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# Model Reduction in Co-Rotated Multi-Body Dynamics Based on the Dual Craig-Bampton Method

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**A new reduction method for dynamical analsis of multi-body systems is presented in this paper. It fundamentally differs from the ones previously published in the way kinematical constraints are handled. Our approach is based on component mode synthesis, but the specificity of articulated mechanism, in which components are assembled through connecting forces, is taken into account from the reduction level. This is achieved by using a set of modes that are consistent with this type of assembly, namely the residual flexibility modes and the free-free vibration modes. In this approach, Lagrange multipliers naturally appear as generalized reduced coordinates together with the vibration modes' amplitudes. An other specific difficulty pertaining to articulated systems is the configuration dependency of the residual flexibility modes. In order to overcome this obstacle a new strategy to handle kinematical constraints has been developed, enabling us to ignore the current direction of the constraint forces at the reduction level by splitting the constraint into local and global contributions. The derived reduced matrices are remarkably simple and the procedure is easy to implement. The validity of the approach will be investigated using the crank-slider benchmark.**



## I. Introduction

A common way to include flexible components in multi-body dynamic analysis consists in generating a linear Finite Element sub-model of the component, reduce it through component mode synthesis, then embed it in a floating frame in order to allow large overall displacements and rotations. Such a co-rotated super-element approach is very efficient and accurate when the deformations in the component remain small: it is often used in simulations of robots or for actuated aeronautical and aerospace systems (flapping wings, landing gears or deployable antennas).

In all co-rotated super-element techniques, the modal synthesis of the reduced component is based on dynamic modes and attachment modes, such that the reduced sub-model is described by internal general coordinates and physical displacements (and rotations) at its interface.[1,2] When assembled to neighboring components in the multi-body dynamic analysis, the interface degrees of freedom are usually connected through Lagrange multipliers, namely by applying interface forces in order to impose the inter-body constraints (revolute joints, sliders ...).

In this contribution we show that a natural way to apply mode synthesis for components of multi-body systems consists in using free interface modes and attachments modes so that the interface degrees of freedom of the reduced component are interface forces instead of interface displacements and rotations. Such a reduction method, called Dual Craig-Bampton reduction, was proposed earlier in[3] for linear systems and will be extended here to multi-body systems. The Dual Craig-Bampton reduction is in fact well suited for multi-bodies since the reduction basis corresponds to the response of the component to interface forces, namely the Lagrange multipliers in the case of multi-body dynamics. In this contribution we show that the reduced matrices (both locally and after co-rotation in the global frames) exhibit a high degree of sparsity thanks to the fact that the local free-interface modes and the rigid co-rotations are uncoupled in the equations of motion.

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In order to extend the Dual Craig-Bampton to multi-body simulations, we introduce a new variational principle based on a four field formulation of the interface constraints. Such a formulation allows us to separately consider the local and global degrees of freedom at the interface, thereby simplifying the application of model reduction in the local frame of the component. This formulation and the extension of the Dual Craig-Bampton to multi-body analysis leads to a clean, simple and accurate formulation of flexible multi-body dynamics. To the best of our knowledge, such a formulation has not been proposed previously.

In the remainder of this extended abstract we first recall how the dynamic equation of a single co-rotated flexible body (locally linear) can be derived. To simplify the discussion, the outline is given for motion in the two-dimensional space, the theory applying as well to the general three-dimensional case. Then in section II we introduce a four-field formulation allowing to distinguish between local and global interface degrees of freedom. In section III we apply the Dual Craig-Bampton reduction to a co-rotated component and show how the assembled equations simplifies for the dynamics analysis of a multi-body systems. The simplicity of the equations physically originates from the fact that our choice of reduction basis automatically leads to choosing the Tisserand reference axis for the floating frame.

In section IV we show results obtained by applying the proposed approach to a crank-slider benchmark. We compare the results so obtained to the outcome of simulations when no reduction is applied. The obtained results illustrate the efficiency and the accuracy of the proposed strategy when only a small number of deformation modes are considered in the local reduction basis.



## II. Notations and conventions

For greater clarity, let us start by introducing the conventions adopted by the authors. Throughout this article, vectors of Euclidian space will be underlined once, tensors underlined twice, matrices bolded, and uni-column matrices represented between braces:{}. For the sake of readability, the braces of the set of generalised coordinates will be dropped, and will instead be represented in bold character. Unless specified differently in the beginning of a section; superscript $s$ designate entities pertaining to the component $s$, whereas the collection of all the components' entities are represented without superscripts. For instance, if we denote $N_s$ the number of components, the stiffness matrix $\mathbf{K}$ and the generalised coordinate vector $\mathbf{p}$ write as follow: $\mathbf{K} = \begin{bmatrix} \mathbf{K}^1 & 0 & 0 \\ & \ddots & 0 \\ sym & & \mathbf{K}^{N_s} \end{bmatrix}$ and $\mathbf{p} = \begin{Bmatrix} \mathbf{p}^1 \\ \vdots \\ \mathbf{p}^{N_s} \end{Bmatrix}$.

## III. Equations of motion for a single unconstrained component

We recall, in this first section, the governing equations of motion for a single component free of kinematical constraints. The kinematics' description based on a co-rotational formalism is first briefly recalled in the following subsection. Based on this formulation the derivation of the equations of motion for a single component free of kinematical constraints is next shortly outlined. For a more detailed presentation the reader can refer to.[1,2]

We will in this study focus on two-dimensional flexible mechanisms. The more general case of flexible multi-body systems in three dimensions can also be treated with the proposed approach. However, for clarity, we outline the fundamental ideas of the reduction approach only in the two-dimensional case.

In this section, the superscript $s$ will be omitted as only a single component is considered.

### A. Floating frame of reference: kinematics

The key idea of the floating frame of reference approach relies on the use of a local frame associated to each flexible component. In the framework of linear mechanisms those frames are required to be such that the elastic displacements measured in the reference frame are small enough to use a linearised expression of strain. Let us assume that an inertial frame has been defined, and designate it by $\mathfrak{R}_0$ (we might later refer to it as the global frame). As depicted in Fig. 1, the position of a material point of the component can be expressed in terms of the global and local variables:

$$\underline{x} = \underline{R} + \mathbf{A}(\theta)\underline{w} = \underline{R} + \mathbf{A}(\theta)\left(\underline{r} + \mathbf{N}\{u_e\}\right) \tag{1}$$

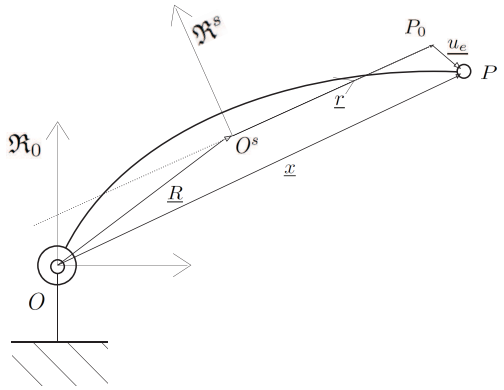


**Figure 1. Inertia and reference frames**



where the global variables $\underline{R}$ and $\theta$ denote the position and orientation of the floating frame. $\mathbf{A}(\theta)$ represents the rotation matrix transforming coordinates in the local frame into coordinates in the global inertia frame. Assuming the elastic displacements to be approximated by a finite element approach, the position $\underline{w}$ in the local frame can be approximated by $\underline{r} + \mathbf{N}\{u_e\}$, where $\underline{r}$ locates the position of the current point $P$ in the undeformed configuration, $\mathbf{N}$ designates the finite element shape function matrix, and $\{u_e\}$ the nodal elastic displacements. The expression of the velocity follows from Eq. (2) by deriving once according to time.

$$\underline{\dot{x}} = \underline{\dot{R}} + \dot{\theta}\mathbf{A}(\theta + \frac{\pi}{2})\underline{w} + \mathbf{A}(\theta)\underline{\dot{w}} \tag{2}$$

If we gather the generalised coordinates $\underline{R}$, $\theta$, and $\{u_e\}$ in $\mathbf{p}$:

$$\mathbf{p} = \left\{ \begin{array}{c} p_r \\ u_e \end{array} \right\} \qquad \text{with} \qquad p_r = \left\{ \begin{array}{c} R \\ \theta \end{array} \right\} \tag{3}$$

the position and velocity then write:

$$\underline{x} = \left[ \begin{array}{ccc} \mathbf{I} & 0 & \mathbf{A}(\theta)\mathbf{N} \end{array} \right] \mathbf{p} + \mathbf{A}(\theta)\underline{r} \tag{4}$$

$$\underline{\dot{x}} = \left[ \begin{array}{ccc} \mathbf{I} & \mathbf{A}(\theta + \frac{\pi}{2})\underline{w} & \mathbf{A}(\theta)\mathbf{N} \end{array} \right] \dot{\mathbf{p}} \tag{5}$$

Let us denote $n_e$ the number of elastic degrees of freedom; the generalised coordinate $\mathbf{p}$ is of dimension $n_e + 3$ . The three extra degrees of freedom represent the degrees of freedom describing the motion of the reference frame, which is not yet linked to the component. There is indeed an infinity of frames suitable for the floating frame reference approach, and associated elastic displacement . One has to make a choice by specifying how the frame is linked to the component, by imposing 3 kinematical constraints between the reference frame and the component. Once the reference is connected to the component, the dimension of $\mathbf{p}$ reduces to $n_e$, and the coordinates $p_r$ are representative of the overall motion of the component.

