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Theory for 1D full waveform inversion of surface GPR data

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Abstract—In one dimension, full waveform inversion is shown to be a linear problem under several conditions. I show that if the magnetic permeability can be assumed constant and electric conductivity to be zero, measuring the magnetic field at the surface or in the air suffices as input data. I present the theory using integral equations that describe the electric field inside the medium in terms of contrast sources. The electric field inside the medium can be computed from the measured magnetic field by solving a Marchenko equation. Once this field is known only the contrast function is unknown and can be found by matrix inversion. If the electric field is also measured the inverse problem can be solved recursively. In one dimension depth is intrinsically unknown and I use recording time as a replacing coordinate. After the electric permittivity is known as a function of one-way travel time from surface to a depth level inside the medium, the depth level can be found by an integral. This produces electric permittivity as a function of depth and full waveform inversion is complete. A simple numerical example demonstrates the method.

Index Terms—1D, full waveform inversion, autofocusing, GPR.

I. INTRODUCTION

Acoustic and electromagnetic 1D inversion have a long history. In fact the inverse problem was solved during the 1950's and 1960's [1]–[4]. They mathematically solved the inverse problem in one dimension and retrieved the potential that occurs in the Schrödinger equation from the measured scattered field. The numerical solution was not very stable, because they inverted for the scattering potential. In the late 1960's to early 1980's this solution method created a lot of activity in exploration geophysics and electromagnetic transmission line investigations [5]–[7]. They found that by scaling the field quantities and stretching the depth axis to a one-way travel time axis, the classical 1D wave equations for acoustic, elastic and electromagnetic problems, turn into a Schrödinger like equation. By inverting for the impedance, or separate parameters rather than the scattering potential, a numerically stable solution was obtained. Unfortunately, the solution could be found only when the zero-frequency was recorded in the data. This does not happen in practise and consequently, interest dwindled.

Recently, the Marchenko equation was reintroduced to geophysics to use the measured reflection response to construct a virtual experiment [8], [9]. This semi-virtual experiment is obtained by redatuming the receivers from the acquisition

surface to a location of choice inside the heterogeneous medium. The source remains at the acquisition surface but the receiver has become a virtual receiver, hence the name semi-virtual experiment. This result of this semi-experiment is the field necessary to solve an integral representation of the field in a contrast source formulation. For ground-penetrating radar it is important to include conductivity or more general dissipation/dispersion effects, for which a general theory for acoustic waves and electromagnetic waves can be found in [10], [11]. Retrieving virtual vertical radar profiles in a dissipative medium requires access at two side of the medium and the full scattering matrix as input data. This formulation avoids the occurrence of artefacts that would occur when the measured surface reflection response would be used to make an image using a standard (model-driven) linear sampling [12], [13] and linearised inversion [14]. A similar approach to inversion was developed for 1D acoustic waves but with a different solution strategy that requires much more computation time [15].

First I formulate an integral equation with travel time as a coordinate rather than depth that is linear in the impedance contrast function. The fields are scaled electric and magnetic fields such that they have the same units. I then derive a Volterra integral equation with the aid of the transmission response. This constitutes an efficient formulation for solving the inverse problem recursively. This is possible if the impulse response as if generated by a source at the surface and received by a virtual receiver inside the medium. I then show how these necessary fields can be obtained by filtering the data through Marchenko redatuming. Finally, the procedure for the inversion is given and illustrated with a numerical example from which conclusions are drawn.

II. INTEGRAL EQUATION FOR IMPEDANCE CONTRAST

Let the medium have depth dependent electric permittivity and magnetic permeability between $0 < z < z_m$, with $z = 0$ denoting the depth level of the acquisition surface and z_m , the depth level below the deepest reflecting boundary of interest. Above the surface, for $z < 0$, and below the bottom reflector, for $z > z_m$, the half spaces are homogeneous. The conductivity is assumed zero throughout the medium. The field satisfies the following 1D wave equation

$$\partial_z \begin{pmatrix} \hat{\mathcal{E}} \\ \hat{\mathcal{H}} \end{pmatrix} + i\omega \begin{pmatrix} 0 & \mu(z) \\ \varepsilon(z) & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathcal{E}} \\ \hat{\mathcal{H}} \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{\mathcal{J}} \end{pmatrix}, \quad (1)$$

