{q}(i\omega) \end{pmatrix} \right) \geq 0 \quad \forall q \in \mathcal{L}_2^{n_q}.$$

Remark 3.1 *Theorem 3.1 generalizes earlier observations on robust stability that were made in Section 2.1. Recall that (2.9)-(2.10) were shown to be sufficient conditions for robust stability of the loop in Figure 3.1, assuming that the uncertainty is an LTI operator.*

3.1 Robust quadratic performance

In controlled systems, one is usually interested in achieving a guaranteed closed loop robust performance level in addition to robust stability. The quadratic performance criterion, of which a definition can be found in Appendix A, is parameterized by the index matrix P_p . It can characterize many different performance measures, of which the induced \mathcal{L}_2 -gain has proven particularly useful. In fact, this quantity equals

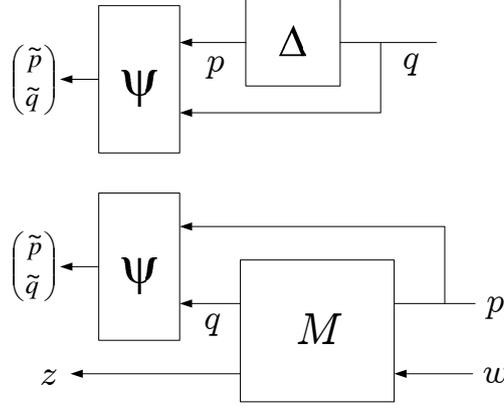


Figure 3.2: Using dynamic multipliers of the form (3.5) can be interpreted as filtering the signals p, q .

the \mathcal{H}_∞ -norm, if applied to an LTI system. Based on Theorem 3.1, the following characterization of robust quadratic performance can be derived.

Proposition 3.1 *Suppose M is stable and let property (i) and (ii) in Theorem 3.1 be satisfied. Then, the feedback interconnection of M with Δ is robustly stable and satisfies robust quadratic performance in the channel $w \rightarrow z$, if there exists a multiplier Π that satisfies (3.2) as well as*

$$(\dots)' \left(\begin{array}{c|c} \Pi(i\omega) & 0 \\ \hline 0 & P_p \end{array} \right) \left(\begin{array}{c|c} I & 0 \\ \hline M_{qp}(i\omega) & M_{qw}(i\omega) \\ 0 & I \\ \hline M_{zp}(i\omega) & M_{zw}(i\omega) \end{array} \right) \prec 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}. \quad (3.11)$$

Proof. Robust stability immediately follows from the left upper block in (3.11) and applying Theorem 3.1. The extension to performance is not difficult, see for example [185]. ■

Similar to the fact that the multiplier structure $\Pi(i\omega) = \Psi(i\omega)^* Q \Psi(i\omega)$ in (3.5) enabled us to characterize robust stability in terms of the realization matrices, we can turn condition (3.11) into an LMI constraint by applying the KYP lemma. In fact, with the realization

$$M = \left[\begin{array}{cc|c} A & B_p & B_w \\ \hline C_q & D_{qp} & D_{qw} \\ C_z & D_{zp} & D_{zw} \end{array} \right]$$

and the realization of Ψ in (3.7), it follows that

$$\left(\begin{array}{c|c} \Psi & 0 \\ \hline 0 & I \end{array} \right) \left(\begin{array}{cc} I & 0 \\ \hline M_{qp} & M_{qw} \\ M_{zp} & M_{zw} \end{array} \right) = \left[\begin{array}{cc|cc} A_\Psi & B_{\Psi 2} C_q & B_{\Psi 1} + B_{\Psi 2} D_{qp} & B_{\Psi 2} D_{qw} \\ 0 & A & B_p & B_w \\ \hline C_\Psi & D_{\Psi 1} C_q & D_{\Psi 1} + D_{\Psi 2} D_{qp} & D_{\Psi 2} D_{qw} \\ 0 & C_z & D_{zp} & D_{zw} \end{array} \right] \quad (3.12)$$

which is abbreviated as

$$\left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ \hline \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \tilde{C}_z & D_{zp} & D_{zw} \end{array} \right].$$

Note that D_{zp}, D_{zw} are not modified by the composition with Ψ , which motivates to drop the symbol $\tilde{\cdot}$. Then, condition (3.11) holds if and only if there exists some \mathcal{X} and $Q \in \mathcal{Q}$ that satisfy

$$\left(\begin{array}{ccc} I & 0 & 0 \\ \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \hline 0 & 0 & I \\ \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} \end{array} \right)' \left(\begin{array}{ccc|c} 0 & \mathcal{X} & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 \\ \hline 0 & 0 & Q & 0 \\ 0 & 0 & 0 & P_p \end{array} \right) \left(\begin{array}{ccc} I & 0 & 0 \\ \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \hline 0 & 0 & I \\ \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} \end{array} \right) \prec 0. \quad (3.13)$$

3.2 Robust \mathcal{H}_2 -performance analysis

Given an LTI system, the \mathcal{H}_2 -norm of its transfer matrix is a well-known performance measure in the context of disturbance rejection problems. In fact, one can view LQG optimal control as the \mathcal{H}_2 -norm minimization of a weighted closed-loop transfer matrix. It is also well-understood that several signal-based interpretations of the \mathcal{H}_2 -norm exist, that can be extended to general non-linear or time-varying systems. In Appendix A, one can find a stochastic and an impulse response interpretation of the \mathcal{H}_2 -norm.

As shown in [135, 13], a natural extension of the \mathcal{H}_2 -norm performance measure exists for linear time-varying systems. For general non-linear time-varying systems there exist multiple extensions of the signal-based interpretations of the \mathcal{H}_2 -norm, that do not lead to the same performance measure. In view of this possible ambiguity, the terminology “ \mathcal{H}_2 -performance measure” should be used with care if the system is neither LTI nor LTV. In view of this ambiguity, it is an interesting research topic to understand which measure of performance is most suitable (and numerically tractable) when an uncertain system is driven by a white noise source.

Remark 3.2 *The fact that LPV systems are a parameterized family of LTV systems implies that the worst-case \mathcal{H}_2 -performance analysis of LPV systems can be unambiguously used. In Section 4.2.2, we will construct sufficient conditions for robust \mathcal{H}_2 -performance analysis in terms of robust SDPs.*

In the remaining part of this section, an LMI characterization of a robust performance measure is presented, which is based on the impulse response interpretation of the \mathcal{H}_2 -norm. This performance measure is motivated by the fact that a particular initial condition can be generated by an impulse response to the plant. Moreover, by adding a suitable filter to the plant input, the impulsive input can also represent specific disturbance signals.

Let δ_D denote Dirac's delta-distribution and define the inputs

$$w^\eta = \delta_D e^\eta, \quad (3.14)$$

where e^η is the vector

$$e^\eta = (0 \quad \dots \quad 0 \quad 1 \quad 0 \dots \quad 0)', \quad (3.15)$$

with "1" being located at the η^{th} position. For any fixed non-linear time-varying uncertain operator $\Delta \in \mathbf{\Delta}$, let us denote the response of the interconnection in Figure 3.1 to w^η by $z^{\eta, \Delta} = z^\eta(w^\eta, \Delta)$. Then, the following robust performance measure is considered:

$$\sum_{\eta=1}^{n_w} \|z^{\eta, \Delta}\|_2^2 \leq \gamma^2, \quad \forall \Delta \in \mathbf{\Delta}, \quad (3.16)$$

in which $\|\cdot\|$ is the standard \mathcal{L}_2 -norm that measures the energy of a signal. For time-invariant uncertainties Δ this definition can be shown to coincide with the worst case \mathcal{H}_2 -norm in the usual sense.

Theorem 3.2 *Consider the interconnection of Figure 3.1 and let M as defined in (3.1) be stable and proper. The interconnection is robustly stable and the \mathcal{H}_2 -norm of channel $w \rightarrow z$ is at most γ if $\tilde{D}_{qw} = 0, D_{zw} = 0$ and there exists matrices $Z, \mathcal{X}, Q \in \mathcal{Q}$ that satisfy $\text{Tr}(Z) \leq \gamma^2$,*

$$\tilde{B}'_w \mathcal{X} \tilde{B}_w < Z, \quad (3.17)$$

$$\begin{pmatrix} I & 0 \\ \tilde{A} & \tilde{B}_p \\ \tilde{C}_q & \tilde{D}_{qp} \\ \tilde{C}_z & D_{zp} \end{pmatrix}' \begin{pmatrix} 0 & \mathcal{X} & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ \tilde{A} & \tilde{B}_p \\ \tilde{C}_q & \tilde{D}_{qp} \\ \tilde{C}_z & D_{zp} \end{pmatrix} < 0. \quad (3.18)$$

Proof. Robust stability follows from the observation that

$$\begin{pmatrix} \tilde{C}_z & D_{zp} \end{pmatrix}' \begin{pmatrix} \tilde{C}_z & D_{zp} \end{pmatrix} \succeq 0,$$

by which feasibility of (3.18) implies (3.9). Since the latter condition is equivalent to (3.4) by the KYP lemma, robust stability follows from Theorem 3.1.

Remark 3.3 *The presented notion of robust performance (3.16) coincides with the robust \mathcal{H}_2 -performance measure in Section 4.2.2 if the uncertainty is parametric and time-invariant.*

In order to prove robust \mathcal{H}_2 -performance, let us choose some $\Delta \in \mathbf{\Delta}$ and close the loop, i.e. we set $p = \Delta(q)$. Further, we choose some impulsive input $w^\eta(t)$ of the form (3.14), and let $x^\eta(t), p^\eta(t), q^\eta(t), z^\eta(t)$ be the corresponding trajectories. Furthermore, let $x_\Psi(t)$ denote the state of the filter Ψ in (3.10) and let again \tilde{p}, \tilde{q} be defined as the filtered versions of p, q respectively. Then, the response of the weighted system (3.12) is given by

$$\begin{aligned}\dot{\xi} &= \tilde{A}\xi + \tilde{B}_p p, & \xi(0) &= \tilde{B}_w e^\eta, \\ z &= \tilde{C}_z \xi + D_{zp} p,\end{aligned}$$

as well as

$$\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \tilde{C}_q \xi + \tilde{D}_{qp} p,$$

in which the notation $\xi = \text{col}(x, x_\Psi)$ was used, in accordance with realization (3.8). If we now pre- and post multiply (3.17) with the vectors $e^{\eta'}$, e^η respectively, it follows that

$$\xi(0)' \mathcal{X} \xi(0) \leq e^{\eta'} Z e^\eta. \quad (3.19)$$

Further, if we pre- and post multiply (3.18) with

$$\begin{pmatrix} \xi(t) \\ p(t) \end{pmatrix}' \quad \text{and} \quad \begin{pmatrix} \xi(t) \\ p(t) \end{pmatrix}$$

respectively, for some $t \geq 0$, we arrive at

$$\xi(t)' \mathcal{X} \dot{\xi}(t) + \dot{\xi}(t)' \mathcal{X} \xi(t) + \begin{pmatrix} \tilde{p}(t) \\ \tilde{q}(t) \end{pmatrix}' Q \begin{pmatrix} \tilde{p}(t) \\ \tilde{q}(t) \end{pmatrix} + z(t)' z(t) \leq 0.$$

Using the fact that the system is robustly stable and, hence, all signals have finite \mathcal{L}_2 -norm, and using also (3.19), we can integrate the latter expression and get

$$\int_0^\infty z(t)' z(t) dt - e^{\eta'} Z e^\eta + \int_0^\infty \begin{pmatrix} \tilde{p}(t) \\ \tilde{q}(t) \end{pmatrix}' Q \begin{pmatrix} \tilde{p}(t) \\ \tilde{q}(t) \end{pmatrix} dt \leq 0.$$

If we denote by \hat{p}, \hat{q} and \hat{z} the Fourier transformed signals of the \mathcal{L}_2 -signals \tilde{p}, \tilde{q} , and z respectively, this can alternatively be written as

$$\frac{1}{2\pi} \int_{-\infty}^\infty \hat{z}(i\omega)' \hat{z}(i\omega) d\omega - e^{\eta'} Z e^\eta + \frac{1}{2\pi} \int_{-\infty}^\infty \begin{pmatrix} \hat{p}(i\omega) \\ \hat{q}(i\omega) \end{pmatrix}' Q \begin{pmatrix} \hat{p}(i\omega) \\ \hat{q}(i\omega) \end{pmatrix} d\omega \leq 0.$$

Now note that the following relation holds for all $\omega \in \mathbb{R}$

$$\begin{aligned} \begin{pmatrix} \hat{p}(i\omega) \\ \hat{q}(i\omega) \end{pmatrix}' Q \begin{pmatrix} \hat{p}(i\omega) \\ \hat{q}(i\omega) \end{pmatrix} &= \begin{pmatrix} \hat{p}(i\omega) \\ \hat{q}(i\omega) \end{pmatrix}' \Psi(i\omega)' Q \Psi(i\omega) \begin{pmatrix} \hat{p}(i\omega) \\ \hat{q}(i\omega) \end{pmatrix} = \\ &= \begin{pmatrix} \widehat{\Delta(\tilde{q})}(i\omega) \\ \hat{q}(i\omega) \end{pmatrix}' \Psi(i\omega)' Q \Psi(i\omega) \begin{pmatrix} \widehat{\Delta(\tilde{q})}(i\omega) \\ \hat{q}(i\omega) \end{pmatrix}. \end{aligned}$$

From the fact that Δ satisfies the IQC defined by $\Pi = \Psi' Q \Psi$, we get (leaving out the factor $\frac{1}{2\pi}$)

$$\int_{-\infty}^{\infty} \hat{z}(i\omega)' \hat{z}(i\omega) d\omega - e^{\eta'} Z e^{\eta} \leq 0.$$

which implies $\|\hat{z}\| = \|z\| \leq e^{\eta'} Z e^{\eta}$.

Let us finally remind the fact that this relation has been derived for a single impulsive inputs w^η and it actually holds for any other input w^1, \dots, w^{n_w} of the form (3.14). Hence, by summing over η and using the fact that $\sum e^{\eta'} Z e^{\eta} = \text{Tr}(Z) \leq \gamma^2$, we have proven that the system satisfies the performance measure (3.16). ■

3.3 IQC-analysis for time-varying parameters

In order to illustrate the use of dynamic IQC multipliers of the form (3.5) in describing a multitude of different uncertainties, suppose that $\Delta(q) = \delta(t)q(t)$ with $\delta(\cdot)$ being a time-varying parameter. Alternative techniques for analyzing these LPV systems that make use of Lyapunov theory will be developed in Section 4.2.3. With the purpose of comparing the computed upper bound values with results obtained in later sections, we continue our discussion in discrete time. However, a slightly different class of IQC multipliers can be proposed for systems with uncertain time-varying parameters in continuous time.

The IQC analysis result that is discussed rather briefly can be found in [113] in full detail. It is based on first extending the generalized plant, after which the swapping lemma can be applied, see [99, 87]. As a result, one arrives at a standard block diagonal uncertainty structure, as illustrated in Figure 3.3. This structure enables us to characterize a class of admissible multipliers for some given bounds on the parameter and its variation.

Suppose a single time-varying parameter δ_k , $k = 1, 2, \dots$ is characterized by some given region of variation $\mathcal{R} \subset \mathbb{R}^2$, according to Definition 4.1, i.e.

$$(\delta_k, \delta_{k+1} - \delta_k) \in \mathcal{R} \quad \text{for all } k \geq 0.$$

In order to incorporate the parameter variation $\nu_k = \delta_{k+1} - \delta_k$, a new diagonal uncertainty block structure is introduced.

A major observation is the following. The interconnected system on the left of Figure 3.3, in which δ represents any \mathcal{R} -admissible parameter sequence, is equivalent to stability of the interconnected system on the right, where the block diagonal uncertainty is parametric and time-invariant, with $(\delta, \nu) \in \mathcal{R}$.

Along the lines in [113], the structure of the multiplier is chosen as

$$\Pi = \Psi^* Q \Psi \quad \text{with} \quad \Psi = (\Psi_1 | \Psi_2) = \left(\begin{array}{c|c} H_1 & 0 \\ \hline 0 & H_2 \end{array} \right), \quad (3.20)$$

in which $Q \in \mathcal{Q}$ is a parameterized set of matrices that satisfies

$$\left(\begin{array}{cc} \delta I_l & 0 \\ 0 & \nu I_k \\ \hline I & 0 \\ 0 & I \end{array} \right)' Q \left(\begin{array}{cc} \delta I_l & 0 \\ 0 & \nu I_k \\ \hline I & 0 \\ 0 & I \end{array} \right) \succeq 0 \quad \text{for all } (\delta, \nu) \in \mathcal{R}. \quad (3.21)$$

Note that this is a robust LMI constraint of the form (2.3), similar as was obtained in Section 2.1.1. The transfer matrices H_1, H_2 that fix the dynamics of the multiplier Π are constructed as follows. First, a fixed (stable) basis matrix H is defined as

$$H = I_r \otimes \left(\begin{array}{c} 1 \\ (z + \lambda)^{-1} \\ \vdots \\ (z + \lambda)^{-\beta} \end{array} \right) = \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array} \right], \quad (3.22)$$

for some chosen pole location λ in the open unit disc and some degree β . This defines the dimensions k, l in (3.21) as $l = (\beta + 1)r$ and $k = r\beta$. Assume that the realization of H is minimal. Then, the extended transfer matrices are defined as

$$H_1 = \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \\ A_H & B_H \end{array} \right], \quad H_2 = \left[\begin{array}{c|c|c} A_H & B_H & I \\ \hline C_H & D_H & 0 \\ 0 & 0 & I \end{array} \right]. \quad (3.23)$$

Finally, introduce the extensions $M_{qp,e} = (M_{qp} \ 0)$, $M_{zp,e} = (M_{zp} \ 0)$ in which k zero columns are appended to M_{qp}, M_{zp} and let the composed transfer matrix have the following realization:

$$\left(\begin{array}{c|c} \Psi & 0 \\ \hline 0 & I \end{array} \right) \left(\begin{array}{cc} I & 0 \\ M_{qp,e} & M_{qw} \\ \hline 0 & I \\ M_{zp,e} & M_{zw} \end{array} \right) = \left[\begin{array}{c|c|c} \tilde{A} & \tilde{B}_{p,e} & \tilde{B}_w \\ \hline \tilde{C}_q & \tilde{D}_{qp,e} & \tilde{D}_{qw} \\ \tilde{C}_z & \tilde{D}_{zp,e} & D_{zw} \end{array} \right].$$

With these preparations, applying the standard IQC analysis result of Theorem 3.1 on the extended plant leads us to the following result.

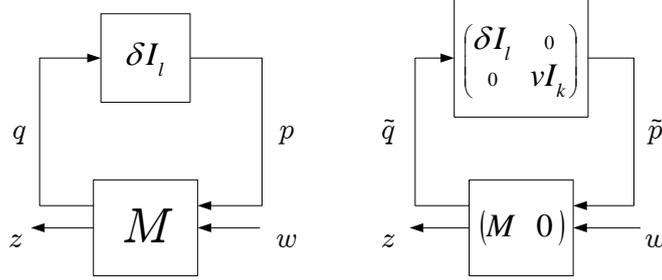


Figure 3.3: Robust stability analysis using the swapping lemma. Robust stability of the left and right interconnection are equivalent.

Theorem 3.3 Consider the interconnection of Figure 3.1 and let M as defined in (3.1) be stable and proper. Let $\Delta(p)_k = \delta_k p_k$ for $k = 0, 1, \dots$ in which the sequence δ is \mathcal{R} -admissible. Choose parameters $\beta, \lambda, |\lambda| < 1$ and construct H_1, H_2 as defined in (3.22) and let the multiplier be parameterized as in (3.20). Then, the interconnection is robustly stable and satisfies quadratic performance on the channel $w \rightarrow z$ if there exist solutions \mathcal{X} and $Q \in \mathcal{Q}$ for which the following LMI holds:

$$(\dots)' \left(\begin{array}{cc|cc} -\mathcal{X} & 0 & 0 & 0 \\ 0 & \mathcal{X} & 0 & 0 \\ \hline 0 & 0 & Q & 0 \\ 0 & 0 & 0 & P_p \end{array} \right) \left(\begin{array}{ccc} I & 0 & 0 \\ \hline \tilde{A} & \tilde{B}_{p,e} & \tilde{B}_w \\ \hline \tilde{C}_q & \tilde{D}_{qp,e} & \tilde{D}_{qw} \\ \hline 0 & 0 & I \\ \tilde{C}_z & \tilde{D}_{zp,e} & D_{zw} \end{array} \right) \prec 0. \quad (3.24)$$

Proof. Robust stability follows from Theorem 3.1. A proof as well as more details can be found in [113, 115]. ■

Condition (3.21) is generally non-tractable and relaxation schemes are needed for implementing the inclusion $Q \in \mathcal{Q}$. In the previous chapter we have extensively discussed this issue, and we have noticed that it is relatively easy to construct relaxation schemes for polytopic regions \mathcal{R} by using convexity arguments. If \mathcal{R} is described by polynomial inequalities, as in case of ellipsoidal regions, sum-of-squares relaxations are needed see also Section 2.2 and [61, 158].

Remark 3.4 Observe that \mathcal{X} enters condition (3.24) in a different fashion than we have seen so far. This is because of the fact that the frequency domain inequality, as it appears in Theorem 3.1, is imposed on the unit circle, rather than on the imaginary axis. This again corresponds to the fact that the analysis conditions have been formulated in discrete time. As shown in [115], a continuous time version of Theorem 3.3 exists when using slightly different transfer matrices H_1, H_2 .

3.4 Numerical example

Let δ denote an uncertain parameter sequence and consider the discrete-time LTI system

$$\begin{pmatrix} x_{k+1} \\ z_k \\ q_k \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ -0.5 & 0.5 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) \begin{pmatrix} x_k \\ w_k \\ p_k \end{pmatrix}, \quad (3.25)$$

that is interconnected with δ . With the relation

$$p_k = \delta_k q_k, \quad \delta_k \in [-r, r] \quad \text{for } k = 1, 2, \dots,$$

the uncertain system corresponds to the LPV system

$$\begin{aligned} x_{k+1} &= \begin{pmatrix} 0 & 1 \\ -0.5 & -0.5 + \delta_k \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w_k, \quad x_0 = \bar{x}_0 \\ z_k &= \begin{pmatrix} 0 & 1 \end{pmatrix} x_k. \end{aligned} \quad (3.26)$$

The goal in this example is to compute upper bounds γ on the worst-case l_2 -gain from $w \rightarrow z$ for different parameter bounds $|\delta| \leq r$. The parameter is allowed to vary between the extreme points of $\delta = [-r, r]$ without any additional constraints on the variation $\delta_{k+1} - \delta_k$. Following the lines of the previous section, we apply Theorem 3.3 for the performance index matrix

$$P_p = \begin{pmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma} I \end{pmatrix}.$$

The dynamic multiplier in (3.20) is constructed for pole-location $\lambda = 0$. The inclusion $Q \in \mathcal{Q}$, needed to parameterize the multiplier class, is implemented by employing a convex hull relaxation scheme from Section 2.3.1. That is, we require the lower right block of Q to be negative definite and impose (3.21) on the four generators

$$\begin{aligned} (\delta, \nu) &= (r, -2r), & (\delta, \nu) &= (r, 0), \\ (\delta, \nu) &= (-r, -2r), & (\delta, \nu) &= (-r, 0). \end{aligned} \quad (3.27)$$

Note that this region displays the fact that, for discrete time LPV systems, the variation ν is trivially bounded, see also Figure 4.1.

Upper bounds on the worst-case l_2 -gain are shown in Figure 3.4 for a number of multiplier degrees β in (3.22). In addition, the upper bound values for the parameter value $p = 0.45$ are listed in Table 3.1. Observe that the upper bounds improve when the degree β is increased. However, it turns out that even if setting $\beta = 10$, the upper bounds of the IQC analysis test do not get lower than about 18.4. The use of less conservative Pólya relaxations does not provide any help. At this point,

it is unknown how we can compute lower bound values, though we will come back to this issue in Section 4.3.

	β	# LMIs	# Vars	CPU time [s]	γ for $r = 0.45$
UB-0	-	4	8	2.80	88.56
UB-1	-	6	47	3.24	16.78
IQC-1	1	6	32	3.07	39.62
IQC-2	2	6	77	3.02	18.44
IQC-3	3	6	142	3.43	18.43

Table 3.1: l_2 -gain upper bounds based on Theorem 3.3 ($\lambda = 0$) and comparison with relaxations UB-0, UB-1 from Section 4.3.

In Table 3.1 we have also listed the CPU computation times, as they are obtained by using Sedumi [170] on a Pentium 4 with 2.4 GHz. The numbers do not include the computation time that is needed for constructing the LMI constraints that define the relaxation scheme, which is in the order of 1-2 seconds.

The labels UB-0 and UB-1 correspond to the Lyapunov based analysis approach of Section 4.3, which addresses the same numerical example. The analysis test UB-0 involves a parameter independent Lyapunov matrix, whereas a parameter dependent Lyapunov matrix was used in UB-1. In view of the less conservative bounds provided by UB-1, we can conclude that none of the upper bound values that are computed in this section are exact.

In order to demonstrate the importance of specifying a bound on the parameter variation, even if it is the trivial one in (3.27), we finally consider the region defined by the generators

$$\begin{aligned} (\delta, \nu) &= (r, -1000), & (\delta, \nu) &= (r, 0), \\ (\delta, \nu) &= (-r, -1000), & (\delta, \nu) &= (-r, 0). \end{aligned}$$

With a convex hull relaxation similar to the one used in IQC-1, the relaxation scheme is referred to as IQC-1000. The resulting upper bound values are shown in Figure 3.4 and are conservative, as expected. Moreover, the computed values happen to be exactly equal to the upper bounds obtained in relaxation UB-0, a relaxation scheme which didn't incorporate any parameter variation bounds. Thus, it is important to incorporate known bounds on the parameter variation, both in the IQC framework, as well as in the Lyapunov function approach that is covered in our next chapter.

Remark 3.5 *Theorem 3.3 provides a sufficient condition for stability and performance. As mentioned in [115], the family of relaxations parameterized by the degree β is asymptotically exact in case Δ is a dynamically time-varying operator. For time-varying parametric uncertainties, which are memoryless, it is unknown whether such guarantees on asymptotic exactness can be given.*

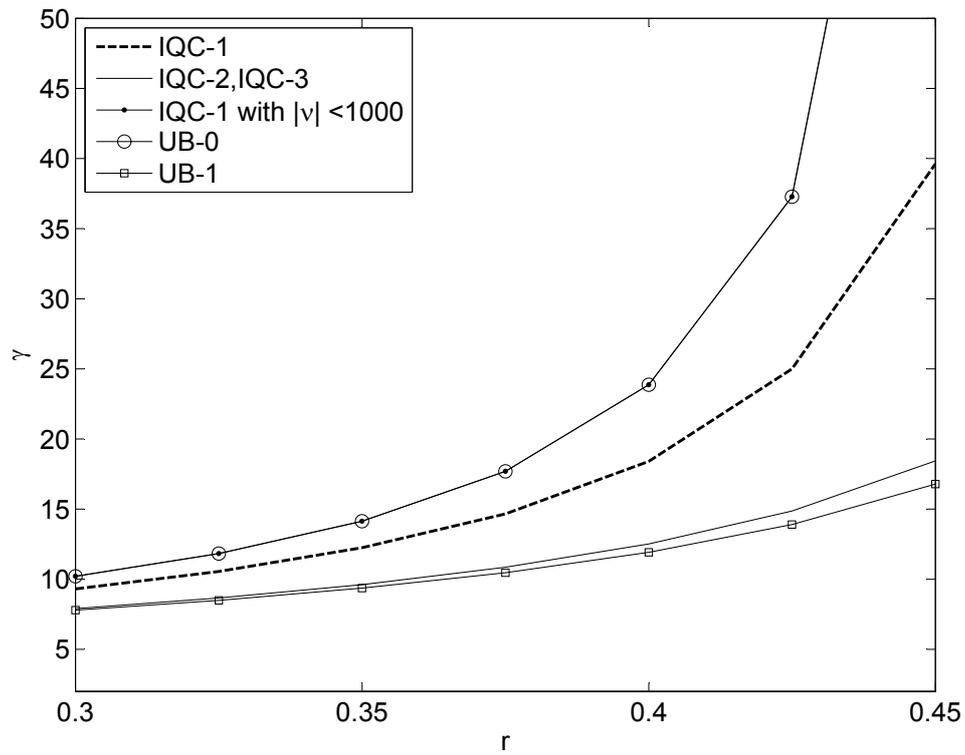


Figure 3.4: Computed upper bounds with IQC analysis and comparison with relaxation UB-0,UB-1, that will be computed in Section 4.3. All values are obtained by using a convex hull relaxation with full block multipliers.

3.5 Summary

We have given a brief recap on the theory on integral quadratic constraints for the analysis of uncertain systems. In the IQC analysis approach, the uncertain system consists of an LTI plant that is interconnected with an uncertain element. The uncertain operator is described via an integral quadratic constraint and a set of admissible multipliers. The framework allows for general non-linear and time-varying uncertainties.

Analysis conditions for robust stability and performance were given in terms of frequency domain inequalities as well as LMI feasibility problems. For the particular case of having time-varying parametric uncertainties only, a suitable class of multipliers was presented that was taken from the literature. This particular multiplier class has been applied on an academic example, in which the goal was to compute upper bounds on the induced l_2 -gain of a discrete-time LPV system.

Chapter 4

Analysis of discrete-time LPV systems

Linear parameter varying (LPV) systems naturally arise in engineering applications. By considering a linear system whose system matrices depend on some time-varying parameters, we can capture uncertain time-varying or, to some extent, non-linear behavior, see e.g. [152]. As mentioned in the introduction, there are many open questions concerning LPV systems. This chapter addresses the stability and performance analysis problem for LPV systems that are described as

$$\begin{aligned}x_{k+1} &= A(\delta_k)x_k + B(\delta_k)w_k & x_0 &= \bar{x}_0, & k &= 0, 1, 2, \dots, \\z_k &= C(\delta_k)x_k + D(\delta_k)w_k,\end{aligned}\tag{4.1}$$

in which $x_k \in \mathbb{R}^n$ denotes the state, $w_k \in \mathbb{R}^{n_w}$ the disturbance, $z_k \in \mathbb{R}^{n_z}$ the controlled output and $\delta_k = (\delta_{k,1}, \dots, \delta_{k,s}) \in \mathbb{R}^s$ the (time-varying) parameter vector. As will become clear in this chapter, treating LPV systems in the discrete time setting has certain advantages, in particular what concerns the search for destabilizing or worst-case parameter sequences.

Since the behavior of the LPV system (4.1) may heavily depend on the parameter sequence that is considered, it is essential that we characterize the nature of the parameter variation. In this thesis, this is done in terms of the sequence ν with $\nu_k = \delta_{k+1} - \delta_k \in \mathbb{R}^s$, $k = 0, 1, \dots$. A more complex characterization of the set of admissible parameter sequences, e.g. by including bounds on $\delta_{k+j} - \delta_k$ for values $j > 1$, is possible, though this will strongly affect the complexity of the resulting numerical optimization problem.

Definition 4.1 (Admissible parameter variation in discrete-time) *For a given compact set $\mathcal{R} \subset \mathbb{R}^{2s}$, called a region of variation, any sequence $(\delta_k)_{k=1,2,\dots}$ with rate-of-variation $\nu_k = \delta_{k+1} - \delta_k$ is called \mathcal{R} -admissible if $(\delta_k, \nu_k) \in \mathcal{R}$ for all $k \geq 0$.*

Clearly, by imposing $\nu_k = 0$ for all k the behavior of system (4.1) becomes time-invariant. In case that no bound on the rate-of-variation of the parameters is specified we will alternatively write $\delta \in \mathcal{D}$, with \mathcal{D} defined as

$$\mathcal{D} := \{\delta \in \mathbb{R}^s \mid (\delta, \cdot) \in \mathcal{R}\}, \quad (4.2)$$

rather than using the more cumbersome notation $(\delta, \cdot) \in \mathcal{R}$.

Assumption 4.1 *The region of variation \mathcal{R} is compact.*

For discrete-time LPV systems with the parameter values taken from a compact set, this assumption is automatically satisfied. In fact, for component-wise intervals $\mathcal{D}_i = [-\alpha_i, \alpha_i]$ with $\alpha_i \in \mathbb{R}$, this is easy to see since $|\nu_{k,i}| = |\delta_{k+1,i} - \delta_{k,i}| < 2\alpha_i$ for all $i = 1, \dots, s$. Hence, a discrete-time LPV system exhibits a natural bound on the rate of variation.

In view of the fact that the parameter variation is characterized in terms of the difference of two sequential parameter values, an alternative description of the region of variation \mathcal{R} is in terms of the variables (δ, θ)

$$\hat{\mathcal{R}} := \{(\delta, \theta) \mid (\delta, \theta - \delta) \in \mathcal{R}\}. \quad (4.3)$$

It immediately follows that any tuple (δ_k, δ_{k+1}) , $k \geq 0$ taken from an \mathcal{R} -admissible parameter sequence $(\delta_k)_{k=1,2,\dots}$ is an element of $\hat{\mathcal{R}}$. Note that it is merely a matter of parametrization, any region \mathcal{R} can be transformed into its counterpart $\hat{\mathcal{R}}$ and vice versa. For a particular polytopic region the two parametrizations are shown in Figure 4.1. In Section 4.1.3, we will further motivate why we prefer to use $\hat{\mathcal{R}}$ rather than \mathcal{R} .

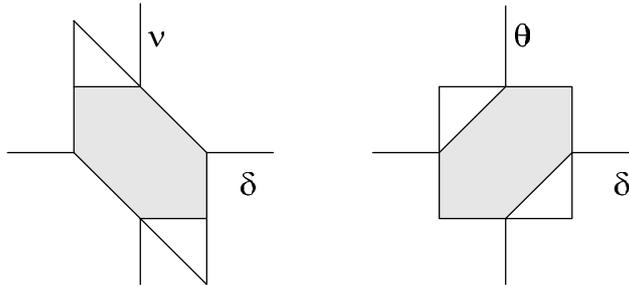


Figure 4.1: Two different descriptions of region of parameter variation: \mathcal{R} (left) and $\hat{\mathcal{R}}$ (right). The grey box corresponds to a rate of variation $\nu \leq \alpha$, if α represents the bound $|\delta| \leq \alpha$.

Let us recall the definition of exponential stability for LTV systems in discrete time, which involves the autonomous system $x_{k+1} = A_k x_k$. For a reference, see for example [105, 85].

Definition 4.2 *The autonomous system $x_{k+1} = A_k x_k$ is exponentially stable if there exist positive scalars m and $\lambda < 1$ such that all sequences $(x_k)_{k=0,1,\dots}$ of the system satisfy*

$$\|x_k\| \leq m\lambda^k \|x_0\|$$

for all $k \geq 0$.

Let us outline this chapter. Lyapunov-based analysis techniques will be developed for the analysis of discrete-time LPV systems. In the next section, a sufficient condition for stability of the LPV system (4.1) is given in terms of a robust LMI. With only few results on Lyapunov converse theorems available, we typically construct conditions for stability that are sufficient only.

In Section 4.1.3, we propose a conceptually simple alternative method based on the so-called monodromy matrix. A family of necessary and sufficient stability conditions will be developed, each in the form of a robust SDP. One of the main results, found in Section 4.1.4, is a generalization of the joint spectral radius, which has been derived for switched systems, to general LPV systems. In addition, we propose sufficient conditions for instability of the LPV system in Section 4.1.5 that are similar to the generalized spectral radius, which has originally been introduced for switched systems, see [116, 180]. Finally, the performance analysis problem is considered in Section 4.2, and a numerical example will be used to illustrate the method.

4.1 Lyapunov stability analysis

As mentioned in the example of Section 2.1, stability of an uncertain LTI system can be captured in terms of the structured singular value. Unfortunately, for time-varying systems it is only known how to derive (a priori) sufficient conditions for stability. In other words, numerical computations often give an imprecise answer to the analysis problem, even if the relaxation gap of a particular relaxation scheme is reduced to zero. As a consequence, the \mathcal{R} -stability margin for a given LPV system, defined as the largest r for which stability can be proven for the scaled region $r\mathcal{R}$, will depend on the stability condition that is chosen.

Consider the following autonomous LPV system that depends on a time-varying parameter vector $(\delta_k)_{k=1,2,\dots}$, with $\delta_k = (\delta_{k,1}, \dots, \delta_{k,s})$, that is characterized by the region of variation $\mathcal{R} \subset \mathbb{R}^{2s}$:

$$x_{k+1} = A(\delta_k)x_k \quad \text{with} \quad x_0 = \bar{x}_0. \quad (4.4)$$

Already for the case that $A(\cdot)$ is an affine matrix function, analyzing robust stability of system (4.4) is a hard problem. A sufficient condition for asymptotic stability of an uncertain system is, roughly speaking, the existence of a single quadratic form $V(x) = x^T X x$ that is positive definite and strictly decreases along all solutions $(x_k)_{k=1,2,\dots}$ that obey the system equations. Any function with these two properties

is called a Lyapunov function for the considered system. Our notion of stability of an LPV system is based on Definition 4.2, assuming that the constants $m, \lambda < 1$ do not depend on the way the parameter varies in time, which is referred to as *uniform exponential stability*.

Lemma 4.1 *Let a compact set $\delta \subset \mathbb{R}^s$ be given. The LPV system (4.4) with parameter sequences $(\delta_k)_{k=1,2,\dots}$ satisfying $\delta_k \in \delta$ for all $k \geq 0$ is uniformly exponentially stable if there exists $X \succ 0$ with*

$$A(\delta)'XA(\delta) - X \prec 0 \quad \forall \delta \in \delta. \quad (4.5)$$

Remark 4.1 *If (4.5) is satisfied for some $X \succ 0$, the system is often called quadratically stable. With $X \succ 0$ being a solution of (4.5), the quadratic form $V(x) = x^T X x$ serves as a Lyapunov function for any \mathcal{R} -admissible parameter sequence, which clarifies the terminology.*

Note that the sufficient condition for stability in Lemma 4.1 does not involve the parameter variation ν . In fact, it is not difficult to show that the LPV system is indeed uniformly exponentially stable for arbitrary time-varying parameter sequences, see [54].

Condition (4.5) is a robust SDP constraint in the decision variable X , and can be rewritten as

$$\begin{pmatrix} I \\ A(\delta) \end{pmatrix}' \begin{pmatrix} -X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I \\ A(\delta) \end{pmatrix} \prec 0, \quad \forall \delta \in \delta. \quad (4.6)$$

As explained in Chapter 2, feasibility of (4.6) is numerically tractable in some particular cases, e.g. if $A(\cdot)$ is affine and if $\delta = \delta_1 \times \delta_2 \cdots \times \delta_s$, for some specified intervals δ_i for $i = 1, \dots, s$. In general, relaxation schemes need to be constructed for solving (4.6), and conservatism is introduced.

Apart from a possibly non-zero relaxation gap, the stability margin that is found by applying Lemma 4.1 is often very conservative. This is due to the fact that not every robustly stable uncertain system admits a quadratic Lyapunov function $x^T X x$, see for example [50]. Hence, one typically searches over an enlarged class of Lyapunov functions $V(x, k) := x^T X(\delta_k)x$, in which the Lyapunov matrix $X(\delta)$ depends on the parameters.

Lemma 4.2 *The LPV system (4.4) is uniformly exponentially stable if there exists matrix function $X(\cdot)$ (not necessarily continuous) such that $X(\delta) \succ 0$ for all $\delta \in \delta$ and*

$$\begin{pmatrix} I \\ A(\delta) \end{pmatrix}' \begin{pmatrix} -X(\delta) & 0 \\ 0 & X(\theta) \end{pmatrix} \begin{pmatrix} I \\ A(\delta) \end{pmatrix} \prec 0, \quad \forall (\delta, \theta) \in \mathcal{R}. \quad (4.7)$$

Proof. With $X(\cdot) \succ 0$ being a solution to (4.7), the function $V(x, k) = x^T X(\delta_k)x$ becomes a Lyapunov function for the LPV system (4.4), see for example [77]. ■

Note that variation of the parameters is explicitly taken into account once the Lyapunov function depends on the parameters. In order to render (4.7) numerically tractable, we must always parameterize $X(\cdot)$. For instance, with the monomials $\mu_j(\delta) = \delta_1^{\alpha_1} \delta_2^{\alpha_2} \dots \delta_s^{\alpha_s}$ with multi-degrees $\alpha_1, \dots, \alpha_m$, a possible parameterized polynomial Lyapunov matrix reads as

$$X(\delta) = X_0 + X_1\mu_1(\delta) + \dots + X_{s_d}\mu_m(\delta). \quad (4.8)$$

We will elaborate further on this topic in Section 4.1.6.

Remark 4.2 [*Polytopic system*] *The parametrization of $X(\cdot)$ is often chosen to 'mirror' the corresponding system structure. For example, in polytopic systems with N generators, i.e. $\delta = \text{co}\{\delta^1, \dots, \delta^N\}$, the Lyapunov function is defined with $X(\delta) = \sum_{i=1}^N X_i \delta_i$ in which $\delta_i \geq 0$ and $\delta_1 + \dots + \delta_N = 1$.*

4.1.1 A survey on parameter dependent Lyapunov functions

One of the first results on parameter dependent Lyapunov functions is perhaps [47], in which multi-affine Lyapunov matrices were considered for a family of matrices of the form $A_0 + A_1\delta_1 + \dots + A_s\delta_s$ with A_1, \dots, A_s having rank one. In [77], the rank constraint on A_1, \dots, A_s was dropped and LMI conditions are derived for the existence of a multi-affine Lyapunov function. The proposed relaxation was based on multi-convexity arguments, which are a variation of the convex hull arguments proposed in Section 2.3.1. In a similar setting, Pólya's relaxation was applied in [129]. A nice survey article on multi-affine parameter dependent Lyapunov functions and implementation issues is [5].

Polynomially dependent Lyapunov matrices for polytopic systems have been considered in [119, 139, 22]. In [64], the Lyapunov matrix is allowed to depend on the parameters in a rational fashion, provided that a particular parametrization is used. As for multi-affine parameter dependent Lyapunov matrices, the stability conditions still require the construction of a relaxation scheme, and it is unlikely that an (a priori known) exact solution in terms of an LMI problem exists.

Although non-quadratic Lyapunov functions are discussed in more detail in the next section, the recent work done in [36] is worth mentioning. As an extension to the stability condition (4.6), the class of Lyapunov functions are higher order (homogeneous) in the state as well as higher order (homogeneous) in the parameters. For polytopic LPV systems the analysis conditions could be turned into a robust SDP.

4.1.2 Non-conservative Lyapunov-based analysis

So far, we have seen how to construct sufficient conditions for stability by making use of quadratic-in-the-state Lyapunov functions. The exact stability radius of an LPV

system is difficult to compute. Even by employing parameter dependent Lyapunov matrices, the stability conditions can be excessively conservative. In this section we address the use of non-quadratic Lyapunov functions for analyzing stability of the LPV system.

In order to have a decisive answer to the question as to whether the outcome of stability analysis tests based on Lyapunov functions is exact, so-called Lyapunov converse theorems have been explored. These theorems involve the definition of a class of Lyapunov functions for a given system or family of systems, which is theoretically proven to be non-restrictive. An example of a well-known non-restrictive class of functions for LTI systems is the family of all quadratic forms $V(x) = x^T X x$ with $X \succ 0$. Moreover, for the uncertain LTI system $\dot{x} = A(\delta)x$ with $\delta \in \mathcal{D}$ being a time-invariant parameter vector in some compact set \mathcal{D} and $A(\cdot)$ being continuous, the class of quadratic forms $V(x, \delta) = x^T X(\delta)x$ with *polynomial* $X(\cdot)$ can be proven to be non-restrictive as well, see [23]. For non-linear and time-varying uncertain systems, however, converse theorems are rare and their applicability is limited by the lack of efficient numerical algorithms. Let us discuss in somewhat more detail two function classes that appear in the literature.

Remark 4.3 *It is stressed that, for a fixed parameter sequence, it is known how to construct time-varying Lyapunov functions, see [105]. The difficulty of the LPV analysis problem is caused by the fact that parameters are uncertain as well as time-varying.*

Polyhedral norm and piecewise quadratic Lyapunov functions

Motivated by the stability analysis problem for the absolute stability problem of a Lure system, a non-restrictive class of polyhedral norm Lyapunov functions has been proposed in [32, 108, 127]. Polyhedral norms have recently been also used in the context of robust control, see [20, 21].

A manageable parametrization of a subclass of polyhedral norms is proposed in [127] (part I), in terms of piecewise quadratic Lyapunov functions, in which finitely many disjoint sectors partition the state-space. Unfortunately, even for the simplest polytopic LPV systems, the search over the class of polyhedral norms is non-convex. Although an efficient numerical implementation is lacking, a procedure based on linear programming has been proposed in [142, 187] and references therein.

Polyhedral norms can be viewed as piecewise quadratic Lyapunov functions. These have independently been considered by many authors. In [47], piecewise quadratic Lyapunov functions have been used for the Lur'e system whereas in [141] a single non-linear autonomous system was addressed. In the context of uncertain systems, see for example [20, 189, 186].

