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The Set-Invariance Paradigm in Fuzzy Adaptive DSC Design of Large-Scale Nonlinear Input-Constrained Systems

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***Abstract*—This paper proposes a novel set-invariance adaptive dynamic surface control (DSC) design for a larger class of uncertain large-scale nonlinear input-saturated systems. The peculiarity of this class is that no *a priori* bound on the continuous control gain functions is assumed (i.e., their boundedness cannot be assumed before obtaining system stability). This requires a new design. Differently from the available methods, the proposed design involves the construction of appropriate invariant sets for the closed-loop trajectories, which allows to remove the restrictive assumption of *a priori* bounds of the control gain functions. Furthermore, we show that such set-invariance design can handle input constraints in the form of input saturation. In line with the DSC methodology, semi-globally uniformly ultimate boundedness is proven: however, differently from the standard methodology, stability analysis requires the combination of Lyapunov and invariant set theories.**

***Index Terms*—Adaptive fuzzy control, dynamic surface control (DSC), input constraints, invariant set theory.**

I. Introduction

IN RECENT decades, much attention has been devoted to the area of neural networks-based and fuzzy logic-based adaptive control, which makes it possible to approximate unknown continuous nonlinear functions with little *a priori* knowledge about the controlled system [1]–[12]. Moreover, global stability for various kinds of uncertain nonlinear dynamic systems has been proven via the adaptive backstepping method [13]–[18]. However, repeated differentiations of the intermediate control laws during backstepping generates the problem of "explosion of complexity." The dynamic surface control (DSC) technique was proposed to tackle this difficulty. This technique has been successfully applied to several classes of (strict-feedback) nonlinear systems, e.g., large-scale, multi-input/multi-output (MIMO) and input-constrained systems. To list a few, a robust adaptive fuzzy control design based on RBF-NN is presented in [19] for a class of MIMO nonlinear systems. In [20], a fuzzy DSC method was proposed for large-scale interconnected strict-feedback nonlinear systems. An approximation-based adaptive control method was proposed in [21] for a class of large-scale nonlinear systems in the presence of input saturation. A neural networks-based adaptive control design was proposed in [22] for large-scale strict-feedback nonlinear systems with unknown time delays. Further works involving strict-feedback nonlinear systems can be found in [23]–[30] and in the references therein.

However, two problems are worth mentioning: the first is that, for all aforementioned designs [19]–[30] to work, lower and upper bounds of the control gain functions must be assumed to exist *a priori* (i.e., before obtaining system stability) [31]. Even though some efforts have been made to get rid of this restrictive assumption, such as [32], it is still required the control gain functions to be bounded by a positive term, which is expressed as an unknown positive constant multiplying a known positive function. Clearly, in many practical control systems, *a priori* bounds of the control gain functions are difficult to be known, or such bounds may be nonexistent [33].

The second problem is that, since the approximation of the nonlinear functions is valid as long as the states are inside a compact set, adaptive control laws should not push the states outside this set. This consideration, often ignored in most works [19]–[30], requires to combine the DSC technique with invariant set theory, as recently done by some of the authors in [33]. The open problem in this paper is how to adopt the set-invariance paradigm in such a way to handle large-scale nonlinear systems and input constraints whose effects are known to severely degrade the control performance [34]–[36]. Therefore, the open questions answered by this paper are: how to relax the assumption on the control gain functions for large-scale strict-feedback nonlinear systems in the presence of interconnection and saturation effects? And, most importantly, how to extend the set-invariance design in such a setting? These questions motivate this paper.

In view of the aforementioned discussion, the main innovations of this paper are given below.

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1) In contrast with all existing works [19]–[30], the boundedness assumption for control gain functions is no longer required, on the contrary, the control gain functions of large-scale nonlinear systems are only required to be positive instead of bounded by positive terms. The main challenge arising from this setting is that the functions cannot be assumed to be bounded *a priori* before obtaining the system stability.
2) A novel set-invariance fuzzy adaptive design is carried out for input-saturated large-scale nonlinear systems. The challenge of this design is to construct appropriate compact sets via invariant set theory, which guarantee that the states of the closed-loop system will stay in those sets all the time, even in the presence of input saturation.
3) It is worth mentioning that, consistently with DSC theory, the resulting stability is semi-globally uniformly ultimate boundedness (SGUUB). This means that the design parameters depend on the initial conditions. However, different from the standard method, Lyapunov stability is enhanced via invariant set theory to prove convergence of the tracking errors to an arbitrarily small neighborhood of the origin after choosing appropriate design parameters.

The rest of this paper is structured as follows. The considered class of systems is presented in Section II. The proposed DSC design procedure, and system stability analysis are presented in Section III. In Section IV, simulation results are given. The conclusions are given in Section V.

II. Problem Formulation and Preliminaries

Consider the following large-scale nonlinear system with input saturation [19]–[21]:

$$\begin{cases} \dot{x}_{j,i_j} = f_{j,i_j}(\bar{x}_{j,i_j}) + g_{j,i_j}(\bar{x}_{j,i_j})x_{j,i_j+1} + \Delta_{j,i_j}(x,t) \\ \qquad 1 \le i_j \le \rho_j - 1 \\ \dot{x}_{j,\rho_j} = f_{j,\rho_j}(\bar{x}_{j,\rho_j}) + g_{j,\rho_j}u_j(v_j(t)) + \Delta_{j,\rho_j}(x,t) \\ y_j = x_{j,1} \qquad j = 1,\dots,m \end{cases} \tag{1}$$

where $x_{j,i_j} \in \mathbb{R}$ is the state of the jth subsystem, $x = [\bar{x}_{1,\rho_1}^T, \dots, \bar{x}_{j,\rho_j}^T, \dots, \bar{x}_{m,\rho_m}^T]^T \in \mathbb{R}^N$ represents the state vector of the whole system ($N = \rho_1 + \dots + \rho_m$), where $\bar{x}_{j,\rho_j} = [x_{j,1}, \dots, x_{j,\rho_j}]^T \in \mathbb{R}^{\rho_j}$ and ρ_j is the order of the jth subsystem. $\bar{x}_{j,i_j} = [x_{j,1}, \dots, x_{j,i_j}]^T \in \mathbb{R}^{i_j}$, $y_j \in \mathbb{R}$ is the output of the jth subsystem. $f_{j,i_j}(\cdot)$ and $g_{j,i_j}(\cdot)$ are unknown continuous functions, $\Delta_{j,i_j}(x,t)$, $i_j = 1,\dots,\rho_j$, and $j = 1,\dots,m$ are uncertain terms, comprising external disturbances and dynamical coupling terms, which might depend on the full-system state x, and $u_j(v_j(t))$ is the saturated input of the jth system, which is expressed as follows:

$$u_j(v_j(t)) = \text{sat}(v_j(t)) = \begin{cases} \text{sign}(v_j(t))u_{j,M}, & |v_j(t)| \ge u_{j,M} \\ v_j(t), & |v_j(t)| < u_{j,M} \end{cases} \tag{2}$$

where $u_{j,M}$ is the bound of $u_j(v_j(t))$. To handle the saturation $u_j(v_j(t))$ in the control design, it follows from [15] that (2) can be approximated by the smooth function:

$$h_j(v_j) = u_{j,M}\tanh\left(\frac{v_j}{u_{j,M}}\right) = u_{j,M}\frac{e^{v_j/u_{j,M}} - e^{-v_j/u_{j,M}}}{e^{v_j/u_{j,M}} + e^{-v_j/u_{j,M}}}. \tag{3}$$

In particular, $\text{sat}(v_j(t))$ in (2) can be rewritten as

$$\text{sat}(v_j) = h_j(v_j) + d_j(v_j) \tag{4}$$

where $|d_j(v_j)| \le u_{j,M}(1 - \tanh(1)) = D_j$, with $D_j > 0$ being an unknown constant.

Invoking the mean value theorem, $h_j(v_j)$ can be given as

$$h_j(v_j) = h_j(v_j^*) + \partial h_j(\cdot)/\partial v_j\Big|_{v_j = v_j^{\theta_{j0}}}(v_j - v_j^*) \tag{5}$$

where $v_j^{\theta_{j0}} = \theta_{j0} + (1-\theta_{j0})v_j^*$ with $0 < \theta_{j0} < 1$. Let $v_j^* = 0$: then we have

$$h_j(v_j) = \partial h_j(\cdot)/\partial v_j\Big|_{v_j = v_j^{\theta_{j0}}}(v_j - v_j^*) = g_{j0}(v_j^{\theta_{j0}})v_j. \tag{6}$$

Remark 1: The mean value theorem is commonly adopted in the literature to handle input saturation (see [14], [15], [21]). According to the definition of $h_j(v_j)$ in (3), it holds that $0 < \underline{g}_{j0} < g_{j0}(v_j^{\theta_{j0}}) \le 1$ for every $v_j \in R$ with $\underline{g}_{j0}$ a constant.

Then, system (1) can be rewritten as

$$\begin{cases} \dot{x}_{j,i_j} = f_{j,i_j}(\bar{x}_{j,i_j}) + g_{j,i_j}(\bar{x}_{j,i_j})x_{j,i_j+1} + \Delta_{j,i_j}(x,t) \\ \qquad 1 \le i_j \le \rho_j - 1 \\ \dot{x}_{j,\rho_j} = f_{j,\rho_j}(\bar{x}_{j,\rho_j}) + g_{j,\rho_j}(\bar{x}_{j,\rho_j})g_{j0}(v_j^{\theta_{j0}})v_j \\ \qquad + g_{j,\rho_j}(\bar{x}_{j,\rho_j})d_j(v_j) + \Delta_{j,\rho_j}(x,t) \\ y_j = x_{j,1} \qquad j = 1,\dots,m. \end{cases} \tag{7}$$

The following assumption on the control-gain functions sensibly relaxes the assumptions in the existing literature.

Assumption 1: The control gain functions satisfy $g_{j,i_j}(\bar{x}_{j,i_j}) > 0$ for $i_j = 1,2,\dots,\rho_j$ and $j = 1,\dots,m$.

Remark 2: In all existing methods, such as [19]–[30], the control gain functions $g_{j,i_j}(\bar{x}_{j,i_j})$ are assumed to satisfy $\underline{g}_{j,i_j} \le g_{j,i_j}(\bar{x}_{j,i_j}) \le \bar{g}_{j,i_j}$, with $\underline{g}_{j,i_j}$ and $\bar{g}_{j,i_j}$ being positive constants. In fact, this assumption guarantees controllability of system (1). However, this assumption $\underline{g}_{j,i_j} \le g_{j,i_j}(\bar{x}_{j,i_j}) \le \bar{g}_{j,i_j}$ is too restrictive since the lower bound $\underline{g}_{j,i_j}$ and upper bound $\bar{g}_{j,i_j}$ of $g_{j,i_j}(\bar{x}_{j,i_j})$ may be nonexistent. Take $g_{j,i_j}(\bar{x}_{j,i_j}) = e^{x_{j,i_j}}$ as an example, then, the condition $\underline{g}_{j,i_j} \le g_{j,i_j}(\bar{x}_{j,i_j}) \le \bar{g}_{j,i_j}$ is not satisfied because $\underline{g}_{j,i_j}$ and $\bar{g}_{j,i_j}$ do not exist for all states: however, Assumption 1 holds since $e^{x_{j,i_j}} > 0$ for all states $\bar{x}_{j,i_j}$.

Remark 3: Obviously, the states $\bar{x}_{j,i_j}$ cannot be assumed to be bounded *a priori* before obtaining the system stability. Therefore, in view of Assumption 1, the control gains cannot be taken bounded *a priori* before obtaining the system stability. In the existing methods, the system stability is achieved under the *a priori* bounded condition for the control gain functions. Therefore, the absence of *a priori* bounds requires a new control design going beyond the existing literature [19]–[30].

The following two assumptions are standard in [20] and [22] among others.

Assumption 2: The desired trajectory $y_{j,d}(t)$ is a sufficiently smooth function, and $y_{j,d}$, $\dot{y}_{j,d}$ and $\ddot{y}_{j,d}$ are bounded, there exists a constant $D_{j0} > 0$ satisfying $\Omega_{j0} := \{[y_{j,d}, \dot{y}_{j,d}, \ddot{y}_{j,d}]^T | (y_{j,d})^2 + (\dot{y}_{j,d})^2 + (\ddot{y}_{j,d})^2 \le D_{j0}\}$.

