Electromagnetism in a rotating frame of reference

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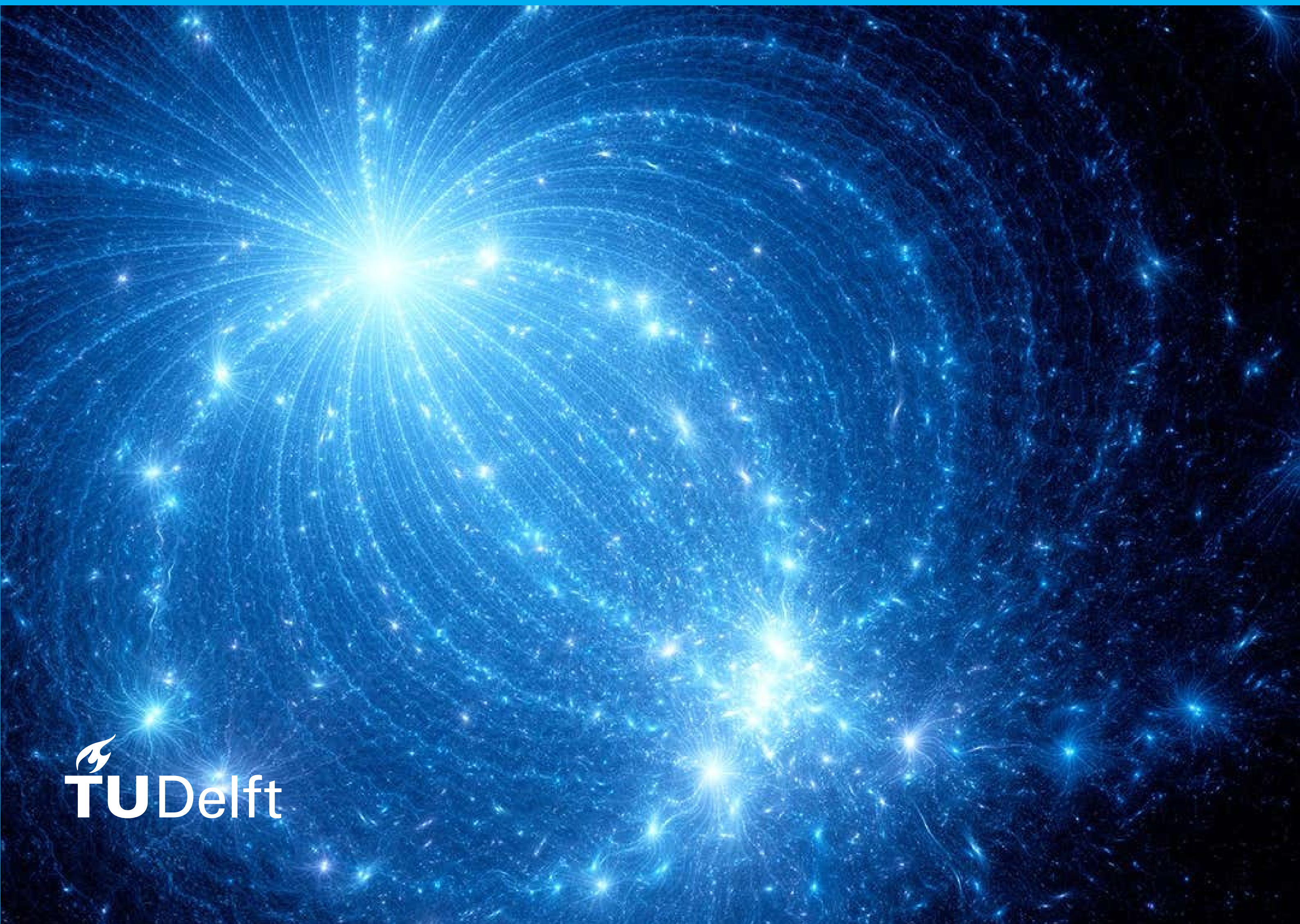

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by

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Abstract

In this thesis two topics are discussed. The covariant formulation of Maxwell's equations of electromagnetism and the formulation of said equations in the context of a rotating frame of reference.
Through the development of the necessary theories of Differential Geometry and Special Relativity we show how to formulate Maxwell's equations in terms of the covariant derivative and in terms of the hodge operator and differential operator. Using this formulation we study the transformation properties of these equations under Lorentz transformations and we conclude that they remain invariant under said transformations.
Secondly, we discuss literature concerning the problem of electromagnetism in the context of a rotating frame of reference. We show that the method of direct transformation to a rotating frame results in an unobservable frame of reference and we show that it is impossible to reconstruct Coriolis like forces in an observable frame of reference.

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Introduction

The earths magnetic field is generated by convecting currents of liquid iron in the earth's outer core [4]. The north and south pole of the field reverse roughly every 10^5 to 10^6 years and it is estimated that the last reversal occurred roughly 774 thousand years ago [23]. Temporary reversals such as the Laschamp excursion, which happened roughly 42,000 years ago, however occur more frequently and there is evidence that the effects of this reversal have caused the Neanderthals to go extinct [5]. Over the past 170 years the magnetic field of the earth has been weakened by approximately 9%, indicating that a field reversal might be upon us.
The field that studies the magnetic field of the earth and other celestial objects is called dynamo theory. A lot of effort has been put in developing mathematical models of these phenomena and there have been self-consistent dynamo models that predict both the fluid motion and the magnetic field since 1995 [16]. These models have been able to predict pole reversals, but results vary greatly between models and with the available observational data. One limiting factor in the modelling of geodynamos is the computational complexity and the requirement of supercomputers to run simulations. As such many efforts have been made to improve in these aspects and recent proposals to reduce computation time include the use of spectral methods [3].
Recently there have been advancements in the theory of Rossby-Haurwitz planetary waves. By viewing wave modes as irreducible representations of the SO(3) lie algebra and by transforming to a frame of reference in which these wave modes stand still, it was found that the phase velocity of these wave models is independent of its orientation with respect to the axis of rotation of the planet [26]. It is hypothesised that similar techniques can be employed to gain new insights into the generation of planetary dynamos but in order to do so, a good understanding of electromagnetism in a rotating frame of reference is needed. As noted by Z. Zawistowski [27], this is a non trivial problem that is largely unnoticed by the scientific community as for example in the treatment of the hyper fine structure of hydrogen in Introduction to quantum mechanics by D. Griffiths [7]. A correct understanding and application however, could lead to great benefits in both engineering and science [27].
In this report we review literature available on the problem of transforming the electromagnetic field between inertial and rotating frames of reference. In order to do so, we first develop the required tools of differential geometry and special relativity that will help us to formulate the laws of electromagnetism in a covariant form. Covariant meaning unchanged under Lorentz Transformations. Using these tools we will be able to discuss literature concerning electromagnetism in a rotating frame of reference.
In the first chapter of this report we will give a general introduction to the theory of differential geometry and explain the most important concepts that will be used in the remainder of the report. Chapter 2 will lay out the theory of special relativity which will be the basis for our understanding of later chapters. Subsequently, in chapter 3 we will develop and discuss the theory of electromagnetism in a relativistic setting and derive a covariant formulation of the equations of Maxwell. Extending the results found in the previous chapter, we will discuss the theory of electromagnetism in the context of a rotating frame of reference in chapter 4. In chapter 5 we will discuss the results found in the previous sections and give recommendations for future research. Chapter 6 will contain the conclusion of this report.

1

An introduction to differential geometry

The physics that will be discussed in this report are described by the mathematical field of differential geometry. Through this field it is possible to accurately define for example electromagnetic fields or take the derivative of a function defined on a space that does not look like the euclidean space that we are so familiar with. Below I will present a rough outline of the necessary theory, but leave out many of the finer details. Appendix A contains a more detailed account but is still lacking some of the finer details. For a rigorous exposition of the entire theory I refer to the lecture notes of the course AM3580, *Introduction to differential geometry* at Delft university of technology [15].

The two-sphere To gain an understanding of differential geometry we will look at the 2-sphere $\mathbb{S}^2$ as a concrete example. The two-sphere $\mathbb{S}^2$ is the sphere in three dimensional space that we are all familiar with. An example in everyday life is our planet earth. Even though our planet is not a perfect sphere, it is sufficiently close for our purposes here to illustrate as an example. Before the rise of technology and devices such as the smartphone, people had to rely on charts to navigate from one place to another. These charts show a part of the earth on a flat piece of paper through which it is possible to figure out your location or the direction you are going in. Charts overlap slightly at their borders to make sure that one can transition from one chart to the next and an entire collection of charts is called an atlas. Useful as these charts are, they are not perfect. Look at the chart in figure 1.1 below. It shows the earth projected on a flat chart by the Mercator projection [1].

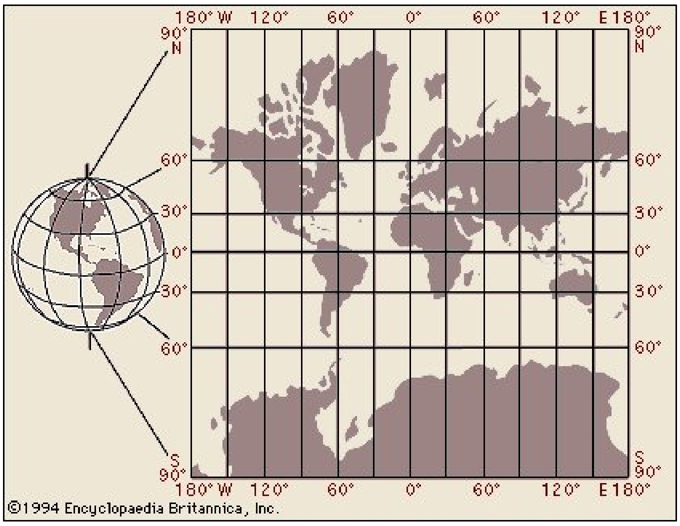

Figure 1.1: A chart showing the Mercator projection of the earth

Save for two small regions at the north and south pole we can see the entire earth. However, if we would try to measure distances using a ruler, our answer would become increasingly erroneous once we move closer to the north or south pole. This is due to the fact that it is impossible to flatten a sphere whilst retaining correct

distances and angles [15]. In case of the Mercator projection, distances are stretched more the further away we move from the equator. Hence it seems as if Russia is a larger country, whereas in fact China has a larger surface area. Fortunately it is possible to use several charts such that we always have sufficient accuracy for the task at hand.
Above description of planet earth, together with its charts that cover the entire surface, is a very informal description of the more general smooth manifold. Any space in which we can construct charts such as described above to project the space onto is a smooth manifold. In chapter 2 we will see that the space-time manifold is a smooth manifold as well.

Tangent vectors Continuing with our example of the two-sphere, at any point on the sphere we can define a so-called tangent vector. As shown in figure 1.2 these can be visualised as arrows emanating from a point on the sphere in a direction that is parallel to the surface. The grey plane in figure 1.2 is called the tangent

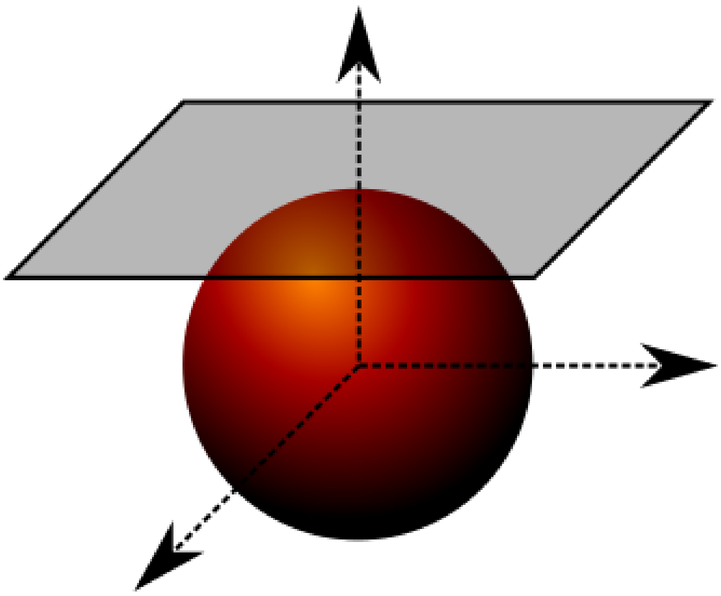

Figure 1.2: The tangent space of the 2-sphere at $(0,0,1)$

space at the point $(0,0,1)$ and constitutes of all tangent vectors at the point $(0,0,1)$. In the context of planet earth, a tangent vector might represent the flow of air through the atmosphere at a certain location. Now if we were to assign a tangent vector to every point on the surface of the two-sphere we get a vector field which might represent, as above, the flow of air, but now on the entire globe instead of at just one point. Figure 1.3 shows such a vector field on the two sphere.

tensors Tensors will be the main stage of all the physics that will be described below. They are a generalisation of tangent vectors and its entries are elements of the tangent space of a manifold. Later on we will see that they fundamentally describe relativistic electromagnetism.
We distinguish two different types of tensors, the first being the contravariant tensor. its defining property is the way in which it transforms under a change of coordinates. For example, take a vector in three dimensional Euclidean space

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{1.1}$$

Now change to a different basis with basis vectors that are twice as long as the original basis vectors. In terms of the new basis we have

$$\boldsymbol{v} = \begin{bmatrix} v_1/2 \\ v_2/2 \\ v_3/2 \end{bmatrix} \tag{1.2}$$

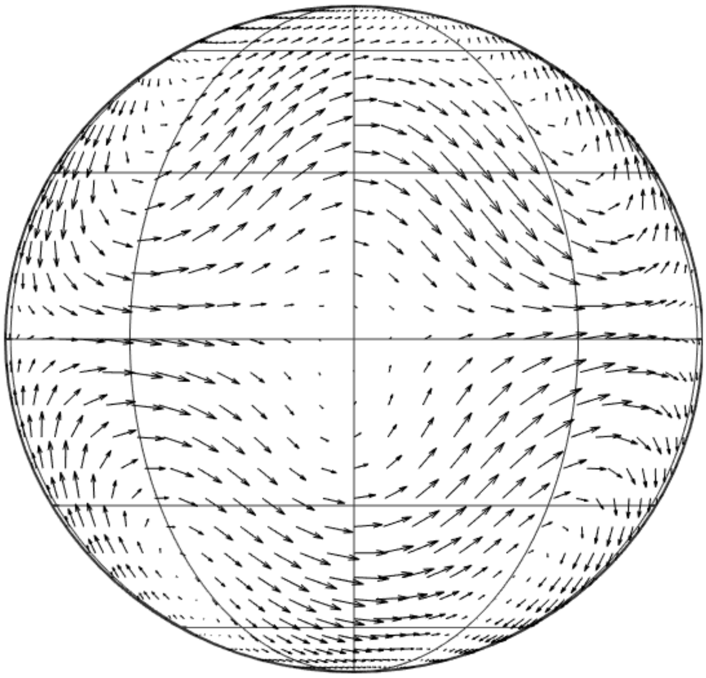

Figure 1.3: A vector field on the two-sphere. Intuitively this might resemble the flow of air around the globe under the assumption that the atmosphere is two-dimensional or flat.

