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LINEARIZED THEORY OF FLOW WITH FINITE CAVITIES ABOUT A WING.

by

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1. Introduction.

In the design of ship propellor blade sections cavitation phenomena play a dominant part. It is well known (ref. (1), chap. II) that a rigorous mathematical theory of cavitational flow, based on the concept of free boundaries, is rather complicated and moreover inadequate for quantitative results. Since propellor blade sections are extremely thin, it lies at hand to replace the exact hodograph plane theory by linearized theory which is far more simple. This method seems proper to a treatment of film cavitation, where the evaporated water spreads along a finite part of the blade like a thin film.

In this paper the theory is given for the case of a plane wing at a small angle of attack, where the calculations are relatively simple. In ref. (2) the results of measurements are given concerning the pressure distributions along Karman-Trefftz profiles in a cavitation tunnel. From these results a rough approximation for the actual cavitation length can be derived. These quantities are used for comparison with the theoretical results. The cavitation length is not determined uniquely as a function of the cavitation number $\sigma = \dfrac{p_\infty - p}{\frac{1}{2}\rho U_\infty^2}$ and the angle of incidence by the condition ~~alone~~ that the cavitation bubble must be closed. Another requirement is needed, for which two alternative possibilities are introduced. The first is the strong Kutta condition. The second is the requirement, that the pressure should be continuous at the rear end of the cavitation bubble. This amounts to the condition, that the tangent to the cavitation bubble there should be directed along the profile.

2. Mathematical formulation of the problem.

Neglecting thickness effects we replace the blade section by its camber line, the chord of which has its centre at the origin of a rectangular coordinate system, is of length 2 and makes an angle $-\alpha$ with the positive x axis. The undisturbed velocity is directed along the positive x axis and has a magnitude U_∞ (see fig. 1).

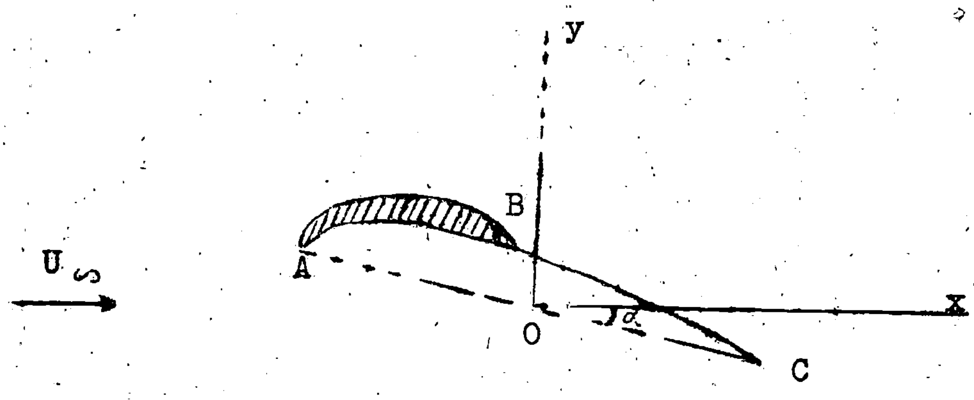


fig. 1.

The blade section causes a disturbance velocity field (u,v), which satisfies the following equations

$u_x + v_y = 0$ continuity equation for incompressible flow.

$u_y - v_x = 0$ irrotational flow.

The boundary conditions are :

i) $u \rightarrow 0$, $v \rightarrow 0$ at infinity

ii) on the cavitation bubble the pressure is a constant.
In linearized theory the condition is satisfied on the projection of the cavitation bubble on the x axis. This projection is assumed to extend from $x = -1$ (leading edge) to a point $x = \ell$ on the suction side.

iii) at the remaining part of the blade the total velocity must be tangential to the contour. This condition too is satisfied on the projection of the camber line on the x axis, extending from $x = -1$ (leading edge) to $x = +1$ (trailing edge).

As to condition iv) two possibilities are investigated in this paper i.e.

a) at the trailing edge the velocity must satisfy the strong Kutta condition: $u = 0$ at $x = +1$, or

b) at the rear end of the cavitation bubble the velocity must be tangential to the contour. This turns out to be aequivalent to the condition that the pressure must be continuous at that point (see later).
In case a) the velocity has a direction normal to the contour and the pressure is discontinuous.