Piecewise quadratic Lyapunov functions are typically represented as

$$V(x, P) = x^T P x \quad P \in \mathcal{P}, \quad \text{with } \mathcal{P} \text{ a compact set of matrices.}$$

In (4.8) we have already seen an explicit method for describing the set \mathcal{P} by a matrix function and a parameter set δ . Depending on the problem, an alternative parametrization is sometimes preferred that involves a partitioning of the state-space into a finite number of segments.

Summarizing, converse theorems provide function classes that, at least theoretically, allow for an exact computation of the stability margin. Their practical use is often limited by the lack of an efficient implementation. Furthermore, it is generally unknown how to obtain a priori bounds on the conservatism for a given finite parametrization of Lyapunov function candidates. From a computational point of view, the currently existing converse theorems have, therefore, little value for the analysis of LPV systems.

Higher order in the state Lyapunov functions

The so-called homogeneous, polynomial in the state Lyapunov functions constitute a function class for which stability analysis can be turned into LMI conditions, see [189, 35]. More recently, in [36], it was shown to be relatively easy to also allow for higher order polynomial dependence in both the parameter and the state. This class of functions is known to reduce conservatism in particular examples, though it is unknown whether it consist of a non-restrictive class of Lyapunov function candidates.

Remark 4.4 *Switched systems form a subclass of LPV systems and belong to the class of hybrid systems. These kind of systems have mixed continuous and discrete dynamics which typically arise from the multiple modes in which the system can operate. In robotic manipulators that either move freely in space or are in contact with a surface such modes are naturally present. Alternatively, a transition between different operation modes can be due to human intervention as it is common in most computer controlled systems like cars, aircraft or copy machines, for reasons of safety. A special class of thoroughly studied switched systems is described by a finite set of matrices, see for example [43, 31, 186, 50, 120] and reads as*

$$x_{k+1} = A_k x_k, \quad A_k \in \mathcal{A} \quad \text{for} \quad k = 0, 1, 2 \dots$$

When \mathcal{A} is a finite and bounded set of matrices, this system can easily be represented as an LPV system. In fact, there always exists a set of s -dimensional vectors $\delta_1, \delta_2, \dots$ and a continuous mapping $A(\cdot)$ that satisfies $A_i = A(\delta_i)$ for all $i \in \mathcal{I}$. By defining the compact set $\delta = \{\delta_i, i \in \mathcal{I}\}$, the switched system is shown to be a particular instance of the LPV system (4.4).

It is stressed that autonomous switched systems, in which the system dynamics

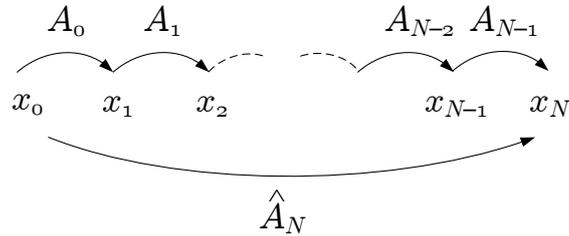


Figure 4.2: The lifting approach for horizon N .

change when the state hits a certain boundary, are typically treated differently than systems in which an external (manual) source is responsible for the switching behavior. In these kind of systems, the parameters δ are a function of the state, which is sometimes referred to as a quasi-LPV systems, see [31]. In this thesis, parameter sequences are always assumed to be determined externally.

So far, stability conditions have been formulated directly in terms of the system matrix $A(\delta)$. As will be shown in the next section, less conservative Lyapunov-based stability analysis conditions can be derived by considering the so-called lifted system. We will actually derive a necessary and sufficient condition for stability, without the need to introduce non-quadratic Lyapunov functions, by constructing a family of robust LMI problems based on the monodromy matrix.

4.1.3 Stability of LPV systems using a lifting approach

In this section, sufficient conditions for stability of general discrete-time LPV systems are derived in terms of the monodromy matrix \hat{A}_N . The approach stems from a well-known lifting procedure for input/output systems that is discussed in Section 4.2, where the robust performance analysis problem is addressed. As we will show, the analysis of \hat{A}_N also extends the notion of the joint spectral radius, as originally defined for switched system, towards LPV systems with general regions of variation.

The notion of lifting is not new and provides an equivalence between discrete-time periodic systems and LTI systems, see [73, 125] and references therein. Due to the strong link between periodic LTV and LTI systems, the design tools originally developed for LTI system became available for the analysis and design of periodic systems. In particular, the lifting technique has resulted in a clear understanding of the analysis and design sampled-control systems, see [11, 107, 126] and references therein.

For the given autonomous LPV system (4.4), define $\xi_t = x_{Nt}$ for $t = 0, 1, \dots$ as well as

$$\hat{\delta}_t = \begin{pmatrix} \delta_{Nt} \\ \delta_{Nt+1} \\ \vdots \\ \delta_{N(t+1)-1} \end{pmatrix} = \begin{pmatrix} \hat{\delta}_{t,1} \\ \hat{\delta}_{t,2} \\ \vdots \\ \hat{\delta}_{t,N} \end{pmatrix}, \quad (4.9)$$

for any given \mathcal{R} -admissible sequence $(\delta_k)_{k=0,1,2,\dots}$. Note that, in the sequel, any vector $\hat{\delta} \in \mathbb{R}^{Ns}$ is assumed to be partitioned as $\hat{\delta} = \text{col}(\hat{\delta}_1, \dots, \hat{\delta}_N)$ with $\hat{\delta}_i \in \mathbb{R}^s$ for $i = 1, \dots, N$. For a given region of variation \mathcal{R} let us further introduce the set

$$\hat{\mathcal{R}}_N = \{ \hat{\delta} \in \mathbb{R}^{Ns} \mid (\hat{\delta}_{i+1}, \hat{\delta}_{i+1} - \hat{\delta}_i) \in \mathcal{R}, \quad i = 1, \dots, N-1 \}. \quad (4.10)$$

Clearly, this construction implies that a sequence $(\delta_k)_{k=0,1,\dots}$ is \mathcal{R} -admissible if and only if the corresponding sequence $(\hat{\delta}_t)_{t=1,2,\dots}$ is $\hat{\mathcal{R}}_N$ -admissible. Note that in view of earlier definitions (4.2) and (4.3), it follows that $\hat{\mathcal{R}} = \hat{\mathcal{R}}_1$ and $\hat{\mathcal{R}} = \hat{\mathcal{R}}_2$. Finally, we define the following matrix function, which is also known as the *monodromy* matrix:

$$\hat{A}_N(\hat{\delta}) := A(\hat{\delta}_N) \cdots A(\hat{\delta}_2)A(\hat{\delta}_1). \quad (4.11)$$

Then, the following system is referred to as the *N-lifted system*:

$$\xi_{t+1} = \hat{A}_N(\hat{\delta}_t)\xi_t, \quad \hat{\delta}_t \in \hat{\mathcal{R}}_N, \quad \xi_0 = \bar{x}_0, \quad t = 0, 1, \dots \quad (4.12)$$

Remark 4.5 For box regions $\hat{\mathcal{R}}$, the set $\hat{\mathcal{R}}_N$ is exactly equal to the direct product of N copies of $\hat{\mathcal{R}}$ for which an explicit characterization can be readily computed. For general polytopic regions with rate-of-variation bounds, $\hat{\mathcal{R}}_N$ is an (implicitly described) subset of $\hat{\mathcal{R}} \times \dots \times \hat{\mathcal{R}}$. It may not always be easy to find reliable algorithms for the computation of a set of generators of $\hat{\mathcal{R}}_N$.

The next proposition contains a sufficient condition for stability of LPV systems, which is based on the N -lifted system. The use of the monodromy matrix \hat{A}_N for the analysis of stability of LPV systems has been considered in for example [57], and more recently in [52]. It is further illustrated in Figure 4.2, where the abbreviation $A(\hat{\delta}_i) = A_i$, for $i = 0, \dots, N-1$ has been used.

Proposition 4.1 *The LPV system (4.4) is uniformly exponentially stable if for some $N \in \mathbb{N}$ there exists some λ such that*

$$\max_{\hat{\delta} \in \hat{\mathcal{R}}_N} \|\hat{A}_N(\hat{\delta})\| \leq \lambda < 1. \quad (4.13)$$

Moreover, stability of the systems (4.4) and (4.12) are equivalent.

Proof. Let us be given an \mathcal{R} -admissible $(\delta_k)_{k=0,1,2,\dots}$ and an initial condition x_0 . Construct the sequences $(\hat{\delta}_t)_{t=0,1,\dots}$ in (4.9) and $(\xi_t)_{t=0,1,\dots}$ with $\xi_t = x_{Nt}$ for all

$t \geq 0$. Then, condition (4.13) implies that there exists some $\epsilon > 0$ such that

$$\hat{A}_N(\hat{\delta}_t)' \hat{A}_N(\hat{\delta}_t) - I \preceq -\epsilon I \quad (4.14)$$

holds for all $t = 0, 1, \dots$. Multiplying from left and right with ξ_t', ξ_t respectively, we get

$$\xi_t' \hat{A}_N(\hat{\delta}_t)' \hat{A}_N(\hat{\delta}_t) \xi_t - \xi_t' \xi_t \leq -\epsilon \xi_t' \xi_t, \quad t = 0, 1, \dots$$

which implies

$$\|\xi_{t+1}\|^2 \leq (1 - \epsilon) \|\xi_t\|^2, \quad t = 0, 1, \dots$$

and hence $\|\xi_t\| \leq (1 - \epsilon)^{t/2} \|\xi_0\|$. Note that by compactness of $\hat{\mathcal{R}}_N$, there exists $\kappa \in \mathbb{R}$ for which

$$\|\hat{A}_i(\hat{\delta})\| \leq \kappa, \quad \text{for all } \hat{\delta} \in \hat{\mathcal{R}}_N, \quad i = 0, 1, \dots, N.$$

Let us now consider any x_k in the sequence $(x_k)_{k=0,1,2,\dots}$ and observe that there always exists N and $i < N$ such that $k = Nt + i$, with t, i being integers. It then follows that

$$\|x_k\| = \|x_{Nt+i}\| \leq \kappa \|x_{Nt}\| = \kappa \|\xi_t\| \leq (1 - \epsilon)^{t/2} \|\xi_0\| = (1 - \epsilon)^{t/2} \|x_0\|,$$

which proves that the sequence $(x_k)_{k=0,1,\dots}$ converges exponentially to zero. Since the parameter sequence $(\delta_k)_{k=1,2,\dots}$ and the initial condition \bar{x}_0 were chosen arbitrarily, we have proven uniform exponential stability of the system (4.4). Stability of (4.4) clearly implies stability of (4.12), since $(\xi_k)_{k=1,2,\dots}$ is a subsequence of $(x_k)_{k=1,2,\dots}$. ■

Uniform exponential stability of the N -lifted system (4.12) can be turned into a robust SDP in several ways. One possible approach is to directly impose (4.13), or equivalently, the robust LMI constraint

$$\begin{pmatrix} I & \hat{A}_N(\hat{\delta})' \\ \hat{A}_N(\hat{\delta}) & I \end{pmatrix} \succ 0 \quad \text{for all } \hat{\delta} \in \hat{\mathcal{R}}_N. \quad (4.15)$$

Once an LFR of the original matrix function $A(\cdot)$ is given, the matrix function $\hat{A}_N(\cdot)$ can be computed for any fixed value of N and (4.15) becomes a robust LMI constraint.

Instead of using a norm bound for proving stability of the lifted system (4.12), one can improve the analysis condition and verify stability of the lifted system by standard Lyapunov arguments. In fact, the N -lifted system is uniformly exponentially stable if there exists some $X(\hat{\delta})$ such that

$$\begin{pmatrix} I \\ \hat{A}_N(\hat{\delta}) \end{pmatrix}' \begin{pmatrix} -X(\hat{\delta}) & 0 \\ 0 & X(\hat{\theta}) \end{pmatrix} \begin{pmatrix} I \\ \hat{A}_N(\hat{\delta}) \end{pmatrix} \prec 0 \quad \text{for all } (\hat{\delta}, \hat{\theta}) \in \hat{\mathcal{R}}_{2N}. \quad (4.16)$$

The pair $(\hat{\delta}, \hat{\theta})$ represents a tuple of $2N$ parameter vectors that satisfies the parameter variation bounds in (4.10). The analysis condition (4.16) reduces to a robust SDP, provided that a linear parametrization of $X(\delta)$ is chosen, as for example in (4.8).

Remark 4.6 *An alternative way to describe parameter variation bounds is to introduce variables $\hat{\nu}_k = \hat{\delta}_{k+1} - \hat{\delta}_k$, similar as was done in our definition of \mathcal{R} in Definition 4.1. Although the stability analysis results in this chapter still hold if the lifted region of variation is described in terms of $(\hat{\delta}, \hat{\nu})$, this representation is unnatural. Note that the vector $\hat{\nu}_{k,i} = \delta_{N(k+1)+1} - \delta_{Mk+i}$ no longer represents parameter variation in the sense of Definition 4.1, leading to a more complicated description of the region of variation.*

As discussed in Chapter 2, (approximate) solutions of a robust SDP problem are provided by so-called relaxation schemes. The complexity of such schemes heavily depends on the specified parameter region and grows exponentially in the horizon length N . In the next section, we examine whether the proposed stability condition is not only sufficient but also necessary for stability of the LPV system (4.4).

4.1.4 Asymptotic exactness of the lifted approach to stability analysis of LPV systems

In this section we derive an exact characterization of stability of the LPV system (4.4) in terms of the N -lifted system. This is done by proving that (4.13) becomes both necessary and sufficient if we let N go to infinity.

First, let us formally define an N -tuple $(\hat{\delta}_1, \dots, \hat{\delta}_N)$ to be \mathcal{R} -admissible if there exists an \mathcal{R} -admissible sequence $(\delta_k)_{k=1, \dots, \infty}$ for which $\delta_i = \hat{\delta}_i, i = 1, \dots, N$ holds. For notational simplicity we allow the argument of $\hat{A}_N(\cdot)$ to be the whole sequence, realizing the fact that $\hat{A}_N(\cdot)$ only depends on the first N elements in the sequence.

Assumption 4.2 [*Product-boundedness*] *There exists $\kappa < \infty$ for which*

$$\max_{\hat{\delta} \in \hat{\mathcal{R}}_N} \|\hat{A}_N(\hat{\delta})\| < \kappa \quad \text{for all } N \geq 0. \quad (4.17)$$

Clearly, if (4.17) does not hold, the LPV system cannot be asymptotically stable. A more precise statement is given in our next theorem, which says that asymptotic stability of the LPV system (4.4) is equivalent to having all infinite product of matrices corresponding to \mathcal{R} -admissible parameter sequences converge to the zero matrix.

Lemma 4.3 [*Stability with matrix products*] *Let Assumption 4.2 hold. Then, the LPV system is uniformly exponentially stable if and only if the limit*

$$\hat{A}_\infty(\delta) := \lim_{N \rightarrow \infty} \hat{A}_N(\delta) \quad (4.18)$$

exists and is zero, uniformly in all \mathcal{R} -admissible parameter sequences $\delta = (\delta_k)_{k=1,2,\dots}$.

Proof. Suppose $\hat{A}_\infty(\delta) = 0$ for all \mathcal{R} -admissible parameter sequences. Then, by uniform convergence in δ , there exists some N for which

$$\max_{\hat{\delta} \in \hat{\mathcal{R}}_N} \|\hat{A}_N(\hat{\delta})\| < \frac{1}{2}.$$

If combining this fact with (4.17), the following observation can be made. For arbitrary $k \geq N$, there exists integers i and $j < N$ satisfying $k = iN + j$, such that the following relations hold:

$$\begin{aligned} \max_{\hat{\delta} \in \hat{\mathcal{R}}_k} \|\hat{A}_k(\hat{\delta})\| &\leq \\ \max_{\hat{\delta} \in \hat{\mathcal{R}}_k} \|\hat{A}_j(\hat{\delta}_{iN+j}, \dots, \hat{\delta}_{iN+1}) \hat{A}_N(\hat{\delta}_{iN}, \dots, \hat{\delta}_{(i-1)N+1}) \cdots \hat{A}_N(\hat{\delta}_N, \dots, \hat{\delta}_1)\| &\leq \\ \max_{\hat{\delta} \in \hat{\mathcal{R}}_j} \|\hat{A}_j(\hat{\delta})\| \cdot \left(\max_{\hat{\delta} \in \hat{\mathcal{R}}_N} \|\hat{A}_N(\hat{\delta})\| \right)^i &\leq \kappa \frac{1}{2^i}. \end{aligned} \quad (4.19)$$

Note that any N -tuple which is taken from a \mathcal{R} -admissible k -tuple, $k > N$ is guaranteed to be \mathcal{R} -admissible, see again the definition of $\hat{\mathcal{R}}$ in (4.3). Let us now be given any \mathcal{R} -admissible sequence δ and initial condition x_0 . It then follows that for all $k = iN + j$,

$$\|x_k\| \leq \|\hat{A}_k(\delta)\| \|x_0\| \leq \kappa \frac{1}{2^i} \|x_0\|,$$

which proves uniform exponential stability, since the constants $m = \kappa$ and $\lambda = \frac{1}{2}$ in Definition 4.2 are independent of from the parameter sequence δ .

Conversely, suppose that $\hat{A}_\infty(\delta) \neq 0$ for some \mathcal{R} -admissible sequence or that the limit in (4.18) does not exist. Then, there exists an \mathcal{R} -admissible sequence $(\delta_k)_{k=1,2,\dots}$ and a subsequence $(k_\nu)_{\nu=1,2,\dots}$ such that for some $\epsilon > 0$,

$$\|\hat{A}_{k_\nu}(\delta)\| \geq \epsilon \quad \text{for all } \nu.$$

For each ν one can find an initial condition v_ν with $\|v_\nu\| = 1$ such that

$$\|\hat{A}_{k_\nu}(\delta)v_\nu\| \geq \epsilon \quad \text{for all } \nu.$$

Consider the sequence $v = (v_1, v_2, \dots)$ which converges without loss of generality to \bar{v} . Thus, we get

$$\|\hat{A}_{k_\nu}(\delta)\bar{v}\| = \|\hat{A}_N(\delta)v_\nu\| - \|\hat{A}_{k_\nu}(\delta)(\bar{v} - v_\nu)\| \geq \epsilon - \kappa\|\bar{v} - v_\nu\|$$

Since v_ν converges to \bar{v} , it follows that

$$\limsup_{\nu \rightarrow \infty} \|\hat{A}_{k_\nu}(\delta)\bar{v}\| \geq \epsilon.$$

Hence, with the initial condition $x_0 = \bar{v}$ and the chosen parameter sequence $(\delta_k)_{k=1,2,\dots}$, the sequence $(x_k)_{k=1,2,\dots}$, does not converge to zero. This clearly means that the origin is not an attractor, i.e. the LPV system is not exponentially stable. ■

In view of the goal of this section, which is to prove that the condition in Proposition 4.1 is also necessary for uniform exponential stability of the LPV system as $N \rightarrow \infty$, let us define the function

$$\bar{\sigma}_k(\mathcal{R}) = \max_{\hat{\delta} \in \hat{\mathcal{R}}_k} \|\hat{A}_k(\hat{\delta})\|, \quad (4.20)$$

with $\|\cdot\|$ being any matrix norm. Although $\bar{\sigma}_k(\mathcal{R})$ is actually a function of both the region \mathcal{R} and the matrix function $A(\cdot)$, the latter dependence is omitted. As a preparation for our next theorem, we first prove the following fact.

Lemma 4.4 *For any region \mathcal{R} that satisfies Assumption 4.1, the following equality holds:*

$$\lim_{k \rightarrow \infty} (\bar{\sigma}_k(\mathcal{R}))^{\frac{1}{k}} = \inf_{k \in \mathbb{N}} (\bar{\sigma}_k(\mathcal{R}))^{\frac{1}{k}}. \quad (4.21)$$

Proof. Since \mathcal{R} is compact, we have that

$$\bar{\sigma}_k(\mathcal{R}) \geq 0 \quad \text{is finite for all } k \in \mathbb{N}. \quad (4.22)$$

Moreover,

$$\begin{aligned} \bar{\sigma}_{k+m}(\mathcal{R}) &= \max_{\hat{\delta} \in \hat{\mathcal{R}}_{k+m}} \|\hat{A}_{k+m}(\hat{\delta})\| \leq \\ &\max_{\hat{\delta} \in \hat{\mathcal{R}}_m} \|\hat{A}_m(\hat{\delta})\| \max_{\hat{\delta} \in \hat{\mathcal{R}}_k} \|\hat{A}_k(\hat{\delta})\| \leq \bar{\sigma}_k(\mathcal{R}) \bar{\sigma}_m(\mathcal{R}). \end{aligned} \quad (4.23)$$

If $\bar{\sigma}_k(\mathcal{R}) = 0$ for some k , $\bar{\sigma}_N(\mathcal{R}) = 0$ for all $N \geq k$ and the equality (4.21) immediately follows. If $\bar{\sigma}_k(\mathcal{R}) > 0$ for all k , let us introduce $a_k = \log(\bar{\sigma}_k(\mathcal{R}))$ and consider the sequence $(a_k)_{k=0,1,2,\dots}$ defined by for $k = 0, 1, \dots$, omitting the argument \mathcal{R} in $\bar{\sigma}_k(\mathcal{R})$. The properties (4.22)-(4.23) imply that the sequence is sub-additive, i.e. $a_{k+m} \leq a_k + a_m$. By Fekete's Lemma [72], it follows that

$$\lim_{k \rightarrow \infty} \frac{a_k}{k} = \inf_{k \in \mathbb{N}} \frac{a_k}{k},$$

by which we infer

$$\lim_{k \rightarrow \infty} \log \bar{\sigma}_k^{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{a_k}{k} = \inf_{k \in \mathbb{N}} \frac{a_k}{k} = \inf_{k \in \mathbb{N}} \log \bar{\sigma}_k^{\frac{1}{k}}.$$

Since $\log(n)$ is a monotonously increasing function, the relation (4.21) follows. ■

In view of the relation (4.21), we are allowed to define the following function

$$\bar{\sigma}(\mathcal{R}) := \lim_{k \rightarrow \infty} (\bar{\sigma}_k(\mathcal{R}))^{\frac{1}{k}}. \quad (4.24)$$

This function is a generalization of the *joint spectral radius* of a set of matrices, which is well-known in the context of switched systems. As shown in our next theorem, it completely determines stability of the LPV system (4.4).

Theorem 4.1 *The LPV system (4.4) with region of variation \mathcal{R} is uniformly exponentially stable if and only if $\bar{\sigma}(\mathcal{R}) < 1$.*

Proof. If $\bar{\sigma}(\mathcal{R}) < 1$ it follows that there exists for each N an $\alpha < 1$ such that $\bar{\sigma}_N(\mathcal{R}) \leq \alpha < 1$. By definition of $\bar{\sigma}_N(\mathcal{R})$ in (4.20), the hypothesis in Proposition 4.1 are satisfied, thus proving uniform exponential stability of the LPV system.

For the converse, let $\bar{\sigma}(\mathcal{R}) \geq 1$. We will prove that the LPV system is not exponentially stable by constructing a suitable parameter sequence $\bar{\delta}$. In order to do so, let us choose tuples $\delta^0, \hat{\delta}^1, \hat{\delta}^2, \dots$ that satisfy

$$\begin{aligned} \delta^0 &= \delta_0^0 && \rightarrow \|A(\delta^0)\| &\geq 1 \\ \hat{\delta}^1 &= (\delta_0^1, \delta_1^1) && \rightarrow \|\hat{A}_2(\hat{\delta}^1)\| &\geq 1 \\ \hat{\delta}^2 &= (\delta_0^2, \delta_1^2, \delta_2^2) && \rightarrow \|\hat{A}_3(\hat{\delta}^2)\| &\geq 1 \\ &\vdots && & \end{aligned} \tag{4.25}$$

Such tuples can always be found since Lemma 4.4 implies that

$$1 \leq \bar{\sigma}(\mathcal{R}) = \inf_{k \in \mathbb{N}} (\bar{\sigma}_k(\mathcal{R}))^{\frac{1}{k}}.$$

Now define the sequences τ^0, τ^1, \dots as follows:

$$\begin{aligned} \tau^0 &= (\delta_0^0, \delta_1^0, \delta_2^0, \delta_3^0, \dots) \\ \tau^1 &= (0, \delta_1^1, \delta_2^1, \delta_3^1, \dots) \\ \tau^2 &= (0, 0, \delta_2^2, \delta_3^2, \dots) \\ &\vdots && \vdots \end{aligned} \tag{4.26}$$

Since the elements of τ^0 are taken from a compact set $\boldsymbol{\delta}$, the sequence τ^0 has a converging subsequence s_0 , the limit point of which is denoted $\bar{\delta}_0$. Similarly, one can find a subsequence s_1 of τ^1 , being also a subsequence of s_0 , which converges to $\bar{\delta}_1$. The sequence $\bar{\delta} = (\bar{\delta}_0, \bar{\delta}_1, \bar{\delta}_2, \dots)$ defined in this fashion can be shown to satisfy $\|\hat{A}_N(\bar{\delta})\| \geq \frac{1}{\kappa}$ for all N and $\kappa > 0$ in (4.17). In order to prove this fact, let us fix N , and suppose that a subsequence $(l_\nu)_{\nu=1,2,\dots}$ has been constructed that satisfies

$$\lim_{\nu \rightarrow \infty} \tau_{l_\nu}^i = \bar{\delta}_i, \quad i = 0, 1, 2, \dots, N.$$

Note that there always exists ν_0 such that $l_\nu \geq N$ for all $\nu \geq \nu_0$. By construction of the sequence $(l_\nu)_{\nu=1,2,\dots}$, it certainly holds that

$$\|A_{l_\nu}(\delta^{l_\nu})\| \geq 1 \quad \forall \nu \geq \nu_0.$$

By using (4.17) it hence follows that

$$1 \leq \|A(\delta_{l_\nu}^{l_\nu})A(\delta_{l_\nu-1}^{l_\nu}) \cdots A(\delta_{N+1}^{l_\nu})A(\delta_N^{l_\nu}) \cdots A(\delta_0^{l_\nu})\| \leq \kappa \|A(\delta_N^{l_\nu}) \cdots A(\delta_0^{l_\nu})\|, \quad (4.27)$$

for all $\nu \geq \nu_0$, which implies that

$$\|A(\delta_N^{l_\nu}) \cdots A(\delta_0^{l_\nu})\| \geq \frac{1}{\kappa}, \quad \forall \nu \geq \nu_0.$$

Since N was arbitrary, the constructed parameter sequence $(\bar{\delta}_k)_{k=0,1,2,\dots}$ proves that the LPV system is not exponentially stable. A corresponding suitable initial condition x_0 can be constructed along the lines of the proof of Lemma 4.3. ■

The condition $\bar{\sigma}(\mathcal{R}) < 1$ for verifying stability of an LPV system can be viewed as an extension of a famous result, known as Gelfand's formula, which proves that the spectral radius of a matrix is equivalent to the limit of a particular sequence of norms.

Theorem 4.2 *Let $\|\cdot\|$ be a matrix norm. Then*

$$\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{\frac{1}{k}} \quad (4.28)$$

for any matrix $A \in \mathbb{R}^{n \times n}$.

Proof. A proof can be found in [94]. ■

The limit expression in (4.28) is not very useful in the context of the analysis of stability of the LTI system $x_{k+1} = Ax_k$, since efficient algorithms for computing the spectral radius of a matrix are available. By contrast, if considering the uncertain LTI system

$$x_{k+1} = A(\delta)x_k \quad x(0) = x_0, \quad \delta \in \boldsymbol{\delta} \quad k = 0, 1, \dots, \quad (4.29)$$

a systematic solution for computing the worst-case spectral radius of $A(\delta)$ is provided not so obvious. A numerically tractable approach is based on the following result.

Corollary 4.1 *Let $A(\delta)$ be a real-valued continuous function of $\delta \in \boldsymbol{\delta}$, and let $\boldsymbol{\delta}$ be a compact set. Then*

$$\max_{\delta \in \boldsymbol{\delta}} \rho(A(\delta)) = \lim_{k \rightarrow \infty} \max_{\delta \in \boldsymbol{\delta}} \|A(\delta)^k\|^{\frac{1}{k}}$$

Consequently, the uncertain LTI system (4.29) is uniformly exponentially stable if and only if

$$\lim_{k \rightarrow \infty} \max_{\delta \in \boldsymbol{\delta}} \|A(\delta)^k\|^{\frac{1}{k}} < 1.$$

Proof. Since Theorem 4.1 applies to general sets \mathcal{R} satisfying Assumption 4.1, it certainly holds for the specific choice $\mathcal{R} = \boldsymbol{\delta} \times \{0\}$. ■

Theorem 4.1 thus provides a proof that Gelfand’s formula holds for parameter dependent matrices, if computing the worst-case spectral radius. For general regions \mathcal{R} in which the parameter is time-varying, the condition $\bar{\sigma}(\mathcal{R}) < 1$ can be viewed as a generalization of Gelfand’s formula, see also [181].

The computation of $\bar{\sigma}(\mathcal{R})$ for general compact sets \mathcal{R} is a non-tractable problem. However, for fixed N , and choosing the standard singular value norm, the computation of $\bar{\sigma}_N(\mathcal{R})$ for fixed N can be translated into a robust SDP problem (4.15) or (4.16), for which suitable relaxation schemes can be constructed. Existing algorithms for (approximately) computing the joint spectral radius of a set of matrices can be found in [25] and [137], the latter of which is based on sum-of-squares programming.

4.1.5 Proving instability via periodic parameter sequences

Although the characterization in Theorem 4.1 is both necessary and sufficient for the stability of the considered LPV system, computations are performed for some finite value of N . Therefore, the conditions for stability are often sufficient only. In this section, we address the converse question of proving instability by constructing \mathcal{R} -admissible destabilizing parameter sequences. It will provide direct information on the level of conservatism that exists if applying Theorem 4.1.

Recall that the LPV system (4.4) is said to be unstable if there exists an admissible parameter sequence for which the corresponding time-varying system is unstable. The following sufficient condition for instability is an analogue to Proposition 4.1.

Proposition 4.2 *Let a region of variation \mathcal{R} be given for the LPV system (4.4) and let $\hat{\mathcal{R}}_N, \hat{A}_N$ be defined in (4.10) and (4.11) respectively. Define the following function*

$$\bar{\rho}_N(\mathcal{R}) := \max_{\hat{\delta} \in \hat{\mathcal{R}}_N} \rho(\hat{A}_N(\hat{\delta})), \quad (4.30)$$

in which $\rho(\cdot)$ denotes the spectral radius. Then, the LPV system is unstable if $\bar{\rho}_N(\mathcal{R}) \geq 1$ for some $N \in \mathbb{N}$.

Proof. Suppose that $\bar{\rho}_N(\mathcal{R}) \geq 1$ for some N . Then, for some \mathcal{R} -admissible N -tuple $\hat{\delta} = (\hat{\delta}_1, \dots, \hat{\delta}_N)$, the matrix $\hat{A}_N(\hat{\delta})$ has an eigenvalue $\lambda \geq 1$. Hence, with the sequence δ defined as

$$\delta = (\hat{\delta}_1, \dots, \hat{\delta}_N, \hat{\delta}_1, \dots, \hat{\delta}_N, \hat{\delta}_1, \dots), \quad (4.31)$$

we have

$$\rho(\hat{A}_{Nk}(\delta)) \geq 1 \quad \text{for } k = 1, 2, \dots$$

By using Lemma 4.3, we have proven that the LPV system is unstable. ■

Note that the condition in Proposition 4.2 is again sufficient only. In order to come to a necessary condition for instability of the LPV system, we introduce the function

$$\bar{\rho}(\mathcal{R}) = \limsup_{k \rightarrow \infty} \bar{\rho}_k(\hat{\mathcal{R}})^{\frac{1}{k}}.$$

Observe the analogy with the definition of $\bar{\sigma}(\mathcal{R})$ in (4.24). In the case of switched systems, $\bar{\rho}(\mathcal{R})$ is called the *generalized spectral radius* of a set of matrices. As conjectured in [48] and [19], the equality $\bar{\sigma}(\mathcal{R}) = \bar{\rho}(\mathcal{R})$ holds true in case \mathcal{R} is a bounded set of matrices and no bounds on the variation is imposed, e.g. $\mathcal{R} = \delta \times (-\infty, \infty)$.

So far, our results do not imply that $\bar{\rho}(\mathcal{R}) = \bar{\sigma}(\mathcal{R})$ for general sets \mathcal{R} . This would require, for instance, a proof of the fact that $\bar{\rho}(\mathcal{R}) < 1$ is both necessary and sufficient for the LPV system to be unstable. For continuous-time LPV systems with general regions of variations, the fact $\bar{\rho}(\mathcal{R}) = \bar{\sigma}(\mathcal{R})$ has been proven by Wirth [181]. At the end of this section, in remark 4.7, we will explain that necessity of $\bar{\rho}(\mathcal{R}) < 1$ is anyhow irrelevant from a practical perspective.

The relation $\bar{\rho}(\mathcal{R}) = \bar{\sigma}(\mathcal{R})$ can be viewed as a generalization of Gelfand's formula. Moreover, the following relation holds for arbitrary compact regions \mathcal{R} :

$$\bar{\rho}_k(\mathcal{R})^{\frac{1}{k}} \leq \bar{\rho}(\mathcal{R}) = \bar{\sigma}(\mathcal{R}) \leq \bar{\sigma}_k(\mathcal{R})^{\frac{1}{k}} \quad \text{for any } k \geq 1.$$

In the next section we will discuss how to implement the characterizations for stability and instability of LPV system as given in Theorem 4.1 and Proposition 4.2.

4.1.6 A numerical procedure for computing stability margins

In this section, we address the computation of the \mathcal{R} -stability margin of an LPV system by using the sufficient conditions for stability and instability, as given by Theorem 4.1 and Proposition 4.2 respectively. Recall that this margin is defined as the largest r for which the system is stable for all $r\mathcal{R}$ -admissible parameter sequences. As illustrated in Figure 4.3, the suggested procedure verifies for some fixed N , whether $\bar{\sigma}_N(\mathcal{R}) < 1$ or $\bar{\rho}_N(\mathcal{R}) \geq 1$ holds.

First, let us concentrate on computing a lower bound on the \mathcal{R} -stability margin by implementing condition (4.16). That is, for some fixed $N \in \mathbb{N}$ and $r \in \mathbb{R}$, the following optimization problem is considered:

$$\begin{aligned} & \text{Minimize } \gamma \text{ subject to } X(\hat{\delta}) \succ 0 \quad \text{and} \\ & \begin{pmatrix} I \\ \hat{A}_N(\hat{\delta}) \end{pmatrix}' \begin{pmatrix} -X(\hat{\delta}) & 0 \\ 0 & X(\hat{\theta}) \end{pmatrix} \begin{pmatrix} I \\ \hat{A}_N(\hat{\delta}) \end{pmatrix} \prec \gamma I, \quad \forall (\hat{\delta}, \hat{\theta}) \in r\hat{\mathcal{R}}_{2N}, \quad (4.32) \end{aligned}$$

with optimal value γ_{opt} . Note that the set $r\hat{\mathcal{R}}_{2N}(\mathcal{R})$ is equal to $\hat{\mathcal{R}}_{2N}(r\mathcal{R})$. If the optimal value of (4.32) is negative, the LPV system is guaranteed to be stable for all $r\mathcal{R}$ -admissible parameter sequences. In order to turn this condition into a robust SDP problem, let the Lyapunov matrix be parameterized as

$$X(\hat{\delta}) = X_0 + \sum_{j=1}^m X_j \hat{\delta}^{\alpha_j} \quad (4.33)$$

with multi-degrees $\alpha_1, \dots, \alpha_m \in \mathbb{R}^{N_s}$. As discussed in Chapter 2, a suitable relaxation scheme can be constructed that approximately solves (4.32). Recall that γ_{rel} denotes the optimal value of such a relaxation scheme. With the knowledge that $\gamma_{\text{rel}} \geq \gamma_{\text{opt}}$ holds, we arrive at a lower bound on the stability margin whenever $\gamma_{\text{rel}} < 0$. The largest r for which $\gamma_{\text{rel}} < 0$ can be obtained from a bi-section procedure.

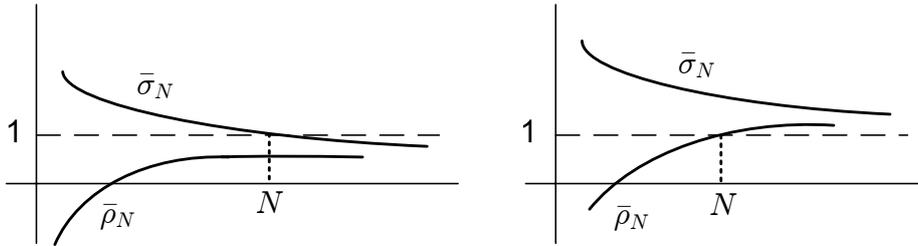


Figure 4.3: The analysis approach based on the quantities $\bar{\sigma}_N$ in (4.20) and $\bar{\rho}_N$ in (4.30). On the left the LPV system is proven stable, on the right it is proven unstable.

Second, let us consider the condition $\bar{\rho}_N(\mathcal{R}) \geq 1$. It is used for the construction of destabilizing parameter sequences and provides an upper bound on the \mathcal{R} -stability margin. Note that the statement in Proposition 4.2 involves a particular \mathcal{R} -admissible sequence for which $\bar{\rho}(\hat{A}_N(\hat{\delta})) > 1$ holds for some $\hat{\delta} \in \hat{\mathcal{R}}_N$.

We stress that the search for such a worst-case $\hat{\delta}$ is fundamentally different from solving robust semi-definite programming problems. At present, the construction of destabilizing parameter sequences is largely an open problem. In view of the fact that we have reduced the analysis problem to a real- μ analysis problem with the uncertainty $\hat{\delta} \in \hat{\mathcal{R}}_N$, one could suggest using a power iteration method, as it is implemented in commercial software [133].

If the parameters can vary in some specified box, and no bounds on the time-variation are imposed, one might possibly benefit from algorithms that have been developed for the construction of destabilizing strategies of switched systems, see [166, 143, 38, 26, 50] and references therein. However, by allowing the parameter to take its values only at the extreme points of a given polytopic region \mathcal{R} , it is likely that conservatism is added to the problem. It is an interesting research question

whether any of these existing algorithms can be extended to general compact sets \mathcal{R} with (non-trivial) parameter variation bounds.

As we will see in the numerical example of Section 4.3, it is possible to find destabilizing N -periodic parameter sequences by computing the maximum eigenvalue of $\hat{A}_N(\hat{\delta})$ for a finite number of values $\hat{\delta} \in \hat{\mathcal{R}}_N$. A more systematic approach for computing destabilizing parameters is needed though, since the computational complexity of such a gridding approach grows exponentially with the lifting horizon N . In this respect, the exactness test of Section 2.4.2 might prove a valuable tool in developing new algorithms for the construction of worst-case parameters.

Extracting destabilizing parameter sequences from relaxation exactness

In Section 2.4.2, we have presented a procedure for extracting worst-case parameters from a computed multiplier-based relaxation scheme, provided that it is exact. Let us illustrate how to apply Theorem 2.6 in the context of stability analysis of LPV systems.

In the sequel, let a parametrization of the Lyapunov matrix be given as shown in (4.33). It is important to realize that instability of the LPV system cannot be concluded from the optimal value of (4.32) being positive. If, however, some multiplier-based relaxation scheme for the robust SDP (4.32) is exact and a single representative pair $(\hat{\delta}^0, \hat{\theta}^0) \in \hat{\mathcal{R}}_{2N}$ exists as well, instability of the LPV system does follow from $\gamma_{\text{rel}} = \gamma_{\text{opt}} \geq 0$.

The crucial observation that enables to extract good lower bound values is the fact that if $(\hat{\delta}^0, \hat{\theta}^0) \in \hat{\mathcal{R}}_{2N}$ is a single representative pair for the robust SDP (4.32), so is the pair $(\hat{\delta}^0, \hat{\delta}^0) \in \hat{\mathcal{R}}_{2N}$. Then, the time-invariant parameter sequence $(\hat{\delta}^0, \hat{\delta}^0, \hat{\delta}^0, \dots)$ destabilizes the N -lifted system (4.12) so that the original LPV system is unstable for the N -periodic parameter sequence $(\hat{\delta}_1^0, \dots, \hat{\delta}_N^0, \hat{\delta}_1^0, \dots, \hat{\delta}_N^0, \hat{\delta}_1^0, \dots)$. Recall that a single representative pair has the property that the LMI problem

$$\gamma_{\text{lb}} := \inf \{ \gamma : \quad \text{subject to} \quad \left(\begin{array}{c} I \\ \hat{A}_N(\hat{\delta}^0) \end{array} \right)' \left(\begin{array}{cc} -X(\hat{\delta}^0) & 0 \\ 0 & X(\hat{\theta}^0) \end{array} \right) \left(\begin{array}{c} I \\ \hat{A}_N(\hat{\delta}^0) \end{array} \right) \prec \gamma I \} \quad (4.34)$$

has optimal value $\gamma_{\text{lb}} = \gamma_{\text{opt}}$, which is why such a pair is said to 'represent' the original robust SDP constraint.

Corollary 4.2 *Suppose that the robust SDP (4.32) for some value of $r \in \mathbb{R}$ has optimal value $\gamma_{\text{opt}} \geq 0$ and let $(\hat{\delta}^0, \hat{\theta}^0) \in r\hat{\mathcal{R}}_{2N}$ be a single representative pair leading to $\gamma_{\text{lb}} = \gamma_{\text{opt}} \geq 0$ with γ_{lb} defined in (4.34). Then,*

$$\rho(\hat{A}_N(\hat{\delta}^0)) \geq 1,$$

and $(\hat{\delta}^0, \hat{\theta}^0) \in r\hat{\mathcal{R}}_{2N}$ is a single representative pair as well.

Proof. Suppose that the pair $(\hat{\delta}^0, \hat{\theta}^0) \in r\hat{\mathcal{R}}_{2N}$ leads to $\gamma_{\text{lb}} = \gamma_{\text{opt}} \geq 0$. Hence, (4.34) is infeasible for $\gamma < 0$. Suppose $\hat{A}_N(\hat{\delta}^0)$ is stable. Then there exist decision variables $X_0, X_1 = 0, \dots, X_m = 0$ for which (4.34) is feasible with $\gamma < 0$. This contradicts the assumption that $(\hat{\delta}^0, \hat{\theta}^0)$ was a single representative pair for the robust SDP (4.32).

Let us now prove that $(\hat{\delta}^0, \hat{\delta}^0)$ is also a representative pair. Since $\gamma_{\text{lb}} \geq 0$, the least γ for which the constraint in (4.32) is feasible for some set of matrices X_0, \dots, X_m , is nonnegative. Hence, with the particular choice $X_0, X_1 = 0, \dots, X_m = 0$, the constraint in (4.32) is feasible only for non-negative values of γ , i.e.

$$X_0 \succ 0, \quad \text{and} \quad \hat{A}_N(\hat{\delta}^0)' X_0 \hat{A}_N(\hat{\delta}^0) - X_0 \prec \gamma$$

has a solution X_0 only if $\gamma \geq 0$, which again proves that $\rho(\hat{A}_N(\hat{\delta}^0)) \geq 1$. Notice that performing the substitution $\hat{\theta}^0 \rightarrow \hat{\delta}^0$ yields the same constraint as if choosing $X_1 = 0, \dots, X_m = 0$. Therefore, $(\hat{\delta}^0, \hat{\delta}^0)$ is a representative pair which finishes the proof. ■

Hence, a destabilizing $r\mathcal{R}$ -admissible parameter sequence that is N -periodic can be extracted from a multiplier relaxation of robust SDP (4.32), once a single representative pair has been found. We stress the fact that it is a priori unknown whether such a pair exists. Theorem 2.6 provides a procedure for verifying exactness of the relaxation as well as constructing a pair $(\hat{\delta}^0, \hat{\theta}^0)$ by solving a suitable system of polynomials.

Remark 4.7 *Let us again point to the fact that the condition in Proposition 4.2 is sufficient for proving instability. It might however happen for a certain region \mathcal{R} , that*

$$\bar{\sigma}(\mathcal{R}) = \bar{\rho}(\mathcal{R}) = 1 \quad \text{while} \quad \bar{\rho}_N(\mathcal{R}) < 1 \quad \text{and} \quad \bar{\sigma}_N(\mathcal{R}) > 1 \quad \forall N \in \mathbb{N}. \quad (4.35)$$

In such cases, we are neither able to prove stability nor instability of the LPV system. A numerical example in which this phenomenon occurs is found in [24].

Fortunately, in the context of the stability analysis of LPV systems, we need not bother much about the pathological case $\bar{\sigma} = \bar{\rho} = 1$. This is caused by the iterative nature of our analysis method. To be precise, we either compute for some fixed N the largest r for which $\bar{\sigma}_N(r\mathcal{R}) < 1$, by solving a sequence of feasibility tests of the form (4.32), or we compute the smallest r for which $\bar{\rho}_N(r\mathcal{R}) \geq 1$, by verifying (4.34) for different values of r . If the situation in (4.35) is encountered, scaling the region \mathcal{R} with any factor $r \neq 1$ will guarantee the existence of $N \in \mathbb{N}$ for which either $\bar{\sigma}_N(r\mathcal{R}) < 1$ or $\bar{\sigma}_N(r\mathcal{R}) \geq 1$.

