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Minimum Pearson Distance Detection Using a Difference Operator in the Presence of Unknown Varying Offset

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***Abstract*—We consider noisy data transmission channels with unknown scaling and varying offset mismatch. Minimum Pearson distance detection is used in cooperation with a difference operator, which offers immunity to such mismatch. Pair-constrained codes are proposed for unambiguous decoding, where in each codeword certain adjacent symbol pairs must appear at least once. We investigate the cardinality and redundancy of these codes.**

***Index Terms*—Unknown scaling and offset, minimum Pearson distance detection, difference operator, pair-constrained code**

I. Introduction

Fix two integers $n \geq 1$, $q \geq 2$, and denote $[q] = \{0, 1, \ldots, q-1\}$. A codebook $\mathcal{S}$ is a subset of $[q]^n$, where $[q]$ serves as the alphabet and n as the codeword length. We consider transmitting a codeword $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ from $\mathcal{S}$. The received vector $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ is given by

$$\mathbf{r} = a(\mathbf{x} + \mathbf{v}) + b\mathbf{1} + c\mathbf{s}, \tag{1}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$ and $\mathbf{s} = (1, 2, \ldots, n)$. The basic premises are that $\mathbf{x}$ is suffering from (i) additive Gaussian noise $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, where $v_i \in \mathbb{R}$ are i.i.d. noise samples with normal distribution $\mathcal{N}(0, \sigma^2)$, where $\sigma^2 \in \mathbb{R}$ denotes the noise variance, (ii) an unknown (positive) scaling factor a, $a > 0$, and (iii) an unknown varying offset, $b\mathbf{1} + c\mathbf{s}$, where $b, c \in \mathbb{R}$.

There are many examples of channels with scaling and varying offset mismatch. In flash memories, physical features like the device temperature will result in rapidly scaling and offset variations of the retrieved signal [1]. Memory cells closer to hotter areas on the chip may lose their charge faster than cells closer to colder areas. For direct conversion receivers, the local oscillator is the main source of dc-offset [2]. With fading and multi-path reception, the received power level can vary rapidly, which results in a time-varying or dynamic dc-offset.

Minimum Pearson distance (MPD) detection [3] has been shown to be intrinsically resistant to the scaling a and offset b, where a and b may change from word to word, but are constant for all transmitted symbols within a codeword. Here, we consider the situation in which the offset varies linearly within a codeword, where the slope of the offset, represented by the parameter c, is unknown. A detection scheme for channels with scaling and such varying offset is investigated in [4], where, for the binary case, MPD detection is used in conjunction with mass-centered codewords, in such a way that the system is insensitive to both scaling and varying offset, i.e., it is (a, b, c)-immune. However, this scheme is very expensive in terms of redundancy.

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In this paper, we show that the combination of MPD and a difference operator is (a, b, c)-immune as well. In addition, pair-constrained codes, where in each codeword certain adjacent symbol pairs must appear at least once, are proposed to achieve unambiguous decoding. The redundancy of pair-constrained codes is much lower than that of prior art mass-centered codes, which makes the new decoding scheme an attractive alternative for practical applications.

We start in Section II with a brief description of the prior art. Section III presents the backbone of the paper, where it is shown how an MPD detector can be used together with the difference operator. In Section IV, we introduce pair-constrained codes and we investigate their cardinality and redundancy. In Section V, we discuss the results of the paper and provide options for future research.

II. Prior Art

For $\mathbf{u} \in \mathbb{R}^n$, let $\bar{\mathbf{u}} = \frac{1}{n}\sum_{i=1}^{n} u_i$ and $\sigma_{\mathbf{u}}^2 = \sum_{i=1}^{n}(u_i - \bar{\mathbf{u}})^2$. The Pearson distance between vectors $\mathbf{u}$ and $\mathbf{w}$ is defined by

$$L_p(\mathbf{u}, \mathbf{w}) = 1 - \rho_{\mathbf{u},\mathbf{w}}, \tag{2}$$

where

$$\rho_{\mathbf{u},\mathbf{w}} = \frac{\sum_{i=1}^{n}(u_i - \bar{\mathbf{u}})(w_i - \bar{\mathbf{w}})}{\sigma_{\mathbf{u}}\sigma_{\mathbf{w}}} \tag{3}$$