We see that the matrix entries have halved and thus they change in a way that is 'opposite to how the basis changes'. It is this property that defines contravariant tensors. The vector above is called a contravariant tensor of rank 1 because its entries change according to

$$v'_i = \frac{\partial \hat{\boldsymbol{x}}'_i}{\partial \hat{\boldsymbol{x}}_j} v_j \tag{1.3}$$

A general contravariant tensor $\tau^{\nu_1,\ldots,\nu_l}$ of rank l is an object whose entries transform under a change of basis according to

$$\sum_{\nu_1}\cdots\sum_{\nu_l}\Big(\frac{\partial x^{\boldsymbol{\nu}_1}}{\partial x^{\nu_1}}\Big)\cdots\Big(\frac{\partial x^{\boldsymbol{\nu}_l}}{\partial x^{\nu_l}}\Big)\tau^{\nu_1,\ldots,\nu_l} = \tau^{\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_l} \tag{1.4}$$

A covariant tensor $\tau_{\mu_1,\ldots,\mu_k}$ is similar, but behaves in the opposite way. Under a change of basis its entries change along with the basis and as such

$$\sum_{\mu_1}\cdots\sum_{\mu_k}\Big(\frac{\partial x^{\mu_1}}{\partial x^{\boldsymbol{\mu}_1}}\Big)\cdots\Big(\frac{\partial x^{\mu_k}}{\partial x^{\boldsymbol{\mu}_k}}\Big)\tau_{\mu_1,\ldots,\mu_k} = \tau_{\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_k} \tag{1.5}$$

Above notation can become quite cumbersome due to the many summations that have to be taken. In order to reduce the notational clutter, the Einstein summation convention is invoked. This states that whenever an index is repeated as an upper and lower index, summation is implied. As such above equation reduces to

$$\Big(\frac{\partial x^{\mu_1}}{\partial x^{\boldsymbol{\mu}_1}}\Big)\cdots\Big(\frac{\partial x^{\mu_k}}{\partial x^{\boldsymbol{\mu}_k}}\Big)\tau_{\mu_1,\ldots,\mu_k} = \tau_{\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_k} \tag{1.6}$$

An important class of tensors that we will be looking at are the contra and covariant tensors of rank 2. They can be represented as an $n \times n$ matrix, where n is the dimension of the vector space that it is defined on. Do note however, that even though we can represent it as a matrix, it does not transform as such. The electromagnetic field tensor that was mentioned above is an example of a rank 2 tensor that is defined on the 4 dimensional space-time manifold (see chapter 2). In contravariant form it is given by

$$F^{\alpha\beta} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix} \tag{1.7}$$

We see that its entries constitute of the well known electric and magnetic field, but the tensor itself is more complicated.

The metric tensor Another special type of tensor that we will be using is the metric tensor g. It is a covariant symmetric tensor of rank 2 that describes the geometry of the manifold it is defined on. As an example, take the 2 dimensional Euclidean metric

$$\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_x \cdot \boldsymbol{e}_x & \boldsymbol{e}_x \cdot \boldsymbol{e}_y \\ \boldsymbol{e}_y \cdot \boldsymbol{e}_x & \boldsymbol{e}_y \cdot \boldsymbol{e}_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.8}$$

The individual entries of the metric tensor tell us how the basis vectors are oriented with respect to one another. In above example we see that the basis vectors are orthogonal and have unit length as we would expect from Euclidean geometry or a flat piece of paper if you will. Now lets do a coordinate transformation to a cylindrical coordinate system defined by

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned}$$

the components of the metric tensor in terms of the new coordinate system are given by

$$g_{ii} = g_{xx}\frac{\partial i}{\partial x}\frac{\partial i}{\partial x} + g_{xy}\frac{\partial i}{\partial x}\frac{\partial i}{\partial y} + g_{yx}\frac{\partial i}{\partial y}\frac{\partial i}{\partial x} + g_{yy}\frac{\partial i}{\partial y}\frac{\partial i}{\partial y} \tag{1.9}$$

Where i is r or θ. Since $g_{xy} = g_{yx} = 0$ we find

$$\begin{bmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{1.10}$$

Raising and lowering indices One important function of the metric tensor is its ability to convert a contravariant tensor into a coavariant tensor. This is done by multiplying the contravariant tensor $\tau^{\nu_1,\ldots,\nu_l}$ of rank l by l times the metric tensor where every metric tensor has one index equal to one of the l indices of $\tau^{\nu_1,\ldots,\nu_l}$. As such

$$\tau_{\mu_1,\ldots,\mu_k} = g_{\mu_1\nu_1}\cdots g_{\mu_l\nu_l}\tau^{\nu_1,\ldots,\nu_l} \tag{1.11}$$

Similarly, the inverse of this operation is given by

$$\tau^{\nu_1,\ldots,\nu_l} = g^{\mu_1\nu_1}\cdots g^{\mu_l\nu_l}\tau_{\mu_1,\ldots,\mu_k} \tag{1.12}$$

Here $g^{\mu\nu}$ is the inverse matrix of the covariant metric tensor.

2

The theory of special relativity

In 1905 Albert Einstein published his paper on the theory of special relativity. Within this paper he formulated his two postulates of special relativity [11, 19]

- The principle of relativity: The laws of physics hold equally in all inertial frames of reference.
- The invariance of the speed of light: light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body.

An inertial frame of reference is defined as a reference frame in which objects behave according to Newton's first law. That is, a frame of reference in which objects standing still or move at a constant velocity will remain doing so unless a force acts upon such objects. A direct consequence of this is that inertial frames of reference move with a constant velocity with respect to one another.
Simple as these two principles are, they do require some careful consideration. Consider the first postulate, what exactly does it mean for a law of physics to hold equally in all inertial frames of reference? It means that any process occurring in one inertial frame of reference can occur in exactly the same way in another inertial frame of reference [19]. That is, the laws of physics written in equation form are exactly the same in every inertial frame of reference. Lets look at an example.
Take Newton's second law, $F = M\frac{\partial^2 x}{\partial t^2}$. Now suppose that we throw an apple straight up into the air and observe what happens. As we all know from experience, it rises until it reaches an apex and starts falling straight down again and will land in the same spot as where we threw it into the air. All of this can be described by Newton's second law. Now imagine the same situation, but whilst observing from another inertial frame that is moving with a constant velocity in the x-direction with respect to initial frame of reference. See figure 2.1 for the orientation of the two frames. We would see the apple following a parabolic trajectory through space and land in a different spot than where it was launched from. Now, even though we see something different in the two references frames, all movement still abides the laws of Newton. The only difference being that in the latter case the apple has an initial velocity that is perpendicular to the force of gravity. Since there are no further forces at play the motion will remain constant and create the parabolic trajectory. Additionally, if we were to throw an apple into the air in the second frame of reference we would observe the exact same trajectory, that is it goes up and down in a straight line, as we did in the first experiment. We conclude that the laws of Newton hold equally in both frames.
To see where this equality fails we repeat the experiment, but instead of moving to a second inertial frame that is moving with constant velocity, we look at it from a frame of reference that is accelerating in the x-direction. From our new perspective the ball would be accelerating in the -x direction and according to Newton's law's a body that is accelerating must experience a force that causes this acceleration. In this case however, there is no force and hence Newton's laws are invalid in this frame of reference. To compensate for the fictitious forces that appear due to the acceleration of the reference frame we must reformulate Newton's laws and add extra terms. Thus they do not hold equally in these frames of reference.
While the first postulate is not necessarily something new, the second postulate forces us to reexamine the very structure of the space and time we live in. Take for example the principle of simultaneity. Two events that happen at the same time in one frame of reference need not happen at the same time in another frame of reference. Another consequence is that objects that are moving close to the speed of light will be contracted

and appear shorter than they are [11]. Due to these consequences moving from one frame of reference to the other is not as simple as in the Galilean transformation that we know from classical mechanics. Another transformation is needed to remain consistent with the postulates of special relativity.

2.1. The Lorentz transformation

The set of equations that quantitatively describes how space and time change when one moves from one inertial frame of reference to another is called the Lorentz transformation [12]. Suppose that we have a reference frame O at rest with respect to the observer with coordinates (x, y, z, t) and another frame of reference O' with coordinates (x', y', z', t') with equal orientation, but moving at velocity v with respect to the observer along the x-direction. See figure 2.1

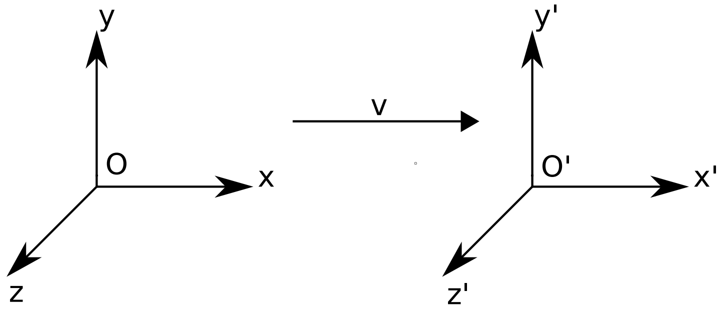

Figure 2.1: Two inertial frames of reference O and O' moving with velocity v with respect to one another

The Lorentz transformations to go from (x, y, z, t) to (x', y', z', t') are given by the following set of equations

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{2.1}$$

$$x' = \gamma(x - vt) \tag{2.2}$$

$$y' = y \tag{2.3}$$

$$z' = z \tag{2.4}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2.5}$$

As we will later see, there are other Lorentz transformations that relate to rotations and translations space that also abide the the postulates of special relativity. Above transformation, in which you move from one inertial frame of reference to another moving at constant velocity with respect tot the initial frame is called a boost.

Derivation of the lorentz transformation

Historically it is said that the Lorentz transformation is derived from the two postulates of relativity. This, however, is not entirely correct. In his original derivation Einstein implicitly used two other principles that intuitive as they are, are essential to the derivation. These principles are [9]

- The Homogeneity of space and time. The laws of physics are invariant under a translation of the origin of coordinates in space and time
- The isotropy of space. The laws of physics are invariant under rotations of the spatial axes in which they are described

Combined with Einstein's postulates of special relativity it is possible to fully derive the Lorentz transformation. In the past there have been many attempts to reduce the number of assumptions needed to develop the theory of special relativity, especially the second postulate of special relativity, the invariance of the speed of light, has been heavily debated. But so far no conclusive arguments has been made to drop it [9].

Remark. Above postulates, intuitive as they may seem at first, are not entirely correct. The theory of general relativity tells us that space and time are in fact inhomogenous and anisotropic. Fortunately these effects are very small in everyday life and hence the theory of special relativity remains very useful [13].

By looking at figure 2.1 we see that $y' = y$ and $z' = z$. From this and the principle of relativity we can conclude that our transformations from x to x' and t to t' must be independent of y, y', z and z'. If this were not the case, translating our reference frame up or down along the y-axis would alter x' or t', violating the principle of relativity. Hence we can write the following system of equations

$$\begin{aligned} x' &= G(x,t) \\ t' &= F(x,t) \end{aligned} \tag{2.6}$$

The next step in our derivation is to show that the homogeneity of space and time implies that these equations are linear [9]. To see this, suppose that we have two events (t_0, x_0, y_0, z_0) and $(t_0 + dt, x_0 + dx, y_0, z_0)$ in our inertial frame O. We immediately see that the difference of the coordinates between these two events is $(dt, dx, 0, 0)$. Now when moving from O to O' our expression for the differences becomes

$$\begin{aligned} dx' &= \frac{\partial F(x_0,t_0)}{\partial x_0}dx + \frac{\partial F(x_0,t_0)}{\partial t_0}dt \\ dt' &= \frac{\partial G(x_0,t_0)}{\partial x_0}dx + \frac{\partial G(x_0,t_0)}{\partial t_0}dt \end{aligned} \tag{2.7}$$

Homogeneity implies that this difference should be independent of our choice of origin and hence it should be independent of x_0 and t_0. From this we can conclude that our partial derivatives should be independent of x_0 and t_0 and thus F and G should be linearly dependent.
Since F and G are linear in x and t we can rewrite system of equations (2.6) in matrix form.

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{2.8}$$

Now, we know that $x' = 0$ implies that $x = vt$ as (x', y', z', t') moves with speed v in the x-direction. Hence we see that $x' = (vA+B)t = 0$ regardless of time t and thus $B = -vA$. Similarly, $x = 0$ is moving with speed v in the $-x$ direction with respect to (x', y', z', t') and here we have $x' = -vt'$. Substituting these values in $x' = A(x-vt)$ we get $t' = -\frac{A}{v}(x-vt) = -\frac{A}{v}x + At$. Comparing this to $t' = Cx + Dt$ we find $A = D$. After introducing $A = \gamma$ and $F = C/A$, our matrix equation becomes

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \gamma \begin{bmatrix} 1 & -v \\ F & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{2.9}$$

In equation (2.9) we have 2 constants, F and γ that are still undetermined. In order to find these two constants, we use the postulate that when transforming from reference frame O to O', where O' moves with velocity v (Which, from now on will be named v_1) with respect to O and transforming from O' to O'' where O'' moves with velocity v_2 with respect to O' we once again have a Lorentz transformation if we apply the two transformations after one another [18]. In matrix form the two transformations can be written as

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \gamma_1 \begin{bmatrix} 1 & -v_1 \\ F_1 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{2.10}$$

$$\begin{bmatrix} x'' \\ t'' \end{bmatrix} = \gamma_2 \begin{bmatrix} 1 & -v_2 \\ F_2 & 1 \end{bmatrix} \begin{bmatrix} x' \\ t' \end{bmatrix} \tag{2.11}$$

Transforming from O to O'' via O' will yield the following system of equations

$$\begin{bmatrix} x'' \\ t'' \end{bmatrix} = \gamma_1\gamma_2 \begin{bmatrix} 1 - F_1 v_2 & -v_1 - v_2 \\ F_1 + F_2 & 1 - F_2 v_1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{2.12}$$

We previously derived that the diagonal elements should be equal. Hence we see that $1 - F_1 v_2 = 1 - F_2 v_1$ or $\frac{v_2}{F_2} = \frac{v_1}{F_1}$. Since both sides of the equation are independent of one another and v_1 and v_2 are chosen arbitrarily, we can conclude that $\frac{v_2}{F_2} = \frac{v_1}{F_1} = a$ and in general $\frac{v}{F} = a$, where a is a constant independent of v. Substituting this result in (2.9) we get

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \gamma \begin{bmatrix} 1 & -v \\ v/a & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{2.13}$$

To find the value of γ we transform from O to O' and back to O. Substituting $v_1 = v$, $v_2 = -v$, $F_1 = v_1/a$ and $F_2 = -v_1/a$ in equation (2.12) gives

$$\begin{bmatrix} x \\ t \end{bmatrix} = \gamma_{-v}\gamma_v \begin{bmatrix} 1+v^2/a & 0 \\ 0 & 1+v^2/a \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{2.14}$$