4.1.7 Evaluation of the proposed analysis method

Let us finally compare the proposed procedure for the stability analysis of LPV systems with other approaches based on non-quadratic Lyapunov functions. The most important observation is the following. Our construction of the asymptotically exact family of analysis conditions that was based on the monodromy matrix \hat{A}_N does not involve non-quadratic Lyapunov function. In fact, there is not even a need for using a parameter dependent Lyapunov matrix, since the family of conditions in Proposition 4.1 is asymptotically exact. Nevertheless, choosing the Lyapunov matrix as being parameter dependent might lead to significant improvements in specific analysis problems.

The proposed analysis method is definitely not suited for LPV systems with a large number of parameters $\delta_1, \dots, \delta_s$, since the resulting robust SDP depends on a total of Ns parameters. For a small number of parameters though, the method has a great advantage over alternative techniques that are based on homogeneous Lyapunov functions, see [34, 189]. Note that it is much harder to modify or update the analysis condition, when tuning of the Lyapunov function involves the selection of basis functions that depend on both the state and parameter.

A comparison of our approach with polyhedral norm Lyapunov functions is more complicated, since these tests do not (yet) translate into robust LMIs, see [142, 187, 127], and is not further investigated here.

4.2 Performance analysis

So far, our discussion focussed on the computation of the \mathcal{R} -stability margin of a given LPV system. This quantity can be obtained from the analysis of the corresponding N -periodic LPV system, as long as N is chosen sufficiently large. In this section, we will demonstrate the prominent role of N -periodic parameter sequences in analyzing performance of LPV systems.

Recall the LPV system in (4.1), which reads as

$$\begin{aligned} x_{k+1} &= A(\delta_k)x_k + B(\delta_k)w_k, & x_0 &= \bar{x}_0, \\ z_k &= C(\delta_k)x_k + D(\delta_k)w_k \end{aligned} \quad (4.36)$$

in which $x_k \in \mathbb{R}^n$ denotes the state, $w_k \in \mathbb{R}^{n_w}$ the disturbance, $z_k \in \mathbb{R}^{n_e}$ the controlled output and $\delta_k = (\delta_{k,1}, \dots, \delta_{k,s}) \in \mathbb{R}^s$ the (time-varying) parameter. The parameter variation is again characterized by the region \mathcal{R} , as in Definition 4.1. Similar to the Chapter 3, the desired behavior is expressed in terms of input/output channels. We will address both the quadratic performance measure and an \mathcal{H}_2 -performance measure.

4.2.1 Quadratic performance

Let us recall the definition of quadratic performance for discrete-time systems referring to Appendix A for the continuous time version, see also Section 3.1. It can capture bounds on the induced energy gain, or other types of performance measures.

Definition 4.3 *The LPV system satisfies quadratic performance with performance index matrix*

$$P_p = \begin{pmatrix} Q_p & S_p \\ S_p' & R_p \end{pmatrix}, \quad R_p \succeq 0,$$

if it is uniformly exponentially stable and if there exists $\epsilon > 0$ such that for $x_0 = 0$, and any \mathcal{R} -admissible parameter sequence, we have

$$\sum_{k=0}^{\infty} \begin{pmatrix} w_k \\ z_k \end{pmatrix}' P_p \begin{pmatrix} w_k \\ z_k \end{pmatrix} \preceq -\epsilon \|w\|^2, \quad \text{for every } w \in l_2. \quad (4.37)$$

Note that this notion of performance is based on a worst-case philosophy, similar as was done in the previous chapter. The following sufficient condition can be used to verify whether an LPV system satisfies a given quadratic performance measure. It is a rather direct extension of the LMI characterization of quadratic performance for LTI systems, which can be found in Appendix A.

Proposition 4.3 For a given region of variation \mathcal{R} defined in (4.3), the LPV system (4.36) is uniformly exponentially stable and admits quadratic performance if there exists a matrix function $X(\delta) = X(\delta)'$ satisfying

$$X(\delta) \succ 0 \quad (4.38)$$

and

$$\left(\begin{array}{cc} I & 0 \\ A(\delta) & B(\delta) \\ 0 & I \\ C(\delta) & D(\delta) \end{array} \right)' \left(\begin{array}{cc|c} -X(\delta) & 0 & 0 \\ 0 & X(\theta) & 0 \\ \hline 0 & 0 & P_p \end{array} \right) \left(\begin{array}{cc} I & 0 \\ A(\delta) & B(\delta) \\ 0 & I \\ C(\delta) & D(\delta) \end{array} \right) \prec 0, \quad (4.39)$$

for all $(\delta, \theta) \in \hat{\mathcal{R}}$.

Proof. Stability follows from the left-upper block of (4.39), i.e.

$$-X(\delta) + A(\delta)'X(\theta)A(\delta) + C(\delta)'R_pC(\delta) \prec 0$$

Since $R_p \succeq 0$, the matrix function $X(\delta)$ also satisfies (4.7) which is known to imply uniform exponential stability. For proving quadratic performance, first note that there exists some $\epsilon > 0$ for which $P_p = \begin{pmatrix} Q_p + \epsilon I & S_p \\ S_p' & R_p \end{pmatrix}$ still satisfies (4.39). Take any \mathcal{R} -admissible parameter sequence $(\delta_k)_{k=1,2,\dots}$ and any input sequence $(w_k)_{k=1,2,\dots}$ in l_2 . Let $(x_k)_{k=1,2,\dots}$ be the resulting evolution of the state with zero initial condition. Then, by multiplying (4.39) from left and right with

$$\begin{pmatrix} x_k \\ w_k \end{pmatrix}' \quad \text{and} \quad \begin{pmatrix} x_k \\ w_k \end{pmatrix}$$

respectively, as well as substituting the \mathcal{R} -admissible pair $(\delta, \theta) = (\delta_k, \delta_{k+1})$, one arrives at

$$-x_k^T X(\delta_k)x_k + x_{k+1}^T X(\delta_{k+1})x_{k+1} + \begin{pmatrix} w_k \\ z_k \end{pmatrix}' P_p \begin{pmatrix} w_k \\ z_k \end{pmatrix} \leq -\epsilon w_k' w_k.$$

Summation from $k = 0$ to $k = N$, using the fact that $x_0 = 0$, we infer

$$x_N^T X(\delta_N)x_N + \sum_{k=0}^N \begin{pmatrix} w_k \\ z_k \end{pmatrix}' P_p \begin{pmatrix} w_k \\ z_k \end{pmatrix} \leq -\epsilon \sum_{k=0}^N w_k^T w_k.$$

Since $X(\delta) \succ 0$ for all $(\delta, \cdot) \in \mathcal{R}$, the term $x_N^T X(\delta_N)x_N$ can be dropped and quadratic performance criterion is satisfied as we let $N \rightarrow \infty$

■

4.2.2 \mathcal{H}_2 -performance

As already mentioned in Section 3.2, an extension of the \mathcal{H}_2 -norm to non-linear time-varying systems is by far unique. In other words, if we extend the deterministic and stochastic interpretation of the \mathcal{H}_2 -norm, as can be found in Appendix A, we usually end up having two different measures of performance. However, for LTV systems a natural extension exists for which the two notions coincide, see for example [13, 107].

Let us sketch how to arrive at the generalization of the \mathcal{H}_2 -norm for LTV systems by adopting the deterministic interpretation. That is, for a SISO system, the \mathcal{H}_2 -norm amounts to the energy, or equivalently, the l_2 -norm of the impulse response. Once the system is time-varying, the response of the system depends on the specific time instant when the impulse is initiated. Hence, we are motivated to take an average over all time-shifts of impulsive sequences. That is, we introduce the inputs

$$w^{\eta,j} = (0, 0, \dots, 0, e^\eta, 0, 0, \dots) \quad \text{for } \eta = 1, \dots, n_w, j = 1, 2, \dots, \quad (4.40)$$

in which the vector e^η is the j^{th} element in the sequence $w^{\eta,j}$, and picks the η^{th} component of input w , as defined earlier in (3.15). Then, for the linear time-varying system defined as

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k w_k, & x_0 &= 0, \\ z_k &= C_k x_k + D_k w_k \end{aligned} \quad (4.41)$$

the \mathcal{H}_2 -performance measure reads as

$$\|G\|_2 := \sqrt{\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=0}^k \sum_{\nu=1}^{n_w} \|z^{\eta,j}\|^2}, \quad (4.42)$$

in which $\|z^{\eta,j}\|$ denotes the l_2 -norm of the response of system (4.41) to the input sequence $w^{\eta,j}$. If G is LTI, no averaging over time-shifts j is needed and the formula reduces to the impulse response interpretation of the standard \mathcal{H}_2 -norm, see also Appendix A.

Since an LPV system can be viewed as a family of parameterized LTV systems, we can adopt the formula (4.42). In fact, in this thesis the \mathcal{H}_2 -performance level of an LPV system is defined as the worst possible value (4.42) that can be achieved over all \mathcal{R} -admissible parameter sequences.

Our next result characterizes the worst-case \mathcal{H}_2 -performance for LPV systems. For convenience we assume $D(\delta) = 0$, a property that is needed for continuous-time systems in order the \mathcal{H}_2 -norm to be finite.

Proposition 4.4 *For a given region \mathcal{R} , let δ and $\hat{\mathcal{R}}$ be defined in (4.2) and (4.3) respectively. The LPV system (4.36) is uniformly exponentially stable and has an \mathcal{H}_2 -performance level smaller than γ if there exists a matrix function $X(\delta) = X(\delta)'$*

and $Z(\delta) = Z(\delta)'$ that satisfy

$$\max_{\delta \in \hat{\mathcal{R}}} \text{Tr}(Z(\delta)) \leq \gamma^2, \quad \begin{pmatrix} Z(\delta) & B'(\delta)X(\delta) \\ X(\delta)B(\delta) & X(\delta) \end{pmatrix} \succ 0, \quad (4.43)$$

and

$$\begin{pmatrix} X(\delta) - A(\delta)'X(\theta)A(\delta) & C(\delta)' \\ C(\delta) & -I \end{pmatrix} \prec 0, \quad (4.44)$$

for all $(\delta, \theta) \in \hat{\mathcal{R}}$.

Proof. Uniform exponential stability immediately follows from the upper-left block in inequality (4.43). The proof for robust \mathcal{H}_2 -performance uses similar arguments as Theorem 3.2.

Let us fix some \mathcal{R} -admissible parameter sequence $\delta = (\delta_0, \delta_1, \dots)$ and choose an input $w^{\eta,j}$ of the form (4.40) for some fixed numbers of η, j . Since the state remains zero until the impulse $w^{\eta,j}$ affects the system at time j , the resulting evolution of the state x^η satisfies

$$\begin{aligned} x_k^\eta &= 0 && \text{for } k = 0, \dots, j-1, \\ x_j^\eta &= B(\delta_j)e^\eta. \end{aligned}$$

Since $D(\delta) = 0$, this also implies $z_k = 0$ for $k = 0, \dots, j-1$.

By applying Schur's Lemma to (4.44), we infer

$$A(\delta)'X(\theta)A(\delta) - X(\delta) + C(\delta)'C(\delta) \prec 0.$$

Let us multiply this expression from left and right with $x_k^{\eta'}$ and x_k^η , and substitute $(\delta, \theta) = (\delta_k, \delta_{k+1})$, in order to get

$$V(x_{k+1}^\eta, k+1) - V(x_k^\eta, k) + \|z_k^{\eta,j}\|^2 \leq 0, \quad \text{for } k = 0, 1, 2, \dots \quad (4.45)$$

in which the notation $V(x, k) = x'X(\delta_k)x$ is used. Applying Schur's complement formula on the second condition in (4.43), we obtain

$$Z(\delta) - B'(\delta)X(\delta)B(\delta) \succ 0, \quad \text{for all } (\delta, \theta) \in \mathcal{R},$$

from which we get

$$x_j^{\eta'}X(\delta_j)x_j^\eta < e^{\eta'}Z(\delta_j)e^\eta. \quad (4.46)$$

Summation of (4.45) over $k = 1, 2, \dots$ yields

$$-x_j^{\eta'}X(\delta_j)x_j^\eta + \|z^{\eta,j}\|_2^2 \leq 0,$$

where we used the fact that the system is uniformly exponentially stable. If we now

exploit (4.46), it follows that

$$-e^{\eta'} Z(\delta_j) e^\eta + \|z^{\eta,j}\|_2^2 \leq 0.$$

Finally, let us take the summation over $\eta = 1, \dots, n_w$ in order to arrive at

$$\sum_{\eta=1}^{n_w} \|z^{\eta,j}\|_2^2 \leq \text{Tr}(Z(\delta_j)).$$

Since the \mathcal{R} -admissible parameter sequence δ was arbitrarily chosen, we can average over $j = 1, 2, \dots$ and take the limit, i.e.

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k \sum_{\eta=1}^{n_w} \|z^{\eta,j}\|_2^2 \leq \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{j=1}^k \text{Tr}(Z(\delta_j)) \leq \max_{\delta \in \bar{\delta}} \text{Tr}(Z(\delta)) \leq \gamma^2,$$

which proves that the robust \mathcal{H}_2 -performance level is bounded by γ . ■

Remark 4.8 *According to the notion of \mathcal{H}_2 -performance that we just presented, analysis and controller synthesis of polytopic LPV systems has been addressed in [71], see also [53] for similar results in continuous time.*

Notice that all conditions in Proposition 4.4 can be formulated as a robust SDP constraint, for any fixed parameterizations of $Z(\delta)$ and $X(\delta)$ that are linear in the coefficient matrices, such as

$$X(\delta) = X_0 + \sum_{j=1}^m X_j \delta^{\alpha_j} \quad \text{and} \quad Z(\delta) = Z_0 + \sum_{j=1}^m Z_j \delta^{\alpha_j},$$

with multi-degrees $\alpha_1, \dots, \alpha_m \in \mathbb{R}^s$. In fact,

$$\max_{\delta \in \bar{\delta}} \text{Tr}(Z(\delta)) \leq \gamma^2$$

can be implemented by constructing a relaxation scheme for the robust linear inequality

$$e^1{}' Z(\delta) e^1 + \dots + e^{n_w}{}' Z(\delta) e^{n_w} \leq \gamma^2.$$

Summarizing this section, we have shown how to determine least upper bounds γ on the \mathcal{H}_2 -performance level in a single optimization step, by fixing a certain parametrization of $Z(\delta)$ and $X(\delta)$ and constructing suitable relaxation schemes.

4.2.3 Lifting of discrete-time LPV systems

In Section 4.1.3, sufficient conditions for stability of discrete-time LPV systems in terms of the monodromy matrix $\hat{A}_N(\hat{\delta})$ were derived, by exploiting the fact that the LPV system is stable if and only if its N -lifted system is stable. In this section, we show that a similar lifting operation exists for input-output systems of the form (4.36). The application of a lifting technique is well-understood in the field of multi-rate systems, see for example [73, 106, 11, 12, 45] and will be employed to derive alternative conditions for computing the robust \mathcal{H}_2 - or l_2 -gain performance level of LPV systems.

Starting from the discrete-time LPV system (4.36), the N -lifted system is formally constructed by first collecting N sequential inputs, outputs and parameter values into the vectors \hat{w}_t, \hat{z}_t and $\hat{\delta}_t$. For $t = 0, 1, 2, \dots$ we therefore define

$$\hat{w}_t = \begin{pmatrix} w_{Nt} \\ w_{Nt+1} \\ \vdots \\ w_{N(t+1)-1} \end{pmatrix}, \hat{z}_t = \begin{pmatrix} z_{Nt} \\ z_{Nt+1} \\ \vdots \\ z_{N(t+1)-1} \end{pmatrix}, \hat{\delta}_t = \begin{pmatrix} \delta_{Nt} \\ \delta_{Nt+1} \\ \vdots \\ \delta_{N(t+1)-1} \end{pmatrix}. \quad (4.47)$$

As before, we will also denote $\hat{\delta}_t = \text{col}(\hat{\delta}_{t,1}, \dots, \hat{\delta}_{t,N})$. An easy computation shows that

$$\begin{aligned} x_1 &= A(\delta_0)x_0 + B(\delta_0)w_0 \\ x_2 &= A(\delta_1)A(\delta_0)x_0 + A(\delta_1)B(\delta_0)w_0 + B(\delta_1)w_1 \\ &\vdots \\ x_N &= A(\delta_{N-1})A(\delta_{N-2})\cdots A(\delta_0)x_0 + A(\delta_{N-1})A(\delta_{N-2})\cdots A(\delta_1)B(\delta_0)w_0 \\ &\quad + \dots + A(\delta_{N-1})B(\delta_{N-2})w_{N-2} + B(\delta_{N-1})w_{N-1}, \end{aligned}$$

and that the outputs of system (4.36) satisfy

$$\begin{aligned} z_0 &= C(\delta_0)x_0 + D(\delta_0)w_0 \\ z_1 &= C(\delta_1)A(\delta_0)x_0 + C(\delta_1)B(\delta_0)w_0 + D(\delta_1)w_1 \\ &\vdots \\ z_{N-1} &= C(\delta_{N-1})\left(A(\delta_{N-2})A(\delta_{N-3})\cdots A(\delta_0)x_0 + \dots + \right. \\ &\quad \left. + A(\delta_{N-2})B(\delta_{N-3})w_{N-3} + B(\delta_{N-2})w_{N-2}\right) + D(\delta_{N-1})w_{N-1}. \end{aligned}$$

If defining the sequence $\xi = (x_0, x_{N-1}, x_{2N-1}, \dots)$, one can consider the system

$$\begin{aligned} \xi_{t+1} &= \hat{A}_N(\hat{\delta}_t)\xi_t + \hat{B}_N(\hat{\delta}_t)\hat{w}_t, \quad \xi_0 = \bar{x}_0, \\ \hat{z}_t &= \hat{C}_N(\hat{\delta}_t)\xi_t + \hat{D}_N(\hat{\delta}_t)\hat{w}_t, \end{aligned} \quad (4.48)$$

in which $\hat{\delta}$ is $\hat{\mathcal{R}}_N$ -admissible as defined in (4.10) and the matrix functions $\hat{A}_N, \hat{B}_N, \hat{C}_N, \hat{D}_N$ are defined as

$$\left(\begin{array}{c|c} \hat{A}_N(\hat{\delta}_t) & \hat{B}_N(\hat{\delta}_t) \\ \hline \hat{C}_N(\hat{\delta}_t) & \hat{D}_N(\hat{\delta}_t) \end{array} \right) = \left(\begin{array}{c|ccc} A_N A_{N-1} \cdots A_1 & A_N \cdots A_2 B_1 & A_N \cdots A_3 B_2 & \cdots & B_N \\ \hline C_1 & D_1 & 0 & \cdots & 0 \\ C_2 A_1 & C_2 B_1 & D_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_N A_{N-1} \cdots A_1 & C_N A_{N-1} \cdots A_2 B_1 & C_N A_{N-1} \cdots A_3 B_2 & \cdots & D_N \end{array} \right),$$

with the abbreviations $A_i = A(\hat{\delta}_{t,i}), B_i = B(\hat{\delta}_{t,i}), C_i = C(\hat{\delta}_{t,i})$ and $D_i = D(\hat{\delta}_{t,i})$ for $i = 1, \dots, N$. The system (4.48) is referred to as the N -lifted system that corresponds to the LPV system (4.36).

The N -lifted system is an equivalent representation of the original LPV system in the following sense. For any \mathcal{R} -admissible parameter sequence δ and any feasible x, z, w that satisfy the system dynamics (4.36), the sequence $\xi = (x_0, x_{N-1}, x_{2N-1}, \dots)$ and \hat{z}, \hat{w} as defined in (4.47) are compatible with the N -lifted system dynamics (4.48). Conversely, given any $\hat{\mathcal{R}}_N$ -admissible $\hat{\delta}$ and sequences ξ, \hat{w}, \hat{z} that are feasible for the system (4.48), one can again construct δ, w, z that are feasible for the original LPV system such that the state x satisfies $x_{Nt} = \xi_t$ for $t = 0, 1, \dots$. In fact, the latter construction amounts to decomposing $\hat{\delta}_t$ into its components $\hat{\delta}_{t,1}, \dots, \hat{\delta}_{t,N} \in \mathbb{R}^s$, and placing them in the sequence

$$(\delta_k)_{k=1,2,\dots} = (\hat{\delta}_{1,1}, \hat{\delta}_{1,2}, \dots, \hat{\delta}_{1,N}, \hat{\delta}_{2,1}, \dots, \hat{\delta}_{2,N}, \hat{\delta}_{3,1}, \dots).$$

The sequences w, z can be formed in a similar fashion.

Remark 4.9 *When the parameter dependent system matrices in (4.36) are given as an LFR, using the formulae from Appendix B, one can directly build the LFR descriptions of $\hat{A}_N, \hat{B}_N, \hat{C}_N, \hat{D}_N$ for any given N . It then naturally follows that the resulting uncertainty structure is block diagonal.*

In view of the one-to-one correspondence between the input and output sequences of the original system and its N -lifted version, a natural question arises as to whether we can analyze the performance of the LPV system through the analysis of the N -lifted system. This is the subject of our next two sections, in which both quadratic performance and the \mathcal{H}_2 -performance measure are addressed.

Analysis of quadratic performance via lifting

The quadratic performance measure for a given index P_p can equivalently be analyzed by considering the N -lifted system, provided that we use an appropriate

performance index matrix. In view of Definition 4.37, the quadratic performance measure can be characterized in terms of the N -lifted system if there exists a matrix \hat{P}_p for which

$$\sum_{k=0}^{\infty} \begin{pmatrix} w_k \\ z_k \end{pmatrix}' P_p \begin{pmatrix} w_k \\ z_k \end{pmatrix} = \sum_{t=0}^{\infty} \begin{pmatrix} \hat{w}_t \\ \hat{z}_t \end{pmatrix}' \hat{P}_p \begin{pmatrix} \hat{w}_t \\ \hat{z}_t \end{pmatrix}$$

holds for any input $w \in l_2$ and initial condition $x_0 = \xi_0 = 0$. It is not difficult to verify that this relation holds for $\hat{P}_p = I_N \otimes P_p$, which brings us to the following result.

Corollary 4.3 *For a given region of variation \mathcal{R} , the LPV system (4.36) is uniformly exponentially stable and satisfies quadratic performance if there exists a matrix function $X(\hat{\delta}) = X(\hat{\delta})'$ that satisfies*

$$X(\hat{\delta}) \succ 0 \quad (4.49)$$

and

$$\begin{pmatrix} I & 0 \\ \hat{A}_N(\hat{\delta}) & \hat{B}_N(\hat{\delta}) \\ 0 & I \\ \hat{C}_N(\hat{\delta}) & \hat{D}_N(\hat{\delta}) \end{pmatrix}' \left(\begin{array}{cc|c} -X(\hat{\delta}) & 0 & 0 \\ 0 & X(\hat{\theta}) & 0 \\ \hline 0 & 0 & I_N \otimes P_p \end{array} \right) \begin{pmatrix} I & 0 \\ \hat{A}_N(\hat{\delta}) & \hat{B}_N(\hat{\delta}) \\ 0 & I \\ \hat{C}_N(\hat{\delta}) & \hat{D}_N(\hat{\delta}) \end{pmatrix} \prec 0 \quad (4.50)$$

for all $(\hat{\delta}, \hat{\theta}) \in \hat{\mathcal{R}}_{2N}$.

Proof. First, note that (4.50) implies

$$-X(\hat{\delta}) + \hat{A}_N(\hat{\delta})' X(\hat{\theta}) A(\hat{\delta}) + \hat{C}_N(\hat{\delta})'(I_N \otimes R_p) \hat{C}_N(\hat{\delta}) \prec 0 \quad \forall (\hat{\delta}, \hat{\theta}) \in \hat{\mathcal{R}}_{2N}.$$

Since $R_p \succeq 0$ the matrix function $X(\cdot)$ serves as a Lyapunov matrix for the N -lifted system (4.48), and proves that it uniformly exponentially stability. By similar arguments as in the proof of Proposition 4.1, the original LPV system is seen to be uniformly exponentially stable.

For proving quadratic performance, we choose some \mathcal{R} -admissible $(\delta_k)_{k=0,1,\dots}$ and some sequence $w \in l_2$. Let $(x_k)_{k=0,1,2,\dots}$ denote the resulting evolution of the state, with $x_0 = 0$, and define the sequence $\xi = (x_0, x_{N-1}, x_{2N-1}, \dots)$. Furthermore, construct the sequences $\hat{\delta}, \hat{w}$ as in (4.47).

Note that for some $\epsilon > 0$ the quadratic cost criterion

$$\hat{P}_p = I \otimes \begin{pmatrix} Q_p + \epsilon I & S_p \\ S_p' & R_p \end{pmatrix}$$

still satisfies (4.50). Using this perturbed quadratic cost criterion, we now multiply (4.50) from left and right with

$$\begin{pmatrix} \xi_t \\ \hat{w}_t \end{pmatrix}' \quad \text{and} \quad \begin{pmatrix} \xi_t \\ \hat{w}_t \end{pmatrix}$$

respectively, also substituting $\hat{\delta} \rightarrow \hat{\delta}_t$, $\hat{\theta} \rightarrow \hat{\delta}_{t+1}$, and get

$$\begin{aligned} -\xi_t' X(\hat{\delta}_t) \xi_t + \xi_{t+1}' X(\hat{\delta}_{t+1}) \xi_{t+1} + \begin{pmatrix} \hat{w}_t \\ \hat{z}_t \end{pmatrix}' (I_N \otimes P_p) \begin{pmatrix} \hat{w}_t \\ \hat{z}_t \end{pmatrix} = \\ -\xi_t' X(\hat{\delta}_t) \xi_t + \xi_{t+1}' X(\hat{\delta}_{t+1}) \xi_{t+1} + \\ \sum_{k=Nt}^{N(t+1)-1} \begin{pmatrix} w_k \\ z_k \end{pmatrix}' P_p \begin{pmatrix} w_k \\ z_k \end{pmatrix} \leq -\epsilon \sum_{k=Nt}^{N(t+1)-1} w_k' w_k. \end{aligned} \quad (4.51)$$

Summation over $t = 0, 1, \dots$, we infer

$$-\xi_0' X(\hat{\delta}_0) \xi_0 + \sum_{k=0}^{\infty} \begin{pmatrix} w_k \\ z_k \end{pmatrix}' P_p \begin{pmatrix} w_k \\ z_k \end{pmatrix} \leq -\epsilon \|w\|_2.$$

Since $X(\hat{\delta}_0) \succ 0$, the first term can be dropped and the LPV system is shown to satisfy the quadratic performance measure. ■

Analysis of \mathcal{H}_2 -performance via lifting

Since the definition given in (4.42) involves a summation over the total number of inputs, it is immediately clear that the \mathcal{H}_2 -norm of a multi-input multi-output system grows unbounded if the number of inputs and/or outputs increases. However, for the N -lifted system, this trouble can be easily overcome by adding a scaling factor, which makes sure that the \mathcal{H}_2 -norm is preserved under lifting.

Following the lines of Section 4.2.2, we notice that at each particular time instant $t = i$ of the lifted system (4.48), a total number of Nn_w different impulsive inputs with which the system can be excited. Let the impulsive inputs be defined

$$\hat{w}^{\nu,i} = (0, 0, \dots, 0, e^\nu, 0, 0, \dots) \quad \text{for } \nu = 1, \dots, Nn_w, i = 1, 2, \dots,$$

and denote $\hat{z}^{\nu,i}$ as the output of the N -lifted system due to input $\hat{w}^{\nu,i}$. It is stressed that the total (infinite) number of different impulsive inputs that can be given to the system stays the same if lifting the system. By definition of the l_2 -norm on vectors, the following relation holds

$$\sum_{i=1}^k \sum_{\nu=1}^{Nn_w} \|\hat{z}^{\nu,i}\|_2^2 = \sum_{j=1}^{Nk} \sum_{\eta=1}^{n_w} \|z^{\eta,j}\|_2^2$$

which implies that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k \sum_{\nu=1}^{Nn_w} \|\hat{z}^{\nu,i}\|_2^2 = N \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k \sum_{\eta=1}^{n_w} \|z^{\eta,i}\|_2^2.$$

Thus, the \mathcal{H}_2 -norm of the N -lifted system is increased by a factor $\frac{1}{N}$. By including the factor N , the following characterization of \mathcal{H}_2 -performance of the LPV system is obtained.

Corollary 4.4 *For a given region of variation \mathcal{R} , the LPV system (4.36) is uniformly exponentially stable and has an \mathcal{H}_2 -performance level smaller than γ if there exists a matrix function $X(\hat{\delta}) = X(\hat{\delta})'$ and $Z(\hat{\delta}) = Z(\hat{\delta})'$ that satisfy*

$$\frac{\text{Tr}(Z(\hat{\delta}))}{N} \leq \gamma^2, \quad \begin{pmatrix} Z(\hat{\delta}) & \hat{B}'_N(\hat{\delta})X(\hat{\delta}) \\ X(\hat{\delta})\hat{B}_N(\hat{\delta}) & X(\hat{\delta}) \end{pmatrix} \succ 0,$$

and

$$\begin{pmatrix} X(\hat{\delta}) - \hat{A}_N(\hat{\delta})'X(\hat{\theta})\hat{A}_N(\hat{\delta}) & \hat{C}_N(\hat{\delta})' \\ \hat{C}_N(\hat{\delta}) & -I \end{pmatrix} \prec 0, \quad (4.52)$$

for all $(\hat{\delta}, \hat{\theta}) \in \hat{\mathcal{R}}_{2N}$.

Proof. The result can be derived along the lines as in Proposition 4.4, using also the arguments in Corollary 4.3. ■

For recent developments on \mathcal{H}_2 -analysis or controller synthesis of linear N -periodic systems, which are obtained from the N -lifted system if the lifted parameter is time-invariant, see [182, 104, 13, 107]. Recently, in [71] the robust \mathcal{H}_2 -synthesis problem was addressed.

Remark 4.10 *In view of the previous discussion, one certainly questions why the standard definition of the \mathcal{H}_2 -norm does not include the scaling factor $1/n_w$. In that case, the \mathcal{H}_2 -norm would naturally be preserved under the lifting operation.*

4.2.4 Asymptotically exact performance analysis

The characterization of the quadratic performance- and \mathcal{H}_2 -performance measure, as given in Propositions 4.3 and 4.4, as well as in Corollaries 4.3 and 4.4, generally provides an upper bound value on the worst case performance level. This is partly due to the fact that the conditions are derived for Lyapunov functions of the form $V(x, k) = x'X(\delta_k)x$, which is generally known to consist of a restrictive function class for proving stability of uncertain time-varying systems. In addition, the results will always depend on the chosen parametrization of $X(\delta)$, $Z(\delta)$ as well as the constructed relaxation scheme.

In view of the asymptotic properties that were derived for the analysis of stability in Section 4.1.4, the following question arises: Do the guaranteed performance levels from the characterizations as given by Corollaries 4.3 and 4.4 converge to the exact robust performance levels of the LPV system, as we let N go to infinity? In order to answer this question on the exactness of the family of performance analysis conditions, we first clarify the link between periodically time-varying and generally time-varying linear systems.

It is a well-known fact that an N -periodic system can be obtained by truncating a given LTV system. From a practical point of view, N periodic systems can approximately describe the LTV system, as long as the period N is chosen large enough. Consider the following periodically time-varying system:

$$\begin{aligned} x_{k+1} &= A_P(k)x_k + B_P(k)w_k, & x_0 &= 0, \\ z_k &= C_P(k)x_k + D_P(k)w_k, \end{aligned} \tag{4.53}$$

in which

$$\begin{aligned} A_P(k + jN) &= A(k), & B_P(k + jN) &= B(k) \\ C_P(k + jN) &= C(k), & D_P(k + jN) &= D(k), \end{aligned}$$

for $j = 0, 1, \dots$

As N increases, the induced l_2 -gain of system (4.53) can be shown to converge to the l_2 -gain of the original LTV system (4.41). A similar fact holds for the \mathcal{H}_2 -performance measure, see for example [13, 107].

This valuable insight proves useful if adopting the usual worst-case notion of performance for LPV systems. Appendix D contains a proof of the fact that the worst-case l_2 -gain of an LPV system can be approximated at any desired accuracy, by restricting the parameter sequence to be periodically time-varying and increasing period N . A similar fact is expected to hold also for the robust \mathcal{H}_2 -performance level of an LPV system, though a proof is not contained in this thesis.

Let us discuss some practical implications. If considering our initial characterization of performance of the LPV system in Sections 4.2.1 and 4.2.2, the least upper bounds on, for example, the l_2 -gain performance level depend on the chosen parametrization of $X(\delta)$ and $Z(\delta)$. It so happens that this source of conservatism typically does not vanish as we increase the total degree of the matrix functions $X(\delta)$, $Z(\delta)$.

On the contrary, if using the suggested family of analysis conditions in Section 4.2.3 this potential source of conservatism does not exist. We actually benefit from the fact that the class of Lyapunov functions of the form $V(x, k) = x'X(\delta_k)x$ is not restrictive for uncertain LTI systems, since analyzing an N -periodic LPV system is equivalent to analyzing the uncertain N -lifted system, which happens to be LTI. Moreover, as shown in [23], the Lyapunov matrix $X(\delta)$ can be chosen to be a polynomial matrix function in δ without loss of generality.

Since numerical computations are performed for some finite value of N , it may still be beneficial to work with parameter dependent matrices $X(\hat{\delta})$ and $Z(\hat{\delta})$. In our experience though, the analysis results often do not improve further by choosing the total degree of $X(\hat{\delta}), Z(\hat{\delta})$ greater than 2. Hence, it makes sense to keep the order of $X(\hat{\delta}), Z(\hat{\delta})$ low while increasing the lifting horizon N . As we already stressed in Section 4.1.7, the lifting approach is most suited for LPV systems with a (possibly) large state dimension and a low number of parameters.

Remark 4.11 *It is an interesting question, whether the performance analysis test based on the N -lifted system becomes non-conservative as $N \rightarrow \infty$, if using a parameter independent Lyapunov function of the form $V(x) = x^T X x$. This fact holds true for the stability analysis problem and directly follows from Theorem 4.1. To be precise, $\bar{\sigma}_N(\mathcal{R}) < 1$ (using the singular value norm) is equivalent to (4.15), which is again identical to (4.16) if the Lyapunov matrix is chosen to be the identity matrix, i.e. $X(\hat{\delta}) = I$. For performance analysis, it is unknown whether $X(\hat{\delta}) = X$ can be assumed without loss of generality.*

Lower bound computations from N -periodic parameter sequences

In order to estimate the level of conservatism and to be able to interpret the computed upper bounds on the performance level, there is a need for computing lower bounds.

The construction of worst-case periodic parameter sequences is a difficult problem. Similar to construction of destabilizing parameter sequences in order to compute the \mathcal{R} -stability margin, we will search among all \mathcal{R} -admissible parameter sequences for an N -periodic sequence that yields poor performance. The sequence δ_{wc} will be called "worst-case" if the performance level of any other \mathcal{R} -admissible parameter sequence is at least as good as the level of performance corresponding to δ_{wc} . In general, a periodic worst-case parameter sequence need not exist.

If the LPV system has only a few parameters and the horizon N is chosen to be small, lower bound values on the performance level can be easily computed by gridding the N -lifted parameter region $\hat{\mathcal{R}}_N$. Analogous to what we have seen in the construction of destabilizing parameter sequences, a more systematic approach is provided by applying Theorem 2.6. Let us briefly sketch this procedure.

Lower bounds from exactness of the relaxation

For the purpose of this section, we consider the l_2 -gain performance measure. Suppose an upper bound on the robust l_2 -gain of the N -lifted LPV system has been computed by implementing a relaxation scheme for the robust SDP constraints (4.49)-(4.50) for some chosen parametrization of $X(\hat{\delta})$. Note that this involves semi-infinite constraints in the lifted parameter vector $(\hat{\delta}, \hat{\theta}) \in \hat{\mathcal{R}}_{2N}$.

Since $(\hat{\delta}, \hat{\theta}) \in \hat{\mathcal{R}}_{2N}$, any upper bound that is obtained from the analysis conditions in Corollaries 4.3 and 4.4 serves as an upper bound on any N -periodic LPV system.

The question arises as to find a suitable $\hat{\delta}$ for which the performance of the N -lifted system is poor.

We further assume that the constructed relaxation scheme is exact with optimal value γ_{rel} , and that a single representative parameter pair $(\hat{\delta}^0, \hat{\theta}^0) \in \hat{\mathcal{R}}_{2N}$ exists. Our goal is to construct a worst-case parameter sequence based on this tuple $(\hat{\delta}^0, \hat{\theta}^0)$.

Let us follow the procedure for constructing destabilizing parameter sequences, as discussed in Section 4.1.6. By definition of $(\hat{\delta}^0, \hat{\theta}^0) \in \hat{\mathcal{R}}_{2N}$ being a single worst-case parameter vector (see Section 2.4.2), we infer that γ_{rel} equals the optimal value of the LMI problem that results from the substitution $(\hat{\delta}, \hat{\theta}) \rightarrow (\hat{\delta}^0, \hat{\theta}^0)$ into (4.49)-(4.50). In contrast to the result in Corollary 4.2, in which any given single representative pair $(\hat{\delta}^0, \hat{\theta}^0) \in \hat{\mathcal{R}}_{2N}$ proves instability of the LPV system, we have to distinguish between the case in which $\hat{\delta}^0 = \hat{\theta}^0$ and $\hat{\delta}^0 \neq \hat{\theta}^0$.

A guaranteed lower bound on the worst-case l_2 -gain is easily obtained in the case of having $\hat{\delta}^0 = \hat{\theta}^0$. In fact, substituting the (time-invariant) parameter sequence $\hat{\delta} = (\hat{\delta}^0, \hat{\delta}^0, \dots)$ into the N -lifted system (4.48) yields an LTI system with an \mathcal{H}_∞ -norm γ_{rel} . Since the l_2 -gain is not affected by lifting, the value $\gamma_{\text{lb}} = \gamma_{\text{rel}}$ serves as a lower bound on the worst-case l_2 -gain of LPV system (4.36), with the \mathcal{R} -admissible N -periodic sequence chosen as $\delta = (\hat{\delta}_1^0, \dots, \hat{\delta}_N^0, \hat{\delta}_1^0, \dots, \hat{\delta}_N^0, \hat{\delta}_1^0, \dots)$.

If $\hat{\delta}^0 \neq \hat{\theta}^0$, the construction of a worst-case parameter sequence is less obvious. An ad hoc solution would be to compute the l_2 -gain for the N -periodic parameter sequence $(\hat{\delta}_1^0, \dots, \hat{\delta}_N^0, \hat{\delta}_1^0, \dots, \hat{\delta}_N^0, \hat{\delta}_1^0, \dots)$ or $(\hat{\theta}_1^0, \dots, \hat{\theta}_N^0, \hat{\theta}_1^0, \dots, \hat{\theta}_N^0, \hat{\theta}_1^0, \dots)$. However, these sequences not necessarily result in poor performance. In order to be able to extract a parameter sequence for which the l_2 -gain equals the upper bound value of the computed relaxation, a different problem should have been solved. That is, exactness is required of a multiplier-based relaxation scheme for the robust l_2 -gain analysis problem of an N -periodic LPV system (4.36), rather than for the generally time-varying LPV system. The corresponding test amounts to first performing the substitution $\hat{\theta} \rightarrow \hat{\delta}$ in the robust SDP constraints (4.49)-(4.50), after which the semi-infinite constraints are imposed on the lifted parameter region $\hat{\mathcal{R}}_N$, rather than $\hat{\mathcal{R}}_{2N}$. In this fashion, we are guaranteed that once the relaxation scheme with optimal value γ_{rel} is exact and $\hat{\delta}^0 \in \hat{\mathcal{R}}_N$ is single-representative, the N -periodic parameter sequence $(\hat{\delta}_1^0, \dots, \hat{\delta}_N^0, \hat{\delta}_1^0, \dots, \hat{\delta}_N^0, \hat{\delta}_1^0, \dots)$ leads to an l_2 gain value γ_{rel} .

In the following section, we present a numerical example that further illustrates the developed analysis method for discrete-time LPV systems.

4.3 Numerical example: l_2 -gain analysis

In order to illustrate the proposed analysis method and compare it with the analysis tools based on IQCs as discussed in the previous chapter. Recall the discrete-time LPV system from Section 3.4:

$$\begin{aligned} x_{k+1} &= \begin{pmatrix} 0 & 1 \\ -0.5 & -0.5 + \delta_k \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w_k, & x_0 = \bar{x}_0 \\ z_k &= \begin{pmatrix} 0 & 1 \end{pmatrix} x_k. \end{aligned} \quad (4.54)$$

The goal in this section is to compute the worst-case l_2 -gain γ_{wc} of channel $w \rightarrow z$ for a given class of parameter trajectories described by a region of variation \mathcal{R} according to Definition 4.1. The considered region of variation $\hat{\mathcal{R}}$ is a box i.e. $\hat{\mathcal{R}} = \boldsymbol{\delta} \times \boldsymbol{\delta} \subset \mathbb{R}^2$ with $\boldsymbol{\delta} = [-1, 1]$, and involves the (trivial) parameter variation bound $|\delta_{k+1} - \delta_k| \leq 2$. We will first apply Proposition 4.3 with a parameter independent or parameter dependent Lyapunov matrix $X(\delta)$ for the original system matrices. Then, the N -lifted system is considered in order to improve upper- and lower bounds.

4.3.1 Stability analysis

Recall that the \mathcal{R} -stability margin is defined as the largest r for which the LPV system is stable for all $r\mathcal{R}$ -admissible parameter sequences. As discussed in Section 4.1.6, a lower bound r on the stability margin is obtained from any relaxation scheme for the robust SDP problem (4.32), whenever the optimal value $\gamma_{\text{rel}} < 0$. Let the Lyapunov matrix have the following structure

$$X(\hat{\delta}) = T_X(\hat{\delta})' X_c T_X(\hat{\delta}), \quad (4.55)$$

in which X_c represents a (full) symmetric coefficient matrix. In particular, we choose

$$T_X(\hat{\delta}) = \begin{pmatrix} I \\ \hat{\delta} \otimes I \end{pmatrix}, \quad (4.56)$$

which is assumed to be given as an LFR, and $\hat{\delta} \in \hat{\mathcal{R}}_N$. The symbol I denotes the identity matrix of size 2, corresponding the state-dimension of system (4.54).

For the horizon $N = 1, \dots, 4$, a convex hull relaxation scheme is employed for robust SDP (4.32), from which the LPV system is proven stable for $|\delta| < 0.57$. Moreover, by gridding the lifted parameter region $\hat{\mathcal{R}}_N$ for various values of N , a 3-periodic destabilizing switching sequence of amplitude $|\delta| = 0.58$ was found, which is identical to the one obtained in [111]. Hence, we conclude that the stability margin lies in the interval $[0.57, 0.58]$.

4.3.2 l_2 -gain analysis

The robust performance analysis problem is performed with a reduced uncertainty size, so to make sure that the LPV system is uniformly exponentially stable. We consider the intervals $\delta \in \boldsymbol{\delta} = [-r, r]$ for a number of values $r \in [0.3, 0.45]$. Based on Proposition 4.3, upper bounds on the worst-case l_2 -gain will be computed. The results will depend on the chosen parametrization for $X(\cdot)$ as well as on the particular relaxation scheme. With $X(\delta)$ of the form (4.56), for some fixed matrix $T_X(\hat{\delta})$ the resulting robust SDP constraints (4.38) and (4.39) should first be written in the general symmetric form (2.2). Then, by making use of the relaxation toolbox [55], one can easily build a whole range of different relaxation schemes from which numerical results can be obtained. For the case $N = 1$, let us show in detail how the robust SDP constraints are put into the form (2.2).