In linearized theory Bernouilli's equation

$p + \frac{1}{2}\rho \left\{(U_\infty + u)^2 + v^2\right\} = p_\infty + \frac{1}{2}\rho U_\infty^2$ takes the simple form

$$\frac{p_\infty - p}{\rho U_\infty} = u \quad .$$

Introducing the cavitation number $\sigma = \dfrac{p_\infty - p}{\frac{1}{2}\rho U_\infty^2}$ we find as the condition ii) on the cavitation bubble $\dfrac{u}{U_\infty} = \dfrac{\sigma}{2}$

The linearized condition iii) can be written as

$\dfrac{v}{U_\infty} = \alpha_{loc}$, where α_{loc} is the local angle of incidence.

Altogether the following boundary conditions must be satisfied.

1) $u \longrightarrow 0, \; v \longrightarrow 0 \qquad \text{for } x^2 + y^2 \longrightarrow \infty$

2) $u = \frac{1}{2}\sigma U_\infty \qquad -1 < x < \ell \qquad y = +0$

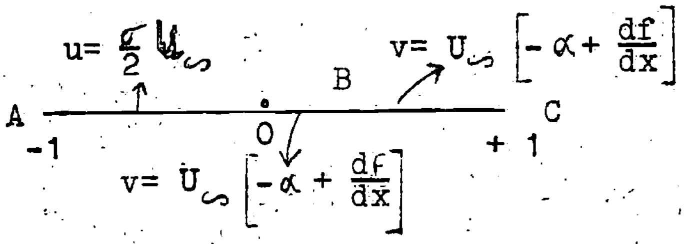
$\quad v = U_\infty \left[-\alpha + \dfrac{df}{dx} \right] \qquad \ell < x < 1 \qquad y = +0$

$\qquad\qquad\qquad\qquad\qquad -1 < x < 1 \qquad y = -0$

where $y = f(x)$ is the equation for the camber line.

3a) $u = 0$ at $x = +1$, $y = 0$

or 3b) $v = 0$? at $x = \ell$, $y = +0$ discontinuïteit van v in $x = \ell$. ? ja



z-plane.

fig. 2.

3. Solution of the problem.

The complex velocity $w = u - iv$ is an analytic function of the complex variable $z = x + iy$, since its real and imaginary part satisfy the Cauchy-Riemann equations.

The z-plane is mapped conformally on the lower half plane of a $\zeta = \xi + i\eta$ plane by the transformation $\zeta = \sqrt{\dfrac{1-z}{1+z}}$. The problem in the ζ plane is now to determine an analytic function $w = u - iv$ in the half plane $\eta < 0$, satisfying the conditions

i) $\eta = 0 \quad \xi < b \; : \; v = \alpha_{loc} = -\alpha + F(\xi)$

$\qquad\qquad\; \xi > b \; : \; u = \dfrac{\sigma}{2}$

ii) $\qquad \zeta = -i \; : \; u = v = 0$

iii) a) $\zeta = 0 \; : \; u = 0$

or b) $\eta = 0 \;\; \xi = b + 0 \; : \; v = \alpha_{loc}$

where $b = \sqrt{\dfrac{1-\ell}{1+\ell}}$ or with $\ell = \cos\gamma$, $b = \operatorname{tg}\dfrac{\gamma}{2}$

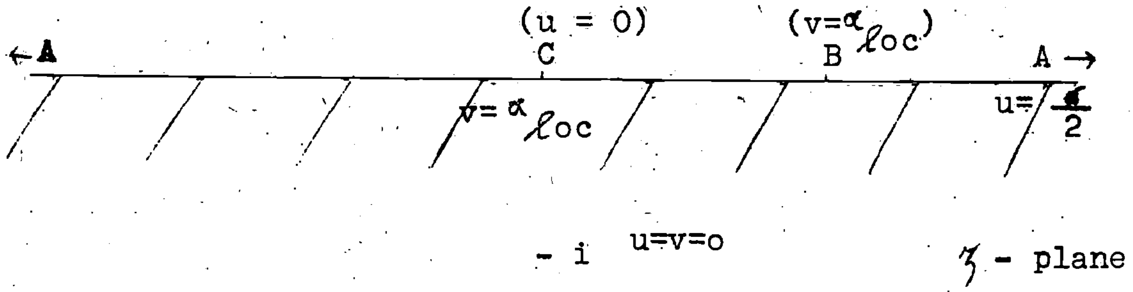


fig. 3.