Assumption 3: For $\forall t > 0$, there exist positive constants Δ^*_{j,i_j} such that $|\Delta_{j,i_j}(x,t)| \le \Delta^*_{j,i_j}$, for $i_j = 1, \ldots, \rho_j$ and $j = 1, \ldots, m$.

The aim of this paper is to design a decentralized robust fuzzy adaptive DSC v_j such that all signals of the interconnected large-scale nonlinear system (7) are SGUUB, and the whole system output $y = [y_1, \ldots, y_m]^T$ follows the desired trajectory $y_d = [y_{1,d}, \ldots, y_{m,d}]^T$ with a tunable bounded tracking error.

The following three lemmas are instrumental to stability analysis.

Lemma 1 [24]: The hyperbolic tangent function fulfills the following inequality for $\forall \varsigma > 0$ and any $q \in R$:

$$0 \le |q| - q \tanh(q/\varsigma) \le 0.2785\varsigma. \tag{8}$$

Lemma 2 [33]: For $\forall (x,y) \in \mathbb{R}^2$, the following inequality holds:

$$xy \le \frac{\varepsilon^2}{\alpha}\|x\|^2 + \frac{1}{\beta\varepsilon^2}\|y\|^2 \tag{9}$$

where $\varepsilon > 0$, $\alpha > 1$, $\beta > 1$, and $(\alpha-1)(\beta-1) = 1$.

Lemma 3 [2]: Consider a continuous function $f(x)$ which is defined in a compact set Ω_x, for any given positive constant ε^*, there exists a fuzzy logic systems $y(x) = W^T\bar{\varphi}(x)$ such that

$$\sup_{x\in\Omega_x} |f(x) - y(x)| \le \varepsilon^*$$

where $\bar{\varphi}(x)$ is a vector of appropriately defined basis functions.

III. Controller Design and Stability Analysis

A. Adaptive Fuzzy DSC Design

In this section, approximator-based adaptive backstepping control method shall be designed for system (7) with the aid of invariant set theory. The recursive design includes ρ_j steps. At the i_jth step ($1 \le i_j \le \rho_j - 1$), the intermediate controller s_{j,i_j} will be designed, while the actual control law v_j is constructed at the final step.

First of all, consider the following change of coordinates:

$$\begin{cases} e_{j,1} = x_{j,1} - y_{j,d} \\ e_{j,i_j} = x_{j,i_j} - \zeta_{j,i_j} \end{cases} \tag{10}$$

with ζ_{j,i_j} being obtained from the following first-order filters:

$$\begin{aligned} \tau_{j,i_j+1}\dot{\zeta}_{j,i_j+1} + \zeta_{j,i_j+1} &= s_{j,i_j} \\ \zeta_{j,i_j+1}(0) &= s_{j,i_j}(0) \end{aligned} \tag{11}$$

where $\tau_{j,i_j+1} > 0$ is a design time constant.

Since $f_{j,i_j}(\bar{x}_{j,i_j})$, $i_j = 1, \ldots, \rho_j$, are unknown continuous functions. Therefore, throughout this note, we use fuzzy logic systems to approximate functions $f_{j,i_j}(\bar{x}_{j,i_j})$ as shown in

$$f_{j,i_j}(\bar{x}_{j,i_j}) = W^T_{j,i_j}\bar{\varphi}_{j,i_j}(\bar{x}_{j,i_j}) + \varepsilon_{j,i_j}, \quad \bar{x}_{j,i_j} \in \Omega_{\bar{x}_{j,i_j}} \tag{12}$$

where $\bar{\varphi}_{j,i_j}(\bar{x}_{j,i_j}) = [\varphi_{j,i_j,1}(\bar{x}_{j,i_j}), \ldots, \varphi_{j,i_j,l_{i_j}}(\bar{x}_{j,i_j})]^T$ with $\varphi_{j,i_j,n}(\bar{x}_{j,i_j})$, for $n = 1, \ldots, l_{j,i_j}$, being Gaussian functions, and ε_{j,i_j} are the approximation errors, satisfying $|\varepsilon_{j,i_j}| \le \varepsilon^*_{j,i_j}$ with ε^*_{j,i_j} unknown positive constants. Let ε, Δ_{j,ρ_j}, g_{j0}, and d_j denote $\varepsilon(Z)$, $\Delta_{j,\rho_j}(x,t)$, $g_{j0}(v_j^{\theta_{j0}})$, and $d_j(v_j)$, respectively.

Step j, 1 (j = 1, ..., m): Using (7), (10), and (12), we obtain the dynamics of $e_{j,1}$ as

$$\dot{e}_{j,1} = W^T_{j,1}\bar{\varphi}_{j,1}(x_{j,1}) + \varepsilon_{j,1} + g_{j,1}(x_{j,1})x_{j,2} + \Delta_{j,1} - \dot{y}_{j,d} \tag{13}$$

where $\varepsilon_{j,1}$ is such that $|\varepsilon_{j,1}| \le \varepsilon^*_{j,1}$ with $\varepsilon^*_{j,1}$ a positive constant.

Consider the quadratic function as follows:

$$V_{e_{j,1}} = \frac{1}{2}e^2_{j,1}. \tag{14}$$

Noting (13), one has

$$\dot{V}_{e_{j,1}} = e_{j,1}\Big(W^T_{j,1}\bar{\varphi}_{j,1}(x_{j,1}) + \varepsilon_{j,1} + g_{j,1}(x_{j,1})x_{j,2} + \Delta_{j,1} - \dot{y}_{j,d}\Big). \tag{15}$$

Defining compact set $\Omega_{j,1} := \{e_{j,1} | V_{e_{j,1}} \le p\}$ with $p > 0$ is a constant. Then, the following Lemma 4 holds.

Lemma 4: The continuous function $g_{j,1}(x_{j,1})$ has maximum and minimum in $\Omega_{j,1} \times \Omega_{j0}$, that is, there exist constants $\underline{g}_{j,1} > 0$ and $\bar{g}_{j,1} > 0$ such that $\underline{g}_{j,1} = \min_{\Omega_{j,1}\times\Omega_{j0}} g_{j,1}(x_{j,1})$ and $\bar{g}_{j,1} = \max_{\Omega_{j,1}\times\Omega_{j0}} g_{j,1}(x_{j,1})$.

Proof: From $e_{j,1} = x_{j,1} - y_{j,d}$, we have $x_{j,1} = y_{j,d} + e_{j,1}$. Thus, one arrives

$$g_{j,1}(x_{j,1}) = \kappa_{j,1}(e_{j,1}, y_{j,d}) \tag{16}$$

with $\kappa_{j,1}(\cdot)$ being a continuous function. Note that $\Omega_{j,1} \times \Omega_{j0}$ is compact because $\Omega_{j,1}$ and Ω_{j0} are compact. As from (19) we have that all the variables of $\kappa_{j,1}(\cdot)$ belong to $\Omega_{j,1} \times \Omega_{j0}$, $\kappa_{j,1}(\cdot)$ has maximum $\bar{g}_{j,1}$ and minimum $\underline{g}_{j,1}$ in $\Omega_{j,1} \times \Omega_{j0}$. Therefore, it holds that

$$0 < \underline{g}_{j,1} \le g_{j,1}(x_{j,1}) \le \bar{g}_{j,1}, \quad x_{j,1} \in \Omega_{j,1} \times \Omega_{j0}. \tag{17}$$

■

Design the following intermediate controller and adaptation laws:

$$\begin{aligned} s_{j,1} = &-k_{j,1}e_{j,1} - \frac{\hat{\theta}_{j,1}e_{j,1}}{2a^2_{j,1}} - \hat{\delta}_{j,1}\tanh\left(\frac{e_{j,1}}{\varsigma_{j,1}}\right) \\ &- \phi_{j,1}\dot{y}_{j,d}\tanh\left(\frac{e_{j,1}\dot{y}_{j,d}}{\varsigma_{j,1}}\right) \end{aligned} \tag{18}$$

$$\dot{\hat{\theta}}_{j,1} = \frac{\eta_{j,1}e^2_{j,1}}{2a^2_{j,1}} - \sigma_{j,1}\eta_{j,1}\hat{\theta}_{j,1} \tag{19}$$

$$\dot{\hat{\delta}}_{j,1} = \gamma_{j,1}e_{j,1}\tanh\left(\frac{e_{j,1}}{\varsigma_{j,1}}\right) - \sigma_{j,1}\gamma_{j,1}\hat{\delta}_{j,1} \tag{20}$$

where $k_{j,1} > 0$, $a_{j,1} > 0$, $\varsigma_{j,1} > 0$, $\eta_{j,1} > 0$, $\sigma_{j,1} > 0$, $\gamma_{j,1} > 0$, and $\phi_{j,1} \ge \underline{g}^{-1}_{j,1}$ are design constants. $\hat{\theta}_{j,1}$ and $\hat{\delta}_{j,1}$ are the estimation values of $\theta_{j,1} = \underline{g}^{-1}_{j,1}\|W_{j,1}\|^2 l_{j,1}$ and $\delta_{j,1} = \underline{g}^{-1}_{j,1}\left(\varepsilon^*_{j,1} + \Delta^*_{j,1}\right)$, respectively, where $l_{j,1}$ is the dimension of $\bar{\varphi}_{j,1}(\bar{x}_{j,1})$. Because (19) and (20) are first-order systems with non-negative input, one has $\hat{\theta}_{j,1}(t) \ge 0$ and $\hat{\delta}_{j,1}(t) \ge 0$ for $\forall t \ge 0$ by selecting $\hat{\theta}_{j,1}(0) = 0$ and $\hat{\delta}_{j,1}(0) = 0$.

Define the filter error $\beta_{j,2} = \zeta_{j,2} - s_{j,1}$, which yields $\dot{\zeta}_{j,2} = -\beta_{j,2}/\tau_{j,2}$ and

$$\dot{\beta}_{j,2} = -\frac{\beta_{j,2}}{\tau_{j,2}} + \chi_{j,2}\Big(e_{j,1}, e_{j,2}, \beta_{j,2}, \hat{\theta}_{j,1}, \hat{\delta}_{j,1}, y_{j,d}, \dot{y}_{j,d}, \ddot{y}_{j,d}\Big) \tag{21}$$

where $\chi_{j,2}(\cdot)$ is a continuous function to be utilized later in the stability analysis.

According to Lemma 2, one has

$$e_{j,1}W_{j,1}^T\bar{\varphi}_{j,1}(x_{j,1}) \le \frac{e_{j,1}^2\|W_{j,1}\|^2}{2a_{j,1}^2}\bar{\varphi}_{j,1}^T(x_{j,1})\bar{\varphi}_{j,1}(x_{j,1}) + \frac{a_{j,1}^2}{2} \tag{22}$$

with $a_{j,1} > 0$ being a design constant. It holds that $\bar{\varphi}_{j,1}^T(x_{j,1})\bar{\varphi}_{j,1}(x_{j,1}) \le l_{j,1}$ since $\bar{\varphi}_{j,1}(x_{j,1}) = [\varphi_{j,1,1}(x_{j,1}),\ldots,\varphi_{j,1,l_{j,1}}(x_{j,1})]^T$ and $|\varphi_{j,1,n}(x_{j,1})| \le 1$, for $n = 1,\ldots,l_{j,1}$. Then, we have

$$e_{j,1}W_{j,1}^T\bar{\varphi}_{j,1}(x_{j,1}) \le \frac{e_{j,1}^2\|W_{j,1}\|^2}{2a_{j,1}^2}l_{j,1} + \frac{a_{j,1}^2}{2}. \tag{23}$$

Noting that $x_{j,2} = e_{j,2} + \beta_{j,2} + s_{j,1}$, $\phi_{j,1}\underline{g}_{j,1} \ge 1$ and (15), one reaches

$$\begin{aligned}\dot{V}_{e_{j,1}} \le& -k_{j,1}\underline{g}_{j,1}e_{j,1}^2 - \frac{\underline{g}_{j,1}\hat{\theta}_{j,1}e_{j,1}^2}{2a_{j,1}^2} + \frac{e_{j,1}^2\|W_{j,1}\|^2}{2a_{j,1}^2}l_{j,1}\\ &- \underline{g}_{j,1}e_{j,1}\hat{\delta}_{j,1}\tanh\left(\frac{e_{j,1}}{\varsigma_{j,1}}\right) + e_{j,1}e_{j,2}g_{j,1}(x_{j,1})\\ &- e_{j,1}\dot{y}_{j,d}\tanh\left(\frac{e_{j,1}\dot{y}_{j,d}}{\varsigma_{j,1}}\right) + e_{j,1}g_{j,1}(x_{j,1})\beta_{j,2}\\ &+ \frac{a_{j,1}^2}{2} + |e_{j,1}|\left(\varepsilon_{j,1}^* + \Delta_{j,1}^*\right) - e_{j,1}\dot{y}_{j,d}.\end{aligned} \tag{24}$$

Take the Lyapunov function as

$$V_{j,1} = V_{e_{j,1}} + \frac{\underline{g}_{j,1}\tilde{\delta}_{j,1}^2}{2\gamma_{j,1}} + \frac{\underline{g}_{j,1}\tilde{\theta}_{j,1}^2}{2\eta_{j,1}} + \frac{1}{2}\beta_{j,2}^2 \tag{25}$$

where $\tilde{\delta}_{j,1} = \delta_{j,1} - \hat{\delta}_{j,1}$ and $\tilde{\theta}_{j,1} = \theta_{j,1} - \hat{\theta}_{j,1}$.