where i is the imaginary unit, $\omega = 2\pi f$ is angular frequency, f being natural frequency, $\varepsilon(z)$ is the electric permittivity, and $\mu(z)$, the magnetic permeability and both can depend on depth, z . In the time domain the source \mathcal{J} emits a time function and operates at $z = 0$ taken in the limit from above such that it is located just above the receivers that are located also at $z = 0$. Because the idea is to connect the equation to reflection measurements, which take place in time, I change the depth coordinate to time with coordinate τ to denote vertical travel time. The modification is $dz = c(\tau)d\tau$ and therefore $\partial_z = \sqrt{\varepsilon_r(\tau)\mu_r(\tau)}\partial_\tau/c_0$. The second modification is to balance field amplitudes such that $\hat{E} = \sqrt{Y}\hat{\mathcal{E}}$ and $\hat{H} = \sqrt{Z}\hat{\mathcal{H}}$, with $Y = 1/Z = \sqrt{\varepsilon/\mu}$, is the plane wave admittance and Z the impedance. Applying the change of coordinate and scaling field amplitude to equation (1) leads to

$$\partial_z \begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix} + i\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{J} \end{pmatrix} + \begin{pmatrix} \chi & 0 \\ 0 & -\chi \end{pmatrix} \begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix}, \quad (2)$$

where $\chi(\tau) = \partial_\tau \log(Y)/2$, in which \log denotes natural logarithm, is the contrast function and can be seen as the generalised reflectivity. The point of this coordinate change is that in 1D there is no depth information to be retrieved from the data by independent means, whereas time is used to make the measurements. Rewriting equation (2) as an integral equation with the aid of the Green's function is well known. We then get

$$\hat{\mathbf{f}}(\tau) = \hat{\mathbf{f}}(\tau_m) \exp[-i\omega(\tau - \tau_m)] - \int_\tau^{\tau_m} \hat{\mathbf{L}}(\tau, \tau') \hat{\mathbf{f}}(\tau') d\tau', \quad (3)$$

and the field at the endpoint is the transmission response, given by $\hat{\mathbf{f}}(\tau_m) = (1, 1)^t \hat{T}(\tau_m)$, where the superscript t denotes matrix transposition. It is the total transmitted field across the layered medium and because we use scaled fields, the transmitted field is the same for each. All scattering is accounted for by the integral expression. The field vector is given by $\hat{\mathbf{f}} = (\hat{E}, \hat{H})^t$ and the fields can be expressed as $\hat{E} = \hat{f}^+ + \hat{f}^-$ and $\hat{H} = \hat{f}^+ - \hat{f}^-$. The matrix operator is given by

$$\hat{\mathbf{L}}(\tau, \tau') = \chi(\tau') \begin{pmatrix} \cos(\omega(\tau' - \tau)) & -i \sin(\omega(\tau' - \tau)) \\ i \sin(\omega(\tau' - \tau)) & -\cos(\omega(\tau' - \tau)) \end{pmatrix} \quad (4)$$

Equation (3) is the electromagnetic field integral equation in 1D that has the total electric and magnetic fields inside the integral. It is not suitable for modelling as it requires knowledge of the transmission response. However, if the transmission response is a minimum phase function, which means it has no zeros and also its inverse has no zeros for any real frequency. Hence for each frequency it is a number we can divide by. After this division the fields have become focusing functions. In that case, when the focusing functions are known, equation (3) is an equation that can be solved recursively for the contrast function χ . Then $\hat{\mathbf{f}}(\tau_m) = (1, 1)^t$ start the recursion if χ is known. Later $\hat{T}(\tau_m)$ can be recovered as

$$\hat{T}(\tau_m) = \frac{2}{\hat{E}(\tau_0) + \hat{H}(\tau_0)}. \quad (5)$$

Given measured electric and magnetic fields only at the acquisition surface $\tau = \tau_0$ the electromagnetic field and the contrast function inside the heterogeneous part are unknown and occur as a product. This is why full waveform inversion through the usual approach is a non-linear, ill-posed problem, with many possible solutions. The usual approach is to estimate a model that produces data that matches the measured data. Improvements of the model estimates are obtained by minimising a scalar. This scalar is computed from a global objective function that involves the difference between the measured data and the forward modelled data and possibly other constraints and/or regularisations. I call this full waveform inversion by iterative forward modelling. If the electric field would be known equation (3) would be an equation with only the contrast function as unknown and it would be a linear problem. The equation could be solved recursively. If the electric field inside the heterogeneous part of the model can be obtained from the measured data, the electric contrast function is obtained from the data by filtering or processing. I call this data-driven full waveform inversion.