Remark: the velocity has been made dimensionless with respect to U_∞

The problem can now be reduced to a Riemann-Hilbert problem. - First the analytic continuation of $w(\zeta)$ into the upper halfplane by means of Schwarz's principle $w(\zeta) = \overline{w(\bar{\zeta})}$ is introduced, where the bar denotes the conjugate complex value. If the limiting values of the function w, when ζ approaches the ξ axis from the upper and lower side, are denoted by w^+ and w^-, this function has to satisfy the following conditions.
w is a holomorphic function in the complex ζ plane with the exception of the real axis. There holds

$$w^+ - w^- = -2i\alpha_{\ell oc} = 2i\left[\alpha - F\right] \qquad \text{for } \xi < b$$
$$w^+ + w^- = 2 \cdot \frac{\sigma}{2} \qquad \text{for } \xi > b$$

Following Muskhelishvili (3) first the homogeneous problem is solved. The boundary conditions are

$$\left.\begin{aligned} w_h^+ - w_h^- &= 0 \qquad \xi < b \\ w_h^+ + w_h^- &= 0 \qquad \xi > b \end{aligned}\right\} \eta = 0$$

The general solution is $w_h = \dfrac{P(\zeta)}{\sqrt{b-\zeta}}$, where $P(\zeta)$ is a polynomial in ζ with real coefficients.
There is assumed that w_h is of finite degree at infinity and that the function is integrable along the ξ - axis.

With the aid of the Plemelj formula the general solution of the original problem is now derived as

$$w(\zeta) = \frac{wh}{2\pi i}\int_{-\infty}^{b} \frac{2i\,F(\xi)\,d\xi}{w_h\,(\xi-\zeta)} + \frac{P(\zeta)}{\sqrt{b-\zeta}} + \frac{\sigma}{2} + i\alpha$$

where $w_h(\zeta) = \sqrt{b-\zeta}$. See Muskhelishvili (3).

The conditions ii) and iii) together with the requirements that the pressure i.e. u should be integrable along the contour and that the cavitation bubble should be closed, lead to equations determining the cavitation length $\frac{1+\ell}{2} = \cos^2 \frac{\gamma}{2}$ as function of α and σ. These calculations will be performed in the next section for the case of a flat plate at a small angle of attack.

4. Application to the case of a flat plate.

For a flat plate the general solution of the inhomogeneous boundary value problem

$$w^+ - w^- = -2i\alpha$$

$$w^+ + w^- = 2 \cdot \frac{\sigma}{2}$$

is easily found to be $w(\zeta) = \frac{P(\zeta)}{\sqrt{b-\zeta}} + \frac{\sigma}{2} + i\alpha$.

At first we argue, that the degree of $P(\zeta)$ cannot be higher than 2. Suppose it has the value n. Since the leading edge of the profile is mapped by the conformal transformation on the point $\zeta = \infty$, it is seen that in the neighbourhood of the leading edge there holds

$$w(\zeta) = O(\zeta^{n-\frac{1}{2}}) = O((1+z)^{-\frac{n}{2}+\frac{1}{4}})$$

But u(x) must be integrable and this leaves open only the values $n = 0,1,2$. Hence we may put

$$w(\zeta) = \frac{A\zeta^2 + B\zeta + C}{\sqrt{b-\zeta}} + \frac{\sigma}{2} + i\alpha .$$