is the well-known Pearson correlation coefficient. It has the property that

$$L_p(\mathbf{u}, \mathbf{w}) = L_p(c_1\mathbf{u} + c_2\mathbf{1}, \mathbf{w}) \tag{4}$$

for all $c_1 > 0$ and $c_2 \in \mathbb{R}$. Hence, the Pearson distance offers immunity to scaling and non-varying offset mismatch, which has lead to the introduction of the minimum Pearson distance (MPD) detector [3], that chooses among all candidate codewords $\hat{\mathbf{x}} \in \mathcal{S}$ the codeword $\mathbf{x}_o$ whose Pearson distance to the received vector $\mathbf{r}$ is smallest, i.e.,

$$\mathbf{x}_o = \arg\min_{\hat{\mathbf{x}} \in \mathcal{S}} L_p(\mathbf{r}, \hat{\mathbf{x}}).$$

In case of varying offset, mass-centered codes in combination with the MPD detector are advocated in [4] for the binary case, where the codebook $\mathcal{S}^* \subseteq [2]^n$ is chosen such that each codeword $\mathbf{x} \in \mathcal{S}^*$ satisfies

$$\sum_{i=1}^{n} \left(i - \frac{n+1}{2} \right) x_i = 0.$$

The error performance of the MPD detector with the employment of mass-centered codes is insensitive to scaling and varying offset mismatch, i.e., (a, b, c)-immune. However, the redundancy is $O(\log n)$ [4]. In this paper, we will propose a less redundant scheme that also guarantees (a, b, c)-immunity.

III. MPD Detection Using a Difference Operator

Define the difference operator of a vector $\mathbf{u} \in \mathbb{R}^n$ as

$$\Delta\mathbf{u} = \mathbf{u}_{2,n} - \mathbf{u}_{1,n-1}, \tag{5}$$

where $\mathbf{u}_{i,j} = (u_i, u_{i+1}, \ldots, u_j)$ for all $1 \le i \le j \le n$.

For any codeword $\mathbf{x} \in \mathcal{S}$, we call $\Delta\mathbf{x}$ its difference codeword. The difference codebook, $\Delta\mathcal{S}$, is defined by $\Delta\mathcal{S} = \{\Delta\mathbf{x} \,|\mathbf{x} \in \mathcal{S}\}$. This is a set of codewords of length $n-1$ over the alphabet $Q' = \{-(q-1), \ldots, -1, 0, 1, \ldots, q-1\}$.

We now show that the use of the difference operator will make Pearson distance based detection (a, b, c)-immune. Upon receipt of a vector $\mathbf{r}$, we find the difference vector $\Delta\mathbf{r}$ and then the MPD detector chooses the member in $\Delta\mathcal{S}$ which has the smallest Pearson distance to $\Delta\mathbf{r}$, i.e.,

$$\Delta\mathbf{x}_o = \underset{\Delta\hat{\mathbf{x}} \in \Delta\mathcal{S}}{\arg\min}\, L_p(\Delta\mathbf{r}, \Delta\hat{\mathbf{x}}), \tag{6}$$

Note that applying the difference operator (5) on the received vector gives

$$\begin{aligned} \Delta\mathbf{r} = \;& \mathbf{r}_{2,n} - \mathbf{r}_{1,n-1} \\ = \;& a(\mathbf{x}_{2,n} + \mathbf{v}_{2,n}) + b\mathbf{1} + c\mathbf{s}_{2,n} \\ & -(a(\mathbf{x}_{1,n-1} + \mathbf{v}_{1,n-1}) + b\mathbf{1} + c\mathbf{s}_{1,n-1}) \\ = \;& a(\mathbf{x}_{2,n} - \mathbf{x}_{1,n-1} + \mathbf{v}_{2,n} - \mathbf{v}_{1,n-1}) + c\mathbf{1} \\ = \;& a(\Delta\mathbf{x} + \Delta\mathbf{v}) + c\mathbf{1}, \end{aligned} \tag{7}$$

where each entry in $\Delta\mathbf{v}$ has the normal distribution $\mathcal{N}(0, 2\sigma^2)$. Based on the discussion in the previous section, now with c in the role of b, we can thus conclude that MPD detection in combination with the difference operator provides (a, b, c)-immunity.