Equation (3) can be expressed in terms of the up- and downgoing fields as

$$\hat{f}_1^\pm(\tau, \tau_m) = (1 \pm 1) \exp[i\omega(\tau_m - \tau)]/2 - \int_\tau^{\tau_m} \exp[\pm i\omega(\tau' - \tau)] \chi(\tau') \hat{f}_1^\mp(\tau', \tau_m) d\tau', \quad (6)$$

Equation (6) is a set of recursive equations coupling the upgoing and downgoing parts of the focusing function. Just below the bottom interface at τ_m the focusing function focuses,

$$\hat{f}_1^+(\tau_m, \tau_m) = 1, \quad \hat{f}_1^-(\tau_m, \tau_m) = 0. \quad (7)$$

A similar expression exists that involves the reflection response as is shown in a discrete model below.

III. OBTAINING THE ELECTRIC FIELD INSIDE THE MEDIUM FROM SURFACE REFLECTION DATA

The electric field that occurs in equations (3) and (6) is required for travel time converted depth levels $\tau_0 < \tau < \tau_m$ inside the heterogeneous part of the model below the acquisition surface. These are to be obtained from the measured data at the acquisition surface. The depth levels, τ , are therefore virtual receiver locations with the source still at acquisition level. The method for carrying out this step has become known as Marchenko redatuming [16], [17]. For our 1D electromagnetic problem in the new coordinate, it is instructive to look at it in a discrete model. In that case the contrast function reduces to the reflection coefficient for a reflecting boundary. let us assume we have an $N + 2$ layered medium embedded in two half spaces, The model has N layers with equal two-way travel time distance equal to the sampling time is Δt , as in the Goupillaud model [18]. This means that at every sample a new reflection can be recorded. In the frequency domain the time delay is represented by $z = \exp(-i\omega\Delta t/2)$. A layer n is defined as $\tau_{n-1} < \tau < \tau_n$ and the field values $\hat{f}_1^\pm(\tau_n)$ inside the layer are taken zero-phase at the bottom of the layer. The

following general representation is valid that connect the fields on either side of a boundary.

$$\begin{pmatrix} \hat{f}_1^+(\tau_{n+1}) \\ \hat{f}_1^-(\tau_{n+1}) \end{pmatrix} = \frac{1}{t_n} \begin{pmatrix} 1/z & r_n z \\ r_n z & 1/z \end{pmatrix} \begin{pmatrix} \hat{f}_1^+(\tau_n) \\ \hat{f}_1^-(\tau_n) \end{pmatrix}, \quad (8)$$

and the reflection and transmission coefficients are given by

$$r_n = (Y_n - Y_{n+1})/(Y_n + Y_{n+1}), \quad t_n = \sqrt{1 - r_n^2}. \quad (9)$$

The equation is relating the focusing functions at either side of a reflector is useful, because if both focusing functions are known the reflection coefficient r_n can be found directly. At a depth level where the Marchenko equation would be solved for $\hat{f}_1^\pm(\tau_{n+1})$ after having solved it for $\hat{f}_1^\pm(\tau_n)$, equation (8) can be used to write the Marchenko equation directly for retrieving r_n . The Marchenko redatuming is initialised at the top with

$$\hat{f}_1^+(\tau_0, \tau_0) = 1, \quad \hat{f}_1^-(\tau_0, \tau_0) = r_0, \quad (10)$$

which is possible, because the first reflection in the data is a primary reflection. Notice that the division by t_0 is not carried out and that it is not necessary. Then at the next level it is given by

$$\hat{f}_1^+(\tau_0, \tau_1) = \hat{f}_1^+(\tau_0, \tau_0)/z + r_1 z \hat{f}_1^-(\tau_0, \tau_0), \quad (11)$$

$$\hat{f}_1^-(\tau_0, \tau_1) = \hat{f}_1^-(\tau_0, \tau_0)/z + r_1 z \hat{f}_1^+(\tau_0, \tau_0). \quad (12)$$