First, a particular version of Schur's Lemma is applied to (4.39), see Appendix A, that will remove the rational dependence on γ . In fact, (4.39) is equivalent to

$$\tilde{F}(\delta, \theta)' \tilde{J}(X_c, \gamma) \tilde{F}(\delta, \theta) \prec 0 \quad \text{for all } (\delta, \theta) \in \hat{\mathcal{R}}, \quad (4.57)$$

with the abbreviations

$$\tilde{J}(X_c, \gamma) = \left(\begin{array}{ccc|ccc} -X_c & 0 & 0 & 0 & 0 & 0 \\ 0 & X_c & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma I \end{array} \right)$$

and

$$\tilde{F}(\delta, \theta) = \left(\begin{array}{ccc} T_X(\delta) & 0 & 0 \\ T_X(\theta)A(\delta) & T_X(\theta)B(\delta) & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \\ C(\delta) & D(\delta) & 0 \\ 0 & 0 & I \end{array} \right).$$

Note that, due to the particular structure of $X(\delta)$, the decision variables can easily be separated from the known parameter dependent terms. Next, construct an LFR $\tilde{F}_1(\delta, \theta) = P_{F_1} F_1(\delta, \theta)$ in which

$$P_{F_1} = \left(\begin{array}{cc} I & 0 \\ D_{F_1} & C_{F_1} \end{array} \right) \quad \text{and} \quad F_1(\delta, \theta) = \left(\begin{array}{c} \Delta_{F_1}(\delta, \theta)(I - A_{F_1} \Delta_{F_1}(\delta, \theta))^{-1} B_{F_1} \\ I \end{array} \right).$$

By defining $J_1(X_c, \gamma) = P_{F_1}' \tilde{J}(X_c, \gamma) P_{F_1}$, we arrive at

$$F_1(\delta, \theta)' J_1(X_c, \gamma) F_1(\delta, \theta) \prec 0 \quad \text{for all } (\delta, \theta) \in \hat{\mathcal{R}}, \quad (4.58)$$

which is precisely the general robust SDP constraint (2.2) from Chapter 2. In a similar fashion, the robust SDP constraint

$$X(\delta) \succ 0 \quad \text{for all } \delta \in \mathcal{D} \quad (4.59)$$

can be turned into the general form (2.2) leading to

$$F_2(\delta, \theta)' J_2(X_c, \gamma) F_2(\delta, \theta) \prec 0 \quad \text{for all } (\delta, \theta) \in \hat{\mathcal{R}}, \quad (4.60)$$

for some appropriate matrices A_{F_2}, B_{F_2}, J_2 and F_2 of the form

$$F_2(\delta, \theta) = \begin{pmatrix} \Delta_{F_2}(\delta, \theta)(I - A_{F_2}\Delta_{F_2}(\delta, \theta))^{-1}B_{F_2} \\ I \end{pmatrix}.$$

All of the relaxation schemes in this section are multiplier-based, and make use of the convexity arguments from Section 2.3.1. If we denote, again for $N = 1$, the four generators of the region $\hat{\mathcal{R}} = [-r, r] \times [-r, r]$ as (δ^ν, θ^ν) , $\nu = 1, \dots, 4$, an upper bound on the l_2 -gain γ_{wc} is obtained by infimizing γ subject to the LMI constraints

$$\begin{pmatrix} I & 0 \\ A_{F_i} & B_{F_i} \end{pmatrix}' \Pi_i \begin{pmatrix} I & 0 \\ A_{F_i} & B_{F_i} \end{pmatrix} + J_i(X_c, \gamma) \prec 0, \quad (4.61)$$

in which full block multipliers $\Pi_i = \Pi_i'$ satisfy

$$\begin{pmatrix} \Delta_{F_i}(\delta^\nu, \theta^\nu) \\ I \end{pmatrix}' \Pi_i \begin{pmatrix} \Delta_{F_i}(\delta^\nu, \theta^\nu) \\ I \end{pmatrix} \succ 0, \quad \nu = 1, \dots, 4 \quad (4.62)$$

and

$$\begin{pmatrix} I \\ 0 \end{pmatrix}' \Pi_i \begin{pmatrix} I \\ 0 \end{pmatrix} \prec 0 \quad (4.63)$$

for $i = 1, 2$.

Discussion of the numerical results

In the first upper bound computation, referred to as UB-0, the Lyapunov matrix is chosen to be a parameter independent matrix, i.e. $X(\delta) = X_c$. As shown above, the condition (4.39) for quadratic performance turns into a robust SDP (4.58). Since X_c does not depend on the parameter, the outer factor $F_1(\delta, \theta)$ in (4.58) no longer depends on θ .

The convex hull relaxation for guaranteeing conditions (4.62)-(4.63) with a single parameter involves a total of 3 LMIs. Moreover, the relaxation is guaranteed to be exact, since we use a full block multiplier, see for a proof [156, 124]. In Figure 4.4 the resulting l_2 -gain upper bound values are shown and a comparison with other relaxation schemes in this section is made in Table 4.1. The upper bounds of UB-0 are conservative, as is shown next.

Parameter dependent Lyapunov matrix

From now on, the Lyapunov matrix is chosen to be of the form (4.55) with $T_X(\delta)$ defined in (4.56). The multiplier-based convex hull relaxation scheme is applied to the robust SDP (4.58) for the region of variation $\hat{\mathcal{R}} = \delta \times \delta$. The resulting values γ_{rel} are plotted in Figure 4.4 and referred to as UB-1. While it is immediately clear that the results improve if using a parameter dependent Lyapunov matrix. Nevertheless, in this example, the upper bounds do not decrease further when adding higher order monomials to the basis $T_X(\delta)$.

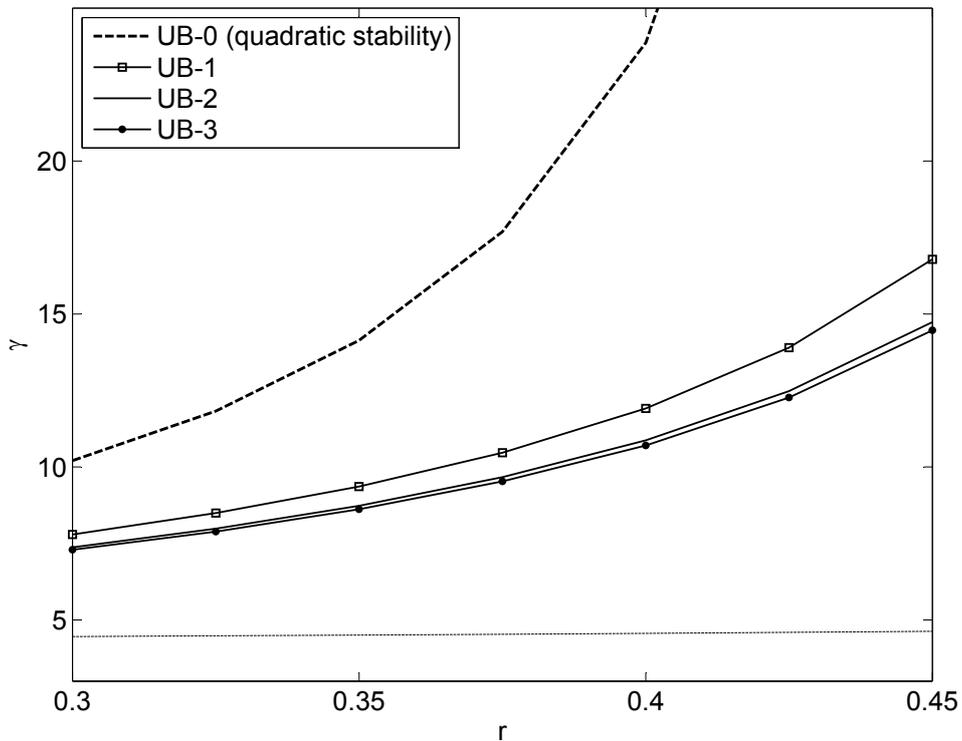


Figure 4.4: l_2 -gain analysis for system (4.54) with time-varying parameters, upper bound computations. Also indicated is the exact lower bound for time-invariant parameters (dashed), which is obtained by sampling the set δ .

Improved upper bounds using the N -lifted system

As explained in Section 4.2.3, analyzing performance of the N -lifted system (with $N > 1$) is expected to improve our results. In view of the fact that the time-varying parameter can vary arbitrarily in the interval $[-r, r]$, the region $\hat{\mathcal{R}}_{2N}$ amounts to a box around the origin in \mathbb{R}^{2N} , see again Figure 4.1. A convex hull relaxation therefore involves $2^{2N} = 4^N$ LMI constraints corresponding to the generators of

$\hat{\mathcal{R}}_{2N}$. The upper bounds obtained for $N = 2, \dots, 6$ are denoted by UB-2, UB-3, UB-4, UB-5 and UB-6, with the corresponding results shown in Table 4.1. We observe that the upper bound on the worst case l_2 -gain no longer improves when choosing N greater than 4.

	$T_X(\delta)$	$\# x_i$	$\#$ LMIs	$\#$ Vars	γ for $r = 0.45$
UB-0	1	1	4 +1	8	88.56
UB-1	$1, \delta_1$	2	6 +4	47	16.78
UB-2	$1, \delta_1$	4	18 +6	193	14.73
UB-3	$1, \delta_1$	6	66 +10	443	14.47
UB-4	$1, \delta_1$	8	256 +18	797	14.41
UB-5	$1, \delta_1$	10	1026 +32	1255	14.41
UB-6	$1, \delta_1$	12	4098 +64	1817	14.41

Table 4.1: l_2 -gain upper bounds for various relaxations. The number of uncertain parameters is indicated by $\# x_i$. The number of the LMIs corresponding the relaxations for (4.58) and (4.60) respectively, is denoted by $\#$ LMIs.

Exactness and computation of lower bounds

As it is proven in Appendix D, the worst-case lower bounds of N -periodic LPV systems will get arbitrarily close to γ_{wc} as the horizon N is increased. Therefore, we now sample the lifted parameter region $\hat{\mathcal{R}}_N$, for values $N = 1, \dots, 4$. Using a uniform grid of 5^N points, the worst-case l_2 -gains that are found by computing the \mathcal{H}_∞ -norm of the N -lifted LTI system have been listed in Table 4.2. Note that the lower bound in Figure 4.4 is compatible with the value $\gamma = 4.63$ at $r = 0.45$.

The lower bound value $\gamma = 14.41$ for the 3-periodic parameter sequence proves exactness of relaxations UB-4, UB-5, UB-6. Unfortunately, this triple $(r, -r, r)$ could not be extracted from the exactness test in Theorem 2.6. In fact, condition (2.69) was infeasible and the polynomial system constructed by an element wise implementation of (2.68) could not be solved due to the complexity of the problem. In order to give an impression, the relaxation UB-3 involves a dual multiplier M in (2.68) of size 22×22 , and $\text{rank}(M)=7$, and the size of Z^ν is 8.

worst-case δ	γ for $r = 0.45$
(r, r, \dots)	4.63
$(r, -r, r, \dots)$	6.69
$(r, r, -r, r, \dots)$	14.41
$(r, -r, r, -r, r, \dots)$	6.69

Table 4.2: N -periodic parameter sequences. The indicated l_2 gain levels γ correspond to periodic time-varying systems and therefore provide a lower bound value on the worst case l_2 -gain of the LPV system.

4.4 Summary

With an initial focus on computing the stability margin of LPV systems, the first analysis test that was considered makes use of a quadratic-in-the-state Lyapunov function which is parameter dependent. A family of robust SDP constraints is obtained by increasing the order of the parameter dependent Lyapunov matrix. Generally speaking, the analysis conditions in this family are sufficient for stability only, since a quadratic in the state Lyapunov function need not exist.

At present, it is unknown how to estimate the level of conservatism of the Lyapunov-based sufficient conditions for stability of uncertain systems. With a lack of efficient numerical algorithms for the so-called “converse theorems”, we have made initial steps towards an alternative framework for stability and performance analysis of general LPV systems in discrete time. As one of the main results, the proposed method provides a systematic reduction of the level of conservatism by constructing a suitable family of robust SDPs.

The family of sufficient conditions for stability is based on the monodromy matrix \hat{A}_N and closely linked to the notion of the joint spectral radius, as it is known for the analysis of switched systems. Theorem 4.1 shows that the constructed family is asymptotically exact, which means that for some large N , the derived stability condition is necessary and sufficient for stability. This fundamental result can be seen as an extension of Gelfand’s formula, providing a numerically tractable solution for analyzing stability of LPV systems with general regions of variation.

Despite the asymptotic exactness of the derived family of stability conditions, it is of practical interest to estimate the level of conservatism in particular numerical computations. For this reason, we discussed periodic destabilizing parameter sequences as a tool for proving instability, leading to upper bounds on the stability margin. When combining the proposed approach with the exactness test in Section 2.4.2, destabilizing parameter sequences can (in principle) be extracted from the multiplier-based relaxations, provided the computed relaxation scheme is exact.

What can be done for stability has been extended to performance by making use of a conceptually simple lifting technique. Analysis conditions have been presented in terms of the N -lifted LPV system for the quadratic performance and \mathcal{H}_2 -performance measures. Once the parametrization of the Lyapunov matrix is fixed, these conditions amount to solving a robust SDP. The crucial role of N -periodic parameter trajectories establishes a close link between our approach to the stability analysis and performance analysis problem.

Finally, the proposed analysis tools have been illustrated on a numerical example, in which the l_2 -gain of a given LPV system was investigated. The very same system was analyzed in Section 3.4 by using the alternative analysis method based on IQCs. For the presented example, the Lyapunov based analysis method showed significant improvements. Exactness of the N -lifted analysis conditions was proven by constructing a worst-case N -periodic parameter sequence, providing a lower bound on the robust l_2 -gain of an LPV system.

Chapter 5

A convex robust synthesis solution for specific generalized plant structures

In the previous two chapters we have discussed two existing approaches to the modeling and analysis of uncertain systems. First, in Chapter 3, we presented the IQC framework, which allows to incorporate all sorts of uncertainties. Second, for the particular class of parametric uncertainties, the analysis of LPV systems based on Lyapunov arguments was addressed in Chapter 4.

In this and the next chapter we further elaborate on the use of these two different modeling environments and focus on the following question: Given an uncertain system, can a certain performance level be achieved by applying feedback control? Second, how to compute such a controller? While the present chapter focusses on controller synthesis in the IQC framework, the next chapter is concerned with the design of scheduled controllers for LPV systems.

Following the analysis approach of Chapter 3, let a parameterized class of multipliers $\Pi \in \mathbf{\Pi}$ be given for a specified set of uncertain operators Δ . As we will see in Section 5.3, robust controller design in the IQC framework involves optimization over both the multiplier variables and the controller matrices. At present, it is unknown whether simultaneous optimization over these variables is possible while keeping the problem convex, which is why iterative schemes have been suggested. Such heuristic methods are based on either fixing the scalings while searching over the controller variables or fixing the controller and finding suitable multipliers. In this chapter, we will show that no such heuristics are needed when the generalized plant has a certain structure, and that optimal robust controllers can be obtained from a single convex optimization problem.

This chapter is organized as follows. In the next section, we give a motivating controller design problem that naturally leads to a particularly structured generalized plant. In Section 5.2 we formally state the problem, after which a brief recap on IQC analysis will be given. Our main robust synthesis result is presented in Section 5.3. We first show how to characterize stability of the nominal closed loop system, and then exploit the problem structure in order to arrive at the LMI synthesis conditions. It is also shown how to eliminate controller variables, similar as it can be done in the nominal output feedback synthesis problem. The numerical example of Section 5.4 concerns the rejection of time-varying sinusoidal disturbances with (a priori) unknown frequency that vary slowly in time. By making use of a suitable multiplier class as proposed in [113], see also Section 3.3, we are able to improve performance by incorporating parameter variation bounds. A closing section will finally summarize the results.

5.1 Motivating example: uncertainty in the disturbance model

The well-known \mathcal{H}_∞ - or \mathcal{H}_2 -synthesis procedure involve adding suitable transfer functions to the plant input/output such that norm minimization of the weighted plant results in satisfactory performance of the closed-loop system. In case that the nature of the disturbance input can be nicely captured by an LTI filter, this filter typically acts as a weight on the input. For instance, measurement noise with certain spectral properties can be effectively dealt with by adding a suitable coloring filter at the plant input. Despite the fact that various parameterized disturbance models are available in the literature, e.g. for describing wind turbulence acting on aircraft, [91, 95], water waves acting on ships, [121], or models of the road roughness used in ride quality analysis of land vehicles, [84], most controller synthesis techniques do not exploit this knowledge as already recognized by some authors, see [49].

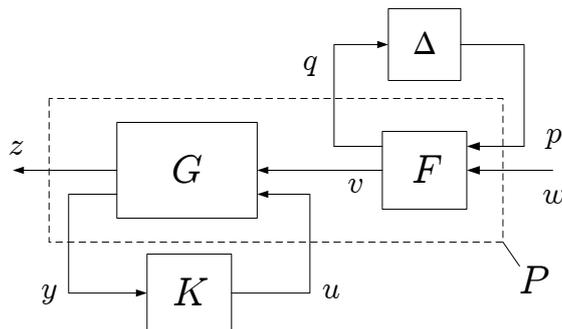


Figure 5.1: System interconnection with uncertain filter F

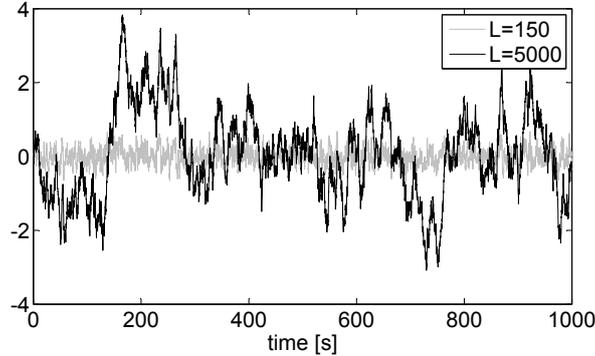


Figure 5.2: Filtered white noise for filter $D(s, \delta)$ in (5.1) with $V = 150$ m/s and two turbulence scales L .

As a particular design example in which the disturbance filter is affected by uncertainty, consider the goal of designing a flight controller to improve passenger comfort or dampen the flexible structure. Referring to the interconnection of Figure 5.1, suppose G represents the LTI model dynamics of an aircraft flying in turbulent air. The gust input, acting horizontal or vertical direction, is denoted by v and z represents the vertical acceleration. The LTI filter F at the plant input represents the atmospheric turbulence which depends on altitude, the speed of the aircraft and the type of weather in which the aircraft is flying. With the control input denoted by u and the measurement output denoted by y , we consider the goal of designing a controller K that achieves the least possible level of vertical acceleration.

Atmospheric turbulence can be viewed as a random process whose power spectrum is known. It is often approximated by a filtered white noise signal, with parameterized filters of the form

$$D(s, \delta) = \frac{\sqrt{\delta}}{\delta s + 1}, \quad (5.1)$$

in which $\delta = \frac{L}{V}$ and where V is the airspeed and L the so-called 'turbulence scale', see [95, 49] and references therein. In Figure 5.2, the output of the filter is shown for a particular realization of a white noise source and for two different values of L .

Now consider the problem of minimizing the worst-case variance of the vertical acceleration for a fixed class of disturbance filters. In view of the fact that both the stochastic and impulse response interpretations of the \mathcal{H}_2 -norm, as given in Appendix A, are equivalent for LTV systems, we will employ the robust \mathcal{H}_2 -performance measure from Section 3.2. Let a minimal realization of G be given as

$$G := \begin{pmatrix} G_{zv} & G_{zu} \\ G_{yv} & G_{yu} \end{pmatrix} = \left[\begin{array}{c|cc} A & B_v & B_u \\ \hline C_z & D_{zv} & D_{zu} \\ C_y & D_{yv} & 0 \end{array} \right]$$

and let the parameter dependent filter be given in LFR form, i.e. $F_\delta(s, \delta) = \Delta(\delta) \star F(s)$ with

$$F := \begin{pmatrix} F_{qp} & F_{qw} \\ F_{vp} & F_{vw} \end{pmatrix} = \left[\begin{array}{c|cc} A_F & B_p & B_w \\ \hline C_q & D_{qp} & D_{qw} \\ C_v & D_{vp} & D_{vw} \end{array} \right]$$

and $\Delta(\delta) = \delta I$. The interconnection of F with Δ is defined as

$$\begin{aligned} p &= \Delta(q) \\ q &= F_{qp}p + F_{qw}w \\ v &= F_{vp}p + F_{vw}w. \end{aligned}$$

Since the dynamics of F are not affected by the control input, all eigenvalues of A_F are assumed to lie in the open left half plane. The generalized plant P is formed by merging the dynamics of F with those of the plant P , that is

$$P = \begin{pmatrix} F_{qp} & F_{qw} & 0 \\ G_{zv}F_{vp} & G_{zv}F_{vw} & G_{zu} \\ G_{yv}F_{vp} & G_{yv}F_{vw} & G_{yu} \end{pmatrix}, \quad (5.2)$$

as indicated by the dashed box in Figure 5.1. Note that the transfer matrix from u to q is zero, which is an immediate consequence of the fact that the uncertainty only affects F and not G . It is precisely this property that will enable us to derive a convex solution to the synthesis problem. We point to the recent work [56], in which our synthesis algorithm in this chapter has been applied to a magnetic bearing system.

Related work on robust disturbance rejection

The problem of disturbance rejection against more specific families of disturbances has been considered in various contexts. First, building on the flight control design example, Davison in [49] has developed a number of frequency domain techniques for SISO systems. Second, inspired by [82], Scherer in [154] considers random disturbance inputs of which the (uncertain) covariance coefficients are specified in an a priori given set. Related work on the control of uncertain systems with stochastic uncertainty can be found in [175, 134] and references therein.

In [103], a class of disturbance signals is directly described in terms of an integral quadratic constraint and a suitably chosen multiplier class. It is an interesting topic of future research to investigate the modeling power of such an approach in solving practical disturbance rejection problems.

Finally, in [27], the robust optimal \mathcal{H}_∞ - or \mathcal{H}_2 - synthesis problem is solved for a so-called “signal polytope”. The uncertain state-space matrices of the disturbance filter are assumed to lie in a given convex polytope which explains the terminology. It is closely related to the general synthesis solution that we propose though our solution allows for general rational parameter dependence of the system matrices of the filter.

5.2 Problem formulation

The convex conditions for the robust controller synthesis problem that are contained in the next section are derived for the generalized plant configuration in Figure 5.3, where

$$P = \begin{pmatrix} P_{qp} & P_{qw} & 0 \\ P_{zp} & P_{zw} & P_{zu} \\ P_{yp} & P_{yw} & P_{yu} \end{pmatrix}. \quad (5.3)$$

It is assumed that P represents an LTI system in continuous time. The measured output and control input are denoted by y and u respectively, and the performance channel is denoted as $w \rightarrow z$. The uncertain operator Δ maps signals $q \in \mathcal{L}_{2,e}^{n_q}$ to $p \in \mathcal{L}_{2,e}^{n_p}$ and affects the nominal plant P in the usual feedback interconnection, defined through the relations

$$\begin{aligned} p &= \Delta(q), \\ q &= P_{qp}p + P_{qw}w. \end{aligned} \quad (5.4)$$

The interconnection of P with Δ is assumed to be well-posed, that is $I - P_{qp}\Delta$ has a causal and bounded inverse for all $\Delta \in \mathbf{\Delta}$, in which $\mathbf{\Delta}$ is a predefined set of uncertain operators, see also Chapter 3.

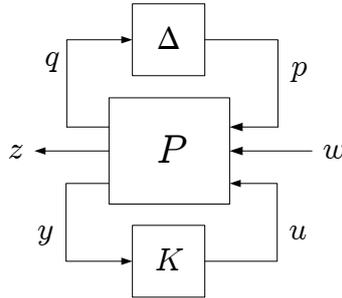


Figure 5.3: Generalized plant

The robust controller synthesis problem can be formulated as follows: *Design an LTI controller, denoted by*

$$K(s) := \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right], \quad (5.5)$$

which achieves, for all $\Delta \in \mathbf{\Delta}$, the desired closed-loop performance as it is specified on the channel w to z .

Despite the fact that dynamic IQCs can capture a vast amount of different types of uncertainties, there are only few results on designing robust controllers in the IQC framework, see e.g. [4]. Although the robust synthesis problem is formulated without specifying the uncertainties, we will more carefully study time-varying parameters.

5.3 Robust synthesis via LMIs

The robust synthesis solution that is developed in this section was first published in our paper [62] for static multipliers, after which the general solution with dynamic multipliers followed in [59, 63].

We will start from an LMI characterization of performance from Chapter 3, that is either a quadratic performance such as the induced \mathcal{L}_2 -gain or some \mathcal{H}_2 -performance criterion. Since the IQC-based analysis result for stability and performance can only be applied if the nominal closed loop system is stable, a recently developed characterization of nominal stability is discussed in Section 5.3.2. By making use of two existing congruence transformations, our main synthesis solution is presented in Section 5.3.3. Finally, the algorithm is illustrated on a numerical example.

5.3.1 From analysis to controller synthesis

The design of a (robust) optimal feedback controller in the LMI framework typically starts with an LMI characterization of the desired performance measure, e.g. the \mathcal{H}_∞ -norm or \mathcal{H}_2 -norm. These are found in Appendix A and in Chapters 3 and 4 for uncertain systems. As for all \mathcal{H}_∞ - or \mathcal{H}_2 - norm based design methodologies, this design method involves choosing suitable weighting functions at the plant input and output.

Since the system matrices depend on to-be-designed controller variables, the conditions for example in Proposition 3.1 are no longer affine in the multiplier Π and the controller variables. Thus, LMI solvers are unable to solve the synthesis problem directly. Let us examine the precise difficulties that one encounters when using an analysis condition for controller design purposes.

For this purpose, let us introduce the following realization for the closed loop system

$$P(s) \star K(s) = \begin{pmatrix} \mathcal{P}_{qp}(s) & \mathcal{P}_{qw}(s) \\ \mathcal{P}_{zp}(s) & \mathcal{P}_{zw}(s) \end{pmatrix} = \left[\begin{array}{c|cc} \mathcal{A} & \mathcal{B}_p & \mathcal{B}_w \\ \hline \mathcal{C}_q & \mathcal{D}_{qp} & \mathcal{D}_{qw} \\ \mathcal{C}_z & \mathcal{D}_{zp} & \mathcal{D}_{zw} \end{array} \right]. \quad (5.6)$$

In the next section, we give the explicit dependence of these matrices on the controller variables A_K, B_K, C_K, D_K . For the moment, we do not yet introduce a realization of P , and keep the details for later. The following analysis result is a direct consequence of Proposition 3.1 in Section 3.1. Recall the fact that the dynamic multiplier was factorized as $\Pi = \Psi'Q\Psi$ in which $A_\Psi, C_\Psi, B_{\Psi_i}, D_{\Psi_i}$ for $i = 1, 2$ denoted the realization matrices of Ψ .

Corollary 5.1 *The closed-loop system (5.6) with performance channel $w \rightarrow z$ satisfies quadratic performance for all $\Delta \in \Delta$ if \mathcal{A} is stable and, for some $Q \in \mathcal{Q}$ there*

exists \mathcal{X} , partitioned as

$$\mathcal{X} = \begin{pmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{pmatrix}, \quad (5.7)$$

such that

$$\left(\begin{array}{cc|cc} 0 & \mathcal{X} & 0 & 0 \\ \mathcal{X} & 0 & 0 & 0 \\ \hline 0 & 0 & Q & 0 \\ \hline 0 & 0 & 0 & P_p \end{array} \right) \left(\begin{array}{cc|cc} A_\Psi & B_{\Psi_1}C_q & B_{\Psi_1}D_{qp} + B_{\Psi_2} & B_{\Psi_1}D_{qw} \\ 0 & \mathcal{A} & \mathcal{B}_p & \mathcal{B}_w \\ \hline C_\Psi & D_{\Psi_1}C_q & D_{\Psi_1}D_{qp} + D_{\Psi_2} & D_{\Psi_1}D_{qw} \\ \hline 0 & 0 & 0 & I \\ 0 & C_z & \mathcal{D}_{zp} & \mathcal{D}_{zw} \end{array} \right) \prec 0. \quad (5.8)$$

Proof. The result directly follows by considering condition (3.13) for the closed loop plant, rather than the given LTI plant M . In particular, the outer factor in (5.8) is defined by the realization matrices of

$$\left(\begin{array}{cc|cc} \Psi_1 & \Psi_2 & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \end{array} \right) \left(\begin{array}{cc} I & 0 \\ \mathcal{P}_{qp} & \mathcal{P}_{qw} \\ \hline 0 & I \\ \mathcal{P}_{zp} & \mathcal{P}_{zw} \end{array} \right). \quad (5.9)$$

■

If Theorem 3.1 is to be used for controller synthesis purposes, there are two main issues that need to be solved in order to render the synthesis conditions convex in all variables. First, due to the multiplication of the controller variables in $\mathcal{A}, \mathcal{B}_p, \dots$ with the matrices \mathcal{X} and Q , it is not possible to simultaneously optimize over all decision variables by semi-definite programming. A suitable congruence transformation will resolve this problem, by exploiting the fact that $P_{qu} = 0$.

Second, the condition in Corollary 5.1 proves robust performance under the assumption that \mathcal{A} is stable. Clearly, for an a priori known system this can easily be verified. In order to guarantee that the resulting synthesis solution has the property that the closed loop system is stable for $\Delta = 0$, a suitable constraint must be added, which is the topic of our next section.

5.3.2 A new characterization of nominal stability

The purpose of this section is to give an elementary proof for a recently developed characterization of stability of \mathcal{A} in (5.8) in terms of an LMI constraint. Note that $Q \in \mathcal{Q}$ is generally an indefinite matrix, which implies that positivity of the Lyapunov matrix \mathcal{X} is no longer the appropriate condition for characterizing closed loop stability, see [8, 162] for a detailed discussion. Recently, a full characterization of stability has been presented in [161]. As compared to the proof given there, we provide an alternative and elementary proof that does not rely on a particular structure of the realization matrices of Ψ .

Recall the notation from Chapter 3 and suppose that we are given a multiplier $\Pi = \Psi^* Q \Psi$ for which the IQC (3.2) holds for all $\Delta \in \mathbf{\Delta}$. Further, let the realization of Ψ be given as

$$\Psi = \begin{pmatrix} \Psi_1 & \Psi_2 \end{pmatrix} = \left[\begin{array}{c|cc} A_\Psi & B_{\Psi_1} & B_{\Psi_2} \\ \hline C_\Psi & D_{\Psi_1} & D_{\Psi_2} \end{array} \right] \quad (5.10)$$

with stable $A_\Psi \in \mathbb{R}^{n_\Psi \times n_\Psi}$. Under the assumption that 0 is contained in the set $\mathbf{\Delta}$, it follows from (3.2) that

$$\Pi_{22}(i\omega) \succeq 0 \quad \text{for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

If Π satisfies (3.11), note that $\Pi_\epsilon = \Pi + \begin{pmatrix} 0 & 0 \\ 0 & \epsilon I \end{pmatrix}$ is admissible and still satisfies (3.11) for some small $\epsilon > 0$. Therefore, without loss of generality, we can assume

$$\Pi_{22}(i\omega) \succ 0 \quad \text{for all } \omega \in \mathbb{R} \cup \{\infty\}.$$

This translates into

$$\Psi_2(i\omega)^* Q \Psi_2(i\omega) \succ 0 \quad \text{for all } \omega \in \mathbb{R} \cup \{\infty\} \quad (5.11)$$

Then, by the KYP lemma, the frequency domain inequality (5.11) is equivalent to the existence of $\check{X} = \check{X}^T$ such that

$$\begin{pmatrix} I & 0 \\ A_\Psi & B_{\Psi_2} \\ C_\Psi & D_{\Psi_2} \end{pmatrix}' \begin{pmatrix} 0 & \check{X} & 0 \\ \check{X} & 0 & 0 \\ 0 & 0 & Q \end{pmatrix} \begin{pmatrix} I & 0 \\ A_\Psi & B_{\Psi_2} \\ C_\Psi & D_{\Psi_2} \end{pmatrix} \succ 0. \quad (5.12)$$

In the next theorem, we characterize robust performance, without assuming that the closed loop system is stable, as opposed to Corollary (5.1). Stability of \mathcal{A} can be captured by an additional LMI constraint.

Theorem 5.1 *Consider the generalized plant in Figure 5.3, and let a controller K be connected, resulting in the realization (5.6). Suppose $Q \in \mathcal{Q}$ and \mathcal{X} , partitioned as in (5.7), are solutions to (5.8) and \check{X} a solution to (5.12). Then, the matrix \mathcal{A} is Hurwitz if and only if*

$$\begin{pmatrix} \mathcal{X}_{11} - \check{X} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{pmatrix} \succ 0. \quad (5.13)$$

Proof. If considering (5.8), that is

$$(\dots)' \left(\begin{array}{cccc|cc} 0 & 0 & \mathcal{X}_{11} & \mathcal{X}_{12} & 0 & 0 \\ 0 & 0 & \mathcal{X}_{21} & \mathcal{X}_{22} & 0 & 0 \\ \mathcal{X}_{11} & \mathcal{X}_{12} & 0 & 0 & 0 & 0 \\ \mathcal{X}_{21} & \mathcal{X}_{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & P_p \end{array} \right) \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ A_\Psi & B_{\Psi_1}C_q & B_{\Psi_1}D_{qp} + B_{\Psi_2} & B_{\Psi_1}D_{qw} \\ 0 & \mathcal{A} & \mathcal{B}_p & \mathcal{B}_w \\ \hline C_\Psi & D_{\Psi_1}C_q & D_{\Psi_1}D_{qp} + D_{\Psi_2} & D_{\Psi_1}D_{qw} \\ \hline 0 & 0 & 0 & I \\ 0 & \mathcal{C}_z & \mathcal{D}_{zp} & \mathcal{D}_{zw} \end{array} \right) \prec 0,$$

let us use (5.12) in order to get

$$(\dots)' \left(\begin{array}{cccc|cc} 0 & 0 & -\check{X} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\check{X} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -Q & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & P_p \end{array} \right) \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ A_\Psi & B_{\Psi_1}C_q & B_{\Psi_1}D_{qp} + B_{\Psi_2} & B_{\Psi_1}D_{qw} \\ 0 & \mathcal{A} & \mathcal{B}_p & \mathcal{B}_w \\ \hline C_\Psi & D_{\Psi_1}C_q & D_{\Psi_1}D_{qp} + D_{\Psi_2} & D_{\Psi_1}D_{qw} \\ \hline 0 & 0 & 0 & I \\ 0 & \mathcal{C}_z & \mathcal{D}_{zp} & \mathcal{D}_{zw} \end{array} \right) \preceq 0.$$

By adding these two LMIs, the terms that multiply with Q cancel, and we arrive at

$$\left(\begin{array}{cc} I & 0 \\ 0 & I \\ A_\Psi & B_{\Psi_2}C \\ 0 & \mathcal{A} \\ 0 & \mathcal{C}_z \end{array} \right)' \left(\begin{array}{cccc|c} 0 & 0 & \mathcal{X}_{11} - \check{X} & \mathcal{X}_{12} & 0 \\ 0 & 0 & \mathcal{X}_{21} & \mathcal{X}_{22} & 0 \\ \mathcal{X}_{11} - \check{X} & \mathcal{X}_{12} & 0 & 0 & 0 \\ \mathcal{X}_{21} & \mathcal{X}_{22} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & R_p \end{array} \right) \left(\begin{array}{cc} I & 0 \\ 0 & I \\ A_\Psi & B_{\Psi_2}C \\ 0 & \mathcal{A} \\ 0 & \mathcal{C}_z \end{array} \right) \prec 0.$$

Recall the fact that

$$P_p = \begin{pmatrix} Q_p & S_p \\ S_p' & R_p \end{pmatrix}, \quad \text{with } R_p \succeq 0.$$

Note that by Lyapunov, any matrix A is stable if there exists $X \succ 0$ for which $XA + A'X \prec 0$. Hence, we conclude that the matrix

$$\begin{pmatrix} A_\Psi & B_{\Psi_2}C \\ 0 & \mathcal{A} \end{pmatrix} \quad (5.14)$$

is stable if and only if (5.13) is satisfied. The proof finishes by the observation that the eigenvalues of (5.14) equal the union of the eigenvalues of \mathcal{A} and A_Ψ , the latter of which is stable by assumption. \blacksquare

If using static multipliers, which corresponds to $\Psi = I$, (5.12) is dropped and (5.13) reduces to $\mathcal{X} \succ 0$, which is the usual characterization of stability.

5.3.3 Derivation of the convex synthesis conditions

With the IQC analysis result of Theorem 3.1 and the characterization for nominal stability in Theorem 5.1, our main synthesis result is established by exploiting the structure in the plant.

First, recall the fact that (5.8) in Corollary 5.1 is non-linear in the decision variables \mathcal{X}, Q and the controller variables A_K, B_K, C_K, D_K . By following the arguments from [159, 122], in which the nominal output feedback control problem was rendered convex, we can overcome the bi-linearity in \mathcal{X} and A_K, B_K, C_K, D_K , after which we will handle the product of the controller matrices with $Q \in \mathcal{Q}$.

Without loss of generality we can consider the following realization of the generalized plant P in (5.3):

$$P = \left[\begin{array}{cc|ccc} A_1 & A_{12} & B_{p1} & B_{w1} & B_u \\ 0 & A_2 & B_{p2} & B_{w2} & 0 \\ \hline 0 & C_q & D_{qp} & D_{qw} & 0 \\ C_{z1} & C_{z2} & D_{zp} & D_{zw} & D_{zu} \\ C_{y1} & C_{y2} & D_{yp} & D_{yw} & 0 \end{array} \right]. \quad (5.15)$$

In fact, the zero in $(0 \ C_q)$ follows immediately from $P_{qu} = 0$ when (A_1, B_u) is chosen to be a controllable pair.

We will first merge the generalized plant matrices with the dynamics of Ψ before closing the loop with the controller. As a result, the composite transfer matrix (5.9) of the closed loop system, as it is needed in Corollary 5.1, can alternatively be obtained by interconnecting the controller $K(s)$ with the weighted open-loop plant

$$\left(\begin{array}{cc|cc} \Psi_1 & \Psi_2 & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \end{array} \right) \left(\begin{array}{ccc} I & 0 & 0 \\ P_{qp} & P_{qw} & 0 \\ \hline P_{zp} & P_{zw} & P_{zu} \\ P_{yp} & P_{yw} & P_{yu} \end{array} \right) = \left(\begin{array}{ccc} \Psi_1 P_{qp} + \Psi_2 & \Psi_1 P_{qw} & 0 \\ \hline P_{zp} & P_{zw} & P_{zu} \\ P_{yp} & P_{yw} & P_{yu} \end{array} \right), \quad (5.16)$$

for which the following realization is determined

$$\left[\begin{array}{ccc|ccc} A_1 & 0 & A_{12} & B_{p1} & B_{w1} & B_u \\ 0 & A_\Psi & B_{\Psi_2} C_q & B_{\Psi_1} + B_{\Psi_2} D_{qp} & B_{\Psi_2} D_{qw} & 0 \\ 0 & 0 & A_2 & B_{p2} & B_{w2} & 0 \\ \hline 0 & C_\Psi & D_{\Psi_1} C_q & D_{\Psi_1} + D_{\Psi_2} D_{qp} & D_{\Psi_2} D_{qw} & 0 \\ \hline C_{z1} & 0 & C_{z2} & D_{zp} & D_{zw} & D_{zu} \\ C_{y1} & 0 & C_{y2} & D_{yp} & D_{yw} & 0 \end{array} \right]. \quad (5.17)$$

For this composite transfer matrix, we introduce the following abbreviation

$$\tilde{P} = \left[\begin{array}{c|c|c|c} \tilde{A} & \tilde{B}_p & \tilde{B}_w & \tilde{B} \\ \hline \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} & 0 \\ \hline \tilde{C}_z & D_{zp} & D_{zw} & D_{zu} \\ \hline \tilde{C} & D_{yp} & D_{yw} & 0 \end{array} \right]. \quad (5.18)$$

By the usual computations, the realization matrices of $\tilde{P}(s) \star K(s)$ are given by

$$\begin{aligned} \left(\begin{array}{c|c|c} \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ \hline \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \hline \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} \end{array} \right) &= \left(\begin{array}{c|c|c} \tilde{A} & 0 & \tilde{B}_p & \tilde{B}_w \\ \hline 0 & 0 & 0 & 0 \\ \hline \tilde{C}_q & 0 & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \hline \tilde{C}_z & 0 & D_{zp} & D_{zw} \end{array} \right) + \\ &\left(\begin{array}{c|c} 0 & \tilde{B} \\ \hline I & 0 \\ \hline 0 & 0 \\ \hline 0 & D_{zu} \end{array} \right) \left(\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right) \left(\begin{array}{c|c|c} 0 & I & 0 & 0 \\ \hline \tilde{C} & 0 & D_{yp} & D_{yw} \end{array} \right). \quad (5.19) \end{aligned}$$

The following intermediate result illustrates that (5.8) can be transformed into a more convenient form, in which the Lyapunov matrix \mathcal{X} no longer multiplies with the controller variables. Rather than assuming that the nominal closed loop system is stable, as was done in Corollary 5.1, our next result makes use of the stability characterization in Theorem 5.1.

Lemma 5.1 *The following conditions for robust stability and quadratic performance of the closed loop system (5.6) are equivalent:*

- *There exist controller matrices A_K, B_K, C_K, D_K and matrices $\mathcal{X}, \check{X}, Q \in \mathcal{Q}$ such that the system matrix \tilde{A} in (5.19) is stable, (5.12) holds and the inequalities*

$$\left(\begin{array}{c|c|c|c} 0 & \mathcal{X} & 0 & 0 \\ \hline \mathcal{X} & 0 & 0 & 0 \\ \hline 0 & 0 & Q & 0 \\ \hline 0 & 0 & 0 & P_p \end{array} \right)' \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ \hline \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \hline 0 & 0 & I \\ \hline \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} \end{array} \right) \prec 0, \quad (5.20)$$

and

$$\mathcal{X} - \left(\begin{array}{c|c} E(\check{X}) & 0 \\ \hline 0 & 0 \end{array} \right) \quad (5.21)$$

are satisfied. Here, the block-structured matrix

$$E(\tilde{X}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{X} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.22)$$

is compatible with the structure of \tilde{A} in (5.17)-(5.18), i.e. the size of the diagonal blocks of \tilde{X} are compatible with the size of A_1 , A_Ψ and A_2 respectively.

- There exist matrices $X, Y, \tilde{X}, K, L, M, N$ and $Q \in \mathcal{Q}$ for which (5.12) and the following inequalities hold:

$$\left(\begin{array}{c|c|c|c} \dots & & & \end{array} \right)' \left(\begin{array}{c|c|c|c} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q & 0 \\ 0 & 0 & 0 & P_p \end{array} \right) \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline \mathbf{A} & \mathbf{B}_p & \mathbf{B}_w \\ \mathbf{C}_q & \mathbf{D}_{qp} & \mathbf{D}_{qw} \\ \hline 0 & 0 & I \\ \mathbf{C}_z & \mathbf{D}_{zp} & \mathbf{D}_{zw} \end{array} \right) \prec 0 \quad (5.23)$$

and

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} - \begin{pmatrix} YE(\tilde{X})Y & YE(\tilde{X}) \\ E(\tilde{X})Y & E(\tilde{X}) \end{pmatrix} \succ 0. \quad (5.24)$$

Here, the boldface symbols are defined as

$$\begin{aligned} \left(\begin{array}{c|c|c} \mathbf{A} & \mathbf{B}_p & \mathbf{B}_w \\ \hline \mathbf{C}_q & \mathbf{D}_{qp} & \mathbf{D}_{qw} \\ \mathbf{C}_z & \mathbf{D}_{zp} & \mathbf{D}_{zw} \end{array} \right) &= \left(\begin{array}{c|c|c|c} \tilde{A}Y & \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ 0 & X\tilde{A} & X\tilde{B}_p & X\tilde{B}_w \\ \hline \tilde{C}_qY & \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \tilde{C}_zY & \tilde{C}_z & D_{zp} & D_{zw} \end{array} \right) + \\ &+ \left(\begin{array}{c|c} 0 & \tilde{B} \\ I & 0 \\ \hline 0 & 0 \\ 0 & D_{zu} \end{array} \right) \begin{pmatrix} K & L \\ M & N \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & \tilde{C} & D_{yp} & D_{yw} \end{pmatrix} \end{aligned} \quad (5.25)$$

Proof. The result follows by applying a congruence transformation proposed in [159]. From Corollary 5.1 it is immediately clear that the existence of \mathcal{X}, \tilde{X} , A_K , B_K , C_K , D_K and $Q \in \mathcal{Q}$ which satisfy (5.12) and (5.20)-(5.21) implies robust quadratic performance. Note that there always exists ϵ for which $\mathcal{X} + \epsilon I$ is invertible while still satisfying (5.20) and (5.21). Provided that the controller order is chosen large enough, the following parametrization of \mathcal{X} exists:

$$\mathcal{X} = \begin{pmatrix} X & U \\ U^T & * \end{pmatrix}, \quad \text{and} \quad \mathcal{X}^{-1} = \begin{pmatrix} Y & V \\ V^T & * \end{pmatrix}, \quad (5.26)$$

with block dimensions compatible with the closed-loop system matrix $\tilde{\mathcal{A}}$ in (5.19).