Hence we find that $\gamma_{-v}\gamma_v = \frac{1}{1+v^2/a}$. Now, the isotropy of space states that the laws of physics are invariant under spatial rotations [18] and hence we can conclude that $\gamma_v = \gamma_{-v}$. We finally arrive at

$$\gamma = \frac{1}{\sqrt{1+v^2/a}} \tag{2.15}$$

and

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \frac{1}{\sqrt{1+v^2/a}} \begin{bmatrix} 1 & -v \\ v/a & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{2.16}$$

$a = +/-\infty$, The first two cases, $a = +/-\infty$ can be discarded as these would result in the ordinary Galilean transformations with no constant speed of light.
Above we still have an undefined quantity a with dimension velocity squared. We have two possible cases. $a > 0$ and $a < 0$. The first case $a > 0$ is invalid as we would obtain an imaginary speed of light. To see this, rewrite $a = \sigma^2$. Now suppose that we have an object flying at the hypothetical speed of light called V in the x-direction, the object must have an equal speed in any other reference frame. If the object started at $t = 0$, $x = 0$ we obtain $x = Vt$. Now if we transform to a reference frame that is moving at velocity v in the x-direction, we find, using equation (2.16)

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \frac{1}{\sqrt{1+v^2/\sigma^2}} \begin{bmatrix} 1 & -v \\ v/\sigma^2 & 1 \end{bmatrix} \begin{bmatrix} Vt \\ t \end{bmatrix} = \frac{1}{\sqrt{1+v^2/\sigma^2}} \begin{bmatrix} Vt - vt \\ vVt/\sigma^2 + t \end{bmatrix} \tag{2.17}$$

Since the velocity of our object should be equal in the second reference frame, we must have $x' = Vt'$ and hence $Vt - vt = vV^2 t/\sigma^2 + Vt$. Solving for V yields $V = \frac{+}{-}\frac{\sqrt{-t/\sigma^2}}{t/\sigma^2}$. This is an imaginary velocity and hence we can discard this as unphysical. This leaves $a < 0$ as our final options. Writing $a = -c^2$ we find by a method similar to the case $a > 0$ that an object traveling at velocity c is travelling at that speed in all reference frames and using the second postulate of relativity we can conclude that this is the speed of light.
Note that in everyday life the velocities we deal with are far slower than that of the speed of light. That is, $v >> c$. If we take the limit $\frac{v}{c} \to 0$ (Or $c \to \infty$) in the Lorentz transformation we obtain the Galilean transformation that we know from classical physics. This explains why in everyday life we rarely see any kind of relativistic effects.
If we include the trivial transformation for y and z, write $\gamma = 1/\sqrt{1+v^2/c^2}$, $\beta = v/c$ and use ct instead of t in our vector, we obtain the Lorentz transformation

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{2.18}$$

The above result is obtained for a boost in the x-direction. its generalisation to a boost in an arbitrary direction is given by

$$\begin{bmatrix} ct' \\ \boldsymbol{x}'_{||} \\ \boldsymbol{x}'_{\perp} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ \boldsymbol{x}_{||} \\ \boldsymbol{x}_{\perp} \end{bmatrix} \tag{2.19}$$

Later we will see that we are free to rotate our axis in any direction we like, hence we will use equation (2.18) unless specified otherwise.

2.2. The structure of space-time

In ordinary mechanics, moving from one frame of reference to the other by the Galilean transformation leaves coordinate differences $(\Delta t, \Delta x, \Delta y, \Delta z)$ unchanged. They are called invariant under Galilean transformations.

From the Lorentz transform we can conclude that these coordinate differences are no longer invariant under Lorentz transformations and another quantity is needed. This quantity is called the spacetime interval and for two events happening in one reference frame it is defined by

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{2.20}$$

Moving from one inertial frame of reference to another using the Lorentz transformation leaves the spacetime interval unchanged [12]. If $(\Delta s)^2 = 0$, then the two events are separated by information travelling at the speed of light and the interval is called light-like. Similarly, $(\Delta s)^2 > 0$ implies that the spatial part is dominant and that the two events are separated by a distance that is larger than light can travel in their time difference. These events are called spacelike and the physical implication is that for two events separated by a spacelike interval there exists a Lorentz transformation such that the two events happen at the same time. Analogously, an interval with $(\Delta s)^2 > 0$ is called timelike and for events with timelike separation, there exists a Lorentz transformation to a reference frame in which the two events happen at the same place but at different times. Equation (2.20) can be written more compactly if a 4 dimensional scalar product, called the Minkowski scalar product, is introduced.

$$\eta(A^\mu, B^m u) = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \tag{2.21}$$

for contravariant vectors $A^\mu = (a^0, a^1, a^2, a^3)$ and $B^\mu = (b^0, b^1, b^2, b^3)$ with contravariant entries.
A^μ and B^μ can also be written in terms of its covariant entries $A^\mu = (a_0, a_1, a_2, a_3)$ where $a_0 = -a^0$ and $a_\mu = a^\mu$ for $\mu = 1, 2, 3$, transformation from the covariant to the contravariant tuples happens via the Minkowski metric $\eta_{\mu\nu}$.

$$a_\mu = \eta_{\mu\nu} a^\nu \tag{2.22}$$

and

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.23}$$

Using the covariant, contravariant notation, the Minkowski product can be written as $\eta(A^\mu, B^\mu) = a^\mu b_\mu$. Here the upper and lower case μ imply Einstein summation.
As such, the spacetime interval of any two events can be written as a vector product

$$(\Delta s)^2 = \eta(\Delta X^\mu, \Delta X^\mu) = \Delta x_\mu \Delta x^\mu = g_{\mu\nu} \Delta x^\nu \Delta x^\mu \tag{2.24}$$

where the entries of ΔX are given by $(c\Delta t, \Delta x, \Delta y, \Delta z)$.

Space Time as a 4-dimensional Pseudo-Riemannian manifold In the paragraph above it is hinted at already, but the 4-dimensional space $\mathbb{R}^4$ (3 dimensions for space, 1 for time), equipped with the Minkowski scalar product, is in fact a Pseudo-Riemannian manifold as defined in equation (A.4.3) [15].

Four-vectors Above we have been talking primarily about the tuple $(c\Delta t, \Delta x, \Delta y, \Delta z)$ and the way in which it transforms under the Lorentz transformation. However, just like electric charge is conserved when moving from one frame of reference to another in classical physics, there exist quantities on the tangent space of a manifold that transform under the Lorentz transformation whilst leaving their Minkowski scalar product unchanged. We call such tuples 4-vectors. Similar to $X^\mu = (c\Delta t, \Delta x, \Delta y, \Delta z)$, also called the four-position, arbitrary 4-vectors leave their Minkowski scalar product invariant under Lorentz transformations. Formally speaking, a tuple $P^\mu = (P^0, P^1, P^2, P^3)$ is a 4-vector if

$$\eta(P^\mu - Q^\mu, P^\mu - Q^\mu) = \eta(\Lambda(P^\mu - Q^\mu), \Lambda(P^\mu - Q^\mu)) \tag{2.25}$$

where Λ is a Lorentz transformation and P and Q are different entities of the same type of 4-vector.
Examples of 4-vectors are the four-current $J^\mu = (\rho c, \boldsymbol{j})$. Here ρ is the charge density (electrical charge is a conserved quantity, the charge density however, is not conserved due to space contraction) and $\boldsymbol{j}$ the current density. Another example is the Four-velocity $U^\mu = \frac{dX^\mu}{d\tau}$. Here $d\tau = dt/\gamma$, also called the proper time, and X is the four-position [12]. Do note that even though the four-position transforms like a four-vector it is in fact not a four vector as it is not defined on the tangent space of the spacetime manifold but it designates an event on the manifold.

The Poincaré group Transformation (2.18) is not the only transformation that leaves the minkowsky scalar product unchanged. The collection of transformations that leaves equation (2.25) invariant is called the Poincaré group. Mathematically speaking, a group is a set G with an operation $G \times G \to G$ $((a,b) \mapsto a \circ b)$ such that the following three requirements are met [15].

G1 (Associativity of $\circ$) For all $a, b, c \in G$ we have that $a \circ (b \circ c) = (a \circ b) \circ c$

G2 (Existence of an identity element) There is an $e \in G$ such that $e \circ a = a \circ e = 0$ for all $a \in G$

G3 (Existence of an inverse) For every $a \in G$ there exists an element $a^* \in G$ such that $a \circ a^* = a^* \circ a = e$

The Poincaré group includes three distinct types of transformations [15]

1. Translations in space and time.
2. rotations in space
3. Boosts.

The third type, the boosts, are the transformations that were derived at the beginning of this section and are given by equation (2.18). The first type, the translations in space and time, simply moves both events in space or time from one inertial frame of reference to another frame of reference that is standing still with respect to the other. The transformations of the second type, the rotations in space, leave all distances intact and only changes the orientation with respect to the origin. Hence the Euclidean interval of the spatial part remains invariant and thus the spacetime interval is invariant as well.

3

Relativistic electromagnetism

In the previous chapter we have seen that quantities such as the 4-current are 4-vectors and behave such that its Minkowski scalar product remains invariant under Lorentz transformations. We now shift our focus to a more complicated situation, the electromagnetic field. Consider a wire through which a current is flowing. The laws of electromagnetism tell us that both an electric field and a magnetic field are present. Now if we Lorentz transform to an inertial frame of reference in which the charge carriers are standing still, with thus no current flowing through the wire, we see that the magnetic field has disappeared. Somehow the electric field and magnetic field are intrinsically linked to one another and one can transform into the other. The way in which the two fields are related is through the field tensor [12].

3.1. The field tensor

The field tensor is an anti-symmetric covariant tensor of rank 2, which in matrix form is given by

$$F_{\alpha\beta} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix} \tag{3.1}$$

Derivation of the field tensor

In order to derive the field tensor, We start with Maxwell's equations in vacuum.

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0}\rho \tag{3.2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{3.3}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{3.4}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{3.5}$$

Since, according to equation (3.4), the magnetic field $\boldsymbol{B}$ has no divergence, we can write it as the curl of a vector potential $\boldsymbol{A}$ [12].

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{3.6}$$

Now if we insert this into equation (3.3) and rearrange, we get

$$\nabla \times (\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}) = 0 \tag{3.7}$$

In brackets we have a quantity with no curl and hence we can rewrite this as the gradient of a scalar [12].

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla V$$

Rewriting this expression, we get

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} \tag{3.8}$$

We now have found expressions for the magnetic field $\boldsymbol{B}$ and the electric field $\boldsymbol{E}$ in terms of two potentials. Using the definitions of these potentials, as in equations (3.6) and (3.8), Maxwell's equations (3.2) until (3.5) can be reduced to

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{A}) = -\frac{1}{\varepsilon_0}\rho \tag{3.9}$$

$$\Big(\nabla^2\boldsymbol{A} - \mu_0\varepsilon_0\frac{\partial^2\boldsymbol{A}}{\partial t^2}\Big) - \nabla\Big(\nabla\cdot\boldsymbol{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\Big) = -\mu_0\boldsymbol{j} \tag{3.10}$$

To obtain the second equation, the following identity is used [12]

$$\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} \tag{3.11}$$

The four Maxwell equations have been reduced to a set of two equations in addition to the definition of the two potentials.

Gauge transformation In equation 3.6 we have defined the magnetic field in terms of the curl of a vector potential and the electric field in terms of a scalar potential for which the curl vanishes. As will be shown below, the definition of these potentials leave a degree of freedom in their choice which can be exploited to simplify equations 3.9 & 3.10. The transformations between different equally valid potentials are called gauge transformations. For vector potential $\boldsymbol{A}$ we define

$$\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\alpha} \tag{3.12}$$

Since both vector potentials must yield the same magnetic field $\boldsymbol{B}$, We require that $\nabla\times\boldsymbol{A}' = \nabla\times(\boldsymbol{A}+\boldsymbol{\alpha}) = \nabla\times\boldsymbol{A}$ and hence we can conclude that $\nabla\times\boldsymbol{\alpha} = 0$. Consequently, by choosing $\boldsymbol{\alpha}$ in a clever way, we can adjust the divergence of $\boldsymbol{A}$ in such a way that our expressions become somewhat easier to work with. A vector potential without curl can be written as the gradient of some scalar [12] and thus we can write $\boldsymbol{\alpha} = \nabla\Phi$.
Similarly, we can add an extra term β that is independent of space and time to V

$$V' = V + \beta \tag{3.13}$$

If we replace **A** and V in equation (3.8) by their gauge transformations we find that

$$\nabla\beta + \frac{\partial\boldsymbol{\alpha}}{\partial t} = 0 = \nabla\Big(\beta + \frac{\partial\Phi}{\partial t}\Big) \tag{3.14}$$

And thus

$$\beta = -\frac{\partial\Phi}{\partial t} + c(t) \tag{3.15}$$

For convenience we absorb $c(t)$ into the partial derivative of Φ.

Lorentz gauge One choice for the gauge transformation is the so-called Lorenz gauge

$$\nabla\cdot\boldsymbol{A} = -\mu_o\varepsilon_0\frac{\partial V}{\partial t} \tag{3.16}$$

This gauge is chosen to simplify equations 3.9 & 3.10 and we are left with

$$-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} + \nabla^2 V = -\frac{1}{\varepsilon_0}\rho \tag{3.17}$$

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\boldsymbol{A} = -\mu_0\boldsymbol{j} \tag{3.18}$$

The Field tensor If we define $A^\mu = (\frac{V}{c}, \boldsymbol{A})$, we can rewrite equations (3.17) and (3.18) into

$$-\frac{1}{c^2}\frac{\partial^2 A^\mu}{\partial t^2} + \nabla^2 A^\mu = -\mu_0 J^\mu \tag{3.19}$$

Where J^μ is the 4-current. Now, if we define the d'alembertian operator

$$\Box^2 = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \tag{3.20}$$

Equation 3.19 reduces to

$$\Box^2 A^\mu = -\mu_0 J^\mu \tag{3.21}$$

From this equation we can conclude that A^μ is in fact a 4-vector. To see why, first note that the right hand side of above equation is the 4-current and thus Lorentz invariant. Now if the d'Alembertian operator is invariant under Lorentz transformation, we can conclude that A^μ is a 4-vector as well. Fortunately, this is precisely the case.

Define the covariant four-derivative as $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$. That is, we differentiate with respect to the contravariant coordinates and this yields a covariant object since, according to equation (3.22) differentiation with respect to a contravariant coordinate transforms in a covariant manner [8]

$$\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta} \tag{3.22}$$

we see that $\Box^2 = \partial_\mu \partial^\mu$, where Einstein summation is implied, and thus the d'Alembert operator is indeed a Lorentz invariant scalar.