As investigated in [3] and [5] for the case of $(a, b, 0)$-immunity, the codebook should satisfy certain properties in order to allow the use of MPD detection and to prevent ambiguous decoding options. For the case of (a, b, c)-immunity, a new class of codes with the required properties will be presented in the next section.

IV. Pair-Constrained Codes

In order to work well with an MPD detector, the codebook should satisfy the following two requirements [3], [5]: (i) it should not contain vectors $\mathbf{u}$ with $\sigma_{\mathbf{u}} = 0$, since it follows from (3) that the Pearson distance is undefined for such $\mathbf{u}$; (ii) the presence of a vector $\mathbf{w}$ in the codebook implies that all vectors $c_1\mathbf{w} + c_2\mathbf{1}$ with $c_1 > 0$, $c_2 \in \mathbb{R}$, and $(c_1, c_2) \neq (1, 0)$, should not appear in the codebook because of (4). In our case, these requirements must hold for $\Delta\mathcal{S}$, since the MPD detector operates on the difference codebook. Furthermore, we have the obvious additional requirement that (iii) the codebook should be designed in such a way that the difference operator is a one-to-one map from $\mathcal{S}$ to $\Delta\mathcal{S}$. In conclusion, we have the following three properties to be satisfied.

Property 1: $k\mathbf{1} \notin \Delta\mathcal{S}$ for all $k \in \mathbb{R}$.

Property 2: If $\Delta\mathbf{x} \in \Delta\mathcal{S}$, then $c_1\Delta\mathbf{x} + c_2\mathbf{1} \notin \Delta\mathcal{S}$ for all $c_1, c_2 \in \mathbb{R}$ with $(c_1, c_2) \neq (1, 0)$ and $c_1 > 0$.

Property 3: $\Delta : \mathcal{S} \to \Delta\mathcal{S}$ is a bijection.

We propose a code satisfying these properties. Pair-constrained codes consist of q-ary n-length codewords, where one or more reference adjacent symbol pairs (s, t), $s, t \in [q]$, must appear at least once, i.e., for each codeword $\mathbf{w}$ there is an i, $1 \le i \le n-1$ such that $w_i = s$ and $w_{i+1} = t$. In this paper, we use a specific set of pair-constrained codes denoted by $\mathcal{S}^p$. The set $\mathcal{S}^p$ contains all the vectors where both the adjacent symbol pairs $(0, q-1)$ and $(q-1, 0)$ appear at least once, i.e., for each codeword $\mathbf{w}$ there are i and j, $1 \le i, j \le n-1$ such that $w_i = 0$, $w_{i+1} = q-1$, $w_j = q-1$, and $w_{j+1} = 0$. This ensures that both the symbols '$q-1$' and '$-(q-1)$' appear at least once in each vector in $\Delta\mathcal{S}^p$. This observation is key in showing that the proposed code satisfies the three properties mentioned above, which we will do next.

Proof. Property 1 follows immediately from the fact that each word in $\Delta\mathcal{S}^p$ contains the symbols '$q-1$' and '$-(q-1)$'.

Property 2 follows by a similar argument as used in [3] for so-called T-constrained codes, which we adapt here to our setting for completeness. Suppose that both $\Delta\mathbf{x} \in \Delta\mathcal{S}^p$ and $c_1\Delta\mathbf{x} + c_2\mathbf{1} \in \Delta\mathcal{S}^p$ for some c_1 and c_2 as indicated in the property statement. Since $c_1 > 0$, the fact that both vectors contain the maximum symbol value '$q-1$' implies that $c_1(q-1) + c_2 = q-1$, while the fact that both vectors contain the minimum symbol value '$-(q-1)$' implies that $-c_1(q-1) + c_2 = -q+1$. Solving these two equations, we find $c_1 = 1$ and $c_2 = 0$ as the unique solution, which gives a contradiction and thus shows the result.