The matrices U, V with V having full row rank, are chosen such that $I - XY = UV^T$ holds. It immediately follows that

$$\begin{pmatrix} Y & V \\ I & 0 \end{pmatrix} \mathcal{X} = \begin{pmatrix} I & 0 \\ X & U \end{pmatrix}. \quad (5.27)$$

We will now apply a congruence transformation \mathcal{Y} to (5.20), with \mathcal{Y} defined as

$$\mathcal{Y} = \left(\begin{array}{cc|cc} Y & I & 0 & 0 \\ V^T & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \end{array} \right). \quad (5.28)$$

If we define new controller parameters (K, L, M, N) as

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix} = \begin{pmatrix} U & X\tilde{B} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} \begin{pmatrix} V^T & 0 \\ \tilde{C}Y & I \end{pmatrix} + \begin{pmatrix} XAY & 0 \\ 0 & 0 \end{pmatrix}, \quad (5.29)$$

the boldface matrices in (5.25) are precisely equal to

$$\mathcal{Y}' \begin{pmatrix} \mathcal{X}\tilde{A} & \mathcal{X}\tilde{B}_p & \mathcal{X}\tilde{B}_w \\ \tilde{C}_q\mathcal{X} & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \tilde{C}_z\mathcal{X} & \tilde{D}_{zp} & \tilde{D}_{zw} \end{pmatrix} \mathcal{Y}. \quad (5.30)$$

Hence, by applying the congruence transformation \mathcal{Y} to (5.20) we arrive at (5.23). Moreover, with \mathcal{X} being factorized as in (5.26), it follows that

$$\begin{pmatrix} Y & I \\ V^T & 0 \end{pmatrix}$$

has full column rank, and hence

$$\begin{pmatrix} Y & I \\ V^T & 0 \end{pmatrix}' \mathcal{X} \begin{pmatrix} Y & I \\ V^T & 0 \end{pmatrix} = \begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0.$$

With the observation that

$$\begin{pmatrix} Y & I \\ V^T & 0 \end{pmatrix}' \begin{pmatrix} E(\tilde{X}) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Y & I \\ V^T & 0 \end{pmatrix} = \begin{pmatrix} YE(\tilde{X})Y & YE(\tilde{X}) \\ E(\tilde{X})Y & E(\tilde{X}) \end{pmatrix},$$

it is shown that (5.21) implies (5.24).

In order to prove the reverse statement, suppose that $X, Y, \tilde{X}, K, L, M, N$ and $Q \in \mathcal{Q}$ satisfy (5.12), (5.23), and (5.24). Then, there always exists ϵ for which both $X_\epsilon = X + \epsilon I, Y_\epsilon = Y + \epsilon I$ as well as $I - X_\epsilon Y_\epsilon$ are non-singular, while still satisfying (5.23)-(5.24). Hence, we can factorize $I - X_\epsilon Y_\epsilon = UV'$ for some square and non-singular

U, V . Since $\begin{pmatrix} I & 0 \\ X & U \end{pmatrix}$ in (5.27) and \mathcal{Y} in (5.28) are square and invertible as well, we can choose $\mathcal{X}, A_K, B_K, C_K, D_K$ that satisfy (5.27) and (5.29). By construction, the expression (5.30) is equal to (5.25). Note also, that the controller has the same order of the plant. If applying the congruency transformations

$$\mathcal{Y}^{-1} \quad \text{and} \quad \begin{pmatrix} Y & I \\ V' & 0 \end{pmatrix}^{-1} \quad \text{to (5.23) and (5.24) respectively,}$$

we arrive at (5.20) and (5.21), which finishes the proof. \blacksquare

Since $P_{qu} = 0$ implies $\mathbf{D}_{qp} = \tilde{D}_{qp}$, $\mathbf{D}_{qw} = \tilde{D}_{qw}$, the only non-linearity that remains in (5.23) is caused by the term $\mathbf{C}'_q Q \mathbf{C}'_q$. In order to arrive at the synthesis solution, we need to get rid of this non-linearity. Moreover, we have to see how to render the coupling condition (5.24) convex at the same time. It turns out that by applying a suitable congruence transformation S_1 , we can at the same time remove all the decision variables from \mathbf{C}'_q by enforcing $\tilde{C}'_q Y S_1 = \tilde{C}'_q$, as well as convexify (5.24).

The synthesis solution based on Corollary 5.1 is developed by first partitioning the matrices X, Y according to \tilde{A} in (5.17), i.e.

$$X = \left(\begin{array}{c|cc} X_{11} & X_{12} & X_{13} \\ \hline X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{array} \right), \quad Y = \left(\begin{array}{c|cc} Y_{11} & Y_{12} & Y_{13} \\ \hline Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{array} \right), \quad (5.31)$$

in which X_{11}, Y_{11} are of compatible size with A_1 , while the same holds for X_{22}, Y_{22} and A_Ψ , X_{33}, Y_{33} and A_2 , respectively. Moreover, we let the matrix T be partitioned in a similar fashion, i.e.

$$T = \left(\begin{array}{c|cc} T_{11} & T_{12} & T_{13} \\ \hline T_{12}^T & T_{22} & T_{23} \\ T_{13}^T & T_{23}^T & T_{33} \end{array} \right). \quad (5.32)$$

Our next theorem is based on Lemma 5.1 and contains the main synthesis result of this paper.

Theorem 5.2 *The closed-loop system (5.6) with performance channel $w \rightarrow z$ satisfies quadratic performance for all $\Delta \in \mathbf{\Delta}$ if for some $Q \in \mathcal{Q}$ there exist $T, X, \tilde{X}, \bar{K}, L, \bar{M}, N$ for which*

$$\left(\dots \right)' \left(\begin{array}{c|cc|c} 0 & I & 0 & 0 \\ \hline I & 0 & 0 & 0 \\ 0 & 0 & Q & 0 \\ \hline 0 & 0 & 0 & P_p \end{array} \right) \left(\begin{array}{c|cc} I & 0 & 0 \\ \hline \mathbf{A} & \mathbf{B}_p & \mathbf{B}_w \\ \hline C_e & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \hline 0 & 0 & I \\ \hline \mathbf{C}_z & \mathbf{D}_{zp} & \mathbf{D}_{zw} \end{array} \right) \prec 0, \quad (5.33)$$

hold with $C_e = \begin{pmatrix} \tilde{C}_q & \tilde{C}_q \end{pmatrix}$, as well as (5.12) and the coupling condition

$$\mathbf{X} := \left(\begin{array}{ccc|ccc} T_{11} & 0 & 0 & I & T_{12} & T_{13} \\ 0 & T_{22} - \check{X} & T_{23} & 0 & T_{22} - \check{X} & T_{23} \\ 0 & T'_{23} & T_{33} & 0 & T'_{23} & T_{33} \\ \hline I & 0 & 0 & X_{11} & X_{12} & X_{13} \\ T'_{12} & T_{22} - \check{X} & T_{23} & X'_{12} & X_{22} - \check{X} & X_{23} \\ T'_{13} & T'_{23} & T_{33} & X'_{13} & X'_{23} & X_{33} \end{array} \right) \succ 0 \quad (5.34)$$

are satisfied. The boldface symbols depend on the decision variables in an affine fashion. With the structured matrices

$$S_1 = \begin{pmatrix} I & 0 & 0 \\ T'_{12} & T_{22} & T_{23} \\ T'_{13} & T'_{23} & T_{33} \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} T_{11} & -T_{12} & -T_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad (5.35)$$

these are defined as

$$\begin{aligned} \left[\begin{array}{c|cc} \mathbf{A} & \mathbf{B}_p & \mathbf{B}_w \\ \mathbf{C}_z & \mathbf{D}_{zp} & \mathbf{D}_{zw} \end{array} \right] &= \begin{pmatrix} S'_1 \tilde{A} S_2 & S'_1 \tilde{A} & S'_1 \tilde{B}_p & S'_1 \tilde{B}_w \\ 0 & X \tilde{A} & X \tilde{B}_p & X \tilde{B}_w \\ \tilde{C}_z S_2 & \tilde{C}_z & D_{zp} & D_{zw} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \tilde{B} \\ I & 0 \\ 0 & D_{zu} \end{pmatrix} \begin{pmatrix} \bar{K} & L \\ \bar{M} & N \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & \tilde{C} & D_{yp} & D_{yw} \end{pmatrix}. \end{aligned} \quad (5.36)$$

Remark 5.1 The existence of $T, X, \check{X}, \bar{K}, L, \bar{M}, N$ and $Q \in \mathcal{Q}$ for which (5.12), (5.33)-(5.34) is equivalent to any of the two sets of conditions in Lemma 5.1.

Proof. Similar as in Lemma 5.1, we can perturb X, T such that X and T , and hence also S_1, S_2 , are non-singular. Moreover, without loss of generality we assume that $I - X S_2 S_1^{-1}$ are invertible as well. Defining the matrices

$$Y = S_2 S_1^{-1}, \quad K = \bar{K} S_1^{-1}, \quad M = \bar{M} S_1^{-1}, \quad (5.37)$$

we have found matrices $X, \check{X}, Y, K, L, M, N$ and $Q \in \mathcal{Q}$ which satisfy the conditions (5.23)-(5.24) and (5.12). By following the arguments in the proof of Lemma 5.1, it the closed loop system is shown to satisfy robust quadratic performance.

Second, let us prove that the existence of $X, \check{X}, Y, K, L, M, N$ and $Q \in \mathcal{Q}$ that satisfy (5.12), (5.23) and (5.24) implies the existence of $T, X, \check{X}, \bar{K}, L, \bar{M}, N$ and $Q \in \mathcal{Q}$ satisfying (5.12), (5.33)-(5.34). As mentioned in Lemma 5.1, the matrices X, Y and $I - XY$ can be taken to be non-singular, possibly after a small perturbation. Let Y

be factorized such that

$$Y \begin{pmatrix} I & 0 & 0 \\ T'_{12} & T'_{22} & T'_{23} \\ T'_{13} & T'_{23} & T'_{33} \end{pmatrix} = \begin{pmatrix} T_{11} & -T_{12} & -T_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

for some non-singular T in (5.32) and let the structured matrices S_1, S_2 be defined in (5.35). Then, with $\bar{V} := S'_1 V$, define $\bar{\mathcal{Y}}$ as

$$\bar{\mathcal{Y}} = \mathcal{Y} \begin{pmatrix} S_1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} = \begin{pmatrix} S_2 & I & 0 & 0 \\ \bar{V}' & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad (5.38)$$

in which \mathcal{Y} is given in (5.28). Note that the particular structure of S_2 has been chosen to enforce $\tilde{C}_q S_2 = \tilde{C}_q$. In fact, left- and right multiplication with $\bar{\mathcal{Y}}', \bar{\mathcal{Y}}$ will transform (5.20) into (5.33) with the substitutions as in (5.36). Note that, as we transform \mathbf{C}_q in (5.25) into $C_e = \begin{pmatrix} \tilde{C}_q & \tilde{C}_q \end{pmatrix}$, the expression (5.33) becomes affine in all decision variables $\{T, X, \tilde{X}, \bar{K}, L, \bar{M}, N, \}$ and $Q \in \mathcal{Q}$.

Finally, it can be verified that the coupling condition (5.24) yields (5.34) by applying the congruence transformation $\begin{pmatrix} S_1 & 0 \\ 0 & I \end{pmatrix}$. This finishes our proof. \blacksquare

Remark 5.2 *A numerical implementation requires to render condition (5.33) affine in $\mathbf{C}_z, \mathbf{D}_{zp}$, which is made possible by Schur's lemma, see also Section 5.3.5.*

Controller reconstruction

The controller matrices can be reconstructed from the solution matrices $X, T, \tilde{X}, \bar{K}, L, \bar{M}, N$ in the following fashion. First, we construct matrices K, M, Y as in (5.37) and find square matrices U, V such that $UV^T = I - XY$. Once X, Y, U, V are available, the controller matrices can be obtained as

$$\begin{aligned} D_K &:= N \\ C_K &:= (M - D_K \tilde{C} X) U^{-T} \\ B_K &:= V^{-1} (L - Y \tilde{B} D_K) \\ A_K &:= V^{-1} (K - V B_K \tilde{C} X \\ &\quad - Y \tilde{B} C_K U^T - Y (\tilde{A} + \tilde{B} D_K \tilde{C}) X) U^{-T}. \end{aligned} \quad (5.39)$$

An alternative approach is to first compute \mathcal{X} from X, T , and then to solve (3.24) for the controller matrices A_K, B_K, C_K, D_K , while keeping \mathcal{X} fixed. Note that the controller variables enter the closed loop matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ in an affine fashion.

5.3.4 Robust \mathcal{H}_2 -synthesis

Similar to the synthesis result regarding the quadratic performance measure, one can start from the characterization of robust \mathcal{H}_2 -performance that was shown in Theorem 3.2.

Theorem 5.3 *Consider the realization (5.15) and let $D_{zw} = 0$ and $D_{zu} = 0$. Then, there exists a controller such that the \mathcal{H}_2 -performance level of the closed loop channel $w \rightarrow z$ is smaller than γ if there exist $T, X, \tilde{X}, \tilde{K}, L, \tilde{M}, N$ and $Q \in \mathcal{Q}$ for which $\text{Tr}(Z) \leq \gamma^2$,*

$$\left(\begin{array}{c|c} I & 0 \\ \mathbf{A} & \mathbf{B}_p \\ \hline C_e & \tilde{D}_{qp} \\ \mathbf{C}_z & \mathbf{D}_{zp} \end{array} \right)' \left(\begin{array}{c|c|c|c} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & Q & 0 \\ \hline 0 & 0 & 0 & I \end{array} \right) \left(\begin{array}{c|c} I & 0 \\ \mathbf{A} & \mathbf{B}_p \\ \hline C_e & \tilde{D}_{qp} \\ \mathbf{C}_z & \mathbf{D}_{zp} \end{array} \right) \prec 0, \quad (5.40)$$

with $C_e = (\tilde{C}_q \quad \tilde{C}_q)$, as well as

$$\left(\begin{array}{c|c} Z & \mathbf{B}'_w \\ \hline \mathbf{B}_w & \mathbf{X} \end{array} \right) \succ 0 \quad (5.41)$$

and condition (5.12) are satisfied.

Remark 5.3 *Note that $\mathbf{X} \succ 0$ need not be separately imposed, since it already follows from (5.41). Again, one must first apply Schur's lemma in order to render condition (5.40) affine in $\mathbf{C}_z, \mathbf{D}_{zp}$.*

Proof. A sketch of the proof will be given by following the same steps as were used in the derivation of Theorem 5.2. Note that considering (3.18) for the closed loop system leads to (5.40). Moreover, condition (3.17), as considered for the closed loop system, reads as

$$\tilde{\mathbf{B}}'_w \mathcal{X} \tilde{\mathbf{B}}'_w - Z \prec 0. \quad (5.42)$$

Since \mathcal{X} is not necessarily a positive definite matrix, we cannot immediately apply Schur's Lemma and resolve the non-linearity in (5.42). However, in view of the stability characterization of Theorem 5.1, the relation $\mathbf{X} \succ 0$ must be imposed, which motivates us to write (5.42) alternatively as

$$\tilde{\mathbf{B}}'_w (\mathcal{X} - E) \tilde{\mathbf{B}}_w + \tilde{\mathbf{B}}'_w E \tilde{\mathbf{B}}_w - Z \prec 0 \quad (5.43)$$

with $E = E(\tilde{X})$ defined in (5.22). Due to the structure of $\tilde{\mathbf{B}}_w$ in (5.19), it immediately follows that $E \tilde{\mathbf{B}}_w = 0$. If now applying the Schur complement formula to (5.43), we finally arrive at

$$\left(\begin{array}{c|c} Z & \tilde{\mathbf{B}}'_w (\mathcal{X} - E) \\ \hline (\mathcal{X} - E) \tilde{\mathbf{B}}_w & \mathcal{X} - E \end{array} \right) = \left(\begin{array}{c|c} Z & \tilde{\mathbf{B}}'_w \mathcal{X} \\ \hline \mathcal{X} \tilde{\mathbf{B}}_w & \mathcal{X} - E \end{array} \right) \succ 0.$$

This matrix inequality can be rendered convex, analogously to the congruence transformation (5.38). In fact, if we multiply this condition from left- and right with

$$\begin{pmatrix} I & 0 & 0 \\ 0 & Z & I \\ 0 & \bar{V}^T & 0 \end{pmatrix}' \quad \text{and} \quad \begin{pmatrix} I & 0 & 0 \\ 0 & Z & I \\ 0 & \bar{V}^T & 0 \end{pmatrix},$$

respectively, we arrive at condition (5.41) and have finished the proof. \blacksquare

5.3.5 Elimination of the controller parameters

In order to reduce the number of variables, thereby speeding up the numerical computations, one can eliminate (part of) the transformed controller variables \bar{K}, L, \bar{M}, N . In the context of the nominal output feedback synthesis problem, this was first observed in [159, 122]. In this section we show that similar arguments allow for elimination in the robust synthesis problem as well. Recall our earlier notation for the augmented plant in (5.17)-(5.18).

The elimination of variables requires that we first turn the synthesis conditions affine in the decision variables $X, T, \bar{K}, L, \bar{M}, N$ and γ . Thus, by applying the Schur complement formula on (5.33), using the index P_p as

$$P_p = \begin{pmatrix} -\gamma & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix},$$

which represents an the \mathcal{L}_2 -induced gain to be bounded by γ , we arrive at

$$\begin{pmatrix} \mathbf{A} + \mathbf{A}' & \mathbf{B}_p & \mathbf{B}_w & 0 \\ \mathbf{B}'_p & 0 & 0 & 0 \\ \mathbf{B}'_w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \mathbf{C}'_z \\ 0 & 0 & 0 & \mathbf{D}'_{zp} \\ 0 & 0 & -\gamma I & \mathbf{D}'_{zw} \\ \mathbf{C}_z & \mathbf{D}_{zp} & \mathbf{D}_{zw} & -\gamma I \end{pmatrix} + \begin{pmatrix} \mathbf{C}'_e \\ \tilde{\mathbf{D}}'_{qp} \\ \tilde{\mathbf{D}}'_{qw} \\ 0 \end{pmatrix} Q \begin{pmatrix} \mathbf{C}_e & \tilde{\mathbf{D}}_{qp} & \tilde{\mathbf{D}}_{qw} & 0 \end{pmatrix} \prec 0, \quad (5.44)$$

with boldface symbols defined in (5.36). We are further motivated to introduce the following matrices. Let

$$\text{Ker} \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \tilde{B} & 0 & 0 & 0 & D_{zu} \end{pmatrix}, \quad \text{Ker} \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & \tilde{C} & D_{yp} & D'_{yw} & 0 \end{pmatrix} \quad (5.45)$$

have the basis matrices

$$\begin{pmatrix} \Phi_1 \\ 0 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}$$

respectively and define $\Phi' = (\Phi'_1 \quad \Phi'_2 \quad \Phi'_3)$ and $\Psi' = (\Psi'_1 \quad \Psi'_2 \quad \Psi'_3)$.

Theorem 5.4 *The synthesis conditions in Theorem 5.2 are satisfied if and only if there exists X, \tilde{X}, T and $Q \in \mathcal{Q}$ such that conditions (5.12) and (5.34) hold, as well as*

$$\Phi'(U'QU + J_1)\Phi \prec 0 \quad \text{and} \quad \Psi'(U'QU + J_2)\Psi \prec 0, \quad (5.46)$$

in which

$$J_1 = \text{sym} \begin{pmatrix} X\tilde{A} & X\tilde{B}_p & X\tilde{B}_w & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\gamma}{2}I & 0 \\ \tilde{C}_z & D_{zp} & D_{zw} & -\frac{\gamma}{2}I \end{pmatrix},$$

$$J_2 = \text{sym} \begin{pmatrix} S'_1\tilde{A}S_2 & S'_1\tilde{B}_p & S'_1\tilde{B}_w & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\gamma}{2}I & 0 \\ \tilde{C}_zS_2 & D_{zp} & D_{zw} & -\frac{\gamma}{2}I \end{pmatrix},$$

and

$$U = (\tilde{C}_q \mid \tilde{D}_{qp} \mid \tilde{D}_{qw} \quad 0).$$

Here, S_1, S_2 are as defined in (5.35).

Proof. By using the boldface symbols defined in (5.36) and the matrices Ψ, Φ from (5.45), we can alternatively write (5.44) as

$$\text{sym} \begin{pmatrix} 0 & \tilde{B} \\ I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & D_{zu} \end{pmatrix} \begin{pmatrix} \bar{K} & L \\ \bar{M} & N \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & \tilde{C} & D_{yp} & D'_{yw} \\ 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$\Upsilon(X, T, Q, \gamma) \prec 0, \quad (5.47)$$

in which $\Upsilon(X, T, Q, \gamma)$ is defined as

$$(\dots)' \left(\begin{array}{ccc|ccc|c} 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & Q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma I & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma I \end{array} \right) \left(\begin{array}{cc|cc|c} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ S'_1 \tilde{A} S_2 & S'_1 \tilde{A} & S'_1 \tilde{B}_p & S'_1 \tilde{B}_w & S'_2 \tilde{C}'_z \\ 0 & X \tilde{A} & X \tilde{B}_p & X \tilde{B}_w & \tilde{C}'_z \\ \hline \tilde{C}'_q & \tilde{C}'_q & \tilde{D}_{qp} & \tilde{D}_{qw} & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & D'_{zp} \\ 0 & 0 & 0 & 0 & D'_{zw} \\ \hline 0 & 0 & 0 & 0 & I \end{array} \right). \quad (5.48)$$

From the projection lemma, see for example [163], it follows that (5.47) holds for some $\bar{K}, L, \bar{M}, N, X, T, Q$ and γ if and only if for some X, T, Q and γ

$$\left(\begin{array}{c} \Phi_1 \\ 0 \\ \Phi_2 \\ \Phi_3 \end{array} \right)' \Upsilon \left(\begin{array}{c} \Phi_1 \\ 0 \\ \Phi_2 \\ \Phi_3 \end{array} \right) \prec 0 \quad \text{and} \quad \left(\begin{array}{c} 0 \\ \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{array} \right)' \Upsilon \left(\begin{array}{c} 0 \\ \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{array} \right) \prec 0.$$

It is easily verified that these two conditions amount to (5.46), which finishes the proof. ■

When solving the eliminated version of the synthesis conditions, an additional step is required for computing optimal (transformed) controller variables. In fact, if X, T are solutions to (5.46), one can substitute these variables in (5.33) and solve for \bar{K}, L, \bar{M}, N , from which the controller matrices can be reconstructed as discussed in Section 5.3.3.

Remark 5.4 *The conditions for robust \mathcal{H}_2 -performance only allow for elimination of variables \bar{K} and L . Detailed arguments can be found in [163], for the nominal output feedback control problem, which easily extends to the robust case.*

5.3.6 Other interconnections of uncertain systems

In view of the synthesis result for a structured generalized plant with $P_{qu} = 0$, the natural question arises whether our controller synthesis result can be extended to more general interconnections of uncertain systems. An interconnection which is closely related to our initial configuration in Section 5.1 is shown in Figure 5.4. Uncertainty only enters a filter at the plant output, and a motivation for this problem is given next.

In \mathcal{H}_∞ - and \mathcal{H}_2 -synthesis, weighting functions that are placed at the plant output can effectively represent performance specifications. For instance, in the manufac-

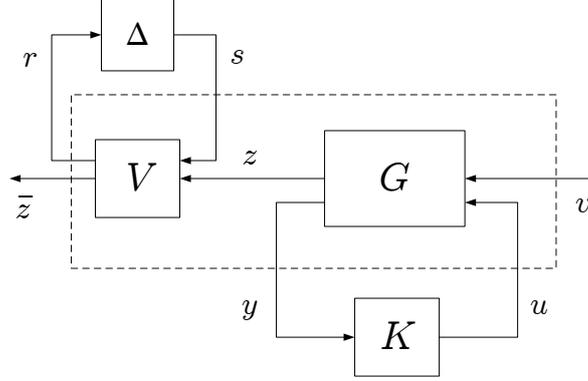


Figure 5.4: System interconnection corresponding to generalized plant (5.49)

turing of cars or aircraft, the allowable intensity of the vibrations at the passenger seats strongly depends on frequency. It is also not hard to imagine a parameter dependent weight in order to reflect multiple (parameterized) criteria.

In a general standard S/KS design scheme in which the performance weight $W_p(\delta)$ on the sensitivity S is parameter dependent, the weight W_u which penalizes the control input typically does not depend on the parameter. That is, the typical mixed sensitivity robust controller design problem reads as

$$\min_{K \text{ stabilizing}} \max_{\delta \in \mathcal{D}} \left\| \begin{array}{c} W_p(\delta)S(K) \\ W_u K S(K) \end{array} \right\|_{\infty},$$

for some compact set \mathcal{D} . Such controller design problems naturally lead to the system interconnection in Figure 5.4.

Let us mention the related work done in [64], in which an LPV synthesis approach was proposed for designing a so-called *trade-off* controller. Such controllers are re-tuned *in situ*, i.e. after the controller has been implemented. This feature is desired if the system characteristics change, e.g. due to aging of components, or a change in the temperature coefficients.

We will now show in detail the robust synthesis solution corresponding the interconnection in Figure 5.4. Let the following realization of the generalized plant P be given:

$$P = \left[\begin{array}{c|ccc} \tilde{A} & \tilde{B}_s & \tilde{B}_v & \tilde{B} \\ \hline \tilde{C}_r & D_{rs} & D_{rv} & D_{ru} \\ \tilde{C}_{\bar{z}} & D_{\bar{z}s} & D_{\bar{z}v} & D_{\bar{z}u} \\ \tilde{C} & 0 & D_{yv} & 0 \end{array} \right] = \left[\begin{array}{cc|ccc} A_1 & A_{12} & B_s & B_{v1} & B_{u1} \\ 0 & A_2 & 0 & B_{v2} & B_{u2} \\ \hline C_{r1} & C_{r2} & D_{rs} & D_{rv} & D_{ru} \\ C_{\bar{z}1} & C_{\bar{z}2} & D_{\bar{z}s} & D_{\bar{z}v} & D_{\bar{z}u} \\ 0 & C_y & 0 & D_{yv} & 0 \end{array} \right], \quad (5.49)$$

in which $\tilde{A}, \tilde{B}, \tilde{C}$ are of course different from those seen in (5.18). Moreover, we restrict ourselves to static multipliers, i.e. $\Psi = I$ and consider parametric uncertainties, i.e. $\mathbf{\Delta}$ is a set of (structured) matrices. The set of admissible symmetric multipliers is denoted by $\mathbf{\Pi}$. That is, for any $\Pi \in \mathbf{\Pi}$ it holds that

$$\begin{pmatrix} \Delta \\ I \end{pmatrix}' \Pi \begin{pmatrix} \Delta \\ I \end{pmatrix} \succeq 0 \quad \text{for all } \Delta \in \mathbf{\Delta}. \quad (5.50)$$

Then, the synthesis condition (5.33), as considered for the plant (5.49) reads as

$$\left(\dots \right)' \left(\begin{array}{c|c|c|c} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & \Pi & 0 \\ 0 & 0 & 0 & P_p \end{array} \right) \left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline \mathbf{A} & \mathbf{B}_s & \mathbf{B}_v \\ 0 & I & 0 \\ \hline \mathbf{C}_r & D_{rs} & \mathbf{D}_{rv} \\ 0 & 0 & I \\ \hline \mathbf{C}_{\bar{z}} & D_{\bar{z}s} & \mathbf{D}_{\bar{z}v} \end{array} \right) \prec 0, \quad (5.51)$$

with the substitutions

$$\begin{aligned} \mathbf{A} &\rightarrow \begin{pmatrix} \tilde{A}Y + \tilde{B}M & \tilde{A} + \tilde{B}N\tilde{C} \\ K & X\tilde{A} + L\tilde{C} \end{pmatrix}, \\ \mathbf{B}_s &\rightarrow \begin{pmatrix} \tilde{B}_s \\ X\tilde{B}_s \end{pmatrix}, \\ \mathbf{B}_v &\rightarrow \begin{pmatrix} \tilde{B}_v + \tilde{B}ND_{yv} \\ X\tilde{B}_v + LD_{yv} \end{pmatrix}, \\ \mathbf{C}_r &\rightarrow \begin{pmatrix} \tilde{C}_rY + D_{ru}M & \tilde{C}_r + D_{ru}N\tilde{C} \end{pmatrix}, \\ \mathbf{C}_{\bar{z}} &\rightarrow \begin{pmatrix} \tilde{C}_{\bar{z}}Y + D_{\bar{z}u}M & \tilde{C}_{\bar{z}} + D_{\bar{z}u}N\tilde{C} \end{pmatrix}, \\ \mathbf{D}_{rv} &\rightarrow D_{rv} + D_{ru}ND_{yv}, \\ \mathbf{D}_{\bar{z}v} &\rightarrow D_{\bar{z}v} + D_{\bar{z}u}ND_{yv}. \end{aligned} \quad (5.52)$$

The following lemma leads to the solution of the controller synthesis problem.

Lemma 5.2 *Let $M \in \mathbb{R}^{n \times m}$ and $\Phi \in \mathbb{R}^{(n+m) \times k}$ be given and let $P \in \mathcal{S}^{n+m}$ be non-singular. Moreover, let $\text{im} \begin{pmatrix} I \\ M \end{pmatrix}$ and $\text{im}(\Phi)$ be complementary subspaces whose sum equal \mathbb{R}^{n+m} . Then*

$$\begin{pmatrix} I \\ M \end{pmatrix}' P \begin{pmatrix} I \\ M \end{pmatrix} \prec 0 \quad \text{and} \quad \Phi' P \Phi \succeq 0$$

is equivalent to

$$\begin{pmatrix} -M' \\ I \end{pmatrix}' P^{-1} \begin{pmatrix} -M' \\ I \end{pmatrix} \succ 0 \quad \text{and} \quad \tilde{\Phi}' P^{-1} \tilde{\Phi} \preceq 0,$$

in which $\tilde{\Phi}$ is any basis matrix of the orthogonal complement of $\text{im}(\Phi)$.

Proof. A proof of this dualization lemma can be found in [163]. ■

The outer factor in (5.51) has the form $\begin{pmatrix} I \\ M \end{pmatrix}$ after a suitable permutation of the rows. Hence, a basis matrix of the orthogonal complement can be explicitly written down. If we define $\tilde{\Pi} = \Pi^{-1}$ and $\tilde{P}_p = P_p^{-1}$ and apply Lemma 5.2, defining also $\Psi = \text{Ker}(D_{rs})$, it follows that (5.51) together with

$$(\dots)' \left(\begin{array}{c|c|c|c} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{\Pi} & 0 \\ \hline 0 & 0 & 0 & \tilde{P}_p \end{array} \right) \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & \Psi & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix} \succeq 0 \quad (5.53)$$

is equivalent to

$$(\dots)' \left(\begin{array}{c|c|c|c} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{\Pi} & 0 \\ \hline 0 & 0 & 0 & \tilde{P}_p \end{array} \right) \begin{pmatrix} -\mathbf{A}^T & -\mathbf{C}_q^T & -\mathbf{C}_z^T \\ I & 0 & 0 \\ \hline -\mathbf{B}_s^T & -D_{rs}^T & -D_{zs}^T \\ 0 & I & 0 \\ \hline -\mathbf{B}_v^T & -\mathbf{D}_{rv}^T & -\mathbf{D}_{zv}^T \\ 0 & 0 & I \end{pmatrix} \succ 0 \quad (5.54)$$

and

$$(\dots)' \left(\begin{array}{c|c|c|c} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{\Pi} & 0 \\ \hline 0 & 0 & 0 & \tilde{P} \end{array} \right) \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & \tilde{\Psi} & 0 \\ \hline 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \preceq 0, \quad (5.55)$$

in which $\tilde{\Psi}$ is a basis matrix of the orthogonal complement of $\text{Im}(\Psi)$.

Our first observation concerns the fact that $\mathbf{A}^T, \mathbf{B}_s^T$ in (5.52) have a similar structure as \mathbf{A}, \mathbf{C}_q in condition (5.18). By following the arguments in the proof of Theorem 5.2, we are therefore able to resolve the non-linear coupling terms between $\tilde{\Pi}$ and $-\mathbf{B}_s^T$. Hence, a tractable synthesis solution for the robust controller design problem depicted in Figure 5.4 is obtained once we can construct a class of admissible inverse multipliers $\tilde{\Pi} \in \tilde{\Pi}$, described by finitely many LMI constraints, provided that both (5.53) and (5.55) are guaranteed to hold.

Note that (5.53) implies

$$\begin{pmatrix} 0 \\ I \end{pmatrix}' P_p \begin{pmatrix} 0 \\ I \end{pmatrix} \succeq 0 \quad \text{and} \quad \begin{pmatrix} 0 \\ \Psi \end{pmatrix}' \Pi \begin{pmatrix} 0 \\ \Psi \end{pmatrix} \succeq 0, \quad (5.56)$$

and (5.55) implies

$$\begin{pmatrix} I \\ 0 \end{pmatrix}' \tilde{P}_p \begin{pmatrix} I \\ 0 \end{pmatrix} \preceq 0 \quad \text{and} \quad \begin{pmatrix} I \\ \tilde{\Psi} \end{pmatrix}' \tilde{\Pi} \begin{pmatrix} I \\ \tilde{\Psi} \end{pmatrix} \preceq 0. \quad (5.57)$$

Here, the constraint on P_p and \tilde{P}_p are naturally satisfied if using the \mathcal{L}_2 -gain performance measure, see (A.6). By contrast, the conditions on Π and $\tilde{\Pi}$ must follow from the multiplier class $\tilde{\mathbf{\Pi}}$. Suppose the set $\tilde{\mathbf{\Pi}}$ is defined as

$$\begin{aligned} \tilde{\mathbf{\Pi}} = \{ \tilde{\Pi} \in \mathcal{S}^{n_p+n_a} : & \begin{pmatrix} I \\ -\Delta^T \end{pmatrix}' \tilde{\Pi} \begin{pmatrix} I \\ -\Delta^T \end{pmatrix} \prec 0 \quad \text{for all } \Delta \in \mathbf{\Delta}, \\ & \text{and } \begin{pmatrix} 0 \\ I \end{pmatrix}' \tilde{\Pi} \begin{pmatrix} 0 \\ I \end{pmatrix} \succ 0 \}. \end{aligned} \quad (5.58)$$

It then follows, again by applying Lemma 5.2 for any fixed $\Delta \in \mathbf{\Delta}$, that $\tilde{\Pi}^{-1}$ satisfies (5.50), and is therefore guaranteed to be an admissible multiplier.

As a result, for any chosen set $\tilde{\mathbf{\Pi}}$ of the form (5.58), the existence of X, Y, K, L, M, N and $\tilde{\Pi} \in \tilde{\mathbf{\Pi}}$ that satisfy (5.54) and the coupling condition

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0, \quad (5.59)$$

implies that the closed-loop system with performance channel $w \rightarrow z$ satisfies robust quadratic performance, provided that (5.56)-(5.57) hold for the chosen $P_p, \tilde{P}_p = P_p^{-1}$. Due to the structure of $\mathbf{A}, -\mathbf{B}_s^T$ in (5.54), Theorem 5.2 applies and the synthesis conditions can be rendered convex. Note that the stability characterization (5.24) reduces to (5.59) if using static scalings.

Remark 5.5 *If the output filter uncertainty is described by dynamic (rather than static) IQCs, or the problem involves non-parametric uncertainties, characterizing a suitable set of inverse multipliers $\tilde{\mathbf{\Pi}}$ for a given uncertainty set $\mathbf{\Delta}$ is more difficult.*

A conjecture on robust output feedback controller synthesis

The synthesis result in previous sections addressed two different configurations. If restricting ourselves to parametric uncertainties, these correspond to following generalized plant structures

$$\begin{pmatrix} G_{zw}(s, \delta) & G_{zu}(s) \\ G_{yw}(s, \delta) & G_{yu}(s) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} G_{zw}(s, \delta) & G_{zu}(s, \delta) \\ G_{yw}(s) & G_{yu}(s) \end{pmatrix}.$$

It is our belief that an LMI solution to the robust synthesis problem exists for all systems of the form

$$G = \begin{pmatrix} G_{zw}(s, \delta) & G_{zu}(s, \delta) \\ G_{yw}(s, \delta) & G_{yu}(s) \end{pmatrix}.$$

Note that the channel $u \rightarrow y$ of the uncertain system is not affected by uncertainty. We can always construct an LFR of G from an LFR description of each of the four blocks of G , and arrive at $G = \Delta(\delta) \star P$ with $\Delta(\delta)$ being a diagonal block and P of the form

$$P = \begin{pmatrix} P_{qp,11} & P_{qp,12} & P_{qw,1} & P_{qu} \\ 0 & P_{qp,22} & P_{qw,2} & 0 \\ P_{zp,1} & P_{zp,2} & P_{zw} & P_{zu} \\ 0 & P_{yp} & P_{yw} & P_{yu} \end{pmatrix}. \quad (5.60)$$

Conjecture 5.1 *For the structured generalized plant (5.60), an LMI solution exists for the robust output feedback synthesis problem with general sets of dynamic multipliers that admit a convex parametrization.*

The LMI synthesis solution is intended not to be based on Lemma 5.2, since the dualization arguments is not expected to be promising in extending the synthesis result towards more general interconnections and uncertainty sets.

5.4 Case study: parameter dependent disturbance filter

In our final section, the synthesis algorithm of Theorem 5.2 is applied to a design example that is similar to the problem sketched in Section 5.1. Here, the focus is on sinusoidal disturbances, which prevail in applications containing rotational mechanics such as helicopters, CD players or disk drives, see [118] and references therein. The period/frequency of the sinusoidal disturbance changes in time, which leads to non-stationary sinusoidal signals. As is shown below, such signals can be effectively modeled as the output of a parameter dependent oscillator, see also [114] and references therein.

Similar as was shown in Section 3.3 for the discrete-time case, let $\delta(t)$ denote the single time-varying parameter for a continuous-time LPV system. For specified bounds on the parameter $\delta(t)$ and its rate-of-variation $\dot{\delta}(t)$, a family of suitable classes of dynamic multipliers have been proposed in [112] and references therein. We will show that synthesis result based on static D/G scalings is conservative as compared to synthesis results based on dynamic multipliers. The example thus demonstrates the importance of taking a bound on the rate-of-variation into account.

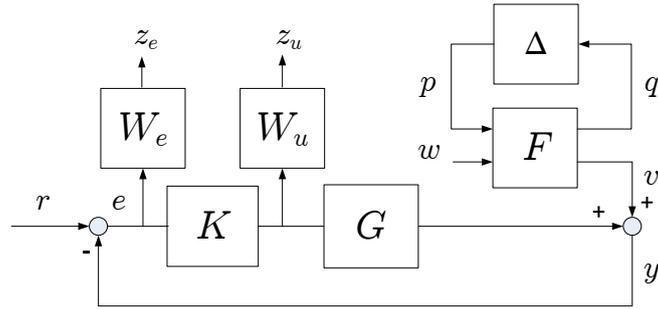


Figure 5.5: System interconnection for the disturbance rejection problem.

Consider the interconnection in Figure 5.5 in which G is the plant model given as

$$G(s) = \frac{s + 0.1}{(s + 0.2)(s + 0.5)},$$

and W_e, W_u are given weighting functions at the output. Adopting the well-known S/KS methodology for solving the disturbance rejection problem, let the performance output be defined as $z = \text{col}(z_e, z_u)$, in which z_e is the weighted tracking error. With an additional weight W_u at the control output and a weight W_e on the tracking error, the generalized plant P becomes

$$P = \begin{pmatrix} -W_e & -W_e G \\ 0 & W_u \\ -I & -G \end{pmatrix}.$$

The disturbances v are (non-stationary) sinusoidal signals with nominal frequency ω_0 that are generated as the second state of the system

$$\begin{aligned} \dot{\xi} &= \begin{pmatrix} 0 & \omega_0(1+\delta) \\ -\omega_0(1+\delta) & 0 \end{pmatrix} \xi, & \xi(0) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ v &= \begin{pmatrix} 0 & 1 \end{pmatrix} \xi \end{aligned} \quad (5.61)$$

in which δ is a time-varying parameter, bounded by $\bar{\delta}$. Since the algorithm outlined in this chapter requires the filter to be stable, we slightly perturb the system matrix in (5.61) by adding a non-zero damping term $\zeta \in \mathbb{R}, \zeta \neq 0$. To be precise, the following realization is chosen to represent our disturbances:

$$\begin{aligned} \dot{\xi} &= \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{pmatrix} \xi + \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{pmatrix} p + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w \\ q &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xi \\ v &= \begin{pmatrix} 0 & 2\omega_0 \end{pmatrix} \xi + \begin{pmatrix} 0 & 2\omega_0 \end{pmatrix} p + \kappa w, \end{aligned} \quad (5.62)$$

and the parameter trajectory $\delta(t)$ satisfies $(\delta(t), \dot{\delta}(t)) \in [-\bar{\delta}, \bar{\delta}] \times [-\bar{\nu}, \bar{\nu}]$. Closing the uncertainty channel amounts to setting

$$p(t) = \begin{pmatrix} \delta(t) & 0 \\ 0 & \delta(t) \end{pmatrix} q(t).$$

If the parameters are time-invariant, i.e. $\delta(t) = \delta$ for some $\delta \in [-\bar{\delta}, \bar{\delta}]$, the dynamical system (5.62) corresponds to the uncertain filter

$$F_\delta(s, \delta) = \delta I \star F(s) = \kappa + \frac{2\zeta\omega_\delta s}{s^2 + 2\zeta\omega_\delta s + \omega_\delta^2}, \quad (5.63)$$

in which $\omega_\delta = \omega_0(1 + \delta)$. The term κ can be used to include a certain level of excitation over all frequencies. In electrical applications, this may be caused by a noise source.

Numerical results

Upper bounds on the worst case \mathcal{L}_2 -gain have been computed from the weighted input u to the weighted output $z = \text{col}(z_e, z_u)$, based on the algorithm of Theorem 5.2. The weight on the control input u is chosen to be $W_u = 1$ and the weight on e pushes down the closed-loop gain at low frequencies, and is chosen to be

$$W_e = \frac{0.5s + 0.35}{s + 0.01}.$$

We take the nominal frequency $\omega_0 = 0.05$ with 30% error, i.e. $\bar{\delta} = 0.3$. The damping coefficient ζ need not be chosen large and is set to $\zeta = 0.005$. The direct feed through

term κ was found to significantly influence the designs and was merely treated as a tuning variable, and was chosen to be $\kappa = 0.2$.

Our first design is denoted by K_{DG} and is based on using static D/G scales, similar as has been done in [62]. It therefore provides stability and performance guarantees against arbitrarily fast time-varying parameters. With a synthesis optimal value of $\gamma = 1.25$, the resulting closed-loop sensitivity from $v \rightarrow e$, that is without the disturbance filter, is shown in Figure 5.6,

Now let us apply Theorem 5.2, making use of a class of dynamic multipliers as proposed in Theorem 3.3. Following the arguments in Section 3.3, considered in continuous-time, we have constructed a dynamic multiplier class for $\beta = 1$ and $\lambda = -1$, i.e. the pole-location of the stable (continuous-time) filter H , see again [112]. By employing a convex hull relaxation for implementing the inclusion $Q \in \mathcal{Q}$ in (3.21), the controllers denoted by K_1, K_2 are obtained by choosing the bounds $|\dot{\delta}| < 1$ and $|\dot{\delta}| < 0.06$ respectively. The synthesis optimal values are $\gamma = 1.32$ and $\gamma = 1.28$ for K_1, K_2 respectively.

Time-domain simulations have been performed with a non-stationary sinusoidal disturbance input v generated as the initial response of the system (5.62) in which $\zeta = \kappa = 0$ and collected in Figure 5.7. It is clearly visible from these plots that the best performance is achieved by controller K_2 .

We finally comment on our observations. We have been able to reduce the conservatism in handling non-stationary sinusoidal disturbance signals by using a class of dynamic multipliers in designs K_1, K_2 . As a result, the disturbance rejection performance was improved. In particular, if the parameter is allowed to vary arbitrarily fast, as is the case in K_{DG} , the resulting closed-loop frequency response involves a notch close to the highest frequency in the interval $[\omega_0(1 - \bar{\delta}), \omega_0(1 + \bar{\delta})]$, which does not effectively account for the fact that the sinusoidal frequency is specified in an interval. Once the rate-of-variation is bounded, this notch shifts to a lower frequency, as shown by design K_1 . In case the bound on $\dot{\delta}$ is further reduced, the sharp notch eventually vanishes. Moreover, the design K_2 shows that the closed loop gain from $v \rightarrow e$ is lower in an average sense, at the cost of a high gain at frequencies greater than 1 rad/s.

Remark 5.6 *Since numerical algorithms (generally) provide upper bound values on the worst case \mathcal{L}_2 -gain, reducing the bound on $\dot{\delta}$ need not always result in lower optimal values.*

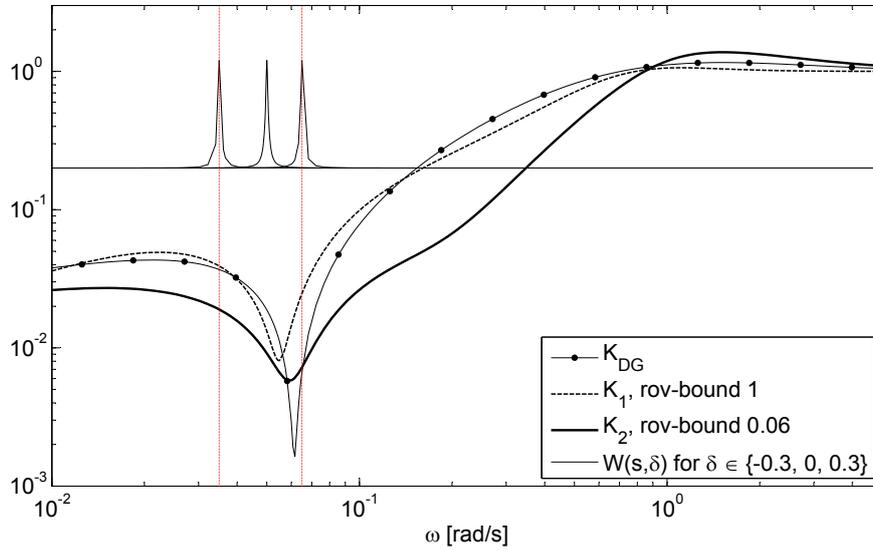


Figure 5.6: Closed-loop sensitivity $v \rightarrow e$. Also shown is F_δ for parameter values $\delta \in \{-0.3, 0, 0.3\}$.