Using the 4-derivative, we can rewrite the Lorenz Gauge as

$$\partial_\mu A^\mu = 0 \tag{3.23}$$

Note that above in the definition of the four-derivative the superscript μ only implies that the four-derivative is given in contravariant form whereas in equation (3.23) it implies the index of the components of the four-derivative and hence Einstein summation is implied

Now we return to equation (3.6) and (3.8). Expressing the $\hat{\boldsymbol{x}}$-component of each equation explicitly yields

$$E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} = \partial^0 A^1 - \partial^1 A^0 \tag{3.24}$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \partial^2 A^3 - \partial^3 A^2 \tag{3.25}$$

This in combination with the 4 equations corresponding to the fields in the $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ direction result in a set of 6 equations that can be taken as the components of a rank 2, antisymmetric field-strength tensor defined by $F^{\alpha\beta} = \partial^\beta A^\alpha - \partial^\alpha A^\beta$. In covariant matrix form this can be written as

$$F_{\alpha\beta} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix} \tag{3.26}$$

We can raise the indices by $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$ to obtain the contravariant form of the field tensor

$$F^{\alpha\beta} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \tag{3.27}$$

This yields the lorentz invariant quantity $F^{\alpha\beta}F_{\alpha\beta} = 2(|B|^2 - \frac{|E|^2}{c^2})$ [12].

By contracting the field tensor with the hodge star operator $*$ (see section 3.3) we obtain the so-called dual tensor

$$(*F)_{\alpha\beta} = \frac{1}{2}\varepsilon_{\gamma\lambda\alpha\beta}F^{\gamma\lambda} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z/c & E_y/c \\ B_y & E_z/c & 0 & -E_x/c \\ B_z & -E_y/c & E_x/c & 0 \end{bmatrix} \tag{3.28}$$

From this we obtain another Lorentz invariant quantity $F^{\alpha\beta}(*F)_{\alpha\beta} = -\frac{4}{c}(\boldsymbol{E}\cdot\boldsymbol{B})$ [12]

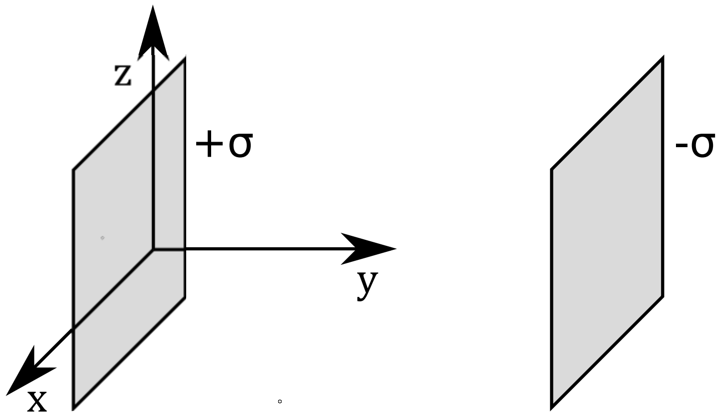

Figure 3.1: Two infinite parallel plates with opposite charges $+\sigma$ and $-\sigma$

The electromagnetic field in between two charged parallel plates To see how we can use the field tensor to calculate the electric and magnetic field in different frames of reference we look at an example. In figure 3.1 we see two infinite charged plates oriented in the x-z plane. Both carry an equal but opposite charge density σ and they are separated by a finite distance. The electric field in between these two plates is given by $\boldsymbol{E} = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{y}}$ and no magnetic field is present. Now suppose that we perform a Lorentz transformation in the $\hat{\boldsymbol{x}}$ direction with velocity v in the $-\hat{\boldsymbol{x}}$ direction. From our point of view the charged plates are now moving in the $\hat{\boldsymbol{x}}$ direction with velocity v as in figure 3.2 and we expect to see a magnetic field.

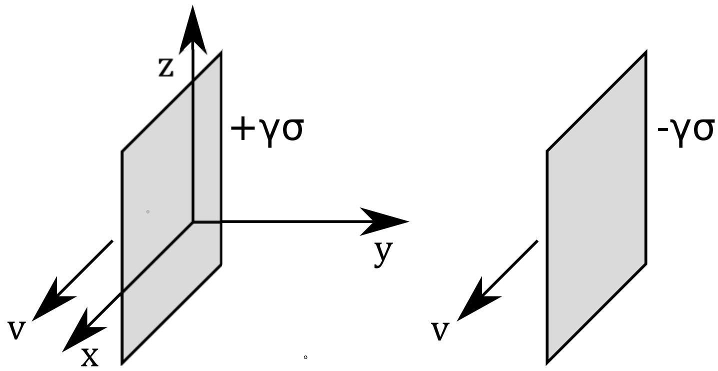

Figure 3.2: Two infinite parallel plates with opposite charges $+\sigma$ and $-\sigma$ both moving in the positive $\hat{\boldsymbol{x}}$ direction.

Using the field tensor we can easily calculate the electromagnetic field in our new frame of reference by boosting the field tensor. Boosting along the $\hat{\boldsymbol{x}}$-direction with velocity v yields Lorentz transformation

$$\Lambda_\mu^{\ \alpha} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.29}$$

and since $\boldsymbol{E} = (0, E_y, 0)$ and $\boldsymbol{B} = (0,0,0)$ we find the field tensor

$$F^{\mu\nu} = \begin{bmatrix} 0 & 0 & -E_y/c & 0 \\ 0 & 0 & 0 & 0 \\ E_y/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.30}$$

This results in, using Einstein summation and equation (A.15)

$$F'^{\alpha\beta} = \Lambda_\mu^{\ \alpha}\Lambda_\nu^{\ \beta}F^{\mu\nu} = \begin{bmatrix} 0 & 0 & -\gamma E_y/c & 0 \\ 0 & 0 & \gamma v E_y/c^2 & 0 \\ \gamma E_y/c & -\gamma v E_y/c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.31}$$

And thus we find that $E'_x = \gamma E_x$ and that $B'_z = \gamma\beta E_x$. We see that as predicted a magnetic field has arisen due to the charges now moving with velocity v. Furthermore we see that the electric field has gained a factor γ. This is due to the lorenz contraction in the $\hat{\boldsymbol{x}}$-direction which results in a higher charge density.
We see that by using the field tensor calculating electromagnetic fields in different inertial frames of reference is simply a matter of boosting the field tensor with the correct lorentz transformation.

3.2. The Maxwell equations in tensor notation

Taking the four-derivative of the field tensor, we obtain

$$\partial_\alpha F^{\alpha\beta} = \begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} = \tag{3.32}$$

$$\begin{bmatrix} \frac{1}{c}\frac{\partial E_x}{\partial x} + \frac{1}{c}\frac{\partial E_y}{\partial y} + \frac{1}{c}\frac{\partial E_z}{\partial z} \\ -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\ -\frac{1}{c^2}\frac{\partial E_y}{\partial t} - \frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} \\ -\frac{1}{c^2}\frac{\partial E_z}{\partial t} + \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{1}{c}\nabla\cdot\boldsymbol{E} \\ \nabla\times\boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \end{bmatrix} \tag{3.33}$$

If we set this equal to $\mu_0 J^\beta$, we get

$$\partial_\alpha F^{\alpha\beta} = \begin{bmatrix} \frac{1}{c}\nabla\cdot\boldsymbol{E} \\ \nabla\times\boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \end{bmatrix} = \begin{bmatrix} \mu_0 c\rho \\ \mu_0 \boldsymbol{j} \end{bmatrix} \tag{3.34}$$

and we obtain the inhomogenous Maxwell equations $\nabla\cdot\boldsymbol{E} = \varepsilon_0\rho$ and $\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j} + \mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$.
Similarly, we can take the four-derivative of the dual field tensor and obtain the following expression

$$\partial_\alpha(*F)^{\alpha\beta} = \begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z/c & E_y/c \\ B_y & E_z/c & 0 & -E_x/c \\ B_z & -E_y/c & E_x/c & 0 \end{bmatrix} = \begin{bmatrix} \nabla\cdot\boldsymbol{B} \\ \frac{1}{c}\nabla\times\boldsymbol{E} + \frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} \end{bmatrix} \tag{3.35}$$

Setting this equal to zero yields

$$\partial_\alpha(*F)^{\alpha\beta} = \begin{bmatrix} \nabla\cdot\boldsymbol{B} \\ \frac{1}{c}\nabla\times\boldsymbol{E} + \frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.36}$$

And thus we have found an expression for the homogeneous Maxwell equations $\nabla\cdot\boldsymbol{B} = 0$ and $\nabla\times\boldsymbol{E} + \frac{\partial\boldsymbol{B}}{\partial t} = 0$ in terms of the field tensor. Combined this gives

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta \tag{3.37}$$

$$\partial_\alpha(*F)^{\alpha\beta} = 0 \tag{3.38}$$

3.3. Alternative derivation of the Field tensor and Maxwell's equation

In the above derivation of the field tensor and the reformulation of Maxwell's equations we have made extensive use of the Cartesian coordinate system. In the derivation below we will use a less coordinate dependent method that will rely on two abstract operations that are called the hodge-star and the differential. Additionally, we will derive the covariant formulation for Maxwell's equations in matter rather than in vacuum as above.

The differential operator

Before we can define the Hodge-star and the differential, we first need to define the Levi-Cevita symbol.

Definition 3.3.1 (Levi-Cevita symbol)**.** The levi-Cevita symbol $\varepsilon_{i_1 i_2 \ldots i_n}$ is the alternating object defined by $\varepsilon_{i_1 i_2 \ldots i_n} = 1$ and $\varepsilon_{i_1 \ldots i_p \ldots i_q \ldots i_n} = -\varepsilon_{i_1 \ldots i_q \ldots i_p \ldots i_n}$. The number of indices n is, unless specified, the dimension of the manifold on which it is defined.

we see that interchanging any two of its indices changes the sign of the object. Hence if any two indices are equal, we must have $\varepsilon_{i_1\dots i_p\dots i_p\dots i_n} = -\varepsilon_{i_1\dots i_p\dots i_p\dots i_n} = 0$. In order to use ε in tensor analysis we introduce the Levi-Cevita tensor $\sqrt{|g|}\varepsilon_{i_1 i_2\dots i_n}$. Here $g = \det(g_{ij})$ where g_{ij} is the metric. This object transforms as an alternating tensor of rank n under coordinate transformations if g is positive [10].

Definition 3.3.2 (hodge star)**.** Given an alternating covariant tensor $T_{i_1\dots i_k}$ of rank k, we define the alternating tensor $*T$ of rank $(n-k)$, where n is the dimension of the vector space, by

$$(*T)_{i_{k+1}\dots i_n} = \frac{1}{k!}\sqrt{|g|}\varepsilon_{i_1\dots i_n} T^{i_1\dots i_k} \tag{3.39}$$

The symbol $*$ is called the hodge star.

Since $\sqrt{|g|}\varepsilon_{i_1 i_2\dots i_n}$ is an alternating tensor of rank n, we can conclude that $(*T)_{i_{k+1}\dots i_n}$ is alternating as well since swapping two indices will only swap two indices greater than k in $\varepsilon_{i_1\dots i_n}$.

Theorem 3.3.1. The inverse of $*$ is $(-1)^{k(n-k)}*$ [10]

Proof. Let $T_{i_1\dots i_k}$ be a covariant tensor of rank k and let n be the dimension of the vector space.
First note that if $*T$ is a tensor of rank $(n-k)$, then $*(*T)$ is a tensor of rank $n-(n-k) = k$ □

Definition 3.3.3 (differential)**.** Let $T_{i_1\dots i_k}$ be an alternating tensor defined on an n-dimensional space with coordinates $x^1,\dots,x^n$. The differential $dT_{j_1\dots j_k j_{k+1}}$ is an alternating covariant tensor of rank $k+1$ with components defined by

$$(dT)_{j_1\dots j_{k+1}} = \sum_{q=1}^{k+1} \frac{\partial T_{j_1\dots j_{q-1} j_{q+1}\dots j_{k+1}}}{\partial x^{j_q}} (-1)^{q-1} \tag{3.40}$$

The differential acts as a generalization of many concepts known from calculus. Take for example the covector $T = T_i$. that is, a covariant tensor of rank 1. According to definition 3.3.3 we obtain the curl of the covector field

$$(dT)_{ij} = \frac{\partial T_j}{\partial x^i} - \frac{\partial T_i}{\partial x^j}$$

Using above definitions we can rewrite Maxwell's equations in a coordinate invariant form. That is, we can rewrite the equation in such a form that it becomes clear that they are form invariant when going from one set of coordinates to the next. To do so however, we first need to express the divergence and curl of a vector field in terms of the differential and the Hodge star.

Theorem 3.3.2. The divergence of a vector field T_i three-dimensional euclidean space is given by $*^{-1}d*T$.

Proof.

$$
\begin{aligned}
*T_{ij} &= \sqrt{|g|}\varepsilon_{ijk}T^k \\
d(*T)_{ijk} &= \big(\frac{\partial * T_{jk}}{\partial x^i} - \frac{\partial * T_{ik}}{\partial x^j} + \frac{\partial * T_{ij}}{\partial x^k}\big) \\
d(*T)^{ijk} &= g^{ii'}g^{jj'}g^{kk'}d(*T)_{i'j'k'} \\
&= g^{ii'}g^{jj'}g^{kk'}\big(\frac{\partial * T_{j'k'}}{\partial x^{i'}} - \frac{\partial * T_{i'k'}}{\partial x^{j'}} + \frac{\partial * T_{i'j'}}{\partial x^{k'}}\big) \\
&= g^{ii'}g^{jj'}g^{kk'}\sqrt{|g|}\big(\varepsilon_{j'k'i'}\frac{\partial T^{i'}}{\partial x^{i'}} - \varepsilon_{i'k'j'}\frac{\partial T^{j'}}{\partial x^{j'}} + \varepsilon_{i'j'k'}\frac{\partial T^{k'}}{\partial x^{k'}}\big) \\
&= g^{ii'}g^{jj'}g^{kk'}\sqrt{|g|}\varepsilon_{i'j'k'}\big(\frac{\partial T^{i'}}{\partial x^{i'}} + \frac{\partial T^{j'}}{\partial x^{j'}} + \frac{\partial T^{k'}}{\partial x^{k'}}\big) \\
&= \sqrt{|g|}\varepsilon^{ijk}\big(\frac{\partial T^{i'}}{\partial x^{i'}} + \frac{\partial T^{j'}}{\partial x^{j'}} + \frac{\partial T^{k'}}{\partial x^{k'}}\big)
\end{aligned}
$$

$$
\begin{aligned}
*^{-1}d(*T) &= \frac{1}{3!}\sqrt{|g|}\varepsilon_{ijk}d(*T)^{ijk} \\
&= \frac{1}{3!}|g|\varepsilon_{ijk}\varepsilon^{ijk}\big(\frac{\partial T^{i'}}{\partial x^{i'}} + \frac{\partial T^{j'}}{\partial x^{j'}} + \frac{\partial T^{k'}}{\partial x^{k'}}\big) \\
&= |g|\big(\frac{\partial T^{i'}}{\partial x^{i'}} + \frac{\partial T^{j'}}{\partial x^{j'}} + \frac{\partial T^{k'}}{\partial x^{k'}}\big)
\end{aligned}
$$

□

Theorem 3.3.3. The curl of a vector field T_i in three dimensional euclidean space is given by $*dT = \nabla \times T$

Proof.

$$(dT)_{ij} = \frac{\partial T_i}{\partial x^j} - \frac{\partial T_i}{\partial x^j} \tag{3.41}$$

$$(*dT)_k = \frac{1}{2}\sqrt{|g|}\varepsilon_{ijk}(dT)^{ij} \tag{3.42}$$

$$(*dT)^k = \frac{1}{2}\sqrt{|g|}\varepsilon^{ijk}(dT)_{ij} \tag{3.43}$$

$$= \frac{1}{2}\sqrt{|g|}\big(\frac{\partial T_i}{\partial x^j} - \frac{\partial T_i}{\partial x^j} - \frac{\partial T_i}{\partial x^j} + \frac{\partial T_i}{\partial x^j}\big) \tag{3.44}$$

$$= \sqrt{|g|}\big(\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}\big) \tag{3.45}$$

Since $|g| = 1$, we have proven the result. □

Maxwell's equations in terms of the differential and hodge derivative

Using above theorems we can rewrite Maxwell's equations in matter

$$\nabla \cdot \boldsymbol{D} = 4\pi\rho \tag{3.46}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \tag{3.47}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{3.48}$$

$$\nabla \times \boldsymbol{H} = \frac{1}{c}\big(4\pi \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}\big) \tag{3.49}$$

into

$$*_{3\mathrm{D}}^{-1}\, d *_{3\mathrm{D}} \boldsymbol{D} = 4\pi\rho \tag{3.50}$$

$$*_{3\mathrm{D}}\, d\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \tag{3.51}$$

$$*_{3\mathrm{D}}^{-1}\, d *_{3\mathrm{D}} \boldsymbol{B} = 0 \tag{3.52}$$

$$*_{3\mathrm{D}}\, d\boldsymbol{H} = \frac{1}{c}\big(4\pi \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}\big) \tag{3.53}$$

Do note that Gaussian units are used in above equations [8] and to distinguish the 3 dimensional hodge star operator from the 4 dimensional that will be used later on, we added the 3D subscript.