Using the adaptation laws (19) and (20), and Lemma 1 yields

$$\begin{aligned}\dot{V}_{j,1} \le& -k_{j,1}\underline{g}_{j,1}e_{j,1}^2 + e_{j,1}e_{j,2}g_{j,1}(x_{j,1})\\ &+ \sigma_{j,1}\underline{g}_{j,1}\left(\tilde{\theta}_{j,1}\hat{\theta}_{j,1} + \tilde{\delta}_{j,1}\hat{\delta}_{j,1}\right) - \frac{\beta_{j,2}^2}{\tau_{j,2}}\\ &+ 0.2785\varsigma_{j,1}\left(\varepsilon_{j,1}^* + \Delta_{j,1}^* + 1\right) + e_{j,1}g_{j,1}(x_{j,1})\beta_{j,2}\\ &+ \left|\beta_{j,2}\chi_{j,2}(\cdot)\right| + \frac{a_{j,1}^2}{2}.\end{aligned} \tag{26}$$

Step j, i_j ($2 \le i_j \le \rho_j - 1$, $j = 1,\ldots,m$): The design process for step i_j follows recursively from step 1. From $e_{j,i_j} = x_{j,i_j} - \zeta_{j,i_j}$ and (12), the dynamics of e_{j,i_j} can be written as

$$\dot{e}_{j,i_j} = W_{j,i_j}^T\bar{\varphi}_{j,i_j}(\bar{x}_{j,i_j}) + \varepsilon_{j,i_j} + g_{j,i_j}(\bar{x}_{j,i_j})x_{j,i_j+1} + \Delta_{j,i_j} - \dot{\zeta}_{j,i_j} \tag{27}$$

where ε_{j,i_j} is such that $|\varepsilon_{j,i_j}| \le \varepsilon_{j,i_j}^*$ with ε_{j,i_j}^* a positive constant.

Choose the quadratic function as follows:

$$V_{e_{j,i_j}} = \frac{1}{2}e_{j,i_j}^2. \tag{28}$$

Using (27), the time derivative of $V_{e_{j,i_j}}$ is

$$\begin{aligned}\dot{V}_{e_{j,i_j}} =& e_{j,i_j}\Big(W_{j,i_j}^T\bar{\varphi}_{j,i_j}(\bar{x}_{j,i_j}) + g_{j,i_j}(\bar{x}_{j,i_j})x_{j,i_j+1}\\ &+ \Delta_{j,i_j} + \varepsilon_{j,i_j} - \dot{\zeta}_{j,i_j}\Big).\end{aligned} \tag{29}$$

We can now design the intermediate controller and adaptation laws as

$$\begin{aligned}s_{j,i_j} =& -k_{j,i_j}e_{j,i_j} - \frac{\hat{\theta}_{j,i_j}e_{j,i_j}}{2a_{j,i_j}^2} - \hat{\delta}_{j,i_j}\tanh\left(\frac{e_{j,i_j}}{\varsigma_{j,i_j}}\right)\\ &- \phi_{j,i_j}\frac{\beta_{j,i_j}}{\tau_{j,i_j}}\tanh\left(\frac{e_{j,i_j}\beta_{j,i_j}}{\tau_{j,i_j}\varsigma_{j,i_j}}\right)\end{aligned} \tag{30}$$

$$\dot{\hat{\theta}}_{j,i_j} = \frac{\eta_{j,i_j}e_{j,i_j}^2}{2a_{j,i_j}^2} - \sigma_{j,i_j}\eta_{j,i_j}\hat{\theta}_{j,i_j} \tag{31}$$

$$\dot{\hat{\delta}}_{j,i_j} = \gamma_{j,i_j}e_{j,i_j}\tanh\left(\frac{e_{j,i_j}}{\varsigma_{j,i_j}}\right) - \sigma_{j,i_j}\gamma_{j,i_j}\hat{\delta}_{j,i_j} \tag{32}$$

where $k_{j,i_j} > 0$, $a_{j,i_j} > 0$, $\varsigma_{j,i_j} > 0$, $\eta_{j,i_j} > 0$, $\sigma_{j,i_j} > 0$, $\gamma_{j,i_j} > 0$, and $\phi_{j,i_j} \ge \underline{g}_{j,i_j}^{-1}$ are design parameters. $\hat{\theta}_{j,i_j}$ and $\hat{\delta}_{j,i_j}$ are the estimates of $\theta_{j,i_j} = \underline{g}_{j,i_j}^{-1}\|W_{j,i_j}\|^2 l_{j,i_j}$ and $\delta_{j,i_j} = \underline{g}_{j,i_j}^{-1}(\varepsilon_{j,i_j}^* + \Delta_{j,i_j}^*)$, respectively, where l_{j,i_j} is the dimension of $\bar{\varphi}_{j,i_j}(\bar{x}_{j,i_j})$.

Define the filter errors $\beta_{j,i_j+1} = \zeta_{j,i_j+1} - s_{j,i_j}$. Invoking (11), we have $\dot{\zeta}_{j,i_j+1} = -\beta_{j,i_j+1}/\tau_{j,i_j+1}$ and

$$\begin{aligned}\dot{\beta}_{j,i_j+1} =& -\frac{\beta_{j,i_j+1}}{\tau_{j,i_j+1}} + \chi_{j,i_j+1}\\ &\times\left(\bar{e}_{j,i_j+1}, \bar{\beta}_{j,i_j+1}, \bar{\hat{\theta}}_{j,i_j}, \bar{\hat{\delta}}_{j,i_j}, y_{j,d}, \dot{y}_{j,d}, \ddot{y}_{j,d}\right)\end{aligned} \tag{33}$$

where $\chi_{j,i_j+1}(\cdot)$ is a continuous function whose arguments are defined later.

Along similar lines as Lemma 4, from $e_{j,i_j} = x_{j,i_j} - \zeta_{j,i_j}$ and $\beta_{j,i_j} = \zeta_{j,i_j} - s_{j,i_j-1}$, we have $x_{j,i_j} = e_{j,i_j} + \beta_{j,i_j} + s_{j,i_j-1}$. Observing (30), it can be seen that s_{j,i_j-1} is a continuous function with respect to e_{j,i_j-1}, $\hat{\theta}_{j,i_j-1}$, $\hat{\delta}_{j,i_j-1}$, and β_{j,i_j-1}. Therefore, the continuous function $g_{j,i_j}(\bar{x}_{j,i_j})$ can be expressed as

$$g_{j,i_j}(\bar{x}_{j,i_j}) = \kappa_{j,i_j}\left(\bar{e}_{j,i_j}, \bar{\beta}_{j,i_j}, \bar{\hat{\theta}}_{j,i_j-1}, \bar{\hat{\delta}}_{j,i_j-1}, y_{j,d}\right) \tag{34}$$

where $\kappa_{j,i_j}(\cdot)$ is a continuous function and $\bar{e}_{j,i_j} = [e_{j,1}, e_{j,2},\ldots,e_{j,i_j}]^T$, $\bar{\beta}_{j,i_j+1} = [\beta_{j,2},\ldots,\beta_{j,i_j+1}]^T$, $\bar{\hat{\theta}}_{j,i_j-1} = [\hat{\theta}_{j,1},\ldots,\hat{\theta}_{j,i_j-1}]^T$, and $\bar{\hat{\delta}}_{j,i_j-1} = [\hat{\delta}_{j,1},\ldots,\hat{\delta}_{j,i_j-1}]^T$.

Define the compact sets Ω_{j,i_j} as follows:

$$\begin{aligned}\Omega_{j,i_j} := \Bigg\{&\left[\bar{e}_{j,i_j}^T, \bar{\beta}_{j,i_j}^T, \bar{\hat{\theta}}_{j,i_j-1}^T, \bar{\hat{\delta}}_{j,i_j-1}^T\right]^T | e_{j,i_j}^2\\ &+ \sum_{k=1}^{i_j-1}\left(e_{j,k}^2 + \beta_{j,k+1}^2 + \frac{\underline{g}_{j,i_j}\tilde{\delta}_{j,i_j}^2}{\gamma_{j,i_j}} + \frac{\underline{g}_{j,i_j}\tilde{\theta}_{j,i_j}^2}{\eta_{j,i_j}}\right) \le 2p\Bigg\}\end{aligned}$$

where $p > 0$ is the same design constant after (15). The following Lemma 5 holds for Ω_{j,i_j} and $g_{j,i_j}(\bar{x}_{j,i_j})$.

Lemma 5: The continuous function $g_{j,i_j}(\bar{x}_{j,i_j})$ has maximum and minimum in $\Omega_{j,i_j} \times \Omega_{j0}$, that is, there exist constants $\underline{g}_{j,i_j} > 0$ and $\bar{g}_{j,i_j} > 0$ such that $\underline{g}_{j,i_j} = \min_{\Omega_{j,i_j}\times\Omega_{j0}} g_{j,i_j}(\bar{x}_{j,i_j})$ and $\bar{g}_{j,i_j} = \max_{\Omega_{j,i_j}\times\Omega_{j0}} g_{j,i_j}(\bar{x}_{j,i_j})$.

Proof: $\Omega_{j,i_j} \times \Omega_{j0}$ is compact because Ω_{j,i_j} and Ω_{j0} are compact. From (34), it can be known that all the variables of $\kappa_{j,i_j}(\cdot)$ belong to $\Omega_{j,i_j} \times \Omega_{j0}$. Therefore, the continuous function $\kappa_{j,i_j}(\cdot)$

has maximum $\bar{g}_{j,i_j} > 0$ and minimum $\underline{g}_{j,i_j} > 0$ in $\Omega_{j,i_j} \times \Omega_{j0}$ and the following inequality holds:

$$\underline{g}_{j,i_j} \leq g_{j,i_j}(\bar{x}_{j,i_j}) \leq \bar{g}_{j,i_j},\ \bar{x}_{j,i_j} \in \Omega_{j,i_j} \times \Omega_{j0}. \tag{35}$$

∎

Take the Lyapunov function as

$$V_{j,i_j} = V_{e_{j,i_j}} + \frac{\underline{g}_{j,i_j}\tilde{\delta}^2_{j,i_j}}{2\gamma_{j,i_j}} + \frac{\underline{g}_{j,i_j}\tilde{\theta}^2_{j,i_j}}{2\eta_{j,i_j}} + \frac{1}{2}\beta^2_{j,i_j+1} \tag{36}$$

where $\tilde{\delta}_{j,i_j} = \delta_{j,i_j} - \hat{\delta}_{j,i_j}$ and $\tilde{\theta}_{j,i_j} = \theta_{j,i_j} - \hat{\theta}_{j,i_j}$.

Using Lemma 2 and similarly to step 1, one gets

$$e_{j,i_j} W^T_{j,i_j}\bar{\varphi}_{j,i_j}(\bar{x}_{j,i_j}) \leq \frac{e^2_{j,i_j}\|W_{j,i_j}\|^2}{2a^2_{j,i_j}} l_{j,i_j} + \frac{a^2_{j,i_j}}{2} \tag{37}$$

where $a_{j,i_j} > 0$ is a design constant and l_{j,i_j} is the dimension of $\bar{\varphi}_{j,i_j}(\bar{x}_{j,i_j}) = [\varphi_{j,i_j,1}(\bar{x}_{j,i_j}), \ldots, \varphi_{j,i_j,l_{j,i_j}}(\bar{x}_{j,i_j})]^T$ with $|\varphi_{j,i_j,n}(\bar{x}_{j,i_j})| \leq 1$, for $n = 1, \ldots, l_{j,i_j}$.