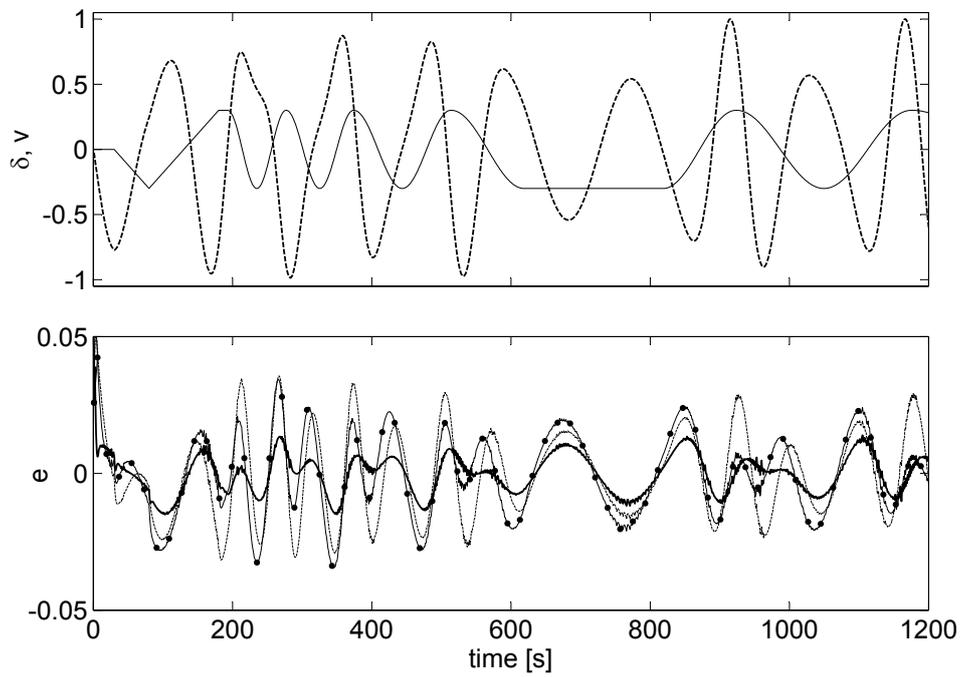


Figure 5.7: Tracking error e for designs K_{DG} (with dots), K_1 and K_2 (bold). On top, signals δ and v (dashed) are shown.

5.5 Summary

Motivated by a robust disturbance rejection problem, in which disturbances are described by an uncertain filter at the plant input, we have given an LMI solution for the corresponding robust output feedback controller synthesis problem. The method adopts the IQC framework for the analysis of uncertain systems, and allows for general dynamic IQC multipliers. Throughout this chapter, a class of admissible multipliers, parameterized by finitely many LMI constraints along the lines in Chapter 3, was assumed to be known.

In order to infer robust performance of the closed loop system by applying an IQC-based analysis test from Chapter 3, it is essential that the nominal closed loop system is stable. An LMI characterization of this property has recently been developed, for which a new elementary proof has been given. The synthesis solution boils down to a combination of two congruence transformations from the existing literature. One of these transformations is well-known, since it solves the nominal output feedback synthesis problem. Synthesis conditions were presented that guarantee a worst-case bound on the \mathcal{L}_2 -gain or \mathcal{H}_2 -performance level.

It is expected that an LMI solution to the robust output feedback synthesis problem exists for many other interconnections of uncertain systems. As a preliminary study, we have considered the dual problem in which uncertainty enters a filter at the plant output. A solution to this problem could be obtained, when parametric uncertainties are involved and a class of static multipliers is used. In view of the fact that the general robust output feedback problem is largely open, it is a challenging research question which kind of interconnections of uncertain systems can be handled by convex optimization in the LMI framework. It is conjectured that the only structural property needed for a synthesis solution to exist in terms of LMIs, is the fact that the transfer matrix from control input to measurement output is not affected by uncertainty. Hence, the interconnections that can be handled in this fashion will cover a specific class of controller design problems only. In particular, it will not include the disturbance rejection problem for a single-input single-output system in which the plant model is affected by an additive uncertain element.

In our final section, we have presented a numerical example that illustrates the algorithm, making use of the multiplier class from Section 3.3 and showing the benefit of incorporating the parameter rate-of-variation, which agrees with our earlier observations on the analysis of uncertain systems.

In the next chapter, the unknown parameters are on-line measurable, and it is our aim to design scheduled (rather than robust) controllers for LPV systems. By making use of a Lyapunov function rather than IQC multipliers, this synthesis problem can be turned into a robust SDP.

Chapter 6

Scheduled controller synthesis

Control theory of linear parameter-varying (LPV) systems has originally been motivated by the limitations of existing gain-scheduling methods [152]. The classical approach towards gain-scheduling was a time-consuming process which involved the interpolation of a collection of local controller designs. In modern LPV controller synthesis methods, most of the time and effort are spent on building a trustworthy LPV model of the plant. The LPV synthesis algorithms then provide a gain-scheduled controller at one shot, providing guaranteed robust stability and performance of the closed-loop system, at least if numerical aspects are left out for the moment.

Existing solutions for LPV synthesis are variations and extensions of two closely related solutions. The first, developed by [130] and [3], assumes the system matrices to be given in LFR form and accounts for the uncertainties by using the so-called D -scalings. By taking the to-be-designed controller in an LFR, the robust optimal control problem can be transformed into a convex optimization problem. In order to reduce the conservatism that is involved when D -scalings are used, more general classes of multipliers have been proposed, see for example [86, 165, 155, 183].

The second approach as proposed by [2] does not start from an LFR structure of the plant and controller. On the contrary, it is derived from an existing synthesis result in [159, 122], which considers the nominal output feedback problem for LTI systems. The sufficient conditions are formulated in terms of a robust SDP. Initial algorithms [2, 155] relied on sampling the parameter dependent LMI constraint on a finite grid of parameter values. As an immediate consequence of this method, the synthesis solutions no longer guarantee robust stability or performance of the closed-loop system, and an additional analysis test is needed. If the design specifications are not met, the parameter grid is refined and a new controller is computed.

Recently, in [184], the controller synthesis solution could be directly formulated in terms of a robust SDP, for general rational dependence in the problem data, therefore avoiding the need for gridding the parameter space.

The controller synthesis approach as presented in this chapter is based on [184, 61] and has been published in [63] in a slightly different form. Our focus is on minimizing the worst-case induced \mathcal{L}_2 -gain of the closed loop system. Making use of the relaxation tools from Chapter 2, LPV controllers are found by means of standard LMI optimization. The synthesis result is presented for continuous time LPV systems, though a discrete-time version of the derived synthesis result is quite straightforward, see [184].

In the next section, a characterization of \mathcal{L}_2 -gain performance is presented that is analogue to the formulation in Proposition 4.3 for discrete-time LPV systems. After the synthesis conditions are derived, it is then shown how to eliminate the controller parameters in order to reduce computational complexity. A numerical example taken from [96] will be presented to illustrate the potential benefits of the LPV synthesis approach as well as the benefits of using sum-of-squares relaxations.

6.1 LPV synthesis by robust semi-definite programming

Consider the following LPV system in continuous time

$$\begin{aligned} \dot{x} &= A(\delta(t))x + B_1(\delta(t))w + B_2(\delta(t))u \\ z &= C_1(\delta(t))x + D_{11}(\delta(t))w + D_{12}(\delta(t))u \\ y &= C_2(\delta(t))x + D_{21}(\delta(t))w, \end{aligned} \tag{6.1}$$

in which $\delta(t)$ is a time-varying parameter, $w \rightarrow z$ represents the performance channel, u is the control input and y the measured output. The size and the rate-of-variation of admissible parameter trajectories are assumed to satisfy $(\delta(t), \dot{\delta}(t)) \in \mathcal{R}$ for all $t \geq 0$ for some given compact set \mathcal{R} . Analogous to the definition in Chapter 4, a trajectory is admissible if $(\delta(t), \dot{\delta}(t)) \in \mathcal{R}$ holds for all $t \geq 0$. Further, it is assumed that $D_{22} = 0$ without loss of generality.

Under the assumption that the parameter $\delta(t)$ is on-line measurable, the LPV controller synthesis problem amounts to designing continuous mappings $A_k(\cdot), B_k(\cdot), C_k(\cdot), D_k(\cdot)$ such that the controller

$$\begin{aligned} \dot{x}_k &= A_k(\delta(t), \dot{\delta}(t))x_k + B_k(\delta(t))y \\ u &= C_k(\delta(t))x_k + D_k(\delta(t))y \end{aligned} \tag{6.2}$$

interconnected with system (6.1) guarantees closed-loop stability and performance for all admissible parameter trajectories. The fact that only $A_k(\cdot, \cdot)$ is a function of $\dot{\delta}$ will follow naturally from the presented synthesis algorithm. Assuming the system

matrices of the plant and controller are continuous functions of the parameter, the realization of the interconnected system reads as

$$\begin{pmatrix} \mathcal{A}(\delta(t)) & \mathcal{B}(\delta(t)) \\ \mathcal{C}(\delta(t)) & \mathcal{D}(\delta(t)) \end{pmatrix} = \left(\begin{array}{cc|c} A(\delta(t)) & 0 & B_1(\delta(t)) \\ 0 & 0 & 0 \\ \hline C_1(\delta(t)) & 0 & D_{11}(\delta(t)) \end{array} \right) + \\ \left(\begin{array}{cc} 0 & B_2(\delta(t)) \\ \hline I & 0 \\ 0 & D_{12}(\delta(t)) \end{array} \right) \begin{pmatrix} A_K(\delta(t)) & B_K(\delta(t)) \\ C_K(\delta(t)) & D_K(\delta(t)) \end{pmatrix} \begin{pmatrix} 0 & I & 0 \\ \hline C_2(\delta(t)) & 0 & D_{21}(\delta(t)) \end{pmatrix}. \quad (6.3)$$

Recall the quadratic performance criterion from Appendix A. Along similar arguments as used in Section 4.2 in discrete time, the closed-loop system can be shown to satisfy quadratic performance with respect to the performance index P_p if there exists a continuously differentiable $\mathcal{X}(\cdot)$ that satisfies for all $(\delta, \nu) \in \mathcal{R}$ the inequalities

$$\mathcal{X}(\delta) \succ 0, \quad (6.4)$$

$$\begin{pmatrix} I & 0 \\ \hline \mathcal{A}(\delta) & \mathcal{B}(\delta) \\ 0 & I \\ \hline \mathcal{C}(\delta) & \mathcal{D}(\delta) \end{pmatrix}' \begin{pmatrix} \partial \mathcal{X}(\delta, \nu) & \mathcal{X}(\delta) & 0 \\ \hline \mathcal{X}(\delta) & 0 & 0 \\ 0 & 0 & P_p \end{pmatrix} \begin{pmatrix} I & 0 \\ \hline \mathcal{A}(\delta) & \mathcal{B}(\delta) \\ 0 & I \\ \hline \mathcal{C}(\delta) & \mathcal{D}(\delta) \end{pmatrix} \prec 0, \quad (6.5)$$

in which $\partial \mathcal{X}(\delta, \nu)$ is defined as

$$\partial \mathcal{X}(\delta, \nu) = \sum_{j=1}^s \frac{\partial \mathcal{X}}{\partial \delta_j}(\delta) \nu_j. \quad (6.6)$$

A proof can be found for example in [185]. The scheduled controller design problem amounts to computing matrix functions $\mathcal{X}(\cdot)$ and $A_K(\cdot), B_K(\cdot), C_K(\cdot), D_K(\cdot)$ that satisfy (6.4) and (6.5). Although these functions must certainly be parameterized for rendering the problem computationally tractable, let us for the moment concentrate on resolving the non-linear dependence of (6.5) on these matrix functions. The following arguments strongly resemble the ones used in Section 5.3.3.

As initially developed in [159, 122], the nominal output feedback synthesis problem is solved by a suitable congruence transformation and a change of variables. An extension of this result that can be applied to parameter dependent systems was developed in [2]. As a first step, let the Lyapunov matrix \mathcal{X} be partitioned according to \mathcal{A} in (6.3) and denote

$$\mathcal{X} = \begin{pmatrix} X & U \\ U^T & * \end{pmatrix}, \quad \mathcal{X}^{-1} = \begin{pmatrix} Y & V \\ V^T & * \end{pmatrix}.$$

We assume U, V to be square matrices, which corresponds to a controller order that equals the order of the plant. The dependence on δ is omitted for notational convenience.

nience. We apply on both (6.4) and (6.5) a particular congruence transformation, that will be clarified in the next theorem. With

$$\mathcal{Y} = \begin{pmatrix} Y & I \\ V^T & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{Z} = \begin{pmatrix} I & 0 \\ X & U \end{pmatrix}$$

we obtain the identities

$$\mathcal{Y}^T \mathcal{X} = \mathcal{Z} \quad \text{and} \quad I - XY = UV^T$$

The differential operator ∂ as defined in (6.6) is now applied to the first functional identity, by which we arrive at $(\partial\mathcal{Y})^T \mathcal{X} + \mathcal{Y}^T (\partial\mathcal{X}) = \partial\mathcal{Z}$. Right-multiplying this relation by \mathcal{Y} leads to

$$\begin{aligned} \mathcal{Y}^T (\partial\mathcal{X}) \mathcal{Y} &= (\partial\mathcal{Z}) \mathcal{Y} - (\partial\mathcal{Y})^T \mathcal{Z}^T = \\ &= \begin{pmatrix} -\partial Y & -(\partial Y)X - (\partial V)U^T \\ (\partial X)Y + (\partial U)V^T & \partial X \end{pmatrix}. \end{aligned} \quad (6.7)$$

Moreover, let v denote functional matrices

$$v(\delta, \nu) = \{K(\delta, \nu), L(\delta), M(\delta), N(\delta), X(\delta), Y(\delta)\} \quad (6.8)$$

with K, L, M, N being defined through the relation

$$\begin{aligned} \begin{pmatrix} K & L \\ M & N \end{pmatrix} &= \begin{pmatrix} U & XB_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} \begin{pmatrix} V^T & 0 \\ C_2 Y & I \end{pmatrix} \\ &+ \begin{pmatrix} XAY + (\partial X)Y + (\partial U)V^T & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (6.9)$$

With these preparations, the existence of $\mathcal{X}(\cdot), A_K(\cdot), B_K(\cdot), C_K(\cdot), D_K(\cdot)$ that satisfy (6.4)-(6.5) can be turned into the following synthesis conditions that are convex in the new functional variables $v(\delta, \nu)$.

Theorem 6.1 (LPV synthesis) *Let \mathcal{R} be a compact set that characterizes the admissible parameter trajectories $\delta(\cdot)$, i.e. $(\delta(t), \dot{\delta}(t)) \in \mathcal{R}$ for all $t \geq 0$. There exists an LPV controller of the form (6.2) that robustly stabilizes the LPV system (6.1) and satisfies quadratic performance with respect to P_p for any \mathcal{R} -admissible parameter trajectory, if there exist matrix-valued functions (6.8) for which*

$$\mathbf{X}(v) \succ 0, \quad (6.10)$$

$$\begin{pmatrix} I & 0 \\ \mathbf{A}(v) & \mathbf{B}(v) \\ 0 & I \\ \mathbf{C}(v) & \mathbf{D}(v) \end{pmatrix}^T \begin{pmatrix} \mathbf{Z}(v) & I & 0 \\ I & 0 & 0 \\ 0 & 0 & P_p \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathbf{A}(v) & \mathbf{B}(v) \\ 0 & I \\ \mathbf{C}(v) & \mathbf{D}(v) \end{pmatrix} \prec 0 \quad (6.11)$$

holds for all $(\delta, \nu) \in \mathcal{R}$, where we use the abbreviations

$$\begin{aligned}
\mathbf{A}(v) &= \begin{pmatrix} \partial Y + \text{sym}(AY + B_2M) & (A + B_2NC_2) + K^T \\ (A + B_2NC_2)^T + K & \partial X + \text{sym}(AX + LC_2) \end{pmatrix}, \\
\mathbf{B}(v) &= \begin{pmatrix} B_1 + B_2ND_{21} \\ XB_1 + LD_{21} \end{pmatrix}, \\
\mathbf{C}(v) &= (C_1Y + D_{12}M \quad C_1 + D_{12}NC_2), \\
\mathbf{D}(v) &= (D_{11} + D_{12}ND_{21}), \\
\mathbf{Z}(v) &= \begin{pmatrix} -\partial Y & 0 \\ 0 & \partial X \end{pmatrix}, \quad \mathbf{X}(v) = \begin{pmatrix} Y & I \\ I & X \end{pmatrix},
\end{aligned} \tag{6.12}$$

and $\text{sym}(P) \equiv P + P^T$.

Remark 6.1 As we already addressed in Chapter 5, Schur's Lemma is needed to render (6.11) affine in the functional variables $v(\delta, \nu)$.

Proof. The proof can be found in [185, 188] and is a direct extension of the LMI solution of the nominal output feedback problem as discussed in [159]. A sketch of the proof is as follows. By using the definitions in (6.9), it follows that

$$\begin{pmatrix} \mathcal{Y} & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} \mathcal{X}\mathcal{A} + \frac{1}{2}\partial\mathcal{X} & \mathcal{X}\mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \mathcal{Y} & 0 \\ 0 & I \end{pmatrix}$$

equals

$$\begin{aligned}
& \left(\begin{array}{cc|c} AY - \frac{1}{2}\partial Y & A & B_1 \\ 0 & XA + \frac{1}{2}\partial X & XB_1 \\ \hline C_1Y & C_1 & D_{11} \end{array} \right) + \begin{pmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{pmatrix} \begin{pmatrix} K & L \\ M & N \end{pmatrix} \begin{pmatrix} I & 0 & | & 0 \\ 0 & C_2 & | & D_{21} \end{pmatrix} \\
& = \begin{pmatrix} \mathbf{A}(v) + \frac{1}{2}\mathbf{Z}(v) & \mathbf{B}(v) \\ \mathbf{C}(v) & \mathbf{D}(v) \end{pmatrix}. \tag{6.13}
\end{aligned}$$

Brief calculations show, using also the variable definition (6.9), that $\mathbf{X}(v) = \mathcal{Y}^T \mathcal{X} \mathcal{Y}$ and (6.11) is obtained by left- and right multiplication of (6.5) with

$$\begin{pmatrix} \mathcal{Y} & 0 \\ 0 & I \end{pmatrix}^T, \quad \begin{pmatrix} \mathcal{Y} & 0 \\ 0 & I \end{pmatrix}$$

respectively. ■

We observe from (6.9) and the definition of $\partial\mathcal{X}, \partial\mathcal{Y}$, that L, M, N depend on δ only, whereas K depends on ν as well. In fact, if the system matrices do not depend on

ν , the matrix function $K(.,.)$ has the structure

$$K(\delta, \nu) = K_0(\delta) + \sum_{j=1}^s K_j(\delta) \nu_j,$$

which is fully defined by the matrix functions $K_0(\delta), \dots, K_s(\delta)$. Note that from (6.9) it follows that $A_K(\delta, \dot{\delta})$, which had already been displayed in (6.2).

Hence, with a priori chosen functions $f_i, i = 1, \dots, m$ assumed continuously differentiable on δ , and linear parametrization of the form

$$\begin{aligned} K_j(\delta) &= \sum_{i=1}^m K_i^j f_i(\delta), \quad j = 0, 1, \dots, s, \\ L(\delta) &= \sum_{i=1}^m L_i f_i(\delta), \quad M(\delta) = \sum_{i=1}^m M_i f_i(\delta), \quad N(\delta) = \sum_{i=1}^m N_i f_i(\delta), \\ X(\delta) &= \sum_{i=1}^m X_i f_i(\delta), \quad Y(\delta) = \sum_{i=1}^m Y_i f_i(\delta), \end{aligned} \tag{6.14}$$

the synthesis conditions (6.10)-(6.11) both become robust SDP constraints in the decision variables $X_i, Y_i, K_i^j, L_i, M_i, N_i$, for which the relaxations methods from Chapter 2 can be employed. How to select the functions f_1, \dots, f_m , as well as the number m , requires further investigation, which is why it is currently a matter of experience.

Controller reconstruction

One should not underestimate the difficulty of implementing an LPV controller in practice. Clearly, for matrix functions $v(\delta, \nu)$ that satisfy the synthesis conditions, controller matrices can be obtained as

$$\begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} = \begin{pmatrix} U & X B_2 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} K - X A Y - [(\partial X) Y + (\partial U) V^T] & L \\ M & N \end{pmatrix} \begin{pmatrix} V^T & 0 \\ C_2 Y & I \end{pmatrix}.$$

At each time-step, one must compute the matrices $A_K(\cdot), B_K(\cdot), C_K(\cdot), D_K(\cdot)$ for the measured parameter values $(\delta(t), \dot{\delta}(t))$. The reconstruction of these constant matrices has already been discussed in Section 5.3.3, see also [2, 159, 122]. Since the operations involve matrix inversion, it is a challenging problem to develop reliable and fast LPV controller implementation schemes.

An heuristic approach is proposed in [81], and uses a linear approximation of the (generally) rational matrix functions A_K, B_K, C_K, D_K . Before implementing the modified LPV controller, an additional analysis test is then needed in order to prove closed-loop stability and performance.

Remark 6.2 *If system matrices are affine functions on the parameter and \mathcal{R} is a convex polytope, the matrix functions in (6.14) are often chosen to be affine also. The resulting robust LMIs then depend quadratically on the parameter, which enables the construction of standard relaxation schemes based on multi-convexity arguments, see [77, 174, 5].*

6.1.1 Elimination of parameters

Rather than directly solving (6.11) for a fixed parametrization of the functional variables $v(\delta, \nu)$, the variables $K(\cdot), L(\cdot), M(\cdot), N(\cdot)$ can be eliminated in order to reduce the problem size. This result is provided by the so-called projection lemma, see [76] for a reference, under the following hypotheses:

Assumption 6.1 $(B_2^T(\delta) \ D_{12}^T(\delta))$ and $(C_2(\delta) \ D_{21}(\delta))$ have full row rank for all $(\delta, \cdot) \in \mathcal{R}$.

With quadratic performance index P_p , we denote

$$P_p^{-1} = \begin{pmatrix} Q_p & S_p \\ S_p' & R_p \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{Q}_p & \tilde{S}_p \\ \tilde{S}_p' & \tilde{R}_p \end{pmatrix}.$$

As shown in [188, 14], conditions (6.10)- (6.11) in Theorem 6.1 are equivalent to the existence of smooth symmetric matrix functions $Y(\cdot)$ and $X(\cdot)$ which satisfy

$$U_Y(\delta)^T \left(\begin{array}{cc|cc} -\partial Y(\delta, \nu) & Y(\delta) & & 0 \\ Y(\delta) & 0 & & \\ \hline & 0 & -\tilde{Q}_p & \tilde{S}_p \\ & & \tilde{S}_p' & -\tilde{R}_p \end{array} \right) U_Y(\delta) \prec 0 \quad (6.15)$$

$$U_X(\delta)^T \left(\begin{array}{cc|cc} 0 & X(\delta) & & 0 \\ X(\delta) & \partial X(\delta, \nu) & & \\ \hline & 0 & Q_p & S_p \\ & & S_p' & R_p \end{array} \right) U_X(\delta) \prec 0 \quad (6.16)$$

$$\begin{pmatrix} Y(\delta) & I \\ I & X(\delta) \end{pmatrix} \succ 0 \quad (6.17)$$

for all $(\delta, \nu) \in \mathcal{R}$. Here, we used the abbreviations

$$U_Y(\delta) = \begin{pmatrix} I & 0 \\ A^T(\delta) & C_1^T(\delta) \\ 0 & I \\ B_1^T(\delta) & D_{11}^T(\delta) \end{pmatrix} \mathcal{N}_Y(\delta),$$

$$U_X(\delta) = \begin{pmatrix} A(\delta) & B_1(\delta) \\ I & 0 \\ C_1(\delta) & D_{11}(\delta) \\ 0 & I \end{pmatrix} \mathcal{N}_X(\delta), \quad (6.18)$$

in which $\mathcal{N}_Y(\delta), \mathcal{N}_X(\delta)$ are basis matrices for the kernels of $(B_2^T(\delta) \ D_{12}^T(\delta))$ and $(C_2(\delta) \ D_{21}(\delta))$ respectively. Due to the full rank property in Assumption 6.1, an LFR description of $\mathcal{N}_Y(\delta)$ and $\mathcal{N}_X(\delta)$ can be constructed from the LFR description

of the system matrices, see Appendix B. It is stressed that this essential observation was made in [184], which leads to an LPV synthesis solution that no longer requires the introduction of a parameter grid.

In view of the relaxation methods in Chapter 2, it is convenient to choose

$$Y(\delta) = T_Y(\delta)^T P T_Y(\delta), \quad X(\delta) = T_X(\delta)^T Q T_X(\delta) \quad (6.19)$$

for some fixed polynomial (or rational) basis matrices $T_Y(\delta)$, $T_X(\delta)$, with P and Q the to-be-computed symmetric coefficient matrices. With these choices, the synthesis conditions (6.15)-(6.17) can easily be expressed in the general form

$$F_i(\delta, \nu)^T J_i(P, Q) F_i(\delta, \nu) \prec 0, \quad i = 1, 2, 3, \quad (6.20)$$

as used in Chapter 2. The matrix functions J_i depend affinely on the decision variables $y = (P, Q)$ and F_i is a known matrix function in the parameters $(\delta, \nu) \in \mathcal{R}$ of which an LFR description can be constructed. The robust LMIs in (6.20) correspond to the conditions presented in [184].

Remark 6.3 *As mentioned in [163], it is possible to extend the LPV synthesis approach to \mathcal{H}_2 -control and to the other quadratic performance specifications.*

Remark 6.4 *In order to arrive at convex synthesis conditions, it turns out to be essential that the parameters can be measured online. For non-parametric uncertainties or parameters that cannot be measured online, the output feedback control problem appears to be non-convex, as already mentioned in Chapter 5. In case parameters are measurable, but the measurements are corrupted by noise, an LPV controller approach involves filtering the parameter before actually using it for gain-scheduling. At present, a systematic and numerically tractable solution for this modified LPV synthesis problem has not been found.*

6.2 Numerical example

In order to illustrate the LPV controller synthesis approach, we construct an LPV system with a system matrix that is based on the μ -analysis example of Section 2.1. By adding a control and performance channel, we consider the following family of system, parameterized by $p \in [0.6, 2]$:

$$\begin{pmatrix} \dot{x} \\ z_u \\ z \\ y \end{pmatrix} = \left(\begin{array}{cccccc|cc} -1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 1 & 0 \\ 0 & 2p & 0 & p & 0 & p & 0 & 0 \\ 0 & 0 & -2p & 0 & -p & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \begin{pmatrix} x \\ w_u \\ w \\ u \end{pmatrix} \quad (6.21)$$

For each fixed value of p , we will apply the LPV synthesis as outlined in the previous section. The parameters that are used for scheduling are δ_1, δ_2 , which enter the plant in a feedback configuration, defined by the relation $w_u = \Delta(\delta_1, \delta_2)z_u$, in which

$$\Delta(\delta_1(t), \delta_2(t)) = \left(\begin{array}{cc|ccc} \delta_1(t) & 0 & 0 & 0 & 0 \\ 0 & \delta_1(t) & 0 & 0 & 0 \\ \hline 0 & 0 & \delta_2(t) & 0 & 0 \\ 0 & 0 & 0 & \delta_2(t) & 0 \\ 0 & 0 & 0 & 0 & \delta_2(t) \end{array} \right). \quad (6.22)$$

One can easily verify that the interconnection of (6.21) with Δ is well-posed for all $\delta \in \boldsymbol{\delta}$. The time-varying parameters $\delta(t) = (\delta_1(t), \delta_2(t))$ are assumed to satisfy $\delta(t) \in \boldsymbol{\delta}$ for all $t \geq 0$, with

$$\boldsymbol{\delta} = [-0.4, 0.4] \times [-0.4, 0.4], \quad (6.23)$$

and no bound on $\dot{\delta}_1(t), \dot{\delta}_2(t)$ is imposed.

The goal in this section is to design an LPV controller $K(\delta) : u \rightarrow y$ that robustly stabilizes the closed loop system and minimizes a bound γ on the worst case \mathcal{L}_2 -gain of the channel $w \rightarrow z$. Our focus is on computing the achievable performance, and not on reconstructing the controller, by solving the LPV synthesis conditions (6.15)-(6.17). The Lyapunov matrix is chosen to be parameter independent, i.e. $X(\delta) = Q$, $Y(\delta) = P$, which reduces (6.17) to a genuine LMI. Note that Q and P are scalar variables, since the state-dimension is one.

Following the arguments from Section 6.1.1, we introduce the abbreviations

$$M_X(\delta) = \begin{pmatrix} A(\delta) & B_1(\delta) \\ I & 0 \\ C_1(\delta) & D_{11}(\delta) \\ 0 & I \end{pmatrix}, \quad M_Y(\delta) = \begin{pmatrix} I & 0 \\ A^T(\delta) & C_1^T(\delta) \\ 0 & I \\ B_1^T(\delta) & D_{11}^T(\delta) \end{pmatrix}.$$

By applying Lemma B.1, we construct parameter-dependent matrices $K_X(\delta), K_Y(\delta)$ that satisfy

$$(C_2(\delta) \ D_{21}(\delta))K_X(\delta) \equiv 0 \quad \text{and} \quad (B_2^T(\delta) \ D_{12}^T(\delta))K_Y(\delta) \equiv 0. \quad (6.24)$$

Since $X(\delta) = Q$ and $Y(\delta) = P$, the parameter dependent LMI constraints (6.15)-(6.16) are in the general form of Chapter 2, i.e.

$$F_i(\delta)' J_i(P, Q) F_i(\delta) \prec 0, \quad i = 1, 2, \quad (6.25)$$

if defining $F_1 = M_X(\delta)K_X(\delta)$ and $F_2 = M_Y(\delta)K_Y(\delta)$, as well as

$$J_1(P, Q) = \left(\begin{array}{cc|cc} 0 & Q & 0 & 0 \\ Q & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & \gamma^{-1} \end{array} \right), \quad J_2(P, Q) = \left(\begin{array}{cc|cc} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ \hline 0 & 0 & \gamma^{-1} & 0 \\ 0 & 0 & 0 & -\gamma \end{array} \right). \quad (6.26)$$

It can be verified that $F_1(\delta), F_2(\delta)$ are both rational matrix functions of δ . The corresponding LFRs that are needed for the construction of multiplier based relaxation schemes are denoted by

$$F_i(\delta) = D_i + C_i \Delta_i(\delta) (I - A_i \Delta_i(\delta))^{-1} B_i, \quad \text{for } i = 1, 2,$$

for some affine matrix functions Δ_1, Δ_2 . As we pointed out in Chapter 2, any robust LMI that depends rationally on the parameters can be transformed into an equivalent polynomially robust LMI. In the sequel, we compare the direct approach of Section 2.2.3 with the S-procedure-based method as discussed in Section 2.3.

Multiplier relaxation

The first and most common approach of handling the rational parameter dependence is the S-procedure of Section 2.3. That is, a set of admissible multipliers Π_i is parameterized for (6.25), such that the semi-infinite constraint

$$\begin{pmatrix} \Delta_i(\delta) \\ I \end{pmatrix}' \Pi_i \begin{pmatrix} \Delta_i(\delta) \\ I \end{pmatrix} \succ 0, \quad \forall \delta \in \delta$$

is satisfied for all $\Pi_i \in \Pi_i$, for $i = 1, 2$.

Although it was not explicitly stated in [184], the LPV synthesis approach relies on $F_1(\delta)$ and $F_2(\delta)$ both being well-posed on the set δ . It is important to realize that well-posedness of $K_Y(\delta)$ and $K_X(\delta)$ is not guaranteed by the formula in Lemma B.1. Indeed, for $p > 1.4$, some element of the matrix functions $F_1(\delta)$ and $F_2(\delta)$ becomes unbounded on the domain $[-0.4] \times [0.4]$. This is also indicated in Table 6.1.

Recall the fact that if $K_X(\delta)$ and $K_Y(\delta)$ had been chosen to be polynomial matrix functions, a well-posed LFR always exists, see [39] for a possible algorithm. Here, both $K_X(\delta)$ and $K_Y(\delta)$ have a single column, which is why we can simply multiply with suitably chosen scalar polynomials $d_X(\delta)$ and $d_Y(\delta)$, and arrive at the polynomial matrix functions

$$\tilde{K}_X(\delta) = K_X(\delta)d_X(\delta), \quad \tilde{K}_Y(\delta) = K_Y(\delta)d_Y(\delta).$$

For the purpose of illustration, the following polynomials have been used at $p = 1.5$:

$$d_Y(\delta) = \frac{3}{38}\delta_1^2\delta_2^3 + \frac{27}{38}\delta_1\delta_2^3 - \frac{15}{19}\delta_2^3 - \frac{13}{38}\delta_1^2\delta_2^2 + \frac{1}{19}\delta_1\delta_2^2 + \delta_2^2 - \frac{2}{19}\delta_1^2\delta_2 - \frac{4}{19}\delta_1\delta_2 + \frac{2}{19}\delta_1^2 - \frac{4}{19},$$

$$d_X(\delta) = \frac{1}{2}\delta_1^2\delta_2^3 - \frac{1}{2}\delta_1^2\delta_2^2 - \frac{2}{9}\delta_1^2\delta_2 + \frac{2}{9}\delta_1^2 + \frac{1}{2}\delta_1\delta_2^3 + \frac{1}{3}\delta_1\delta_2^2 - \delta_2^3 + \delta_2^2 + \frac{4}{9}\delta_2 - \frac{4}{9}.$$

If we define

$$\tilde{F}_1(\delta) = M_X(\delta)K_X(\delta)d_X(\delta) \quad \text{and} \quad \tilde{F}_2(\delta) = M_Y(\delta)K_Y(\delta)d_Y(\delta), \quad (6.27)$$

and $J_1(P, Q)$, $J_2(P, Q)$ as in (6.26), the robust LMI constraints (6.15)-(6.16) amount to

$$\tilde{F}_i(\delta)' J_i(P, Q) \tilde{F}_i(\delta) \prec 0, \quad i = 1, 2, \quad (6.28)$$

with the LFR now written as

$$\tilde{F}_i(\delta) = \tilde{D}_i + \tilde{C}_i \tilde{\Delta}_i(\delta) (I - \tilde{A}_i \tilde{\Delta}_i(\delta))^{-1} \tilde{B}_i, \quad i = 1, 2,$$

for some affine matrix functions $\tilde{\Delta}_1, \tilde{\Delta}_2$. Assuming without loss of generality that the LFR of the polynomial matrix $\tilde{F}_1(\delta)$ and $\tilde{F}_2(\delta)$ is well-posed on δ , the LPV synthesis conditions can be implemented by constructing relaxation schemes based on the S-procedure. Note that the points $\delta \in \delta$ at which the LFR of $K_X(\delta)$ or $K_Y(\delta)$ is not defined consists of a set of measure zero. By continuity argument, it is therefore justified to impose the robust LMIs on the whole domain δ .

A convex hull relaxation from Section 2.3.1 has been employed for characterizing a set of admissible multipliers Π_i , for which the robust LMI constraint

$$\begin{pmatrix} \tilde{\Delta}_i(\delta) \\ I \end{pmatrix}' \Pi_i \begin{pmatrix} \tilde{\Delta}_i(\delta) \\ I \end{pmatrix} \succ 0, \quad \forall \delta \in \delta,$$

is satisfied for all $\Pi_i \in \mathbf{\Pi}_i$, for $i = 1, 2$. With full block multipliers and a partial (or

multi-) convexity argument, see e.g. in [5], the relaxation is referred to as 'CH-PC'.

	0.6	0.8	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$K_X(\delta)$	+	+	+	+	+	+	-	-	-	-	-	-	-
$\dim \tilde{\Delta}_1(\delta)$	10	10	10	10	10	10	40	40	40	40	40	40	40
$K_Y(\delta)$	+	+	-	-	-	-	-	-	-	-	-	-	-
$\dim \tilde{\Delta}_2(\delta)$	10	10	43	51	50	51	51	51	51	51	51	51	48

Table 6.1: Well-posedness of the LFRs $K_X(\delta)$ and $K_Y(\delta)$ on the δ in (6.23) for different values of $p \in [0.6, 2]$. A '+' indicates that the LFR is well-posed. Also indicated are the sizes of the blocks $\tilde{\Delta}_i(\delta)$ of the LFR $\tilde{F}_i(\delta)$.

In Figure 6.1, the minimal achievable γ -level (of the closed loop system) is shown for different values of the parameter p . As shown in Table 6.2, the number of variables grows excessively as we increase the value of p . This is due to the increased sizes of $\tilde{F}_X(\delta)$ and $\tilde{F}_Y(\delta)$, as indicated in Table 6.2, as polynomials $d_X(\delta)$, $d_Y(\delta)$ are added to the problem. Let us point out that the number of variables for given multiplier based relaxation scheme therefore depends on the data matrices in (6.21). In view of the fact that it is currently unknown how to construct the most compact LFRs (in multiple variables), the indicated numbers in Table 6.1 depend on the algorithms that are employed.

p	Multiplier-based convex hull	Direct sum-of-squares
0.6	424	1099
0.8	424	1099
1	3955	1099
1.1	5467	1099
1.2	5264	1099
1.3	5467	1099
1.4	8497	1099
1.5	8497	1099
1.6	8497	1099
1.7	8497	1099
1.8	8497	1099
1.9	8497	1099
2	7900	1099

Table 6.2: Number of LMI variables in the multiplier based relaxation scheme convex hull with partial convexity arguments (CH-PC) and the direct sum-of-squares relaxation (SOS).

Remark 6.5 Note that if using the LFT calculus in Appendix B for computing the product $A(\delta)B(\delta)$, the resulting LFR is only well-posed if both $A(\delta)$ and $B(\delta)$ are. If

the product $A(\delta)B(\delta)$ is a polynomial matrix, an LFR that is well-posed can always be constructed, see again [39]. As compared to the formula (B.4), this typically results in an increased size of the Δ -block.

Direct sum-of-squares relaxation

Motivated by the computational complexity that arises if employing multiplier relaxations, we now construct relaxations that are not based on the S-procedure. Similarly to the construction of relaxation CH-PC, the parameter dependent matrices $K_X(\delta), K_Y(\delta)$ satisfying (6.24) are obtained from Lemma B.1. We again multiply with suitable denominator polynomials $d_X(\delta), d_Y(\delta)$ in order to arrive at polynomial expressions.

For a polynomial robust LMI constraint, we can directly employ matrix-sum-of-squares techniques. The obtained robust LMIs actually correspond to the polynomial constraint (2.35), in which the matrix $\tilde{P}(y)$ represents the $J_1(P, Q)$ (or $J_2(P, Q)$) whereas $U(x)$ represents $\tilde{F}_1(\delta)$ (or $\tilde{F}_2(\delta)$) in (6.28). The following two constraints are used to bound the parameters:

$$V(\delta_i)' \tilde{G} V_i(\delta_i) = \begin{pmatrix} 1 \\ \delta_i \end{pmatrix}' \begin{pmatrix} -0.4^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \delta_i \end{pmatrix} \leq 0, \quad \text{for } i = 1, 2. \quad (6.29)$$

Following the lines of Section 2.2.3, it turns out that the relaxation scheme (2.39) is infeasible if including all possible $T_j(\delta)$ with monomial elements of total degree at most 3. If, however, the monomials as depicted in Table 6.3 are employed, the sum of squares relaxation is feasible. In fact, as shown in Figure 6.1, the same guaranteed \mathcal{L}_2 -gain bounds are obtained as for CH-PC, with less number of variables. The computation time of CH-PC at $p = 1.5$ is in the order of hours, whereas the relaxation SOS takes only minutes to solve. Note that the computational complexity is also influenced by the number and size of the LMI constraints, which are in the same order of magnitude for relaxations CH-PC and SOS.

Robust LMI	monomials in basis $T_j(\delta), j = 1, 2$
$\tilde{F}_1(\delta)' J_1(P, Q) \tilde{F}_1(\delta) \prec 0$	$1, \delta_1, \delta_2, \delta_1 \delta_2, \delta_1^2, \delta_2^2$
$\tilde{F}_2(\delta)' J_2(P, Q) \tilde{F}_2(\delta) \prec 0$	$1, \delta_1, \delta_2, \delta_1 \delta_2, \delta_1^2, \delta_2^2, \delta_1 \delta_2^2, \delta_2 \delta_1^2, \delta_1^3, \delta_2^3, \delta_1 \delta_2^3, \delta_1^3 \delta_2$

Table 6.3: Monomials terms of sum-of-squares basis T_j in (2.32), for the two robust LMIs in (6.28). The indicated monomial bases are used for each of the constraints in (6.29).

If compared to multiplier relaxations with convex hull arguments, matrix sum of squares techniques offer a way to systematically reduce conservatism, with the extra benefit of allowing for far more general parameter regions than polytopes or boxes. The example of this section shows that the direct relaxation based on matrix sum-

of-squares is to be preferred, if the parameter dependent LMI constraints involves high order rational or polynomial functions.

Remark 6.6 *It is stressed that the suggested matrix sum of squares relaxation technique can be applied to any of the multitude of problems in robust control which can be translated into the generic formulation (2.3). In particular, different versions of LPV synthesis conditions can be chosen as a starting-point. For example, as pointed out in [5], one can employ Finsler's Lemma to obtain robust LMIs for synthesis which avoids the projection onto the null-spaces of $(B_2^T(\delta) D_{12}^T(\delta))$ and $(C_2(\delta) D_{21}(\delta))$ respectively.*

Remark 6.7 *In CH-PC, the construction of the LFR relies on automated procedures in Matlab, using the Symbolic Toolbox and the Robust Control Toolbox. It is expected that an LFR of smaller size typically exists.*

Remark 6.8 *Note that since the parameter region is a convex polytope, a direct Pólya relaxation method is also possible.*

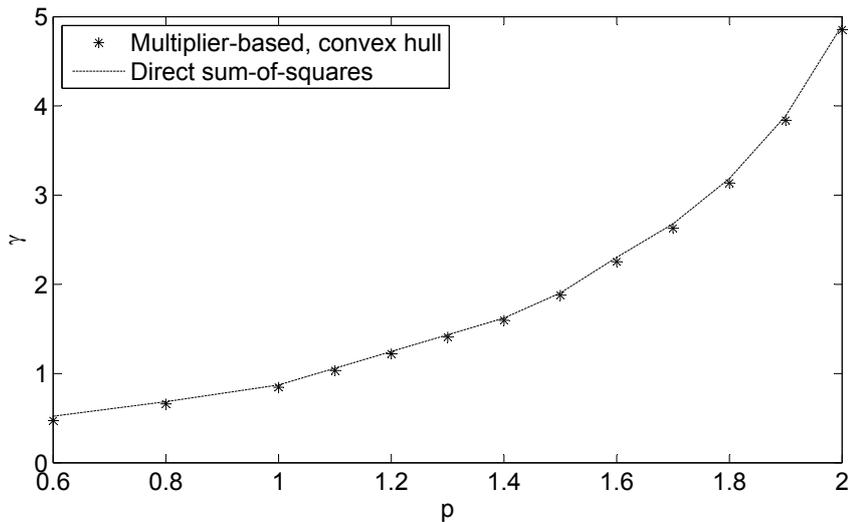


Figure 6.1: Guaranteed \mathcal{L}_2 -gain levels for the LPV synthesis problem for two different relaxation methods.

6.3 Summary

A solution to the LPV controller synthesis presented and amounts to a set of robust LMIs, for which relaxation schemes can be found in Chapter 2. As opposed to relaxations based on convexity arguments, which are typically employed in the literature, matrix sum-of-squares relaxations enable to describe the parameter region by general semi-algebraic constraints. In view of the fact that, at least in principle, the relaxation gap can be reduced to zero, the only source of conservatism in the LPV presented synthesis approach is caused by the fact that it is based on a (possibly) restricted class of quadratic-in-the-state Lyapunov functions.

It has been demonstrated how the transformed controller variables can be eliminated from the synthesis conditions for the robust quadratic performance measure. In a numerical example, the computational complexity could be significantly reduced by making use of direct sum-of-squares relaxations instead of the more common S-procedure based approach.

We again stress the versatility of the framework, which allows us to (approximately) solve the LPV synthesis problem by constructing any of the relaxation schemes from Chapter 2. Moreover, the approach can handle both continuous-time as well as discrete-time LPV synthesis.

We also note that the LPV synthesis algorithm could also be applied to an N -lifted system in discrete time, as discussed in Chapter 4. Note however, that the purpose of such an approach would be to get to know the limits of performance, more than controller design itself, since the controller that results from an N -lifted system runs slower (the sampling frequency is reduced by a factor of N) and has an increased number of inputs and outputs. Whether such controllers can be used in practice requires further investigation.