## B. Equations of motion

Using the kinematical description introduced in the previous section, the derivation of the equations of motion for a single component is briefly outlined in this subsection. The kinematical constraints linking the components are not considered yet. So that in the inertial frame $\mathfrak{R}_0$, the component is submitted only to internal stresses due to elastic deformations, and externally applied forces.

We denote $\Omega$ the domain delimiting this component, on which external volume forces $F_v$ are applied. It's boundary $\partial\Omega$ is assumed to be partitioned into a Neumann boundary $\partial_F\Omega$ on which external surface forces $F_s$ are applied, and a Dirichlet boundary $\partial_u\Omega$ on which the displacement field $\underline{u}_d$ is applied: $\partial\Omega = \partial_F\Omega \bigcup \partial_u\Omega$. The Lagrangian for this component writes:

$$\mathcal{L}(x, \dot{x}, t) = \mathcal{T} - (\mathcal{W}_F + \mathcal{W}_e) \tag{6}$$

$\mathcal{T}$ denotes the kinetic energy, $\mathcal{W}_F$ the work of applied forces, and $\mathcal{W}_e$ the work of the internal elastic stresses.

$$\mathcal{T} = \int_\Omega \rho \dot{\underline{x}}^2 dV \tag{7}$$

$$\mathcal{W}_e = \int_\Omega \frac{1}{2} tr[\underline{\underline{\sigma}} : \underline{\underline{\epsilon}}] dV \tag{8}$$

$$\mathcal{W}_F = -\left( \int_\Omega \underline{u}.\underline{\underline{F_v}} dV + \int_{\partial_F \Omega} \underline{u}.\underline{\underline{F_s}} dS \right) \tag{9}$$

The equations of motion in terms of the generalised coordinates $\mathbf{p}$, can be easily derived from the Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \mathbf{p}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{p}}} = 0 \tag{10}$$

The contributions of kinetic energy, strain energy, and applied forces' work being

$$\begin{aligned} \frac{\partial \mathcal{T}}{\partial \mathbf{p}} - \frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{p}}} &= -\mathbf{M}(\mathbf{p})\ddot{\mathbf{p}} + \{f(\mathbf{p}, \dot{\mathbf{p}})\} \\ -\frac{\partial \mathcal{W}_F}{\partial \mathbf{p}} &= \{F(\mathbf{p})\} \\ -\frac{\partial \mathcal{W}_e}{\partial \mathbf{p}} &= -\mathbf{K}\mathbf{p} \end{aligned}$$

The equations governing the motion of a single component non-constrained kinematically are

$$\mathbf{M}(\mathbf{p})\ddot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \{F(\mathbf{p})\} + \{f(\dot{\mathbf{p}}, \mathbf{p})\} \tag{11}$$

where $\mathbf{M}(\mathbf{p})$ and $\mathbf{K}$ are the mass and stiffness matrices. $\{f(\mathbf{p}, \dot{\mathbf{p}})\}$ represents the generalised Coriolis forces and part of the generalised transport forces, and $\{F(\mathbf{p})\}$ stands for the generalised applied forces.
The contribution of the strain energy and applied forces' work are easily derivable, and are therefore not reminded in this article. The contribution of the kinematic energy being less straightforward, details of it's derivation are described in appendix A. The expressions of the mass matrix and the generalised inertia forces are there recalled for the most general case.
However, in the remainder of this article, we will assume that the reference frame is initially located at the center of mass. It has also been assumed that the elastic displacements are small enough to neglect the second order terms in $\{u_e\}$. Within those assumptions the mass and stiffness matrices and the generalised inertia forces have the following expressions:

$$\mathbf{M}(\mathbf{p}) = \begin{bmatrix} m_{tot}\mathbf{I}_d & \mathbf{A}(\theta + \frac{\pi}{2})\mathbf{L}_1\{u_e\} & \mathbf{A}(\theta)\mathbf{L}_1 \\ & \mathbf{I}_z + 2\mathbf{I}_1\{u_e\} & \mathbf{I}_3 + \{u_e\}^T\mathbf{L}_3 \\ sym & & \mathbf{M}_{ee} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{RR} & \mathbf{M}_{R\theta} & \mathbf{M}_{Re} \\ & \mathbf{M}_{\theta\theta} & \mathbf{M}_{\theta e} \\ sym & & \mathbf{M}_{ee} \end{bmatrix} \tag{12}$$

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{K}_{ee} \end{bmatrix} \tag{13}$$

$$\{f(\mathbf{p}, \dot{\mathbf{p}})\} = \begin{Bmatrix} f_R \\ f_\theta \\ f_e \end{Bmatrix} = \dot{\theta}^2 \begin{Bmatrix} \mathbf{A}(\theta)\mathbf{L}_1\{u_e\} \\ 0 \\ \mathbf{I}_1^T + \mathbf{M}_{ee}\{u_e\} \end{Bmatrix} - 2\dot{\theta} \begin{bmatrix} \mathbf{A}(\theta + \frac{\pi}{2})\mathbf{L}_1 \\ \mathbf{I}_1 + \{u_e\}^T\mathbf{M}_{ee} \\ \mathbf{L}_3^T \end{bmatrix} \{\dot{u}_e\} \tag{14}$$

using the following notations:

$\mathbf{M}_{ee}$, $\mathbf{K}_{ee}$: Linear finite element mass and stiffness matrices in the local axis

$$
\begin{aligned}
m_{tot}\ \underline{O'G_i} &= \int_\Omega \rho \underline{r} dV & \mathbf{I}_z &= \int_\Omega \rho \underline{r}^2 dV \\
\mathbf{I}_1 &= \int_\Omega \rho \underline{r}^T \mathbf{N} dV & \mathbf{L}_1 &= \int_\Omega \rho \mathbf{N} dV \\
\mathbf{I}_3 &= \int_\Omega \rho \underline{r}^T \mathbf{A}(-\frac{\pi}{2}) \mathbf{N} dV & \mathbf{L}_3 &= \int_\Omega \rho \mathbf{N}^T \mathbf{A}(-\frac{\pi}{2}) \mathbf{N} dV
\end{aligned}
$$

## IV. Components assembly using a 4 field formulation

The equations governing the motion of a component undergoing possibly large translation and rotations within its unconstrained motion have been recalled in the previous section. In this section, we now consider the kinematical constraints linking each single component to form the assembled mechanism. The approach proposed hereafter significantly differs from the classical assembly method. The scope of our study is restrained to holonomic constraints that are usually expressed in the following form

$$\left\{h(\underline{x}_b, t)\right\} = 0 \tag{15}$$

where in accordance with our convention, $\{h\}$ collects the set of all the kinematical constrains at the joints, and $\underline{x}_b$ the set of all the positions of the components' boundaries.

In multi-body dynamics, the constraints are commonly enforced in a dual fashion, i.e., through interconnecting forces by introducing Lagrange multipliers. In section III., it will be shown that in order to be in line with this assembly method, the elastic displacement should be expanded on the Dual Craig-Bampton modal basis.

The residual flexibility modes, which belong to this set, depend on the elastic part of the Jacobian of the constraints, $\mathbf{B}_e^s = \frac{\partial h}{\partial \underline{u}_e^s}$. While in linear sub-structuring (or domain decomposition) analysis, $\mathbf{B}_e^s$ is nearly always a signed boolean, therefore constant, matrix. It is unfortunately no longer the case for articulated mechanism. The Jacobian of the constraints indeed contains the directions of the connecting forces, that obviously depend on the configuration of the mechanism. To avoid the recursiveness entailed by

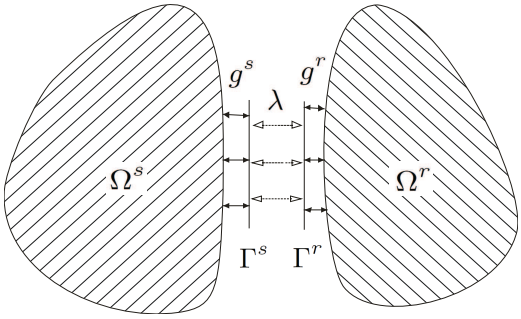


**Figure 2. Local and global interfaces**

such a dependency, we hereby propose an alternative way to handle these constraints, that is based on two observations:

- At the reduction step, the current direction of the loads at the interface does not need to be known. The knowledge of a basis containing the possible directions for the loads is in fact sufficient. The amplitude of the loads on each direction can then be computed at each time step.
- At the global level the deformations at the interface and the orientations of the concerned component have to be known to enforce the constraints.

The constraint equations have thus been split into local and global parts, by introducing a local interface $\Gamma^s$ for each component $s$ and a global interface $\Gamma$ per joint. See Fig. 2.