From (29), (33), (37), and $\phi_{j,i_j}\underline{g}_{j,i_j} \geq 1$, we can further have

$$\begin{aligned}
\dot{V}_{j,i_j} \leq & -k_{j,i_j}\underline{g}_{j,i_j} e^2_{j,i_j} + e_{j,i_j} g_{j,i_j}(\bar{x}_{j,i_j})\beta_{j,i_j+1} \\
& + e_{j,i_j} e_{j,i_j+1} g_{j,i_j}(\bar{x}_{j,i_j}) + |\beta_{j,i_j+1}\chi_{j,i_j+1}(\cdot)| \\
& + \left[\left|\frac{e_{j,i_j}\beta_{j,i_j}}{\tau_{j,i_j}}\right| - \frac{e_{j,i_j}\beta_{j,i_j}}{\tau_{j,i_j}}\tanh\left(\frac{e_{j,i_j}\beta_{j,i_j}}{\varsigma_{j,i_j}\tau_{j,i_j}}\right)\right] \\
& + \left(\varepsilon^*_{j,i_j} + \Delta^*_{j,i_j}\right)\left[|e_{j,i_j}| - e_{j,i_j}\tanh\left(\frac{e_{j,i_j}}{\varsigma_{j,i_j}}\right)\right] \\
& - \frac{\underline{g}_{j,i_j}}{\gamma_{j,i_j}}\tilde{\delta}_{j,i_j}\left[\dot{\hat{\delta}}_{j,i_j} - \gamma_{j,i_j} e_{j,i_j}\tanh\left(\frac{e_{j,i_j}}{\varsigma_{j,i_j}}\right)\right] \\
& - \frac{\underline{g}_{j,i_j}}{\eta_{j,i_j}}\tilde{\theta}_{j,i_j}\left[\dot{\hat{\theta}}_{j,i_j} - \frac{\eta_{j,i_j} e^2_{j,i_j}}{2a^2_{j,i_j}}\right] - \frac{\beta^2_{j,i_j+1}}{\tau_{j,i_j+1}} + \frac{a^2_{j,i_j}}{2}.
\end{aligned} \tag{38}$$

Substituting the adaptation laws (31) and (32) into (38) and invoking Lemma 1 yields

$$\begin{aligned}
\dot{V}_{j,i_j} \leq & -k_{j,i_j}\underline{g}_{j,i_j} e^2_{j,i_j} + e_{j,i_j} g_{j,i_j}(\bar{x}_{j,i_j})\beta_{j,i_j+1} - \frac{\beta^2_{j,i_j+1}}{\tau_{j,i_j+1}} \\
& + \sigma_{j,i_j}\underline{g}_{j,i_j}\left(\tilde{\theta}_{j,i_j}\hat{\theta}_{j,i_j} + \tilde{\delta}_{j,i_j}\hat{\delta}_{j,i_j}\right) + \frac{a^2_{j,i_j}}{2} \\
& + \left|\beta_{j,i_j+1}\chi_{j,i_j+1}(\cdot)\right| + e_{j,i_j} e_{j,i_j+1} g_{j,i_j}(\bar{x}_{j,i_j}) \\
& + 0.2785\varsigma_{j,i_j}\left(\varepsilon^*_{j,i_j} + \Delta^*_{j,i_j} + 1\right).
\end{aligned} \tag{39}$$

Step j, ρ_j (j = 1, ..., m): Using (7), (12), and $e_{j,\rho_j} = x_{j,\rho_j} - \zeta_{j,\rho_j}$, the dynamics of e_{j,ρ_j} are given by

$$\begin{aligned}
\dot{e}_{j,\rho_j} = & W^T_{j,\rho_j}\bar{\varphi}_{j,\rho_j}(\bar{x}_{j,\rho_j}) + \varepsilon_{j,\rho_j} + g_{j,\rho_j}(\bar{x}_{j,\rho_j}) g_{j0} v_j \\
& + g_{j,\rho_j}(\bar{x}_{j,\rho_j}) d_j + \Delta_{j,\rho_j} - \dot{\zeta}_{j,\rho_j}.
\end{aligned} \tag{40}$$

Consider the quadratic function as

$$V_{e_{j,\rho_j}} = \frac{1}{2}e^2_{j,\rho_j}. \tag{41}$$

Using (40), it yields that

$$\begin{aligned}
\dot{V}_{e_{j,\rho_j}} = & e_{j,\rho_j}\Big(W^T_{j,\rho_j}\bar{\varphi}_{j,\rho_j}(\bar{x}_{j,\rho_j}) + g_{j,\rho_j}(\bar{x}_{j,\rho_j}) g_{j0} v_j \\
& + g_{j,\rho_j}(\bar{x}_{j,\rho_j}) d_j + \Delta_{j,\rho_j} - \dot{\zeta}_{j,\rho_j} + \varepsilon_{j,\rho_j}\Big).
\end{aligned} \tag{42}$$

Similarly to step i_j, it follows from (30) and $x_{j,\rho_j} = e_{j,\rho_j} + \beta_{j,\rho_j} + s_{j,\rho_j-1}$ that the continuous function $g_{j,\rho_j}(\bar{x}_{j,\rho_j})$ can be rewritten as

$$g_{j,\rho_j}(\bar{x}_{j,\rho_j}) = \kappa_{j,\rho_j}\left(\bar{e}_{j,\rho_j}, \bar{\beta}_{j,\rho_j}, \bar{\hat{\theta}}_{j,\rho_j-1}, \bar{\hat{\delta}}_{j,\rho_j-1}, y_{j,d}\right) \tag{43}$$

where $\kappa_{j,\rho_j}(\cdot)$ is a continuous function.

Similar to the reasoning in Lemma 5, we know that, for the compact set $\Omega_{j,\rho_j} \times \Omega_{j0}$, there exist constants $\underline{g}_{j,\rho_j} > 0$ and $\bar{g}_{j,\rho_j} > 0$ such that

$$\underline{g}_{j,\rho_j} \leq g_{j,\rho_j}(\bar{x}_{j,\rho_j}) \leq \bar{g}_{j,\rho_j},\ \bar{x}_{j,\rho_j} \in \Omega_{j,\rho_j} \times \Omega_{j0}. \tag{44}$$

Let us now design the actual control law v_j and adaptation laws as

$$\begin{aligned}
v_j = & -\phi_{j,0}\left[c_{j,\rho_j} e_{j,\rho_j} + \frac{\hat{\theta}_{j,\rho_j} e_{j,\rho_j}}{2a^2_{j,\rho_j}} + \hat{\delta}_{j,\rho_j}\tanh\left(\frac{e_{j,\rho_j}}{\varsigma_{j,\rho_j}}\right)\right] \\
& - \phi_{j,0}\phi_{j,\rho_j}\frac{\beta_{j,\rho_j}}{\tau_{j,\rho_j}}\tanh\left(\frac{e_{j,\rho_j}\beta_{j,\rho_j}}{\tau_{j,\rho_j}\varsigma_{j,\rho_j}}\right)
\end{aligned} \tag{45}$$

$$\dot{\hat{\theta}}_{j,\rho_j} = \frac{\eta_{j,\rho_j} e^2_{j,\rho_j}}{2a^2_{j,\rho_j}} - \sigma_{j,\rho_j}\eta_{j,\rho_j}\hat{\theta}_{j,\rho_j} \tag{46}$$

$$\dot{\hat{\delta}}_{j,\rho_j} = \gamma_{j,\rho_j} e_{j,\rho_j}\tanh\left(\frac{e_{j,\rho_j}}{\varsigma_{j,\rho_j}}\right) - \sigma_{j,\rho_j}\gamma_{j,\rho_j}\hat{\delta}_{j,\rho_j} \tag{47}$$

where $k_{j,\rho_j} > 0$, $a_{j,\rho_j} > 0$, $\varsigma_{j,\rho_j} > 0$, $\eta_{j,\rho_j} > 0$, $\sigma_{j,\rho_j} > 0$, $\gamma_{j,\rho_j} > 0$, and $\phi_{j,\rho_j} \geq \underline{g}^{-1}_{j,\rho_j}$ are design constants. $\hat{\theta}_{j,\rho_j}$ and $\hat{\delta}_{j,\rho_j}$ are the estimation values of $\theta_{j,\rho_j} = \underline{g}^{-1}_{j,\rho_j}\|W_{j,\rho_j}\|^2 l_{j,\rho_j}$ and $\delta_{j,\rho_j} = \underline{g}^{-1}_{j,\rho_j}(\varepsilon^*_{j,\rho_j} + \Delta^*_{j,\rho_j})$, respectively, where l_{j,ρ_j} is the dimension of $\bar{\varphi}_{j,\rho_j}(\bar{x}_{j,\rho_j})$.

Take the following Lyapunov function candidate:

$$V_{j,\rho_j} = V_{e_{j,\rho_j}} + \frac{\underline{g}_{j,\rho_j}\tilde{\delta}^2_{j,\rho_j}}{2\gamma_{j,\rho_j}} + \frac{\underline{g}_{j,\rho_j}\tilde{\theta}^2_{j,\rho_j}}{2\eta_{j,\rho_j}} \tag{48}$$

where $\tilde{\delta}_{j,\rho_j} = \delta_{j,\rho_j} - \hat{\delta}_{j,\rho_j}$ and $\tilde{\theta}_{j,\rho_j} = \theta_{j,\rho_j} - \hat{\theta}_{j,\rho_j}$.

From (42), (48), and Assumption 3, it holds that

$$\begin{aligned}
\dot{V}_{j,\rho_j} \leq & e_{j,\rho_j} W^T_{j,\rho_j}\bar{\varphi}_{j,\rho_j}(\bar{x}_{j,\rho_j}) + e_{j,\rho_j} g_{j,\rho_j}(\bar{x}_{j,\rho_j}) g_{j0} v_j \\
& + e_{j,\rho_j} g_{j,\rho_j}(\bar{x}_{j,\rho_j}) d_j - e_{j,\rho_j}\dot{\zeta}_{j,\rho_j} \\
& + |e_{j,\rho_j}|\left(\varepsilon^*_{j,\rho_j} + \Delta^*_{j,\rho_j}\right) - \frac{\underline{g}_{j,\rho_j}}{\gamma_{j,\rho_j}}\dot{\hat{\delta}}_{j,\rho_j}\tilde{\delta}_{j,\rho_j} - \frac{\underline{g}_{j,\rho_j}}{\eta_{j,\rho_j}}\dot{\hat{\theta}}_{j,\rho_j}\tilde{\theta}_{j,\rho_j}.
\end{aligned} \tag{49}$$

Following the similar steps as in (22) and (37), one arrives:

$$e_{j,\rho_j} W_{j,\rho_j}\bar{\varphi}_{j,\rho_j}(\bar{x}_{j,\rho_j}) \leq \frac{e^2_{j,\rho_j}\|W_{j,\rho_j}\|^2}{2a^2_{j,\rho_j}} l_{j,\rho_j} + \frac{a^2_{j,\rho_j}}{2} \tag{50}$$

where $a_{j,\rho_j} > 0$ is design constant and l_{j,ρ_j} is the dimension of $\bar{\varphi}_{j,\rho_j}(\bar{x}_{j,\rho_j}) = [\varphi_{j,\rho_j,1}(\bar{x}_{j,\rho_j}), \ldots, \varphi_{j,\rho_j,l_{j,\rho_j}}(\bar{x}_{j,\rho_j})]^T$ with $|\varphi_{j,\rho_j,n}(\bar{x}_{j,\rho_j})| \leq 1$, for $n = 1, \ldots, l_{j,\rho_j}$.