The homogenous Maxwell equations Now if we introduce the coordinates $x^4 = ct$ and multiply equation 3.51 by $*_{3D}^{-1}$ we obtain

$$d\mathbf{E} + \frac{\partial *_{3D}^{-1}\mathbf{B}}{\partial x^4} = 0 \tag{3.54}$$

Do note that in above equation $d\mathbf{E}$ and $*_{3D}^{-1}\mathbf{B}$ are both tensors of rank 2 in 3d space and hence our equation makes sense. Furthermore, note that since x^4 was chosen to be a time coordinate, the spatial indices i, j and k that we will use below, will have values 1, 2 and 3.
Writing out the individual components of above equation for $i \neq j$ and noting that $(*_{3D}^{-1}\boldsymbol{B})_{ij} = \sqrt{|g|_{3D}}\varepsilon_{ijk}B^k$ yields

$$(d\mathbf{E})_{ij} + (\frac{\partial *_{3D}^{-1}\mathbf{B}}{\partial x^4})_{ij} = 0 \tag{3.55}$$

$$\frac{\partial}{\partial x^i}E_j - \frac{\partial}{\partial x^j}E_i + \frac{\partial}{\partial x^4}(\sqrt{|g|_{3D}}\varepsilon_{ijo}B^o) = 0 \tag{3.56}$$

Where the index o designates the spatial index that is unequal to both i and j. Additionally, equation (3.52) and theorem 3.3.2 show that

$$\frac{\partial}{\partial x^1}B^1 + \frac{\partial}{\partial x^2}B^2 + \frac{\partial}{\partial x^3}B^3 = 0 \tag{3.57}$$

above equation can be recognized as components of the differential of a 4-dimensional alternating tensor of rank 3. To see this, let F be such a tensor. In matrix form this corresponds to

$$F_{\alpha\beta} = \begin{bmatrix} 0 & F_{12} & F_{13} & F_{14} \\ -F_{12} & 0 & F_{23} & F_{24} \\ -F_{13} & -F_{23} & 0 & F_{34} \\ -F_{14} & -F_{24} & -F_{34} & 0 \end{bmatrix} \tag{3.58}$$

Now taking the differential of F, we find that its components are given by

$$(dF)_{ijk} = \frac{\partial}{\partial x^i}F_{jk} - \frac{\partial}{\partial x^j}F_{ik} + \frac{\partial}{\partial x^k}F_{ij} \tag{3.59}$$

Note that dF has only 4 distinct components, dF_{123}, dF_{124}, dF_{134} and dF_{234}. All other components are 0 (if either $i = j$, $j = k$ or $i = k$) or equal to another component up to its sign.
If we compare equation (3.59) to (3.57) we see that if we choose $F_{12} = B_3$, $F_{13} = -B_2$ and $F_{23} = B_1$ we obtain equal expressions by taking $(dF)_{123} = 0$. Here we also set $|g|_{3D} = 1$ Similarly, comparing (3.59) to (3.56), we see that setting $F_{14} = E_1$, $F_{24} = E_2$ and $F_{34} = E_3$ gives equation (3.56) after setting $(dF)_{ij4} = 0$. Combining all above we see that $dF = 0$ is equivalent to equations (3.51) and (3.52) where F is give by

$$F_{\alpha\beta} = \begin{bmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{bmatrix} \tag{3.60}$$

In above derivation we labeled our spatial coordinates with 1, 2 and 3 and our temporal coordinate with 4. In our other derivation of the field tensor we have used a different convention where our temporal coordinate has label 1 and the spatial coordinates have labels 2, 3 and 4. Hence the field tensor in (3.60) is different from (3.26). We obtain (3.26) by adding one unit to the index of every element and calculate modulo 4 (That is, a 5 becomes a 1). Visually we shift all entries one down and one right. If the entry is out of range it loops back to the other side of the matrix.

The inhomogenous equations Equations (3.50) and (3.53) can be recast in similar fashion. We first apply the three-dimensional Hodge-star to both sides, introduce $x^4 = ct$ and rearrange to obtain

$$(d *_{3D} \boldsymbol{D})_{ijk} = 4\pi(*_{3D}\rho)_{ijk} \tag{3.61}$$

$$(d\boldsymbol{H})_{ij} - \frac{\partial(*_{3D}\boldsymbol{D})_{ij}}{\partial x^4} = \frac{4\pi}{c}(*_{3D}\boldsymbol{j})_{ij} \tag{3.62}$$

In a similar fashion to the derivation of F above, we can identify the left hand side of these equations with the elements of the differential of an antisymmetric covariant tensor of rank 2 which will be denoted by M. Do note however, that the parity of the $*\boldsymbol{D}$ term in equation (3.62) is flipped when compared to the homogenous equations and hence the values of $\boldsymbol{H}$ have an extra minus sign in front.

$$M_{\alpha\beta} = \begin{bmatrix} 0 & D_3 & -D_2 & -H_1 \\ -D_3 & 0 & D_1 & -H_2 \\ D_2 & -D_1 & 0 & -H_3 \\ H_1 & H_2 & H_3 & 0 \end{bmatrix} \tag{3.63}$$

And we find

$$(dM)_{123} = \frac{\partial(D^1)}{\partial x^1} - \frac{\partial(-D^2)}{\partial x^2} + \frac{\partial D^3}{\partial x^4} = \frac{4\pi}{c}\sqrt{|g|_{3\mathrm{D}}}c\rho \tag{3.64}$$

$$(dM)_{124} = \frac{\partial(-H_2)}{\partial x^1} - \frac{\partial(-H_1)}{\partial x^2} + \frac{\partial\sqrt{|g|_{3\mathrm{D}}}\varepsilon_{123}D^3}{\partial x^4} = -\frac{4\pi}{c}\sqrt{|g|_{3\mathrm{D}}}J^3 \tag{3.65}$$

$$(dM)_{134} = \frac{\partial(-H_3)}{\partial x^1} - \frac{\partial(-H_1)}{\partial x^3} + \frac{\partial\sqrt{|g|_{3\mathrm{D}}}\varepsilon_{132}D^2}{\partial x^4} = \frac{4\pi}{c}\sqrt{|g|_{3\mathrm{D}}}J^2 \tag{3.66}$$

$$(dM)_{234} = \frac{\partial(-H_j)}{\partial x^2} - \frac{\partial(-H_2)}{\partial x^3} + \frac{\partial\sqrt{|g|_{3\mathrm{D}}}\varepsilon_{231}D^1}{\partial x^4} = -\frac{4\pi}{c}\sqrt{|g|_{3\mathrm{D}}}J^1 \tag{3.67}$$

We would like to rewrite above equations into one single equation. In order to do so, note that the determinant of the 3 dimensional metric $|g|_{3\mathrm{D}}$ is equal to the determinant of the 4 dimensional Minkowski metric $|g|_{4\mathrm{D}}$. Comparing the expression of the four-dimensional hodge star acting on the 4-current $J^\alpha = (\boldsymbol{j}, c\rho)$ we see that

$$*_{4\mathrm{D}}^{-1}\begin{pmatrix} \boldsymbol{j} \\ c\rho \end{pmatrix} = \sqrt{|g|_{4\mathrm{D}}}\varepsilon_{\alpha\beta\gamma\sigma}J^\alpha \tag{3.68}$$

is equal to the right hand side of equations (3.64) until (3.67) and thus we can conclude that

$$dM = \frac{4\pi}{c} *_{4\mathrm{D}}^{-1} J \tag{3.69}$$

or

$$*_{4\mathrm{D}}\, dM = \frac{4\pi}{c} J \tag{3.70}$$

Above we have found a formulation of Maxwell's equations in a coordinate independent manner by using the Hodge star and the differential. We can conclude that as long as the Hodge star in 4 dimensions remains the same, which is the case for any Lorentz transformation from one inertial frame of reference to the next, Maxwell's equations look like

$$*_{4\mathrm{D}} dM = \frac{4\pi}{c} J \tag{3.71}$$

$$dF = 0 \tag{3.72}$$

3.4. Looking ahead

In above section we have seen how the laws of electromagnetism are formulated and behave in the context of special relativity. Although this allows us to describe a multitude of phenomena, there are some restrictions. For our purpose the main restriction is that above theory only works for inertial frames of reference. Once we start looking at accelerating or rotating frames, above theory does not work anymore as our core assumptions are invalid. In rotating frames of reference the spacetime that we observe is no longer governed by the Minkowski metric and we will have to develop other techniques.

4

Electromagnetism in a rotating frame of reference

Direct translation to a rotating frame of reference

P. Arendt [2] and others [22, 27] have adopted a direct and straightforward method to obtain a formulation of the laws of electromagnetism in a rotating frame of reference. Below we will present and discuss the results found in these papers. The method employed uses a frame of reference equipped with Cartesian coordinates (t', x', y', z') that rotates around the z-axis of inertial frame of reference (t, x, y, z). The relation between the coordinate systems is defined by

$$t' = t \tag{4.1}$$

$$z' = z \tag{4.2}$$

$$x' = x\cos(\Omega t) + y\sin(\Omega t) \tag{4.3}$$

$$y' = y\cos(\Omega t) - x\sin(\Omega t) \tag{4.4}$$

By transforming the diagonal Minkowski metric using above relations to our new coordinate system we obtain the metric tensor in our rotating coordinate system. Note that although our metric is no longer the Minkowski metric, we are still dealing with flat Minkowski space-time as we have not added any sources of a gravitational field.

$$g_{\mu\nu} = \begin{bmatrix} -(1-\Omega^2(x'^2+y'^2)) & -\Omega y' & \Omega x' & 0 \\ -\Omega y' & 1 & 0 & 0 \\ \Omega x' & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.5}$$

And its inverse is

$$g^{\mu\nu} = \begin{bmatrix} -1 & -\Omega y' & \Omega x' & 0 \\ -\Omega y' & 1-\Omega^2 y'^2 & \Omega^2 x' y' & 0 \\ \Omega x' & \Omega^2 x' y' & 1-\Omega^2 x'^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.6}$$

Using the metric it is possible to transform the field tensor as defined in equation (3.1) to the new system of coordinates. This results in, using Gaussian units

$$F_{\mu'\nu'} = \begin{bmatrix} 0 & -\tilde{E}_{x'} & -\tilde{E}_{y'} & -\tilde{E}_{z'} \\ \tilde{E}_{x'} & 0 & B_{z'} & -B_{y'} \\ \tilde{E}_{y'} & -B_{z'} & 0 & B_{x'} \\ \tilde{E}_{z'} & B_{y'} & -B_{x'} & 0 \end{bmatrix} \tag{4.7}$$

Here $\tilde{\boldsymbol{E}} = \boldsymbol{E} + (\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{B}$. Furthermore $\tilde{E}_{i'}$ and $B_{i'}$ are fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in inertial system (t, x, y, z) projected on the i' axis.

Raising the indices using the metric we find the contravariant tensor

$$F^{\mu'\nu'} = \begin{bmatrix} 0 & E_{x'} & E_{y'} & E_{z'} \\ -E_{x'} & 0 & \tilde{B}_{z'} & -\tilde{B}_{y'} \\ -E_{y'} & -\tilde{B}_{z'} & 0 & \tilde{B}_{x'} \\ -E_{z'} & \tilde{B}_{y'} & -\tilde{B}_{x'} & 0 \end{bmatrix} \tag{4.8}$$

Where $\tilde{\boldsymbol{B}} = \boldsymbol{B} - (\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{E}$ and we use the same kind of projection of the fields to obtain $E_{i'}$ and $\tilde{B}_{i'}$.
In order to obtain Maxwell's equations in (t', x', y', z') the four current is defined as $J^{\alpha'} = (\rho', \boldsymbol{j}')$. Here $\rho' = \rho$ and $\boldsymbol{j}' = \boldsymbol{j} - \rho(\boldsymbol{\Omega} \times \boldsymbol{r})$.
Using (4.7) as the field tensor in the covariant notation of Maxwell's equations (3.37) and (3.38), and by taking J^{α} as the four current, the following equations are found to represent Maxwell's equations in rotating coordinate system (t', x', y', z') [2]

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho' \tag{4.9}$$

$$\nabla \times \tilde{\boldsymbol{E}} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{4.10}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{4.11}$$

$$\nabla \times \tilde{\boldsymbol{B}} = 4\pi \boldsymbol{j}' + \frac{\partial \boldsymbol{E}}{\partial t} \tag{4.12}$$

Using equation (4.9) and (4.12) we find that

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot \boldsymbol{j}' = 0 \tag{4.13}$$

And thus a charge conservation law has been found. Additionally, P. Arendt notes that the two Lorentz invariants that were obtained in chapter 3 are recovered when calculating $F^{\mu'\nu'}F_{\mu'\nu'}$ and $F^{\mu'\nu'}(*F)_{\mu'\nu'}$, which is convenient but not surprising since invariants of a tensor do not change under a change of coordinates [20]. The final result P. Arendt states is a set of equations of motions for a particle with charge q and mass m moving in (t', x', y', z'). We will not derive these equations here, but in their final form they are

$$\frac{d\gamma}{dt'} = \frac{q}{m}(\boldsymbol{v}' + (\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \boldsymbol{E} \tag{4.14}$$

$$\frac{d(\gamma \boldsymbol{v}')}{dt'} = \gamma(-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) - 2\boldsymbol{\Omega} \times \boldsymbol{v}') + \frac{q}{m}(\tilde{\boldsymbol{E}} + \boldsymbol{v}' \times \boldsymbol{B} - (\boldsymbol{v}' \cdot \boldsymbol{E})(\boldsymbol{\Omega} \times \boldsymbol{r})) \tag{4.15}$$

In above equations we have defined $\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{\Omega} \times \boldsymbol{r}$, the speed of the particle as measured in the rotating frame of reference. The effects of the centrifugal force and the Coriolis force are seen in the terms $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})$ and $-2\boldsymbol{\Omega} \times \boldsymbol{v}'$ respectively. The other terms are contributions of the Lorentz force acting on the charged particle.
Using above results we would like to obtain a definition of the electric $\boldsymbol{E}'$ and magnetic field $\boldsymbol{B}'$ in (t', x', y', z'). Looking at equation (4.7) it seems that $\boldsymbol{E}' = \tilde{\boldsymbol{E}}$ and $\boldsymbol{B}' = \boldsymbol{B}$ are logical choices. Equation (4.8) however, implies that $\boldsymbol{E}' = \boldsymbol{E}$ and $\boldsymbol{B}' = \tilde{\boldsymbol{B}}$ should be chosen as the electric and magnetic fields. This duality is also seen in Maxwell's equations (4.9) until (4.12). Both the ordinary fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, and the adjusted fields, $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$, are present. Thus it is difficult to give an unambiguous definition of the electric and magnetic fields [2].