A set of local constraints defined for each component $s$, attaches the interface $\Gamma^s$ to the component's boundary.

$$\mathbf{L}_b^s\{u_e^s\} - \{u_\Gamma^s\} = 0 \text{ on the local interface } \Gamma^s \tag{16}$$

$\mathbf{L}_b^s$ denotes the boolean localization matrix that returns the components boundary displacements: $\{u_b^s\} = \mathbf{L}_b^s\{u_e^s\}$. The nodal displacements of the local boundary $\Gamma$ are denoted $\{u_\Gamma^s\}$, they initially coincides with boundary in the undeformed configuration.

A second set of global constraints then connects the local boundaries $\Gamma^s$ to the joints.

$$\big\{h(\underline{x}_b, t)\big\} = \big\{h(p_r, u_\Gamma, t)\big\} = 0 \text{ at the global joints } \Gamma \tag{17}$$

where $\underline{x}_b^s$ now writes: $\underline{x}_b^s = \underline{R}^s + \mathbf{A}(\theta^s)(\underline{r}_b^s + \mathbf{N}\{u_\Gamma^s\})$

Two sets of multipliers are associated to each type of constraints, the local multipliers are denoted $g^s$, and the global ones $\lambda$ (see Fig. 2).

The components' boundary displacements values are transferred to the global level via $\{u_\Gamma^s\}$, whereas the current directions of the loads at the joints are transferred via $g^s$ to the local components' level. In our approach, four different types of fields are thus needed to enforce the kinematical constraints. Namely the degrees of freedom $u_b^s$ and $u_\Gamma^s$, and the forces fields $g^s$ and $\lambda$. Therefore we will refer to this formulation as a *four-field formulation*.

The Lagrangian for the assembled system now writes:

$$\mathcal{L} = \sum_s \Big[\mathcal{T}^s - (\mathcal{W}_e^s + \mathcal{W}_F^s + \mathcal{W}_{cons\ loc}^s)\Big] - \mathcal{W}_{cons\ glob} \tag{18}$$

where $\mathcal{W}_{cons\ loc}^s$ and $\mathcal{W}_{cons\ glob}$ are the work introduced by the local and global constraints forces:

$$\mathcal{W}_{cons\ loc}^s = -\left(\mathbf{L}_b^s\{u_e\}^s - \{u_\Gamma^s\}\right)^T\{g^s\} \tag{19}$$

$$\mathcal{W}_{cons\ glob} = -\big\{h(p_r, u_\Gamma, t)\big\}^T\{\lambda\} \tag{20}$$

The expression of the kinematic energy and strain energy have been derived in the previous section. So that only the contributions of the work of global constraints $\mathcal{W}_{cons\ glob}$ and local constraints $\mathcal{W}_{cons\ loc}$ need to be computed. We denote $\mathcal{W}_{cons\ loc} = \sum_s \mathcal{W}_{cons\ loc}^s = -\left(\mathbf{L}_b\{u_e\} - \{u_\Gamma\}\right)^T\{g\}$.
The contribution of the global and local constraint forces' work being

$$\begin{aligned}
-\frac{\partial \mathcal{W}_{cons\ loc}}{\partial u_e} &= {\mathbf{L}_b}^T\{g\} & -\frac{\partial \mathcal{W}_{cons\ glob}}{\partial u_e} &= 0\\
-\frac{\partial \mathcal{W}_{cons\ loc}}{\partial p_r} &= 0 & -\frac{\partial \mathcal{W}_{cons\ glob}}{\partial p_r} &= {\mathbf{B}_r}^T\{\lambda\}\\
-\frac{\partial \mathcal{W}_{cons\ loc}}{\partial u_\Gamma} &= -\{g\} & -\frac{\partial \mathcal{W}_{cons\ glob}}{\partial u_\Gamma} &= {\mathbf{B}_\Gamma}^T\{\lambda\}\\
-\frac{\partial \mathcal{W}_{cons\ loc}}{\partial g} &= \mathbf{L}_b\{u_e\} - \{u_\Gamma\} & -\frac{\partial \mathcal{W}_{cons\ glob}}{\partial g} &= 0\\
-\frac{\partial \mathcal{W}_{cons\ loc}}{\partial \lambda} &= 0 & -\frac{\partial \mathcal{W}_{cons\ glob}}{\partial \lambda} &= \big\{h(p_r, u_\Gamma, t)\big\}
\end{aligned}$$

we obtain the following set of equations of motion for the flexible multi-body system:

$$\mathbf{M}(\mathbf{p})\ddot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \{F(\mathbf{p})\} + \{f(\dot{\mathbf{p}}, \mathbf{p})\} + \begin{Bmatrix} {\mathbf{B}_r}^T\{\lambda\} \\ {\mathbf{L}_b}^T\{g\} \end{Bmatrix} \tag{21}$$

$$0 = {\mathbf{B}_\Gamma}^T\lambda - \{g\} \tag{22}$$

$$0 = \mathbf{L}_b\{u_e\} - \{u_\Gamma\} \tag{23}$$

$$0 = \big\{h(p_r, u_\Gamma, t)\big\} \tag{24}$$

where the matrices $\mathbf{B}_r$ and $\mathbf{B}_\Gamma$ are parts of the Jacobian of the constrains corresponding to variables $\{p_r\}$ and $\{u_\Gamma\}$. Equation Eq. (21), corresponds to the dynamic equilibrium for each of the components of the flexible

multi-body system, that are submitted to external, inertia and constraint forces. Note that elastic degrees of freedom are constrained by local multipliers $\{g\}$ whereas rigid body degrees of freedom are constrained by global multipliers $\{\lambda\}$. Eq .(22) expresses the equilibrium between local and global constraint forces . Eq. (23) and Eq. (24) respectively corresponds to the local and global constraints as stated earlier.

The set of differential and algebraic equations of motion can also be rewritten in the following matrix form:

$$\underbrace{\begin{bmatrix} \mathbf{M}_{rr}(\mathbf{p}) & \mathbf{M}_{re}(\mathbf{p}) & 0 & 0 \\ \mathbf{M}_{re}^T(\mathbf{p}) & \mathbf{M}_{ee} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{M}_a} \begin{Bmatrix} \ddot{p}_r \\ u_e \\ g \\ u_\Gamma \end{Bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbf{K}_{ee} & -\mathbf{L}_b^T & 0 \\ 0 & -\mathbf{L}_b & 0 & \mathbf{I} \\ 0 & 0 & \mathbf{I} & 0 \end{bmatrix}}_{\mathbf{K}_a} \underbrace{\begin{Bmatrix} p_r \\ u_e \\ g \\ u_\Gamma \end{Bmatrix}}_{\mathbf{p}_a} = \underbrace{\begin{Bmatrix} F_r(\mathbf{p}) \\ F_e(\mathbf{p}) \\ 0 \\ 0 \end{Bmatrix}}_{F_a} + \underbrace{\begin{Bmatrix} f_r(\mathbf{p}, \dot{\mathbf{p}}) \\ f_e(\mathbf{p}, \dot{\mathbf{p}}) \\ 0 \\ 0 \end{Bmatrix}}_{f_a} + \underbrace{\begin{bmatrix} \mathbf{B}_r^T \\ 0 \\ 0 \\ \mathbf{B}_\Gamma^T \end{bmatrix}}_{\mathbf{B}^T} \{\lambda\}$$

$$\{h(p_r, u_\Gamma, t)\} = 0 \tag{25}$$

where $p_a$ is the new set of generalised coordinates for the assembled system. Although the third block-line is not a real dynamic equilibrium equation but rather a constraint equation, we will refer to $\mathbf{M}_a$ and $\mathbf{K}_a$ as the generalised mass and stiffness matrices of the assembled system. Similarly, $\{F_a\}$ and $\{f_a\}$ designate the generalised applied and inertia forces for the assembled system.

**Remark:**
If the Jacobian $\mathbf{B}$ wouldn't be configuration dependent, one could find a null space for equations Eq. (22) and Eq. (23) that wouldn't be configuration dependent. A projection on this subspace would thereafter eliminate the last two block-lines of Eq. (25), along with the last two block-columns of the generalised assembled mass and stiffness matrices. Thereby leading to the classical assembled set of equations of motion in structure dynamic analysis.

## V. Dual Craig-Bampton

The Craig-Bampton method[4] has been successfully used in structural analysis over the past 40 years, and is nowadays among the most widely used Component Mode Synthesis (CMS) methods for flexible multi-body systems, together with the MacNeal and Rubin's method. Its computational efficiency mainly relies on the sparse structure of the reduced matrices.

In,[3] a new component mode synthesis method has been proposed, where substructures are seen as being attached through interconnecting forces at the boundaries. Based on this vision, the system is consistently assembled, leading to a method totally dual to Craig-Bampton. A similarly sparse structure is indeed obtained, but with a similar modal basis as MacNeal and Rubin's approaches.

If the Craig-Bampton method is well suited for structure analysis using finite elements, we believe that the Dual Craig-Bampton is the best suited for articulated mechanisms. As in the framework of multi-body systems, it features the advantages of the afore mentioned approaches. Namely, the sparse structure of the Craig-Bampton reduced matrices, and the same modes as in both Mac Neal and Rubin's reduction which are the most suitable for flexible multi-body systems.

In the present section, an extension of the Dual Craig-Bampton to flexible multi-body analysis is proposed, the choice of the reduction basis is first justified and its properties recalled. The reduced matrices are then derived.