Using (44) and substituting the actual control law (45) into (49) yields

$$\begin{aligned}\dot{V}_{j,\rho_j} \le& -c_{j,\rho_j}\underline{g}_{j,\rho_j}e^2_{j,\rho_j} + |e_{j,\rho_j}|\bar{g}_{j,\rho_j}D_j + \frac{a^2_{j,\rho_j}}{2}\\ &+ \left[\left|\frac{e_{j,\rho_j}\beta_{j,\rho_j}}{\tau_{j,\rho_j}}\right| - \frac{e_{j,\rho_j}\beta_{j,\rho_j}}{\tau_{j,\rho_j}}\tanh\left(\frac{e_{j,\rho_j}\beta_{j,\rho_j}}{\tau_{j,\rho_j}\varsigma_{j,\rho_j}}\right)\right]\\ &+ \left(\varepsilon^*_{j,\rho_j} + \Delta^*_{j,\rho_j}\right)\left[\left|e_{j,\rho_j}\right| - e_{j,\rho_j}\tanh\left(\frac{e_{j,\rho_j}}{\varsigma_{j,\rho_j}}\right)\right]\\ &- \frac{\underline{g}_{j,\rho_j}}{\gamma_{j,\rho_j}}\tilde{\delta}_{j,\rho_j}\left[\dot{\hat{\delta}}_{j,\rho_j} - \gamma_{j,\rho_j}e_{j,\rho_j}\tanh\left(\frac{e_{j,\rho_j}}{\varsigma_{j,\rho_j}}\right)\right]\\ &- \frac{\underline{g}_{j,\rho_j}}{\eta_{j,\rho_j}}\tilde{\theta}_{j,\rho_j}\left[\dot{\hat{\theta}}_{j,\rho_j} - \frac{\eta_{j,\rho_j}e^2_{j,\rho_j}}{2a^2_{j,\rho_j}}\right]. \end{aligned} \quad (51)$$

From Lemma 2, it follows that $|e_{j,\rho_j}|\bar{g}_{j,\rho_j}D_j \le (c_{j,0}/2) + [(e^2_{j,\rho_j}\bar{g}^2_{j,\rho_j}D^2_j)/2c_{j,0}]$, with $c_{j,0}$ being a positive constant. Substituting the adaptation laws (46) and (47) into (51), we can get

$$\begin{aligned}\dot{V}_{j,\rho_j} \le& -\left(c_{j,\rho_j}\underline{g}_{j,\rho_j} - \frac{\bar{g}^2_{j,\rho_j}D^2_j}{2c_{j,0}}\right)e^2_{j,\rho_j} + \frac{a^2_{j,\rho_j}}{2}\\ &+ \underline{g}_{j,\rho_j}\sigma_{j,\rho_j}\left(\hat{\delta}_{j,\rho_j}\tilde{\delta}_{j,\rho_j} + \hat{\theta}_{j,\rho_j}\tilde{\theta}_{j,\rho_j}\right) + \frac{c_{j,0}}{2}\\ &+ 0.2785\varsigma_{j,\rho_j}\left(\varepsilon^*_{j,\rho_j} + \Delta^*_{j,\rho_j} + 1\right). \end{aligned} \quad (52)$$

Let $c_{j,\rho_j} \ge \underline{g}^{-1}_{j,\rho_j}[(\bar{g}^2_{j,\rho_j}D^2_j)/2c_{j,0}] + k_{j,\rho_j}$ with k_{j,ρ_j} being positive design constant. We finally have

$$\begin{aligned}\dot{V}_{j,\rho_j} \le& -k_{j,\rho_j}\underline{g}_{j,\rho_j}e^2_{j,\rho_j} + 0.2785\varsigma_{j,\rho_j}\left(\varepsilon^*_{j,\rho_j} + \Delta^*_{j,\rho_j} + 1\right)\\ &+ \underline{g}_{j,\rho_j}\sigma_{j,\rho_j}\left(\hat{\delta}_{j,\rho_j}\tilde{\delta}_{j,\rho_j} + \hat{\theta}_{j,\rho_j}\tilde{\theta}_{j,\rho_j}\right) + \frac{a^2_{j,\rho_j}}{2} + \frac{c_{j,0}}{2}. \end{aligned} \quad (53)$$

B. Closed-Loop Stability Analysis

Theorem 1: Consider the closed-loop systems composed by the intermediate controllers (18) and (30), the actual control law (45), the parameter adaptation laws (19), (20), (31), (32), (46), and (47), and the filters (11). Let Assumptions 1–3 hold. For any given $p > 0$, $\hat{\theta}_{j,i_j}(0) \ge 0$, $\hat{\delta}_{j,i_j}(0) \ge 0$, and $V_j(0) \le p$, with V_j defined in (54), there exist adjustable parameters k_{j,i_j}, a_{j,i_j}, ς_{j,i_j}, η_{j,i_j}, σ_{j,i_j}, γ_{j,i_j}, ϕ_{j,i_j}, τ_{j,i_j}, and $\phi_{j,0}$ ($1 \le i_j \le \rho_j$, $j = 1, \ldots, m$) such that: 1) $\Omega_{j,\rho_j} \times \Omega_{j0}$ is an invariant compact set, that is, $V_j(t) \le p$ holds for $\forall t > 0$, and all signals of system (7) are SGUUB and 2) system output tracking error $e_{j,1}$ is such that $\lim_{t\to\infty}|e_{j,1}(t)| \le \mu_{j,1}$ with $\mu_{j,1} > 0$ a constant. Furthermore, the whole system output tracking errors $e_1 = [e_{1,1}, \ldots, e_{m,1}]^T$ are such that $\lim_{t\to\infty}\|e_1(t)\| \le \mu_1$, where μ_1 is a positive constant.

Proof: First, consider the following Lyapunov function candidate:

$$V_j = \frac{1}{2}\sum_{i_j=1}^{\rho_j}\left(e^2_{j,i_j} + \frac{\underline{g}_{j,i_j}}{\gamma_{j,i_j}}\tilde{\delta}^2_{j,i_j} + \frac{\underline{g}_{j,i_j}}{\eta_{j,i_j}}\tilde{\theta}^2_{j,i_j}\right) + \frac{1}{2}\sum_{i_j=1}^{\rho_j-1}\beta^2_{j,i_j+1}. \quad (54)$$

After summing (26), (39), and (53), we can obtain

$$\begin{aligned}\dot{V}_j \le& \sum_{i_j=1}^{\rho_j}\left[-k_{j,i_j}\underline{g}_{j,i_j}e^2_{j,i_j}\right] + \sum_{i_j=1}^{\rho_j-1}\left[\left|\beta_{j,i_j+1}\chi_{j,i_j+1}(\cdot)\right|\right]\\ &+ \sum_{i_j=1}^{\rho_j-1}\left[-\frac{\beta^2_{j,i_j+1}}{\tau_{j,i_j+1}} + \bar{g}_{j,i_j}\left(\left|e_{j,i_j+1}\right| + \left|\beta_{j,i_j+1}\right|\right)\left|e_{j,i_j}\right|\right]\\ &+ \sum_{i_j=1}^{\rho_j}\left[\sigma_{j,i_j}\underline{g}_{j,i_j}\left(\tilde{\theta}_{j,i_j}\hat{\theta}_{j,i_j} + \tilde{\delta}_{j,i_j}\hat{\delta}_{j,i_j}\right) + b_{j,i_j}\right] + \frac{c_{j,0}}{2} \end{aligned} \quad (55)$$

where $b_{j,i_j} = 0.2785\varsigma_{j,i_j}(\varepsilon^*_{j,i_j} + \Delta^*_{j,i_j} + 1) + (a^2_{j,i_j}/2)$.

By Lemma 2, it has

$$\begin{aligned}\left|\beta_{j,i_j+1}\chi_{j,i_j+1}(\cdot)\right| &\le \frac{\beta^2_{j,i_j+1}\chi^2_{j,i_j+1}(\cdot)}{2c_{j,1}} + \frac{c_{j,1}}{2}\\ \bar{g}_{j,i_j}\left|e_{j,i_j+1}\right|\left|e_{j,i_j}\right| &\le \frac{\bar{g}_{j,i_j}e^2_{j,i_j+1}}{2} + \frac{\bar{g}_{j,i_j}e^2_{j,i_j}}{2}\\ \bar{g}_{j,i_j}\left|e_{j,i_j}\right|\left|\beta_{j,i_j+1}\right| &\le \frac{c_{j,2}\bar{g}^2_{j,i_j}\beta^2_{j,i_j+1}}{2} + \frac{e^2_{j,i_j}}{2c_{j,2}} \end{aligned} \quad (56)$$

where $c_{j,1}$ and $c_{j,2}$ are positive constants.

Substituting inequalities (56) into (55) yields

$$\begin{aligned}\dot{V}_j \le& \sum_{i_j=1}^{\rho_j}\left[-k_{j,i_j}\underline{g}_{j,i_j}e^2_{j,i_j} - \frac{1}{2}\sigma_{j,i_j}\underline{g}_{j,i_j}\left(\tilde{\theta}^2_{j,i_j} + \tilde{\delta}^2_{j,i_j}\right)\right]\\ &+ \sum_{i_j=1}^{\rho_j-1}\left[-\frac{1}{\tau_{j,i_j+1}} + \frac{\chi^2_{j,i_j+1}(\cdot)}{2c_{j,1}} + \frac{c_{j,2}\bar{g}^2_{j,i_j}}{2}\right]\beta^2_{j,i_j+1}\\ &+ \sum_{i_j=1}^{\rho_j-1}\frac{e^2_{j,i_j}}{2c_{j,2}} + \sum_{i_j=1}^{\rho_j-1}\left[\frac{\bar{g}_{j,i_j}}{2}\left(e^2_{j,i_j+1} + e^2_{j,i_j}\right)\right] + C_j \end{aligned} \quad (57)$$

where $C_j = (1/2)\sum_{i_j=1}^{\rho_j}\sigma_{j,i_j}\underline{g}_{j,i_j}(\theta^2_{j,i_j} + \delta^2_{j,i_j}) + (c_{j,0}/2) + \sum_{i_j=1}^{\rho_j}b_{j,i_j} + ([(\rho_j - 1)c_{j,1}]/2)$.

Following the same reasoning as DSC design. we have that $|\chi_{j,i_j+1}(\cdot)|$ has maximum $M_{j,i_j+1} > 0$ in $\Omega_{j,i_j+1} \times \Omega_{j0}$.

Let $k_{j,i_j} \ge \underline{g}^{-1}_{j,i_j}(\bar{G}_j + (1/2c_{j,2}) + \alpha_j)$ and $[1/(\tau_{j,i_j+1})] \ge ([M^2_{j,i_j+1}(\cdot)]/2c_{j,1}) + [(c_{j,2}\bar{g}^2_{j,i_j})/2] + \alpha_j$ with $\bar{G}_j = \max\{\bar{g}_{j,1}, \ldots, \bar{g}_{j,\rho_j}\}$ and α_j being positive constants. Hence, the time derivative of V_j can be given by

$$\begin{aligned}\dot{V}_j \le& -\sum_{i_j=1}^{\rho_j}\left(\alpha_j e^2_{j,i_j}\right) - \frac{1}{2}\sigma_{j,i_j}\underline{g}_{j,i_j}\sum_{i_j=1}^{\rho_j}\left(\tilde{\theta}^2_{j,i_j} + \tilde{\delta}^2_{j,i_j}\right)\\ &- \sum_{i_j=1}^{\rho_j-1}\left(\alpha_j\beta^2_{j,i_j+1}\right) + C_j. \end{aligned} \quad (58)$$

We further have

$$\dot{V}_j \le -\vartheta_j V_j + C_j \quad (59)$$

where $\vartheta_j = \min\{2\alpha_j, \sigma_{j,i_j}\gamma_{j,i_j}, \sigma_{j,i_j}\eta_{j,i_j}\} > 0$, for $i_j = 1, \ldots, \rho_j$ and $j = 1, \ldots, m$.

Remark 5 after this proof explains that we can obtain $C_j/\vartheta_j \le p$. It follows from $C_j/\vartheta_j \le p$ and (59) that $\dot{V}_j \le 0$

on the level set $V_j = p$: As a consequence, the compact set $\Omega_{j,\rho_j} \times \Omega_{j0}$ is an invariant set and all signals are SGUUB.

Multiply (59) by $e^{\vartheta_j t}$ results in

$$V_j(t) \le \big[V_j(0) - \Gamma\big] e^{-\vartheta_j t} + \Gamma \quad (60)$$

where $\Gamma = C_j/\vartheta_j$ is a positive constant.

From (60), we know that $\lim_{t\to\infty} V_j(t) \le \lim_{t\to\infty} V_j(0)e^{-\vartheta_j t} + \Gamma \le \Gamma$, which leads to

$$\lim_{t\to\infty} |e_{j,1}| \le \lim_{t\to\infty} \sqrt{2V_j(t)} \le \sqrt{2\Gamma} = \mu_{j,1}. \quad (61)$$

Now we can extend the stability properties from the jth subsystem to the whole system (7). Take the Lyapunov function candidate $V = \sum_{j=1}^m V_j$. It follows from (59) that:

$$\dot{V} = \sum_{j=1}^{m} \dot{V}_j \le \sum_{j=1}^{m} \big[-\vartheta_j V_j + C_j\big] \le -\lambda V + \Lambda \quad (62)$$

with $\lambda = \min\{\vartheta_1, \ldots, \vartheta_m\}$ and $\Lambda = \sum_{j=1}^m C_j$. Then, we have

$$V(t) \le [V(0) - \Xi]e^{-\lambda t} + \Xi \quad (63)$$

where $\Xi = \Lambda/\lambda$ is a positive constant.