The constants A, B and C are now determined from the conditions, that at

$$\zeta = -i, \quad u = v = 0$$

and that the cavitation bubble must be closed.
This last condition takes the form

$$\oint v\,dx = \oint (v+\alpha)\,dx = \int_{-1}^{\ell} (v+\alpha)\,dx = \int_{-1}^{\ell} \frac{dg}{dx}\,dx = 0$$

where $y = g(x)$ is the equation for the cavitation bubble and the first two integrations are performed along the segment $-1 < x < +1$ at both sides and in clockwise direction. With the help of contour integration this is equivalent to

$$-\operatorname{Im} \oint w\,dz = -\operatorname{Im} 2\pi i . \left\{ \text{residue } w \text{ at } z = \infty \right\} = 0$$

or $\operatorname{Re} \left\{ \text{residue } w \text{ at } z = \infty \right\} = 0$

Expressions for the lift L and the moment M with respect to the origin are found in an analogous manner.

$$L = \int_{-1}^{+1} (p_- - p_+)dx = -\rho\, U_\infty^2 \int_{-1}^{+1} (u_- - u_+)dx =$$

$$= \rho U_\infty^2 \int_{-1}^{+1} (u_+ - u_-)dx = \rho\, U_\infty^2 \oint udx = \mathrm{Re}\,\rho U_\infty^2 \oint wdz =$$

$$= \mathrm{Re}\; 2\pi i \rho U_\infty^2 \left\{ \text{res.} w \text{ at } z = \infty \right\} = -\, \mathrm{Im}\; 2\pi\rho\; U_\infty^2 \left\{ \text{res.} w \text{ at } z = \infty \right\}.$$

Thus

$$C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 . 2} = -\ \mathrm{Im}\; 2\pi \left\{ \text{res.} w \text{ at } z = \infty \right\}$$

$$M = \int_{-1}^{+1} (p_- - p_+)xdx = \rho\, U_\infty^2 \oint u\; x\; dx =$$

$$= -\ \mathrm{Im}\; 2\pi\, \rho U_\infty^2 \left\{ \text{res.} wz \text{ at } z = \infty \right\}$$

$$C_M = \frac{M}{\frac{1}{2}\rho U_\infty^2 . 4} = -\ \mathrm{Im}\, \pi \left\{ \text{res. } wz \text{ at } z = \infty \right\}$$

Further area cavitation bubble $= \int_{-1}^{l} g(x)dx =$

$$= \oint \left[g(x) - \alpha x \right] dx = -\oint x \left[\frac{dg}{dx} - \alpha \right] dx =$$

$$= -\oint v\; x\; dx = +\ \mathrm{Im} \oint w\; z\; dz =$$

$$= +\ \mathrm{Re}\; 2\pi \left\{ \text{res. } wz \text{ at } z = \infty \right\}$$

Therefore it appears that for a calculation of these quantities only a Taylor expansion of $w(z)$ in the vicinity of $z = \infty$ is needed :

$$w(z) = \frac{\sigma}{2} + i\alpha + \frac{-A-Bi+C}{\sqrt{b+i}} + \left[2A+Bi+(-A-Bi+C)\frac{1+bi}{2(1+b^2)} \right] \times$$

$$\times \frac{1}{\sqrt{b+i}\,(z+1)} + \left[(-A-Bi+C)\left(\frac{1+bi}{4(1+b^2)} + \frac{3(1+bi)^2}{8(1+b^2)^2}\right) + (2A+Bi)\frac{1+bi}{2(1+b^2)} \right.$$

$$\left. + \frac{Bi}{2} \right] \frac{1}{\sqrt{b+i}\,(z+1)^2} + \ldots \qquad \text{where } \sqrt{b+i} = \frac{1}{\sqrt{\cos\frac{\gamma}{2}}}\, e^{\frac{\pi-\gamma}{4} i}$$

From the above result the following equations determining the real constants A, B and C :

$$\frac{-A-Bi+C}{\sqrt{b+i}} + \frac{\sigma}{2} + i\alpha = 0$$

$$\text{Re}\left\{ \left[2A+Bi+(-A-Bi+C)\,\frac{1+bi}{2(1+b^2)} \right] \times \frac{1}{\sqrt{b+i}} \right\} = 0$$