Notice that in this step the time advance is carried out, but it is not necessary for the recursion to remain valid. Let us write the Marchenko equation for depth level τ_1 from which we can find the focusing functions from the reflection data,

$$f_1^-(\tau_0, \tau_1, t) = \int_{-\tau_1}^t f_1^+(\tau_0, \tau_1, t') R(\tau_0, t - t') dt' \quad (13)$$

$$f_{1m}^+(\tau_0, \tau_1, t) = \int_{-\tau_1}^t f_1^-(\tau_0, \tau_1, -t') R(\tau_0, t - t') dt' \quad (14)$$

for $t < \tau_1$. The function f_{1m}^+ has the first arrival removed from f_1^+ . From equations (11) and (12) we can write

$$\begin{aligned} f_1^+(\tau_0, \tau_1, t) &= f_1^+(\tau_0, \tau_0, t + \Delta t) + r_1 f_1^-(\tau_0, \tau_0, t - \Delta t), \\ f_1^-(\tau_0, \tau_1, t) &= f_1^-(\tau_0, \tau_0, t + \Delta t) + r_1 f_1^+(\tau_0, \tau_0, t - \Delta t). \end{aligned} \quad (15)$$

Because $f^\pm(\tau_0, \tau_0, t)$ are known, the only unknown in equations (13) and (14) is r_1 and only the convolution equation is necessary to find it. This scheme can recurse down into the layered medium. It is noted that a full Marchenko redatuming is possible anywhere in the medium with the aid of equations (13)-(14) after which a reflection coefficient can be found just below it. To find impedance as a function of one-way vertical travel time, the full recursion has to be performed.

IV. FINDING IMPEDANCE FROM THE REFLECTION COEFFICIENT AND TIME TO DEPTH CONVERSION

The inversion is carried out according to the above scheme to obtain the reflection coefficient as a function of one-way

travel time. The relative electric permittivity can then be found in an integral sense as

$$\varepsilon_r(\tau) = \exp\left(-4 \int_{\tau_0}^{\tau} r(\xi) d\xi\right), \quad (16)$$

given that $\varepsilon_r(\tau_0) = 1$, or in the discrete sense as

$$\varepsilon_r(n\Delta\tau) = \left(\prod_{m=0}^n \frac{1 - r_m}{1 + r_m}\right)^2. \quad (17)$$

Because depth and one-way travel time τ are related to each other as $dz = c_0 d\tau / \sqrt{\varepsilon_r(\tau)}$, I can recover the depth from

$$z = c_0 \int_{\xi=0}^{\tau} (\varepsilon_r(\xi))^{-1/2} d\xi. \quad (18)$$

With this step the inversion is complete.

V. NUMERICAL RESULTS

The numerical model eight reflecting boundaries over a stretch of just over 11 m. The source emits a known Ricker wavelet with 250 MHz centre frequency. The reflection response is shown in Figure 1. This data is used as $R(\tau_0, t)$ in equation (13) using the time advanced traveling impulse as initial estimate of $f_1^+(\tau_0, \tau_0, t)$ of equation (11) and the time delayed initial first reflection as estimate of $f_1^-(\tau_0, \tau_0, t)$ of equation (12). This first reflection strength and its timing can be found automatically from the input data. Then the scheme can start with time sampling of the data as constant travel time increment in depth direction. Equation (14) can then be used to test the accuracy of the obtained result and adjustments to retrieved reflection coefficient can be made if necessary. In the process of going into the layered medium, the focusing functions and reflection coefficients are obtained simultaneously as a function of one-way travel time in depth. Figure 2 shows the retrieved electric permittivity is shown as a function of vertical travel time (top) and depth (bottom). The exact relative permittivity profiles are shown in black solid

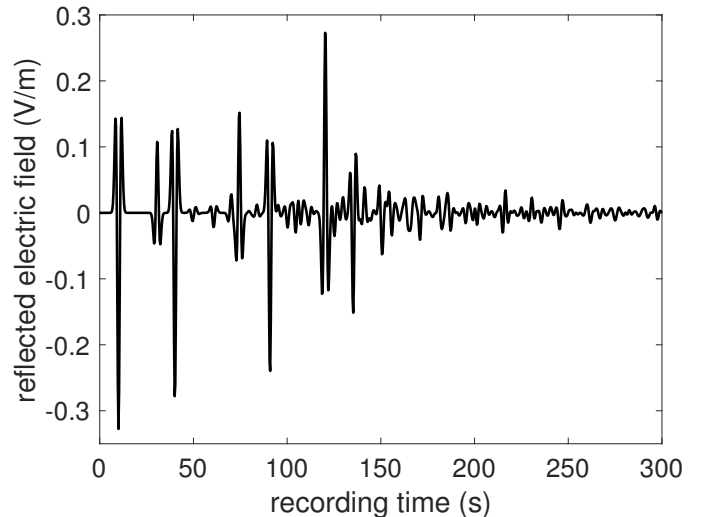


Fig. 1. The reflection response data used as input for inversion.