Similarly, we obtain $\lim_{t\to\infty} V(t) \le \lim_{t\to\infty} V(0)e^{-\lambda t} + \Xi \le \Xi$, which gives rise to

$$\lim_{t\to\infty} \|e_1(t)\| \le \lim_{t\to\infty} \sqrt{2V(t)} \le \sqrt{2\Xi} = \mu_1. \quad (64)$$

Following a similar analysis way as in [19], we conclude from (63) that the signals e_{j,i_j}, $\tilde{\delta}_{j,i_j}$, $\tilde{\theta}_{j,i_j}$, and β_{j,i_j+1}, along with v_j, s_{j,i_j}, and ζ_{j,i_j} in the closed-loop control system, $i_j = 1, \ldots, \rho_j$, $j = 1, \ldots, m$, are also SGUUB.

This completes the proof of Theorem 1. ■

Remark 4: It is worth remarking that the system stability analysis has been acquired with the help of (17), (35), and (44). Differently from the control gain function of [19]–[30], such inequalities are defined *a posteriori* on appropriately designed compact sets. Specifically, (17) only holds on $\Omega_{j,1} \times \Omega_{j0}$, (35) only holds on $\Omega_{j,i_j} \times \Omega_{j0}$, and (44) only holds on $\Omega_{j,\rho_j} \times \Omega_{j0}$. In other words, we have removed the assumption on *a priori* boundedness of $g_{j,i_j}(\bar{x}_{j,i_j})$ after making the most of the fact that $g_{j,i_j}(\bar{x}_{j,i_j})$ are bounded in $\Omega_{j,i_j} \times \Omega_{j0}$. Furthermore, it is also worth mentioning that $\Omega_{j,\rho_j} \subset \Omega_{j,\rho_j-1} \times \mathbb{R}^4 \subset \cdots \subset \Omega_{j,3} \times \mathbb{R}^{4(\rho_j-3)} \subset \Omega_{j,2} \times \mathbb{R}^{4(\rho_j-2)} \subset \Omega_{j,1} \times \mathbb{R}^{4(\rho_j-1)}$. Consequently, (17), (35), and (44) also hold in $\Omega_{j,\rho_j} \times \Omega_{j0}$ for all the time. This is because $\Omega_{j,\rho_j} \times \Omega_{j0}$ is an invariant compact set.

Remark 5: It should be noticed that C_j/ϑ_j can be made arbitrarily small by decreasing σ_{j,i_j}, a_{j,i_j}, and ς_{j,i_j}, and meanwhile increasing α_j, γ_{j,i_j}, and η_{j,i_j}. Therefore, the tracking error can be made arbitrarily small by appropriate choice of the design parameters. This will be further shown in the following numerical example.

IV. Simulation Examples

A. Numerical Example

Consider the following large-scale input-saturated nonlinear systems [19]:

$$\begin{cases} \dot{x}_{1,1} = x_{1,1}e^{-0.5x_{1,1}} + \left(1 + e^{x_{1,1}^2}\right)x_{1,2} + \Delta_{1,1}(t,x) \\ \dot{x}_{1,2} = \left|\cos\left(x_{1,1}\right)\right| x_{1,2}^2 + (3 + e^{x_{1,1}x_{1,2}})u_1(v_1(t)) \\ \qquad + \Delta_{1,2}(t,x) \\ \dot{x}_{2,1} = \left(2 + \sin\left(x_{1,2}x_{2,1}\right)^3\right) + e^{x_{1,1}}x_{2,2} + \Delta_{2,1}(t,x) \\ \dot{x}_{2,2} = x_{2,1}x_{2,2} + x_{1,1}x_{1,2} + (2 + e^{x_{1,1}x_{2,1}})u_2(v_2(t)) \\ \qquad + \Delta_{2,2}(t,x) \\ y_1 = x_{1,1}, y_2 = x_{2,1} \end{cases} \quad (65)$$

where $\Delta_{1,1} = 0.5\cos(x_{1,1}^2 x_{2,1}x_{2,2})\sin(t)$, $\Delta_{1,2} = 0.2\cos(x_{2,1}^2 + x_{1,2}^2)\cos(t)$, $\Delta_{2,1} = 0.6\sin(x_{1,1}x_{2,1}x_{1,2})\sin(t)$, and $\Delta_{2,2} = 0.5\sin(x_{2,1}^2 + x_{2,2}^2)(\sin(t))^2$. The desired tracking trajectories are $y_{1,d} = 0.5(\sin(t) + \sin(0.5t))$ and $y_{2,d} = \sin(t)$. Note that $f_{1,2} = |\cos(x_{1,1})|x_{1,2}^2$ is nondifferentiable at $x_{1,1} = (\pi/2)$ and the control gain functions $g_{1,1} = (1 + e^{x_{1,1}^2})$, $g_{1,2} = (3 + e^{x_{1,1}x_{1,2}})$, $g_{2,1} = e^{x_{1,1}}$, and $g_{2,2} = (2 + e^{x_{1,1}x_{2,1}})$ cannot be bounded *a priori*, but they obviously satisfy Assumption 1. Therefore, where existing approaches cannot be used, our approach can be applied to the nonlinear system (65). The inputs $u_1(v_1(t))$ and $u_2(v_2(t))$ are defined as in (2) with $u_{1,M} = u_{2,M} = 2$.

In accordance with Theorem 1, the intermediate controllers and actual controller are designed as

$$\begin{aligned} s_{1,1} &= -8e_{1,1} - \frac{\hat{\theta}_{1,1}e_{1,1}}{2 \times 0.25^2} - \hat{\delta}_{1,1}\tanh\left(\frac{e_{1,1}}{0.25}\right) \\ &\quad - 5\dot{y}_{1,d}\tanh\left(\frac{e_{1,1}\dot{y}_{1,d}}{0.25}\right) \\ s_{2,1} &= -3e_{2,1} - \frac{\hat{\theta}_{2,1}e_{2,1}}{2 \times 0.25^2} - \hat{\delta}_{2,1}\tanh\left(\frac{e_{2,1}}{0.25}\right) \\ &\quad - 3\dot{y}_{2,d}\tanh\left(\frac{e_{2,1}\dot{y}_{2,d}}{0.25}\right) \\ v_1 &= -3\left(5e_{1,2} + \frac{\hat{\theta}_{1,2}e_{1,2}}{2 \times 0.25^2} + \hat{\delta}_{1,2}\tanh\left(\frac{e_{1,2}}{0.5}\right)\right) \\ &\quad - 3\left(2\dot{\zeta}_{1,2}\tanh\left(\frac{e_{1,2}\dot{\zeta}_{1,2}}{0.5}\right)\right) \\ v_2 &= -5\left(3e_{2,2} + \frac{\hat{\theta}_{2,2}e_{2,2}}{2 \times 0.25^2} + \hat{\delta}_{2,2}\tanh\left(\frac{e_{2,2}}{0.5}\right)\right) \\ &\quad - 5\left(2\dot{\zeta}_{2,2}\tanh\left(\frac{e_{2,2}\dot{\zeta}_{2,2}}{0.5}\right)\right) \end{aligned}$$

where $e_{1,1} = x_{1,1} - y_{1,d}$, $e_{1,2} = x_{1,2} - \zeta_{1,2}$, $e_{2,1} = x_{2,1} - y_{2,d}$, and $e_{2,2} = x_{2,2} - \zeta_{2,2}$, and the adaptation laws are provided by (31), (32), (46), and (47) with design parameters $\eta_{1,1} = \eta_{1,2} = 2$, $\eta_{2,1} = \eta_{2,2} = 1.5$, $\sigma_{1,1} = \sigma_{1,2} = 0.1$, $\sigma_{2,1} = \sigma_{2,2} = 0.1$, $\gamma_{1,1} = 2$, $\gamma_{1,2} = \gamma_{2,2} = 1.5$, and $\gamma_{2,1} = 2.5$. Let the initial conditions be $[x_{1,1}(0), x_{1,2}(0), x_{2,1}(0), x_{2,2}(0)]^T = [-0.1, 0, -0.1, 0]^T$, $\hat{\theta}_{1,1}(0) = \hat{\theta}_{1,2}(0) = \hat{\theta}_{2,1}(0) = \hat{\theta}_{2,2}(0) = 0$, and $\hat{\delta}_{1,1}(0) = \hat{\delta}_{1,2}(0) = \hat{\delta}_{2,1}(0) = \hat{\delta}_{2,2}(0) = 0$. The resulting simulation results are presented in Figs. 1–5.

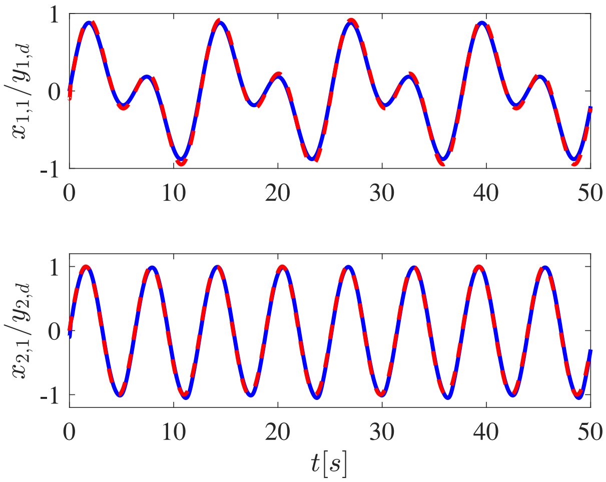


Fig. 1. Outputs $y_1(x_{1,1})$ and $y_2(x_{2,1})$ (dashed), and desired trajectories $y_{1,d}$ and $y_{2,d}$ (solid).

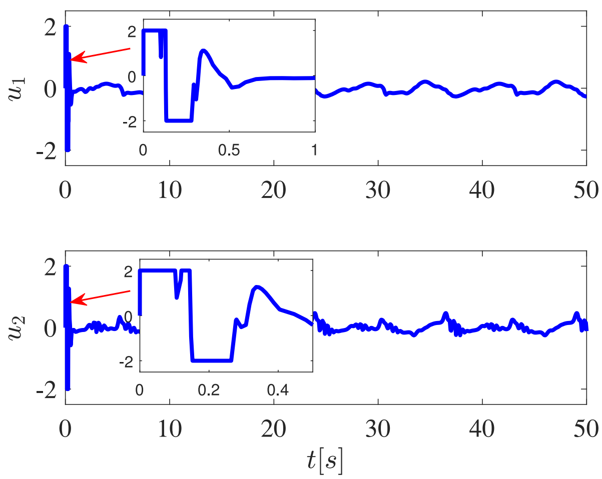


Fig. 2. System inputs u_1 and u_2.

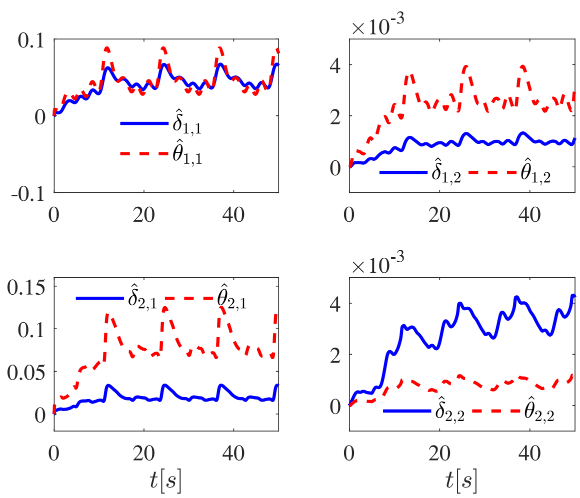


Fig. 3. Adaptation parameters $\hat{\delta}_{1,1}$, $\hat{\delta}_{1,2}$, $\hat{\delta}_{2,1}$, and $\hat{\delta}_{2,2}$.