Discussion The time coordinate used by P. Arendt in his definition of (t', x', y', z') is equal to the time coordinate of inertial system (t, x, y, z). Now if we were an actual observer rotating around a central axis, our time clock would be with us and not be part of inertial system (t, x, y, z). Hence the representation of the physics taking place is not that of an intrinsic observer and the ambiguity of our electric and magnetic field does not have to be a problem for an actual observer. Furthermore, looking at metric (4.7) we see that it has off-diagonal components g_{tx} and g_{ty}. This means that the time component of our spacetime is not orthogonal to our spatial coordinates. Asking oneself the question "What would this look like to me as an observer?", you might have trouble imagining such a non-orthogonal time-axis, I certainly do. As a matter of fact, an observer will always have an orthonormal set of coordinates [13] and hence a diagonal metric. Thus we see that the metric used by P. Arendt could never correspond to that of an observer.

4.1. Christoffel symbols of a diagonal metric

Above we have seen that trouble arises when transforming to a rotating frame of reference. Since our metric is no longer diagonal we are unable to speak of an observer and an ambiguity in the electrica and magnetic field arises. This raises the question as to whether it is possible to have a rotating frame of reference, including the Coriolis and centrifugal forces, that is described by a diagonal metric. Since we can find the equations of motion using the geodesic equations as defined in appendix B.1, and these are largely governed by the Christoffel symbols, the diagonal metric that we are looking for should give rise to the Christoffel symbols that account for the Coriolis and centrifugal forces. That is, the Christoffel symbols of our diagonal metric should at least contain, $\Gamma^x_{tt} = -\Omega^2 x$, $\Gamma^y_{tt} = -\Omega^2 y$, $\Gamma^x_{yt} = \Gamma^x_{ty} = -\Omega$ and $\Gamma^y_{xt} = \Gamma^y_{tx} = \Omega$ [2]. Appendix B.1 shows all possible Christoffel symbols of a diagonal metric. Comparing the expressions above to the Christoffel symbols found in appendix B.1, we find that the Christoffel symbols responsible for the centrifugal force can be reconstructed using a diagonal metric. The Coriolis force however, cannot be reconstructed via a diagonal metric, as appendix B.1 contains no Christoffel symbols with three different indices. Therefore, it is impossible to find a set of coordinates that converts metric (4.5) into a global, that is on the entire space, diagonal metric that still gives rise to similar equations of motion. Do note that, most importantly, the time coordinate must be orthogonal to the spatial coordinates. The spatial coordinates themselves need not be orthogonal to one another as we can use any curvilinear coordinate system we like.

4.2. An observer rotating around his own axis

Above we have seen how one can derive a set of equations that describes the electromagnetic field in a rotating frame of reference. Noted was, however, that the underlying metric is off-diagonal and thus does not correspond to the frame of reference of an observer. Furthermore, we have seen that it is impossible to construct a purely diagonal metric that is capable of reproducing a Coriolis force. Hence, we need off-diagonal elements. There seems to be an apparant paradox here. On the one hand we need off-diagonal elements to correctly describe the curvature of spacetime and on the other hand we need to use a diagonal metric to measure quantities as a physical observer. Below we will show a method on how to deal with an observer rotating around his own local axis.

H. Stephani studies in his book, General Relativity, An introduction to the Theory of the Gravitational Field, the way an observer observes particle dynamics, while he himself is subject to an acceleration $\boldsymbol{a}$ and rotates around his own axis with angular velocity ω. The rotation ω is different from the rotation as used by P. Arendt, as Arendt rotates around a global axes whereas Stephani rotates round the observers own axes. The used metric is given by [25]

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta - 2\varepsilon_{\beta\nu\mu} x^\nu \frac{\omega^\mu}{c} dx^1 dx^\beta - (1 + 2\frac{a_\nu x^\nu}{c^2})(dx^1)^2 \tag{4.16}$$

Note that α, β, ν and μ only run over spatial indices 2, 3 and 4. We see that this metric is indeed very similar to the one that is used in equation (4.5) by P. Arendt. The major difference is the g_{11} component that now has a term dependent on a rather than ω^2. This is because H. Stephani gives the observer an arbitrary acceleration rather than the centrifugal acceleration dependent on Ω that P. Arendt uses.

If we assume that the rotation is only in the x^4 direction, we obtain in matrix form

$$g_{ij} = \begin{bmatrix} -1 - \frac{2a_\nu x^\nu}{c^2} & -\frac{\omega^4}{c} y & \frac{\omega^4}{c} x & 0 \\ -\frac{\omega^4}{c} y & 1 & 0 & 0 \\ \frac{\omega^4}{c} x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.17}$$

We see that in the origin of the coordinate system the metric becomes the Minkowski metric and we have an observable frame of reference. Also note that all components of the metric that do not correspond to the Minkowski metric contain a $1/c$ or $1/c^2$ factor. Since all velocities on earth are small compared to that of the speed of light (e.g. $y\omega^4 << c$), the off-diagonal components are minute and we obtain a near diagonal metric. Due to this it is very difficult to measure the effects of this non-diagonal metric on earth such as the curvature of light. Do note however, that this is a partial explanation as we have not taken into account the effect of gravity on the curvature of spacetime.

Stephani shows that the equations of motion for a particle as measured by the rotating and accelerating observer is given by [25]

$$\ddot{\boldsymbol{r}} = -\boldsymbol{a} - 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \frac{2\boldsymbol{a}\dot{\boldsymbol{r}}}{c^2}\dot{\boldsymbol{r}} \tag{4.18}$$

Here we see that the third term is a relativistic correction for accelerations at very high velocities or accelerations that approach the speed of light. Furthermore we recover the Coriolis force in the second term, note however, that no relativistic correction is present here. This is due to the fact that we are working locally around the observer as our metric would not be the flat Minkowski metric anymore if we strayed too far away from our original position.

5

Discussion and Recommendations

Electromagnetism in a rotating frame of reference

As we have seen in chapter 4, we can formulate Maxwell's equations in a rotating frame of reference by direct transformation to a set of rotating coordinates. This resulted in an ambiguity in our definition of the electric and magnetic field which could be clarified by noting that we are working in a non-intrinsic frame of reference with off-diagonal metric components.
This result shows a fundamental difference between the fields of (Mathematical) Physics and Mathematics. The method employed by P. Arendt is mathematically justified, as we can define any coordinate system we like and use tensor calculus to transform to such a frame of reference. However, in the fields of Physics and Mathematical Physics, most of these coordinate systems lead to issues in the interpretation of phenomena, as these coordinate systems do not correspond to the coordinate system of an observer. Hence the results of such coordinate systems, mathematically sound as they may be, need not say much about the physical world that we observe around us.

In the light of this, we must address the following question. In chapter 3 we have found that Maxwell's equations of electromagnetism can be formulated as equation (3.71) and (3.72). These equations will look exactly the same so long as we observe from an inertial frame of reference that is equipped with the Minkowski metric. In other words, they are invariant under Lorentz transformations. In chapter 4 available literature [2, 27] is reviewed that attempts to reformulate these equations such that they represent what an observer moving in a non-inertial frame that is rotating around a central axis might observe. As noted above, this leads to a multitude of problems and the question arises as to whether it is possible at all to formulate these equations in a way that leaves their form invariant under transformations from one intrinsically rotating frame of reference to the next.
In order to have Coriolis-like forces we need to have off-diagonal elements in our metric and hence our time-coordinate can't be orthogonal to our spatial coordinates. The latter however, is essential for the very definition of an observable frame of reference.
The duality in these two requirements indicate that it might be impossible to obtain such a formulation of the laws of electromagnetism in a rotating frame of reference.

Recommendations for future research

There is literature available on the problem of an observer rotating around a global axis and there have been proposed solutions to solve the problem of a non-diagonal metric by using the formalism of orthogonal tetrads [6, 14, 24]. Within this formalism a local observer in an inertial frame of reference is constructed that is able to measure local quantities and relate them to the rotating frame of reference via a transformation matrix. Using said formalism it is possible to derive a set of equations that describe the electromagnetic field in a rotating frame of reference. The equations found, using orthogonal tetrads, predict observations that differ from the equations that were found by P. Arendt [24]. Using a so-called electromagnetic gyroscope it might be possible to experimentally verify whether these results are correct or not. Fully developing the theory needed for these results was beyond the scope of this bachelor thesis, hence they are not presented in chapter 4 but rather given here as a suggestion for future research.

The following must be noted however. R. Klauber says in his paper *Relativistic Rotation: A Comparison of Theories* that no conclusive theory has been found yet that describes an observer rotating around a central axis in general relativity [17]. Furthermore, he postulates that the traditionally used method of locally co-moving inertial frames (which is equivalent to the method of orthogonal tetrads) has some inconsistencies which have not yet been resolved [17]. If this is true, then the foundation of [14] is faulty and another method needs to be adopted.

6
Conclusion

In this report, relevant literature on the topic of the formulation of Maxwell's equation in rotating non-inertial reference systems is discussed. Since Maxwell's equation of electromagnetism have been developed through the theory of Special relativity, which deals with inertial non-rotating systems of reference this is a non-trivial task.
Through the development of the necessary theories of Differential Geometry and Special Relativity we have been able to formulate Maxwell's equations of electromagnetism in a covariant form that remains unchanged under Lorentz transformations. Using these results we have discussed a direct approach using tensor calculus to obtain a rotating frame of reference. Applying this frame of reference to Maxwell's equations in covariant notation lead to an ambiguity in our choice of the electric and magnetic field while the frame of reference no longer corresponded to that of a physical observer. Hence it was no longer possible to speak of an electric or magnetic field in this frame of reference. An attempt was made to solve this problem by studying the Christoffel-symbols of a diagonal metric but no method was found to reconstruct Coriolis-like forces in such a space.
For further research it is recommended that other approaches to the problem of rotating frames of reference in electromagnetism such as the methods of orthogonal tetrads or locally co-moving inertial frames are studied to see whether these resolve the ambiguity problems in the choice of the electric and magnetic frame of reference. In addition, the validity of these methods in the broader context of General Relativity needs to be verified as there is still debate regarding the tools available for rotating bodies.

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A

Differential geometry

The starting point for the definition of a smooth manifold is a Hausdorff topological space, which we designate by L. Loosely speaking, in a Hausdorff topological space we have sufficient structure to talk about notions of openness and closedness. And with these notions we can define what it means for a function to be continuous and define limits in these spaces. The Hausdorff property ensures that limits of sequences are in fact unique and converge to only one 'point'. For the exact definitions I refer to [15]. A Hausdorff topological space allows one to talk about continuity, but there is no notion of differentiability or smoothness. For that we will need something extra. The text below is heavily inspired by the lecture notes of the course AM3580, Introduction to differential geometry at Delft university of technology [15]. The account I give here is far from rigorous but intended to give a complete idea of all concepts and definitions needed to understand the mathematical background of what happens in chapter 2. Whenever I skip over some of the details I will refer to the course-notes where an interested reader can do some further reading.

A.1. The Smooth manifold

Definition A.1.1 (Chart)**.** The chart (U', ϕ') is a homeomorphism (a smooth mapping that is one to one or bijective)

$$\phi' : L \supset U' \to \phi(U') \subseteq \mathbb{R}^n \tag{A.1}$$

Here U' is an open subset of L and $\phi'(U')$ is an open subset of $\mathbb{R}^n$.

Definition A.1.2. transition functions Let (U', ϕ') and (U'', ϕ'') be two charts on L such that there exists a $p \in L$ with $p \in U' \cap U''$. Then the transition function $\kappa^{\prime\to\prime\prime}$ is defined as

$$\kappa^{\prime\to\prime\prime} : \phi'(U' \cap U'') \to \phi''(U' \cap U'') \quad \text{with} \quad \kappa^{\prime\to\prime\prime} := \phi'' \circ \phi'^{-1}$$

Definition A.1.3 (Smooth manifold M)**.** A smooth manifold of dimension n is a Hausdorff topological space L, equipped with charts $\phi' : U' \to \phi(U') \subseteq \mathbb{R}^n$ such that $L = \bigcup' U'$, and such that all transition functions $\kappa^{\prime\to\prime\prime}$ are smooth.