### A. Reduction basis and its properties

Let us consider the equations governing the dynamics of the nodal elastic displacements of the component $s$, $\{u_e^s\}$. The equations of interest are the following $n_e$ lines of system Eq. (25):

$$\mathbf{M_{ee}^s}\{\ddot{u}_e^s\} + \mathbf{K_{ee}^s}\{u_e^s\} = \mathbf{L}_b^{sT}\{g^s\} + \{F_e^s(\mathbf{p})\} + \{f_e^s(\mathbf{p}, \dot{\mathbf{p}})\} - \mathbf{M_{er}^s}\{\ddot{x}_r^s\} \tag{26}$$

The local constraint forces $\mathbf{L}_b^{sT}\{g^s\}$, the inertia forces $\{f_e^s(\mathbf{p}, \dot{\mathbf{p}})\} - \mathbf{M_{er}^s}\{\ddot{x}_r^s\}$ and generalised applied forces $\{F_e^s(\mathbf{p})\}$, are seen as excitations at the component's nodes. The solution of this second order differential equation splits into a particular and homogeneous solution.

The homogenous solution belongs to the subspace spanned by the eigenmodes of the component free at its interface, those modes (the set of $n_k^s$ kept ones) are collected in the columns of the matrix $\mathbf{\Phi}^s$.

The particular solution is generally more difficult to obtain. Although, if one could easily compute it, there would be no need to reduce the system in the view of solving it. However, in the static case the particular solution is easy to compute, it solves the following equation:

$$\mathbf{K_{ee}^s}\{u_e^s\} = \mathbf{L}_b^{sT}\{g^s\} + \{F_e^s(\mathbf{p})\} \tag{27}$$

Note, that the generalised inertia and applied forces have not been taken into account. Since the generalised inertia forces are usually well distributed and of relatively low frequency, they should be well represented by quasi-static modes. However, if the modal approximation basis representing the quasi-static solution is too poor, one can choose to enrich it using Modal Truncation Augmentation (MTA) modes for instance, more details can be found in.[5] The particular solution in the static case is hence $\{u_e^s\} = \mathbf{K_{ee}^s}^{-1}\mathbf{L}_b^{sT}\{g^s\}$.

Note that when interconnecting boundaries are left free, if the component admits rigid body motions, the matrix $\mathbf{K_{ee}^s}$ becomes singular. In that case, a generalised inverse should be used and the rigid body modes added to the static solution, see[3] for details. Moreover, the inverse of the stiffness matrix $\mathbf{K_{ee}^s}^{-1}$ can be expanded on the free eigenmodes, the part that is not spanned by the $n_k^s$ kept ones is represented by the residual flexibility matrix: $\mathbf{G}_{res}^s = \mathbf{K_{ee}^s}^{-1} - \sum_i^{n_k^s} \frac{\phi_i^s \phi_i^{sT}}{\omega_i^{s\,2}}$

Consequently, the nodal elastic displacement for a component attached to the system via interconnecting forces can be spanned on a basis containing the free vibration modes and the residual flexibility modes:

$$\{u_e^s\} = \mathbf{\Phi^s}\{\eta^s\} + \mathbf{G_{res}^s}\mathbf{L_b^s}^T\{g^s\} \tag{28}$$

Some of the remarkable properties of the modes $\mathbf{\Phi}$ and $\mathbf{G}_{res}$, are hereunder recalled

- Orthogonality between the free-free modes of vibration and the residual flexibility modes:

$$\mathbf{\Phi^s}^T\mathbf{K_{ee}^s}\mathbf{G_{res}^s} = 0 \tag{29}$$

$$\mathbf{\Phi^s}^T\mathbf{M_{ee}^s}\mathbf{G_{res}^s} = 0 \tag{30}$$

- Normalised free-free modes of vibration:

$$\mathbf{\Phi^s}^T\mathbf{M_{ee}^s}\mathbf{\Phi^s} = \mathbf{I} \quad \text{and} \quad \mathbf{\Phi^s}^T\mathbf{K_{ee}^s}\mathbf{\Phi^s} = \mathbf{\Omega^{s2}} \tag{31}$$

- Properties of $\mathbf{G_{res}^s}$:

$$\mathbf{G_{res}^s}^T = \mathbf{G_{res}^s} \quad \text{and} \quad \mathbf{G_{res}^s}^T\mathbf{K_{ee}^s}\mathbf{G_{res}^s} = \mathbf{G_{res}^s} \tag{32}$$

- Orthogonality between rigid body modes and free-free vibration modes:
  Although, orthogonality between purely translational rigid body modes and free vibration modes always holds; for rigid body modes involving rotations, the orthogonality holds for small rotations only. this will be further discussed next.

Let usdetail the orthogonality property between rigid and elastic eigenmodes. Consider an arbitrary rigid body motion of component $s$. If $\delta\underline{R}^s$ and $\delta\underline{\alpha}^s$ are the translation and rotation increment, we deduce from Eq. (1) the expression of the new position

$$\underline{x}^s + \delta\underline{x}^s = \underline{R}^s + \delta\underline{R}^s + \mathbf{A}(\theta^s + \delta\alpha^s)\underline{r}^s \tag{33}$$

and subsequently, the displacement within a rigid body motion:

$$\underline{u}_r^s = \delta\underline{x}^s = \delta\underline{R}^s + (\mathbf{A}(\theta^s + \delta\alpha^s) - \mathbf{A}(\theta^s))\underline{r}^s \tag{34}$$

For small rotation increments, the first order approximation of displacement is

$$\underline{u}_r^s = \delta\underline{R}^s + \delta\alpha^s\mathbf{A}(\theta^s + \frac{\pi}{2})\underline{r}^s \tag{35}$$

We can now write the orthogonality relation between rigid modes $\underline{u}_r^s$ and free-free modes $\underline{\phi}^s$, expressed in the local frame

$$\int_{\Omega^s} \rho^s \underline{u}_r^{s\,T} \mathbf{A}(\theta^s).\underline{\phi}^s dV = 0 \tag{36}$$

The orthogonality with the purely translational rigid body displacement is therefore for any free-free mode, that in the finite element approximation writes $\mathbf{N}^s\{\phi^s\}$:

$$\delta\underline{R}^{s\,T} \mathbf{A}(\theta^s) \int_{\Omega^s} \rho^s \mathbf{N}^s dV\{\phi^s\} = 0 \quad \forall \theta^s, \delta\underline{R}^s \qquad \Longleftrightarrow \qquad \mathbf{L}_1^s\{\phi^s\} = 0 \tag{37}$$

and for the purely rotational rigid body displacements fulfilling the small displacement assumption:

$$\delta\alpha^s \int_{\Omega^s} \rho^s \underline{r}^{s\,T} \mathbf{A}\left(-\frac{\pi}{2}\right) \mathbf{N}^s dV\{\phi^s\} = 0 \quad \forall \delta\alpha^s \qquad \Longleftrightarrow \qquad \mathbf{I}_3^s\{\phi^s\} = 0 \tag{38}$$

Each colones of the matrices $\mathbf{\Phi^s}$ and $\mathbf{G_{res}^s}$ being in the space of of the free-free elastic modes, we have:

$$\mathbf{L}_1^s \mathbf{\Phi^s} = 0 \qquad \mathbf{L}_1^s \mathbf{G_{res}^s} = 0 \tag{39}$$

$$\mathbf{I}_3^s \mathbf{\Phi^s} = 0 \qquad \mathbf{I}_3^s \mathbf{G_{res}^s} = 0 \tag{40}$$

Contrarily to equations Eq. (29-32), Eq. (36) sets a relation between rigid and elastic displacements (for planar systems, 3 scalar relations in total). In fact, since we do not include the rigid body modes in the approximation Eq. (28) of the local displacements, it dictates how the reference frame (that is initially located at the center of mass and oriented collinear to the inertia axis) has to move in order to follow the current center of mass and the orientation of the inertia axis.

It has already been emphasised in earlier works,[6,7] that the choice of the reference frames is closely related to the choice of the modal basis. We here see how the Dual Craig and Bampton modal basis is naturally associated to the Tisserand axis, which is one of most interesting reference frame, as it brings the mass matrix to it's simplest expression. We will indeed see in the following subsection how the inertia terms considerably simplifies. This results from the fact that we chose the frame that is physically the most relevant for flexible multi-body systems. We also did not introduce any cumbersomeness by trying to enforce constraints in a primal fashion when they are already used in their dual form in the governing equations.