It can be seen from Fig. 1 that the outputs y_1 and y_2 can follow the desired trajectories $y_{1,d}$ and $y_{2,d}$ with good tracking performance. Fig. 2 shows that the proposed controller works

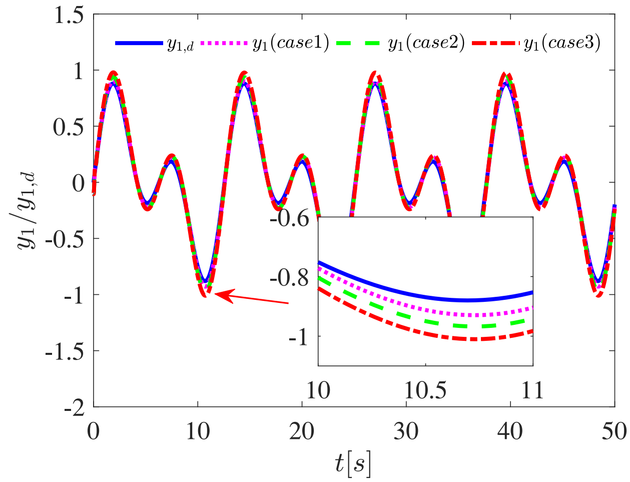


Fig. 4. Output y_1 under three cases.

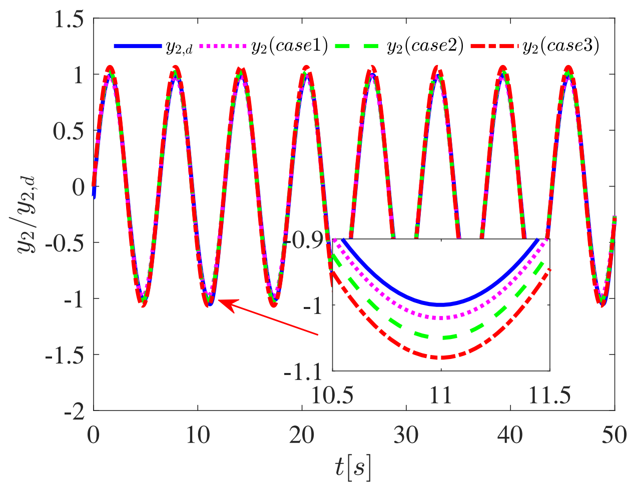


Fig. 5. Output y_2 under three cases.

well. Moreover, the adaptation parameters $\hat{\delta}_{j,i_j}$ and $\hat{\theta}_{j,i_j}(j = 1,2, i_j = 1,2)$ are presented in Fig. 3.

In order to further verify the effectiveness of the developed scheme with different design parameters, three different sets of parameters are taken into account.

Case 1: $\sigma_{1,1} = \sigma_{1,2} = \sigma_{2,1} = \sigma_{2,2} = 0.1$, $a_{1,1} = a_{1,2} = 0.25$, $a_{2,1} = a_{2,2} = 0.2$, $\varsigma_{1,1} = \varsigma_{1,2} = 0.35$; $\varsigma_{2,1} = \varsigma_{2,2} = 0.5$; $k_{1,1} = k_{2,1} = 8$, $k_{1,2} = k_{2,2} = 3$, $\gamma_{1,1} = \gamma_{1,2} = \gamma_{2,1} = \gamma_{2,2} = 2.5$, $\eta_{1,1} = \eta_{1,2} = 3$, $\eta_{2,1} = \eta_{2,2} = 2.5$, $\phi_{1,1} = \phi_{1,2} = \phi_{2,1} = \phi_{2,2} = 2$, and $\tau_{1,2} = \tau_{2,2} = 0.05$.

Case 2: $\sigma_{1,1} = \sigma_{1,2} = \sigma_{2,1} = \sigma_{2,2} = 0.25$, $a_{1,1} = a_{1,2} = 0.4$, $a_{2,1} = a_{2,2} = 0.5$, $\varsigma_{1,1} = \varsigma_{1,2} = 0.5$, $\varsigma_{2,1} = \varsigma_{2,2} = 0.75$, $k_{1,1} = k_{2,1} = 6$, $k_{1,2} = k_{2,2} = 2$, $\gamma_{1,1} = \gamma_{1,2} = \gamma_{2,1} = \gamma_{2,2} = 1.5$, $\eta_{1,1} = \eta_{1,2} = 2$, $\eta_{2,1} = \eta_{2,2} = 1.5$, $\phi_{1,1} = \phi_{1,2} = \phi_{2,1} = \phi_{2,2} = 2$, and $\tau_{1,2} = \tau_{2,2} = 0.05$.

Case 3: $\sigma_{1,1} = \sigma_{1,2} = \sigma_{2,1} = \sigma_{2,2} = 0.5$, $a_{1,1} = a_{1,2} = 0.5$,$a_{2,1} = a_{2,2} = 0.75$, $\varsigma_{1,1} = \varsigma_{1,2} = 0.7$, $\varsigma_{2,1} = \varsigma_{2,2} = 0.75$, $k_{1,1} = k_{2,1} = 4$, $k_{1,2} = k_{2,2} = 1.5$, $\gamma_{1,1} = \gamma_{1,2} = \gamma_{2,1} = \gamma_{2,2} = 1$, $\eta_{1,1} = \eta_{1,2} = 1$, $\eta_{2,1} = \eta_{2,2} = 0.5$, $\phi_{1,1} = \phi_{1,2} = \phi_{2,1} = \phi_{2,2} = 2$, and $\tau_{1,2} = \tau_{2,2} = 0.05$.

The system output responses are given in Figs. 4 and 5, which demonstrate the considerations in Remark 5 (tracking

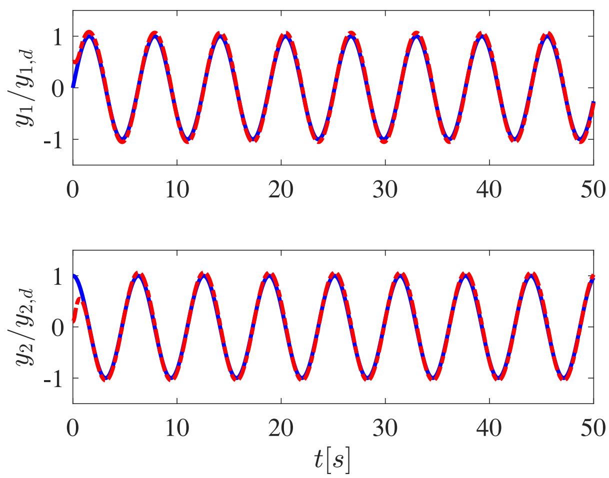


Fig. 6. Outputs y_1 and y_2 (dashed), and desired trajectories $y_{1,d}$ and $y_{2,d}$ (solid).

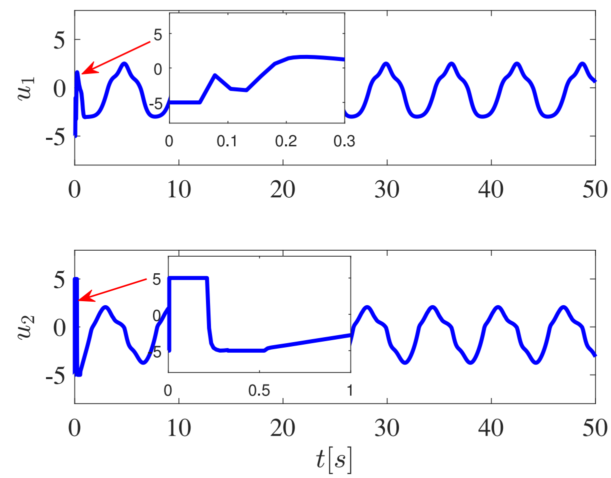


Fig. 7. System inputs u_1 and u_2.

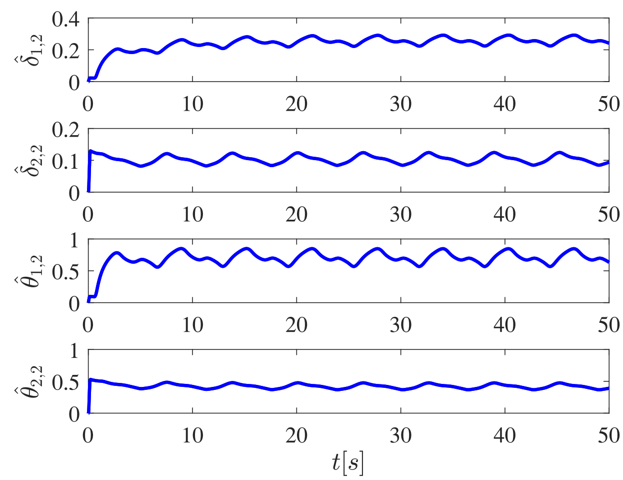


Fig. 8. Adaptation parameters $\hat{\delta}_{1,2}$, $\hat{\delta}_{2,2}$, $\hat{\theta}_{1,2}$, and $\hat{\theta}_{2,2}$.

errors is smaller after decreasing σ_{j,i_j}, a_{j,i_j}, and ς_{j,i_j}, and meanwhile increasing k_{j,i_j}, η_{j,i_j}, and γ_{j,i_j}).

B. Practical Example

To further validate the applicability of the proposed approach, we take the two inverted pendulums as a practical example as described in [11] and [12]. The input to each pendulum is the torque u_i $(i = 1, 2)$ with input saturation value $u_{1,M} = u_{2,M} = 5$. Define the state vectors as $[x_{1,1}, x_{1,2}]^T = [\theta_1, \dot{\theta}_1]^T$ (rad, rad/s) and $[x_{2,1}, x_{2,2}]^T = [\theta_2, \dot{\theta}_2]^T$ (rad, rad/s). The dynamic equations of the two inverted pendulums are [11], [12]

$$\begin{cases} \dot{x}_{1,1} = x_{1,2}, \ \ \dot{x}_{2,1} = x_{2,2} \\ \dot{x}_{1,2} = \left(\dfrac{m_1 g r}{J_1} - \dfrac{k r^2}{4 J_1}\right) \times \sin(x_{1,1}) \\ \qquad + \dfrac{k r}{2 J_1}(l - b) + \dfrac{u_1(v_1(t))}{J_1} + \dfrac{\sqrt{|x_{1,1}|}}{4 + x_{1,1}^2} \\ \dot{x}_{2,2} = \left(\dfrac{m_2 g r}{J_2} - \dfrac{k r^2}{4 J_2}\right) \times \sin(x_{2,1}) + \dfrac{k r}{2 J_2}(l - b) \\ \qquad + \dfrac{u_2(v_2(t))}{J_2} + \dfrac{\sqrt{3 x_{2,1} \sin(x_{2,1})}}{1 + x_{2,1}^2} \\ y_1 = x_{1,1}, \ \ y_2 = x_{2,1} \end{cases}$$

where $m_1 = 2$ kg and $m_2 = 2$ kg denote the inverted pendulums end masses, $k = 10$ N/m represents the spring constant. $J_1 = 1$ kg and $J_2 = 1$ kg are the moments of inertia, $r = 0.1$ m is the pendulum height, the natural length of the spring is $l = 0.5$ m, $g = 9.81$ m/s^2, and $b = 0.4$ m. The desired trajectories are $y_{1,d} = \sin(t)$ and $y_{2,d} = \cos(t)$.

According to Theorem 1, we design the intermediate controller as $s_{1,1} = -2e_{1,1} + \dot{y}_{1,d}$ and $s_{2,1} = -2e_{2,1} + \dot{y}_{2,d}$. The actual control laws are

$$v_1 = -1.5\left(2e_{1,1} + \frac{\hat{\theta}_{1,1} e_{1,1}}{2 \times 0.25^2} + \hat{\delta}_{1,1} \tanh\left(\frac{e_{1,1}}{0.5}\right)\right) - 1.5\left(2\dot{\zeta}_{1,2} \tanh\left(\frac{e_{1,1}\dot{\zeta}_{1,2}}{0.5}\right)\right)$$

$$v_2 = -3\left(3e_{2,1} + \frac{\hat{\theta}_{2,1} e_{2,1}}{2 \times 0.2^2} + \hat{\delta}_{2,1} \tanh\left(\frac{e_{2,1}}{0.5}\right)\right) - 3\left(2\dot{\zeta}_{2,2} \tanh\left(\frac{e_{2,1}\dot{\zeta}_{2,2}}{0.5}\right)\right)$$

where $e_{1,1} = x_{1,1} - y_{1,d}$ and $e_{2,1} = x_{2,1} - y_{2,d}$, and the adaptation laws are provided by (31), (32), (46), and (47), with design parameters $\eta_{1,2} = 1.5$, $\eta_{2,2} = 4$, $\sigma_{1,2} = \sigma_{2,2} = 0.1$, $\gamma_{1,2} = 1.5$, and $\gamma_{2,2} = 4$. Let the initial conditions be $[x_{1,1}(0), x_{1,2}(0), x_{2,1}(0), x_{2,2}(0)]^T = [0.5, 0.2, \ 0.1, 0.2]^T$, $\hat{\theta}_{1,2}(0) = \hat{\theta}_{2,2}(0) = 0$, and $\hat{\delta}_{1,2}(0) = \hat{\delta}_{2,2}(0) = 0$. Because the control gain functions are *a priori* bounded, this system is amenable for some comparisons with existing approaches. For comparison purposes, two approaches are considered: the method proposed here (scheme 1) and the hybrid output feedback controller of [11] (scheme 2). The simulation results are shown as Figs. 6–8 for the proposed approach, while the comparison on the tracking error is provided in Fig. 9. For scheme 2, the same design parameters provided in [11] have been adopted.