After substitution of tg $\frac{\gamma}{2}$ for b, the solution is found as

$$A = -\frac{2\alpha\cos\frac{\gamma}{2}\,(1+\sin\frac{\gamma}{2}) - \sigma(1-\sin\frac{\gamma}{2})\sin\frac{\gamma}{2}}{8\sqrt{\cos\frac{\gamma}{2}}\,\cos\frac{\pi-\gamma}{4}}$$

$$B = \frac{\sigma\sin\frac{\pi-\gamma}{4} + 2\alpha\cos\frac{\pi-\gamma}{4}}{2\sqrt{\cos\frac{\gamma}{2}}}$$

$$C = \frac{2\alpha\cos\frac{\gamma}{2}\,(1-\sin\frac{\gamma}{2}) - \sigma\,(2+\sin\frac{\gamma}{2}+\sin^2\frac{\gamma}{2})}{8\sqrt{\cos\frac{\gamma}{2}}\,\cos\frac{\pi-\gamma}{4}}$$

With the help of these values for A, B and C expressions for C_L , C_M and the area of the cavitation bubble are derived

$$C_L = \pi\left[\alpha\,(1+\sin\frac{\gamma}{2}) + \sigma\,\frac{(1-\sin\frac{\gamma}{2})^2}{2\cos\frac{\gamma}{2}}\right]$$

$$C_M = \frac{\pi}{8}\left[\alpha\left\{-2-2\sin\frac{\gamma}{2}+\cos^2\frac{\gamma}{2}+2\sin\frac{\gamma}{2}\cos^2\frac{\gamma}{2}-2\cos^4\frac{\gamma}{2}\right\}\right.$$
$$+\frac{\sigma}{\cos\frac{\gamma}{2}}\left\{-4+4\sin\frac{\gamma}{2}+5\cos^2\frac{\gamma}{2}-3\sin\frac{\gamma}{2}\cos^2\frac{\gamma}{2}-\cos^4\frac{\gamma}{2}-\right.$$
$$\left.\left.-\sin\frac{\gamma}{2}\cos^4\frac{\gamma}{2}\right\}\right]$$

area cavitation bubble =

$$=\frac{\pi}{8}\left[4\alpha\cos^3\frac{\gamma}{2}\,(1+\sin\frac{\gamma}{2})+\sigma\left\{6-6\sin\frac{\gamma}{2}-5\cos^2\frac{\gamma}{2}+\right.\right.$$
$$\left.\left.+2\sin\frac{\gamma}{2}\cos^2\frac{\gamma}{2}-2\cos^4\frac{\gamma}{2}\right\}\right]$$

At last the condition 3,iii) a) or b) is used to determine $\frac{\gamma}{2}$ as a function of α and σ.

a) gives

$$\frac{\alpha}{\sigma} = \frac{3+\sin\frac{\gamma}{2} - \cos^2\frac{\gamma}{2} - 4\sqrt{\sin\frac{\gamma}{2}}\cos\frac{\pi-\gamma}{4}}{2\cos\frac{\gamma}{2}\left(1-\sin\frac{\gamma}{2}\right)}$$

b) gives

$$\frac{\alpha}{\sigma} = \frac{\left(1-\sin\frac{\gamma}{2}\right)\left(2+\sin\frac{\gamma}{2}\right)}{2\cos\frac{\gamma}{2}\left(1+\sin\frac{\gamma}{2}\right)}$$

In fig. 4 the cavitation length $\frac{1+\ell}{2} = \cos^2\frac{\gamma}{2}$ is plotted against $\frac{\alpha}{\sigma}$. There are two curves corresponding to the two hypotheses a) and b). The isolated points represent values derived from the experimental results of Balhan concerning the pressure distribution along Karman-Trefftz profiles. As cavitation length has been taken the length of the part of the profile, along which the pressure was nearly constant, divided by the length of the profile.

References:

(1) G. Birkhoff, Hydrodynamics, Princeton 1950.

(2) J. Balhan, Metingen aan enige bij scheepsschroeven gebruikelijke profielen in vlakke stroming met en zonder cavitatie, Diss. Delft, 1951.