Property 3 easily follows by observing that for any $\mathbf{u}, \mathbf{w}$ in any code $\mathcal{S}$ it holds that $\Delta\mathbf{u} = \Delta\mathbf{w} \iff \mathbf{u} = \mathbf{w} + k\mathbf{1}$ for some $k \in \mathbb{R}$. In case $\mathcal{S} = \mathcal{S}^p$, the fact that $\mathbf{w}$ contains the minimum symbol value '0' implies that if $k > 0$ then $u_i = w_i + k > 0\ \forall i$, and if $k < 0$ then there exists a position j such that $w_j = 0$ and $u_j = w_j + k < 0$. These observations contradict that $\mathbf{u} \in \mathcal{S}^p$, which implies $k = 0$ and thus shows the bijective property for $\mathcal{S}^p$. □

It should be noted that not all pair-constrained codes are suitable to cooperate with an MPD detector. For example, when choosing the pairs $(1, 2)$ and $(2, 3)$ rather than $(0, q-1)$ and $(q-1, 0)$, the resulting code does not satisfy Property 3 if $q \ge 5$ and $n \ge 4$, since, e.g., both $(0, 1, 2, 3, 3, \ldots, 3)$ and $(1, 2, 3, 4, 4 \ldots, 4)$ have the same difference vector $(1, 1, 1, 0, 0, \ldots, 0)$.

A. Cardinality

The cardinality of $\mathcal{S}^p$ is denoted by $N(n)$. For the binary case, $q = 2$, we simply find that $N(n) = 2^n - 2n$, since $\mathcal{S}^p$ consists of all sequences in $\{0, 1\}^n$, except the sequences without or with only one transition of the $0 \to 1$ or $1 \to 0$ type, i.e., $(0,\dots,0,1,\dots,1)$, $(1,\dots,1,0,\dots,0)$, $(0,\dots,0)$, and $(1,\dots,1)$.

In general, we can calculate the number $N(n)$ as follows. Consider the complement set $\bar{\mathcal{S}}^p$ of $\mathcal{S}^p$ in $[q]^n$, and let $M(n) = \left|\bar{\mathcal{S}}^p\right|$ denote the cardinality of this complement set. We have

$$M(n) = |K_2| + |K_3| - |K_1|,$$

where

$$K_1 = \{\mathbf{x} \in [q]^n \,|(x_{i-1}, x_i) \notin \{(0, q-1), (q-1, 0)\}, \forall i = 2, \dots, n\},$$

$$K_2 = \{\mathbf{x} \in [q]^n \,|(x_{i-1}, x_i) \neq (0, q-1), \forall i = 2, \dots, n\},$$

$$K_3 = \{\mathbf{x} \in [q]^n \,|(x_{i-1}, x_i) \neq (q-1, 0), \forall i = 2, \dots, n\}.$$

Let $a_n = |K_1|$. We consider the following partition of K_1:

$$K_1{}^* = \{\mathbf{x} \in K_1 \,|x_n \in \{1, \dots, q-2\}\},$$

$$K_1{}^\circ = \{\mathbf{x} \in K_1 \,|x_n \in \{0, q-1\}\},$$

and let $a_n^* = |K_1^*|$ and $a_n^\circ = |K_1^\circ|$. Then we have the recursive relations $a_n^* = (q-2)(a_{n-1}^* + a_{n-1}^\circ)$ and $a_n^\circ = 2a_{n-1}^* + a_{n-1}^\circ$, from which it follows for all $n \geq 2$ that

$$\begin{aligned} a_n &= a_n^* + a_n^\circ \\ &= (q-2)(a_{n-1}^* + a_{n-1}^\circ) + 2a_{n-1}^* + a_{n-1}^\circ \\ &= (q-1)(a_{n-1}^* + a_{n-1}^\circ) + (q-2)(a_{n-2}^* + a_{n-2}^\circ) \\ &= (q-1)a_{n-1} + (q-2)a_{n-2} \end{aligned} \tag{8}$$

with initial conditions $a_0 = 1$ and $a_1 = q$.

Let $b_n = |K_2|$. Using the same method, we find for all $n \geq 2$ that

$$b_n = qb_{n-1} - b_{n-2} \tag{9}$$

with initial conditions $b_0 = 1$ and $b_1 = q$. The number of sequences in K_3 follows the same recurrence scheme as in K_2.