The charts defined above allow us to transport the mathematics that we want to do on our smooth manifold from the manifold itself to the well-known structure of $\mathbb{R}^n$. For example, a function $f : M \to \mathbb{R}$ is smooth at $p \in M$ if for some chart (U', ϕ') containing p we have that $f \circ \phi'^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth at $\phi'(p)$. The smoothness of the transition functions ensures that regardless of the chart we use, we find the same result regarding the smoothness of a function $f : M \to \mathbb{R}$. That is, if f is smooth in the chart (U', ϕ'), then it is smooth in (U'', ϕ'') [15].
We now know what a smooth manifold is and what it means for a function to be smooth on M. However, we do not yet know how we can take, for example, the derivative of a function $F : M \to \mathbb{R}$ on a manifold. For this we will need the concept of the tangent space of a point on a manifold. Before we look at the exact definitions of the tangent space, As a concrete example of what a tangent space looks like, we look at the 2-sphere defined by $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1 \mid (x, y, x) \in \mathbb{R}^3\}$. The tangent space at the point $(0, 0, 1)$ can be visualised as the plane tangent to $\mathbb{S}^2$ at $(0, 0, 1)$. See figure A.1

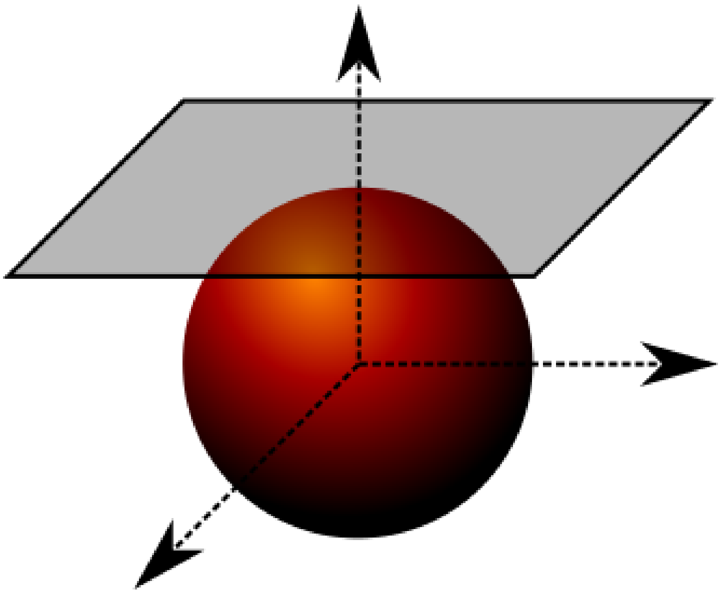

Figure A.1: The tangent space of the 2-sphere at $(0,0,1)$

Any vector in the tangent plane at $(0,0,1)$ that originates from this point is called a tangent vector at $(0,0,1)$. In this example, there is one caveat though. The tangent plane as visualised in figure A.1 lies outside of $\mathbb{S}^2$ itself. This is possible because $\mathbb{S}^2$ is embedded in $\mathbb{R}^3$ and hence we can use the additional structure of $\mathbb{R}^3$ to show the tangent plane. For a general manifold M however, there need not be a manifold in which it is embedded and hence the definition that we will use is slightly more nuanced than the picture painted above. At its core lie smooth curves $\gamma : \mathbb{R} \supseteq I \to M$.

Definition A.1.4 ($\sim_p$)**.** Two curves γ and l through p in M are equivalent, $\gamma \sim_p l$, if there exists a chart (U', ϕ') around p such that $\dot{\gamma}'^{\mu}(0) = \dot{l}'^{\mu}(0)$ for all μ. Here $\gamma' = \phi' \circ \gamma$, the function γ with respect to the chart (U', ϕ') and μ designates the μ'th coordinate.

Remark. The equivalence relation is independent of our choice of charts [15]. That is, if two tangent vectors are equivalent with respect to one set of charts, they will also be equivalent with respect to another chart.

Definition A.1.5 (Tangent space)**.** A tangent vector at $p \in M$ is an equivalence class $v_p = [\gamma]$ of curves through p with respect to the relation $\sim_p$. The set T_pM of all tangent vectors at p is called the tangent space of M at p.

For a tangent vector $v \in T_pM$, we call $(v'^1, \ldots, v'^n)$, the coordinate expression of v with respect to chart (U', ϕ'). These coefficients uniquely determine a tangent vector as the map

$$\phi'_* : T_pM \to \mathbb{R}^n, \ \phi'_*(v) = (v'^1, \ldots, v'^n) \tag{A.2}$$

Is a bijection. Hence we can conclude that T_pM is in fact a vector space where addition of two tangent vectors and scalar multiplication is defined [15].
As a consequence, we can write down a basis for T_pM. Let ∂_μ be the tangent vector in T_pM that, with respect to chart (U', ϕ'), is zero in all coordinates, except for the μ'th one, which has value 1. This is in fact the partial derivative in direction μ, or $\partial_\mu(f) = \frac{\partial}{\partial x^\mu} f'(x^1, \ldots, x^n)$, where f' indicates that we are evaluating f with respect to (U', α') [15]. The collection $\{\partial_\mu | \mu \in \{1, \ldots, n\}\}$ is an orthonormal basis in (U', ϕ') and hence a basis in the tangent space T_pM. Using this definition we can write tangent vector v as

$$v = \sum_{\mu=1}^{n} v'^{\mu} \partial_\mu \tag{A.3}$$

Furthermore, we note that if we express $v = \sum_{\boldsymbol{\mu}} v''^{\boldsymbol{\mu}} \partial_{\boldsymbol{\mu}}$ with respect to a different chart (U'', ϕ''), the coefficients change according to

$$v''^{\boldsymbol{\mu}} = \sum_{\mu} \frac{\partial x^{\boldsymbol{\mu}}}{\partial x^{\mu}} v'^{\mu} \tag{A.4}$$

Here x^{μ} is the transition function $\kappa_{\alpha\beta}$ between the charts.
In order to write expressions such as equation A.3 more compactly we use the so-called Einstein summation convention. This convention states that summation is implied if an index is repeated twice where one of them is a subscript and the other a superscript. As such we can reduce equation A.3 to

$$v = v'^{\mu}\partial_{\mu} \tag{A.5}$$

If we extend the visual depiction of a tangent space as in figure A.1 with definition A.1.4 and A.1.5 we get something that looks like figure A.2. Here the white line represents a smooth curve on M and the black arrow emanating from $(0,0,1)$ represents a tangent vector. Furthermore, due to the bijectivity of map A.2 we see that our crude visualisation of the tangent space at $(0,0,1)$ of $\mathbb{S}^2$ as a tangent plane was not as bad as initially thought.

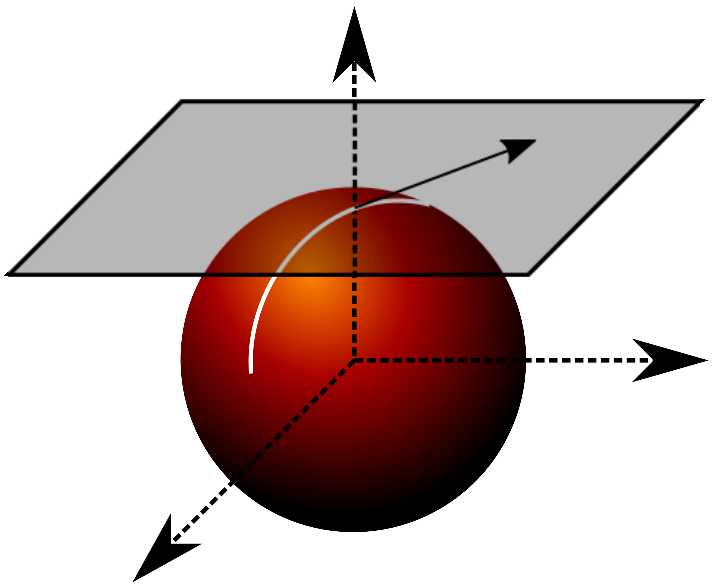

Figure A.2: The tangent space of the 2-sphere at $(0,0,1)$ with a curve on the two-sphere shown in white. The tangent vector at $(0.0,1)$ is given by the black arrow lying in the tangent plane of the 2-sphere at $(0,0,1)$

Using above definition, we can define the tangent bundle of M as the disjoint union over the tangent spaces at $p \in M$.

$$TM := \bigsqcup_{p\in M} T_pM \tag{A.6}$$

The tangent bundle is an extremely useful concept as it allows us to define a vector field on a general manifold.

Definition A.1.6 (Vector field)**.** A vector field v on M is a smooth map

$$v : M \to TM$$

such that $\pi \circ v = \mathrm{Id}_M$, where π is the canonical projection. The space of all vector fields on M is denoted by $\mathrm{Vec}(M)$

The constraint $\pi \circ v = \mathrm{Id}_M$ ensures that the vector v_p assigned to $p \in M$ is an element of the tangent space T_pM. Visually, a vector field assigns a tangent vector such as in figure A.2 to every point on manifold M. Continuing with our example from the two-sphere, a vector field can be visualised as in figure A.3 Now if we interpret the 2-sphere as planet earth the flow in figure A.3 can represent the flow of air around the globe. Do note that this is a simplification of reality as the atmosphere is three-dimensional, whereas the 2-sphere is two-dimensional. In other words, we have squished the atmosphere flat.

A.2. The covariant tensor

Definition A.2.1 (Multilinear map)**.** If V and W are vector spaces, then a multilinear map or k-linear map

$$F : \underbrace{V \times \ldots \times V}_{\text{k times}} \to W \tag{A.7}$$

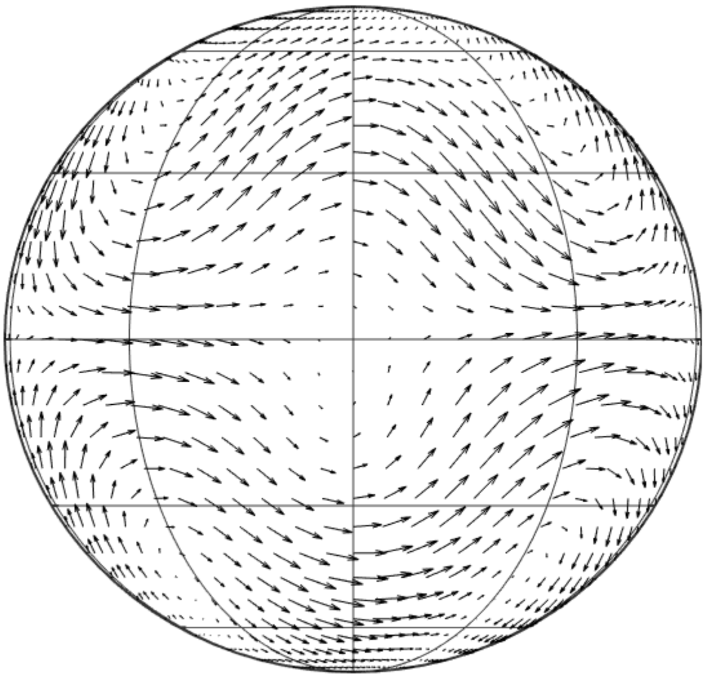

Figure A.3: A vector field on the tangent bundle of the two-sphere. Intuitively this might resemble the flow of air around the globe under the assumption that the atmosphere is two-dimensional or flat.

is a map which is linear in each of its k entries. That is,

$$F(v_1,\dots,\alpha v_i+\beta v_i',\dots,v_k)=\alpha F(v_1,\dots,v_i,\dots,v_k)+\beta F(v_1,\dots,v_i',\dots,v_k)$$

for all $v_i, v_i' \in V$, and for all $\alpha, \beta \in \mathbb{R}$

Remark. If $k=2$, then F is called a bilinear map.

Definition A.2.2 (Covariant Tensor)**.** a covariant tensor of rank k at p $\in M$ is a multilinear map

$$\tau_p : \underbrace{T_pM \times \ldots \times T_pM}_{\text{k times}} \to \mathbb{R} \tag{A.8}$$

The set of all tensors of rank k at $p \in M$ is denoted by $T_k(M)$.

Just as we could write down a basis for tangent space T_pM as done in equation A.5, we can express a covariant tensor $\tau(v_1,\dots,v_k)$ in terms of ∂_μ. For this we rewrite $\tau(\partial_{\mu_1},\dots,\partial_{\mu_k})=\tau_{\mu_1,\dots,\mu_k}$ and employ the linearity of τ in all of its entries.

$$\begin{aligned}\tau(v_1,\dots,v_k)&=\tau(v_1^{\mu_1}\partial_{\mu_1},\dots,v_k^{\mu_k}\partial_{\mu_k})\\&=v_1^{\mu_1}\cdots v_k^{\mu_k}\tau(\partial_{\mu_1},\dots,\partial_{\mu_k})\\&=v_1^{\mu_1}\cdots v_k^{\mu_k}\tau_{\mu_1,\dots,\mu_k}\end{aligned} \tag{A.9}$$

Note the repeated indices. We sum over all μ_i's and can conclude that our tensor is determined by a set of n^k coefficients with respect to coordinate basis ∂_μ.
If we express tensor $\tau(v_1,\dots,v_k)$ with respect to charts (U',ϕ') and (U'',ϕ''), we obtain, by writing $\partial_{\boldsymbol{\mu}}$ as the coordinate basis of τ with respect to (U'',ϕ'')

$$v_1^{\mu_1}\cdots v_k^{\mu_k}\tau_{\mu_1,\dots,\mu_k}=\tau(v_1,\dots,v_k)=v_1^{\boldsymbol{\mu}_1}\cdots v_k^{\boldsymbol{\mu}_k}\tau_{\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_k} \tag{A.10}$$

Using equation A.4 we can rewrite above equation to

$$v_1^{\boldsymbol{\mu}_1}\cdots v_k^{\boldsymbol{\mu}_k}\frac{\partial x^{\mu_1}}{\partial x^{\boldsymbol{\mu}_1}}\cdots\frac{\partial x^{\mu_k}}{\partial x^{\boldsymbol{\mu}_k}}\tau_{\mu_1,\dots,\mu_k}=v_1^{\boldsymbol{\mu}_1}\cdots v_k^{\boldsymbol{\mu}_k}\tau_{\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_k} \tag{A.11}$$

Now since this holds for all tensors we can conclude that

$$\frac{\partial x^{\mu_1}}{\partial x^{\boldsymbol{\mu}_1}} \cdots \frac{\partial x^{\mu_k}}{\partial x^{\boldsymbol{\mu}_k}} \tau_{\mu_1,\dots,\mu_k} = \tau_{\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_k} \tag{A.12}$$

Similarly to the tangent space, we can define the tensor bundle $T_k(M)$ as the disjoint union of all $T_k(M)_p$

$$T_k(M) := \bigsqcup_{p\in M} T_k(M)_p \tag{A.13}$$

We can uniquely describe a covariant tensor in $T_k(M)$ by a set of $n+n^k$ coefficients. The first n coefficients describe the location on manifold M and the n^k coefficients give the entries of the tensor itself as described by equation A.9.
Two important types of tensors which we will be using extensively throughout this text are the alternating tensor of rank 2 and the symmetric tensor of rank 2.

Definition A.2.3 (alternating tensor of rank 2)**.** $\tau(v_1, v_2) : T_p(M) \times T_p(M) \to \mathbb{R}$ is alternating if

$$\tau(v_1, v_2) = -\tau(v_2, v_1)$$

It can be represented by a skew-symmetric matrix $\tau_{\mu\nu} = -\tau_{\nu\mu}$

Definition A.2.4 (symmetric tensor of rank 2)**.** $\tau(v_1, v_2) : T_p(M) \times T_p(M) \to \mathbb{R}$ is symmetric if

$$\tau(v_1, v_2) = \tau(v_2, v_1)$$

It can be represented by a skew-symmetric matrix $\tau_{\mu\nu} = \tau_{\nu\mu}$

Just as we are able to generalize tangent vectors to covariant tensors, we can generalize a vector field to a covariant tensor field

Definition A.2.5 (Covariant tensor field)**.** A covariant tensor field on M is a smooth map $\tau : M \to T_k(M)$ such that $\pi \circ \tau = Id_M$

Here $\pi : T_k(M) \to M$ is the canonical projection. And just as with vector fields the constraint that $\pi \circ \tau = Id_M$ ensures that every point is mapped to a covariant tensor from its own tensor space.