### B. Reduced set of equations of motion

We showed in the previous subsection (Eq. (28)) that the nodal elastic displacements can be expressed in terms of the reduced generalised coordinates $\{\eta\}$ and $\{g\}$. The assembled generalised coordinates $\mathbf{p}$ can be represented in terms of the assembled reduced set of generalised coordinates $\mathbf{q}$ using the Rayleigh-Ritz transformation: $\mathbf{p_a} = \mathbf{T}\mathbf{q}$, where:

$$\mathbf{p_a} = \begin{Bmatrix} p_r \\ u_e \\ g \\ u_\Gamma \end{Bmatrix}, \mathbf{q} = \begin{Bmatrix} p_r \\ \eta \\ g \\ u_\Gamma \end{Bmatrix}, \mathbf{T} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{\Phi} & \mathbf{G_{res}}\mathbf{L_b}^T & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{I} \end{bmatrix}$$

Thanks to the assembly method introduced in section III., the reduction matrix $\mathbf{T}$ is constant. We therefore also have $\dot{\mathbf{p}} = \mathbf{T}\dot{\mathbf{q}}$, and we can use the following chain rule to derive the equations of motion starting from Eq. (25):

$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \mathbf{T}^T \left( \frac{\partial \mathcal{L}}{\partial \mathbf{p}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{p}}} \right) \tag{41}$$

So the reduced set of equation for the assembled system is given by:

$$\underbrace{\mathbf{T}^T \mathbf{M}_a(\mathbf{q}) \mathbf{T}}_{\mathbf{M}_{red}} \ddot{\mathbf{q}} + \underbrace{\mathbf{T}^T \mathbf{K}_a \mathbf{T}}_{\mathbf{K}_{red}} \mathbf{q} = \underbrace{\mathbf{T}^T \{F_a(\mathbf{q})\}}_{F_{red}} + \underbrace{\mathbf{T}^T \{f_a(\mathbf{q}, \dot{\mathbf{q}})\}}_{f_{red}} + \mathbf{B}^T\{\lambda\} \tag{42}$$

$$\left\{h(p_r^s, u_\Gamma^s, t)\right\} = 0$$

Recalling that the inertia terms writes:

$$\mathbf{M}_a(\mathbf{p}) = \begin{bmatrix} m_{tot}\mathbf{I}_d & \mathbf{A}(\theta+\frac{\pi}{2})\mathbf{L}_1\{u_e\} & \mathbf{A}(\theta)\mathbf{L}_1 & 0 & 0 \\ & \mathbf{I}_z + 2\mathbf{I}_1\{u_e\} & \mathbf{I}_3 + \{u_e\}^T\mathbf{L}_3 & 0 & 0 \\ sym & & \mathbf{M}_{ee} & 0 & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$$

$$\{f_a(\mathbf{p},\dot{\mathbf{p}})\} = \dot{\theta}^2 \begin{Bmatrix} \mathbf{L}_1\{u_e\} \\ 0 \\ \mathbf{I}_1^T + \mathbf{M}_{ee}\{u_e\} \\ 0 \\ 0 \end{Bmatrix} - 2\dot{\theta} \begin{bmatrix} \mathbf{A}(\theta+\frac{\pi}{2})\mathbf{L}_1 \\ \mathbf{I}_1 + \{u_e\}^T\mathbf{M}_{ee} \\ \mathbf{L}_3^T \\ 0 \\ 0 \end{bmatrix} \{\dot{u}_e\}$$

and using the orthogonality properties Eq. (29-32) and Eq. (39,40) we obtain:

$$\mathbf{M}_{red}(\mathbf{q}) = \begin{bmatrix} m_{tot}\mathbf{I}_d & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I}_z + 2\mathbf{I}_1(\mathbf{\Phi}\{\eta\} + \mathbf{G}_{res}\mathbf{L}_b^T\{g\}) & \{\eta\}^T\mathbf{\Phi}^T\mathbf{L}_3\mathbf{\Phi} & \{g\}^T\mathbf{L}_b\mathbf{G}_{res}\mathbf{L}_3\mathbf{G}_{res}\mathbf{L}_b^T & 0 \\ 0 & -\mathbf{\Phi}^T\mathbf{L}_3\mathbf{\Phi}\{\eta\} & \mathbf{I} & 0 & 0 \\ 0 & -\mathbf{L}_b\mathbf{G}_{res}\mathbf{L}_3\mathbf{G}_{res}\mathbf{L}_b^T\{g\} & 0 & \mathbf{M}_{res} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{43}$$

$$\mathbf{K}_{red} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{\Omega}^2 & -\mathbf{\Phi}^T\mathbf{L}_b^T & 0 \\ 0 & 0 & -\mathbf{L}_b\mathbf{\Phi} & -\mathbf{F}_{res} & \mathbf{I} \\ 0 & 0 & 0 & \mathbf{I} & 0 \end{bmatrix} \tag{44}$$

where $\mathbf{M}_{res} = \mathbf{L_b}\mathbf{G}_{res}\mathbf{M}\mathbf{G}_{res}\mathbf{L_b}^T$ and $\mathbf{F}_{res} = \mathbf{L_b}\mathbf{G}_{res}\mathbf{L_b}^T$.

$$\{f_{red}(\mathbf{q},\dot{\mathbf{q}})\} = \dot{\theta}^2 \begin{Bmatrix} 0 \\ 0 \\ \mathbf{\Phi}^T\mathbf{I}_1^T + \{\eta\} \\ \mathbf{L}_b\mathbf{G}_{res}\mathbf{I}_1^T + \mathbf{M}_{res}\{g\} \\ 0 \end{Bmatrix} - 2\dot{\theta} \begin{bmatrix} 0 \\ \mathbf{I}_1\mathbf{\Phi} + \{\eta\}^T \\ \mathbf{\Phi}^T\mathbf{L}_3^T\mathbf{\Phi} \\ \mathbf{L}_b\mathbf{G}_{res}\mathbf{L}_3^T\mathbf{\Phi} \\ 0 \end{bmatrix} \{\dot{\eta}\} - 2\dot{\theta} \begin{bmatrix} 0 \\ \mathbf{I}_1\mathbf{G}_{res}\mathbf{L}_b^T + \{g\}^T\mathbf{M}_{res} \\ \mathbf{\Phi}^T\mathbf{L}_3^T\mathbf{G}_{res}\mathbf{L}_b^T \\ \mathbf{L}_b\mathbf{G}_{res}\mathbf{L}_3^T\mathbf{G}_{res}\mathbf{L}_b^T \\ 0 \end{bmatrix} \{\dot{g}\} \tag{45}$$

where we also made use of the anti-symmetry property of $L_3$.

$$\{F_{red}(\mathbf{q})\} = \begin{Bmatrix} F_R \\ F_\theta \\ \mathbf{\Phi}^T F_e \\ \mathbf{L}_b\mathbf{G}_{res}F_e \\ 0 \end{Bmatrix} \tag{46}$$

The reduced vectors and matrices can easily be computed from the system's global vectors and matrices using the transformation matrix $\mathbf{T}$ that can be formed using independent finite element software. The matrices Eq. (43-46) do not contain the inter-body constraints. Those constraints are enforced by the $\lambda$ as indicated in Eq. (25). Hence the matrices Eq. (43-46) can be constructed at the level of each body separately exactly like the contributions of substructures in usual sub-structuring approaches. Elements of the reduced matrices can also be build in the finite element package, before being assembled to form the reduced system and solved using software dedicated to multi-body systems.

**Remarks**

- In the Dual Craig-Bampton method, here applied to flexible multi-bodies, the reduction space for the displacement is in fact identical to the basis used in the methods of Rubin and MacNeal.[8,9] However, whereas in those methods the reduction basis is further transformed to eliminate the interface forces, the Dual Craig-Bampton method retains the interface forces as generalised degrees of freedom and reduces the problem assembled in its dual form. This also implies that the compatibility on the interface Eq. (16) is weakened in this approach. For more details see.[3]

- In case all the free-free modes are kept, the operator $\mathbf{T}$ no longer represents a reduction, but merely a change of basis. In that case $\mathbf{G_{res}} = 0$ as well as $\mathbf{F_{res}} = \mathbf{M_{res}} = 0$ and the interface compatibility is strongly enforced. In that case the present method is equivalent to formulations sometimes found in commercial multi-body codes, like Adams.

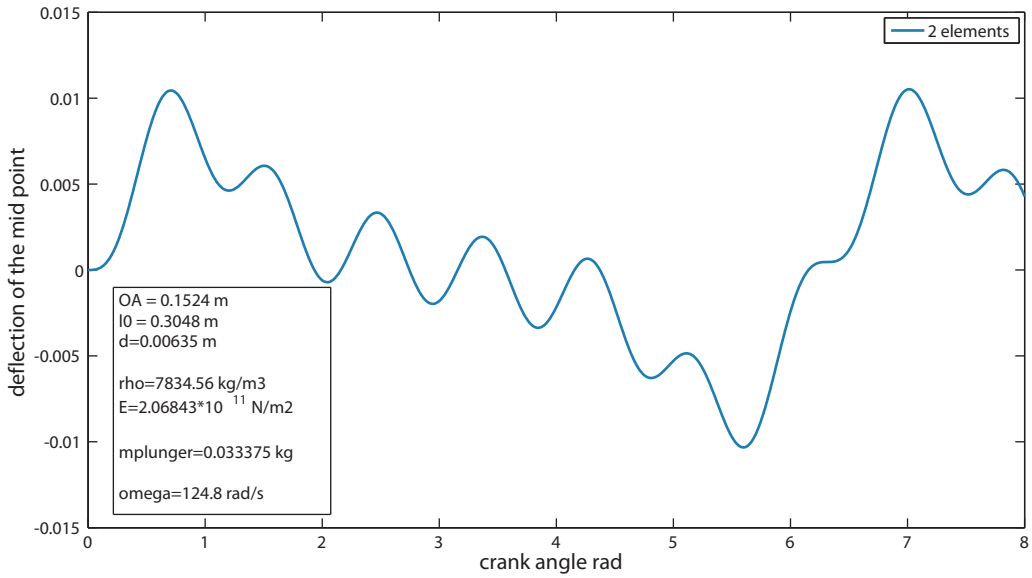


**Figure 3. Dimensionless deflection of the midpoint of the connecting rod, computed without reduction using the dimensions of[10]**

## VI. Test on the slider crank system

We have introduced in the previous sections a Dual Craig-Bampton reduction method for multi-body systems. In the present section, the transient response of a slider-crank mechanism is studied in order to investigate the validity of the proposed model. This simple problem having been widely studied, an abundant collection of data is available. We will here refer to the work of Meijaard[10] and Schwab.[11]

The mechanism consists of a rigid crank $OA$, a flexible connecting rod of uniform section $AB$, and a sliding block $B$. Dimensions and material properties of the system have been chosen identical to those in the references. They are reminded, together with the rotation speed, on each figure.