Since $N(n) = q^n - M(n)$ and $M(n) = |K_2| + |K_3| - |K_1| = 2b_n - a_n$, we have

$$N(n) = q^n + a_n - 2b_n, \tag{10}$$

from which we can derive the recursive relation

$$\begin{aligned} N(n) \;\; = \;\; & (2q-1)N(n-1) - (q^2 - 2q + 3)N(n-2) \\ & -(q^2 - 3q + 1)N(n-3) \\ & +(q-2)N(n-4) + 2q^{n-4} \end{aligned} \tag{11}$$

for all $n \geq 4$, with initial conditions $N(0) = 0$, $N(1) = 0$, $N(2) = 0$, and $N(3) = 2$. Relation (11) can be shown by replacing all $N(i)$, $n-4 \leq i \leq n$, by $q^i + a_i - 2b_i$, according to (10), and then (repeatedly) applying (8) and (9) on the a_i and b_i, $n-2 \leq i \leq n$, until expressions containing only a_{n-4}, a_{n-3}, b_{n-4}, and b_{n-3} are left. The results for the left-hand and right-hand sides are the same, which proves the claim.

Table I shows results of computations of $N(n)$ for binary and ternary codes. Also, for comparison purposes, it includes the sizes $N_o(n)$ of the binary mass-centered codes [4] mentioned in Section II. Since the all-zero and all-one sequences should be excluded, actually $N_o(n) - 2$ is presented. Note that the remaining binary mass-centered sequences are all in the binary pair-constrained code of the same length. However, this code contains many other sequences as well, and therefore $N(n)$ considerably exceeds $N_o(n) - 2$ in the binary case.

Table I
CODEBOOK SIZES $N_o(n) - 2$ AND $N(n)$.

n	$N_o(n) - 2$	$N(n), q = 2$	$N(n), q = 3$
4	2	8	12
5	6	22	54
6	6	52	214
7	18	114	790
8	16	240	2786
9	50	494	9516
10	46	1004	31746

B. Redundancy

Since the redundancy of $\mathcal{S}^p$ is equal to

$$r(n) = n - \log_q N(n), \tag{12}$$

it would be convenient for evaluation purposes to have an explicit expression for $N(n)$ rather than a recursive one. Here we will derive such an expression using generating functions, which are described in, e.g., [6].

We start by rewriting the recurrence (8) using the Kronecker delta symbol, such that it is valid for all $n \geq 0$ (assuming $a_n = 0$ for all $n < 0$):

$$a_n - (q-1)a_{n-1} - (q-2)a_{n-2} - \delta_{n0} - \delta_{n1} = 0. \tag{13}$$

Let the ordinary generating function of a_n be denoted by $A(z) = \sum_{n=0}^{\infty} a_n z^n$. Then we derive $A(z)$ by multiplying (13) by z^n and summing over n, which gives

$$\sum_{n=0}^{\infty} a_n z^n - (q-1) \sum_{n=0}^{\infty} a_{n-1} z^n - (q-2) \sum_{n=0}^{\infty} a_{n-2} z^n - 1 - z = 0.$$

We can rewrite the above equation as

$$A(z) - (q-1)zA(z) - (q-2)z^2 A(z) = 1 + z.$$

Hence, we have

$$A(z) = \frac{1+z}{1 - (q-1)z - (q-2)z^2}. \tag{14}$$

Similarly, we can rewrite (9) as

$$b_n - qb_{n-1} + b_{n-2} - \delta_{n0} = 0, \tag{15}$$

for all $n \geq 0$ (assuming $b_n = 0$ for all $n < 0$), which leads to the ordinary generating function of b_n being

$$B(z) = \sum_{n=0}^{\infty} b_n z^n = \frac{1}{1 - qz + z^2}. \tag{16}$$

Next, we find the power series of $A(z)$ and $B(z)$ by applying Taylor's theorem, in which a_n and b_n, respectively, appear as the coefficients of z^n. This results in

$$a_n = \frac{(q + \lambda - 1)^n (\lambda + q + 1) + (q - \lambda - 1)^n (\lambda - q - 1)}{2^{n+1} \lambda},$$

where

$$\lambda = \sqrt{q^2 + 2q - 7},$$

$$N(n) = q^n - 2U_n(q/2) + \frac{(q+\lambda-1)^n(\lambda+q+1)+(q-\lambda-1)^n(\lambda-q-1)}{2^{n+1}\lambda} \tag{17}$$

$$r(n) \approx \left[\frac{2U_n(q/2)}{q^n} - \frac{(q+\lambda-1)^n(\lambda+q+1)+(q-\lambda-1)^n(\lambda-q-1)}{2^{n+1}q^n\lambda}\right] / \ln q \tag{18}$$