A.3. The contravariant tensor

Definition A.3.1 (covector)**.** A covector at $p \in M$ is a linear map $\alpha_p : T_pM \to \mathbb{R}$

The set of all covectors at $p \in M$ is called the cotangent space and is denoted by T_p^*M. Just like tangent vectors and covariant tensors, we can write down a basis dx^μ for T_p^*M that is defined by

$$dx^\mu(\partial_\nu) = \delta^\mu_\nu \tag{A.14}$$

Theorem A.3.1. The basis of a contravariant tensor changes according to

$$dx^{\boldsymbol{\mu}} = \left(\frac{\partial x^{\boldsymbol{\mu}}}{\partial x^\mu}\right) dx^\mu \tag{A.15}$$

Proof.

$$\begin{aligned}
dx^{\boldsymbol{\mu}}(\partial_{\boldsymbol{\nu}}) &= \delta^{\boldsymbol{\mu}}_{\boldsymbol{\nu}} && \text{(A.16)}\\
&= \frac{\partial x^{\boldsymbol{\mu}}}{\partial x^{\boldsymbol{\nu}}} && \text{(A.17)}\\
&= \frac{\partial x^{\boldsymbol{\mu}}}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\boldsymbol{\nu}}} && \text{(A.18)}\\
&= \frac{\partial x^{\boldsymbol{\mu}}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\boldsymbol{\nu}}}\frac{\partial x^\mu}{\partial x^\nu} && \text{(A.19)}\\
&= \frac{\partial x^{\boldsymbol{\mu}}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\boldsymbol{\nu}}}\delta^\mu_\nu && \text{(A.20)}\\
&= \frac{\partial x^{\boldsymbol{\mu}}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\boldsymbol{\nu}}} dx^\mu(\partial_\nu) && \text{(A.21)}\\
&= \frac{\partial x^{\boldsymbol{\mu}}}{\partial x^\mu} dx^\mu(\partial_\nu \frac{\partial x^\nu}{\partial x^{\boldsymbol{\nu}}}) && \text{(A.22)}\\
&= \frac{\partial x^{\boldsymbol{\mu}}}{\partial x^\mu} dx^\mu(\partial_{\boldsymbol{\nu}}) && \text{(A.23)}
\end{aligned}$$

□

Note that a covector is a covariant tensor of rank 1. The concept of a covector is something that we have all used hundreds of times without even realising it. On the manifold $\mathbb{R}^n$, the input of a covector is a tangent vectors x which can be represented by a column vector. A covector f acting on x can be written down as

$$f(x) = a_1 x_1 + \ldots + a_n x_n$$

And this can be represented as the dot product of two vectors by

$$f(x) = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

We could also have written down f in terms of the coordinate basis as $f = a_\mu dx^\mu$.

Definition A.3.2 (Contravariant tensor)**.** A Contravariant tensor of rank l at $p \in M$ is a multilinear map

$$\tau : \underbrace{T_p^* M \times \ldots \times T_p^* M}_{\text{l times}} \to \mathbb{R} \tag{A.24}$$

Furthermore, we denote the vector space of contravariant tensors of rank l at $p \in M$ by $T^l(M)_p$

Similar to our line of reasoning with A.9, a contravariant tensor is determined by a set of n^l coefficients. If we look at the entire bundle $T^l(M) := \bigsqcup_{p \in M} T^l(M)_p$ we need an additional n coefficients to uniquely determine the tensor. By equation A.15, the multilinearity of τ and defining $\tau(dx^{\nu_1}, \ldots, dx^{\nu_l}) = \tau^{\nu_1,\ldots,\nu_l}$, we can write

$$\frac{\partial x^{\boldsymbol{\nu}_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x^{\boldsymbol{\nu}_l}}{\partial x^{\nu_l}} \tau^{\nu_1,\ldots,\nu_l} = \tau^{\boldsymbol{\nu}_1,\ldots,\boldsymbol{\nu}_l} \tag{A.25}$$

And we conclude that the coordinate transformations are smooth and hence $T^l(M)$ is a smooth manifold. Note that even though we only have upper indices Einstein summation is still implied as $\frac{\partial}{\partial x^\nu} = \partial_\nu$. Again, just like with covariant tensors, we can define a covariant tensor field of rank l.
Note that in above definitions of the covariant and contravariant the first uses lower case indices whereas the latter uses upper case indices. This is done on purpose to differentiate between these types of tensors, even though they have the same symbol τ. Later we will see that it is possible to transform a covariant tensor into a contravariant tensor and vice versa using a so-called Riemannian metric.

A.4. The metric tensor

Definition A.4.1 (Inner product)**.** An inner product is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{L}$, where V is a vector space and $\mathbb{L}$ is a field (Either $\mathbb{R}$ or $\mathbb{C}$), that is

1. Linear in the first argument: $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$
2. Conjugate symmetric: $\langle x, y\rangle = \overline{\langle y, x\rangle}$
3. positive definite: $\langle x, x\rangle > 0$

Since we will only be dealing with real vector spaces and inner products, we can relax condition two, the conjugate symmetry, to $\langle x, y\rangle = \langle y, x\rangle$.

Definition A.4.2 (Riemannian metric)**.** A Riemannian metric g on M is a covariant tensor field of rank 2 such that for every point $p \in M$, the bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ is an inner product.

In local coordinates with $v = v^\mu \partial_\mu$ and $w = w^\nu \partial_\nu$ we can write the Riemannian metric acting on v and w as

$$g_p(v, w) = g_p(v^\mu \partial_\mu, w^\nu \partial_\nu) = v^\mu w^\nu g_p(\partial_\mu, \partial_\nu) = v^\mu w^\nu g_{\mu\nu}(p) \tag{A.26}$$

Here we have defined the matrix $g_{\mu\nu}(p) = g_p(\partial_\mu, \partial_\nu)$. Now, since g is an inner product it must be positive definite ($g_p(v,w) = g_{\mu\nu}(p)v^\mu w^\nu > 0$). Transforming from one coordinate system to the other yields the known transformation formula for rank 2 covariant tensors as given in equation A.12.
As an example, the most well known Riemannian metric is the euclidean metric with respect to the Cartesian coordinates defined as

$$\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{A.27}$$

Raising and lowering indices We can use the Riemannian metric to 'raise' or 'lower' the indices of a tensor. In other words, we can transform a covariant tensor to a contravariant tensor and vice versa. This operation makes sense as the riemannian metric g_p induces an isomorphism between T_pM and T_p^*M [15]. As such there exists a one to one mapping from the tangent space of $p \in M$ to its cotangent space and vice versa and we can indeed identify a covariant tensor with a contravariant counterpart. Given a tangent vector $v \in T_pM$, we identify its cotangent vector under g_p by letting g_p act on v and $w \in T_pM$. that is, $v^\flat(w) = g_p(v,w)$. When working in local coordinates we can explicitly find the coefficients of $v^\flat$ by $v^\flat_\nu = v^\flat(\partial_\nu) = g(v, \partial_\nu)$. As written above, contra- and covariant tensors can be distinguished by their upper and lower indices respectively. Hence we can leave out the $\flat$ index and write, using equation A.26 and the fact that ∂_ν is part of an orthonormal basis,

$$v_\nu = g_{\mu\nu} v^\mu \tag{A.28}$$

Note that by applying the Riemannian metric we have converted a contravariant tensor into a covariant tensor and thus transformed the upper index into a lower index. Hence, we 'lower the index'.
Similar to lowering the index, we can raise the index by making use of the inverse of $g_{\mu\nu}$, denoted by $g^{\mu\nu}$. This inverse matrix has the property that $g_{\mu\nu}g^{\nu\eta} = \delta^\eta_\mu$. The operation itself in local coordinates can be written down as $v^\mu = g^{\mu\nu} g_\nu$. Formally, for a covector $v^* \in T_p^*M$ we identify its corresponding tangent vector $v \in T_pM$ under g by the property that $v^*(w) = g_p(v,w)$ for all $w \in T_pM$.

Definition A.4.3 (Pseudo-Riemannian metric)**.** A Pseudo-Riemannian metric is a Riemannian metric with the relaxation that the biliniar form $g_p : T_pM \times T_pM \to \mathbb{R}$ need not be an inner product but only must be non-singular.

Since g does not have to be an inner product anymore, we can relax the property of g being positive definite and hence g can yield negative scalars. The constraint that g_p must be non-singular indicates that the determinant of the matrix representation of g_p can not be equal to 0. As we see in chapter 2, the Minkowski metric is a Pseudo-Riemannian metric.

B

The Geodesic equation and Christoffel symbols

In classical mechanics the equations of motion are governed by Newton law's

1. An object will not change its motion unless a force acts on it
2. The force of an object is equal to its mass times its acceleration
3. when two objects interact, they apply forces to each other of equal magnitude and opposite direction

Put together, we can determine the direction a particle will flow in while it interacts with other particles. The first law tells us that if an object experiences no force whatsoever, it will continue to move as it did. An object standing still will remain doing so, whereas a moving object will keep on moving in a straight line. In general relativity, the underlying space need not be flat and thus the notion of a straight line becomes problematic. Instead of talking about straight lines, we look at the shortest path length. In three-dimensional and even 4-dimensional Minkowski geometry, the shortest paths are equal to straight lines, but for general manifolds this need not be true. For the remainder of the discussion we will limit ourselves to 4-dimensional spacetime. Hence we have 4 coordinates, one time coordinate and three spatial coordinates. Their orientation with respect to one another is described by the metric tensor and this will allow us to determine the length of a curve through spacetime.

Definition B.0.1 (Length of a curve through spacetime)**.** Given two points $\boldsymbol{x}_1 = (t_1, x_1, y_1, z_1)$ and $\boldsymbol{x}_2 = (t_2, x_2, y_2, z_2)$, and a curve $\gamma(s)$ connecting these two points such that $\gamma(s_1) = \boldsymbol{x}_1$ and $\gamma(s_2) = \boldsymbol{x}_2$, the length of the curve γ is defined as

$$S(\gamma) = \int_{s_1}^{s_2} \sqrt{-g_{\mu\nu}\dot{\gamma}^{\nu}\dot{\gamma}^{\mu}}ds \tag{B.1}$$

Note that the minus sign under the squareroot implies that the curve is timelike, just as we have seen in chapter 2. It implies that a particle could travel along this trajectory with a speed that is lower than the speed of light. Above arc length can be minimized using techniques from variational calculus. The proof will not be stated here, but the resulting geodesic equations give us the equations of motions, necessary to determine the trajectory of a particle in spacetime. In four dimensional spacetime the geodesic equations are [13]

$$\frac{d^2x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\beta\gamma}\frac{dx^{\beta}}{d\tau}\frac{dx^{\gamma}}{d\tau} = 0 \tag{B.2}$$

Here the Christoffel symbol $\Gamma^{\alpha}_{\beta\gamma}$ is

$$\Gamma^{\alpha}_{\beta\gamma} = g^{\alpha\nu}\frac{1}{2}(\frac{\partial g_{\nu\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\nu\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\nu}}) \tag{B.3}$$

As an example, consider Minkowski spacetime with metric

$$g^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{B.4}$$

Since the metric is constant, all Christoffel symbols are zero and we are left with

$$\frac{d^2ct}{d\tau^2} = 0 \tag{B.5}$$

$$\frac{d^2x}{d\tau^2} = 0 \tag{B.6}$$

$$\frac{d^2y}{d\tau^2} = 0 \tag{B.7}$$

$$\frac{d^2z}{d\tau^2} = 0 \tag{B.8}$$

with solutions

$$ct(\tau) = ct(o) + p_0\tau \tag{B.9}$$

$$x(\tau) = x(o) + p_1\tau \tag{B.10}$$

$$y(\tau) = y(o) + p_2\tau \tag{B.11}$$

$$z(\tau) = z(o) + p_3\tau \tag{B.12}$$

And we have recovered the straight lines. Since we require $g_{\mu\nu}\dot{\gamma}^\nu\dot{\gamma}^\mu < 0$, we find that $-p_0^2 + p_1^2 + p_2^2 + p_3^2 < 0$ and hence we have a timelike trajectory.

B.1. Christoffel symbols of a diagonal metric

Take the diagonal metric defined by

$$ds^2 = A(dx^1)^2 + B(dx^2)^2 + C(dx^3)^2 + D(dx^4)^2 \tag{B.13}$$

Here dx^1, dx^2, dx^3 and dx^4 are arbitrary coordinates and A, B, C and D are arbitrary functions that have arbitrary coordinate dependence. Furthermore, defined are

$$\bar{P} \equiv \frac{1}{2P} \qquad P_i \equiv \frac{\partial P}{\partial x^i}$$

$$P_{ij} \equiv \frac{\partial^2 P}{\partial x^i \partial x^j} \qquad \text{etc.}$$

Then the christoffel symbols are given on the next page [21]

$$\Gamma^1_{11} = \bar{A}A_1, \qquad \Gamma^1_{12} = \Gamma^1_{21} = \bar{A}A_2, \qquad \Gamma^1_{13} = \Gamma^1_{31} = \bar{A}A_3, \qquad \Gamma^1_{14} = \Gamma^1_{41} = \bar{A}A_4$$

$$\Gamma^1_{22} = -\bar{A}B_1, \qquad \Gamma^1_{33} = -\bar{A}C_1, \qquad \Gamma^1_{44} = -\bar{A}D_1, \qquad \text{others} = 0$$

$$\Gamma^2_{21} = \Gamma^2_{12} = \bar{B}B_1, \qquad \Gamma^2_{22} = \bar{B}B_2, \qquad \Gamma^2_{23} = \Gamma^2_{32} = \bar{B}B_3, \qquad \Gamma^2_{24} = \Gamma^2_{42} = \bar{B}B_4$$

$$\Gamma^2_{11} = -\bar{B}A_2, \qquad \Gamma^2_{33} = -\bar{B}C_2, \qquad \Gamma^2_{44} = -\bar{B}D_2, \qquad \text{others} = 0$$

$$\Gamma^3_{31} = \Gamma^3_{13} = \bar{C}C_1, \qquad \Gamma^3_{32} = \Gamma^3_{23} = \bar{C}C_2, \qquad \Gamma^3_{33} = \bar{C}C_3, \qquad \Gamma^3_{34} = \Gamma^3_{43} = \bar{C}C_4$$

$$\Gamma^3_{11} = -\bar{C}A_3, \qquad \Gamma^3_{22} = -\bar{C}B_3, \qquad \Gamma^3_{44} = -\bar{C}D_3, \qquad \text{others} = 0$$

$$\Gamma^4_{41} = \Gamma^4_{14} = \bar{D}D_1, \qquad \Gamma^4_{42} = \Gamma^4_{24} = \bar{D}D_2, \qquad \Gamma^4_{43} = \Gamma^4_{34} = \bar{D}D_3, \qquad \Gamma^4_{44} = \bar{D}D_4$$

$$\Gamma^4_{11} = -\bar{D}A_4, \qquad \Gamma^4_{22} = -\bar{D}B_4, \qquad \Gamma^4_{33} = -\bar{D}C_3, \qquad \text{others} = 0$$

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