The LPV synthesis approach has been implemented and added to the developed Matlab-toolbox [55]. Once the data is given and the problem is formulated, robust controller solutions can immediately be computed.

Chapter 7

Conclusions

Let us summarize the contributions of this thesis and outline some directions for future research. Recall the two main objectives formulated at the beginning.

Objective 1: Improve the usability of the LMI framework for solving robust control problems.

Here, we have been concerned with the analysis of LPV systems in discrete-time and the synthesis of robust controllers in the IQC framework.

An overview of the IQC approach towards stability and performance analysis of uncertain systems was given in Chapter 3. Analysis conditions were shown for robust stability, robust quadratic performance as well as for robust \mathcal{H}_2 -performance, as measured in the impulse response definition of Appendix A. As already stressed in [135], it is unclear how to characterize a bound on the \mathcal{H}_2 -performance level of an uncertain system, if the stochastic interpretation is chosen. The IQC analysis tools were applied to an LPV system with a single time-varying parameter, for which upper bounds on the worst-case \mathcal{L}_2 -gain were computed by constructing various relaxation schemes. Within the considered class of dynamic multipliers, the relaxation gap could not be brought to zero.

Under the assumption that a class of dynamic IQC multipliers is parameterized in terms of finitely many LMI constraints, we have been able to solve the robust controller synthesis problem by exploiting the structure of the generalized plant, see Chapter 5. The LMI synthesis conditions provide an elegant solution to the robust disturbance rejection problem in which disturbances are characterized by an uncertain disturbance filter. Our algorithm has been illustrated on a small numerical example and was recently applied on a model of a magnetically levitated bearing system, see [56].

The presented synthesis result applies to a somewhat specific (though well-motivated) structure of the generalized plant. In addition, if uncertainty is only

caused by an uncertain filter that is positioned at the plant output, we were able to also derive a convex solution. As a matter of fact, we expect that robust synthesis can be convexified for general interconnections of uncertain systems, as long as the control channel of the generalized plant is not affected by uncertainty. As it seems very hard (if not impossible) to solve the output feedback robust synthesis problem in its full generality, one should exploit the problem structure that occurs in practically relevant system interconnections. The following interesting research topics are mentioned:

- The comparison of our robust synthesis algorithm with alternative techniques as mentioned in Section 5.1, in the context of optimal robust disturbance rejection against a class of uncertain disturbance signals. It is particularly interesting to investigate which approach requires the least effort in terms of the number design iterations and computational cost.
- The development of the discrete-time analogue of the stability characterization given in Theorem 5.1, which is not expected to be too difficult. Then, the controller synthesis algorithm with dynamic IQCs can be applied to discrete-time systems as well.
- The development of a convex solution to the robust output feedback controller synthesis problem as suggested in Conjecture 5.1, in which the system interconnection can be general, provided that control channel is not affected by uncertainty. As a first step, a solution is needed for the problem sketched in Section 5.3.6, that does not rely on the dualization Lemma 5.2.
- The development of analysis and synthesis techniques for effectively handling mixtures of parametric and dynamic uncertainties. In particular, it is unknown how to design an LPV controller for a plant of which not all uncertain elements are parametric and on-line measurable.

Concerning the analysis of the class of discrete-time LPV systems in Chapter 4, it has been shown that sufficient conditions for stability and performance are provided by Lyapunov theory. Moreover, a family of sufficient conditions could be formulated in terms of robust LMIs by making use of a well-known lifting technique. This family of stability tests is asymptotically exact, which implies that exact analysis results can be obtained by using a quadratic-in-the-state Lyapunov function, as long as the lifting horizon is chosen large enough. If combining this remarkable fact with an asymptotically exact family of LMI relaxations (i.e. Pólya, sum-of-squares), a systematic procedure for analyzing LPV systems is obtained. As opposed to most alternative Lyapunov function-based approaches to the analysis of non-linear and uncertain systems, our method is constructive. In other words, the analysis conditions are formulated in terms of tractable LMI relaxation schemes. The method is also marked by a moderate growth in computational complexity, as the state dimension increases, as compared to existing algorithms that rely on higher order in the

state Lyapunov functions. Note, however, that the success of the analysis approach of Chapter 4 will always depend on the particular system that is considered, and a general statement is hard to make.

Finally, Chapter 6 addresses the LPV controller synthesis problem. By following a well-known solution approach that results based on a robust LMI problem, we have shown that computational complexity can be reduced if direct matrix sum-of-squares arguments are employed, rather than relaxations based on the full block S-procedure. Again, we stress the versatility of the LMI based approach. A list of interesting research topics concerning LPV systems is given next.

- The construction of destabilizing parameter trajectories or trajectories with worst-case performance for continuous-time LPV systems.
- A numerical example that illustrates the potential use of lifted systems for LPV controller synthesis.
- An extension of the stability analysis framework such that rationally parameter dependent Lyapunov functions can be employed. Although such functions have been considered, for example in [64, 54], it is yet unknown how to turn (4.7) into a robust SDP, in case of having parameter variation bounds unequal to zero, and without a priori fixing the denominator polynomials in $X(\delta)$.
- A proof of the fact that $\bar{\rho}(\mathcal{R}) > 1$ is necessary for instability of the discrete-time LPV system, analogous to what was done by Wirth [181] in the continuous-time case. (see Section 4.1.5)
- The development of a reliable procedure for building and validating LPV models.

Objective 2: Develop a unified framework for constructing LMI relaxations.

Relaxation schemes allow one to compute feasible solutions of (non-tractable) robust LMI problems. In Chapter 2, three types of relaxation schemes were presented in a unified setting: relaxations based on convex hull arguments, on Pólya's theorem and on matrix sum-of-squares decompositions. The flexibility of constructing all sorts of LMI relaxation schemes has been illustrated on a number of academic examples. A more detailed discussion on the implementation of the relaxation schemes from Chapter 5 can be found in the developed Matlab toolbox [55].

Two families of relaxation schemes, both of which are asymptotically exact, were constructed based on Pólya's theorem or sum-of-squares decompositions. Hence, we are able, at least theoretically speaking, to come arbitrarily close to the genuine optimum of a robust LMI problem.

A new implementation of matrix sum-of-squares relaxations has been developed that does not involve linear equation constraints and therefore results in a genuine LMI optimization problem. Although a numerical comparison with the previous approach by Scherer and Hol [160] was not carried out, our proposed method is expected to perform better, since existing LMI solvers do not explicitly account for linear equation constraints.

It is emphasized that any given robust LMI constraint which is rational in the parameters can be transformed into an equivalent polynomial one. An elegant and often numerically efficient tool that can be used to perform this task is the so-called full block S-procedure. Relaxation schemes based on the full-block S-procedure are referred to as *multiplier-based*.

Relaxation schemes generally provide approximate solutions to a robust SDP problem, which explains the importance of estimating the level of conservatism. For the class of so-called multiplier-based relaxations, a condition for verifying exactness was proposed in the recent paper [157]. This test involves a solution to a system of polynomials. An extension to problems with multiple robust SDP constraints has been given in Section 2.4.2. In the case of having only scalar constraints, which correspond to the class of so-called robust linear programming problems, an efficient implementation of this test was obtained by reducing the number of polynomials and variables. As one of the main contributions in Chapter 2, a new algorithm for solving polynomial systems was proposed, which does not rely on structural knowledge on the polynomial system in terms of a Gröbner basis.

It remains an important research topic how to make use of information from previous computations in order to construct new, less conservative, relaxation schemes. The reader is referred to [96] for some practical experience on the choice of basis functions in using sum-of-squares relaxations. The following problems all relate to the goal of solving robust LMI problems in a systematic fashion:

- Efficient and reliable tools are needed to estimate the level of conservatism of relaxation schemes in general robust SDPs. A powerful test for verifying exactness of multiplier-based relaxations is given in Theorem 2.6. Similar to what could be done for robust linear programming problems, there is a need for an efficient numerical implementation of condition (2.68) for general robust SDPs.
- Without having a means to systematically reduce conservatism, the guaranteed convergence of the family of sum-of-squares relaxations has limited practical value. The number of possible relaxation schemes that can be constructed for a given bound on the total degree d of the sum-of-squares basis $T_j(x)$ grows combinatorially with d . Based on existing computations, one should be able to decide which monomials should be added in order to arrive at less conservative relaxation schemes. This also involves the issue whether, for

a given relaxation scheme, certain monomial terms can be removed without degrading the computed upper bounds.

- If employing the relaxation schemes based on the S-procedure of Section 2.3, block-structured multipliers can be used in order to reduce the size of the problem. Surprisingly, adding such a block-structure need not worsen the results as compared to the full block case, as was shown in Section 2.4.3. Similar to the argument in the previous item, tools are needed that systematically reduce the number of multiplier variables, without adding conservatism to the problem.

Let us finally discuss the important issues to overcome so that the tools as presented in this thesis become easily accessible to a control engineer.

7.1 Towards a systematic and practical design procedure

As in all engineering problems, the design of a control system is an iterative process. In classical PID control, the designer directly tunes over the controller parameters whereas in \mathcal{H}_∞ -synthesis he iteratively adjusts weighting functions at the input/output of the plant.

The process of translating design specifications into suitable weighting functions in an \mathcal{H}_∞ - or \mathcal{H}_2 - synthesis problem requires training, but is a relatively straightforward task. The difficulty typically arises from a large number of inputs and outputs of the system. For a synthesis approach based on robust SDPs, the presence of robust LMI constraints complicates the design process even for small system dimensions. Let us address some of the troubles that are caused by the fact that robust LMIs can only be solved approximately.

- The inability to distinguish "good" designs from "bad" designs.
In \mathcal{H}_∞ -synthesis, the synthesis-optimal-value equals the \mathcal{H}_∞ -norm of the closed-loop weighted plant. This number indicates the performance that can be expected. Based on this norm, one decides whether to reconstruct the controller in order to perform a more detailed evaluation of the design.
The algorithms for robust controller design problems typically involve relaxation schemes for robust SDPs. Hence, in view of the possibly large relaxation gap, a high synthesis optimal value (γ_{rel}) no longer indicates that the weighting functions were inappropriately chosen.
- The increased cost of the intermediate analysis.
As the designer reaches the limits of performance, the weighting functions can only be successfully modified if one knows the level of conservatism. This requires the construction of multiple relaxation schemes, lower bound computations and/or verification of exactness. Hence, the intermediate analysis

not only involves the analysis of the designed controller, but also of the LMI solution itself. This clearly slows down the design process.

- The increased number of tuning variables
The improved performance and the flexibility of modern optimal control algorithms is, to some extent, based on an increase in the degrees of freedom (controller order) over which the design takes place. However, too many tuning knobs make it difficult to find a suitable controller. In particular, the Lyapunov based methods of Chapter 4 and 6 rely on the pre-defined structure of the Lyapunov matrix, as well as on the chosen relaxation scheme, whereas the IQC-based method in Chapter 3 and 5 depend on the chosen parametrization of a class of the suitable multipliers and the chosen relaxation scheme, if relaxation is required.

A very important and essential aspect, that has not been mentioned yet, concerns the numerical computation of LMI relaxations. Despite the significant amount of literature on LMI problems in control, see for example [29, 69], it is important to make LMI-based controller design techniques more accessible for industry. Hence, there is a need for dedicated reliable interior point solvers that numerically solve LMI problems arising from the systems and control field, in particular for solving LMI relaxations. Let us finish with the following two considerations.

- The outcome of existing LMI algorithms applied to control related problems often depends on the different numerical representations of the data. New LMI solvers should exploit the particular problem structure as seen in control applications and should probably pre-condition the data. The development of new interior point algorithms should be intertwined with the construction of relaxation schemes in order to arrive at reliable solvers.
- An LMI constraint is not always strictly feasible. Since existing solvers cannot handle non-strict LMIs, such cases are likely to cause numerical troubles. In order to fully benefit from the power of sum-of-squares relaxations, a general understanding is needed of how to a priori guarantee that the constructed LMI constraints are strictly feasible. Only then, the selection of basis functions T_1, \dots, T_M in (2.32) can be done in a systematic and numerically efficient way.

Robust LMIs will definitely play an important role in various engineering design problems. At present, the construction of relaxation schemes involve some technicalities that make the overall procedure inconvenient from a numerical point of view. Once numerically efficient and reliable computation of the relaxation gap has been established, as well as a systematic reduction thereof, the potential benefit of robust SDPs will become visible.

Appendix A

Analysis of LTI systems through LMIs

In this appendix we briefly summarize the basic notions on system theory of finite dimensional systems described by differential equations of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bw(t), & x(0) &= x_0, \\ z(t) &= Cx(t) + Dw(t).\end{aligned}\tag{A.1}$$

Here, $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^{n_d}$ the disturbance and $z(t) \in \mathbb{R}^e$ the performance output. The coefficient matrices A, B, C, D are real-valued and the system can alternatively be represented by the corresponding transfer matrix $G(s) = C(sI - A)^{-1}B + D$, provided that the realization is minimal.

A.1 Stability

Stability of system (A.1) is equivalent to stability of the autonomous system

$$\dot{x} = Ax, \quad x(0) = x_0.\tag{A.2}$$

The equilibrium $\bar{x} = 0$ is *asymptotically* stable if the state goes to zero for all initial conditions $x(0) = x_0$, i.e.

$$\lim_{t \rightarrow \infty} \|x(t)\| = 0 \quad \text{for all } x_0 \in \mathbb{R}^n.$$

The system is called *exponentially* stable if for some positive constants α, β , all trajectories of the system satisfy

$$\|x(t)\| \leq \alpha e^{-\beta(t-t_0)} \|x(t_0) - \bar{x}\| \quad \text{for all } t \geq t_0 \geq 0$$

For the LTI system (A.1), exponential stability is equivalent to the property that all eigenvalues of A are in the open left half plane. Note also that due to linearity of the system dynamics, the origin $x = 0$ is a global equilibrium.

A.2 Performance measures

For LTI systems, input-output specifications have traditionally been described in the frequency domain, by suitable norm bounds on the transfer matrix $G(s)$.

A.2.1 \mathcal{H}_∞ -norm

The \mathcal{H}_∞ -norm of a transfer matrix $G(s)$ is defined as

$$\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega)),$$

provided that G is stable. In order to see the relevance of the \mathcal{H}_∞ -norm in characterizing performance, suppose \hat{w}, \hat{z} are the Fourier transformed signals for any given $w(\cdot)$ and $z(\cdot)$ that satisfy the system dynamics. Since $z(i\omega) = G(i\omega)w(i\omega)$, it immediately follows that $\|G\|_\infty$ provides a bound on the worst case steady-state amplification of a sinusoidal input. Recall the vector space of real or complex signals with finite energy, defined as follows:

$$\mathcal{L}_2 := \{x : [0, \infty) \rightarrow \mathbb{R}^m \mid \|x\|_2 < \infty\},$$

with $\|\cdot\|_2$ the standard vector l_2 -norm.

Definition A.1 (Induced \mathcal{L}_2 -gain) *The induced \mathcal{L}_2 -gain of system (A.1) is defined as the smallest γ for which*

$$\|z\|_2 \leq \gamma \|w\|_2 \quad \text{for every } w \in \mathcal{L}_2$$

if the output $z(\cdot)$ corresponding to the zero initial condition $x(0) = 0$ and input $w \in \mathcal{L}_2$ satisfies (A.1).

If the system (A.1) is exponentially stable, it always has a finite \mathcal{L}_2 -induced gain. Moreover, the induced energy gain of an LTI system equals the \mathcal{H}_∞ -norm of the corresponding transfer matrix. That is, the following equality holds:

$$\|G\|_\infty = \sup_{w \in \mathcal{L}_2, w \neq 0} \frac{\|z\|_2}{\|w\|_2}.$$

The \mathcal{H}_∞ -norm is directly linked to robust stability guarantees, by the so-called small gain theorem. In particular, $\|G\|_\infty < 1$ ensures invertibility of $I - G$, which is extremely useful if verifying well-posedness of an interconnected system, as well as

stability. The \mathcal{H}_∞ -norm captures performance in a worst case sense, and is therefore not always the best measure for expressing performance.

A.2.2 \mathcal{H}_2 -norm

Whenever the spectral content of a (random) disturbance signal w is known, the \mathcal{H}_2 -norm is usually a good measure for disturbance rejection performance. It is defined for LTI systems, as

$$\|G\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Tr} (G^*(i\omega)G(i\omega))},$$

in which transfer matrix $G(s)$ with $z = Gw$ is defined as $G(s) = C(sI - A)^{-1}B$, For discrete time systems with realization matrices A_d, B_d, C_d, D_d and transfer matrix $G_d(z) = C_d(zI - A_d)^{-1}B_d + D_d$, the \mathcal{H}_2 -norm is defined as

$$\|G\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Tr} (G^*(i\omega)G(i\omega))},$$

There are several motivations for using the \mathcal{H}_2 -norm as a performance measure, of which the following two interpretations are relevant.

The impulse response interpretation

For scalar inputs, $\|G\|_2^2$ is the energy of the impulse response. In case of multiple inputs, say m , the \mathcal{H}_2 -norm equals an average (or strictly speaking a sum) over all impulsive inputs

$$\|G\|_2^2 = \sum_{i=1}^m \|G_{zw_i} \delta_D\|_2^2$$

in which

$$G = \begin{pmatrix} G_{zw_1} & G_{zw_2} & \cdots & G_{zw_{n_w}} \end{pmatrix}$$

and G_{zw_i} represents the transfer matrix from the i^{th} input to the output z and $\delta_D = \delta_D(t)$ is the Dirac impulse function. The \mathcal{H}_2 -norm can thus be used to measure (typically an error signal of) a transient response due to impulsive inputs or, equivalently, non-zero initial conditions. By adding a suitable shaping filter at the output, one can hence capture any desired trajectory.

Stochastic interpretation

Suppose the input w is a realization of a stochastic process having spectral density $S_w(i\omega)$ and let $S_z(i\omega)$ be the spectral density of the output z . Since the system is LTI, these two spectra are related as

$$S_z(i\omega) = G(i\omega)S_w(i\omega)G(i\omega)^*.$$

The expectation of the asymptotic variance $E(|z(t)|^2)$ can be expressed as

$$\begin{aligned} E(|z(t)|^2) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Tr} G(i\omega) S_w(i\omega) G(i\omega)^* = \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Tr} G(i\omega) G(i\omega)^* S_w(i\omega) = \|G\|_2^2 \quad (\text{A.3}) \end{aligned}$$

Thus, the \mathcal{H}_2 -norm can be interpreted as the asymptotic output variance for any input taken generated by a white noise source, i.e. $S(i\omega) = I$. Similar as in the previous interpretation, an additional filter can be used to shape the input signal characteristics (colored noise).

Although the terminology ' \mathcal{H}_2 -norm' is abusive for systems that are not LTI, one can show that both interpretations of the \mathcal{H}_2 -norm coincide when the system is linear time varying, see [107, 12]. Hence, for LTV systems the \mathcal{H}_2 -norm performance measure can be used unambiguously. For non-linear systems one has to be more careful, since there are multiple extensions possible, as mentioned in [169, 172, 12, 135] and references therein.

Remark A.1 *The standard \mathcal{H}_2 -norm equals the induced \mathcal{L}_2 - \mathcal{L}_∞ norm if the output of the system is scalar, see for example [151]. This interpretation is particularly meaningful in the context of disturbance rejection, and is another reason why the \mathcal{H}_2 -norm is important when it comes to measuring disturbance rejection performance.*

A.2.3 Quadratic performance

The notion of quadratic performance generalizes the induced \mathcal{L}_2 -gain.

Definition A.2 (Quadratic Performance) *The system (A.1) is said to satisfy quadratic performance with respect to performance index*

$$P_p = \begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix}, \quad R_p \succeq 0, \quad (\text{A.4})$$

if it is exponentially stable and if there exists $\epsilon > 0$ such that for $x_0 = 0$ the following property

$$\int_0^\infty \begin{pmatrix} w(t) \\ z(t) \end{pmatrix}^T P_p \begin{pmatrix} w(t) \\ z(t) \end{pmatrix} dt \leq -\epsilon \|w_p\|_2^2, \quad \text{for every } w \in \mathcal{L}_2 \quad (\text{A.5})$$

holds.

It is easy to see that \mathcal{L}_2 -gain performance corresponds to the performance index

$$P_p = \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & I \end{pmatrix} \quad \text{or} \quad P_p = \begin{pmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma} I \end{pmatrix}. \quad (\text{A.6})$$

A.3 LMI characterization of stability and performance

Linear matrix inequalities have long been present in systems theory. They go back to Lyapunov who showed around 1900 that the autonomous LTI system

$$\dot{x} = Ax$$

is exponentially stable if and only if there exists a positive definite solution $X \succ 0$ to the Lyapunov inequality

$$A'X + XA \prec 0. \quad (\text{A.7})$$

The existence of such a solution X implies that A is a stable matrix, i.e. it has all its eigenvalues in the open left half plane. The Lyapunov inequality (A.7) is an example of an LMI for which a solution X can be constructed explicitly.

Formally, an LMI is a matrix inequality of the form

$$P(y) = P_0 + \sum_{i=1}^{n_d} P_i y_i \prec 0 \quad (\text{A.8})$$

in which $y \in \mathbb{R}^{n_d}$ are the decision variables and P_0, P_1, \dots, P_{n_d} are known Hermitian matrices. A convex optimization with LMI constraints of the form

$$\begin{aligned} \text{infimize} \quad & c_1 y_1 + \dots + c_{n_d} y_{n_d} \\ \text{subject to} \quad & P_0 + P_1 y_1 + \dots + P_{n_d} y_{n_d} \prec 0. \end{aligned} \quad (\text{A.9})$$

is called a Semi-Definite-Program (SDP). As shown, the inequality is taken in a strict sense, which means that all eigenvalues of $P(y)$ are non-zero and negative. LMIs allow for a wide range of convex constraints on y and it has become clear that many control problems can be translated into convex optimization problems. LMI's have become so popular over the past two decades mainly due to the fact that they can be efficiently solved by using interior point algorithms. For a nice overview on convex optimization, the relations to linear and quadratic programming and some practical applications, see [30, 29, 17]. Among the many LMI solvers now available, we refer to numerical solvers such as LMILAB, Sedumi, see [170], CSDP, the latter two which are available on the internet.

LMI characterization of quadratic performance

Quadratic performance for an LTI system can be verified by solving an LMI problem. One way of showing this fact starts with the observation that with transfer matrix

$G(s) = D + C(sI - A)^{-1}B$, the semi-infinite constraint

$$\begin{pmatrix} I \\ G(i\omega) \end{pmatrix}^* P_p \begin{pmatrix} I \\ G(i\omega) \end{pmatrix} \prec 0, \quad \text{for all } \omega \in \mathbb{R} \cup \{\infty\}. \quad (\text{A.10})$$

is a sufficient condition for quadratic performance of system (A.1). Let us denote the Fourier transformed signals as \hat{z}, \hat{w} respectively. Then expression (A.10) being strictly negative definite means adding $-\epsilon I$ to the right-hand side is allowed while still being feasible. Left- and right multiplication with $\hat{w}(i\omega)^*, \hat{w}(i\omega)$ respectively, using the fact $\hat{z}(i\omega) = G(i\omega)\hat{w}(i\omega)$, we infer that

$$\int_{-\infty}^{\infty} \begin{pmatrix} \hat{w}(i\omega) \\ \hat{z}(i\omega) \end{pmatrix}^* P_p \begin{pmatrix} \hat{w}(i\omega) \\ \hat{z}(i\omega) \end{pmatrix} d\omega \leq -\epsilon \|\hat{w}\|_2^2, \quad \text{for every } w \in \mathcal{L}_2. \quad (\text{A.11})$$

Finally, by Parseval's relation, the condition for quadratic performance (A.5) is satisfied.

As such, constraint (A.10) is not immediately tractable due to its semi-infinite nature. However, frequency domain inequalities (FDI) of this form can be reduced to an LMI, as the following lemma shows.

Proposition A.1 *The system (A.1) is asymptotically stable and admits quadratic performance if and only if there exists a solution $X = X^T \succ 0$ satisfying*

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}' \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}' P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0. \quad (\text{A.12})$$

Remark A.2 *It now becomes clear why $R_p \succeq 0$ is assumed in the definition of quadratic performance. In fact, the upper-left term in (A.12) yields $A'X + XA + C'R_p C \prec 0$ which implies that A is stable.*

A proof of this proposition is based on the following lemma that relates a semi-infinite constraint on the imaginary axis to a finite-dimensional LMI optimization problem.

Lemma A.1 (Kalman Yakobovich Popov) *Let $G(s) = D + C(sI - A)^{-1}B$ and assume $\det(i\omega - A) \neq 0$ for all $\omega \in \mathbb{R}$. Then, (A.10) holds if and only if there exists (generally indefinite) $X = X^T$ for which (A.12) holds.*

Proof. see [9, 148]. ■

With this lemma we can complete the proof of Proposition A.1.

Proof of Proposition A.1 :

Using the KYP Lemma, (A.10) is equivalent to (A.12). Further, stability of A is inferred by the additional property $X \succ 0$, exploiting the fact $R_p \succeq 0$. An alternative

derivation of (A.12) is based on a Lyapunov function $V(x) = x^T X x$ in combination with the performance indicator (A.5). The proof also extends to linear time-varying and uncertain systems.

The equivalence between the LMI (A.12) and the boundedness of $\|G\|_\infty$ is commonly known as the bounded real lemma, see for example [190]. The following Lemma will be used in order to render matrix inequality affine in the decision variable γ .

Lemma A.2 (Schur) *For any given symmetric matrix $M \in \mathbb{R}^{n \times n}$ partitioned as*

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}$$

define $S = M_{22} - M_{12}^T M_{11}^{-1} M_{12}$, known as the Schur complement of M_{11} in M . Then,

$$M \succ 0 \leftrightarrow M_{11} \succ 0 \quad \text{and} \quad S \succ 0$$

By applying Schur's lemma and the fact that $R_p \succeq 0$, one can equivalently rewrite (A.12) as

$$\begin{pmatrix} XA + A^T X & XB + C^T S_p^T & C^T \\ B^T X + S_p C & \text{sym}(S_p D) + Q_p & D^T \\ C & D & R_p^{-1} \end{pmatrix} \prec 0. \quad (\text{A.13})$$

The induced \mathcal{L}_2 gain performance measure corresponds to $Q_p = \gamma$, $R_p = \gamma^{-1}$ and $S_p = 0$. The application of Schur's lemma in this context is often referred to as the linearization lemma since γ enters (A.13) in an affine fashion.

LMI characterization of \mathcal{H}_2 -norm

Similar as the \mathcal{H}_∞ -norm admits an LMI characterization, the following result shows how the \mathcal{H}_2 -norm of an LTI system can be computed in terms of state-space matrices.

Proposition A.2 *Suppose the system (A.1) is asymptotically and let $G(s)$ denote its transfer function. Then, $\|G\|_2 < \gamma$ if and only if $D = 0$ and the following statements are equivalent*

- $\|G\|_2 < \gamma$
- There exists $K = K^T \succ 0$ and Z such that

$$\begin{pmatrix} A^T K + K A & K B \\ B^T K & -I \end{pmatrix} \prec 0; \quad \begin{pmatrix} K & C^T \\ C & Z \end{pmatrix} \succ 0; \quad \text{Tr}(Z) < \gamma^2$$

- There exists $K = K^T \succ 0$ and Z such that

$$\begin{pmatrix} A K + K A^T & K C^T \\ C K & -I \end{pmatrix} \prec 0; \quad \begin{pmatrix} K & B \\ B^T & Z \end{pmatrix} \succ 0; \quad \text{Tr}(Z) < \gamma^2$$

Appendix B

Linear fractional transformations

In order to be able to reduce LPV analysis and synthesis conditions into a robust SDP, the system matrices of the LPV system must be well-posed for the given set of admissible parameter values. Moreover, the construction of the multiplier based relaxations of Section 2.3 require that a *Linear Fractional Representation* (LFR) of the parameter dependent system matrices exists.

Definition B.1 *Let G be a mapping from the indeterminate variables $\delta_1, \dots, \delta_s$ into $\mathbb{R}^{p \times m}$. Then, we say that the relation*

$$\eta = G(\delta)\xi \tag{B.1}$$

admits the LFR

$$\begin{pmatrix} z \\ \eta \end{pmatrix} = \underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_H \begin{pmatrix} w \\ \xi \end{pmatrix}, \quad w = \Delta(\delta)z \tag{B.2}$$

defined with the constant matrix H with sub-matrices A, B, C, D and with parameter block $\Delta(\delta)$ depending linearly on δ if the following holds: For all δ for which $I - A\Delta(\delta)$ is invertible and for all $\xi \in \mathbb{R}^m, \eta \in \mathbb{R}^p$, the relation (B.1) holds if and only if there exists $w \in \mathbb{R}^{n_w}, z \in \mathbb{R}^{n_z}$ such that (B.2) holds. This relation is pictorially expressed in Figure B.1.

Observe that the existence of an LFR implies the function $G(\delta)$ to be rational. Moreover, it can be proven that any matrix-valued multi-variable rational function without pole in zero admits an LFR. Thus, when the LPV system matrices in (6.1) admit an LFR we can alternatively pull out the uncertainty and equivalently repre-

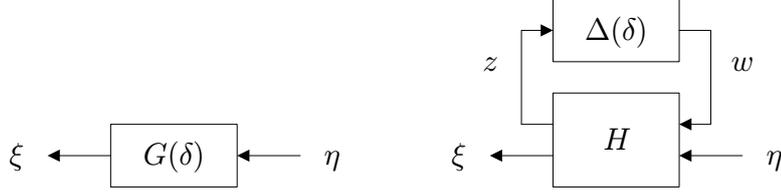


Figure B.1: LFR for rational matrix function $G(\delta)$

sent the LPV system as

$$\begin{pmatrix} \dot{x}(t) \\ z_\delta(t) \\ z(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \bar{A} & \bar{B}_1 & \bar{B}_2 & \bar{B} \\ \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} & \bar{D}_{13} \\ \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22} & \bar{D}_{23} \\ \bar{C} & \bar{D}_{31} & \bar{D}_{32} & \bar{D}_{33} \end{pmatrix} \begin{pmatrix} x(t) \\ w_\delta(t) \\ w(t) \\ u(t) \end{pmatrix}, \quad (\text{B.3})$$

$$w_\delta(t) = \Delta(\delta(t))z_\delta(t) \quad \delta \in \mathcal{D},$$

and appropriately chosen matrices $\bar{A}, \bar{B}_1, \dots$. Moreover, when no specific size of the realization is required, the parameter block $\Delta(\delta), \delta \in \mathbb{R}^s$ can always be chosen of the form

$$\Delta(\delta(t)) = \begin{pmatrix} \delta_1(t)I_{r_1} & & & \\ & \ddots & & \\ & & \delta_s(t)I_{r_s} & \end{pmatrix}.$$

In general, verifying well-posedness of an LFR for a given set \mathcal{D} is a non-tractable problem.

LFT calculus of LTI systems

State space realizations of (proper) transfer functions are a particular type of LFR, i.e. the transfer function for the LTI system (A.1) can be computed as

$$G(s) = \frac{1}{s} I \star \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad s \in \mathbb{C}^0 \cup \mathbb{C}^+.$$

It will not come as a surprise that the LFR can be defined to operate on LTI systems rather than (structured) matrices. An essential tool in the generalized plant framework is the use of linear fractional transformations.

Suppose we are given the following LTI system:

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

with $M_{11} \in RH_\infty^{n_1 \times m_1}, M_{22} \in RH_\infty^{n_2 \times m_2}$ and LTI systems $\Delta_l \in RH_\infty^{m_2 \times n_2}$ and

$\Delta_u \in RH_\infty^{m_1 \times n_1}$. Then, the upper fractional transformation with respect to Δ_u is defined as

$$F_u(M, \Delta_u) = M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}$$

and the lower fractional transformation with respect to Δ_l as

$$F_l(M, \Delta_l) = M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21}$$

where Δ_l, Δ_u are of compatible sizes. Upper- and lower LFTs are actually special cases of the star-product, for a definition see [190]. In our discussion, only F_u and F_l are needed, more compactly written as

$$F_l(M, \Delta_l) = M \star \Delta_l \quad F_u(M, \Delta_u) = \Delta_u \star M.$$

Well-posedness of $F_u(M, \Delta_u)$ in which M, Δ_u are LTI systems, means that $I - M_{22}\Delta_u$ is an invertible transfer matrix. In an analogue fashion, well-posedness of $F_l(M, \Delta_l)$ can be defined.

LFTs provide a powerful data structure for uncertain matrices or systems and can be added, multiplied, and inverted provided that D is invertible.

Example B.1 *Given LFR*

$$\bar{A}(x) = I_n x \star \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

then the product $\bar{A}(x)\bar{A}(y)$ admits the LFR

$$\left(\begin{array}{cc} xI_n & 0 \\ 0 & yI_n \end{array} \right) \star \left(\begin{array}{cc|c} A & BC & BD \\ \hline 0 & A & B \\ C & DC & DD \end{array} \right) \quad (\text{B.4})$$

Moreover, the LFT of LFT is again an LFT, which is easily seen by the fact that

$$\delta_2 \star \left(\delta_1 \star \left(\begin{array}{ccc|c} A_{11} & A_{12} & B_1 & \\ \hline A_{21} & A_{22} & B_2 & \\ C_1 & C_2 & D & \end{array} \right) \right) = \delta_1 \star \left(\delta_2 \star \left(\begin{array}{ccc|c} A_{22} & A_{21} & B_2 & \\ \hline A_{12} & A_{11} & B_1 & \\ C_2 & C_1 & D & \end{array} \right) \right). \quad (\text{B.5})$$

can be written as $\Delta(\delta) \star M$, in which

$$\Delta(\delta) = \left(\begin{array}{cc} \delta_1 & 0 \\ 0 & \delta_2 \end{array} \right), \quad \text{and} \quad M = \left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ \hline A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{array} \right),$$

The final important property of LFTs has been used in Section 6.1.

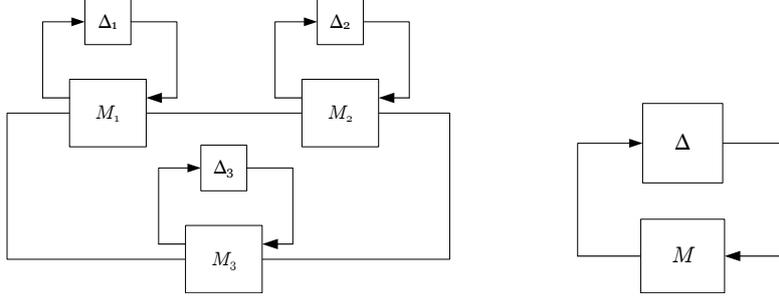


Figure B.2: An interconnected system

Lemma B.1 Consider the following LFT

$$G(\Delta) = \Delta \star \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

If M_{22} has dimension $\mathbb{R}^{n_2 \times m_2}$ and rank n_2 , the SVD of M_{22} is given by $M_{22} = U \begin{bmatrix} \Sigma & 0 \end{bmatrix} V^*$. V is partitioned into the first n_2 and last $(m_2 - n_2)$ columns as $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$. Then, the null space of $G(\Delta)$ can be parameterized via an LFT as follows:

$$\mathcal{N}(G(\Delta)) = \text{Im} \left[\Delta \star \begin{bmatrix} M_{11} - M_{12}V_1\Sigma^{-1}U^*M_{21} & -M_{12}V_2 \\ V_1\Sigma^{-1}U^*M_{21} & V_2 \end{bmatrix} \right].$$

Proof. The proof can be found in [177, 184].

Remark B.1 In view of Definition B.1, the LFR of $G(\delta)$ equals the upper LFT of the systems H with Δ , in which H is a constant gain and Δ is a parametric uncertainty block.

Structured uncertainties and interconnected systems

The main reason for LFTs to play such an important role in system theory, is the way it unifies the notion of interconnected systems. Consider the case of having three uncertain components that are interconnected as shown in Figure B.2. By separating the uncertain elements from the known dynamics M_1, M_2, M_3 , a process sometimes referred to as “pulling out the uncertainties”, the interconnected system is represented by a single loop P , the generalized plant, and the structured uncertainty block

$$\Delta = \begin{pmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{pmatrix}$$

Thus, interconnections of uncertain systems naturally lead to structured uncertainty blocks once unknown blocks are isolated from the known blocks. Note that relation (B.5) implies that interconnecting multiple uncertain component systems can be done in arbitrary order.

Linear fractional representations apply to LTI systems, which includes parametric (static) blocks. In Chapter 5 a framework is developed that enables to consider more general uncertain operators Δ , such as static non-linearities or time-varying parameters. We stress the fact that interconnection of non-linear subsystems is much more precarious and LFT operations should be used with greatest care.

Appendix C

Equivalence of the sum-of-squares approximations

As mentioned in Section 2.2.3, an alternative proof of Theorem 2.4 started from (2.29) rather than (2.32). Although the latter two sum-of-squares programming problems appear to be different, either one can be derived from the other by a suitable change of variables.

Let us start from (2.29), with $P(x, y)$ of size $p \times p$ and $G(x)$ of size $q \times q$ and obtain an explicit formula for

$$(G(x), S(x))_p = \text{Tr}_p((I_p \otimes G(x))S(x))$$

with sum-of-squares matrix $S(x)$ of dimension pq . Note that there exists a polynomial matrix $T(x)$ with pq rows and r columns for which

$$S(x) = T(x)T(x)^T.$$

With basis matrices B_ν , let T be parameterized as

$$T = \sum_{\nu=1}^N \alpha_\nu B_\nu = \sum_{\nu=1}^N \alpha_\nu \begin{pmatrix} b_{\nu,11} & \cdots & b_{\nu,1r} \\ \vdots & \ddots & \vdots \\ b_{\nu,p1} & \cdots & b_{\nu,pr} \end{pmatrix}$$

with q -column vectors $b_{\nu,jk}$, $k = 1, \dots, r$. Then

$$TT^T = \sum_{\substack{\nu=1 \\ \mu=1}}^N \alpha_\nu \alpha_\mu \begin{pmatrix} b_{\nu,11} & \cdots & b_{\nu,1r} \\ \vdots & \ddots & \vdots \\ b_{\nu,p1} & \cdots & b_{\nu,pr} \end{pmatrix} \begin{pmatrix} b_{\mu,11}^T & \cdots & b_{\mu,p1}^T \\ \vdots & \ddots & \vdots \\ b_{\mu,1r}^T & \cdots & b_{\mu,pr}^T \end{pmatrix} = \\ \sum_{\nu,\mu,j} \alpha_\nu \alpha_\mu \begin{pmatrix} b_{\nu,1j} \\ \vdots \\ b_{\nu,pj} \end{pmatrix} (b_{\mu,1j}^T \quad \cdots \quad b_{\mu,pj}^T).$$

Using the previously introduced definition of Tr_p we get

$$\text{Tr}_p((I_p \otimes G)S) = \text{Tr}_p \left(\sum_{\nu,\mu,j} \alpha_\nu \alpha_\mu \begin{pmatrix} Gb_{\nu,1j} \\ \vdots \\ Gb_{\nu,pj} \end{pmatrix} (b_{\mu,1j}^T \quad \cdots \quad b_{\mu,pj}^T) \right) = \\ = \sum_{\nu,\mu,j} \alpha_\nu \alpha_\mu \begin{pmatrix} \text{Tr}(Gb_{\nu,1j}b_{\mu,1j}^T) & \cdots & \text{Tr}(Gb_{\nu,1j}b_{\mu,pj}^T) \\ \vdots & \cdots & \vdots \\ \text{Tr}(Gb_{\nu,pj}b_{\mu,1j}^T) & \cdots & \text{Tr}(Gb_{\nu,pj}b_{\mu,pj}^T) \end{pmatrix}.$$

By using the properties of the trace operator we then arrive at

$$\sum_{\nu,\mu,j} \alpha_\nu \alpha_\mu \begin{pmatrix} b_{\mu,1j}^T Gb_{\nu,1j} & \cdots & b_{\mu,pj}^T Gb_{\nu,1j} \\ \vdots & \cdots & \vdots \\ b_{\mu,1j}^T Gb_{\nu,pj} & \cdots & b_{\mu,pj}^T Gb_{\nu,pj} \end{pmatrix} = \\ = \sum_{\nu,\mu,j} \alpha_\nu \alpha_\mu \begin{pmatrix} b_{\mu,1j}^T Gb_{\nu,1j} & \cdots & b_{\mu,1j}^T Gb_{\nu,pj} \\ \vdots & \cdots & \vdots \\ b_{\mu,pj}^T Gb_{\nu,1j} & \cdots & b_{\mu,pj}^T Gb_{\nu,pj} \end{pmatrix} = \\ = \sum_{\nu,\mu,j} \alpha_\nu \alpha_\mu \begin{pmatrix} b_{\mu,1j}^T \\ \vdots \\ b_{\mu,pj}^T \end{pmatrix} G (b_{\nu,1j} \quad \cdots \quad b_{\nu,pj}) = \\ = \sum_{j=1}^r \left(\sum_{\mu} \alpha_\mu \begin{pmatrix} b_{\mu,1j}^T \\ \vdots \\ b_{\mu,pj}^T \end{pmatrix} \right) G \left(\sum_{\nu} \alpha_\nu (b_{\nu,1j} \quad \cdots \quad b_{\nu,pj}) \right).$$

Finally, by defining

$$T_j = \sum_{\nu} \alpha_\nu (b_{\nu,1j} \quad \cdots \quad b_{\nu,pj}), \quad j = 1, \dots, r, \quad (\text{C.1})$$

we have shown that constraint (2.29) can be turned into (2.32).

In order to show the converse, let M and T_1, \dots, T_M in (2.32) be given. Let $r = M$ and decompose T_j as in (C.1) and further define

$$S = \sum_{\nu, \mu, j} \alpha_\nu \alpha_\mu \begin{pmatrix} b_{\nu, 1j} \\ \vdots \\ b_{\nu, pj} \end{pmatrix} \begin{pmatrix} b_{\mu, 1j}^T & \cdots & b_{\mu, pj}^T \end{pmatrix}.$$

By applying the converse arguments, it follows that

$$(G(x), S(x))_p = \sum_{j=1}^M T_j(x)^T G(x) T_j(x).$$

Thus, it has been shown how to transform (2.29) into (2.29) and vice versa.

When the a priori bound l on the total degree of $T_j(x)$ in (2.34) is increased, without removing any of the monomial terms already represented, the extended relaxation scheme is guaranteed not to be worse.