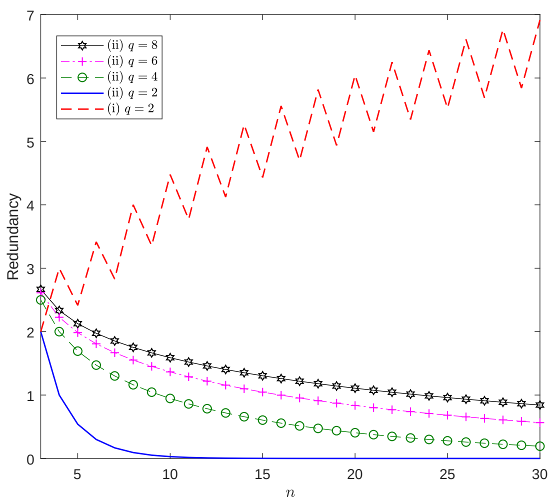

Figure 1. Redundancy versus codeword length n: (i) $r_o(n)$ for $q=2$; (ii) $r(n)$ for $q=2,4,6,8$.

and

$$b_n = U_n(q/2),$$

where

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}$$

is the Chebyshev polynomial of the second kind and $U_n(1) = n+1$. Hence, combining with (10) leads to the explicit expression for $N(n)$ given in (17). For example, we find for $q=3$ that

$$\begin{aligned} N(n) &= 3^n - \frac{(3+\sqrt{5})^{n+1}-(3-\sqrt{5})^{n+1}}{2^n\sqrt{5}} \\ &+\frac{1}{2}(1+\sqrt{2})^{n+1} + \frac{1}{2}(1-\sqrt{2})^{n+1}, \end{aligned}$$

which confirms the values in the most right column of Table I.

From (12) and the fact that $\log_q(1+x) \approx x/\ln q$ for small x, we obtain the approximate expression given in (18) for the redundancy of $\mathcal{S}^p$. Figure 1 shows the redundancy of $\mathcal{S}^p$ as a function of the codeword length n for $q=2,4,6,8$. As we can see, $r(n)$ approaches 0 as the codeword length increases, and the rate of convergence to 0 decreases as q grows. Also included in the figure is the redundancy of binary mass-centered codes, $r_o(n) = n - \log(N_o(n)-2)$, where $N_o(n)-2$ is the number of binary mass-centered sequences of length n without the all-'0' and all-'1' words [4]. Note the significant difference between $r_o(n)$ and $r(n)$ for $q=2$. With the increase of n, $r_o(n) = O(\log n)$ has an upward trend, while $r(n)$ experiences a downward trend to 0. For example, $r_o(10) \approx 4.5$ is more than 100 times larger than $r(10) \approx 0.028$. We conclude that the redundancy of the proposed pair-constrained codes gives a significant improvement compared to the corresponding mass-centered codes.

V. Discussion

We have presented a scheme for channels with unknown scaling and varying offset, where minimum Pearson distance detection is used in conjunction with a difference operator and pair-constrained codes. These codes have significantly less redundancy than the previously proposed mass-centered codes, which makes the new scheme an attractive alternative for practical applications. However, there are still some issues which need to be addressed, as will be discussed next.

The introduction of the difference operator is very effective to deal with the unknown varying offset, but it follows from the analysis in (7) that it doubles the noise power. Hence, in the error analysis, this extra 3 dB loss should be taken into account, and it makes the scheme less suitable for applications in which the noise is dominant over the (varying) offset. An interesting topic for further research is to investigate to which extent the involvement of an error-correcting code into the scheme can help to resolve this.

Another concern is the complexity of the proposed scheme. The use of the difference operator demands extra subtractions, but the major problem is that the minimization operation (6) requires $|\Delta\mathcal{S}|$ computations, which is impractical for codes with large cardinalities. In [3], it has been shown that the number of computations can be significantly reduced by considering the codebook as the union of a number of constant composition codes, which makes, at the expense of extra sorting operations, the number of options in the minimization equal to only the number of such subcodes. Similar complexity reduction could be explored for the setting under consideration here as well.

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