For scheme 1, the system output tracking responses are depicted in Fig. 6. Moreover, the evolution of the system inputs

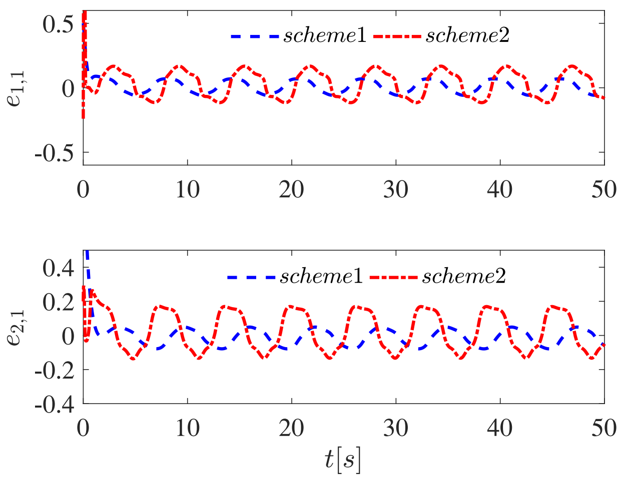


Fig. 9. Tracking errors of two schemes.

u_1 and u_2 and of the adaptation parameters $\hat{\delta}_{1,2}$, $\hat{\delta}_{2,2}$, $\hat{\theta}_{1,2}$, and $\hat{\theta}_{2,2}$ are presented in Figs. 7 and 8, respectively. Output tracking errors under two schemes are presented in Fig. 9. From Fig. 6, we know that good tracking performances have been achieved and the outputs y_1 and y_2 converge rapidly to the desired trajectories $y_{1,d}$ and $y_{2,d}$. From Fig. 9, we see that the proposed scheme 1 can achieve smaller tracking errors than scheme 2, which confirms good tracking performance of our approach.

V. Conclusion

An extended adaptive fuzzy DSC method has been designed for a less restrictive class of large-scale nonlinear systems with possibly unbounded control gain functions and input saturation. As compared with existing approaches in the literature, the restrictive assumption on *a priori* boundedness of the control gain functions has been removed by constructively introducing appropriate compact invariant sets. In other words, boundedness of the control gain function is derived *a posteriori* from the boundedness of the closed-loop state obtained in the control design. We believe that the following points are worth investigating in future research: 1) it is still unclear if set-invariance mechanisms can be adopted in prescribed performance control: studying this point would be relevant to address more general constraints and 2) it is still unclear if set-invariance mechanisms can be adopted in a distributed control setting, when the systems have to minimize a consensus error, in place of a tracking error: studying this point would be relevant to address more general large-scale systems.

References

[1] M. Kristic, I. Kanellakopoulos, and P. V. Kokotovic, *Nonlinear and Adaptive Control Design*. New York, NY, USA: Wiley, 1995.

[2] C. L. P. Chen, C.-Y. Zhang, L. Chen, and M. Gan, “Fuzzy restricted Boltzmann machine for the enhancement of deep learning,” *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 6, pp. 2163–2173, Dec. 2015.

[3] C. L. P. Chen, G.-X. Wen, Y. Liu, and Z. Liu, “Observer-based adaptive backstepping consensus tracking control for high-order nonlinear semi-strict-feedback multiagent systems,” *IEEE Trans. Cybern.*, vol. 46, no. 7, pp. 1591–1601, Jul. 2016.

[4] S. Tong, L. Zhang, and Y. Li, “Observed-based adaptive fuzzy decentralized tracking control for switched uncertain nonlinear large-scale systems with dead zones,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 46, no. 1, pp. 37–46, Jan. 2016.

[5] X. Zhao, H. Yang, and G. Zong, “Adaptive neural hierarchical sliding mode control of nonstrict-feedback nonlinear systems and an application to electronic circuits,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 7, pp. 1394–1404, Jul. 2017.

[6] Y.-J. Liu, S. Lu, D. Li, and S. Tong, “Adaptive controller design-based ABLF for a class of nonlinear time-varying state constraint systems,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 7, pp. 1546–1553, Jul. 2017.

[7] W. Wang, D. Wang, Z. Peng, and T. Li, “Prescribed performance consensus of uncertain nonlinear strict-feedback systems with unknown control directions,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 46, no. 9, pp. 1279–1286, Sep. 2016.

[8] H. Li, J. Yu, C. Hilton, and H. Liu, “Adaptive sliding-mode control for nonlinear active suspension vehicle systems using T–S fuzzy approach,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 60, no. 8, pp. 3328–3338, Aug. 2013.

[9] Y.-J. Liu, S. Lu, and S. Tong, “Neural network controller design for an uncertain robot with time-varying output constraint,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 8, pp. 2060–2067, Aug. 2017.

[10] Z. Liu, X. Dong, W. Xie, Y. Chen, and H. Li, “Adaptive fuzzy control for pure-feedback nonlinear systems with nonaffine functions being semi-bounded and indifferentiable,” *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 2, pp. 395–408, Apr. 2018.

[11] Y. Li, S. Tong, and T. Li, “Hybrid fuzzy adaptive output feedback control design for uncertain MIMO nonlinear systems with time-varying delays and input saturation,” *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 4, pp. 841–853, Aug. 2016.

[12] Y. Li, S. Tong, and T. Li, “Composite adaptive fuzzy output feedback control design for uncertain nonlinear strict-feedback systems with input saturation,” *IEEE Trans. Cybern.*, vol. 45, no. 10, pp. 2299–2308, Oct. 2015.

[13] D. Swaroop, J. K. Hedrick, P. P. Yip, and J. C. Gerdes, “Dynamic surface control for a class of nonlinear systems,” *IEEE Trans. Autom. Control*, vol. 45, no. 10, pp. 1893–1899, Oct. 2000.

[14] M. Chen and S. S. Ge, “Direct adaptive neural control for a class of uncertain nonaffine nonlinear systems based on disturbance observer,” *IEEE Trans. Cybern.*, vol. 43, no. 4, pp. 1213–1225, Aug. 2013.

[15] M. Chen, T. Gang, and B. Jiang, “Dynamic surface control using neural networks for a class of uncertain nonlinear systems with input saturation,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 26, no. 9, pp. 2086–2097, Sep. 2015.

[16] X. Zhao, H. Yang, W. Xia, and X. Wang, “Adaptive fuzzy hierarchical sliding-mode control for a class of MIMO nonlinear time-delay systems with input saturation,” *IEEE Trans. Fuzzy Syst.*, vol. 25, no. 5, pp. 1062–1077, Oct. 2017.

[17] C. Wu, J. Liu, X. Jing, H. Li, and L. Wu, “Adaptive fuzzy control for nonlinear networked control systems,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 8, pp. 2420–2430, Aug. 2017.

[18] Q. Zhou, L. Wang, C. Wu, H. Li, and H. Du, “Adaptive fuzzy control for nonstrict-feedback systems with input saturation and output constraint,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 47, no. 1, pp. 1–12, Jan. 2017.

[19] T. Li, S. Tong, and G. Feng, “A novel robust adaptive-fuzzy-tracking control for a class of nonlinear multi-input/multi-output systems,” *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 1, pp. 150–160, Feb. 2010.

[20] S. I. Han and J. M. Lee, “Partial tracking error constrained fuzzy dynamic surface control for a strict feedback nonlinear dynamic system,” *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 5, pp. 1049–1061, Oct. 2014.

[21] Q. Zhou, P. Shi, Y. Tian, and M. Wang, “Approximation-based adaptive tracking control for MIMO nonlinear systems with input saturation,” *IEEE Trans. Cybern.*, vol. 45, no. 10, pp. 2119–2128, Oct. 2015.

[22] T. Li, R. Li, and D. Wang, “Adaptive neural control of nonlinear MIMO systems with unknown time delays,” *Neurocomputing*, vol. 78, no. 1, pp. 83–88, Feb. 2012.

[23] H. Li, B. Lu, Q. Zhou, R. Lu, and L. Wang, “Adaptive fuzzy control of stochastic nonstrict-feedback nonlinear systems with input saturation,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 8, pp. 2185–2197, Aug. 2017.

[24] X. Zhao, H. Wang, H. R. Karimi, and Y. Zhu, “Adaptive neural control of MIMO nonstrict-feedback nonlinear systems with time delay,” *IEEE Trans. Cybern.*, vol. 46, no. 6, pp. 1337–1349, Jun. 2016.

[25] B. Chen, X. Liu, K. Liu, and C. Lin, "Adaptive fuzzy tracking control of nonlinear MIMO systems with time-varying delays," *Fuzzy Sets Syst.*, vol. 217, pp. 1–21, Apr. 2013.

[26] J. Yu, P. Shi, W. Dong, and C. Lin, "Command filtering-based fuzzy control for nonlinear systems with saturation input," *IEEE Trans. Cybern.*, vol. 47, no. 9, pp. 2472–2479, Sep. 2017.

[27] T. Li, G. Feng, D. Wang, and S. Tong, "Neural-network-based simple adaptive control of uncertain multi-input multi-output non-linear systems," *IET Control Theory Appl.*, vol. 4, no. 9, pp. 1543–1557, Sep. 2010.

[28] S. Han, H. Ha, and J. Lee, "Fuzzy finite-time dynamic surface control for nonlinear large-scale systems," *Int. J. Fuzzy Syst.*, vol. 18, no. 4, pp. 570–584, Aug. 2016.

[29] S. Mehraeen, S. Jagannathan, and M. Crow, "Decentralized control of large scale interconnected systems using adaptive neural network-based dynamic surface control," in *Proc. IEEE Int. Joint Conf. Neural Netw.*, Jun. 2009, pp. 2058–2064.

[30] S. Mehraeen, S. Jagannathan, and M. L. Crow, "Power system stabilization using adaptive neural network-based dynamic surface control," *IEEE Trans. Power Syst.*, vol. 26, no. 2, pp. 669–680, May 2011.

[31] Z. Sun and Y. Liu, "Adaptive state-feedback stabilization for a class of high-order nonlinear uncertain systems," *Automatica*, vol. 43, no. 10, pp. 1772–1783, Oct. 2007.

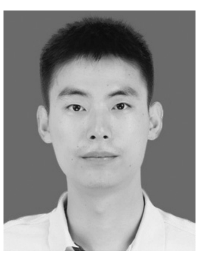
[32] K. Esfandiari, F. Abdollahi, and H. A. Talebi, "Adaptive control of uncertain nonaffine nonlinear systems with input saturation using neural networks," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 26, no. 10, pp. 2311–2322, Oct. 2015.

[33] M. Lv, Y. Wang, S. Baldi, Z. Liu, and Z. Wang, "A DSC method for strict-feedback nonlinear systems with possibly unbounded control gain functions," *Neurocomputing*, vol. 275, pp. 1383–1392, Jan. 2018.

[34] Y. Li, S. Tong, and T. Li, "Adaptive fuzzy output-feedback control for output constrained nonlinear systems in the presence of input saturation," *Fuzzy Sets Syst.*, vol. 248, pp. 138–155, Aug. 2014.

[35] W. Liu, J. Lu, Z. Zhang, and S. Xu, "Observer-based neural control for MIMO pure-feedback non-linear systems with input saturation and disturbances," *IET Control Theory Appl.*, vol. 10, no. 17, pp. 2314–2324, Nov. 2016.

[36] H. Wang, W. Liu, J. Qiu, and P. X. Liu, "Adaptive fuzzy decentralized control for a class of strong interconnected nonlinear systems with unmodeled dynamics," *IEEE Trans. Fuzzy Syst.*, vol. 26, no. 2, pp. 836–846, Apr. 2018.



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