Appendix D

Asymptotic exactness in \mathcal{L}_2 -gain analysis

Referring to Section 4.2.4, we give proof of the fact that by restricting ourselves to N -periodic LPV systems, we can get arbitrarily close to the worst case l_2 -gain. For any \mathcal{R} -admissible parameter trajectory $\delta = (\delta_k)_{k=1,2,\dots}$ and any input $w = (w_k)_{k=1,2,\dots} \in l_2$, let $z = z(w, \delta, x_0)$ be the output of system (4.36) with initial condition $x_0 = 0$. Recall the worst-case l_2 -gain that, defined as

$$\gamma_{\text{wc}} := \sup_{\delta(\cdot) \text{ admissible}} \sup_{w \neq 0} \frac{\|z\|}{\|w\|} \quad (\text{D.1})$$

Define the shift operator $S_N(\cdot)$ as $S_N(z) = (z_{N+1}, z_{N+2}, \dots)$ and remind the fact that $T_N(z)$ is the truncated signal defined as $T_N(z) = (z_1, z_2, \dots, z_N, 0, 0, \dots)$. Moreover, for any \mathcal{R} -admissible parameter sequence $(\delta_k)_{k=0,1,\dots}$, let the mapping P_N generate periodic parameter trajectories in the following fashion:

$$P_N(\delta) = (\delta_1, \delta_2, \dots, \delta_N, \delta_1, \delta_2, \dots).$$

Theorem D.1 *Assume that the LPV system (4.36) is exponentially stable and let γ be the worst case l_2 -gain. Then, for any given $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that*

$$\sup_{\delta(\cdot) \text{ admissible}} \sup_{w \neq 0, \|w\|=1} \frac{\|z(w, P_N(\delta))\|}{\|w\|} > \gamma - \epsilon.$$

Proof. Let $\epsilon > 0$ be given. By definition of γ in (D.1), there exists $\bar{\delta}, \bar{w}$ such that the output $\bar{z} = z(\bar{w}, \bar{\delta})$ satisfies the relation

$$\|\bar{z}\| > \gamma - \frac{\epsilon}{8}, \quad \text{as well as} \quad \|\bar{z}\|^2 > \gamma^2 - \frac{\epsilon^2}{16}. \quad (\text{D.2})$$

Second, choose N such that

$$\|T_N(\bar{z})\|^2 > \|\bar{z}\|^2 - \frac{3\epsilon^2}{16}. \quad (\text{D.3})$$

Note that such an N exists since the term on the left hand side of the inequality converges to $\|\bar{z}\|^2$ as $N \rightarrow \infty$, and $\|\bar{z}\|$ was chosen to satisfy (D.2). Since

$$\|\bar{z}\|^2 = \|S_N(\bar{z})\|^2 + \|T_N(\bar{z})\|^2,$$

we infer that

$$\|S_N(\bar{z})\|^2 \leq \|\bar{z}\|^2 - \|T_N(\bar{z})\|^2 \leq \frac{3\epsilon^2}{16}. \quad (\text{D.4})$$

By definition of the worst case l_2 -gain γ , the energy of $z(\bar{w}, \delta)$ is bounded by γ for any \mathcal{R} -admissible sequence δ , which hence implies

$$\|S_N(z(\bar{w}, \delta))\|^2 \leq \gamma^2 - \|T_N(z(\bar{w}, \delta))\|^2.$$

Let us use this fact for the particular periodic sequence $\delta = P_N(\bar{\delta})$ and observe that $T_N(z(\bar{w}, \bar{\delta})) = T_N(\bar{z})$ since the first N elements of \bar{z} and $z(\bar{w}, \bar{\delta})$ are the same. Thus, we get

$$\begin{aligned} \|S_N(z(\bar{w}, P_N(\bar{\delta})))\|^2 &\leq \gamma^2 - \|T_N(\bar{z})\|^2 \leq \\ &\gamma^2 - \|\bar{z}\|^2 + \frac{3\epsilon^2}{16} \leq \frac{\epsilon^2}{16} + \frac{3\epsilon^2}{16} \leq \frac{\epsilon^2}{4}. \end{aligned} \quad (\text{D.5})$$

Combining (D.4) and (D.5), we get

$$\begin{aligned} \|\bar{z} - z(\bar{w}, P_N(\bar{\delta}))\| &\leq \|S_N(\bar{z}) - S_N(z(\bar{w}, P_N(\bar{\delta})))\| \leq \\ &\|S_N(\bar{z})\| + \|S_N(z(\bar{w}, P_N(\bar{\delta})))\| \leq \sqrt{\frac{3\epsilon^2}{16}} + \frac{\epsilon}{2} \leq \frac{\epsilon}{4} + \frac{\epsilon}{2} = \frac{3\epsilon}{4}. \end{aligned} \quad (\text{D.6})$$

and finally arrive at

$$\|z(\bar{w}, P_N(\bar{\delta}))\| \geq \|\bar{z}\| - \|\bar{z} - z(\bar{w}, P_N(\bar{\delta}))\| \geq \|\bar{z}\| - \frac{3\epsilon}{4} \geq \gamma - \frac{\epsilon}{8} - \frac{3\epsilon}{4} > \gamma - \epsilon,$$

which finishes the proof. ■

Bibliography

- [1] *Stability and Control of Time-Delay Systems*. Lecture Notes in Control and Information Sciences. Springer Verlag, Berlin, 1998.
- [2] P. Apkarian and R.J. Adams. Advanced gain-scheduling techniques for uncertain systems. *IEEE Transactions on Control Systems Technology*, 6(1):21–32, 1998.
- [3] P. Apkarian and P. Gahinet. A convex characterization of gain-scheduled \mathcal{H}_∞ -controllers. *IEEE Transactions on Automatic Control*, 40(5):853–864, 1995.
- [4] P. Apkarian and D. Noll. IQC-analysis and synthesis via nonsmooth optimization. *Systems & Control Letters*, 55(12):971–981, 2006.
- [5] P. Apkarian and H.D. Tuan. Parameterized LMIs in Control Theory. *SIAM Journal on Control and Optimization*, 38(4):1241–1264, 2000.
- [6] E. Artin. Über die zerlegung definiter funktionen in quadrate. In *Collected Papers*, pages 273–288. Springer Verlag, reprint, New York, 1965.
- [7] W. Auzinger and H.J. Stetter. An elimination algorithm for the computation of all zeros of a system of multivariate polynomial equations. In *Numerical Mathematics*, volume 86, pages 11–30. Birkhäuser, Singapore, 1988.
- [8] V. Balakrishnan. Lyapunov functions in complex μ -analysis. *IEEE Transactions on Automatic Control*, 47(9):1466–1479, 2002.
- [9] V. Balakrishnan and L. Vandenberghe. Semidefinite programming duality and linear time-invariant systems. *IEEE Transactions on Automatic Control*, 48(1):30–41, 2003.
- [10] G.J. Balas, J.C. Doyle, K. Glover, A. Packard, and R. Smith. μ -analysis and synthesis toolbox, 1995.
- [11] B. Bamieh and J.B. Pearson. A lifting technique for linear periodic systems with applications to sampled-data control. *Systems & Control Letters*, 17(2):79–88, 1991.

- [12] B. Bamieh and J.B. Pearson. A general framework for linear periodic systems with applications to \mathcal{H}_∞ -sampled-data control. *IEEE Transactions on Automatic Control*, 37(4):418–435, 1992.
- [13] B. Bamieh and J.B. Pearson. The \mathcal{H}_2 -problem for sampled-data systems. *Systems & Control Letters*, 19:1–12, 1992.
- [14] G. Becker and A. Packard. Robust performance of linear parametrically varying systems using parametrically-dependent linear feedback. *Systems & Control Letters*, 23(3):205–215, 1994.
- [15] A. Ben-Tal and A. Nemirovski. Robust solutions of uncertain linear programs. *Operations Research Letters*, 25:1–13, 1999.
- [16] A. Ben-Tal and A. Nemirovski. Robust solutions of linear programming problems contaminated with uncertain data. *Math. Program. Ser. A.*, 88:411–424, 2000.
- [17] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms and engineering applications*. Series on Optimization. SIAM, 2001.
- [18] A. Ben-Tal and A. Nemirovski. N-tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty. *SIAM Journal on Optimization*, 12(3):811–833, 2002.
- [19] M.A. Berger and Y. Wang. Bounded semigroups of matrices. *Linear Algebra and its Applications*, 166:21–27, 1992.
- [20] F. Blanchini. Nonquadratic Lyapunov functions for robust control. *Automatica*, 31(3):451–461, 1995.
- [21] F. Blanchini and S. Miani. Stabilization of LPV systems: state feedback, state estimation, and duality. *SIAM Journal on Control and Optimization*, 42(1):76–97, 2003.
- [22] P.A. Bliman. A convex approach to robust stability of linear systems with uncertain scalar. *SIAM Journal on Control and Optimization*, 42(6):2016–2042, 2004.
- [23] P.A. Bliman. An existence result for polynomial solutions of parameter-dependent LMIs. *Systems & Control Letters*, 51(3-4):165–169, 2004.
- [24] V. D. Blondel, J. Theys, and A.A. Vladimirov. An elementary counterexample to the finiteness conjecture. *SIAM Journal on Matrix Analysis and Applications*, 24(4):963–970, 2003.
- [25] V.D. Blondel and Y. Nesterov. Computationally efficient approximations of the joint spectral radius. *SIAM J. Matrix Anal. Appl.*, 27(1):256–272, 2005.

- [26] V.D. Blondel and J.N. Tsitsiklis. A survey of computational complexity results in Systems and Control. *Automatica*, 36:1249–1274, 2000.
- [27] S. Boyarski and U. Shaked. Discrete-time \mathcal{H}_∞ - and \mathcal{H}_2 -control with structured disturbances and probability-relaxed requirements. *International Journal of Control*, 77(14):1243–1259, 2004.
- [28] S. Boyd and C. Barrat. *Linear Controller Design: Limits of Performance*. Prentice Hall, 1991.
- [29] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM. Philadelphia, 1994.
- [30] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.
- [31] M.S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):475–482, 1998.
- [32] R.K. Brayton and C.H. Tong. Stability of dynamical systems: a constructive approach. *IEEE Transactions on Circuits and Systems*, 26(4):224–234, 1979.
- [33] G.C. Califore and Campi. Uncertain convex programs: randomized solutions and confidence levels. *Mathematical Programming, Ser. A.*, 102:25–46, 2005.
- [34] G. Chesi, A. Garulli, A. Tesi, and A. Vicino. Homogeneous Lyapunov functions for systems with structured uncertainties. *Automatica*, 39(6):1027–1035, 2003.
- [35] G. Chesi, A. Garulli, A. Tesi, and A. Vicino. Robust stability of polytopic systems via polynomially parameter-dependent Lyapunov functions. In *Proceedings of the 42th Conference on Decision and Control*, pages 4670–4675, Maui, Hawaii, 2003.
- [36] G. Chesi, A. Garulli, A. Tesi, and A. Vicino. Robust stability of time-varying polytopic systems via parameter-dependent homogeneous Lyapunov functions. *Automatica*, 43:309–316, 2007.
- [37] G. Chesi, G. Garulli, A. Tesi, and A. Vicino. Characterizing the solution set of polynomial systems in terms of homogeneous forms: an LMI approach. *International Journal of Robust and Nonlinear Control*, 13:1239–1257, 2003.
- [38] G. Chesi and Y.S. Hung. Robust commutation time for switching polynomial systems. In *Proceedings of the 46th IEEE Conference on Decision and Control*, pages 6334–6339, New Orleans, LA, USA, 2007.
- [39] J.C. Cockburn and B.G. Morton. Linear fractional representations of uncertain systems. *Automatica*, 33(7):1263–1271, 1997.

- [40] R.M Corless. Gröbner basis and matrix eigenvalue problems. *Association for Computing Machinery SIGSAM Bulletin*, 30(4):26–32, 1996.
- [41] R.M. Corless, P.M. Gianni, and B.M. Trager. A reordered Schur factorization method for zero-dimensional polynomial systems with multiple roots. In *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation*, pages 133–140, Maui, Hawaii, USA, 1997.
- [42] J. Cox, D. Little and D. O’Shea. *Using Algebraic Geometry*. Springer, 1998.
- [43] J. Daafouz, P. Riedinger, and C. Iung. Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach. *IEEE Transactions on Automatic Control*, 47(11):1883–1887, 2002.
- [44] M.A. Dahleh and I.J. Diaz-Bobillo. *Control of Uncertain Systems: A Linear Programming Approach*. Prentice Hall, 1995.
- [45] M.A. Dahleh and I.J. Diaz-Bobillo. *Control of Uncertain Systems: a Linear Programming Approach*. Prentice-Hall Inc., New Jersey, 1995.
- [46] R. D’Andrea and G.E. Dullerud. Distributed control design for spatially interconnected systems. *IEEE Transactions on Automatic Control*, 49(9):1478–1495, 2003.
- [47] S. Dasgupta, G. Chockalingam, B.D.O. Anderson, and M. Fu. Lyapunov functions for uncertain systems with applications to the stability of time-varying systems. *IEEE Transactions on Circuits and Systems-1: Fundamental Theory and Applications*, 41(2):93–106, 1994.
- [48] I. Daubechies and J.C. Lagarias. Sets of matrices all infinite products of which converge. *Linear Algebra and its Applications*, 161:227–263, 1992.
- [49] D.E. Davison. *Robustness of disturbance attenuation with respect to disturbance model uncertainty*. Phd thesis, University of Michigan, 1995.
- [50] W.P. Dayawansa and C.F. Martin. A converse Lyapunov theorem for a class of dynamical systems which undergo switching. *IEEE Transactions on Automatic Control*, 44(4):751–760, 1999.
- [51] J. de Loera and F. Santos. An effective version of Pólyas theorem on positive definite forms. *Journal of Pure and Applied Algebra*, 108(3):231–240, 1996.
- [52] M.C. de Oliveira, R.C.L.F. Oliveira, and P.L.D. Peres. Schur stability of polytopic systems through positivity analysis of matrix-valued polynomials. In *Proceedings of the 46th Conference on Decision and Control*, New Orleans, 2007.

- [53] C.E. de Souza, A. Trofino, and J. de Oliveira. Parametric Lyapunov function approach to \mathcal{H}_2 -analysis and control of linear parameter-dependent systems. *IEE Proceedings Control Theory & Application*, 150(5):501–508, 2003.
- [54] M. Dettori. *LMI techniques for control with applications to a compact disc player mechanism*. PhD Thesis, Delft University of Technology, 2001.
- [55] S.G. Dietz. Relaxation toolbox for Matlab. Technical report, Delft University of Technology, Delft Center for Systems and Control, 2008.
- [56] S.G. Dietz, H.M.N.K. Balini, and C.W. Scherer. Robust control against disturbance model uncertainty in active magnetic bearings. In *11th International Symposium on Magnetic Bearings*, Nara, Japan, 2008.
- [57] S.G. Dietz and C.W. Scherer. Exponential stability analysis for uncertain linear systems. In *23th Benelux Meeting*, page 71, Helvoirt, 2004.
- [58] S.G. Dietz and C.W. Scherer. Relaxations of robust linear programs and verification of exactness. In *IFAC Symposium ROCOND*, Toulouse, France, 2006.
- [59] S.G. Dietz and C.W. Scherer. A convex robust synthesis solution for particularly structured generalized plants. *International Journal of Robust and Nonlinear Control*, 2008. submitted.
- [60] S.G. Dietz and C.W. Scherer. Verifying exactness of relaxations for robust semi-definite programs by solving polynomial systems. *Linear Algebra and its Applications*, 429:1758–1778, 2008.
- [61] S.G. Dietz, C.W. Scherer, and W. Huygen. Linear parameter-varying controller synthesis using matrix sum-of-squares relaxations. In *Brazilian Automation Conference (CBA)*, 2006.
- [62] S.G. Dietz, C.W. Scherer, and H. Köroğlu. Robust control against disturbance model uncertainty: a convex solution. In *46th IEEE Conference on Decision and Control*, pages 842–847, New Orleans, 2007.
- [63] S.G. Dietz, C.W. Scherer, and H. Köroğlu. Robust controller synthesis for disturbance filter uncertainty described by dynamic integral quadratic constraints. In *17th IFAC World Congress*, Seoul, South-Korea, 2008.
- [64] M. Dinh, G. Scorletti, V. Fromion, and E. Magarotto. Parameter dependent \mathcal{H}_∞ -control by finite dimensional LMI optimization: application to trade-off dependent control. *International Journal of Robust and Nonlinear Control*, 15:383–406, 2005.
- [65] J. Doyle. Guaranteed robustness margins for LQG regulators. *IEEE Transactions on Automatic Control*, 23(4):756–757, 1978.

- [66] J. Doyle, A. Packard, and K. Zhou. Review of LFT's LMI's and μ . In *Proceedings of the IEEE Conference on Decision and Control*, 1991.
- [67] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis. State-space solutions to standard \mathcal{H}_2 - and \mathcal{H}_∞ -control problems. *IEEE Transactions on Automatic Control*, 34(8):831–847, 1989.
- [68] G.E. Dullerud and F. Paganini. *A Course in Robust Control: a Convex Approach*. Texts in Applied Mathematics. Springer Verlag, New York, 2000.
- [69] L. El Ghaoui and S.I. Niculescu. *Advances in Linear Matrix Inequality Methods in Control*. SIAM. 1999.
- [70] L. El Ghaoui, F. Oustry, and H. Lebret. Robust solutions to uncertain semidefinite programs. *SIAM Journal on Optimization*, 9(1):33–52, 1998.
- [71] C. Farges, D. Peaucelle, D. Arzelier, and J. Daafouz. Robust \mathcal{H}_2 -performance analysis and synthesis of linear polytopic discrete-time periodic systems via LMIs. *Systems & Control Letters*, 56:159–166, 2007.
- [72] M. Fekete. Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. *Mathematische Zeitschrift*, 17:228–249, 1923.
- [73] D.S. Flamm. A new shift-invariant representation for periodic systems. *Systems & Control Letters*, 17:9–14, 1991.
- [74] B.A. Francis. *A course in \mathcal{H}_∞ -control theory*, volume 88 of *Lecture Notes in Control and Information Sciences*. Springer, Berlin, 1987.
- [75] M. Fu, S. Dasgupta, and Y.C. Soh. Integral quadratic constraint approach versus multiplier approach. *Automatica*, 41:281–287, 2005.
- [76] P. Gahinet and P. Apkarian. A linear matrix inequality approach to \mathcal{H}_∞ -control. *International Journal of Robust and Nonlinear Control*, 4:421–448, 1994.
- [77] P. Gahinet, P. Apkarian, and M. Chilali. Affine parameter dependent Lyapunov functions and real parametric uncertainty. *IEEE Transactions on Automatic Control*, 41(3):436–442, 1996.
- [78] P. Gahinet, A. Nemirovski, A.J. Laub, and M. Chilali. The LMI control Toolbox . In *Proc. 33th IEEE Conf. Decision and Control*, pages 2038–2041, Orlando, FL, 1994.
- [79] M.A. Goberna and M.A. López. *Linear Semi-Infinite Optimization*. Mathematical Methods in Practice. John Wiley and Sons, 1998.

- [80] M.X. Goemans and D.P. Williamson. .878-approximation for MAX CUT and MAX 2SAT. In *Proceedings of the 26th ACM Symposium on Theory of Computing*, pages 422–431, 1994.
- [81] M. Groot Wassink, M. van de Wal, C.W. Scherer, and O. Bosgra. LPV control for a wafer stage: beyond the theoretical solution. *Control Engineering Practice*, 12(231-245), 2005.
- [82] S.V. Gusev. Minimax control under a bound on the partial covariance sequence of the disturbance. *Automatica*, 31(9):1287–1301, 1995.
- [83] W. Habicht. Über die Zerlegung strikte definiten Formen in Quadrate. *Comment Math. Helv.*, 12:317–322, 1940.
- [84] A. Hać. Suspension optimization of a 2-DOF vehicle model using a stochastic optimal control technique. *Journal of Sound and Vibration*, 100:343–357, 1985.
- [85] W. Hahn. *Stability of Motion*. Springer Verlag, Berlin, 1967.
- [86] A. Helmersson. *Methods for robust gain scheduling*. PhD Thesis, Linköping University, 1995.
- [87] A. Helmersson. An IQC-based Stability Criterion for Systems with Slowly Varying Parameters. Technical report, 1997. Available at <http://www.control.isy.liu.se>.
- [88] D. Henrion and A. Garulli, editors. *Positive Polynomials in Control*. Lecture Notes in Control and Information Sciences. Springer, 2005.
- [89] D. Henrion and J.B. Lasserre. Convergent relaxations of polynomial matrix inequalities and static output feedback. Technical report, LAAS-CNRS, 2004.
- [90] D. Henrion and J.B. Lasserre. Detecting global optimality and extracting solutions in Gloptipoly. In *Positive Polynomials in Control*, Lecture Notes in Information Sciences. 2005.
- [91] F.M. Hoblit. *Gust Loads on Aircraft: Concepts and Applications*. AIAA Education Series. American Institute of Aeronautics and Astronautics, Washington, DC, 1988.
- [92] C. Hol and C.W. Scherer. Sum-of-squares relaxations for polynomial semi-definite programming. In *Proceedings of the 16th International Symposium on MTNS.*, 2004.
- [93] C.W.J. Hol and C.W. Scherer. *Structured controller synthesis for mechanical servo-systems: algorithms, relaxations and optimality certificates*. PhD Thesis, Delft University of Technology, 2006.

- [94] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.
- [95] J.C. Houbolt. Atmospheric turbulence. *AIAA Journal*, 11:421–437, 1973.
- [96] W. Huygen. *Linear parameter varying controller synthesis using matrix sum-of-squares relaxations*. M.Sc. Thesis, Delft University of Technology, 2006.
- [97] T. Iwasaki and G. Shibata. LPV System Analysis via Quadratic Separator for Uncertain Implicit Systems. *IEEE Transactions on Automatic Control*, 46(8):1195–1208, 2001.
- [98] U. Jönsson, C. Kao, A. Megretski, and A. Rantzer. A Guide To IQC β : A MATLAB Toolbox for Robust Stability and Performance Analysis, 2004. available at <http://www.math.kth.se/cykao/>.
- [99] U. Jönsson and A. Rantzer. Systems with uncertain parameters-time-variations with bounded derivatives. In *Proceedings of the 33th Conference on Decision and Control*, pages 3074–3077, Lake Buena Vista, FL, 1994.
- [100] M. Jun. *Robustness Analysis of Uncertain Time-delay Systems*. PhD Thesis, University of Southern California, 2002.
- [101] M. Jun and M.G. Safonov. IQC robustness analysis for time-delay systems. *International Journal of Robust and Nonlinear Control*, 11:1455–1468, 2001.
- [102] C. Kao. *Efficient Computational Methods for Robustness Analysis*. PhD Thesis, MIT, 2002.
- [103] C.Y. Kao, M. Ravuri, and A. Megretski. Control synthesis with dynamic integral quadratic constraints - LMI approach. In *Proceedings of the 39th IEEE Conference on Decision and Control*, pages 1477–1482, Sydney, Australia, 2000.
- [104] V. Kapila and W.M. Haddad. Robust controller synthesis for uncertain discrete-time periodic systems. In *Proceedings of the American Control Conference*, Philadelphia, Pennsylvania, 1998.
- [105] H.K. Khalil. *Nonlinear systems*. Prentice Hall, Upper Saddle River, 2002.
- [106] P.P. Khargonekar, K. Poolla, and A. Tannenbaum. Robust control of linear time-invariant plants using periodic compensation. *IEEE Transactions on Automatic Control*, 30(11):1088–1096, 1985.
- [107] P.P. Khargonekar and N. Sivashankar. \mathcal{H}_2 -optimal control for sampled data systems. *Systems & Control Letters*, 17:425–436, 1992.
- [108] J. Kiendl, H. Adamy and P. Stelzner. Vector norms as Lyapunov functions for linear systems. *IEEE Transactions on Automatic Control*, 37(6):839–842, 1992.

- [109] S. Kim and Y. Moon. Structurally constrained \mathcal{H}_2 -and \mathcal{H}_∞ -control: A rank-constrained LMI approach. *Automatica*, 42(9):1583–1588, 2006.
- [110] M. Kojima and M. Muramatsu. An extension of sums of squares relaxations to polynomial optimization problems over symmetric cones. *Mathematical Programming, ser. A.*, 110(2):315–336, 2007.
- [111] S.R. Kolla, R.K. Yedevalli, and J.B. Farison. Robust stability bounds on time-varying perturbations for state-space models of linear discrete-time systems. *International Journal of Control*, 50(1):151–159, 1989.
- [112] H. Koroğlu and C.W. Scherer. Robust stability analysis for structured uncertainties with bounded variation rates. In *16th IFAC World Congress*, Prague, 2005.
- [113] H. Koroğlu and C.W. Scherer. Robust stability analysis against the perturbations of smoothly time-varying parameters. In *45th IEEE Conference on Decision and Control*, pages 2895–2900, San Diego, 2006.
- [114] H. Koroğlu and C.W. Scherer. An LPV control approach to asymptotic regulation of non-stationary disturbances with guaranteed worst-case performance. In *American Control Conference*, New York City, USA, 2007.
- [115] H. Koroğlu and C.W. Scherer. Robust performance analysis for structured linear time-varying perturbations with bounded rates-of-variation. *IEEE Transactions on Automatic Control*, 52(3):197–211, 2007.
- [116] J.C. Lagarias and Y. Wang. The finiteness conjecture for the generalized spectral radius of a set of matrices. *Linear Algebra and its Applications*, 214:17–42, 1995.
- [117] J.B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM Journal on Optimization*, 11(3):796–817, 2001.
- [118] M.-N. Lee and M.J. Chung. Robust control for uncertain linear systems with sinusoidal disturbance rejection. *IEE Proceedings Control Theory & Application*, 145(6):531–536, 1998.
- [119] V.J.S Leite and P.L.D. Peres. An improved LMI condition for robust D-stability of uncertain polytopic systems. *IEEE Transactions on Automatic Control*, 48(3):500–504, 2003.
- [120] D. Liberzon and A.S. Morse. Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, 19(5):59–70, 1999.
- [121] A.R.J.M. Lloyd. *Seakeeping: Ship Behaviour in Rough Weather*. John Wiley and Sons, New York, 1989.

- [122] I. Masubuchi, A. Ohara, and N. Suda. LMI-based controller synthesis: a unified formulation and solution. *International Journal of Robust and Nonlinear Control*, 8(8):669–686, 1998.
- [123] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, 42(6):819–830, 1997.
- [124] G. Meinsma, Y. Shrivastava, and M. Fu. A dual formulation of mixed μ and the losslessness of (D,G) scaling. *IEEE Transactions on Automatic Control*, 42(7):1032–1035, 1997.
- [125] R.A. Meyer and C.S. Burrus. A unified analysis of multirate and periodically time-varying digital filters. *IEEE Transactions on Circuits and Systems*, 22(3):162–168, 1975.
- [126] L. Mirkin and Z.J. Palmor. Mixed continuous/discrete specifications in sampled data \mathcal{H}_2 -optimal control. *Automatica*, 33(11):1997–2014, 1997.
- [127] A.P. Molchanov and E.S. Pyatnitskii. Lyapunov functions that specify necessary and sufficient conditions of absolute stability of nonlinear nonstationary control systems I, II, III. *Automation and Remote Control*, 47(3, 4, 5):344–354, 443–451, 620–630, 1986.
- [128] Y. Oishi. Asymptotic exactness of parameter-dependent Lyapunov functions: an error bound and exactness verification. In *Proceedings of the 46th IEEE Conference on Decision and Control*, pages 5666–5671, New Orleans, LA, USA, 2007.
- [129] R.C.L.F. Oliveira and P.L.D. Peres. Stability of polytopes of matrices via affine parameter-dependent Lyapunov functions: asymptotically exact LMI conditions. *Linear Algebra and its Applications*, 405(1):209–228, 2005.
- [130] A. Packard. Gain scheduling via linear fractional transformations. *Systems & Control Letters*, 22(2):79–92, 1994.
- [131] A. Packard and J. Doyle. The complex structured singular value. *Automatica*, 29(1):71–109, 1993.
- [132] A. Packard, J. Doyle, and G. Balas. Linear, multivariable robust control with a μ -perspective. *ASME Journal of Dynamic Systems, Measurement and Control*, 50th Anniversary Issue, pages 426–438, 1993.
- [133] Fan M. Packard, A. and J. Doyle. A power method for the structured singular value. In *Proceedings of the 27th Conference on Decision and Control*, pages 2132–2137, Austin, Texas, 1988.
- [134] F. Paganini. Set descriptions of white noise and worst case induced norms. In *Proceedings of the 32th Conference on Decision and Control*, pages 3658–3663, San Antonio, Texas, 1993.

- [135] F. Paganini and E. Feron. LMI Methods for robust \mathcal{H}_2 -analysis: a survey with comparisons. In *Recent Advances on LMI Methods in Control SIAM*. 1999.
- [136] P.A. Parrilo. Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming Ser. B*, 96(2):293–320, 2003.
- [137] P.A. Parrilo and A. Jadbabaie. Approximation of the joint spectral radius of a set of matrices using sum-of-squares. *Linear Algebra and Its Applications*, 428(10):2385–2402, 2008.
- [138] I.C. Paschalidis, S.C. Kang, and K. Li. Robust linear optimization with asymmetric data uncertainty and limited distributional information. In *IFAC 2008*, Seoul, South Korea, 2007.
- [139] D. Peacelle, D. Arzelier, O. Bachelier, and J. Bernussou. A new robust-stability condition for real convex polytopic uncertainty. *Systems & Control Letters*, 40:21–30, 2000.
- [140] I.R. Peterson and C.W. Hollot. A Riccati equation approach to the stabilization of uncertain linear systems. *Automatica*, 22(4):397–411, 1986.
- [141] S. Pettersson and B. Lennartson. A converse theorem for exponential stability using piecewise quadratic Lyapunov functions. Technical report, Control Engineering Lab, 1997. Chalmers University.
- [142] A. Polànski. Lyapunov function construction by linear programming. *IEEE Transactions on Automatic Control*, 42(7):1013–016, 1997.
- [143] A. Polànski. Destabilizing strategies for uncertain linear systems. *IEEE Transactions on Automatic Control*, 45(12):2378–2382, 2000.
- [144] V.M. Popov. Absolute stability of nonlinear systems of automatic control. *Automation and Remote Control*, 22:857–875, 1961.
- [145] V. Powers and B. Reznick. A new bound for Pólya theorem with applications to polynomials positive on polyhedra. *Journal of Pure and Applied Algebra*, 164:221–229, 2001.
- [146] M. Putinar. Positive polynomials on compact semi-algebraic sets. *Indiana University Mathematical Journal*, 42(3):969–984, 1993.
- [147] A. Rantzer. Friction analysis based on integral quadratic constraints. In *Proceedings of the 35th IEEE Conference on Decision and Control*, volume 4, pages 3948–3952, Kobe, Japan, 1996.
- [148] A. Rantzer. On the Kalman-Yakubovich-Popov lemma. *Systems & Control Letters*, 28:7–10, 1996.

- [149] B. Reznick. Some concrete aspects of Hilbert's 17th problem. In *Real Algebraic Geometry and Ordered Structures*, volume 253 of *Comtemp. Math*, pages 251–272. 2000.
- [150] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, 1996.
- [151] M.A. Rotea. The generalized \mathcal{H}_2 -control problem. *Automatica*, 29(2):373–385, 1993.
- [152] W.J. Rugh and J.S. Shamma. Research on gain scheduling. *Automatica*, 26:1401–1425, 2000.
- [153] C.W. Scherer. Design of structured controllers with applications. In *Conference on Decision and Control*, Sydney, 2000.
- [154] C.W. Scherer. Robust controller design by output feedback against uncertain stochastic disturbances. In *IFAC Symposium on Robust Control Design*, 2000.
- [155] C.W. Scherer. LPV control and full block multipliers. *Automatica*, 37:361–375, 2001.
- [156] C.W. Scherer. When are multiplier relaxations exact? In *IFAC Symposium ROCOND*, Milano, Italy, 2003.
- [157] C.W. Scherer. Relaxations for robust linear matrix inequality problems with verifications for exactness. *SIAM Journal on Matrix Analysis and Applications*, 27(2):365–395, 2005.
- [158] C.W. Scherer. LMI relaxations in robust control. *European Journal of Control*, 12:3–29, 2006.
- [159] C.W. Scherer, P. Gahinet, and M. Chilali. Multiobjective output-feedback control via LMI optimization. *IEEE Transactions on Automatic Control*, 42(7):896–911, 1997.
- [160] C.W. Scherer and C.W.J. Hol. Matrix sum-of-squares relaxations for robust semi-definite programs. *Mathematical Programming*, 107(1):189–211, 2005.
- [161] C.W. Scherer and I.E. Köse. Robust \mathcal{H}_2 -estimation with dynamic IQCs: A convex solution. In *Proceedings of the 45th Conference on Decision and Control*, San Diego, California, USA, 2006.
- [162] C.W. Scherer and I.E. Köse. On robust controller synthesis with dynamic D-scalings. In *46th IEEE Conference on Decision and Control*, New Orleans, 2007.
- [163] C.W. Scherer and S. Weiland. Linear matrix inequalities in control: DISC Course notes. 1999.

- [164] Schmüdgen, K. The K-moment problem for semi-algebraic sets. *Mathematische Annalen*, 289:203–206, 1991.
- [165] G. Scorletti and L. El Ghaoui. Improved LMI conditions for gain scheduling and related control problems. *International Journal of Robust and Nonlinear Control*, 8:845–877, 1995.
- [166] R.N. Shorten, F. Ó Cairbre, and P. Curran. On the dynamic instability of a class of switching system. *International Journal of Control*, 79(6):630–635, 2006.
- [167] A.J. Sommese and C.W. Wampler. *The Numerical Solution of Systems of Polynomials arising in Engineering and Science*. World Scientific Publishing, Singapore, 2005.
- [168] H.J. Stetter. *Numerical Polynomial Algebra*. SIAM, 2004.
- [169] A.A. Stoorvogel. The robust \mathcal{H}_2 -control problem: A worst-case design. *IEEE Transactions on Automatic Control*, 38(9):1358–1370, 1993.
- [170] J. Sturm. Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11-12:625–653, 1999. available at <http://sedumi.mcmaster.ca>.
- [171] B. Sturmfels. Polynomial equations and convex polytopes. *American Mathematical Monthly*, 105(10):907–922, 1998.
- [172] M. Sznaier, T. Amishiwa, P.A. Parrilo, and J. Tierno. A convex approach to robust \mathcal{H}_2 -performance analysis. *Automatica*, 38:957–966, 2002.
- [173] R. Tempo, G. Calafiore, and F. Dabbene. *Randomized Algorithms for Analysis and Control of Uncertain Systems*. Springer Verlag, 2004.
- [174] H.D. Tuan and P. Apkarian. Relaxations of parameterized LMIs with control applications. In *Proceedings of the 37th Conference on Decision and Control*, Tampa, Florida USA, 1998.
- [175] V. A. Ugrinovskii and I.R. Petersen. Absolute stabilization and minimax optimal control of uncertain systems with stochastic uncertainty. *SIAM Journal on Control and Optimization*, 37(4):1089–1122, 1999.
- [176] L. Vandenberghe, S. Boyd, and M. Nouralishahi. Robust linear programming and optimal control. In *Proceedings of the 15th IFAC World Congress on Automatic Control*, 2002.
- [177] E.L. Wemhoff. *Signal estimation in structured non-linear systems with unknown functions*. PhD Thesis, University of California, 2003.

- [178] J.C. Willems. Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Transactions on Automatic Control*, 16(6):621–634, 1971.
- [179] J.C. Willems. Dissipative dynamical systems: I: General theory, II: Linear systems with quadratic supply functions. *Archive for Rational Mechanics and Analysis*, 45:321–343, 1972.
- [180] F. Wirth. The generalized spectral radius and extremal norms. *Linear Algebra and its Applications*, 342:17–40, 2002.
- [181] F. Wirth. Parameter dependent extremal norms for linear parameter varying systems. In *Proceedings of the Symposium on Mathematical Theory of Networks and Systems (MTNS)*, South Bend, IN, US, 2002.
- [182] R. Wiśniewski and J. Stoustrup. Generalized \mathcal{H}_2 -control synthesis for periodic systems. In *Proceedings of the American Control Conference*, Arlington, VA, 2001.
- [183] F. Wu. A generalized LPV system analysis and control synthesis framework. *International Journal of Control*, 74(7):745, 2001.
- [184] F. Wu and K. Dong. Gain-scheduling control of LFT systems using parameter-dependent Lyapunov functions. *Automatica*, 42:39–50, 2006.
- [185] F. Wu, X.Y. Yang, A. Packard, and G. Becker. Induced \mathcal{L}_2 -norm control for LPV systems with bounded parameter variation rates. *International Journal of Robust and Nonlinear Control*, 6:983–998, 1996.
- [186] L. Xie, S. Shishkin, and M. Fu. Piecewise Lyapunov functions for robust stability of linear time-varying systems. *Systems & Control Letters*, 31:165–171, 1997.
- [187] C.A. Yfoulis and R. Shorten. A numerical technique for stability analysis of linear switched systems. *International Journal of Control*, 77(11):1019–1039, 2003.
- [188] J. Yu and A. Sideris. \mathcal{H}_∞ -control with parametric Lyapunov functions. *Systems & Control Letters*, 30:57–69, 1997.
- [189] A.L. Zelentsovsky. Nonquadratic Lyapunov functions for robust stability analysis of linear uncertain systems. *IEEE Transactions on Automatic Control*, 39(135-138), 1994.
- [190] K. Zhou, J. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, Upper Saddle River, New Jersey, 1996.
- [191] K. Zhou and P.P. Khargonekar. An algebraic Riccati equation approach to \mathcal{H}_∞ -optimization. *Systems & Control Letters*, 11:85–91, 1988.

Summary

Analysis and Control of Uncertain Systems by Using Robust Semi-Definite Programming

Over the past century, many interesting problems in the field of systems and control have been solved by formulating them as a mathematical optimization problem. Many of these optimal control problems, like the well-known \mathcal{H}_∞ - or \mathcal{H}_2 -optimal control, can be formulated as a linear optimization subject to Linear Matrix Inequalities (LMIs). Typically, these algorithms for controller design rely on the assumption that the underlying system is linear time-invariant (LTI). In practice though, the system dynamics often varies in time and some system components will be non-linear. Nevertheless, simple mathematical models are preferred for controller design purposes in view of the numerical complexity that arises when using high order complex models.

In this thesis, we investigate how to compute optimal robust and scheduled controllers for systems with uncertainty by using LMI methods. The philosophy of robust controller design is to consider a nominal LTI plant model of modest complexity that is augmented with a class of uncertain elements, such that the real system lies within the family of models so constructed. Then, it is possible to analyze, by using linear analysis tools, whether the original system is stable or satisfies a certain performance criterion. However, contrary to the analysis conditions for LTI systems, so-called robust LMIs dominate the problem, which consist of a parameterized family of LMIs. It significantly increases the computational complexity since the parameters take infinitely many values in general. Any (linear) optimization problem, in which the decision variables are constrained by a robust LMI, is called robust semi-definite programming.

The class of robust semi-definite programming problems is introduced in Chapter 2 by means of an elementary example from linear algebra. It arises from a well-known analysis problem for uncertain systems, also known as μ -analysis. It is shown that robust SDPs are approximately solved by constructing so-called relaxation schemes. Throughout the thesis, the proposed relaxation methods have been applied to a number of problems taken from the system's and control field. Emphasis is put on improving existing analysis or controller synthesis methods, since the LMI conditions are often sufficient but not necessary.

In Chapter 3 we give an overview of the analysis method for uncertain systems, that is based on so-called “integral quadratic constraints”. It allows to analyze stability as well as a certain performance level. Moreover, it can incorporate a wide range of uncertainties, such as static non-linearities or time-varying parameters. A numerical example is presented for a discrete-time LPV system with a single time-varying parameter.

In Chapter 4, we consider the analysis of linear parameter varying (LPV) systems. An LPV model can be viewed as an LTI model, of which the system matrices depend on some (time-varying) parameters. The question whether a given LPV system is stable for a given family of allowable parameter trajectories, is typically approached by using a Lyapunov function. This leads to sufficient conditions, and little is known in general about the required Lyapunov function in order for analysis conditions to be exact. For discrete-time LPV systems, we have proposed a systematic procedure, based on a lifting argument, that allows for the construction of a family of asymptotically exact analysis conditions. Whether exact computations can be expected in practical problems essentially depends on the available computational power.

In the Chapters 5 and 6 we aim at designing robust and scheduled controllers. Based on the IQC-analysis methodology, we have given an LMI solution for the corresponding robust output feedback controller synthesis problem. Contrary to existing methods that rely on iterative schemes, it turns out that the problem can be rendered convex if assuming a particular structure of the generalized plant. Although the synthesis solution applies to a specific class of problems, the structure is definitely of practical interest, as a motivating example at the beginning of the chapter shows.

If parameters of an LPV system can be measured online, one typically wishes to design a controller that depends on the parameters of the system. This can significantly improve the performance of the closed loop system. A well-known design approach in this respect is known as LPV synthesis, which forms the topic of Chapter 4. Robust LMIs naturally follow from the design problem, and the relaxation methods from Chapter 2 are again employed in a numerical example.

Finally, conclusions are drawn in Chapter 7. On the one hand, these concern the applicability and issues of implementing the relaxation schemes of Chapter 2. On the other hand, we address the contributions when it comes to turning analysis and controller design problems into a robust SDP problem.

Samenvatting

Het Analyseren en Regelen van Systemen met Onzekerheid door Gebruik te Maken van Robuust Semi-Definiet Programmeren.

In de loop van de vorige eeuw zijn vele regeltechnische problemen opgelost door ze als een wiskundig optimalisatie probleem te formuleren. Veel van deze optimale regelaarontwerp methoden, zoals \mathcal{H}_∞ -of \mathcal{H}_2 -regelaarsynthese, kunnen worden geschreven in termen van lineaire matrix ongelijkheden (LMI's). Deze problemen behoren tot de convexe optimalisatie, waarvoor diverse algoritmen zijn ontwikkeld. Echter, veelal wordt het systeem als lineair tijdinvariant (LTI) verondersteld, ook al zal de dynamica van het systeem variëren met de tijd, of hebben bepaalde componenten een duidelijk niet-lineair karakter. Toch is, met het oog op de numerieke complexiteit die gepaard gaat met het gebruik van hoge orde modellen, een eenvoudig wiskundig model gewenst voor het ontwerpen en analyseren van regelaars.

In dit proefschrift wordt onderzocht hoe er met LMI technieken optimale robuuste en parameter afhankelijke regelaars kunnen worden gevonden voor systemen met onzekerheid. De gedachte achter robuust regelaarontwerp is om gebruik te maken van een nominaal LTI model in combinatie met een te beschrijven onzekerheidsklasse. Door ervoor te zorgen dat de op deze manier vastgelegde verzameling van modellen het daadwerkelijke systeem omvat, kan er met lineaire theorie een uitspraak worden gedaan over stabiliteit en over de haalbare prestatie van een geregeld systeem. Echter, in tegenstelling tot de condities die kunnen worden afgeleid voor LTI systemen, spelen zogenaamde robuuste LMI's een cruciale rol bij het analyseren van systemen met onzekerheid. Deze bestaan uit een geparametriseerde familie van LMI's. Indien voor een gegeven (lineaire) optimalisatie, de beslissingsvariabelen door een robuuste LMI wordt beperkt, spreekt men van robuust semi-definiet programmeren (SDP). De complexiteit van robuuste SDPs wordt veroorzaakt door het feit dat de beslissingsvariabelen door oneindig veel LMI's worden beperkt.

In Hoofdstuk 2 wordt de klasse van robuuste semi-definiete problemen geïntroduceerd aan de hand van een elementaire vraag in de lineaire algebra. Het heeft zijn oorsprong in de analyse van stabiliteit voor een onzeker systeem, beter bekend als μ -analyse. Het numeriek oplossen van robuuste LMI's gaat gepaard met benaderingen,

de zogenoemde relaxaties. In de overige hoofdstukken van dit proefschrift worden de gepresenteerde relaxatie technieken toegepast op problemen die voortkomen uit the regeltheorie. De nadruk ligt dan met name op het verbeteren van condities voor zowel de analyse van onzekere systemen als voor regelaarsynthese. Deze condities zijn in de regel enkel voldoende en niet noodzakelijk.

Hoofdstuk 3 geeft een beknopt overzicht van een methode gebaseerd op zogenaamde “integral quadratic constraints” (IQC), waarmee zowel de stabiliteit als de prestatie van een gegeven systeem met onzekerheid kan worden bepaald. Bovendien kan een breed scala aan onzekerheden in rekening worden gebracht, zoals statische niet-lineariteiten, tijdvariërende parameters etc. Een numeriek voorbeeld laat zien hoe de methode werkt in het geval van een tijdvariërende parameter.

In Hoofdstuk 4 richten we ons eveneens op het analyse vraagstuk, ditmaal voor de klasse van “Linear Parameter Varying”-systemen. Een LPV model heeft eenzelfde structuur als een LTI model, maar de systeem matrices hangen van een aantal (tijdvariërende) parameters af. De vraag of een LPV model robuust stabiel is voor een gegeven familie van parameter variaties, kan worden behandeld door gebruik te maken van een Lyapunov functie. Helaas is over het algemeen weinig bekend over de vereiste structuur van de Lyapunov functie, waarvoor exacte condities worden verkregen. Voor LPV systemen in discrete-tijd zal een systematische procedure worden ontwikkeld, gebaseerd op het zogenaamde “liften” van het systeem. Deze maakt een exacte analyse voor stabiliteit van het LPV systeem mogelijk, echter zal beperkte reken capaciteit dit in de praktijk niet altijd toelaten.

De volgende twee hoofdstukken richten zich op het ontwerpen van een robuuste of parameter afhankelijke regelaar. Gebaseerd op het IQC-raamwerk uit Hoofdstuk 3 wordt in Hoofdstuk 5 een nieuw algoritme gepresenteerd om optimale robuuste regelaars te ontwerpen. Gezien de condities voor robuuste regelaarsynthese tot dusver niet convex zijn bevonden, zijn bestaande synthese technieken gebaseerd op een iteratieve procedure. Het voorgestelde algoritme bevat uitsluitend LMI's, en is daarmee convex in de beslissingsvariabelen. Ook al heeft het resultaat betrekking op interconnecties met een specifieke structuur, het voorbeeld aan het begin van Hoofdstuk 5 laat zien dat deze zeker voor de praktijk relevant kunnen zijn.

Indien de parameters in een LPV systeem kunnen worden gemeten gedurende de tijd dat de regelaar operationeel is, wordt meestal een regelaar ontworpen die zich continue aanpast aan de waarde van de parameter. Een veelgebruikte techniek hiervoor, beter bekend als LPV synthese, wordt in Hoofdstuk 6 beschreven. Het berekenen van een optimale LPV regelaar kan relatief eenvoudig worden geformuleerd als een robuust SDP door gebruik te maken van de LMI condities die eerder al zijn afgeleid voor het nominale regelaar probleem. Wederom worden verschillende relaxatietechnieken uit Hoofdstuk 2 toegepast.

Tot slot wordt in Hoofdstuk 7 een opsomming gegeven van de voornaamste conclusies. Enerzijds worden de gepresenteerde relaxatie technieken vergeleken, anderzijds wordt een blik geworpen op de mogelijkheden nieuwe problemen in de regeltheorie te formuleren als een robuust SDP.

Curriculum Vitae

Sjoerd G. Dietz was born in Eindhoven on June 20, 1978. In 1996 he received his Atheneum certificate at the Gertrudis College in Roosendaal. He studied Aerospace Engineering at the Delft University of Technology and got his M.Sc degree in 2002. He then worked at the German Aerospace Center, Institute of Robotics and Mechatronics for one year, after which he started his PhD project at the Delft Center for Systems and Control. He is currently working as an Engineering Consultant at BERATA GmbH, Germany, which is part of the ALTRAN Group.