As in[10] and,[11] we focus on the deflection of the midpoint, the dimensionless parameter obtained by dividing the transversal displacement of the midpoint by the length of the connecting rod. The results are represented as a function of the crank angle.

The generalized HHT-alpha method has been implemented to time-integrate the assembled set of equations of motion. In the initial configuration both the connecting rod and the crank are horizontal. A constant rotation speed is applied to the crank. In the performed tests, only the connecting rod has been discretised into finite elements. Beam elements were used for the discretisation, each having 6 degrees of freedom (2 translational and 1 rotational at each node).

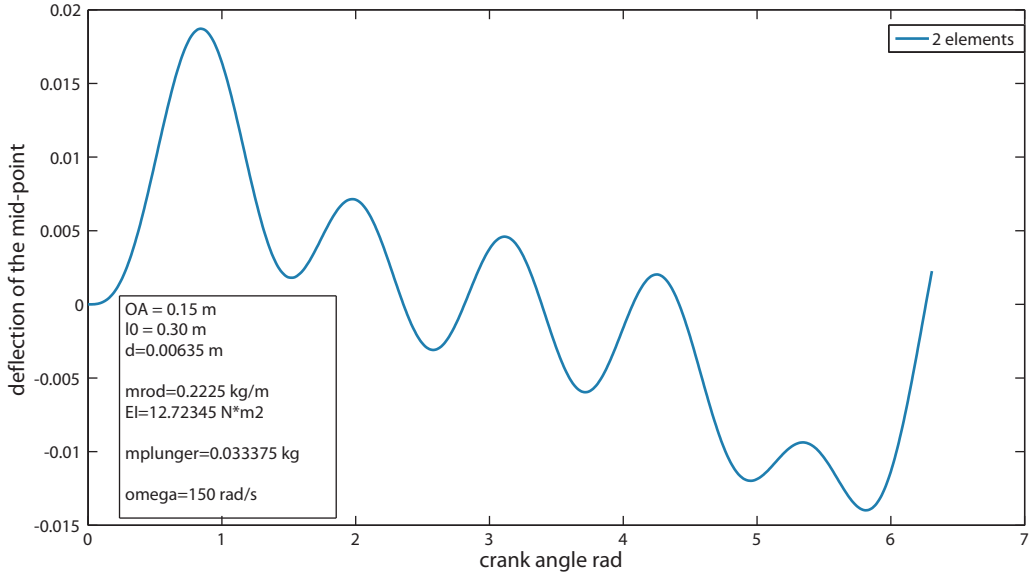


**Figure 4. Dimensionless deflection of the midpoint of the connecting rod, computed without reduction using the dimensions of[11]**

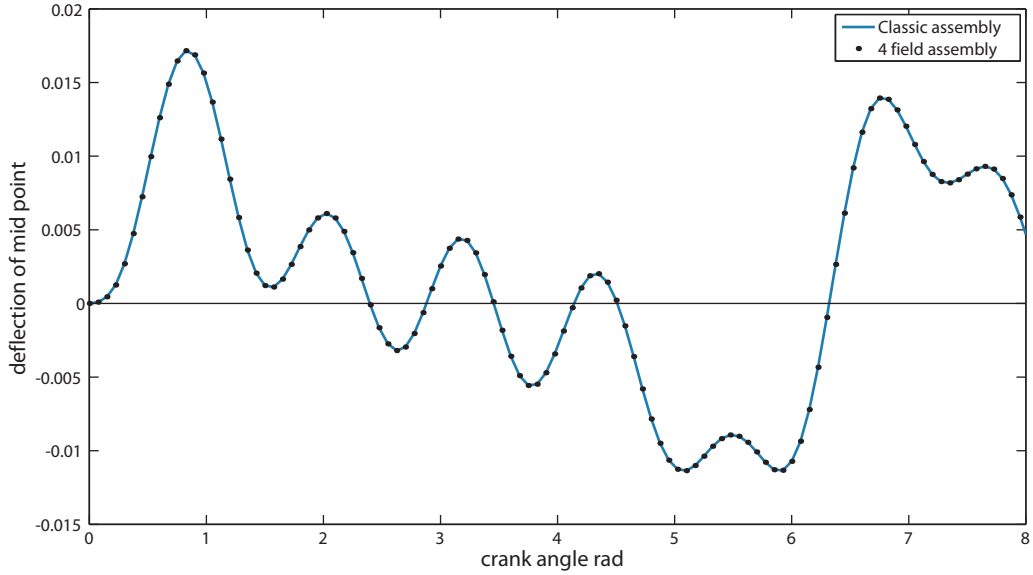


**Figure 5. Comparison of classic assembly and 4 field assembly approaches**

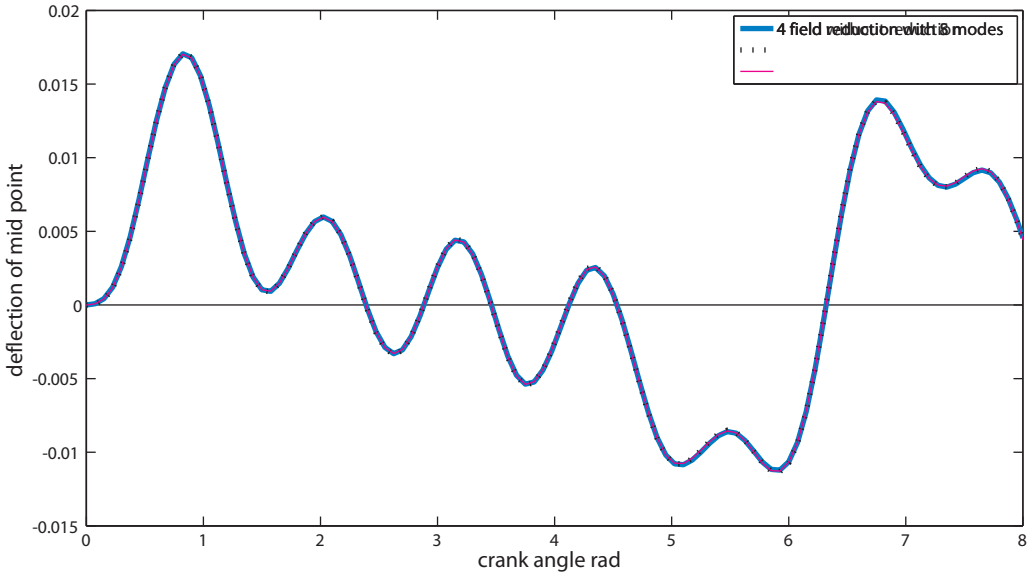


**Figure 6. Comparison of reduced and non reduced responses assembled through a 4 field approach**

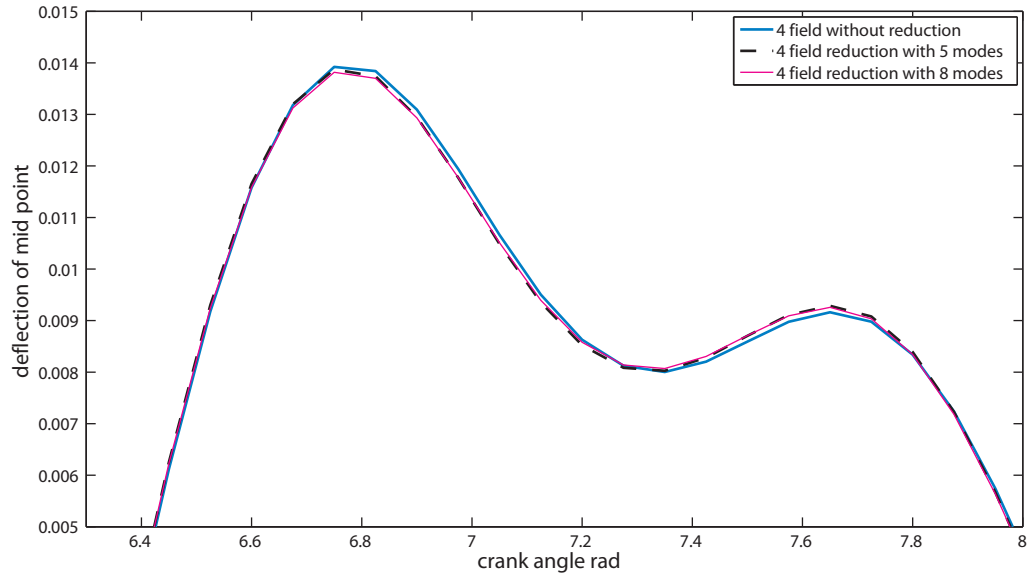


**Figure 7. Close-up view of the reduced and non reduced response assembled through a 4 field approach**

First, few tests were carried out (with classic assembly and no reduction) in order to validate the multi-body dynamic solver and the integration scheme. Fig. 3 and Fig. 4, depict the deflection obtained using the dimensions of respectively[10] and.[11] They are in good agreement with their previous investigations. For those tests, two elements were used to describe the connecting rod.