(3) N.I. Muskhelishvili, Singular integral equations. Groningen 1953.

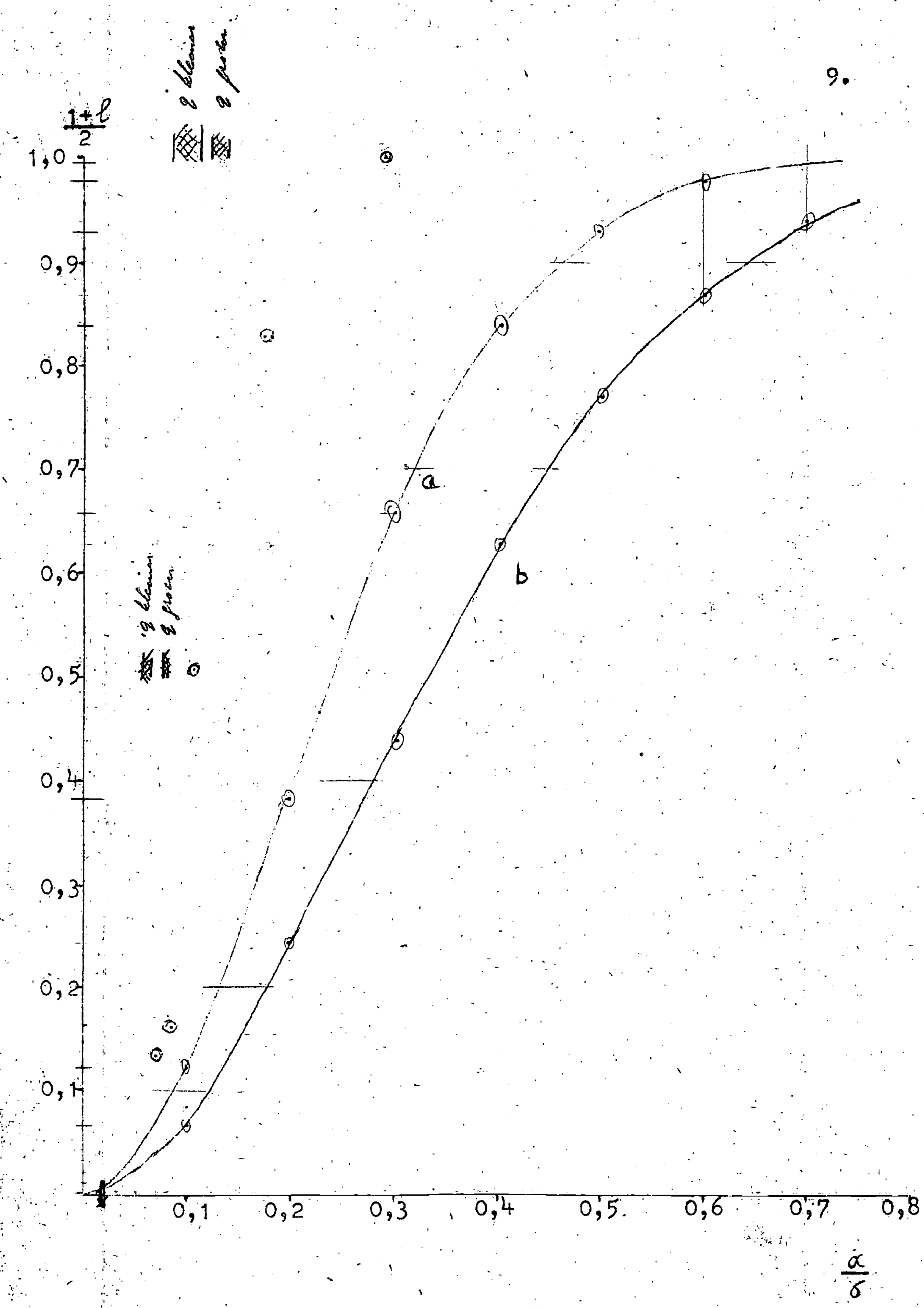

$\frac{1+\ell}{2}$
1,0
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
a
b
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
$\frac{\alpha}{\sigma}$

fig. 4.