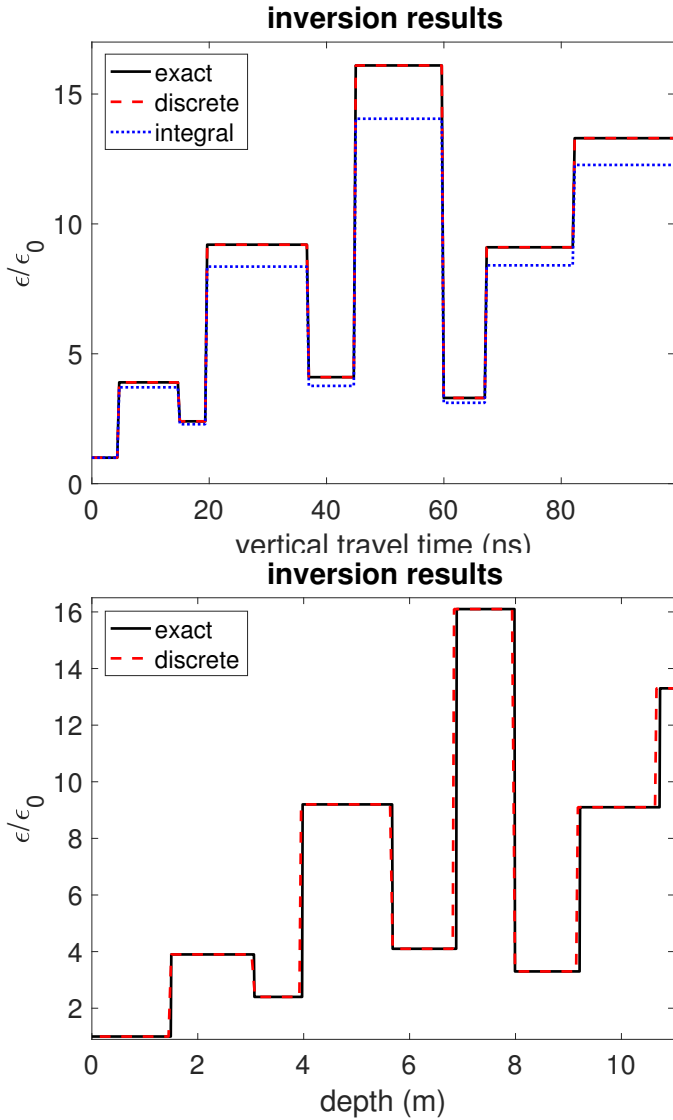


Fig. 2. The retrieved electric permittivity as a function of vertical travel time (top) and as a function of depth (bottom).

lines and the ones obtained by the filtering procedure and equation (17) in red dashed lines. In the plot where the electric permittivity is shown as function of travel time I also show the result in assuming the integral form given in equation (16) can be used. The reflection data was computed using reflectivity code for which reason the result of equation (17) is almost exact while the result of equation (16) shows the effect of discretisation errors, because taking finite difference on the logarithm of the square root of the inverse of impedance as an approximation its the derivative is not the same as computing the actual reflection coefficient. The conversion to depth shows slight deterioration with increasing depth as can be expected.

VI. CONCLUSION

I have shown 1D theory and a numerical example to retrieve the vertical permittivity profile as a function of vertical travel time. In this process the electromagnetic reflection response

generated by an electric dipole source, both located at the surface, is propagated into the subsurface. At each propagation step the local reflection coefficient is computed together with a focusing function. Because it is done recursively, no matrix inversion is required. By assuming only permittivity varies, it is possible to convert travel time to depth. The example shows that the theory works in 1D. Extension to 3D seems possible, which would lead to a potentially efficient data-driven inversion scheme.

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