Next, responses obtained through classical and 4 field assembly approaches have been compared. Results are depicted in Fig. 5, as can be observed, both plots perfectly match. No instabilities were observe during time integration, the two assembly methods seem to be equivalent.

Finally, solutions to the reduced and non reduced equations of movement assembled using the 4 field approach have been compared. As illustrated by Fig. 7, for this simple problem relatively few modes are needed to reliably represent the response of the system. Fig. 7 represents a zoom on the two last peaks of Fig. 6, where reduced and non reduced deflection differ the most. As can be observed, the relative error on the deflection is under the percent when using 5 modes. And is even better when using 8 modes. For those plots the connecting rod was discretised into 4 beam elements, and the time integration performed with the spectral radii in the infinite limit $\rho_\infty$ set to 0.8.

In this section the validity of both the 4 field assembly and the Dual Craig-Bampton based reduction approaches have been proven. Further simulations need to be processed in order to test the performance of the presented reduction method. However, we are confident on the potential of the presented method for reducing multi-body systems.



## VII. Concluding remarks

In this contribution, a new reduction method for multi-body systems has been introduced. As in all co-rotated super-elements techniques, the reduced components were described through internal general coordinates and interface coordinates. However, the components having been reduced using a Dual Craig-Bampton technique, the interface coordinates are now representing attachment forces instead of displacements.

The reduction methods of MacNeal and Rubin, and the Dual Craig-Bampton technique, all share the same modal basis, which consists of free interface modes and residual flexibility (or attachment) modes. This basis being naturally related to the Tisserand frame it allows for a highly sparse expression of the mass matrix.

The difference with the approaches of MacNeal and Rubin mainly resides in the way constraints are handled. In those approaches further transformations are processed in order to enforce the constraints in a primal fashion (through interface displacements). This reveals to be unnecessary when dealing with Multi-body systems, and often causes dispensable cumbersomeness. In order to simplify the formulation, while keeping attachment forces as interface degrees of freedom, a novel assembly approach have been introduced. As result, the sparsity of the reduced mass matrix could be preserved.

However, in our approach the constraints in the reduced system are only weakly enforced, which might causes instability as negative eigen frequencies can arise. It would thus be interesting to carry out further simulations on mechanisms of higher complexity in order to investigate the performances and limitations of the method.

## Appendix: Details on the derivation of the inertia terms

The equations of motion for a single component of a flexible multi-body mechanism have been introduced in section **III.B**. In this appendix we detail the derivation of inertia terms arising from the kinetic energy $\mathcal{T} = \int_\Omega \frac{1}{2}\rho\dot{\underline{x}}^2 d\Omega$.

$$\frac{\partial \mathcal{T}}{\partial \mathbf{p}} - \frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{p}}} \tag{47}$$

denoting $T = \frac{1}{2}\rho\dot{\underline{x}}^2$, we have:

$$\int_\Omega \left(\frac{\partial T}{\partial \mathbf{p}} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{p}}}\right) d\Omega \tag{48}$$

Let us recall the expressions of position, velocity and acceleration:

$$\underline{x} = \underline{R} + \mathbf{A}(\theta)\underline{w} = \underline{R} + \mathbf{A}(\theta)\left(\underline{r} + \mathbf{N}\{u_e\}\right)$$
$$\dot{\underline{x}} = \dot{\underline{R}} + \dot{\theta}\mathbf{A}(\theta + \frac{\pi}{2})\underline{w} + \mathbf{A}(\theta)\dot{\underline{w}}$$
$$\ddot{\underline{x}} = \underbrace{\ddot{\underline{R}} + \ddot{\theta}\mathbf{A}(\theta + \frac{\pi}{2})\underline{w} - \dot{\theta}^2\mathbf{A}(\theta)\underline{w}}_{transport} + \underbrace{2\dot{\theta}\mathbf{A}(\theta + \frac{\pi}{2})\dot{\underline{w}}}_{corriolis} + \underbrace{\mathbf{A}(\theta)\ddot{\underline{w}}}_{relative}$$

They can be expressed in term of the generalized coordinate $\mathbf{p} = \left\{ \begin{array}{c} R \\ \theta \\ u_e \end{array} \right\}$:

$$\underline{x} = \left[ \begin{array}{ccc} \mathbf{I} & 0 & \mathbf{A}(\theta)\mathbf{N} \end{array} \right] \mathbf{p} + \mathbf{A}(\theta)\underline{r} \tag{49}$$
$$\dot{\underline{x}} = \left[ \begin{array}{ccc} \mathbf{I} & \mathbf{A}(\theta + \frac{\pi}{2})\underline{w} & \mathbf{A}(\theta)\mathbf{N} \end{array} \right] \dot{\mathbf{p}} \tag{50}$$
$$\ddot{\underline{x}} = \left[ \begin{array}{ccc} \mathbf{I} & \mathbf{A}(\theta + \frac{\pi}{2})\underline{w} & \mathbf{A}(\theta)\mathbf{N} \end{array} \right] \ddot{\mathbf{p}} - \dot{\theta}^2\mathbf{A}(\theta)\underline{w} + 2\dot{\theta}\mathbf{A}(\theta + \frac{\pi}{2})\dot{\underline{w}} \tag{51}$$

Using the derivation chain rule, the integrand of Eq. (48) also writes:

$$\frac{\partial T}{\partial \mathbf{p}} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{p}}} = \left( \frac{\partial \dot{x}}{\partial \mathbf{p}} - \frac{d}{dt}\frac{\partial \dot{x}}{\partial \dot{\mathbf{p}}} \right) \frac{\partial T}{\partial \dot{x}} - \frac{\partial \dot{x}}{\partial \dot{\mathbf{p}}}\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} \tag{52}$$

Each term of the above equation details as follows:

$$\frac{\partial T}{\partial \dot{x}} = \rho \dot{\underline{x}} \tag{53}$$
$$\frac{\partial \dot{x}}{\partial \dot{\mathbf{p}}} = \left[ \begin{array}{c} \mathbf{I} \\ \left(\mathbf{A}(\theta + \frac{\pi}{2})\underline{w}\right)^T \\ \left(\mathbf{A}(\theta)\mathbf{N}\right)^T \end{array} \right] \tag{54}$$
$$\frac{\partial \dot{x}}{\partial \mathbf{p}} = \left[ \begin{array}{c} 0 \\ -\dot{\theta}\left(\mathbf{A}(\theta)\underline{w}\right)^T + \left(\mathbf{A}(\theta + \frac{\pi}{2})\dot{\underline{w}}\right)^T \\ \dot{\theta}\left(\mathbf{A}(\theta + \frac{\pi}{2})\mathbf{N}\right)^T \end{array} \right] = \frac{d}{dt}\frac{\partial \dot{x}}{\partial \dot{\mathbf{p}}} \tag{55}$$

As $\frac{\partial \dot{x}}{\partial \mathbf{p}} = \frac{d}{dt}\frac{\partial \dot{x}}{\partial \dot{\mathbf{p}}}$, the first term of the RHS vanishes and Eq. (48) simply writes:

$$\int_\Omega \left( \frac{\partial T}{\partial \mathbf{p}} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{p}}} \right) d\Omega = \int_\Omega \rho \left[ \begin{array}{c} \mathbf{I} \\ \left(\mathbf{A}(\theta + \frac{\pi}{2})\underline{w}\right)^T \\ \left(\mathbf{A}(\theta)\mathbf{N}\right)^T \end{array} \right] \ddot{\underline{x}}\, d\Omega \tag{56}$$

which corresponds to the virtual power of the accelerations forces. If the constraints on the system are holonomic, this also corresponds to the virtual work of the inertia forces. The velocity indeed writes: $\dot{x} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial \mathbf{p}}\dot{\mathbf{p}}$ so that $\frac{\partial x}{\partial \mathbf{p}} = \frac{\partial \dot{x}}{\partial \dot{\mathbf{p}}}$. Using Eq. (51), we obtain:

$$\int_\Omega \left( \frac{\partial T}{\partial \mathbf{p}} - \frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{p}}} \right) d\Omega = \int_\Omega \rho \left[ \begin{array}{c} \mathbf{I} \\ \left(\mathbf{A}(\theta + \frac{\pi}{2})\underline{w}\right)^T \\ \left(\mathbf{A}(\theta)\mathbf{N}\right)^T \end{array} \right] \left[ \begin{array}{ccc} \mathbf{I} & \mathbf{A}(\theta + \frac{\pi}{2})\underline{w} & \mathbf{A}(\theta)\mathbf{N} \end{array} \right] d\Omega \ddot{\mathbf{p}} \tag{57}$$
$$- \dot{\theta}^2 \int_\Omega \rho \left[ \begin{array}{c} \mathbf{A}(\theta)\underline{w} \\ 0 \\ \mathbf{N}^T\underline{w} \end{array} \right] d\Omega \tag{58}$$
$$+ 2\dot{\theta} \int_\Omega \rho \left[ \begin{array}{c} \mathbf{A}(\theta + \frac{\pi}{2})\mathbf{N} \\ \underline{w}^T\mathbf{N} \\ \mathbf{N}^T\mathbf{A}(\frac{\pi}{2})\mathbf{N} \end{array} \right] \{\dot{u_e}\}\, d\Omega \tag{59}$$
$$= -\mathbf{M}(\mathbf{p})\ddot{\mathbf{p}} + \{f(\mathbf{p}, \dot{\mathbf{p}})\} \tag{60}$$

where:

$$\mathbf{M}(\mathbf{p}) = \begin{bmatrix} m_{tot}\mathbf{I}_d & m_{tot}\ \mathbf{A}(\theta+\frac{\pi}{2})\underline{O'G} & \mathbf{A}(\theta)\mathbf{L}_1 \\ & \mathbf{I}_z + 2\mathbf{I}_1\{u_e\} + \{u_e\}^T\mathbf{M}_{ee}\{u_e\} & \mathbf{I}_3 + \{u_e\}^T\mathbf{L}_3 \\ sym & & \mathbf{M}_{ee} \end{bmatrix} \tag{61}$$

$$\{f(\mathbf{p},\dot{\mathbf{p}})\} = \begin{Bmatrix} f_R \\ f_\theta \\ f_e \end{Bmatrix} = \dot{\theta}^2 \begin{Bmatrix} m_{tot}\mathbf{A}(\theta)\underline{O'G} \\ 0 \\ \mathbf{I}_1^T + \mathbf{M}_{ee}\{u_e\} \end{Bmatrix} - 2\dot{\theta} \begin{bmatrix} \mathbf{A}(\theta+\frac{\pi}{2})\mathbf{L}_1 \\ \mathbf{I}_1 + \{u_e\}^T\mathbf{M}_{ee} \\ \mathbf{L}_3^T \end{bmatrix} \{\dot{u}_e\} \tag{62}$$

The following notations have been used:

$$\begin{aligned} m_{tot}\ \underline{O'G_i} &= \int_\Omega \rho \underline{r} dV & \mathbf{I}_z &= \int_\Omega \rho \underline{r}^2 dV \\ \mathbf{I}_1 &= \int_\Omega \rho \underline{r}^T \mathbf{N} dV & \mathbf{L}_1 &= \int_\Omega \rho \mathbf{N} dV \\ \mathbf{I}_3 &= \int_\Omega \rho \underline{r}^T \mathbf{A}(-\frac{\pi}{2})\mathbf{N} dV & \mathbf{L}_3 &= \int_\Omega \rho \mathbf{N}^T \mathbf{A}(-\frac{\pi}{2})\mathbf{N} dV \end{aligned}$$

$$m_{tot}\ \underline{O'G} = \textstyle\int_\Omega \rho \underline{w} dV = m_{tot}\ \underline{O'G_i} + \mathbf{L}_1\{u_e\} \tag{63}$$

$\mathbf{M}_{ee}$, $\mathbf{K}_{ee}$: Linear finite element mass and stiffness matrices in the local axis

**Remarks**

- Depending on the choice of the reference frame, the expressions of the mass matrix and the generalized inertia forces simplify. For instance if the reference is initially attached to the center of mass of the undeformed configuration, the term $\underline{O'G_i}$ vanishes.
- In the framework of small deformations, the term $\{u_e\}^T\mathbf{M}_{ee}\{u_e\}$ being a second order term in $\{u_e\}$, can almost always be neglected compared to $\mathbf{I}_z$. This corresponds to neglecting an effect sometimes called "centrifugal stiffening" which typically introduces negative stiffness-like effects.

In this study we assume that the reference frame is initially positioned at the center of mass of the undeformed body, and that the dynamics of the system is such that the second order terms in $\{u_e\}$ can be neglected. The mass matrix and the generalized inertia forces thus have the following form:

$$\mathbf{M}(\mathbf{p}) = \begin{bmatrix} m_{tot}\mathbf{I}_d & \mathbf{A}(\theta+\frac{\pi}{2})\mathbf{L}_1\{u_e\} & \mathbf{A}(\theta)\mathbf{L}_1 \\ & \mathbf{I}_z + 2\mathbf{I}_1\{u_e\} & \mathbf{I}_3 + \{u_e\}^T\mathbf{L}_3 \\ sym & & \mathbf{M}_{ee} \end{bmatrix} \tag{64}$$

$$\{f(\mathbf{p},\dot{\mathbf{p}})\} = \begin{Bmatrix} f_R \\ f_\theta \\ f_e \end{Bmatrix} = \dot{\theta}^2 \begin{Bmatrix} \mathbf{A}(\theta)\mathbf{L}_1\{u_e\} \\ 0 \\ \mathbf{I}_1^T + \mathbf{M}_{ee}\{u_e\} \end{Bmatrix} - 2\dot{\theta} \begin{bmatrix} \mathbf{A}(\theta+\frac{\pi}{2})\mathbf{L}_1 \\ \mathbf{I}_1 + \{u_e\}^T\mathbf{M}_{ee} \\ \mathbf{L}_3^T \end{bmatrix} \{\dot{u}_e\} \tag{65}$$

- When the reference frame is located at all time at the current center of mass, the terms $\underline{O'G}$, $\underline{v}_{G/\mathfrak{R}'} = \mathbf{L}_1\{\dot{u}_e\}$, and $\underline{a}_{G/\mathfrak{R}'} = \mathbf{L}_1\{\ddot{u}_e\}$ vanish. Where $\underline{v}_{G/\mathfrak{R}'}$ and $\underline{a}_{G/\mathfrak{R}'}$ denote the velocity and acceleration of the center of mass relative to the reference frame $\mathfrak{R}'$. This condition corresponds to the one impled by the orthogonality relation between rigid and free vibration modes (see section **V**.**A**).

## References


[1]Shabana, A., "Dynamics of Multibody Systems (3rd edn)," *Cambridge University Press*, 2005.

[2]Geradin, M. and Cardona, A., "Flexible Multibody Dynamics: A Finite Element Approach," *Wiley*, 2001.

[3]Rixen, D., "A dual Craig-Bampton method for dynamic substructuring," *Journal of Computational and Applied Mathematics*, Vol. 168, 2003, pp. 383–391.

[4]Craig, R. and Bampton, M., "Coupling of Substructures for Dynamic Analyses," *AIAA Journal*, Vol. 6, No. 7, 1968, pp. 1313–1319.

[5]Rixen, D., "A Dual Craig-Bampton with Enrichment to Avoid Spurious Modes," *IMAC XXVII: International Modal Analysis Conference*, 9-12 February 2009.

[6]R. Schwertassek, O. W. and Shabana, A., "Flexible multibody simulation and choice of shape functions," *Nonlinear Dynamics*, 1999, pp. 361380.

[7]Shabana, A., "Resonance conditions and deformable body co-ordinate systems," *Journal of Sound and Vibration*, 1996, pp. 389398.

[8]MacNeal, R., "A hybrid method of component mode synthesis," *Computers & Structures*, Vol. 1, 1971, pp. 581–601.

[9]Rubin, S., "Improved component-mode representation for structural dynamic analysis," *AIAA JOURNAL*, Vol. 13, 1975, pp. 995–1006.

[10]Meijaard, J., "Validation of flexible beam element in dynamics programs," *Nonlinear Dynamics*, Vol. 9, 1996, pp. 21–36.

[11]Schwab, A., *Dynamics of flexible multibody systems; small vibration superimposed on a general rigid body motion*, Ph.D. thesis, Delft University of Technology, 2002.

[12]Gradin, M. and Rixen, D., *Thorie des vibrations*, Masson, 1993.

[13]Hurty, W., "Vibrations of structural systems by component-mode synthesis," *Journal of Engineering Mechanics ASCE*, Vol. 86, 1960, pp. 51–69.

[14]Cardona, A. and Geradin, M., "Superelements modelling in flexible Multibody Dynamics," *Multibody System Dynamics*, Vol. 4, 2000, pp. 245–266.

[15]Cardona, A., "Modelling of superelements in mechanism analysis," *International Journal For Numerical Methods in Engineering*, Vol. 32, 1991, pp. 1565–1593.

[16]de Veubeke, B. F., "The Dynamic of Flexible Bodies," *International Journal of Engineering Science*, Vol. 14, 1976, pp. 895–913.

[17]Argawal, P. and Shabana, A., "Dynamic Analysis of Multibody Systems using Component Modes," *International Journal of Engineering Science*, Vol. 14, 1976, pp. 895–913.

[18]Song, J. and Haug, E., "Dynamic Analysis of Planar Flexible Mechanisms," *Computer Methods in Applied Mechanics and Engineering*, Vol. 24, 1980, pp. 359–381.